

# ON THE CONVEXITY OF LEVEL-SETS OF PROBABILITY FUNCTIONS

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**Abstract.** In decision-making problems under uncertainty, probabilistic constraints are a valuable tool to express safety of decisions. They result from taking the probability measure of a given set of random inequalities depending on the decision vector. Even if the original set of inequalities is convex, this favourable property is not immediately transferred to the probabilistically constrained feasible set and may in particular depend on the chosen safety level. In this paper, we provide results guaranteeing the convexity of feasible sets to probabilistic constraints when the safety level is greater than a computable threshold. Our results extend all the existing ones and also cover the case where decision vectors belong to Banach spaces. The key idea in our approach is to reveal the level of underlying convexity in the nominal problem data (e.g., concavity of the probability function) by auxiliary transforming functions. We provide several examples illustrating our theoretical developments.

**Key words.** Probability constraints, Convex analysis, Elliptical distributions, Stochastic optimization

**AMS subject classifications.** 90C15, 90C25, 65K10

## 1. Introduction.

**1.1. Probability constraints and eventual convexity.** We consider a probabilistic constraint built up from the following ingredients: a map  $g : X \times \mathbb{R}^m \rightarrow \mathbb{R}^k$ , where  $X$  is a (reflexive) Banach space, a random vector  $\xi \in \mathbb{R}^m$  defined on an appropriate probability space, and a user-defined safety level  $p \in [0, 1]$ . The probabilistic constraint then reads:

$$\varphi(x) := \mathbb{P}[g(x, \xi) \leq 0] \geq p, \quad (1.1)$$

where  $\varphi : X \rightarrow [0, 1]$  is the associated probability function. The interpretation of (1.1) is simple: one requires the decision  $x$  to be such that the random inequality system  $g(x, \xi) \leq 0$  holds with probability at least  $p$ . Such constraints (also called chance-constraint) often appear in decision-making problems under uncertainty; for the theory and applications of chance-constraints optimization, we refer to [8, 20, 22, 51, 65] and references therein.

In this paper we focus on the “convexity of the probabilistic constraint” (1.1), i.e., the convexity of the set of feasible solutions defined by

$$M(p) := \{x \in X : \mathbb{P}[g(x, \xi) \leq 0] \geq p\}. \quad (1.2)$$

Understanding when  $M(p)$  is a convex set is important for the point of view of optimization, to guarantee that local solutions are also globally optimal and to use numerical solution methods that exploit this convexity (we review the most popular methods in section 1.3). A first result of the convexity of  $M(p)$  follows from Prékopa’s celebrated log-concavity theorem (see [18, Proposition 4] for its infinite dimensional version and [12] for generalizations): the convexity of  $M(p)$  is guaranteed for all  $p \in [0, 1]$ , when  $-g$  is jointly quasi-concave in both arguments and  $\xi$  an appropriate random vector. However, joint-quasi-concavity of  $-g$  is rather “exceptional” and fails in many basic situations. For example, when  $g(x, \xi) = x^\top \xi$  and  $\xi$  is multi-variate Gaussian, it is well known that  $M(p)$  is convex only whenever  $p \geq \frac{1}{2}$  (see e.g., [28]). In this example and many others, we thus observe that if the convexity of  $M(p)$  does not hold for all  $p$ , there still exists a (computable) threshold  $p^* \in [0, 1]$  such that the set  $M(p)$  is convex for all  $p \geq p^*$ . This property is called *eventual convexity* as introduced by [24] (which studies the case where  $g$  is separable and  $\xi$  has independent components). Eventual convexity results are further generalized in [25] by allowing for the components of  $\xi$  to be coupled through a copulæ dependency structure. These results are refined, by allowing for more copulæ and with sharper bounds for  $p^*$  in [58], and extended to all Archimedian copulæ in [61], where also an appropriate solution algorithm is provided. When the mapping  $g$  is non-separable, eventual convexity results are provided in [66] for the special case where  $\xi$  is elliptically symmetrically distributed. Here we will simplify, clarify and extend these results.

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**1.2. Ideas, contributions, and outline of this paper.** In this paper, we build on this line of research about establishing convexity of superlevel-sets of probability functions (1.2) for  $p$  larger than a threshold. We show that a notion of generalized concavity naturally appears in this framework, and allows us to reveal the level of hidden convexity of the data. We formalize a way to analyze separately the convexity inherent to the randomness and the one associated with the optimization model structure. Roughly speaking, our approach is the following. In various contexts, the probability function involves a composition of two functions  $F \circ Q$ , with  $F: \mathbb{R} \rightarrow [0, 1]$  carrying the randomness of the problem and  $Q: \mathbb{R}^n \rightarrow \mathbb{R}$  given by the optimization model. We split the problem of establishing concavity (or at least quasi-concavity) of the composition  $F \circ Q$  by finding an adequate function  $G$  (the inverse of which is denoted  $G^{-1}$ ) to write

$$F \circ Q = F \circ G^{-1} \circ G \circ Q \quad (1.3)$$

such that  $F \circ G^{-1}$  and  $G \circ Q$  satisfy appropriate convexity properties. Thus this approach naturally raises interest in the concavity of function  $G \circ Q$ , which is a “transformable concavity”, formalized by the notion called tranconcavity or  $G$ -concavity by [55] and [2]. Similarly, we will briefly study concavity of terms  $F \circ G^{-1}$  and introduce the counterpart notion that we will call concavity- $G^{-1}$ .

Beyond this intuition, the analysis of the convexity of chance-constrained sets is not trivial:  $F$ , as a distribution function, cannot be concave and one has to be careful with working on appropriate smaller subsets. The size of those subsets turns out to be directly related to the level of probability beyond which convexity is guaranteed. Arguments of this type are implicitly and partially used in [24, 58, 61, 66]. Here we highlight this approach, exploit it in its full generality, and provide a set of tools to apply it in practice. We apply this set of tools to get eventual convexity statements in two general contexts

(A) when  $\xi$  is specific (elliptically distributed) and  $g$  is general (with light geometrical assumptions), or

(B) when  $\xi$  is general (with known associated copula) and  $g$  is specific (of the form  $g(x, \xi) = \xi - h(x)$ )

In these two cases, we prove the existence of the threshold  $p^*$  such that  $M(p)$  is convex for all  $p \geq p^*$ . Beyond the theoretical contribution regarding the existence of  $p^*$ , we also provide a concrete way of numerically evaluating  $p^*$  from the nominal data. Our results are illustrated through various examples (not captured by existing results) of eventual convexity with a specified threshold  $p^*$ .

The contributions of this work are thus the following ones. Our main contribution is to identify the interplay of generalized concavity with the functions used to model the constraints and the uncertainty in chance-constrained optimization problems. This clarification would allow practitioners to refine their models of nonlinearity and readily swapping certain uncertainty distributions by others sharing similar generalized concavity properties. Since we care about practical use of our results, this introduction is meant to be an accessible overview of the state-of-the-art on eventual convexity and its use in practice. We also provide throughout the paper many examples illustrating our results. Finally we specifically list our main technical contributions:

- We slightly extend the existing notion of  $G$ -concavity (by allowing for decreasing functions  $G$ ) and introduce the new notion of concavity- $G^{-1}$  that goes with it. We also extend a useful result of [24] on one-dimensional distribution functions and give necessary and sufficient conditions under which such functions can be composed with monotonous maps to make them concave.
- For the context (A), our results clarify and extend those of [66] allowing us to treat new situations: we use here various forms of  $G$ -concavity and thus can cover a wider range of non-linear mappings  $g$ .
- For the context (B), we first refine the results of [69] by providing a better threshold. More importantly, our results cover new situations as they tackle the most general case with nonlinear mappings and copulae, while [58, 61] restricts to  $\alpha$ -concavity (a special  $G$ ) and [69] restricts to independent copulae.
- We also extend all these previous results (that consider finite dimensional decision vectors  $x$ ), by analysing and stating our results in Banach spaces. This opens the door to cover recent applications in PDE-constrained optimization (e.g., [18]).

The paper is organized as follows. After section 1.3, which will put this work into a broader practical perspective, our development starts in Section 2, which provides a careful account of useful and used notions of generalized concavity together with calculus rules. Section 3 then provides general eventual convexity statements in the two contexts (A) and (B) in Sections 3.1 and 3.2 respectively. Finally Section 4 is devoted to providing examples covered in our extended framework.

**1.3. Discussion on applicability: practicality & open issues..** Although this work is theoretical, it can be inserted in the bigger picture of solving chance-constrained optimization problems. Of course, a priori, evaluating probability functions is computationally demanding (see e.g., the discussion in [47]), especially if the random vector is highly dimensional. Still recent results (e.g., [6]) indicate that random vectors with dimensions in their hundreds, i.e., practically relevant sizes, can be handled with CPU times hovering roughly around a minute. Those CPU times relate to solving a non-convex optimization problem involving a probability function, evaluated repeatedly. Although this does not alleviate the theoretical or algorithmic difficulties, it does show that, by exploiting the structure present in applications, one can solve applications of relevant size. The existing numerical solution methods use sample-based approximations of the probabilistic constraint or treat probability functions (or a surrogate) by non-linear optimization techniques; we briefly review the main methods here. Notice that many of these methods rely on some convexity properties of the chance-constrained problems, bringing interest to the results of this paper.

Popular numerical methods for dealing with probabilistic constraints are sample-based approximations, e.g., [38–42, 48] with various strengthening procedures, e.g., [32, 68] or investigation of convexification procedures [1]. We can also mention boolean approaches, e.g., [30, 31, 35, 36],  $p$ -efficient point based concepts, e.g., [9–11, 37, 60], robust optimization [4], penalty approach [14], scenario approximation [7, 52], convex approximation [47], or yet other approximations [19, 26]. Aside from this rich literature, the non-linear constraint (1.1) can also be dealt with directly as such, from the study of (generalized) differentiability of probability functions and the development of readily implementable formulæ for gradients. Such formulæ can be further improved by using well known “variance reduction” techniques, such as Quasi-Monte Carlo methods (e.g., [5]) or importance sampling (e.g., [3]). For further insights on differentiability, we refer to e.g., [23, 29, 43, 53, 56, 57, 64]. Nonlinear programming methods using these properties include sequential quadratic programming [6] and the promising bundle methods [61, 67].

Practical probabilistic constrained problems also involve several other constraints, that can be represented as an abstract subset  $S \subset X$ . Important questions concern, in fact, the constrained set  $M(p) \cap S$ , for which the results presented in this paper might be used. As a brief observation we do write  $p$  and not  $p^*$ , since  $p$  is the user chosen safety level and thus what is practically relevant.

- Is  $M(p) \cap S$  convex? Convexity of  $S$  is achieved in many practical cases: in a significant share of applications  $S$  is polyhedral, or easily seen to be convex. The difficulty in establishing convexity of  $M(p) \cap S$  therefore lies in checking whether  $M(p)$  is a convex set, which is the aim of this paper. As a sufficient condition, the user would check  $p \geq p^*$ , when applicable, in order to have a global guarantee on computed solutions. Since, occasionally,  $p^*$  may depend adversely on random vector dimension, if the test fails, this does not necessarily imply that  $M(p) \cap S$  is not convex. The results of this paper rather indicate that a computed “solution” should not necessarily be taken as a global solution, because convexity of  $M(p) \cap S$  is no longer guaranteed. Such an information is still useful for the user, who may decide to invest additional effort in running the local optimization solver with multiple starting points, or calling another (more expensive) global optimization solver.
- Is  $M(p) \cap S$  non-empty? The safety level  $p$  is chosen by the user, who, as a modeler, is responsible for ensuring that a well-posed model is formulated. In practice, we can expect a reasonable convex optimization solver to return an infeasibility flag when the set  $M(p) \cap S$  is empty (in the case that  $M(p) \cap S$  is convex as attested by the previous point). In such a case, the user can examine his data, and subsequently formulate a better model. Answering the feasibility question without convexity is of course an entirely different matter, and such a theoretically and algorithmically difficult problem goes largely beyond the scope and the setting of this paper. Below we mention some relevant heuristic procedures that have worked well in our experience.

The feasibility question regarding probabilistic restrictions is related to the question of maximizing the probability function over  $X$  or  $S$ , which recently has received special attention; see e.g., [15, 16, 44]. Indeed, if the maximal probability thus found is greater than or equal to  $p$ , then  $M(p) \cap S$  is ensured to be a nonempty set. Of course, finding the global solution of this probability maximization problem is a hard task in general, because the probability function need not be concave. Still, there are situations where this (global) maximization can be carried out in polynomial time [45]!

