1. Define a collection C of subsets of $\mathbb{R}$ by

$$C := \{A \subseteq \mathbb{R} : \textit{ either } A \textit{ is countable or } A^c \textit{ is countable}\}$$

show that $\sigma(C) \neq \mathcal{B}(\mathbb{R})$, i.e., the Borel field $\sigma-$field is Not generated by subsets of $\mathbb{R}$ that are countable.

answer:
First let's show that C is a $\sigma-$field;

- $\mathbb{R}$ belongs to C because $\mathbb{R}^c = \emptyset$
- A belong to C, meaning A is countable or A^c is countable. If A^c is countable then A and A^c are both in C. if A is countable, then we know that $A = (A^c)^c$ is countable, hence A^c is in C. $\implies$ if $A \in C$ then $A^c \in C$.
- let $A_i \in C$, if all A_i are countable then $\bigcup_{i=1}^{\infty} A_i \in C$. If there exists an A_{no} such that A_{no} is not countable, then A^c_{no} is countable. $(\bigcup_{i=1}^{\infty} A_i)^c = \bigcap_{i=1}^{\infty} A_i^c \subset A^c_{no}$, hence countable. $\implies \bigcup_{i=1}^{\infty} A_i \in C$.

we just showed $\sigma(C) = C$, lets take the interval (0,1), $(0,1) \notin C$ because it is not countable and neither its complement, but (0,1) is in $\mathcal{B}(\mathbb{R})$. Hence, $\sigma(C) \neq \mathcal{B}(\mathbb{R})$

2. Let's consider the set $E_n = \{\omega \in \Omega : \mathbb{P}(\{\omega\}) > 0\}$

answer:
Every set has less than n elements, hence finite. Or $\bigcup_{i=1}^{\infty} E_n$ is the set off elements with positive probability. $(\bigcup_{i=1}^{\infty} E_n$ is a finite set because it is the union of finite, countable sets. $\implies$ $\mathbb{P}$ can only assign positive integer to at most countable singletons.

3. Apply a similar method as the one explained in class to solve the problem: if a fair coin is flipped infinitely many times, what is the probability that successive three heads never occur?

   (a) For each $n \in N$. Define A_n to be the event that successive 3 heads never occur in the first n flips, and a_n to be the number of sample points in An, i.e.,a_n is the number of ways of flipping a coin n times without having successive 3 heads in the outcomes. First write down $a_1,\ a_2 \textit{ and } a_3$. Then argue that

$$a_n = a_{n-1} + a_{n-2} + a_{n-3}$$

answer:
$a_1 = 2$, either H or T. $a_2 = 4$, HH, HT, TH or TT. $a_3 = 7$, all the possible outcomes except HHH, that's $8 - 1 = 7$.
First let's take a look at a_n, If the first toss is Tails, then we don't want 3 successive Heads in the remaining n-1 tosses, but if the first Toss is Heads, then we take a look at the second toss, now if the 2nd toss is Tails, then we dont want any 3 successive Heads in the remaining n-2 tosses, but if it is Heads, then the 3rd toss must be tails and then no successive 3 Heads in the remaining n-3 tosses.
$\implies a_n = a_{n-1} + a_{n-2} + a_{n-3}$

(b) Show that for every $n \geq 4$, $a_n < (1.9)^n$, and argue that the probability of never having successive 3 heads when flipping a fair coin infinitely many times is zero.

answer:
For the first part, proof by induction; Q(n) = $a_n < (1.9)^n$.
Base case: For n = 4, $a_4 = a_3 + a_2 + a_1 = 7 + 4 + 2 = 13$ or $1.9^4 = 13.03 \implies a_4 < (1.9)^4$.
$a_5 = 24 < 24.76 \approx (1.9)^5$
$a_6 = 44 < 47.04 \approx (1.9)^6$
Let's assume that indeed Q(n) holds. Let's prove that for Q(n+1).

Induction step: $a_{n+1} = a_n + a_{n-1} + a_{n-2} < (1.9)^n + (1.9)^{n-1} + (1.9)^{n-2}$
$\rightarrow \mathrm{a}_n + a_{n-1} + a_{n-2} < (1.9)^{n+1}\left(\frac{1}{1.9} + \frac{1}{1.9^2} + \frac{1}{1.9^3}\right) \approx 0.95 \times (1.9)^{n+1} < (1.9)^{n+1}$
Hence $a_{n+1} < (1.9)^{n+1}$
$\implies a_n < (1.9)^n$

We know that $\mathbb{P}(A_n) = \frac{a_n}{2^n}$, then $\mathbb{P}(A_n) < \frac{(1.9)^n}{2^n} = \left(\frac{1.9}{2}\right)^n$. When $n \to 0$, $\mathrm{P}(A_n)$ goes to Zero as well, because $\frac{1.9}{2} < 1$
$\implies$ the probability of never having successive 3 heads when flipping a fair coin infinitely many times is zero.

