Practice Problems Midterm

Tiago Salvador (`tiago.saldanhasalvador@mail.mcgill.ca`)

1. What does the binary number 1001001 correspond to in base 10 digits, i.e., human style.

> **Solution:** The binary number 1001001 converts to
>
> $$(2^6 + 2^3 + 2^0)_{10} = (64 + 8 + 1)_{10} = (73)_{10}.$$

2. How accurately do we need to know e to be able to compute e^{-1} with five correct decimals?

> **Solution:** The idea is to Taylor expand $(e + \varepsilon)^{-1}$ about $\varepsilon = 0$. Here, ε expresses the accuracy by which we know e or equivantely the error from the exact value of e. For simplicity, we assume $\varepsilon > 0$. We then have
>
> $$\frac{1}{e+\varepsilon} = \frac{1}{e} - \frac{\varepsilon}{(e+\xi)^2},$$
>
> where $\xi \in (0, \varepsilon)$. Thus we have to choose ε sufficiently small such that
>
> $$\left| \frac{1}{e+\varepsilon} - \frac{1}{e} \right| \leq \frac{|\varepsilon|}{(e+\xi)^2} \leq \frac{|\varepsilon|}{e^2} < 10^{-5}$$
>
> which is equivalent to $|\varepsilon| < 10^{-5} e^2 \approx 7.389 \times 10^{-5}$. Therefore six decimals are enough.

3. Find the third Taylor polynomial $P_3(x)$ for the function $f(x) = \sqrt{x+1}$ about $x_0 = 0$.

> **Solution:** We start by computing f', f'' and $f^{(3)}$:
>
> $$f'(x) = \frac{1}{2\sqrt{x+1}} \quad f''(x) = -\frac{1}{4(1+x)^{3/2}} \quad f^{(3)}(x) = \frac{3}{8(1+x)^{5/2}}.$$
>
> The third Taylor polynomial $P_3(x)$ is then given by
>
> $$\begin{aligned} P_3(x) &= f(0) + xf'(0) + \frac{x^2}{2} f''(0) + \frac{x^3}{3!} f^{(3)}(0) \\ &= 1 + x\left(\frac{1}{2}\right) + \frac{x^2}{2}\left(-\frac{1}{4}\right) + \frac{x^3}{3!}\left(\frac{3}{8}\right) \\ &= 1 + \frac{x}{2} - \frac{x^2}{8} + \frac{x^3}{16}. \end{aligned}$$

4. Consider the iteration with $g(x) = x + \frac{1}{2}(2 - e^x)$.

(a) Show that the iteration has a fixed point $x^* = \log(2)$.

Solution: The fixed point x^* is such that $x^* = g(x^*)$. We then have

$$g(x) = x \Longleftrightarrow x + \frac{1}{2}(2 - e^x) = x \Longleftrightarrow 2 - e^x = 0 \Longleftrightarrow x = \log(2)$$

and so we are done.

(b) Show that the scheme satisfies all the conditions of the Fixed Point Theorem on the interval $[0, 1]$.

Solution: We have $g'(x) = 1 - \frac{1}{2}e^x$ and $g'(x) = 0 \Longleftrightarrow x = \log(2)$. Hence

x	0		$\log(2)$		1
$g'(x)$	$+$	$+$	0	$-$	$-$
$g(x)$	$\frac{1}{2}$	$\nearrow$	$\log(2)$	$\searrow$	≈ 0.64

and therefore $g(x) \in [0, 1]$ for all $x \in [0, 1]$. This means that the first condition of the Fixed Point Theorem is satisfied. We now observe that $g''(x) = -\frac{e^x}{2} < 0$, therefore the maximum of $|g'|$ in $[0, 1]$ is attained at either $x = 0$ or $x = 1$. Thus

$$|g'(x)| \leq \max\{g'(0), g'(1)\} = \frac{1}{2} := k$$

for all $x \in [0, 1]$. Since k is such that $0 < k < 1$, the second condition of the Fixed Point Theorem is also satisfied.

(c) What is the order of convergence of the scheme? State the asymptotic error constant.