Heuristically, feasibility can also be addressed by considering a sample based variant of probabilistic maximization problem, with few samples, i.e.,

$$\begin{aligned} \min_{z_1, \dots, z_N} \quad & \sum_{i=1}^N z_i \\ \text{s.t.} \quad & g(x, \xi_i) \leq M z_i \\ & x \in S, z_i \in \{0, 1\}, \end{aligned}$$

where  $M$  is an appropriate “big- $M$ ” constant,  $S$  is the deterministic constraints set, and  $\xi_1, \dots, \xi_N$  are i.i.d. samples of  $\xi$ . The last problem can in principle be solved with fairly few samples (small  $N$ ) and with low accuracy (e.g., 10% MIP-gap) to produce  $\bar{x}$ . An a posteriori evaluation of the probability function can then assert feasibility of  $\bar{x}$  for the true probabilistic constraint. Indeed, when  $g$  is convex in  $x$  and  $S$  is convex, one can solve the last program with the methodology laid out in [63]. The sample can also be exploited in a “scenario approach” like problem (the asymptotics with respect to  $N$  are well studied in for instance [52]). Finally, the approach (maximization of copula structured probability not requiring convexity) in [62] can also be employed. It consists of minimizing a lower- $C^2$  function (requiring easily verified differentiability, whenever the copula is Archimedian) with tools from nonsmooth optimization.

**2. Generalized concavity, propagation of concavity, and cumulative distribution functions.** In this section, we gather the tools on generalized concavity that we will use in next sections to reveal the underlying concavity of nominal data in probabilistic constraints. Section 2.1 briefly reviews the definitions and useful properties of  $G$ -concavity (also called transconcavity (see [2])) and Section 2.2 introduces the right counterpart of concavity- $G^{-1}$ . We provide new technical lemmas, including a characterization of generalized concavity of cumulative distribution functions.

**2.1. Discussions on  $G$ -concavity.** We start by recalling the notion of  $G$ -concavity introduced by [55] and presented in the book [2] under the name transconcavity. We just add here the possibility of  $G$  being strictly decreasing, which will turn out to be useful in our context.

**DEFINITION 2.1.** *Let  $X$  be a Banach space and  $C$  be a convex subset of  $X$ . We say that a function  $f: C \rightarrow \mathbb{R}$  is  $G$ -concave if there exists a continuous and strictly monotone\* function  $G: f(C) \rightarrow \mathbb{R}$  such that*

$$f(\lambda x + (1 - \lambda)y) \geq G^{-1}(\lambda G \circ f(x) + (1 - \lambda)G \circ f(y))$$

*holds for all  $x, y \in C$  and  $\lambda \in [0, 1]$ .*

Note that when  $G$  is increasing,  $G$ -concavity of  $f$  is just the concavity of the map  $G \circ f$ . When  $G$  is decreasing,  $G$ -concavity of  $f$  is simply the convexity of  $G \circ f$ . A given function  $f$  can be “ $G$ -concave” for several different mappings  $G$ . It will be convenient however to pin down a specific choice and subsequently speak of  $G$ -concavity of  $f$  for such a specific choice. The naming “transconcavity” would then refer to an unspecified, yet implicitly assumed to exist, mapping  $G$  for which  $f$  is  $G$ -concave.

**Example 2.2 (Special family).** *A particularly well studied set of choices for  $G$  is that of the family*

$$G_\alpha: t \mapsto t^\alpha \quad \text{for } \alpha \in \mathbb{R} \setminus \{0\} \quad \text{and} \quad G_0: t \mapsto \ln(t). \quad (2.1)$$

*This family has several properties that help to measure a “level of generalized concavity” of a function  $f$ , as used below in the definition of  $\alpha$ -concavity.*

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\*In this paper, we note the inverse of a map  $G$  by  $G^{-1}$ , and the division by a map, whenever well defined, by  $\frac{1}{G}$ .

We introduce the following mapping  $m_\alpha : \mathbb{R}_+ \times \mathbb{R}_+ \times [0, 1] \rightarrow \mathbb{R}$  (for a given  $\alpha \in [-\infty, \infty)$ ) defined as follows:

$$\text{if } ab = 0 \text{ and } \alpha \leq 0, \quad m_\alpha(a, b, \lambda) = 0 \quad (2.2)$$

else, for  $\lambda \in [0, 1]$ , we let:

$$m_\alpha(a, b, \lambda) = \begin{cases} a^\lambda b^{1-\lambda} & \text{if } \alpha = 0 \\ \min\{a, b\} & \text{if } \alpha = -\infty \\ (\lambda a^\alpha + (1-\lambda)b^\alpha)^{\frac{1}{\alpha}} & \text{else} \end{cases} \quad (2.3)$$

This enables us to define the known notion of  $\alpha$ -concavity as a “particular case” of  $G$ -concavity.

**DEFINITION 2.3** ( $\alpha$ -concave function). *Let  $X$  be a Banach space and  $C$  be a convex subset of  $X$ . We say that a function  $f : C \rightarrow \mathbb{R}_+$  is  $\alpha$ -concave if*

$$f(\lambda x + (1-\lambda)y) \geq m_\alpha(f(x), f(y), \lambda), \quad (2.4)$$

for all  $x, y \in C$  and  $\lambda \in [0, 1]$ .

Observe that for  $\alpha \neq 0$ ,  $\alpha$ -concavity of  $f$  is indeed equivalent with  $G_\alpha$ -concavity of  $f$  (in the sense of Definition 2.1 with  $G_\alpha$  of (2.1)). Notice that 1-concavity coincides with the usual notion of concavity.

We also note that our definition of the mapping  $m_\alpha$  differs slightly of that found in [8, Def 4.7], in so much that we have appended the condition  $\alpha \leq 0$  to condition (2.2). The reason for this is that otherwise the definition does not match up with what is expected whenever  $a = 0$  or  $b = 0$  and  $\alpha > 0$ . In particular consider  $\alpha = 1$  and the usual definition of concavity for a function  $f$ , two points  $x, y \in C$  with  $f(y) = 0$  for instance. Since our definition slightly differs from the classical one, we provide the proof of the following technical lemma used to establish the hierarchy of  $\alpha$ -concavity.

**LEMMA 2.4.** *Let  $a, b \in \mathbb{R}_+$ ,  $\lambda \in [0, 1]$  be given and fixed. Then for  $\alpha, \beta \in [-\infty, \infty)$ ,  $m_\beta(a, b, \lambda) \leq m_\alpha(a, b, \lambda)$  holds true when  $\beta \leq \alpha$ . Moreover the map  $\alpha \mapsto m_\alpha(a, b, \lambda)$  is continuous.*

*Proof.* Let  $a, b, \lambda$  be as in the statement. Since  $\lambda$  is arbitrary and  $m_\alpha(a, b, \lambda) = m_\alpha(b, a, 1-\lambda)$ , we may without loss of generality assume  $a \leq b$ . Furthermore, let also  $\beta \leq \alpha$  be given but fixed. We will proceed by a case distinction.

**First case:** If  $\alpha \geq \beta > 0$  or  $\alpha > 0 > \beta$  then the mapping  $t \mapsto t^{\frac{\alpha}{\beta}}$  is convex on  $\mathbb{R}_+$ . So, we have

$$\lambda u^{\frac{\alpha}{\beta}} + (1-\lambda)v^{\frac{\alpha}{\beta}} \geq (\lambda u + (1-\lambda)v)^{\frac{\alpha}{\beta}}, \quad \text{for } u, v \geq 0$$

which since  $\alpha > 0$ , and hence  $t \mapsto t^{\frac{1}{\alpha}}$  strictly increasing on  $\mathbb{R}_+$ , is equivalent to  $(\lambda u^{\frac{\alpha}{\beta}} + (1-\lambda)v^{\frac{\alpha}{\beta}})^{\frac{1}{\alpha}} \geq (\lambda u + (1-\lambda)v)^{\frac{1}{\beta}}$ . The desired result follows by substituting  $u = a^\beta$  and  $v = b^\beta$ .

**Second case:** If  $0 > \alpha \geq \beta$ , then we have  $-\beta \geq -\alpha > 0$  and for any  $u, v \geq 0$ , we can apply the previous case to obtain the inequality:

$$(\lambda u^{-\beta} + (1-\lambda)v^{-\beta})^{\frac{-1}{\beta}} \geq (\lambda u^{-\alpha} + (1-\lambda)v^{-\alpha})^{\frac{-1}{\alpha}}.$$

The latter is equivalent with

$$(\lambda(\frac{1}{u})^\beta + (1-\lambda)(\frac{1}{v})^\beta)^{\frac{1}{\beta}} \leq (\lambda(\frac{1}{u})^\alpha + (1-\lambda)(\frac{1}{v})^\alpha)^{\frac{1}{\alpha}},$$

provided that  $u, v > 0$  hold. Assuming  $a > 0$  (and hence  $b > 0$ ), we may substitute  $u = \frac{1}{a}$  and  $v = \frac{1}{b}$  to obtain the desired inequality. When  $ab = 0$ , since both  $\alpha$  and  $\beta \leq 0$ , the desired inequality holds.

**Third case:** To treat a case where  $\alpha = 0$  or  $\beta = 0$ , we first establish continuity of  $\alpha \mapsto m_\alpha$  around 0. To this end, consider the following Taylor expansions:

$$\begin{aligned} a^\alpha &= e^{\alpha \ln(a)} = 1 + \alpha \ln(a) + o(\alpha) \\ \frac{1}{\alpha} \ln(\lambda a^\alpha + (1-\lambda)b^\alpha) &= \frac{1}{\alpha} \ln(1 + \alpha(\lambda \ln(a) + (1-\lambda)\ln(b)) + o(\alpha)) = \lambda \ln(a) + (1-\lambda)\ln(b) + o(1) \end{aligned}$$



Consequently, we get at the limit when  $\alpha \rightarrow 0$

$$\exp\left(\frac{1}{\alpha} \ln(\lambda a^\alpha + (1-\lambda)b^\alpha)\right) = \exp(\lambda \ln(a) + (1-\lambda) \ln(b) + o(1)) \rightarrow a^\lambda b^{1-\lambda} = m_0(a, b, \lambda),$$

Now assume  $\beta = 0$  and  $\alpha > 0$ . If  $m_0(a, b, \lambda) \leq m_\alpha(a, b, \lambda)$  were not to hold true, then it would follow that  $m_\alpha(a, b, \lambda) < m_0(a, b, \lambda)$ . We may pick a sequence  $\alpha_k \downarrow 0$  and for  $k$  large enough it holds  $0 < \alpha_k \leq \alpha$ . By the already established order, we have  $m_{\alpha_k}(a, b, \lambda) \leq m_\alpha(a, b, \lambda) < m_0(a, b, \lambda)$ . But this contradicts the just established continuity of  $\alpha \mapsto m_\alpha(a, b, \lambda)$  near 0. The situation  $\alpha = 0$  can be established along similar lines of argument.

**Fourth case:** The situation wherein  $\beta < 0 < \alpha$  follows by invoking the third case twice, since indeed  $m_\beta(a, b, \lambda) \leq m_0(a, b, \lambda) \leq m_\alpha(a, b, \lambda)$ .

**Last case:** When  $\beta = -\infty$ . The inequality  $m_{-\infty}(a, b, \lambda) \leq m_\alpha(a, b, \lambda)$  holds trivially whenever  $ab = 0$ . By combining the previous cases we may assume  $\alpha < 0$  as well as  $a, b > 0$ . We observe that  $a, b \geq m_{-\infty}(a, b, \lambda) = \min\{a, b\}$  and since  $\alpha < 0$ , it holds  $a^\alpha, b^\alpha \leq m_{-\infty}(a, b, \lambda)^\alpha$ . Consequently too,

$$\lambda a^\alpha + (1-\lambda)b^\alpha \leq m_{-\infty}(a, b, \lambda)^\alpha.$$

Since  $t \mapsto t^{\frac{1}{\alpha}}$  is strictly decreasing, we get  $m_{-\infty}(a, b, \lambda) \leq m_\alpha(a, b, \lambda)$ .

We finish by proving the continuity. Since both terms  $\lambda a^\alpha$  and  $(1-\lambda)b^\alpha$  are nonnegative, we have that:

$$\max(\lambda a^\alpha, (1-\lambda)b^\alpha) \leq \lambda a^\alpha + (1-\lambda)b^\alpha,$$

and consequently

$$m_{-\infty}(a, b, \lambda) \leq (\lambda a^\alpha + (1-\lambda)b^\alpha)^{\frac{1}{\alpha}} \leq \max(\lambda a^\alpha, (1-\lambda)b^\alpha)^{\frac{1}{\alpha}} = [\max(\lambda^{\frac{1}{1-\alpha}} \frac{1}{a}, (1-\lambda)^{\frac{1}{1-\alpha}} \frac{1}{b})]^{-1}. \quad (2.5)$$

Hence by passing to the limit:

$$\lim_{\alpha \rightarrow -\infty} [\max(\lambda^{\frac{1}{1-\alpha}} \frac{1}{a}, (1-\lambda)^{\frac{1}{1-\alpha}} \frac{1}{b})]^{-1} = [\max(\frac{1}{a}, \frac{1}{b})]^{-1} = \min(a, b),$$

This gives the continuity at  $-\infty$ .  $\square$

The above property of the map  $m_\alpha$  allows us to establish an entire hierarchy of “concavity” immediately, as formalized by the next corollary.