4. Assume that there are n students in a class. Also assume that one year has 365 days, $n \leq 365$

(a) What is the probability that AT LEAST two students have the same birthday, First determine the exact formula (in terms of n), and then use a calculator/computer to compute the (approximated) numerical value of that probability for n= 5,n= 23 and n= 65.
answer:
Let the event A_n: At least two people have the same birthday.
$\overline{A_n}$: n people having disticnt birthday.
$\mathbb{P}(A_n) = 1 - \mathbb{P}(\overline{A})$

$\mathbb{P}(\overline{A_n}) = \frac{356}{356} \times \frac{364}{356} \times \frac{363}{356} \times ... \times \frac{365-n+1}{365} = \frac{365!}{(365-n)! \times 365^n}$
$\mathbb{P}(A_5) \approx 0.0271$
$\mathbb{P}(A_23) \approx 0.507$
$\mathbb{P}(A_65) \approx 0.998$

(b) What is the probability that at least three students have the same birthday? Leave your answer in the form of an exact formula (in terms of n).
answer:
We can our partition into: Distinct birthdays, 1 pair share a birthday, 2 pairs share a birthday, ... $\frac{n}{2}$ *or* $(\frac{n-1}{2}\ ; n\ odd)$ pairs share a birthday. A: at least 3 people sharing a birthday. B: 2 people share birthday from 1 to n/2 pairs. C: Distinct birthdays The probability of k pairs sharing a birthday is:
$P(B) = \frac{C(n,2)C(n-2,2)...C(n-2k+2,2)P(365-k,n-2k)}{365^n}$.
Hence $\mathbb{P}(A) = 1 - \mathbb{P}(B) - \mathbb{P}(C)$
$= 1 - \frac{P(365,n)}{365^n} - \sum_{n=1}^{\lfloor n/2 \rfloor} \frac{C(n,2)C(n-2,2)...C(n-2k+2,2)P(365-k,n-2k)}{365^n}$

5. Let b,r and c be three positive integers. An urn contains b blue balls and r red balls. One ball is drawn at random from the urn, its color noted, and the ball itself plus c extra balls of the same color are put back into the urn; the process is then repeated.

(a) Determine the probability that the result of the first three draws areblue, blue, red, i.e., the first two balls are blue and the third is red.
answer:
Let E_1: first ball is blue.
Let E_2: second ball is blue.
Let E_3: third ball is red.

$\mathbb{P}(E_1 \cap E_2 \cap E_3) = \mathbb{P}(E_1) \times \mathbb{P}(E_2|E_1) \times \mathbb{P}(E_3|E_2 \cap E_3)$
$\mathbb{P}(E_1) = \frac{b}{r+b}$
$\mathbb{P}(E_2|E_1) = \frac{b+c}{r+b+c}$
$\mathbb{P}(E_3|E_1 \cap E_2) = \frac{r}{r+b+2c}$

$\mathbb{P}(E_1 \cap E_2 \cap E_3) = \mathbb{P}(E_1) \times \mathbb{P}(E_2|E_1) \times \mathbb{P}(E_3|E_2 \cap E_3) =$
$\frac{b}{r+b} \times \frac{b+c}{r+b+c} \times \frac{r}{r+b+2c}$

(b) Show by induction that the probability of the nth ball drawn being red is r/r+b.
answer:
For n = 1 (First Draw), $\mathbb{P}(R_1) = \frac{r}{r+b}$

Let's assume this holds for (n-1), $\mathbb{P}(R_{n-1}) = \dfrac{r}{r+b}$.
Let's prove it for $P(R_n)$

$$\mathbb{P}(R_n) = \mathbb{P}(R_n|B_{n-1})\mathbb{P}(B_{n-1}) + \mathbb{P}(R_n|R_{n-1})\mathbb{P}(R_{n-1})$$

(Note, $\mathbb{P}(B_{n-1}) = 1 - \mathbb{P}(R_{n-1}) = \dfrac{r+b-r}{r+b} = \dfrac{b}{r+b}$)

Hence, $\mathbb{P}(R_n) = (\dfrac{r}{r+b+c})(\dfrac{b}{r+b}) + (\dfrac{r+c}{r+b+c})(\dfrac{r}{r+b})$,

Then $\mathbb{P}(R_n) = \dfrac{r}{r+b} \times \dfrac{b+(r+c)}{r+b+c} = \dfrac{r}{r+b}$

6. There are n urns marked 1,2,..,n, and each of n urns contains 4 white and 6 black balls. There is another urn, marked (n+1), containing5 white and 5 black balls. An urn is chosen at random from the (n+1) urns, and two balls are drawn at random from that urn, both being black. The probability that 5 white and 3 black balls are left in the chosen urn is 1/7. Determine the value of n.
answer:
let A: The event 2 black balls chosen randomly from a randomly chosen urn.
and B_j: The jth urn is chosen.

