Supplementary Notes 5

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Abstract

We discuss spline interpolation.

Contents

1 Spline interpolation

Spline interpolation is a form of interpolation where the interpolant is a special type of piecewise polynomial with global smoothness called a spline. The idea behind the use of splines is that we can make the interpolation error small even when using low degree polynomials. Additionally, they avoid the Runge's phenomenon which occurs when we use high degree polynomials due to their oscillatory nature.

The simplest piecewise polynomial approximation is piecewise linear interpolation: the data points $\{x_i, f(x_i)\}_{i=0}^n$ are joined by straight lines. The disadvantage of their use is that there will most likely be no differentiability at the nodes, meaning that the interpolation function won't be "smooth".

We will then focus on approximations using piecewise polynomials that require no derivative information, except perhaps at the end points of the interval on which the function is being approximated.

2 Linear spline interpolation

Example 2.1. *Consider the data $\{(0,1),(1,-1),(3,0)\}$ (Hence $x_0 = 0$, $x_1 = 1$, $x_2 = 3$ and $f(x_0) = 1$, $f(x_1) = -1$, $f(x_2) = 0$.). We want to look for linear spline. This means looking for a function that is piecewise linear in the subintervals formed by the nodes:*

$$S_L(x) = \begin{cases} a_0 + b_0(x - x_0) & x \in [0,1) \\ a_1 + b_1(x - x_1) & x \in [1,3] \end{cases} = \begin{cases} a_0 + b_0 x & x \in [0,1) \\ a_1 + b_1(x - 1) & x \in [1,3] \end{cases}$$

where a_0, a_1, b_0, b_1 are constants. We want S_L to be an interpolation function meaning that it must match the function at the nodes, i.e., $S_L(x_i) = f(x_i)$ for $i = 0,1,2$. This leads to

$$\begin{aligned} &S_L(0) = 1 \iff a_0 = 1 \\ &S_L(1) = -1 \iff a_1 = -1 \\ &S_L(2) = 0 \iff -1 + b_1(3-1) = 0 \iff b_1 = \frac{1}{2} \end{aligned}$$

Finally, we impose that our linear spline must be continuous:

$$S_L(1^-) = S_L(1^+) \iff 1 + b_0 = -1 \iff b_0 = -2.$$

Hence

$$S_L(x) = \begin{cases} 1 - 2x & x \in [0, 1), \\ -1 + \frac{1}{2}(x-1) & x \in [1, 3]. \end{cases}$$

In general we have the following definition for a linear spline.

Definition 2.1 (linear spline interpolant)**.** *Given $\{x_0, \ldots, x_n\}$ and $\{f(x_0), \ldots, f(x_n)\}$, linear spline $S_L(x)$ is a function*

$$S(x) = \begin{cases} a_0 + b_0(x - x_0) & \text{if } x \in [x_0, x_1), \\ a_1 + b_1(x - x_1) & \text{if } x \in [x_1, x_2), \\ \quad \vdots & \\ a_{n-1} + b_{n-1}(x - x_{n-1}) & \text{if } x \in [x_{n-1}, x_n), \end{cases}$$

such that

(a) $S_L(x_i) = f(x_i)$ for $i = 0, \ldots, n$ (interpolation conditions)

(b) $S_L(x_i^-) = f(x_i^+)$ for $i = 1, \ldots, n-1$ (continuity conditions)

Notice that S_L is a continuous functions and is uniquely determined: the number of degrees of freedom is $2n$ (there are $2n$ parameters in total) and the number of constraints is also $2n$ (there $n+1$ interpolation conditions and $n-1$ continuity conditions).

Remark 2.2. *$S_L(x)$ has sharp corners which is not desirable in some applications. Moreover, it has low order approximation.*

3 Cubic spline interpolation

A general cubic polynomial involves four constants. This allows for sufficient flexibility in the cubic spline procedure to ensure the interpolant is not only continuous differentiable on the interval but also has a continuous second derivative. The construction of the cubic spline does not however assume that the derivatives of the interpolant agree with those of the function we are approximating even at the nodes.

Definition 3.1 (Cubic spline interpolation)**.** *Given a function f defined on $[a, b]$ and a set of nodes $a = x_0 < x_1 < \ldots < x_{n-1} < x_n = b$, a cubic spline interpolant S for f is a function*

$$S(x) = \begin{cases} S_0(x) & \text{if } x \in [x_0, x_1), \\ S_1(x) & \text{if } x \in [x_1, x_2), \\ \quad \vdots & \\ S_{n-1}(x) & \text{if } x \in [x_{n-1}, x_n), \end{cases}$$

where the S_i are cubic polynomials of the form

$$S_i(x) = a_i + b_i(x - x_i) + c_i(x - x_i)^2 + d_j(x - x_i)^3$$

such that:

(a) $S(x_i) = f(x_i)$ for $i = 0, \ldots, n$ ($n+1$ interpolation conditions).