**COROLLARY 2.5 (Hierarchy of  $\alpha$ -concavity).** *Let  $X$  be a Banach space and  $C$  be a convex subset of  $X$ . Let the map  $f : C \rightarrow \mathbb{R}_+$ , together with  $\alpha, \beta \in [-\infty, \infty)$ , be given. If  $f$  is  $\alpha$ -concave, it is also  $\beta$ -concave when  $\alpha \geq \beta$ . In particular  $f$  is quasi-concave.*

The family of mappings  $\{G_\alpha\}_\alpha$  of (2.1) allows us to distinguish the level of generalized concavity of a function  $f$  ranging from quasi-concavity ( $\alpha = -\infty$ ) to classic concavity ( $\alpha = 1$ ). Intuitively, the greater  $\alpha$  is, the “more concave”,  $f$  will be. All the practical examples in this paper will use these functions to quantify and extract underlying convexity. Let us mention though that this family does not capture completely the subtle notion of transconcavity (see Example 2.6 below) and that alternative families of functions could be considered, such as the exponential family of functions  $G_r : t \mapsto -e^{-rt}$ , for varying values of  $r$ , extensively studied in [2, Chap.8].

**Example 2.6 (transconcavity does not imply  $\alpha$ -concavity).** *Let us provide an example of a mapping  $h : \mathbb{R} \rightarrow \mathbb{R}_+$  that is not  $\alpha$ -concave for any  $\alpha \in \mathbb{R}$ , but is  $G$ -concave for an appropriate choice of a map  $G$ . We will show that  $h : \mathbb{R} \rightarrow \mathbb{R}_+$  defined as  $h(x) = \exp(-x^3)$  is not  $\alpha$ -concave for any  $\alpha < 0$  (and then by Corollary 2.5 that  $h$  can not be  $\alpha$ -concave for any  $\alpha$ ). Indeed, let  $\alpha < 0$  be arbitrary. Then  $\alpha$ -concavity of  $h$  is equivalent to convexity of  $h^\alpha$ . Now by differentiating twice we obtain:*

$$\frac{d}{dx}(h^\alpha)(x) = -3x^2\alpha(h^\alpha)(x) \quad \text{and} \quad \frac{d^2}{dx^2}(h^\alpha)(x) = h(x)\alpha x(9\alpha x^3 - 6).$$

For  $x < 0$ ,  $\alpha x > 0$  and moreover  $x \mapsto 9\alpha x^3 - 6$  has unique (negative root)  $x^r = (\frac{6}{9\alpha})^{\frac{1}{3}}$ . Consequently  $9\alpha x^3 - 6 > 0$  for  $x < x^r$  and  $9\alpha x^3 - 6 < 0$  for  $x > x^r$ . By combining with the above, we establish that  $\frac{d^2}{dx^2}(h^\alpha)(x) < 0$  must hold for  $x \in (x^r, 0)$ , implying that  $h^\alpha$  can not be convex, i.e.,  $h$  is not  $\alpha$ -concave.

Now define  $G : \mathbb{R}_+ \rightarrow \mathbb{R}_+$  as  $G(x) = \exp(-(\ln(x))^{\frac{1}{3}})$ , then by direct computation  $\frac{d}{dx}G(x) = -\frac{1}{3x(\ln(x))^{\frac{2}{3}}}G(x)$  at any  $x \neq 1$ . We now readily verify that  $G$  is strictly decreasing. Moreover,  $G(h(x)) = \exp(x)$ ; hence, by definition,  $h$  is  $G$ -concave.

We now recall and extend a lemma from [2] that enables us to propagate the property of  $G$ -concavity.

**LEMMA 2.7** (Propagation of generalized concavity). *Let  $X$  be a Banach space and  $C$  be a convex subset of  $X$ . Let the map  $f : X \rightarrow \mathbb{R}$  be a  $G_1$ -concave function for an appropriate choice  $G_1 : f(C) \rightarrow \mathbb{R}$ . Let  $G_2$  be a continuous and strictly monotonic function over  $f(C)$ . If  $G_1^{-1}$ , the inverse function of  $G_1$  is  $G_2$ -concave over  $G_1 \circ f(C)$ , then  $f$  is also  $G_2$ -concave over  $C$ .*

*Proof.* By assumption we have, for any  $x, y \in C, \lambda \in [0, 1]$  that

$$f(\lambda x + (1 - \lambda)y) \geq G_1^{-1}(\lambda G_1 \circ f(x) + (1 - \lambda)G_1 \circ f(y))$$

and for any  $u, v \in G_1 \circ f(C), \lambda \in [0, 1]$ , we have

$$G_1^{-1}(\lambda u + (1 - \lambda)v) \geq G_2^{-1}(\lambda G_2 \circ G_1^{-1}(u) + (1 - \lambda)G_2 \circ G_1^{-1}(v)).$$

Hence, if we fix  $x, y \in C, \lambda \in [0, 1]$  and we set  $u = G_1 \circ f(x), v = G_2 \circ f(y)$ , we get from these two inequalities:

$$\begin{aligned} f(\lambda x + (1 - \lambda)y) &\geq G_1^{-1}(\lambda u + (1 - \lambda)v) \geq G_2^{-1}(\lambda G_2 \circ G_1^{-1}(u) + (1 - \lambda)G_2 \circ G_1^{-1}(v)) \\ &= G_2^{-1}(\lambda G_2 \circ G_1^{-1}(G_1 \circ f(x)) + (1 - \lambda)G_2 \circ G_1^{-1}(G_1 \circ f(y))) \\ &\geq G_2^{-1}(\lambda G_2 \circ f(x) + (1 - \lambda)G_2 \circ f(y)), \end{aligned}$$

which gives the result.  $\square$

**2.2. Study of concavity- $G^{-1}$ .** We introduce in this section the notion of concavity- $G^{-1}$  which is the right counterpart of the classical  $G$ -concavity recalled previously. In view of (1.3), the two complementary notions will be useful in the sequel, in particular because we establish that many cumulative distribution functions are concave- $G^{-1}$ . Along the way, we generalize a result of [24].

**DEFINITION 2.8** (concave- $G^{-1}$  functions). *Let  $F : \mathbb{R} \rightarrow \mathbb{R}$  and  $G : \mathbb{R} \rightarrow \mathbb{R}$  be continuous and strictly monotonic mappings. The map  $F$  is said to be concave- $G^{-1}$  on an interval  $I \subset \mathbb{R}$  if  $F \circ G^{-1}$  is concave on the interval  $I$ . By extension of Definition 2.8, we also speak of a  $G_1$ -concave- $G_2^{-1}$  function  $F$  if  $G_1 \circ F \circ G_2^{-1}$  is concave.*

This definition can be specialized as follows when considering the family  $\{G_\alpha\}_\alpha$  of Example 2.2. Let  $\alpha \in (-\infty, 1]$  be given. We say that  $f : \mathbb{R} \rightarrow \mathbb{R}$  is concave- $\alpha$  on an interval  $I \subset \mathbb{R}$  if (i)  $f$  is increasing and  $t \mapsto f(t^{\frac{1}{\alpha}})$  is concave on  $I$ , or (ii)  $f$  is decreasing and  $t \mapsto f(t^{\frac{1}{\alpha}})$  is convex on  $I$ . ( $t \mapsto f(\exp(t))$  has to be understood whenever  $\alpha = 0$  is chosen. Let us provide a positive example of concave- $\alpha$  functions in our context.

**Example 2.9 (Cumulative distribution functions).** *Let  $F : \mathbb{R}_+ \rightarrow [0, 1]$  be the cumulative distribution function of a  $\chi$ -random variable with  $m$  degrees of freedom. Then for any  $\alpha$ ,  $F$  is concave- $\alpha$  on the interval  $I = (0, (m - \alpha)^{\frac{\alpha}{2}}]$  if  $\alpha < 0$ ,  $I = [\ln(m)/2, \infty)$  if  $\alpha = 0$  and  $I = [(m - \alpha)^{\frac{\alpha}{2}}, \infty)$  if  $\alpha \in (0, 1]$ . This can be established by direct computation or as a result of [66, Lemma 3.1]. Further positive examples can be found in [24, Table 1].*

As we will shortly see, concavity- $G^{-1}$  and concavity- $\alpha$  of cumulative distribution functions can be conveniently related to specific properties of their density functions (provided they exist). To this end, we introduce the following concept.

**DEFINITION 2.10** ( $G$ -decreasing functions). *Let  $G : \mathbb{R}_+ \rightarrow \mathbb{R}_+$  be a strictly decreasing (resp. increasing) continuously differentiable map with finitely many critical points. A mapping  $f$  is said to be  $G$ -decreasing (resp.  $G$ -increasing) if there exists*

$$t_G^* > [\max \{t : G'(t) = 0\}]_+$$

*such that the ratio  $r(t) := \frac{f(t)}{G'(t)}$  is strictly decreasing (increasing) on the set  $t \geq t_G^*$ . Here  $[t]_+ := \max \{t, 0\}$  is the positive part.*

The instantiation of this definition, related to the family of mappings  $\{G_\alpha\}_\alpha$  was already introduced in [24] under the notion of  $\alpha$ -decreasing functions (only considering the situation  $\alpha < 0$ ). We now provide a key result relating concavity- $G^{-1}$  of a given distribution function with its density being  $G$ -decreasing.

**PROPOSITION 2.11** (concave- $G^{-1}$  cumulative distribution functions and  $G$ -decreasing densities). *Let  $F : \mathbb{R} \rightarrow [0, 1]$  be the cumulative distribution function of a random variable with associated (continuously differentiable) density function  $f$ . Consider the statements:*

1. *the density  $f$  is  $G$ -decreasing (see Definition 2.10) with associated parameter  $t^*$  ;*
2. *the mapping  $F$  is concave- $G^{-1}$  on the interval  $I = (0, G(t^*))$  if  $G$  is strictly decreasing, and on  $I = [G(t^*), \infty)$  if  $G$  is strictly increasing, i.e.,  $z \mapsto F(G^{-1}(z))$  is concave on  $I$  ;*

*Then 1. implies 2. and if moreover  $G$  is twice continuously differentiable, then 2. also implies 1..*

*Proof. 1.  $\Rightarrow$  2..* We note that the proof of this implication follows closely the proof of [24, Lemma 3.1] as well as [69, Lemma 4]. Let  $G : \mathbb{R}_+ \rightarrow \mathbb{R}_+$  be a strictly increasing (decreasing) map and  $t^*$  be such  $t \mapsto \frac{f(t)}{G'(t)}$  is strictly decreasing (increasing) on the set  $t \geq t^*$ .

Let us begin by considering the situation wherein  $G$  is strictly decreasing. Then for  $z \in (0, G(t^*))$ , we have  $G^{-1}(z) \geq t^*$ . Now, the map  $z \mapsto F(G^{-1}(z))$  that we will call  $\chi$  can be written:

$$\begin{aligned} \chi(z) &= \int_{-\infty}^{G^{-1}(z)} f(s)ds = F(t^*) + \int_{t^*}^{G^{-1}(z)} f(s)ds \\ &= F(t^*) + \int_{G(t^*)}^z \frac{f(G^{-1}(u))}{G'(G^{-1}(u))} du = F(t^*) - \int_z^{G(t^*)} \frac{f(G^{-1}(u))}{G'(G^{-1}(u))} du, \end{aligned}$$

where we have carried out the substitution  $u = G(s)$ . The ratio appearing in the integral is a continuous map, making  $\chi$  (continuously) differentiable. Moreover,

$$\chi'(z) = \frac{f(G^{-1}(z))}{G'(G^{-1}(z))},$$

which together with  $z \in (0, G(t^*))$  implies  $G^{-1}(z) \geq t^*$  so that  $\chi'$  is strictly decreasing. As a consequence  $\chi$  is indeed concave.

Let us now consider the case wherein  $G$  is strictly increasing. Then for  $z \in (G(t^*), \infty)$  it also holds true that  $G^{-1}(z) \geq t^*$ . We can write  $\chi$  as

$$\chi(z) = \int_{-\infty}^{G^{-1}(z)} f(s)ds = F(t^*) + \int_{t^*}^{G^{-1}(z)} f(s)ds = F(t^*) + \int_{G(t^*)}^z \frac{f(G^{-1}(u))}{G'(G^{-1}(u))} du$$

where we have carried out the substitution  $u = G(s)$ . Now

$$\chi'(z) = \frac{f(G^{-1}(z))}{G'(G^{-1}(z))},$$

which together with  $z \in (G(t^*), \infty)$  implies  $G^{-1}(z) \geq t^*$  so that  $\chi'$  is strictly decreasing. As a consequence  $\chi$  is indeed concave.

**2.  $\Rightarrow$  1.** Let us assume to begin with that  $F$  is concave- $G^{-1}$ , on the interval  $[G(t^*), \infty)$  for a strictly increasing map  $G$  and define  $\chi(z) = F \circ G^{-1}(z)$ . We first note that  $G$  is strictly increasing and (continuously differentiable) and hence by the classic inverse function Theorem (e.g., [13, Theorem 1A.1],  $G^{-1}$  is also continuously differentiable and the identity  $(G^{-1})'(x) = \frac{1}{G' \circ G^{-1}(x)}$  holds true.

