Assignment 1

Honours Probability

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Proof. $\sigma(\mathcal{C})$ is the smallest $\sigma-$algebra containing all the sets in $\mathcal{C}$. We will show that $\sigma(\mathcal{C}) = \mathcal{C}$. To do this, we will show that $\mathcal{C}$ is a $\sigma-$algebra, and must therefore contain $\sigma(\mathcal{C})$. It is clear the empty set is in $\mathcal{C}$, and that by definition it is closed under complements, so we only need to check the third property of $\sigma-$algebras, that is it closed under countable unions, therefore if $A_i \in \mathcal{C}, i \in \mathbb{N}$,

$$\bigcup_{i\in\mathbb{N}} A_i \in \mathcal{C}$$

If the A_i are all countable, their union is too, so their union will be in $\mathcal{C}$. Now assume without loss of generality that for some $k \in \mathbb{N}$, A_k is uncountable. Then $\overline{A_k}$ must be countable. So

$$\overline{\bigcup_{i\in\mathbb{N}} A_i} = \bigcap_{i\in\mathbb{N}} \overline{A_i} = \overline{A_k} \cap \left(\bigcap_{i\in\mathbb{N}\setminus\{k\}} \overline{A_i} \right)$$

Therefore, as a subset of a countable set, $\overline{\bigcup_{i\in\mathbb{N}} A_i}$ is countable and so $\bigcup_{i\in\mathbb{N}} A_i \in \mathcal{C}$. We have shown that $\mathcal{C}$ is closed under countable unions, so $\mathcal{C}$ is a $\sigma-$algebra and $\sigma(\mathcal{C}) = \mathcal{C}$. However, $\mathcal{C}$ does not contain any open intervals since both they and their complement are uncountable, so $\sigma(\mathcal{C}) \neq \mathcal{B}(\mathbb{R})$. □

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Proof. Let $E_n = \left\{\omega \in \Omega : \mathbb{P}(\{\omega\}) > \frac{1}{n}\right\}$. Each E_n must contain finitely many singletons, else the sum of the singletons in E_n would be infinite, contradicting $\mathbb{P}(\Omega) = 1$. Note that

$$\bigcup_{n\in\mathbb{N}} E_n$$

contains all singletons with positive probability, since if $\mathbb{P}(\{\omega\}) > 0$, there exists $N \in \mathbb{N}$ such that $\frac{1}{N} < \mathbb{P}(\omega)$. As the countable union of finite sets, $\bigcup_{n\in\mathbb{N}} E_n$ is countable, as desired. □

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(i)

Proof. Define A_n to be the event such that HHH does not occur in the n first flips, and $|A_n| = a_n$. a_n is the number of ways to flip a coin n times without seeing three consecutive heads. Clearly, $a_1 = 2$, $a_2 = 4$, and $a_3 = 7$. To define a_n as a recurrence, note that if the first flip is T, there are a_{n-1} ways to flip the coin $n-1$ times without getting HHH, if the first two flips are HT, there are a_{n-2} ways to flip the coin $n-2$ times without getting HHH, and if the first three flips are HHT,there are a_{n-3} ways to flip the coin $n-3$ times without getting HHH. Since no other sequence of flips can be in A_n, we have, as desired,

$$a_n = a_{n-1} + a_{n-2} + a_{n-3}$$

□

(ii)

Proof. We will use strong induction. For the three bases case we require, $a_4 = 13 < 1.9^4 = 13.0321$, $a_5 = 24 < 1.9^5 = 24.76099$ and $a_6 = 44 < 1.9^6 = 47.045881$. Now let $n \geq 7$, and for the induction hypothesis, assume $a_n < 1.9^n$ for all n from 4 to n, including, notably, a_n, a_{n-1} and a_{n-2}. Then, using this hypothesis,

$$\begin{aligned} a_{n+1} &= a_n + a_{n-1} + a_{n-2} \\ &< 1.9^n + 1.9^{n-1} + 1.9^{n-2} \\ &< 1.9^{n+1} \left(\frac{1}{1.9} + \frac{1}{1.9^2} + \frac{1}{1.9^3} \right) \end{aligned}$$