Solution: To determine the order of convergence of the scheme we need to look at $g'(x^*)$ and $g''(x^*)$:

$$g'(x^*) = 1 - \frac{1}{2}e^{x^*} = 1 - \frac{1}{2}e^{\log(2)} = 0$$

and

$$g''(x^*) = -\frac{1}{2}e^{x^*} = -\frac{1}{2}e^{\log(2)} = -1 \neq 0$$

Thus the scheme has order of convergence 2.
The asymptotic error constant is

$$\frac{|g''(x^*)|}{2!} = \frac{1}{2}.$$

5. Given

i	0	1	2	3
x_i	0	1	2	3
$f(x_i)$	2	3	10	29

construct the appropriate table of divided differences and hence state

i. the polynomial of degree 2 which interpolates f at x_1, x_2 and x_3.

ii. the polynomial of degree 3 which interpolates f at x_0, x_1, x_2 and x_3.

Solution: The table of divided differences is given by

x_i	$f[x_i]$	$f[x_i, x_{i+1}]$	$f[x_i, x_{i+1}, x_{i+2}]$	$f[x_i, x_{i+1}, x_{i+2}, x_{i+3}]$
0	2			
		1		
1	3		3	
		7		1
2	10		6	
		19		
3	29			

We then have

i. $p(x) = 3 + 7(x-1) + 6(x-1)(x-2) = 8 - 11x + 6x^2$.

ii. $p(x) = 8 - 11x + 6x^2 + (x-1)(x-2)(x-3) = 2 + x^3$

6. Explain Runge's phenomenon and how it can be fixed.

Solution: When doing Lagrange interpolation over equally-spaced nodes, in some situations, the Lagrange polynomials exhibit high oscillations near the endpoints of the interval. This is called Runge's phenomenon, and the typical example is given by $f(x) = \frac{1}{1+25x^2}$. This phenomenon can be fixed by an optimal placement of the nodes (Chebyshev interpolation) or by doing spline interpolation.

7. How does Hermite interpolation improve upon Lagrange interpolation?

Solution: Hermite interpolation improves upon Lagrange interpolation by finding a polynomial that not only matches the function value but also the derivative. However, it still suffers from Runge's phenomenon.

8. Given the function $f(x) = \cos(\pi x)$ compute the Hermite interpolation polynomial with nodes $x_0 = 0$ and $x_1 = 1$.

Solution: Let H denote the Hermite interpolation polynomial. We look for H that satisfies

$$\begin{cases} H(0) = f(0) \\ H(1) = f(1) \\ H'(0) = f'(0) \\ H'(1) = f'(1) \end{cases}$$

We have 4 constraints and so we look for a polynomial of degree 3, say

$$H(x) = a + bx + cx^2 + dx^3,$$

where $a, b, c, d \in \mathbb{R}$ are unknowns. Since

$$H'(x) = b + 2cx + 3dx^2,$$

we can rewrite and solve the above linear system:

$$\begin{cases} a = 1 \\ a+b+c+d = -1 \\ b = 0 \\ b+2c+3d = 0 \end{cases} \iff \begin{cases} a = 1 \\ b = 0 \\ c+d = -2 \\ 2c+3d = 0 \end{cases} \iff \begin{cases} a = 1 \\ b = 0 \\ c = -6 \\ d = 4 \end{cases}$$

Hence $H(x) = 1 - 6x^2 + 4x^3$.

9. A clamped cubic spline S for a function f is defined by

$$S(x) = \begin{cases} S_0(x) = 1 + b_0 x + 2x^2 - 2x^3 & x \in [0,1) \\ S_1(x) = 1 + b_1(x-1) - 4(x-1)^2 - 7(x-1)^3 & x \in [1,2] \end{cases}$$

where b_0, b_1 are constants. Find $f'(0)$ and $f'(2)$.

Solution: Let's first compute S':

$$S'(x) = \begin{cases} b_0 + 4x - 6x^2 & x \in [0,1) \\ b_1 - 8(x-1) - 21(x-1)^2 & x \in [1,2] \end{cases}$$

Since S is a clamped cubic spline we have

$$\begin{aligned} f'(0) &= S'(0) = b_0 \\ f'(2) &= S'(2) = b_1 - 8(2-1) - 21(2-1)^2 = b_1 - 29 \end{aligned}$$

We then need to find b_0, b_1. We know that $S \in C^2[0,2]$ and so

$$\begin{cases} S_0(1) = S_1(1) \\ S_0'(1) = S_1'(1) \end{cases} \iff \begin{cases} 1 + b_0 + 2 - 2 = 1 \\ b_0 + 4 - 6 = b_1 \end{cases} \iff \begin{cases} b_0 = 0 \\ b_1 = -2 \end{cases}$$

Hence $f'(0) = 0$ and $f'(2) = -31$.

