Supplementary Notes 3

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Abstract

In this set of notes we discuss Lagrange interpolation through the Lagrange basis polynomials and the Newton divided differences.

Contents

1 Interpolation in general

In engineering and science, one often has a number of data points, obtained by sampling or experimentation which represent the values of a function for a limited number of values of the independent variable. It is often required to interpolate (i.e., estimate) the value of that function for an intermediate value of the independent variable.

Given any continuous function on a closed and bounded interval, there exists a polynomial that is as "close" to the given function as desired. The following theorem states this fact.

Theorem 1.1 (Weierstrass Approximation Theorem)**.** *Let $f \in C[a,b]$. For each $\varepsilon > 0$, there exists a polynomial L such that*

$$|f(x) - L(x)| < \varepsilon \quad \text{for all } x \in [a,b].$$

This is one of the reasons we consider polynomial interpolation. Another important reason is that their derivative and indefinite integral are easy to determine and are also polynomials.

On a first glance, one could assume that polynomial interpolation makes use of Taylor polynomials. However, this is not the case since there are significant differences. The Taylor polynomials are used to approximate a given function near a specific point and they concentrate their accuracy there. On the other hand, a good interpolation polynomial needs to provide a relatively accurate approximation over an entire interval. Additionally, the function being interpolated may be unknown, i.e., we may only know its value at specific points, which is not the case when we use Taylor polynomials.

2 Lagrange interpolation

We are interested in the following problem.

Lagrange interpolation problem: *Given $f \in C[a,b]$ and $n+1$ distinct nodes (x-coordinates) satisfying $a = x_0 < x_1 < \ldots < x_n = b$ find an interpolation polynomial L_n such that*

$$f(x_i) = L_n(x_i) \quad i = 0, \ldots, n.$$

Example 2.1. *Consider the data set $\{(0,0),(1,2),(2,6)\}$. We want to find a polynomial L_2 such that*

$$\begin{cases} L_2(0) = 0, \\ L_2(1) = 3, \\ L_2(2) = 6. \end{cases}$$

We have three equations so we look for a polynomial with three unknowns, i.e., a polynomial of degree 2:

$$L_2(x) = a_2x^2 + a_1x + a_0.$$

We then have

$$\begin{aligned}
L_2(0) = 0 &\iff a_0 \times 0^2 + a_1 \times 0 + a_0 = 0 \iff a_0 = 0 \\
L_2(1) = 3 &\iff a_0 \times 1^2 + a_1 \times 1 + a_0 = 2 \iff a_0 + a_1 + a_2 = 2 \\
L_2(2) = 6 &\iff a_0 \times 2^2 + a_1 \times 2 + a_0 = 6 \iff a_0 + 2a_1 + 4a_2 = 6
\end{aligned}$$

We have a linear system to solve

$$\begin{cases} a_0 = 0 \\ a_0 + a_1 + a_2 = 2 \\ a_0 + 2a_1 + 4a_2 = 6 \end{cases} \iff \begin{cases} a_0 = 0 \\ a_1 = 1 \\ a_2 = 1 \end{cases}$$

We then have $L_2(x) = x^2 + x$.

In general, we look for a polynomial

$$L_n(x) = a_nx^n + a_{n-1}x^{n-1} + \ldots + a_1x + a_0,$$

such that $y_i = f(x_i) = L_n(x_i)$ for $i = 0, \ldots, n$. This leads to the linear system of $n + 1$ equations and $n + 1$ unknowns which we can show to have a unique solution. We then have the following theorem.

Theorem 2.1. *For any set of $n + 1$ distinct nodes $x_0 < x_1 < \ldots < x_n$ and associated function values $f(x_i)$, there exists a unique polynomial L_n of degree at most n such that*

$$f(x_i) = L_n(x_i) \quad i = 0, \ldots n.$$

We are expressing L_n is terms of the basis $\{1, x, x^2, \ldots, x^n\}$. Instead, we can also express L_n in terms of the Lagrange basis polynomials.

