Math 317 Assignment 4

Due in class: November 29th, 2016

Instructions: Submit a hard copy of your solution with your name and student number. (**No name = zero grade!**) You must include all relevant program code, electronic output and explanations of your results. Write your own codes and comment them. Late assignment will not be graded and will receive a grade of zero.

1. Consider the boundary value problem,

$$\begin{cases} -u''(x) + 3u'(x) - 2u(x) = f(x) \quad \text{on } [0,1], \\ u(0) = \alpha, \\ u(1) = \beta. \end{cases} \tag{1}$$

   (a) (10 marks) Design a finite difference method to solve the B.V.P. with the local truncation error of $\mathcal{O}(h^2)$. State the entries of your resulting linear system in the form $A_h\vec{u}_h = \vec{f}_h$.

   (b) (5 marks) Find the exact solution to (1) for $\alpha = 1, \beta = 1 + e$ and $f(x) = 3 - 2x$.

   (c) (20 marks) For the B.V.P. from part (b), perform a convergence analysis of your method (measure the error in the l_2 norm) and deduce the convergence rate. In order to do so write a program to approximate the solution of (1) using the finite difference method define in (a) for $h = 2^{-1}, \ldots 2^{-8}$, plot log(error) versus log(h) and deduce the convergence rate by estimating the slope in the loglog plot.

2. (35 marks) Consider the prescribed curvature problem which arises in modelling beam deflection:

$$\begin{cases} -\frac{u''(x)}{(1+u'(x)^2)^{\frac{3}{2}}} = 2x(1-x) \quad \text{on } [0,1], \\ u(0) = 0, \\ u(1) = 0. \end{cases}$$

   Solve the nonlinear B.V.P. using the shooting method with the Newton's method. Choose initial slope of $s_0 = 0$ and use a stopping tolerance of $\epsilon = 10^{-10}$. Use the forward Euler method to solve the associated I.V.P. with $N = 100$. Plot the successive solutions of different slopes. How many iterations of the Newton method is needed to reach the tolerance?

3. Consider the linear system,

$$\begin{aligned} 3x_1 - x_2 \qquad &= 2 \\ -x_1 + 3x_2 - x_3 &= 2 \\ -x_2 + 3x_3 &= -1 \end{aligned} \qquad \text{with } \boldsymbol{x}_0 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}.$$

   (a) (5 marks) Find the exact solution by Gaussian elimination.

   (b) (5 marks) Compute by hand the first iteration of the Jacobi method and the Gauss-Seidel method.

   (c) (20 marks) Compare the number of iterations required to reach a successive relative error of 10^{-10} in the l_2 norm for the method of Richardson ($\omega = 0.2$), Jacobi, Gauss-Seidel and SOR ($\theta = 1.1$).