Now, by assumption,  $\chi$  is concave on  $[G(t^*), \infty)$ , and for any  $x \in [G(t^*), \infty)$ , we have

$$\begin{aligned} \chi'(x) &= (G^{-1})'(x) \cdot (f \circ G)(x) \\ \chi''(x) &= (G^{-1})''(x) \cdot (f \circ G)(x) + (G^{-1})'(x)^2 \cdot (f' \circ G)(x). \end{aligned}$$

We can rewrite the second derivative as follows:

$$\chi''(x) = \frac{1}{(G' \circ G^{-1})(x)} \left( -\frac{G'' \circ G^{-1}(x)}{(G' \circ G^{-1}(x))^2} f \circ G^{-1}(x) + \frac{f' \circ G^{-1}(x)}{G' \circ G^{-1}(x)} \right),$$



where we have used the identity  $(G^{-1})''(x) = -\frac{G'' \circ G^{-1}(x)}{(G' \circ G^{-1}(x))^3}$  resulting from differentiating twice in the identity  $G(G^{-1}(x)) = x$  holding locally at any  $x > 0$  and thus in particular at any  $x \in [G(t^*), \infty)$  since  $G(t^*) > 0$ . Since  $G$  is strictly increasing, so is  $G^{-1}$  and consequently  $G' \circ G^{-1}(x) > 0$ . Concavity of  $\chi$  implies in turn that  $\chi''(x) \leq 0$ , which can be equivalently stated as:

$$\frac{-G''(t)}{(G'(t))^2}f(t) + \frac{f'(t)}{G'(t)} \leq 0, \quad (2.6)$$

where  $t = G^{-1}(x)$  and  $x \geq G(t^*)$  if and only if  $t \geq t^*$ . By defining  $\psi : t \mapsto \frac{f(t)}{G'(t)}$  and differentiating once, we obtain:

$$\psi'(t) = \frac{f'(t)}{G'(t)} + f(t)(-1)\frac{G''(t)}{G'^2(t)} = -\frac{G''(t)}{G'^2(t)}f(t) + \frac{f'(t)}{G'(t)}.$$

Now by (2.6) it follows that  $\psi'(t) \leq 0$  for all  $t \geq t^*$  and hence by Definition 2.10,  $f$  is  $G$ -decreasing.

This situation wherein  $G$  is strictly decreasing follows upon observing that  $G^{-1}$  is also strictly decreasing and that consequently  $G' \circ G^{-1}(x) < 0$  holds true. Hence, concavity of  $\chi$  on the set  $(0, G(t^*)]$ , implies  $\psi'(t) \geq 0$  as was to be shown.  $\square$

When applying the previous result with the family  $\{G_\alpha\}_{\alpha < 1}$  of (2.1), we obtain the following corollary.

**COROLLARY 2.12** (Characterisation in the case of  $G_\alpha$ ). *Let  $\alpha \in (-\infty, 1)$  be given and  $F : \mathbb{R} \rightarrow [0, 1]$  be the cumulative distribution function of a random variable with continuously differentiable density  $f$ . Then we have the following equivalence*

1.  $f$  is  $G_\alpha$ -decreasing (i.e.,  $f$  is  $(1 - \alpha)$ -decreasing in the sense of Definition 2.2 in [24])
2.  $F$  is concave- $G_\alpha$  (i.e.,  $F$  is  $\alpha$ -revealed-concave in the sense of Definition 3.1 in [66]).

The implication  $1. \Rightarrow 2.$  of Corollary 2.12 for  $\alpha < 0$  was already known and corresponds to Lemma 3.1 in [24]. However both the extension to  $\alpha \in [0, 1)$  and the reverse implication are novel. Especially the latter shows that, in principle, there is no loss of generality in studying the properties of the density instead of the cumulative distribution function  $F$ .

As already mentioned, [24, Table 1] contains a large choice of usual distribution functions (normal, exponential, Weibull, gamma, chi, Maxwell, etc...) with a  $(1 - \alpha)$ -decreasing density function for all  $\alpha < 0$  and an analytic expression for the parameter  $t^*$  indicated in Definition 2.10. Although these results may give the impression that all cumulative distribution functions are concave  $-G_\alpha^{-1}$ , this is not true as the following example shows.

**Example 2.13 (Not concave  $-G_\alpha^{-1}$  distribution function).** *Let  $f : \mathbb{R}_+ \rightarrow \mathbb{R}$  be defined as  $f(t) = (\frac{\sin^2(t)}{t^2})\frac{2}{\pi}$ . Let us first verify that  $f$  does indeed integrate to 1. This follows recalling the identity  $\sin^2(t) = (1 - \cos(2t))/2$  and by using integration by parts, as well as by recalling that  $\lim_{t \rightarrow \infty} \frac{\sin(s)}{s} ds = \frac{\pi}{2}$  (from the properties of the Sine-Integral function).*

Now, should  $f$  be  $G_\alpha$ -decreasing for some  $\alpha < 0$ , then it must hold by (2.6) that

$$-\frac{\alpha(\alpha - 1)}{\alpha^2}t^{-\alpha}f(t) + \frac{1}{\alpha}t^{1-\alpha}f'(t) \leq 0,$$

for  $t \geq t^*$  for some  $t^*$ . Yet the previous inequality is equivalent with  $(1 - \alpha)f(t) + tf'(t) \geq 0$  for  $t \geq t^*$ .

Let us verify that this can not hold. Note that

$$f'(t) = \left( \frac{-2\sin^2(t)}{t^3} + \frac{2\sin(t)\cos(t)}{t^2} \right) \frac{2}{\pi}.$$

We verify the negativeness of expression  $g(t) := (1 - \alpha)f(t) + tf'(t)$ , after algebraic manipulations:

$$g(t) = \frac{2}{\pi} \frac{\sin(t)}{t} \left( (-1 - \alpha) \frac{\sin(t)}{t} + 2\cos(t) \right)$$

Case  $-1 < \alpha < 0$ : Choosing the points  $t_n = \frac{\pi}{2} + 2\pi n$ , for integers  $n > 0$ , we note that  $\cos(t_n) = 0$ ,  $\frac{\sin(t_n)}{t_n} > 0$  and  $(-1 - \alpha)\frac{\sin(t_n)}{t_n} < 0$ . Hence, there is always  $t_n$  such that  $g(t_n) < 0$ .

Case  $\alpha \leq -1$ : Choosing the points  $t_n = \frac{3\pi}{4} + 2\pi n$ , for integers  $n > 0$  sufficient large such that  $t_n > -1 - \alpha$ . By noting that  $\cos(t_n) = -0.71$  and  $\sin(t_n) = 0.71$ , it is easy to verify that  $(-1 - \alpha)\frac{\sin(t_n)}{t_n} + 2\cos(t_n) < 0$ . Again, consequently, there is always  $t_n$  such that  $g(t_n) < 0$ .

Would the requested  $t^*$  exist, we must have for some  $n$  sufficiently large that  $t_n > t^*$  and consequently condition (2.6) must hold in particular at  $t = t_n$ . Yet, we have established that it can not. Hence the density function is not  $G_\alpha$ -decreasing for any  $\alpha < 0$ .

**Example 2.14 (concavity- $G^{-1}$  with  $G \neq G_\alpha$ ).** Let us come back to Example 2.6 and the map  $G : \mathbb{R}_+ \rightarrow \mathbb{R}_+$  defined as  $G(x) = \exp(-\ln(x)^{\frac{1}{3}})$ . Then,  $\Phi$  the cumulative distribution function of a standard normal Gaussian random variable is concave- $G^{-1}$  on the set  $(0, G(t^*))$ , with  $t^* = 1.86$ .

This follows from Proposition 2.11 as soon as the ratio  $r(x) = \frac{1}{\sqrt{2\pi G(x)}} \exp(-\frac{1}{2}x^2)(-3x \ln(x)^{\frac{3}{2}})$  is strictly increasing. This, in turn, can be asserted if  $\exp(-\frac{1}{2}x^2)(-3x \ln(x)^{\frac{3}{2}})$  is strictly increasing on the set  $x \geq t^*$ .

In order to show this, let us define  $f(x) := \exp(-\frac{1}{2}x^2)(-3x \ln(x)^{\frac{3}{2}})$ , which has derivative:  $f'(x) = 3\exp(-\frac{1}{2}x^2)\ln(x)^{\frac{1}{2}}((x^2 - 1)\ln(x) - \frac{3}{2})$ . Observing that for  $x \in (1, \infty)$ ,  $(3\exp(-\frac{1}{2}x^2)\ln(x)^{\frac{1}{2}}) > 0$ , the sign of  $f'(x)$  depends on the term:  $f_2(x) = (x^2 - 1)\ln(x) - \frac{3}{2}$ . The latter has derivative:  $f'_2(x) = 2x \ln(x) + \frac{(x^2 - 1)}{x}$

For  $x > 1$ ,  $f'_2(x) > 0$  ( $f_2(x)$  is strictly increasing for  $x > 1$ ). We find that  $f_2(1.8) = -0.1834$  and  $f_2(1.9) = 0.1752$ , so there is a root of  $f_2$  in the interval  $[1.8, 1.9]$ . So, for  $x \in [1.9, \infty)$ ,  $f_2(x) > 0$  and hence  $f'(x) > 0$ , which in turn implies that  $f(x)$  is strictly increasing. Numerically solving  $f_2(x) = 0$  in  $x$ , we find the solution  $x^* = 1.8528$ .

**3. Interplay of generalized concavity and “convexity” of chance constraints.** In this section, we establish convexity results for feasible sets of probabilistic constraints by employing the set of tools of generalized concavity. Our analysis considers two special structures for (1.1). The first situation, analyzed in Section 3.1, refers to the general case wherein  $g$  is non-linear and relatively arbitrary, but the random vector  $\xi$  is assumed to follow a multi-variate elliptically symmetric distribution. The second situation, analyzed in Section 3.2, is when  $g$  is separable, which boils<sup>†</sup> down to considering  $g(x, z) = z - h(x)$ . The random vector  $\xi$  can be relatively arbitrary in so much that it can have nearly arbitrary marginal distributions and the (joint) dependency structure is pinned down by the choice of a copula.

**3.1. Non-linear couplings of decisions vectors and elliptically distributed random vectors.** In this section we consider the situation of (1.1) wherein the map  $g : X \times \mathbb{R}^m \rightarrow \mathbb{R}^k$  is convex in the first argument and continuous as a function of both arguments. We also assume that the random vector  $\xi$  taking values in  $\mathbb{R}^m$  is elliptically symmetrically distributed with mean  $\mu$ , covariance matrix  $\Sigma$  and generator  $\theta : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ , which is denoted by  $\xi \sim \mathcal{E}(\mu, \Sigma, \theta)$  if and only if its density  $f_\xi : \mathbb{R}^m \rightarrow \mathbb{R}_+$  is given by

$$f_\xi(z) = (\det \Sigma)^{-1/2} \theta \left( (z - \mu)^\top \Sigma^{-1} (z - \mu) \right), \quad (3.1)$$

where the generator function  $\theta : \mathbb{R}_+ \rightarrow \mathbb{R}_+$  must satisfy

$$\int_0^\infty t^{\frac{m}{2}} \theta(t) dt < \infty.$$

We consider  $L$  as the matrix arising from the Choleski decomposition of  $\Sigma$ , i.e.,  $\Sigma = LL^\top$ , it can be shown that  $\xi$  admits a representation as

$$\xi = \mu + \mathcal{R}L\zeta \quad (3.2)$$

<sup>†</sup>A general separable  $g$  would be of the form  $g(x, z) = \psi(z) - h(x)$ . In our case, recalling that  $z$  will be substituted out for the random vector  $\xi$ , there is no loss of generality in assuming  $\psi(z) = z$  since the general case reduces to it when taking  $\tilde{\xi} = \psi(\xi)$  to be the underlying random vector that we study.

where  $\zeta$  has a uniform distribution over the Euclidean  $m$ -dimensional unit sphere  $S^{m-1} := \{z \in \mathbb{R}^m : \sum_{i=1}^m z_i^2 = 1\}$  and  $\mathcal{R}$  possesses a density, which is given by

$$f_{\mathcal{R}}(r) := \frac{2\pi^{\frac{m}{2}}}{\Gamma(\frac{m}{2})} r^{m-1} \theta(r^2), \quad (3.3)$$

with  $\Gamma$  is the usual gamma-function.