10. Find the constants a, b, c such that the finite difference of the first derivative

$$D_h f(x_0) := af(x_0 - h_1) + bf(x_0) + cf(x_0 + h_2)$$

has the highest degree of accuracy possible.

Solution: The idea is for the formula to be exact to the highest degree polynomial possible. Hence, taking $f(x) = 1$ we get

$$D_h f(x_0) = Df(x_0) \iff a + b + c = 0.$$

With $f(x) = x$ we obtain

$$\begin{aligned} D_h f(x_0) = Df(x_0) &\iff a\,(x_0 - h_1) + bx_0 + c(x_0 + h_2) = 1 \\ &\iff x_0(a+b+c) - h_1 a + h_2 c = 1 \\ &\implies -h_1 a + h_2 c = 1, \end{aligned}$$

where we used the first equation. Finally, for $f(x) = x^2$, we get

$$\begin{aligned} D_h f(x_0) = Df(x_0) &\iff a\,(x_0 - h_1)^2 + bx_0^2 + c(x_0 + h_2)^2 = 2x_0 \\ &\iff (a + b + c)x_0^2 + 2x_0(-ah_1 + ch_2) + h_1^2 a + h_2^2 c = 2x_0 \\ &\implies h_1^2 a + h_2^2 c = 0, \end{aligned}$$

where we used the second equation. We thus have

$$\begin{cases} a + b + c = 0 \\ -h_1 a + h_2 c = 1 \\ h_1^2 a + h_2^2 c = 0 \end{cases} \iff \begin{cases} a = -\frac{h_2}{h_2^1 + h_1 h_2} \\ b = \frac{-h_1 + h_2}{h_1 h_2} \\ c = \frac{h_1}{h_1 h_2 + h_2^2} \end{cases}$$

Thus the finite difference formula is given by

$$D_h f(x_0) = -\frac{h_2}{h_2^1 + h_1 h_2} f(x_0 - h_1) + \frac{-h_1 + h_2}{h_1 h_2} f(x_0) + \frac{h_1}{h_1 h_2 + h_2^2} f(x_0 + h_2).$$

11. Consider the integral $I(f) = \int_0^3 f(x)\,dx$. Find a_0, a_1, a_2, a_3 such that the quadrature

$$I_h(f) = a_0 f(0) + a_1 f(1) + a_2 f(2) + a_3 f(3).$$

has the highest degree possible.

Solution: The idea is for the formula to be exact for the highest degree of polynomial possible. We start by taking $f(x) = 1$ which leads to

$$I_h(f) = I(f) \iff a_0 + a_1 + a_2 + a_3 = 3.$$

With $f(x) = x$, we obtain

$$I_h(f) = I(f) \iff a_1 + 2a_2 + 3a_3 = \frac{9}{2}.$$

With $f(x) = x^2$, we get

$$I_h(f) = I(f) \iff a_1 + 4a_2 + 9a_3 = 9.$$

Finally, with $f(x) = x^3$, we get

$$I_h(f) = I(f) \iff a_1 + 8a_2 + 27a_3 = \frac{81}{4}.$$

We have a linear system to solve

$$\begin{cases} a_0 + a_1 + a_2 + a_3 = 3 \\ a_1 + 2a_2 + 3a_3 = \frac{9}{2} \\ a_1 + 4a_2 + 9a_3 = 9 \\ a_1 + 8a_2 + 27a_3 = \frac{81}{4} \end{cases} \iff \begin{cases} a_0 = \frac{3}{8} \\ a_1 = \frac{9}{8} \\ a_2 = \frac{9}{8} \\ a_3 = \frac{3}{8} \end{cases}$$

12. Determine constants a, b, c and d that will produce a quadrature formula

$$\int_{-1}^{1} f(x)\,dx \approx af(-1) + bf(1) + cf'(-1) + df'(1)$$

that has at least degree of precision 3.

Solution: The quadrature needs to be exact for $f(x) = 1, x, x^2, x^3$. We then obtain the following linear system for a, b, c and d:

$$\begin{cases} 2 = a + b \\ 0 = -a + b + c + d \\ \frac{2}{3} = a + b - 2c + 2d \\ 0 = -a + b + 3c + 3d \end{cases}$$

The weights a, b, c and d are then given by

$$\begin{cases} a = 1 \\ b = 1 \\ c = \frac{1}{3} \\ d = -\frac{1}{3} \end{cases}$$