Math 317 Assignment 2

Due in class: October 20, 2016

Instructions: Submit a hard copy of your solution with your name and student number. (**No name = zero grade!**) You must include all relevant program code, electronic output and explanations of your results. Write your own codes and comment them. Late assignment will not be graded and will receive a grade of zero.

1. Using Newton's method, find a zero to the system of equations

$$F(x,y) = \begin{pmatrix} x^2 + y^2 - 4 \\ x + \sin(xy) - y \end{pmatrix},$$

with $\vec{x_0} = (1,1)^T$. Use the stopping criterion of $||F(\vec{x_k})||_2 < 10^{-8}$.

2. Consider the following data set $\{(-1,0),(1,6),(2,9)\}$.

   (a) Compute the Lagrange interpolation polynomial $L_2(x)$:

      i. solving the linear system with the Vandermonde matrix;

      ii. using the Lagrange basis polynomials;

      iii. using Newton's divided differences.

   (b) Add the point $(-2,3)$ to the data set. Compute the Lagrange interpolation polynomial $L_3(x)$.

3. For a general set of points $\{x_0,\ldots,x_n\}$ belonging to $[a,b]$, we derived in class the error bound of Lagrange interpolation,

$$|f(x) - L_n(x)| \le \frac{M_{n+1}}{(n+1)!} \max_{x\in[a,b]} |(x-x_0)\cdots(x-x_n)|,$$

where $M_i = \max_{x\in[a,b]} |f^{(i)}(x)|$. This exercise is to show an error bound for the case when the points are equally-spaced, i.e. $x_k = a + kh$ with $h = \frac{b-a}{n}$.

   (a) For any $x \in [a,b)$, we can write $x = x_0 + (k+t)h$ for some unique $0 \le t < 1$ and $k = 0,1,\ldots,n-1$. Show that,

$$|(x-x_0)\cdots(x-x_n)| = h^{n+1}(k+t)(k-1+t)\cdots(1+t)t(1-t)(2-t)\cdots(n-k-t).$$

   (b) Show that for $0 \le t \le 1$

      i. $t(1-t) \le \frac{1}{4}$

      ii. $(k+t)(k-1+t)\cdots(1+t) \le (k+1)!$

      iii. $(2-t)\cdots(n-k-t) \le (n-k)!$.

   (c) Show that $(k+1)!(n-k)! \le n!$ for any $k = 0,1,\ldots,n-1$ and conclude that for equally-spaced points,

$$|f(x) - L_n(x)| \le M_{n+1}\frac{h^{n+1}}{4(n+1)}.$$

4. Consider the function $f(x) = x + x^4$. The goal of this question is to compute the Hermite interpolation polynomial using two different approaches.

(a) Set up an appropriate linear system and compute the Hermite interpolation polynomial of f with nodes $x_0 = 0$, $x_1 = 1$.

(b) Recall the definition of Hermite basis functions.

Definition. *Given the nodes* $\{x_0, \ldots, x_n\}$, *for* $k = 0, \ldots, n$, *the Hermite basis functions* $h_k(x)$ *and* $\hat{h}_k(x)$ *are polynomials of degree* $2n+1$ *which satisfy*

$$h_k(x_i) = \begin{cases} 0 & \text{if } i \neq k, \\ 1 & \text{if } i = k, \end{cases} \quad \hat{h}_k(x_i) = 0, \quad h'_k(x_i) = 0, \quad \hat{h}'_k(x_i) = \begin{cases} 0 & \text{if } i \neq k, \\ 1 & \text{if } i = k., \end{cases}$$

for $i = 0, \ldots, n$

Compute the Hermite basis functions associated with the nodes $x_0 = 0$, $x_1 = 1$.

(c) Show that for $k = 0, \ldots, n$, the Hermite basis functions can be expressed as

$$h_k(x) = (1 - 2(x - x_k) l'_k(x_k))\, l_k^2(x) \quad \text{and} \quad \hat{h}_k(x) = (x - x_k) l_k^2(x),$$

where l_k is the k-th Lagrange basis polynomial. *Hint: you simply have to check the definition given in (b).*

(d) Use the formulas in (c) to compute the Hermite basis functions associated with the nodes $x_0 = 0$, $x_1 = 1$. Compare your answer with (b).

(e) Compute the Hermite interpolation polynomial of f with nodes $x_0 = 0$, $x_1 = 1$ using the Hermite basis functions computed in (b) and (d).

5. Determine the natural cubic spline S that interpolates the data $f(0) = 3$, $f(1) = 2$ and $f(2) = 9$.

6. Find a, b, c so that the finite difference of the first derivative

$$D_h f(x_0) = a f(x_0) + b f(x_0 + h) + c f(x_0 + 2h)$$

has the highest degree of accuracy possible. State this degree.