(b) $S(x_i^-) = S(x_i^+)$ for $i = 0, \ldots, n-2$. ($n-1$ continuity conditions).

(c) $S'(x_i^-) = S'(x_i^+)$ and $S''(x_i^-) = S''(x_i^+)$ for $i = 0, \ldots, n-2$ ($2n-2$ smoothness conditions).

(d) One of the following sets of boundary conditions (2 extra conditions) is satisfied:

i) $S''(x_0) = S''(x_n) = 0$ (free or natural boundary conditions) *ii)* $S'(x_0) = f'(x_0)$ *and* $S'(x_n) = f'(x_n)$ *(clamped boundary conditions).*

Remark 3.2. *The unknowns are the* a_i, b_i, c_i, d_i *with* $i = 0, 1, \ldots, n-1$*. They can be determined from the conditions (a) to (d) by solving a matrix equation. Note that the* a_i *can be easily found to satisfy* $a_i = f(x_i)$*. Furthermore, the matrix system will be only for the* c_i*. The* b_i *and* d_i *can be found from these.*

Remark 3.3. *It is also possible to define quadratic and quartic splines. They are defined in a similar way by expanding or trimming the list of smoothness conditions.*

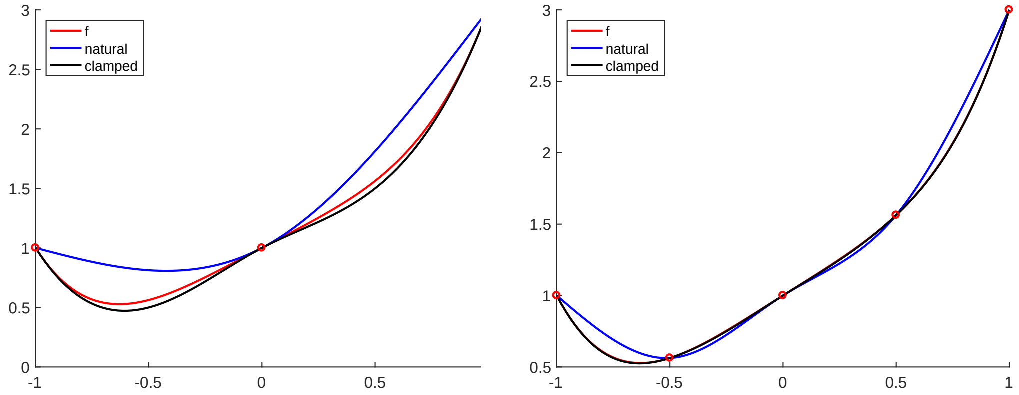

Figure 1: Comparison between clampled and natural cubic spline for $f(x) = x^4 + x + 1$.

Theorem 3.4 (Uniqueness theorem)**.** *Given a function* f *defined on* $[a, b]$ *and a set of nodes* $a = x_0 < x_1 < \ldots < x_{n-1} < x_n = b$*, there is a unique natural spline interpolant* S *for* f*. If* $f'(a)$ *and* $f'(b)$ *are given, there there exists a unique clamped spline as well.*

The proof of this Theorem can be found in [1]. We point out however that a cubic spline problem with $n+1$ points has $4n$ unknowns and the set of conditions given contains exactly $4n$ equations. There is thus an equal amount of equations and unknowns which in general means that the unknowns will be uniquely determined. The boundary conditions are crucial to the uniqueness of the spline and the ones we choose depend on the problem. When using the free boundary conditions, we call the spline a natural or free spline. We think of it as a long flexible rod which is forced to go through the data points $\{(x_0, f(x_0), \ldots, (x_n, f(x_n))\}$. As for the clamped boundary conditions they lead naturally to more accurate approximations, since we are using more information about the function, with the spline being called clamped spline.

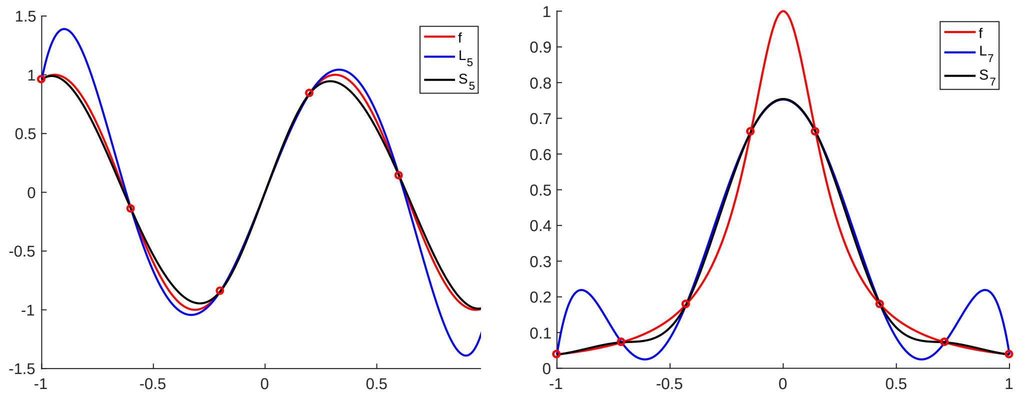

Figure 2: Comparison between clampled cubic spline and Lagrange interpolation on equally space nodes: $f(x) = \sin(5x)$ (left); Runge function (right).