The family of elliptically distributed random vectors include many classical families (see e.g. [17] and [34]): for instance, Gaussian random vectors and Student random vectors (with  $\nu$  degrees of freedom) are elliptical with the respective generators

$$\theta^{\text{Gauss}}(t) = \exp(-t/2)/(2\pi)^{m/2} \quad \text{and} \quad \theta^{\text{Student}}(t) = \frac{\Gamma(\frac{m+\nu}{2})}{\Gamma(\frac{\nu}{2})} (\pi\nu)^{-m/2} \left(1 + \frac{t}{\nu}\right)^{-\frac{m+\nu}{2}}.$$

The advantage of the spherical radial decomposition (3.2) is that it allows one to derive the following attractive form for  $\varphi$  (see e.g. Theorem 2.1 [66]): if  $x \in X$  is such that

1.  $g(x, \mu) \leq 0$  (recall that  $\mu = \mathbb{E}(\xi)$ )
2. for any  $z \in \mathbb{R}^m$  such that  $g(x, z) \leq 0$ , we have

$$g(x, \lambda\mu + (1 - \lambda)z) \leq 0 \quad \forall \lambda \in [0, 1],$$

then  $\varphi$  defined in (1.1) can be written as

$$\varphi(x) = \int_{v \in \mathbb{S}^{m-1}} F_{\mathcal{R}}(\rho(x, v)) d\mu_{\zeta}(v) \quad (3.4)$$

where  $F_{\mathcal{R}}$  is the cumulative distribution function of  $\mathcal{R}$  and  $\rho : X \times \mathbb{S}^{m-1} \rightarrow \mathbb{R}_+ \cup \{\infty\}$  is a mapping defined by

$$\rho(x, v) = \begin{cases} \sup_{t \geq 0} & t \\ \text{s.t.} & g(x, \mu + tLv) \leq 0. \end{cases} \quad (3.5)$$

In particular  $\rho$  is quasi-concave due to convexity of  $g$  in  $x$  (see Lemma 3.2 [66]). Note that if for each  $v$  the map  $\rho(\cdot, v)$  is  $F_{\mathcal{R}}$ -concave, then due to the linearity of the integral such a property would carry over immediately to  $\varphi$ . It is clear that such a request could not hold without restrictions since generally a probability function can not be “concave”. Indeed it is a bounded function by 0 and 1 and usually increasing along a certain “path”. For this reason the analysis is non-trivial. The second difficulty is in conveying these desired properties from  $p$  alone. Let us provide a precise statement.

**THEOREM 3.1** (Convexity of probability functions). *Let  $C$  be a convex subset of  $X$ . Assume that, for any  $v \in \mathbb{S}^{m-1}$ , there exists a continuous function  $G_v : \rho(C \times \{v\}) \rightarrow \mathbb{R}_+$  such that*

- $G_v$  is strictly monotonic on  $\rho(C \times \{v\})$ ,
- $x \mapsto \rho(x, v)$  is  $G_v$ -concave on  $C$  and continuous,
- $F_{\mathcal{R}}$  is concave  $\circ G_v^{-1}$  on  $(G_v \circ \rho)(C \times \{v\})$ ,

where  $\rho$  is defined as in (3.5). Then  $\varphi : X \rightarrow [0, 1]$  defined in (1.1) is concave on  $C$ .

*Proof.* Let us first establish that for any fixed  $v \in \mathbb{S}^{m-1}$  that  $x \mapsto F_{\mathcal{R}} \circ \rho(x, v)$  is concave on  $C$ . To this end pick  $x_1, x_2 \in C$  and  $\lambda \in [0, 1]$  arbitrarily and consider  $x^\lambda = \lambda x_1 + (1 - \lambda)x_2$ . Then by  $G_v$ -concavity of  $\rho(\cdot, v)$  it follows:

$$\rho(x^\lambda, v) \geq G_v^{-1}(\lambda G_v(\rho(x_1, v)) + (1 - \lambda)G_v(\rho(x_2, v))).$$

Now since  $F_{\mathcal{R}}$  is increasing as a distribution function it follows too that

$$F_{\mathcal{R}}(\rho(x^\lambda, v)) \geq F_{\mathcal{R}}(G_v^{-1}(\lambda G_v(\rho(x_1, v)) + (1 - \lambda)G_v(\rho(x_2, v)))).$$

Now consider the set  $I_v \subseteq \mathbb{R}$  defined as  $I_v = G_v(\rho(C, v))$ .

Our claim is that  $I_v$  is an interval, i.e., is convex. To this end, we recall that the continuous image of a connected set is connected and hence  $\rho(C, v)$  is a connected set (recall that  $C$  is a convex set). By applying the argument a second time, since  $G_v$  is continuous, it follows that  $I_v$  is connected. Now since  $I_v$  is a subset of  $\mathbb{R}$ , it is connected if and only if it is convex if and only if it is an interval.

Now since  $I_v$  is an interval and hence convex,  $\lambda G_v(\rho(x_1, v)) + (1 - \lambda)G_v(\rho(x_2, v)) \in I_v$  and we may apply concavity  $-G_v^{-1}$  of  $F_{\mathcal{R}}$  to pursue our development as follows:

$$F_{\mathcal{R}}(\rho(x^\lambda, v)) \geq (F_{\mathcal{R}} \circ G_v^{-1})(\lambda G_v(\rho(x_1, v)) + (1 - \lambda)G_v(\rho(x_2, v))) \geq \lambda F_{\mathcal{R}}(\rho(x_1, v)) + (1 - \lambda)F_{\mathcal{R}}(\rho(x_2, v)),$$

which is what was to be shown. Now by linearity of integrals, we get

$$\begin{aligned} \varphi(x^\lambda) &= \int_{v \in \mathbb{S}^{m-1}} F_{\mathcal{R}}(\rho(x^\lambda, v)) d\mu_\zeta(v) \geq \lambda \int_{v \in \mathbb{S}^{m-1}} F_{\mathcal{R}}(\rho(x_1, v)) d\mu_\zeta(v) + (1 - \lambda) \int_{v \in \mathbb{S}^{m-1}} F_{\mathcal{R}}(\rho(x_2, v)) d\mu_\zeta(v) \\ &= \lambda \varphi(x_1) + (1 - \lambda) \varphi(x_2), \end{aligned}$$

thus concluding the proof.  $\square$

Although Theorem 3.1 allows us to establish concavity of  $\varphi$  on a certain given convex set  $C$ , an important additional difficulty is how to entail that  $M(p) \subseteq C$  holds for  $p$  large enough. Then one can immediately deduce the convexity of  $M(p)$  from concavity of  $\varphi$ . A convenient situation is one when, for all  $x$ , the set  $\mathfrak{M}(x) := \{z \in \mathbb{R}^m : g(x, z) \leq 0\}$  is convex in  $\mathbb{R}^m$ . For this it would be sufficient to request that  $g$  is convex respectively in  $x$  and in  $z$  (but not necessarily jointly). We can however generalize to the situation wherein  $\mathfrak{M}(x)$  is star-shaped with respect to  $\mu$  if the convex hull of the latter sets does “not distort” measurement of length of lines segments  $\{r \geq 0 : \mu + rLv \in \mathfrak{M}(x)\}$  moving through it. In order to make a precise statement, we introduce the following map:  $\rho^{\text{co}} : X \times \mathbb{S}^{m-1} \rightarrow \mathbb{R}_+ \cup \{-\infty, \infty\}$ :

$$\rho^{\text{co}}(x, v) = \begin{cases} \sup_{t \geq 0} t & \\ s.t. & \mu + tLv \in \text{Co}(\mathfrak{M}(x)) \end{cases} \quad (3.6)$$

where  $\text{Co}(\mathfrak{M}(x))$  denotes the convex hull of  $\mathfrak{M}(x)$ .

**THEOREM 3.2** (Eventual convexity of elliptical chance constraints). *Let  $C$  be a given convex subset of  $X$ . In addition to the framework of this section, assume that*

1. *there exists a  $t^* > 0$ , such that  $\{x \in X : \rho(x, v) \geq t^*, \forall v \in \mathbb{S}^{m-1}\} \subseteq C$*
2. *For any  $v \in \mathbb{S}^{m-1}$ , there exists a continuous function  $G_v : \mathbb{R} \rightarrow \mathbb{R}$  such that*
  - *$G_v$  is strictly monotonic on  $\rho(C \times \{v\})$ .*
  - *$x \mapsto \rho(x, v)$  is  $G_v$ -concave and continuous.*
  - *$F_{\mathcal{R}}$  is concave  $-G_v^{-1}$  on  $(G \circ \rho)(C \times \{v\})$ .*
3. *There exists  $p_0 \in [\frac{1}{2}, 1]$  and  $\delta^{\text{nd}} > 0$  such that*

$$\delta^{\text{nd}} \rho(x, v) \geq \rho^{\text{co}}(x, v),$$

*for all  $x \in M(p_0)$  and all  $v \in \mathbb{S}^{m-1}$ ,*

*where  $\rho$  and  $\rho^{\text{co}}$  are defined respectively in (3.5) and (3.6).*

*Then for any  $q \in (0, \frac{1}{2})$  and any  $p \geq \max(p_0, p(t^*, q))$  with*

$$p(t^*, q) = \left(\frac{1}{2} - q\right) F_{\mathcal{R}}\left(\frac{\delta^{\text{nd}} t^*}{\delta(q)}\right) + \frac{1}{2} + q, \quad (3.7)$$

*the set  $M(p)$  defined in (1.2) is convex. Here  $\delta(q)$  is the unique solution (in  $\delta$ ) to the equation*

$$\mathfrak{B}_i\left(\frac{m-1}{2}, \frac{1}{2}, \sin^2(\arccos(\delta))\right) = (1 - 2q) \mathfrak{B}_c\left(\frac{m-1}{2}, \frac{1}{2}\right),$$

*where  $\mathfrak{B}_i$  (resp.  $\mathfrak{B}_c$ ) refers to the incomplete (resp. complete) Beta function.*

*Proof.* We follow here closely the demonstration of the Theorem 4.1 from [66]. Let  $p \in [p_0, 1]$  be given and take any  $x \in M(p)$ . We have, for such  $x$  that  $\frac{1}{2} < p \leq \mathbb{P}[g(x, \xi) \leq 0] \leq \mathbb{P}[\xi \in \text{Co}(\mathfrak{M}(x))]$ . Then corollary 2.1 from [66] gives that  $\mu \in \text{int}(\text{Co}(\mathfrak{M}(x)))$ .

Let us now pick an arbitrary but fixed  $v \in \text{Dom}(\rho(x, \cdot))$ . Note that, by assumption, there is  $\delta^{\text{nd}} > 0$  such that  $\delta^{\text{nd}}\rho(x, v) \geq \rho^{\text{co}}(x, v)$ , i.e.,  $v \in \text{Dom}(\rho^{\text{co}}(x, \cdot))$  as well. Hence,  $\mu + \rho^{\text{co}}(x, v)Lv$  belongs to the boundary of  $\text{Co}(\mathfrak{M}(x))$ .

Therefore, we can separate  $\mu + \rho^{\text{co}}(x, v)Lv$  from the convex set  $\text{Co}(\mathfrak{M}(x))$ , so that there exists a non-zero  $s \in \mathbb{R}^m$  such that for all  $z \in \text{Co}(\mathfrak{M}(x))$ ,

$$s^\top z \leq s^\top (\mu + \rho^{\text{co}}(x, v)Lv) \leq s^\top (\mu + \delta^{\text{nd}}\rho(x, v)Lv).$$

Now define  $c \in \mathbb{R}^m$  and  $\gamma > 0$  as follows:

$$c := \frac{s}{\|L^\top s\|} \text{ and } \gamma = c^\top (\mu + \delta^{\text{nd}}\rho(x, v)Lv),$$

where we recall the  $\|L^\top s\| > 0$  since  $L$  is regular and  $s \neq 0$ . It now follows by construction that,

$$\mathfrak{M}(x) \subset \text{Co}(\mathfrak{M}(x)) \subset \{z \in \mathbb{R}^m : c^\top z \leq \gamma\}.$$

In particular this entails  $\mathbb{P}[g(x, \xi) \leq 0] \leq \mathbb{P}[c^\top \xi \leq \gamma]$ . We can employ Theorem 2.2 of [66] to get the estimate

$$\begin{aligned} p &\leq \mathbb{P}[g(x, \xi) \leq 0] \leq \mathbb{P}[c^\top \xi \leq \gamma] \leq \left(\frac{1}{2} - q\right) F_{\mathcal{R}} \left( \frac{\delta^{\text{nd}}\rho(x, v) \frac{s^\top Lv}{\|L^\top s\|}}{\delta(q)} \right) + q + \frac{1}{2} \\ &\leq \left(\frac{1}{2} - q\right) F_{\mathcal{R}} \left( \frac{\delta^{\text{nd}}\rho(x, v)}{\delta(q)} \right) + q + \frac{1}{2}, \end{aligned}$$

for any  $q \in (0, \frac{1}{2})$  and associated  $\delta(q) > 0$ , where we have used the Cauchy-Schwartz inequality and the monotonicity of  $F_{\mathcal{R}}$ . Since  $F_{\mathcal{R}}$  is increasing and  $(\frac{1}{2} - q) > 0$  as well as  $p \geq \max(p_0, p(t^*, q))$ , we derive  $\rho(x, v) \geq t^*$ . Moreover obviously,  $\rho(x, v) \geq t^*$  for  $v \notin \text{Dom}(\rho(x, \cdot))$ . Hence,  $M(p) \subseteq \{x \in X : \rho(x, v) \geq t^*, \forall v \in \mathbb{S}^{m-1}\} \subseteq C$ . Now, we can apply Theorem 3.1 to establish that  $\varphi$  is concave on  $C$  and therefore  $M(p)$  must be convex.  $\square$

**Remark 3.3 (Abstract theorem at work).** *Though looking abstract, the conditions of the theorem are often present in practice. They can for example be ensured whenever the following conditions hold. We provide examples in Section 4.*