We know that $\mathbb{P}(B_{n+1}|A) = 1/7$

We're gonna use Bayer's rule to solve for n. We have 5 white and 3 blacks ball remaining, then the chosen urn is the (n+1)th.

$$\mathbb{P}(B_{n+1}|A) = \frac{\mathbb{P}(A|B_{n+1})\mathbb{P}(B_{n+1})}{\sum_{k=1}^{n+1}\mathbb{P}(A|B_k)\mathbb{P}(B_k)}$$

.

for j = 1,..,n urns; $\mathbb{P}(A|B_j) = \dfrac{C(6,2)}{C(10,2)} = \dfrac{1}{3}$

$\mathbb{P}(A|B_{n+1}) = \dfrac{C(5,2)}{C(10,2)} = \dfrac{2}{9}$

So, $\mathbb{P}(B_{n+1}|A) = \dfrac{\frac{2}{9}\times\frac{1}{n+1}}{\frac{n}{3(n+1)}+\frac{2}{9(n+1)}} = \dfrac{\frac{2}{9(n+1)}}{\frac{3n+2}{9(n+1)}} = \dfrac{2}{3n+2}$.

So we have $\dfrac{2}{3n+2} = \dfrac{1}{7} \to 3n+2 = 14 \to n = 4$

7. Let $(\Omega, \S, \mathbb{P})$ be a probability space where all the singletons are events. Suppose $\{A_n : n \leq N\}$. S is an countable sequence of independent events, i.e., for any $i_1 < i_2 < ... < i_N$ where $i_k \in N$for k= 1,..,N, events $A_{i1}, Ai2, , A_{iN}$ are independent. Show that if P(An) =12 for all n $\leq$ N, then all the singletons are null sets.
answer:

let $\omega \in \Omega$. Any i $\in \mathbb{N}$, ω_i is in A_i or A_i^c. So, lets consider the following Events E_i which are either, A_i or $A_i^c \rightarrow \mathbb{P}(E_i) = 1/2$. We know that A_i are independent for all i. Then the same goes for E_i for all i in $\mathbb{N}$. Then $\mathbb{P}(\omega) \leq \prod_{i=1}^{N} \mathbb{P}(E_i) = \dfrac{1}{2^N}$. Or N is arbitrarily large, Then $\mathbb{P}(\omega) = 0$

8. Let$(\Omega, \S, \mathbb{P})$ be a probability space and $\{An : n \in N\}$ S a countable sequence of events. Define,

$$A^* = \bigcap_{1 \leq k} \bigcup_{n \geq k} A_n$$

   (a) Show that if $\sum_{n \geq 1} \mathbb{P}(A_n) < \infty \; Then \; \mathbb{P}(A^*) = 0$

   answer:

   Let $B_k = \bigcup_{k \leq n} A_n$. as $k \rightarrow \infty$, $B_k \searrow A^*$, then by continuity from above we get,

$$\mathbb{P}(A^*) = \lim_{k \rightarrow \infty} \mathbb{P}(B_k) \leq \lim_{k \rightarrow \infty} \sum_{n \geq k} \mathbb{P}(A_n) = 0$$

   because $\sum_{n=1} \mathbb{P}(A_n)$is finite, A_n must converges to Zero. Then $\mathbb{P}(A*) = 0$

   (b) Further assume that $\{A_n : n \in \mathbb{N}\}$ is a sequence of independent events, Show that if $\sum_{n \geq 1} \mathbb{P}(A_n) = \infty \; Then \; \mathbb{P}(A^*) = 1$

   answer:

   We use De mogre's law :

$$(A^*)^c = \bigcap_{1 \leq k} \bigcup_{n \geq k} A_n^c$$

   Let $C_k = \bigcap_{k \leq n} A_n^c$. As $k \rightarrow \infty$, $C_k \nearrow A_n^c$, So we get :

$$\mathbb{P}((A^*)^c) = \lim_{k \rightarrow \infty} \mathbb{P}(C_k) = \lim_{k \rightarrow \infty} \prod_{k \leq n} (A^*)^c = \lim_{k \rightarrow \infty} \prod_{k \leq n} (1-(A^*)) \leq \lim_{k \rightarrow \infty} e^{\sum_{n \geq k} \mathbb{P}(A_n)} = e^{-\infty} = 0$$

   Then $\mathbb{P}(A^*) = 1 - \mathbb{P}((A^*)^c) = 1 - 0 = 1$.