McGill University
Department of Mathematics and Statistics
MATH 243 Analysis 2, Winter 2018
Assignment 2

You should carefully work out **all** problems. However, you only have to hand in solutions to **problems 3, 4 and 5(b).**

This assignment is due **Wednesday, January 24, at 10:30am** in class. **Late assignments will not be accepted!**

1. Consider the function $f : \mathbb{R} \to \mathbb{R}$, defined by

$$f(x) := \begin{cases} x + 2x^2 \sin(1/x) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

   (a) Show that f is continuously differentiable on $\mathbb{R} \setminus \{0\}$ and differentiable at 0 with $f'(0) = 1$.

   (b) Prove that, nonetheless, f isn't increasing on any neighborhood of 0 i.e. show that f isn't increasing on $]-\delta, \delta[$ for any $\delta > 0$.
   Hint: Prove that for any $\delta > 0$ there exists an $x \in]-\delta, \delta[$, $x \neq 0$, such that $f'(x) < 0$. Then, using the fact that f' is continuous at x, prove that there exists an $\eta > 0$ such that f is decreasing on $]x - \eta, x + \eta[\subseteq]-\delta, \delta[$.

2. Consider the function $f : \mathbb{R} \to \mathbb{R}$, defined by

$$f(x) := \begin{cases} x^4 \sin(1/x) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

   (a) Show that f is continuously differentiable.

   (b) Let $g(x) := 2x^4 + f(x)$. Show that g has an absolute minimum at 0 but that, nonetheless, there does not exist any $\delta > 0$ such that g is decreasing on $]-\delta, 0[$ and increasing on $]0, \delta[$.

3. Let I be an interval and let $f : I \to \mathbb{R}$ be differentiable on I. Prove that f satisfies a Lipschitz condition on I if and only if f' is bounded on I (recall that a function $f : I \to \mathbb{R}$ is said to satisfy a Lipschitz condition on I if there exists a $K > 0$ such that $|f(x) - f(u)| \leq K|x - u|$ for all $x, u \in I$).

4. Let $c \in [a, b]$ and let

$$f : [a, b] \to \mathbb{R}, \; f(x) := \begin{cases} 1 & \text{if } x = c \\ 0 & \text{if } x \neq c \end{cases}$$

   Prove that f is Riemann integrable on $[a, b]$ and that $\displaystyle\int_a^b f = 0$.

—Please turn over!—

5. (a) Let f and g be Riemann integrable on $[a, b]$ such that $f(x) \leq g(x)$ for all $x \in [a, b]$. Prove that $\displaystyle\int_a^b f \leq \int_a^b g$.

   Hint: Prove first that $S(f; \dot{\mathcal{P}}) \leq S(g; \dot{\mathcal{P}})$ for all tagged partitions $\dot{\mathcal{P}}$ of $[a, b]$.

   (b) Let f be Riemann integrable on $[a, b]$ and let $M \in \mathbb{R}$ be a constant such that $|f(x)| \leq M$ for all $x \in [a, b]$. Prove that $\displaystyle\left|\int_a^b f\right| \leq M(b - a)$.

6. Use induction to prove that if $f_1, \ldots, f_n$ are Riemann integrable on $[a, b]$ and $k_1, \ldots, k_n \in \mathbb{R}$, then the linear combination $f := k_1 f_1 + \ldots k_n f_n$ is Riemann integrable on $[a, b]$ and

$$\int_a^b f = k_1 \int_a^b f_1 + \cdots + k_n \int_a^b f_n$$