- There exist some  $\alpha \in \mathbb{R}$  such that for each  $v \in \mathbb{S}^{m-1}$ , the map  $x \mapsto \rho(x, v)$  is  $\alpha$ -concave (see [66, Proposition 5.1] for an exemple) and the radial distribution function  $F_{\mathcal{R}}$  is concave- $\alpha$ . Here prominent examples are the chi distribution, the Fisher-Snedecor distribution, etc.
- To get a suitable  $t^*$  associated with  $F_{\mathcal{R}}$ , one can use the concavity- $\alpha$  of  $F_{\mathcal{R}}$ . For the example of the chi-distribution, we get  $t^* = \sqrt{m - \alpha}$  and  $C = \{x \in X : \rho(x, v) \geq \sqrt{m - \alpha} \text{ for all } v \in \mathbb{S}^{m-1}\}$ .
- The request of item 3 holds whenever the set  $\mathfrak{M}(x)$  is convex for all  $x \in C$  and in that case  $\delta^{\text{nd}} = 1$ . Such convexity can be ensured whenever the map  $z \mapsto g(x, z)$  is quasi-convex for each  $x \in X$ , and in that case  $p_0 = \frac{1}{2}$  can be taken.
- Note finally that continuity of the map  $\rho$  can be ensured under fairly general conditions, e.g., convexity of  $g$  in the second argument and  $g(x, \mu) < 0$  together with continuous differentiability of  $g$  immediately entail continuity of  $x \mapsto \rho(x, v)$  (even continuity in both arguments). See for instance [21]. Continuity can also be ensured under less restrictive conditions. For instance whenever, the map  $g$  is continuous at any  $\bar{x}, \bar{v}, \bar{r}$  such that  $g(\bar{x}, \bar{r}L\bar{v}) = 0$ , neighbourhoods  $U, V, W$  of  $\bar{x}, \bar{v}, \bar{r}$  respectively can be identified such that for all  $(x, v) \in U \times V$ , the map  $r \mapsto g(x, rLv)$  is monotone. This in turn is related to uniqueness of solutions of perturbed systems  $g(x, rLv) = t$  for  $t$  sufficiently small and not very restrictive. To ensure continuity of  $\rho$  from this assumption, we can just use a general version of the implicit function theorem; see [13, Theorem 1H.3] as well as [27, 33] which are older statements of such a result.

The computed threshold  $p^*$  is thus valid for a large class of nonlinear functions  $g$ . Should it be conservative, refinements might be obtained when studying specific structures; see e.g., [44, 59] and forthcoming Remark 4.5. Beyond being a general guarantee, the threshold  $p^*$  is thus an indication that lower thresholds could be revealed for specific functions.

**3.2. Separable copulae-structured probabilistic constraints.** In this section we will consider the following form of (1.1):

$$\mathbb{P}[\xi \leq h(x)] \geq p,$$

where  $\xi$  taking values in  $\mathbb{R}^m$  is a random vector and  $h : X \rightarrow \mathbb{R}^m$  a given map. By employing Sklar's Theorem [54], we may write (1.1) with the special structure as above in the following form:

$$\mathbb{P}[\xi \leq h(x)] = \mathfrak{C}(F_1(h_1(x)), \dots, F_m(h_m(x))). \quad (3.8)$$

Here  $F_1, \dots, F_m$  are the marginal distribution functions of the random vector  $\xi$  and  $h_i : X \rightarrow \mathbb{R}, i = 1, \dots, m$  refers to the components of the mapping  $h$ . Moreover  $\mathfrak{C} : [0, 1]^m \rightarrow [0, 1]$  is a copula, i.e., a multi-variate distribution function with uniform marginal distributions (e.g., [46] for further details). In what follows, we require specific properties of  $\mathfrak{C}$  and the marginal distribution functions  $F_1, \dots, F_m$  and through these choices pin down the multivariate distribution of  $\xi$ .

We introduce the analogue of concave- $G^{-1}$  functions in this context of copulae.

**DEFINITION 3.4** (concave- $G^{-1}$  copulae). *Let  $\mathfrak{C} : [0, 1]^m \rightarrow [0, 1]$  be a copula and  $G : [0, 1]^m \rightarrow \mathbb{R}^m$  a map such that the  $i$ th component  $G_i$  is continuous and strictly monotonic. The copula  $\mathfrak{C}$  is said to be concave- $G^{-1}$  on the product of intervals  $I = \prod_{i=1}^m I_i$ , with  $I_i \subseteq \mathbb{R}$  if the map*

$$I \ni z \mapsto \mathfrak{C}(G_1^{-1}(z_1), \dots, G_m^{-1}(z_m))$$

*is quasi-concave.*

**Example 3.5 (Relation to other concave copulae).** *Taking  $G_i = G_\gamma$  in the previous definition, a concave- $G^{-1}$  copula corresponds to a  $(-\infty) - \gamma$ -concave copula in the terminology of [58]. For example, all Archimedian copulae are concave- $G_1^{-1}$  (see [61, Theorem 3.3]. Example 3.6 below provides an example of concave- $G^{-1}$  copula which is necessarily  $(-\infty) - \gamma$ -concave. Note also that, in the special case  $\gamma = 0$ , this notion is weaker than the notion of logexp-concave of [25]). Prominent examples of such copulae are for instance the independent, maximum or Gumbel copula. The Clayton copula is an example of concave- $G_0^{-1}$  copula which is not logexp-concave (see Lemma 5.5 of [58]).*

**Example 3.6 (Gaussian case).** *Let  $R$  be a positive definite  $m \times m$  correlation matrix and  $\Phi$  denote the standard normal distribution function. Then the Gaussian copula  $\mathfrak{C}^R : [0, 1]^m \rightarrow [0, 1]$  is defined as*

$$\mathfrak{C}^R(u) = \Phi^R(\Phi^{-1}(u_1), \dots, \Phi^{-1}(u_m)),$$

where  $\Phi^R$  is the multivariate Gaussian distribution function related to correlation matrix  $R$ . Now observe that  $\Phi^{-1} : [0, 1] \rightarrow \mathbb{R}$  is strictly increasing and continuous. Now for any  $z \in I = \mathbb{R}^m$ ,  $\mathfrak{C}^R(\Phi(z_1), \dots, \Phi(z_m)) = \Phi^R(\Phi^{-1}(\Phi(z_1)), \dots, \Phi^{-1}(\Phi(z_m))) = \Phi^R(z_1, \dots, z_m)$ . Recalling that multivariate Gaussian distribution functions are 0-concave (see, e.g., [49]), it follows from Corollary 2.5 that  $\mathfrak{C}^R$  is concave- $\Phi^{-1^{-1}}$ . It was however not clear whether or not  $\mathfrak{C}^R$  is  $(-\infty) - \gamma$ -concave; see the extensive discussion in [58, section 6]). Similar analysis can be carried out with Clayton Copula which can be shown to be concave- $G^{-1}$  with  $G_i(z) = \ln(z)^2$ .

We can now provide the announced eventual convexity result.

**THEOREM 3.7** (Eventual convexity of separable copulae-structured probabilistic constraints). *Let  $h_i : X \rightarrow \mathbb{R}$  be continuous mappings and consider the following identity:*

$$\mathbb{P}[\xi \leq h(x)] = \mathfrak{C}(F_1(h_1(x)), \dots, F_m(h_m(x))), \quad (3.9)$$

where  $\mathfrak{C}$  is a suitable copula and  $F_i$  are the marginal distribution functions of component  $i$  of  $\xi$ ,  $i = 1, \dots, m$ .



Assume that we can find strictly monotonous mappings  $G_i : \mathbb{R} \rightarrow \mathbb{R}$ , such that the functions  $h_i$  are  $G_i$ -concave on a given convex level set  $C = \{x \in X : h_i(x) \geq b_i, i = 1, \dots, m\}$  of  $X$  for appropriate parameters  $b_i \in \mathbb{R}$ ,  $i = 1, \dots, m$ .

Assume moreover that for  $i = 1, \dots, m$  continuous strictly monotonic mappings  $\hat{G}_i : [0, 1] \rightarrow \mathbb{R}$  can be identified such that  $\mathfrak{C}$  is concave  $-\hat{G}^{-1}$  (see Definition 3.4) on the set  $I = \prod_{i=1}^m I_i$ , where

- the interval  $I_i = [\hat{G}_i(b_i), \infty)$  whenever  $\hat{G}_i$  is strictly increasing
- the interval  $I_i = (-\infty, \hat{G}_i(b_i)]$  whenever  $\hat{G}_i$  is strictly decreasing.

Finally, assume that the marginal distribution functions  $F_i : \mathbb{R} \rightarrow \mathbb{R}$  are  $\hat{G}_i$ -concave- $G_i^{-1}$  on

- the interval  $[G_i(b_i), \infty)$  whenever  $G_i$  is strictly increasing
- the interval  $(0, G_i(b_i)]$  whenever  $G_i$  is strictly decreasing.

Then the set  $M(p) := \{x \in X : \mathbb{P}[\xi \leq h(x)] \geq p\}$  is convex for all  $p > p^* := \max_{i=1, \dots, m} F_i(b_i)$ . Convexity can also be asserted for  $p = p^*$  if each individual distribution function  $F_i$ ,  $i = 1, \dots, m$  is strictly increasing.

*Proof.* Pick any  $p > p^*$ ,  $x, y \in M(p)$ ,  $\lambda \in [0, 1]$  and  $i \in \{1, \dots, m\}$  arbitrarily. Define  $x^\lambda := \lambda x + (1 - \lambda)y$ . Since all copulae are dominated by the maximum-copula, we get:

$$F_i(h_i(x)) \geq \min_{j=1, \dots, m} F_j(h_j(x)) \geq \mathfrak{C}(F_1(h_1(x)), \dots, F_m(h_m(x))) \geq p > p^* \geq F_i(b_i). \quad (3.10)$$

Now the latter entails

$$h_i(x) \geq b_i. \quad (3.11)$$

and this in turn means  $M(p) \subseteq C$ . Estimate (3.11) also holds whenever  $p \geq p^*$  and  $F_i$  is strictly increasing for each  $i = 1, \dots, m$ . A similar estimate is obtained for  $y$  clearly. Let us first remark that due to (3.11), we have  $G_i(h_i(x)) \geq G_i(b_i)$ , whenever  $G_i$  is strictly increasing and  $G_i(h_i(x)) \leq G_i(b_i)$  otherwise.

Consequently,  $\lambda G_i(h_i(x)) + (1 - \lambda)G_i(h_i(y)) \leq G_i(b_i)$  whenever  $G_i$  is strictly decreasing, and  $\lambda G_i(h_i(x)) + (1 - \lambda)G_i(h_i(y)) \geq G_i(b_i)$  otherwise. We note that in either situation  $\lambda G_i(h_i(x)) + (1 - \lambda)G_i(h_i(y))$  belongs to the interval associated with  $\hat{G}_i$ -concavity- $G_i^{-1}$  of  $F_i$ . Now, since  $F_i$  is increasing we establish first that

$$F_i(h_i(x^\lambda)) \geq F_i(G_i^{-1}(\lambda G_i(h_i(x)) + (1 - \lambda)G_i(h_i(y)))), \quad (3.12)$$

by  $G_i$ -concavity of  $h_i$  on the set  $C$ . As argued above, related to  $\lambda G_i(h_i(x)) + (1 - \lambda)G_i(h_i(y))$  belonging to an appropriate domain, we may pursue (3.12) by invoking  $\hat{G}_i$ -concavity- $G_i^{-1}$  of  $F_i$ , and obtain

$$\begin{aligned} F_i(h_i(x^\lambda)) &\geq F_i(G_i^{-1}(\lambda G_i(h_i(x)) + (1 - \lambda)G_i(h_i(y)))) \\ &\geq \hat{G}_i^{-1}(\lambda \hat{G}_i(F_i(G_i^{-1}(G_i(h_i(x))))) + (1 - \lambda)\hat{G}_i(F_i(G_i^{-1}(G_i(h_i(x))))) \\ &\geq \hat{G}_i^{-1}(\lambda \hat{G}_i(F_i(h_i(x))) + (1 - \lambda)\hat{G}_i(F_i(h_i(y)))) \end{aligned} \quad (3.13)$$

Since  $i$  was fixed but arbitrary, the above equation (3.13) holds for all  $i = 1, \dots, m$ . A Copula is increasing in its arguments, so we get in turn:

$$\begin{aligned} \mathfrak{C}(F_1(h_1(x^\lambda)), \dots, F_m(h_m(x^\lambda))) &\geq \\ \mathfrak{C}(\hat{G}_1^{-1}(\lambda \hat{G}_1(F_1(h_1(x))) + (1 - \lambda)\hat{G}_1(F_1(h_1(y)))) &\dots, \hat{G}_m^{-1}(\lambda \hat{G}_m(F_m(h_m(x))) + (1 - \lambda)\hat{G}_m(F_m(h_m(y))))) \end{aligned}$$

Moreover arguing as before, for any  $i = 1, \dots, m$  it holds that  $\lambda \hat{G}_i(F_i(h_i(x))) + (1 - \lambda)\hat{G}_i(F_i(h_i(y))) \in I_i$ . We can now employ concavity  $-\hat{G}^{-1}$  of the copula  $\mathfrak{C}$ , to establish:

$$\begin{aligned} \mathfrak{C}(F_1(h_1(x^\lambda)), \dots, F_m(h_m(x^\lambda))) &\geq \\ m_{-\infty}(\mathfrak{C}(F_1(h_1(x)), \dots, F_m(h_m(x))), \mathfrak{C}(F_1(h_1(y)), \dots, F_m(h_m(y))), \lambda) &\geq p, \end{aligned}$$

which is equivalent with  $\mathfrak{C}(F_1(h_1(x^\lambda)), \dots, F_m(h_m(x^\lambda))) \geq p$ , i.e.,  $x^\lambda \in M(p)$  as was to be shown.  $\square$

The conditions given in the above Theorem 3.7 allow for quite some flexibility. We refer to [58, section 5] for examples. In section 4.2 below we provide other examples, not covered by prior results.