Since $(\frac{1}{1.9}+\frac{1}{1.9^2}+\frac{1}{1.9^3})\approx 0.949<1$, we have $a_{n+1}<1.9^{n+1}$, completing the inductive step. Therefore, for all $n\in\mathbb{N}$,

$$a_n < 1.9^n$$

Now, notice

$$\mathbb{P}(A_n)=\frac{a_n}{2^n}<\left(\frac{1.9}{2}\right)^n$$

Thus, we have, as desired

$$\lim_{n\to\infty}\mathbb{P}(A_n)=0$$

□

4

(i)

Let's call the event that at least two people in a class of n students share a birthday A_n. Then $\overline{A_n}$ is the event that no students in the class of n students share a birthday. Clearly, $\mathbb{P}(\overline{A_1})=1$, and by the pigeonhole principle, $\mathbb{P}(\overline{A_{366}})=0$. If all are birthdays uniformly distributed, we have

$$\mathbb{P}(\overline{A_n})=\frac{365!}{(365-n)!\times 365^n}$$

Since $\overline{A_n}$ is complementary to A_n, $\mathbb{P}(A_n)=1-\mathbb{P}(\overline{A_n})$. Computing for a few values of n,

$$\mathbb{P}(A_5)=0.02716$$
$$\mathbb{P}(A_{23})=0.5073$$
$$\mathbb{P}(A_{65})=0.9977$$

(ii)

The event that at least 3 students share a birthday is complementary to to the event that no students or up to $\lfloor\frac{n}{2}\rfloor$ pairs of students have the same birthday. The probability that i pairs of students out of n have the same birthday is

$$\mathbb{P}(B_{i,n})=\frac{365!n!}{i!(365-n+1)!2^i(n-2i)!}$$

Therefore, the probability of i pairs for any i from 0 to $\lfloor\frac{n}{2}\rfloor$ is

$$\mathbb{P}(P_n)=\sum_{i=0}^{\lfloor\frac{n}{2}\rfloor}\mathbb{P}(B_{i,n})=\sum_{i=0}^{\lfloor\frac{n}{2}\rfloor}\frac{365!n!}{i!(365-n+1)!2^i(n-2i)!}$$

Therefore, the probability of at least one triple T_n is

$$\mathbb{P}(T_n)=1-\mathbb{P}(P_n)$$

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(i)

We have that

$$\begin{aligned}\mathbb{P}(B_1\cap B_2\cap R_3)&=\mathbb{P}(B_1)\cdot\mathbb{P}(B_2|B_1)\cdot\mathbb{P}(B_3|(B_2\cap B_1))\\&=\frac{b}{r+b}\cdot\frac{b+c}{r+b+c}\cdot\frac{r}{r+b+2c}\end{aligned}$$

(ii)

PROOF. Let R_n denote the event that the nth ball drawn is red. Clearly $\mathbb{P}(R_1) = \frac{r}{r+b}$, so we have a base case for our induction. Let r_n be the number of red balls after n draws and b_n be the number of blue balls. Let us assume as induction hypothesis $\mathbb{P}(R_n) = \frac{r_n}{r_n+b_n} = \frac{r}{r+b}$. Then

$$\begin{aligned}
\mathbb{P}(R_{n+1}) &= \mathbb{P}(R_{n+1} \cap R_n) + \mathbb{P}(R_{n+1} \cap B_n) \\
&= \mathbb{P}(R_{n+1}|R_n)\mathbb{P}(R_n) + \mathbb{P}(R_{n+1} \cap B_n)\mathbb{P}(B_n) \\
&= \frac{r_n + c}{r_n + b_n + c} \cdot \frac{r_n}{r_n + b_n} + \frac{r_n}{r_n + b_n + c} \cdot \left(1 - \frac{r_n}{r_n + b_n}\right) \\
&= \frac{r_n(r_n + b_n + c)}{(r_n + b_n + c)(r_n + b_n)} \\
&= \frac{r_n}{r_n + b_n} = \frac{r}{r+b}
\end{aligned}$$

So if it is true for n, it is true for $n+1$, and by the principle of induction it is true for all $n \in \mathbb{N}$. $\square$

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Let U_k be the event that the kth urn is chosen and A the event that two balls drawn are black. For $k = 1, \ldots, n$,

$$\mathbb{P}(A|U_k) = \frac{6}{10} \times \frac{5}{9} = \frac{1}{3}$$

For $k = n+1$,

$$\mathbb{P}(A|U_k) = \frac{5}{10} \times \frac{4}{9} = \frac{2}{9}$$

Using that $\mathbb{P}(U_{n+1}|A) = \frac{1}{7}$,

$$\frac{1}{7} = \mathbb{P}(U_{n+1}|A) = \frac{\mathbb{P}(A|U_{n+1})\mathbb{P}(U_{n+1})}{\sum_{i=1}^{n+1} \mathbb{P}(A|U_i)\mathbb{P}(U_i)}$$

This gives us

$$\frac{1}{7} = \frac{\frac{1}{9} \cdot \frac{2}{n+1}}{\frac{n}{3(n+1)} + \frac{2}{9(n+1)}} = \frac{2}{3n+2}$$

Solving, we obtain $n = 4$.

