Math 318: Assignment 3 Solutions

Problem 1

Part (A)

q, p	0	1
0	0	1
1	1	1

This is not a tautology.

Part (B)

q, p	0	1
0	1	0
1	1	1

This is not a tautology.

Part (C)

q, p	0	1
0	1	1
1	1	1

This is a tautology.

Part (D)

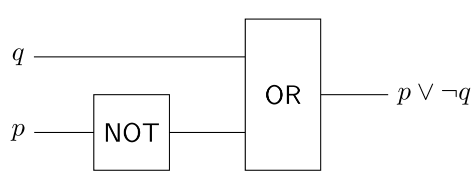
q, p	0	1
0	1	0
1	1	1

This is not a tautology.

Problem 2

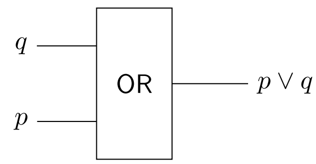
Part (A)

A formula for this truth table is $p \vee \neg q$, which is represented by the following diagram:



Part (B)

A formula for this truth table is $p \vee q$, which is represented by the following diagram:



Problem 3

We have:

$$((p \vee \neg q) \wedge (r \vee p)) \vee r \equiv (p \vee \neg q \vee r) \wedge (r \vee p \vee r) \equiv (p \vee \neg q \vee r) \wedge (p \vee r) \equiv p \vee r$$

This is the DNF and the CNF.

Problem 4

Part (1)

Note that $\neg \phi \equiv \phi \operatorname{NAND} \phi$. Thus:

$$p \to q \equiv \neg p \vee q \equiv p \operatorname{NAND} \neg q \equiv p \operatorname{NAND} (q \operatorname{NAND} q)$$

Part (2)

We have:

$$(p \wedge q) \vee \neg p \equiv (p \vee \neg p) \wedge (q \vee \neg p) \equiv \neg p \vee q \equiv \neg(\neg p \operatorname{NOR} q)$$

By using $\neg \phi \equiv \phi \operatorname{NOR} \phi$, we have:

$$\neg(\neg p \operatorname{NOR} q) \equiv \neg((p \operatorname{NOR} p) \operatorname{NOR} q) \equiv ((p \operatorname{NOR} p) \operatorname{NOR} q) \operatorname{NOR} ((p \operatorname{NOR} p) \operatorname{NOR} q)$$

Problem 5

We will show that $\{\vee, \wedge\}$ is not 1-complete.
Let F be the set of formulas in $\{p, \vee, \wedge\}$ and let $s : \{p\} \to \{0, 1\}$ be the truth assignment $s(p) = 0$. We claim that for any $\phi \in F$, we have $\tilde{s}(\phi) = 0$. We proceed by induction on ϕ. The base case is $\phi = p$, and we have $\tilde{s}(\phi) = \tilde{s}(p) = s(p) = 0$. For the inductive step, suppose we have $\psi, \psi' \in F$ with $\tilde{s}(\psi) = \tilde{s}(\psi') = 0$. Then $\tilde{s}(\psi \vee \psi') = \max\{\tilde{s}(\psi), \tilde{s}(\psi')\} = 0$ and $\tilde{s}(\psi \wedge \psi') = \min\{\tilde{s}(\psi), \tilde{s}(\psi')\} = 0$. Thus the claim holds by induction.
Now $\tilde{s}(\neg p) = 1 - \tilde{s}(p) = 1$, so $\neg p \notin F$. Thus $\{\vee, \wedge\}$ is not 1-complete, and thus it is not complete. $\square$

Problem 6

We will show that $\{\text{XOR}\}$ is not 1-complete.
Let F be the set of formulas in $\{p, \text{XOR}\}$ and let $s : \{p\} \to \{0, 1\}$ be the truth assignment $s(p) = 0$. We claim that for any $\phi \in F$, we have $\tilde{s}(\phi) = 0$. We proceed by induction on ϕ. The base case is $\phi = p$, and we have $\tilde{s}(\phi) = \tilde{s}(p) = s(p) = 0$. For the inductive step, suppose we have $\psi, \psi' \in F$ with $\tilde{s}(\psi) = \tilde{s}(\psi') = 0$. Then $\tilde{s}(\psi \text{ XOR } \psi') = 0$. Thus the claim holds by induction.
As in the previous problem, we have $\neg p \notin F$ and thus $\{\text{XOR}\}$ is not 1-complete, and thus it is not complete. $\square$

