1. *Proof.*

$$f(x) = f(c) + \varphi(x)(x - c)$$

$$g(x) = g(c) + \psi(x)(x - c)$$

So we get the following:

$$f(x)g(x) = f(c)g(c) + f(c)\psi(x)(x - c) + g(c)\varphi(x)(x - c)\varphi(x)\psi(x)(x - c)^2$$

We know by defition that $\varphi(c) = f'(c)$ and $\psi(c) = g'(c)$

so now:

$$lim_{x\to c}\frac{f(x)g(x) - f(c)g(c)}{x - c}$$

$$= lim_{x\to c}f(c)\psi(x) + g(c)\varphi(x) + \varphi(x)\psi(x)(x - c)$$

$$= f(c)\psi(c) + g(c)\varphi(c) + \varphi(c)\psi(c)(c - c)$$

$$= f(c)g'(c) + g(c)f'(c)$$

$\square$

2. *Proof.* Let's consider $g(x) := x^3$ and assume f(x) is differentiable at 0.

we consider $g \circ f$, g is differentiable at f(0) and we assumed that f is diff at 0.

$$(g \circ f)'(x) = (x)' = 1$$

But

$$(g \circ f)'(0) = 3f'(0)f(0)^2$$

We have $f(0) = 0$, hence we arrive at a contradiction. $\square$

3. (a) *Proof.* Let $\epsilon > 0$ and $\delta = \epsilon$.

Let $0 < |x| < \delta$ and $x \in \mathbb{R} - \mathbb{Q}$ then $\dfrac{f(x) - f(0)}{x} = 0 < \epsilon$. Now, let $0 < |x| < \delta$ and $x \in Q$, then

$$\frac{f(x) - f(0)}{x} = \frac{x^2}{x} = |x| < \delta = \epsilon$$

Hence f is diff at 0 and f'(0) = 0. $\square$

(b) we want to show that f is not diff at any $c \neq 0$

Proof. Now for $c \in \mathbb{Q}$, let $(x_n)_{n=1}^{\infty} \subseteq \mathbb{R} \setminus \mathbb{Q} s.t$ x_n converges to c from above (always possible). We get:

$$\frac{f(x_n) - f(c)}{x - c} = \frac{-c^2}{x_n - c}$$

$$lim_{n \to \infty} \frac{-c^2}{x_n - c} = -\infty$$

if the sequence converges to c from below, the result would be ∞. Hence limit doesnt exist.

As for $c \in \mathbb{R} \setminus \mathbb{Q}$, let $(x_n)_{n=1}^{\infty} \subseteq Q$, same idea as above, x_n converges to c from either above or below, we get

$$\frac{f(x_n) - f(c)}{x_n - c} = \frac{x_n^2}{x_n - c}$$

Taking the limit would give us ∞ or $-\infty$. Hence limit doesnt exist.
Hence f is not differentiable at any $c \neq 0$ □