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PROOF. Let $\omega \in \Omega$. For each $n \in \mathbb{N}$, $\omega \in A_n \vee \omega \in \overline{A_n}$. Define a sequence of events B_n such that

$$B_n = \begin{cases} A_n & \text{if } \omega \in A_n \\ \overline{A_n} & \text{otherwise} \end{cases}$$

So $\omega \in B_n$ for all $n \in \mathbb{N}$, and since A_n were independent, B_n are as well. Also, since $\mathbb{P}(A_n) = \frac{1}{2}$ for all $n \in \mathbb{N}$, $\mathbb{P}(B_n) = \frac{1}{2}$ as well. $\{\omega\} \subseteq \bigcap B_n$, so therefore, using the independence of the events,

$$\mathbb{P}(\{\omega\}) \leq \mathbb{P}\left(\bigcap_{n=1}^{N} B_n\right) = \left(\frac{1}{2}\right)^N$$

Since this is true for N arbitrarily large, $\mathbb{P}(\{\omega\}) \leq \frac{1}{2^N}$ for all $N \in \mathbb{N}$, so $\mathbb{P}(\{\omega\}) = 0$. $\square$

8

(i)

Proof. Let $B_k = \bigcup_{n\geq k} A_n$. Note that $B_k \searrow A^*$ as $k \to \infty$. Using continuity from above (since clearly $B_1 \supseteq B_2 \supseteq \cdots$), we see that

$$\lim_{k\to\infty} \mathbb{P}(B_n) = \mathbb{P}(A^*)$$

Using countable subadditivity, we have that

$$\mathbb{P}(A^*) = \lim_{k\to\infty} \mathbb{P}(B_n) = \lim_{k\to\infty} \mathbb{P}\left(\bigcup_{n\geq k} A_n\right) \leq \lim_{k\to\infty} \sum_{n\geq k} \mathbb{P}(A_n)$$

However, $\sum_{n=1}^{\infty} \mathbb{P}(A_n) < \infty$. Since the partial sums converge, the tail sums $(\sum_{n\geq k} \mathbb{P}(A_n))$ must go to 0 as $k \to \infty$. Therefore $\mathbb{P}(A^*) = 0$. $\square$

(ii)

We want to show $\mathbb{P}(\overline{A^*}) = 0$.
Proof. Using De Morgan's laws,

$$\overline{A^*} = \bigcup_{k\geq 1} \bigcap_{n\geq k} \overline{A_n}$$

Define $C_k = \bigcap_{n\geq k} \overline{A_n}$, and notice $C_k \nearrow \overline{A^*}$ as $k \to \infty$. Using continuity from below, we have

$$\lim_{k\to\infty} \mathbb{P}(C_k) = \mathbb{P}(\overline{A^*})$$

Using the independence of the C_k, we have that

$$\lim_{k\to\infty} \mathbb{P}(C_k) = \lim_{k\to\infty} \mathbb{P}\left(\bigcap_{n\geq k} \overline{A_n}\right) = \lim_{k\to\infty} \prod_{n\geq k} \mathbb{P}(\overline{A_n})$$

We know $\mathbb{P}(\overline{A_n}) = 1 - \mathbb{P}(A_n)$, and since $\mathbb{P}(A_n) \geq 0$, using $1 - x \leq e^{-x}$ for all $x \geq 0$

$$\prod_{n\geq k}(1 - \mathbb{P}(A_n)) \leq \prod_{n\geq k} e^{-\mathbb{P}(A_n))} = e^{-\sum_{n\geq k} \mathbb{P}(A_n))}$$

Now, the tails of a diverging series are necessarily infinite, since we are only removing a finite number of terms. So

$$\lim_{k\to\infty} e^{-\sum_{n\geq k} \mathbb{P}(A_n))} = e^{-\infty} = 0$$

This gives us $\mathbb{P}(\overline{A^*}) = 0$, as desired.

$\square$