**4. Selected examples.** In this section, we provide several new examples for which eventual convexity can be asserted with the extended framework built upon  $G$ -concavity. Section 4.1 is concerned with a situation wherein  $g$  is non-linear and Section 4.2 wherein  $g$  separable.

**4.1. Example with a quadratic probabilistic constraint.** Consider the map  $g : \mathbb{R}^n \times \mathbb{R}^m \rightarrow \mathbb{R}$  defined as

$$g(x, z) = z^\top W(x)z + 2 \sum_{i=1}^n a_i w_i^\top z + b \quad (4.1)$$

where  $a_1, \dots, a_n$  are fixed constants,  $w_1, \dots, w_n$  are fixed vectors of  $\mathbb{R}^m$ , and  $W : \mathbb{R}^n \rightarrow S_n^+$  is a “convex” function, in the sense that for any  $(x, y) \in \mathbb{R}^n, \lambda \in [0, 1]$ ,  $\lambda W(x) + (1 - \lambda)W(y) - W(\lambda x + (1 - \lambda)y)$  is positive (semi-)definite. Note that, for all  $z$ ,  $g(\cdot, z)$  is then convex with respect to its first variable. We assume furthermore that  $b < 0$ , which ensures  $g(x, 0) < 0$ . A concrete example of a mapping  $W$  satisfying the above request is for instance  $W(x) = \sum_{i=1}^n x_i W_i$ , with  $W_i$  positive semi-definite and  $x \geq 0$  as an ambient further restriction.

Let  $\xi$  be an elliptically symmetrically distributed random vector with given spherical-radial decomposition  $\xi = \mathcal{R}L\zeta$ . Then, by fixing  $(x, v) \in \mathbb{R}^n \times \mathbb{S}^{m-1}$  and defining

$$h(x, v) := (Lv)^\top W(x)(Lv) \quad \text{and} \quad \beta(v) := \sum_{i=1}^n a_i w_i^\top (Lv),$$

we can explicitly identify the map  $\rho$  of (3.4), being the unique solution to the equation  $g(x, rLv) = 0$ . This solution map is given by the expression:

$$\rho(x, v) = \frac{-\beta(v) + \sqrt{\beta(v)^2 - bh(x, v)}}{h(x, v)}. \quad (4.2)$$

Next, we identify a function  $g_v : \mathbb{R}^+ \rightarrow \mathbb{R}$  such that  $\rho(\cdot, v)$  is  $g_v$ -concave for any  $v \in \mathbb{S}^{m-1}$ .

**LEMMA 4.1.** *For any fixed  $v \in \mathbb{S}^{m-1}$  define the map  $g_v : (0, \infty) \rightarrow \mathbb{R}_-$  as  $g_v(t) = -(\frac{b}{t} + \beta(v))^2$ . Then  $x \mapsto \rho(x, v)$   $g_v$ -concave on  $\mathbb{R}^n$ .*

*Proof.* We first note that  $\rho(x, v) > 0$ , since  $\rho(x, v)$  is the solution to  $g(x, rLv) = 0$ , and  $z \mapsto g(x, z)$  is convex (as a convex quadratic map in  $z$ ) as well as  $g(x, 0) < 0$  so that by continuity any solution (if any) to  $g(x, rLv) = 0$  must satisfy  $r > 0$ . Consequently the composition  $g_v(\rho(x, v))$  is well-defined. Moreover,  $g_v$  is clearly (continuously) differentiable on  $(0, \infty)$ . Now observe that

$$\frac{1}{\rho(x, v)} = \frac{h(x, v)}{-\beta(v) + \sqrt{\beta(v)^2 - bh(x, v)}} = \frac{\beta(v) + \sqrt{\beta(v)^2 - bh(x, v)}}{-b}$$

So that,  $\frac{-b}{\rho(x, v)} - \beta(v) = \sqrt{\beta(v)^2 - bh(x, v)} \geq 0$ , i.e.,

$$g_v(\rho(x, v)) = -\left(\frac{b}{\rho(x, v)} + \beta(v)\right)^2 = -\beta(v)^2 + bh(x, v).$$

Now,  $b < 0$  and  $x \mapsto h(x, v)$  is convex, therefore  $x \mapsto g_v(\rho(x, v))$  is concave. It remains to show that  $g_v$  is strictly increasing on an appropriate domain. To this end, observe that

- if  $\beta(v) \leq 0$ , we have for  $t > 0$ ,  $g'_v(t) = 2\frac{b}{t^2}(\frac{b}{t} + \beta(v))$ . Now, since  $b < 0$  and  $\beta(v) \leq 0$ , it follows that  $(\frac{b}{t} + \beta(v)) < 0$  and evidently  $\frac{b}{t^2} < 0$  so we conclude that  $g'_v(t) > 0$ , as was to be shown. Note that  $g_v$  maps to  $(-\infty, -\beta(v)^2)$  for  $\beta(v) \leq 0$ .
- if  $\beta(v) > 0$ , we still have, for  $t > 0$ ,  $g'_v(t) = (2\frac{b}{t^2}(\frac{b}{t} + \beta(v)))$  and  $\frac{b}{t^2} < 0$ . However  $(\frac{b}{t} + \beta(v)) < 0$  if and only if  $t < \frac{b}{-\beta(v)}$ , so that  $g_v$  is strictly increasing on  $(0, \frac{b}{-\beta(v)})$  in this case only. We will now establish that  $\rho(x, v) \in (0, \frac{b}{-\beta(v)})$  holds true for such  $v$ . To this end recall

$$\frac{1}{\rho(x, v)} = \frac{\beta(v) + \sqrt{\beta(v)^2 - bh(x, v)}}{-b} \iff \rho(x, v) = \frac{-b}{\beta(v) + \sqrt{\beta(v)^2 - bh(x, v)}},$$

which with  $\beta(v) + \sqrt{\beta(v)^2 - bh(x, v)} > \beta(v) \geq 0$ , gives  $\rho(x, v) < \frac{b}{-\beta(v)}$  as desired.

□

We can identify explicitly the inverse function of  $g_v$ , but for this we will need to make a case-distinction.

- For  $v$  such that  $\beta(v) \leq 0$ ,  $g_v^{-1} : (-\infty, -\beta(v)^2) \rightarrow (0, \infty)$  is given by  $g_v^{-1}(t) = \frac{-b}{\sqrt{-t+\beta(v)}}$
- For  $v$  such that  $\beta(v) > 0$ ,  $g_v^{-1} : (-\infty, 0] \rightarrow (0, \frac{b}{-\beta(v)}]$ , is given by  $g_v^{-1}(t) = \frac{-b}{\sqrt{-t+\beta(v)}}$ .

We now employ Lemma 2.7 in order to combine specialized concavity of  $\rho$  with more usual notions.

**LEMMA 4.2.** *Let  $v \in \mathbb{S}^{m-1}$  be given and consider the map  $g_v^{-1}(t) = \frac{-b}{\sqrt{-t+\beta(v)}}$  defined from  $(-\infty, -\beta(v)^2)$  to  $(0, \infty)$  when  $\beta(v) \leq 0$  and from  $(-\infty, 0]$  to  $(0, \frac{b}{-\beta(v)}]$  otherwise. This map is  $-3$ -concave on the set  $(-\infty, -\beta(v)^2]$ .*

*Proof.* We will show this by a direct computation. In order to do so, let  $\alpha > -1$  be given but arbitrary and let  $\psi$  denote the map  $t \mapsto (\frac{1}{g_v^{-1}(t)})^{1+\alpha}$ . The map  $\psi$  is clearly twice differentiable on the appropriate domain and hence establishing the requested generalized concavity of  $g_v^{-1}$  amounts to establishing convexity of  $\psi$ . Now we establish

$$\begin{aligned}\psi(t) &= \left(\frac{-1}{b}\right)^{1+\alpha} (\sqrt{-t} + \beta(v))^{1+\alpha} \\ \psi'(t) &= -\left(\frac{-1}{b}\right)^{1+\alpha} \frac{1+\alpha}{2} (-t)^{-\frac{1}{2}} (\sqrt{-t} + \beta(v))^\alpha \\ \psi''(t) &= \left(\frac{1+\alpha}{4b^2}\right) (-t)^{-\frac{3}{2}} \left(\frac{1}{g_v^{-1}(t)}\right)^{\alpha-1} ((\alpha-1)\sqrt{-t} - \beta(v)).\end{aligned}$$

It is now clear that for  $\beta(v) \leq 0$  and  $\alpha > 1$ , it holds  $\psi''(t) > 0$ , implying the convexity of  $\psi$ , i.e., the  $-1-\alpha$ -concavity of  $g_v^{-1}$ . When  $\beta(v) > 0$  holds, then for  $\alpha \geq 2$ , we observe that  $(\alpha-1)\sqrt{-t} - \beta(v) \geq \sqrt{-t} - \beta(v)$ . The latter is evidently positive whenever  $t \leq -\beta^2(v)$ .

□

**Remark 4.3.** *Let us observe that for any  $v$  and  $x$  with  $g(x, 0) < 0$ ,  $g_v(\rho(x, v)) \in (-\infty, -\beta(v)^2]$  holds true. Indeed, let us set  $-t := -g_v(\rho(x, v)) = (\frac{b}{\rho(x, v)} + \beta(v))^2$ . Then as observed already:  $\sqrt{-t} = -\frac{b}{\rho(x, v)} - \beta(v) = \sqrt{\beta(v)^2 - bh(x, v)} \geq 0$ . And consequently*

$$\sqrt{-t} - \beta(v) = -\frac{b}{\rho(x, v)} - 2\beta(v) = \frac{1}{\rho(x, v)} (-b - 2\beta\rho(x, v)).$$

But,  $\rho(x, v)$  is the unique (positive) solution (in  $r$ ) to the equation  $h(x, v)r^2 + 2\beta(v)r^2 + b = 0$ . Therefore we have  $\sqrt{-t} - \beta(v) = \frac{1}{\rho(x, v)} h(x, v)\rho(x, v)^2 = h(x, v)\rho(x, v) \geq 0$ , thus establishing the claim.

We can now provide an eventual convexity statement for a probability function involving the map  $g$  defined in (4.1).

**PROPOSITION 4.4.** *Consider the probability function  $\varphi(x) := \mathbb{P}[g(x, \xi) \leq 0]$ , where  $x \in \mathbb{R}^n$  is given and  $g$  defined as in (4.1). Let  $\xi$  taking values in  $\mathbb{R}^m$  be an elliptically symmetrically distributed random vector with mean 0 and covariance matrix  $\Sigma$  and associated radial distribution  $F_{\mathcal{R}}$ . Let  $F_{\mathcal{R}}$  be concave- $(-3)$  on the set  $(0, (t^*)^{-3}]$ , with  $t^*$  given by the first assumption of Theorem 3.2.*

*Then for any  $q \in (0, \frac{1}{2})$ , the set  $M(p)$  is convex provided that  $p \geq p^*$  with*

$$p^* := \left(\frac{1}{2} - q\right) F_{\mathcal{R}}\left(\frac{t^*}{\delta(q)}\right) + \frac{1}{2} + q,$$

where  $\delta(q)$  is as in Theorem 3.2.

*Proof.* The set  $C := \{x \in \mathbb{R}^n : \rho(x, v) \geq t^* \forall v \in \mathbb{S}^{m-1}\}$  is a convex set since  $\rho(x, v)$  is quasi-concave in  $x$  as a result of convexity of  $g$  in  $x$ . Moreover evidently requisite 1. of Theorem 3.2 holds true. Next, let us turn our attention to establishing requisite 2.