Problem 7

Define the formulas ϕ_n recursively for $n \geq 1$ by $\phi_1 = p$ and $\phi_{n+1} = \phi_n \to p$. We are trying to find which ϕ_n are tautologies.
We claim that we have:

$$\phi_n \equiv \begin{cases} p & n \text{ odd} \\ \top & n \text{ even} \end{cases}$$

We proceed by induction on $n \geq 1$. The base case is $n = 1$, and we have $\phi_1 = p$ (by definition). For the inductive step, there are two cases:

- If n is even, then we have:
$$\phi_n = \phi_{n-1} \to p \equiv p \to p \equiv \top$$
- If n is odd, then we have:
$$\phi_n = \phi_{n-1} \to p \equiv \top \to p \equiv p$$

Thus the claim holds by induction. Thus ϕ_n is a tautology iff n is even. $\square$

Problem 8

Note that we have:

$$a \leftrightarrow b \equiv (a \to b) \wedge (b \to a) \equiv (b \to a) \wedge (a \to b) \equiv b \leftrightarrow a$$

In other words, thus $\leftrightarrow$ is commutative.
We also have:

$$\begin{aligned} (a \leftrightarrow b) \leftrightarrow c &\equiv ((a \wedge b) \vee (\neg a \wedge \neg b)) \leftrightarrow c \\ &\equiv (((a \wedge b) \vee (\neg a \wedge \neg b)) \wedge c) \vee (\neg((a \wedge b) \vee (\neg a \wedge \neg b)) \wedge \neg c) \\ &\equiv (a \wedge b \wedge c) \vee (\neg a \wedge \neg b \wedge c) \vee ((\neg a \vee \neg b) \wedge (a \vee b) \wedge \neg c) \\ &\equiv (a \wedge b \wedge c) \vee (\neg a \wedge \neg b \wedge c) \vee (\neg a \wedge b \wedge \neg c) \vee (a \wedge \neg b \wedge \neg c) \end{aligned}$$

A similar argument shows that $a \leftrightarrow (b \leftrightarrow c)$ is also equivalent to this, so $(a \leftrightarrow b) \leftrightarrow c \equiv a \leftrightarrow (b \leftrightarrow c)$. In other words, $\leftrightarrow$ is associative. Since $\leftrightarrow$ is commutative and associative, we can ignore the ordering of variables and any parentheses in expressions involving only $\leftrightarrow$.
Now let ϕ be a formula whose only connective is $\leftrightarrow$, and let $\{p_1, \ldots, p_n\}$ be the set of variables used in ϕ. For each $1 \leq i \leq n$, let m_i be the number of times that p_i appears in ϕ. Define ψ_i by:

$$\psi_i = \underbrace{p_i \leftrightarrow \cdots \leftrightarrow p_i}_{m_i \text{ times}}$$

Then we have:

$$\phi \equiv \psi_1 \leftrightarrow \cdots \leftrightarrow \psi_n$$

Note that by an identical argument to Problem 7, we have:

$$\psi_i \equiv \begin{cases} p_i & m_i \text{ odd} \\ \top & m_i \text{ even} \end{cases}$$

Now suppose that each m_i is even. Then we have:

$$\phi \equiv \underbrace{\top \leftrightarrow \cdots \leftrightarrow \top}_{n \text{ times}} \equiv \top$$

So ϕ is a tautology.

On the other hand, suppose that m_k is odd for some $1 \leq k \leq n$. Consider the truth assignment s defined by:

$$s(p_i) = \begin{cases} 0 & i = k \\ 1 & i \neq k \end{cases}$$

Then we have $\tilde{s}(\psi_k) = \tilde{s}(p_k) = 0$ and for $i \neq k$, we have:

$$\tilde{s}(\psi_i) = \begin{cases} \tilde{s}(p_i) & m_i \text{ odd} \\ \tilde{s}(\top) & m_i \text{ even} \end{cases} = \begin{cases} 1 & m_i \text{ odd} \\ 1 & m_i \text{ even} \end{cases} = 1$$

Thus:

$$\tilde{s}(\phi) = 0 \leftrightarrow \underbrace{1 \leftrightarrow \cdots \leftrightarrow 1}_{n-1 \text{ times}} = 0$$

Thus ϕ is not a tautology. $\square$