Definition 2.2. *Given $n+1$ distinct nodes $\{x_0, \ldots, x_n\}$, the Lagrange basis polynomials $\{l_k(x)\}_{k=0}^n$ are n-th order polynomials satisfying*

$$l_i(x_j) = \delta_{ij} = \begin{cases} 0 & \text{if } i \neq j, \\ 1 & \text{if } i = j. \end{cases}$$

One can show that the Lagrange basis polynomials are indeed given by

$$l_i(x) = \prod_{\substack{k=0 \\ k \neq i}}^{n} \frac{(x - x_k)}{(x_i - x_k)}.$$

The polynomial L_n can then be expressed as

$$L_n(x) = \sum_{i=0}^{n} f(x_i) l_i(x).$$

There are some practical difficulties with Lagrange interpolation. The main one is that adding points to the interpolation scheme requires adjusting lots of the already computed values. For example, how can one get L_5 from L_4? Newton's divided difference addresses this issue.

The Newton's divided differences of f with respect to $x_0, x_1, \ldots, x_n$ allows us to express L_n in the form

$$L_n(x) = a_0 + a_1(x - x_0) + a_2(x - x_0)(x - x_1) + \ldots a_n(x - x_0) \ldots (x - x_{n-1})$$

for appropriate constants $a_0, \ldots, a_n$. In other words, we are writing $L_n(x)$ in terms of the basis $\{1, x - x_0, (x - x_0)(x - x_1), \ldots, (x - x_0) \ldots (x - x_{n-1})\}$.

Definition 2.3. *Given $n+1$ distinct nodes $\{x_0, \dots, x_n\}$, the k-th order divided difference is defined as*

$$\begin{aligned} k = 0: &\quad f[x_i] = f(x_i), \\ k > 0: &\quad f[x_i, \dots, x_{i+k}] = \frac{f[x_{i+1}, \dots, x_{i+k}] - f[x_i, \dots, x_{i+k-1}]}{x_{i+k} - x_i}. \end{aligned}$$

One can then show that the interpolation polynomial $L_n(x)$ can then be written as

$$L_n(x) = f[x_0] + \sum_{k=1}^{n} f[x_0, \dots, x_k](x - x_0) \dots (x - x_{k-1}).$$

Notice that the value of $f[x_0, \dots, x_k]$ is independent of the order of the numbers $x_0, \dots, x_k$. The above formula is known as Newton's divided difference interpolation formula.

Usually the divided differences are computed in a table. It's faster and easier. Let us look at the table for four points.

x_i	$f[x_i]$	$f[x_i, x_{i+1}]$	$f[x_i, x_{i+1}, x_{i+2}]$	$f[x_i, x_{i+1}, x_{i+2}, x_{i+3}]$
x_0	$f[x_0]$			
		$f[x_0, x_1]$		
x_1	$f[x_1]$		$f[x_0, x_1, x_2]$	
		$f[x_1, x_2]$		$f[x_0, x_1, x_2, x_3]$
x_2	$f[x_2]$		$f[x_1, x_2, x_3]$	
		$f[x_2, x_3]$		
x_3	$f[x_3]$			

The upper downward diagonal values are enough to determine L_n. The divided differences essentially give some measure of the derivative of the function through the interpolating points.

Exercise 2.1. *Given the measurements $(0,3)$, $(1,-2)$, $(2,5)$ write down the l_j and the full expression of p, the interpolation polynomial. Show $L_2(x) = 6x^2 - 11x + 3$.*

Solution. We have

$$\begin{aligned} l_0(x) &= \frac{(x-1)}{(0-1)}\frac{(x-2)}{(0-2)} = \frac{1}{2}(x-1)(x-2) = \frac{1}{2}x^2 - \frac{3}{2}x + 1 \\ l_1(x) &= \frac{(x-0)}{(1-0)}\frac{(x-2)}{(1-2)} = -x^2 + 2x \\ l_2(x) &= \frac{(x-0)}{(2-0)}\frac{(x-1)}{(2-1)} = \frac{1}{2}x^2 - \frac{1}{2}x \end{aligned}$$

Hence

$$\begin{aligned} L_2(x) &= 3\, l_0(x) + (-2)\, l_1(x) + 5\, l_2(x) \\ &= \frac{3}{2}x^2 - \frac{9}{2}x + 3 + 2x^2 - 4x + \frac{5}{2}x^2 - \frac{5}{2}x \\ &= 6x^2 - 11x + 3. \end{aligned}$$