To this end, let  $v \in \mathbb{S}^{m-1}$  be given and let  $g_v : \mathbb{R}_+ \rightarrow \mathbb{R}_-$  be the map defined in Lemma 4.1. From this Lemma, we know in particular that  $\rho$  is  $g_v$ -concave on the range of  $\rho(\cdot, v)$ . Hence in particular on  $C$ . Let us thus pick  $x_1, x_2 \in C$ ,  $\lambda \in [0, 1]$  arbitrarily. We thus obtain the estimate:

$$\rho(\lambda x_1 + (1 - \lambda)x_2, v) \geq g_v^{-1}(\lambda g_v(\rho(x_1, v)) + (1 - \lambda)g_v(\rho(x_2, v))).$$

Moreover, by Remark 4.3  $g_v(\rho(x_1, v)) \in (-\infty, -\beta(v)^2]$  (and likewise for  $x_2$ ) and therefore similarly  $\lambda g_v(\rho(x_1, v)) + (1 - \lambda)g_v(\rho(x_2, v)) \in (-\infty, -\beta(v)^2]$ . Now, we may apply Lemma 4.2 to derive the further estimate:

$$\begin{aligned} g_v^{-1}(\lambda g_v(\rho(x_1, v)) + (1 - \lambda)g_v(\rho(x_2, v))) &\geq m_{-3}(g_v^{-1}(g_v(\rho(x_1, v))), g_v^{-1}(g_v(\rho(x_2, v))), \lambda) \\ &= m_{-3}(\rho(x_1, v), \rho(x_2, v), \lambda). \end{aligned}$$

Finally, since  $F_{\mathcal{R}}$  is increasing as a distribution function, we may pursue our estimates to obtain:

$$F_{\mathcal{R}}(\rho(\lambda x_1 + (1 - \lambda)x_2, v)) \geq F_{\mathcal{R}}(m_{-3}(\rho(x_1, v), \rho(x_2, v), \lambda)) \geq \lambda F_{\mathcal{R}}(\rho(x_1, v)) + (1 - \lambda)F_{\mathcal{R}},$$

since  $z \mapsto F_{\mathcal{R}}(z^{-\frac{1}{3}})$  is concave on the set  $(0, (t^*)^{-3}]$  by assumption and  $\rho(x_1, v) \geq t^*$  (likewise for  $x_2$ ) so that  $\lambda \rho(x_1, v)^{-3} + (1 - \lambda)\rho(x_2, v)^{-3} \in (0, (t^*)^{-3}]$ . This concludes the proof of requisite 2.

Moreover note furthermore that the map  $g$  defined in (4.1) is convex in the second argument  $z$ . Consequently, for all  $x \in M(\frac{1}{2})$ , we have  $g(x, 0) < 0$  and requisite 3. of Theorem 3.2 holds for  $p_0 = \frac{1}{2}$ . The resulting convexity now follows from applying Theorem 3.2.  $\square$

**Remark 4.5 (Refined threshold).** When  $\xi$  is multi-variate Gaussian, the value  $t^*$  can be established to be  $\sqrt{m+3}$  and moreover the threshold can be strengthened to  $p^* = \Phi(\sqrt{m+3})$ . Note moreover that, in all cases, we do not only have convexity of  $M(p)$  for  $p > p^*$ , but even concavity of  $\varphi$  on this set, which is a rather strong property. Concavity of  $\varphi$  is not required to derive convexity of its level-sets. Indeed, quasi-concavity would suffice.

In view of this, it is of interest to recall results for the specifically structured mapping  $g(x, z) = z - x$ . In this case,  $\rho$  can be seen to be concave [66, Example 3.3]). Using Theorem 3.2, we would derive a strong property: concavity of  $\varphi$  on the sets of the form  $\{x \in \mathbb{R}^m : x \geq \mu + \sqrt{m-1} \|L\| e\}$  (with  $e$  the all-one vector); if  $\xi$  is Gaussian, this is already known (e.g., [50, Theorem 2.1]). Yet, it is also well-known that multivariate Gaussian distribution functions are log-concave and thus have all level sets convex. As a result, the true threshold in that situation would be  $p^* = 0$ .

**Example 4.6.** As a numerical illustration, we propose to take both the decision and random vector in dimension 2, i.e.,  $m = n = 2$ . We let  $\xi$  be a Gaussian random vector with mean  $\mu = 0$  and covariance matrix  $\Sigma = \begin{pmatrix} 0.01125 & 0.00675 \\ 0.00675 & 0.2025 \end{pmatrix}$ . The mapping  $W : \mathbb{R}_+^2 \rightarrow S_2^{++}$  is given by  $W(x_1, x_2) = \begin{pmatrix} x_1^2 + 0.5 & 0 \\ 0 & |x_2 - 1|^3 + 0.2 \end{pmatrix}$ ,  $2(a_1 w_1^T + a_2 w_2^T) = (1, 1)$  and  $b = -1$ .

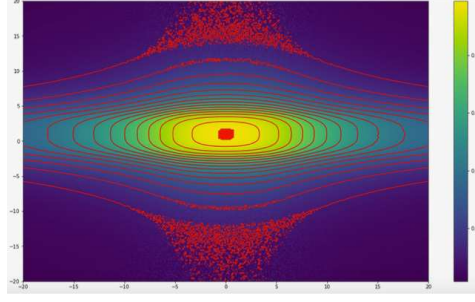
We obtain for these inputs the contour plot of Figure 4.1. The contour lines are regularly drawn from the probability value 0.1 to 0.95. We see on this graph how the level sets tend to get convex as the level increases. The red region in the center is the region where  $\rho(x, v) \geq \sqrt{m+3} = \sqrt{5}$  for any  $v \in \mathbb{S}^{m-1}$ . Proposition 4.4 allows us to establish convexity for  $p^* = \Phi(\sqrt{5}) = 0.9873$ . We see here that the set obtained, is far, from being the largest one where convexity seems to be present. This illustrates the discrepancy between the current available threshold  $p^*$ , depending, in this case, adversely on dimension and the practical situation, which seems to exhibit convexity for thresholds significantly lower!

We gather in a Python toolbox many useful functions for numerical illustrations in this framework. In particular, we provide tools for computing the probability function  $\varphi$  (with associated function  $\rho$ ) and plotting of its level-sets; as in Figure 4.1. We call this toolbox `pychance` and make it publicly available on GitHub at <https://github.com/yassine-laguel/pychance>.

**4.2. Examples with copulae.** We provide here eventual convexity statements based on results of Section 3.2 for examples that are not covered by prior results. We choose these examples with decision vectors of dimension 2 to keep calculus simple without misrepresenting our results; this dimension does not impact our results, valid in infinite dimension.

**Example 4.7 (Convexity statement for a  $m$ -dimensional copula).** Consider any  $m$  dimensional Archimedean copula, and construct the probability functions

$$\varphi(x) = \mathfrak{C}(F_1(h_1(x)), \dots, F_m(h_m(x))),$$

FIG. 4.1. Plot of the probability function  $\varphi$  for the given quadratic problem.

with a  $G$ -concave  $h_1$  and  $\alpha$ -concave  $h_i$  ( $i \geq 2$ ) defined as follows. Given remark 3.5, we know that  $\mathfrak{C}$  is concave  $-G_1^{-1}$ . Let  $h_1 : \mathbb{R}_{++}^2 \rightarrow \mathbb{R}_{++}$  be defined by  $h_1(x, y) = (\frac{y}{x^2})^2$  which is neither concave nor convex, but  $G$ -concave with  $G(x) = -1/\sqrt{x}$  as easily shown<sup>‡</sup>. Let  $h_i : \mathbb{R}_{++}^2 \rightarrow \mathbb{R}_{++}$  be defined from any convex positive function  $f_i$  by  $h_i(x, y) = f_i(x, y)^{\alpha_i}$  with  $\alpha_i \in [-6, 0)$ . Let us define now the marginal distribution functions. We take  $F_1(z) = 1 - \exp(-\lambda z)$  the exponential distribution function of parameter  $\lambda > 0$ ; we easily<sup>§</sup> see that it is concave- $G^{-1}$  on the interval  $I = [b, \infty)$  with  $b = \sqrt{2\lambda}/3$ . We take  $F_i$  to be the Rayleigh distribution function with parameter  $\sigma = 1.5$  for all  $i \geq 2$ .

We can then apply Theorem 3.7 to get the convexity of  $M(p)$  for any  $p > p^*$  with

$$p^* = \max \left\{ F_1 \left( \frac{3}{2\lambda} \right), \max_{i \geq 2} F_i \left( \sqrt{(1 - \alpha_i/2)\sigma} \right) \right\} = 0.7769$$

since  $F_1(\frac{3}{2\lambda}) = 0.7769$  and the max over  $i \geq 2$  does not exceed 0.7364. Thus the obtained threshold is fixed and independent of the dimension of the random vector.

**Remark 4.8 (Independence from the dimension).** The previous example gives a situation wherein the threshold  $p^*$  does not depend on the dimension  $m$  of the random vector. The underlying reason is that all mappings  $h_i$  are generalized concave with a set of parameters that do not degenerate with  $m$ . It should also be observed that the involved parameters (see e.g., [24, Table 1]) depend continuously on the parameters of the chosen marginal distributions and the generalized concavity parameter of  $h$ . Therefore, should these parameters belong to a compact set, the threshold  $p^*$  does not depend on dimension either. It is reasonable to assume that the mappings  $h_i$ , marginal distributions, and their parameters are homogenous. Typically the index  $i$  expresses time and the probability function stems from the desire to incorporate akin features: each component  $i$  is relatively similar.

**Example 4.9 (Convexity statement with functions from the literature).** Let us consider the functions of Example 1 of [69]: the two mappings from  $\mathbb{R}^2$  to  $\mathbb{R}_+$

$$h_1(x, y) = \exp(-(x + y)^3) \quad \text{and} \quad h_2(x, y) = \frac{1}{x^2 + y^2 + 1}$$

and the distributions  $F_1 = \Phi$ , the distribution function of a standard normal Gaussian random variable and  $F_2 = F_\chi$ , the distribution function of a  $\chi$ -random variable with 2 degrees of freedom. From Example 2.6, the map  $h_1$  can be shown to be  $G$ -concave with a given strictly decreasing map  $G$ . Moreover, Example 2.14 gives us that  $\Phi$  is concave- $G^{-1}$  on the set  $(0, G(1.86)]$ . Observe also that the map  $h_2$  is

<sup>‡</sup>We see that  $G(x)$  is strictly increasing. To verify that  $(x, y) \mapsto G(h_1(x, y)) = -\frac{x^2}{y}$  is concave, just compute its Hessian

$$-\frac{2}{y^3} \begin{pmatrix} y^2 & -xy \\ -xy & x^2 \end{pmatrix}$$

which is negative semi-definite by a direct computation (negative trace and zero determinant).

<sup>§</sup>Observe first that  $G^{-1}(z) = z^{-2}$  is convex and strictly increasing. Verify that the composition  $F_1(G^{-1}(z))$  is concave on some subset when  $z$  takes arguments in  $(-\infty, 0)$ , by a direct computation of its second derivative which gives  $2f_1(G^{-1}(z))z^{-4}(3 - 2\lambda z^{-2})$  with the density function  $f_1(z) = \lambda \exp(-\lambda z)$ .



$(-1)$ -concave (indeed,  $h_2^{-1}$  is convex) and  $F_\chi$  is concave  $-(g_{-1})^{-1}$  (see Table 1 in [66]) on the set  $(0, \sqrt{3}^{-1}]$ .

We extend now the situation of Example 1 of [69] which is restricted to the independent copula. We consider here any concave  $-g_1^{-1}$  copula (in particular for all Archimedian copulae, by Example 3.5), and we apply Theorem 3.7 to get that the feasible set  $M(p)$  with probability function

$$\varphi(x, y) = \mathfrak{C}(F_1(h_1(x, y)), F_2(h_2(x, y))),$$

is convex for  $p^* = \max \{\Phi(1.86), F_\chi(\sqrt{3})\} = \max \{0.9686, 0.7769\} = 0.9686$ . Existing theory cannot be applied to this case: [69] requires independent copula, and [58] requires  $\alpha$ -concavity (but  $h_1$  is not  $\alpha$ -concave for any  $\alpha$ ).

**Example 4.10 (Improved previous result for specific copula).** Let us go further with the previous example when  $\mathfrak{C}$  is concave  $-G_0^{-1}$  which subsumes the case of independent copula considered in [69]. In this case,  $z \mapsto \Phi(G^{-1}(z))$  is log-concave on the larger set  $(0, G(t^\star))$ , with  $t^\star = 1.6422$  (this value is numerically identified by employing the principles of [58, section 5.2]), and  $z \mapsto F_\chi(z^{-1})$  is log-concave on an interval of the form  $(0, 0.7563]$  (by Corollary 2.12). Then for any concave  $-G_0^{-1}$  copula (e.g., independent, Gumbel, Clayton), Theorem 3.7 gives the threshold

$$p^* = \max \{\Phi(t^\star), F_\chi(1.3223)\} = \max \{\Phi(1.6422), 0.5828\} = 0.9497.$$

This value is significantly better than the value  $p^* = \Phi(3) = 0.9987$  given in [69, Example 1] for the independent case only.

**Concluding Remarks.** In this paper we have provided general conditions under which probabilistic constraints define a convex set. We have investigated two different structures of the probability functions: (i) a non-separable structure with elliptically random vectors; (ii) a separable structure with the dependency expressed by a given copula. In these situations, we have employed the notion of  $G$ -concavity to reveal the level of underlying convexity in the functions defining the constraints. The obtained results are more general than those appearing in prior results. The provided conditions can be verified from the nominal problem data, as illustrated on various examples. This work raises issues about the application of our convexity results in practice for general chance-constrained optimization problems (non-emptiness and convexity of feasible sets, guarantees of solutions... see the discussion in Section 1.3). A related question is the improvement of the computed thresholds. In some specific situations indeed, it might be possible to lower the computed threshold  $p^*$  by exploiting structure of the problems. More generally, more work is required in order to reduce to gap between observed convexity and guaranteed convexity.

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