We could as well have computed f using the method of divided differences. Pay close attention to how each column is computed from the previous one.

x_i	$f[x_i]$	$f[x_i, x_{i+1}]$	$f[x_i, x_{i+1}, x_{i+2}]$
x_0	3		
		-5	
x_1	-2		6
		7	
x_2	5		

and therefore

$$p(x) = 3 - 5x + 6x(x-1) = 3 - 11x + 6x^2.$$

□

Exercise 2.2. *Find the the interpolating polynomials for the following two sets of data points:*

i) $(0,1)$, $(1,4)$, $(2,1)$

ii) $(0,1)$, $(1,4)$, $(2,1)$, $(3,-2)$

Solution: If we were to use the Lagrange basis polynomials we would have two compute them for each set of data points. By using the method of divided differences we avoid that by computing L_3 using L_2.

x_i	$f[x_i]$	$f[x_i, x_{i+1}]$	$f[x_i, x_{i+1}, x_{i+2}]$	$f[x_i, x_{i+1}, x_{i+2}, x_{i+3}]$
x_0	1			
		3		
x_1	4		-3	
		-3		1
x_2	1		0	
		-3		
x_3	-2			

We then have

$$\begin{aligned} L_2(x) &= f[x_0] + f[x_0, x_1](x - x_0) + f[x_0, x_1, x_2](x - x_0)(x - x_1) \\ &= 1 + 3x + (-3)x(x-1) = 1 + 6x - 3x^2 \end{aligned}$$

and

$$\begin{aligned} L_3(x) &= L_2(x) + f[x_0, x_1, x_2, x_3](x - x_0)(x - x_1)(x - x_2) \\ &= L_2(x) + x(x-1)(x-2) \\ &= 1 + 8x - 6x^2 + x^3. \end{aligned}$$

Exercise 2.3. *Find the interpolating polynomial for the points* $(0,0)$, $(\pi/2, 1)$, $(\pi, 0)$ *and* $(3\pi/2, -1)$ *using the method of divided differences.*

Solution: We compute the divided differences using the table.

x_i	$f[x_i]$	$f[x_i, x_{i+1}]$	$f[x_i, x_{i+1}, x_{i+2}]$	$f[x_i, x_{i+1}, x_{i+2}, x_{i+3}]$
x_0	0			
		$\frac{2}{\pi}$		
x_1	1		$-\frac{4}{\pi^2}$	
		$-\frac{2}{\pi}$		$\frac{8}{3\pi^3}$
x_2	0		0	
		$-\frac{2}{\pi}$		
x_3	-1			

Consequently

$$
\begin{aligned}
L_3(x) &= 0 + \frac{2}{\pi}x - \frac{4}{\pi^2}x\left(x-\frac{\pi}{2}\right) + \frac{8}{3\pi^3}x\left(x-\frac{\pi}{2}\right)(x-\pi) \\
&= \frac{2}{\pi}x - \frac{4}{\pi^2}x\left(x-\frac{\pi}{2}\right) + \frac{8}{3\pi^3}x\left(x-\frac{\pi}{2}\right)(x-\pi) \\
&= \frac{16}{3\pi}x - \frac{8}{\pi^2}x^2 + \frac{8}{3\pi^3}x^3.
\end{aligned}
$$

We can also compute it in the following way:

$$
\begin{aligned}
L_3(x) &= 0 - \frac{2}{\pi}(x-\pi) + 0\,(x-\pi)\left(x-\frac{3\pi}{2}\right) + \frac{8}{3\pi^3}\left(x-\frac{\pi}{2}\right)(x-\pi)\left(x-\frac{3\pi}{2}\right) \\
&= -\frac{2}{\pi}(x-\pi) + \frac{8}{3\pi^3}\left(x-\frac{\pi}{2}\right)(x-\pi)\left(x-\frac{3\pi}{2}\right) \\
&= \frac{16}{3\pi}x - \frac{8}{\pi^2}x^2 + \frac{8}{3\pi^3}x^3.
\end{aligned}
$$

References

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[3] Tutorial notes written by Jan Feys