Remark 3.5. *Cubic spline interpolation has comparable accuracy to Lagrange interpolation, however Runge's phenomenon is avoided on equally-spaced nodes.*

Exercise 3.1. *Determine the natural cubic spline S that interpolates the data $f(0) = 0$, $f(1) = 1$ and $f(2) = 2$*

Solution: We are given three points $x_0 = 0$, $x_1 = 1$ and $x_2 = 2$ and so $n = 2$. Therefore there will be two pieces S_0 and S_1 to the spline. The goal is then to determine $a_0, b_0, c_0, d_0, a_1, b_1, c_1$ and d_1 where

$$S_0(x) = a_0 + b_0(x - x_0) + c_0(x - x_0)^2 + d_0(x - x_0)^3$$

and

$$S_1(x) = a_1 + b_1(x - x_1) + c_1(x - x_1)^2 + d_1(x - x_1)^3.$$

This is done by imposing the eight conditions from the definition of free cubic spline.

- Three interpolation conditions: $S_0(x_0) = f(x_0)$, $S_1(x_1) = f(x_1)$ and $S_1(x_2) = f(x_2)$.

This leads to

$$\begin{cases} a_0 = f(x_0) = 0 \\ a_1 = f(x_1) = 1 \\ 2 = S_1(x_2) = a_1 + b_1(x_2 - x_1) + c_1(x_2 - x_1)^2 + d_1(x_2 - x_1)^3 = 1 + b_1 + c_1 + d_1 \end{cases}$$

- One continuity condition: $S_0(x_1) = S_1(x_1)$.

Thus

$$1 = S_1(x_1) = S_0(x_1) = a_0 + b_0(x_1 - x_0) + c_1(x_1 - x_0)^2 + d_1(x_1 - x_0)^3 = b_0 + c_0 + d_0.$$

- Two smoothness conditions: $S_0'(x_1) = S_1'(x_1)$ and $S_0''(x_1) = S_1''(x_1)$

We then have

$$\begin{cases} b_0 + 2c_0(x_1 - x_0) + 3d_0(x_1 - x_0)^2 = b_1 + 2c_1(x_1 - x_1) + 3d_1(x_1 - x_1)^2 \\ 2c_0 + 6d_0(x_1 - x_0) = 2c_1 + 6d_1(x_1 - x_1) \end{cases}$$

By plugging in all the values we know, we get

$$\begin{cases} b_0 + 2c_0 + 3d_0 = b_1 \\ 2c_0 + 6d_0 = 2c_1 \end{cases}$$

- Free boundary conditions: $S''(x_0) = 0$ and $S''(x_2) = 0$.

We then have

$$\begin{cases} S''(x_0) = 2c_0 = 0 \\ S''(x_2) = 2c_1 + 6d_1(x_1 - x_0) = 2c_1 + 6d_1 = 0 \end{cases}$$

Let us summarize all the equations.

$$\begin{cases} a_0 = 0 \\ a_1 = 1 \\ 1 + b_1 + c_1 + d_1 = 2 \\ b_0 + c_0 + d_0 = 1 \\ b_0 + 2c_0 + 3d_0 = b_1 \\ 2c_0 + 6d_0 = 2c_1 \\ 2c_0 = 0 \\ 2c_1 + 6d_1 = 0 \end{cases} \iff \begin{cases} a_0 = 0 \\ a_1 = 1 \\ b_0 + d_0 = 1 \\ b_1 + c_1 + d_1 = 1 \\ b_0 + 3d_0 = b_1 \\ 6d_0 = 2c_1 \\ c_0 = 0 \\ 2c_1 + 6d_1 = 0 \end{cases}$$

We have

$$\begin{cases} b_0 + d_0 = 1 \\ b_1 + c_1 + d_1 = 1 \\ b_0 + 3d_0 = b_1 \\ 6d_0 = 2c_1 \\ 2c_1 + 6d_1 = 0 \end{cases} \iff \begin{cases} b_0 - d_1 = 1 \\ b_1 - 2d_1 = 1 \\ b_0 - 3d_1 = b_1 \\ d_0 + d_1 = 0 \\ c_1 = -3d_1 \end{cases} \iff \begin{cases} b_0 = 1 + d_1 \\ b_1 = 1 + 2d_1 \\ 1 + d_1 - 3d_1 = 1 + 2d_1 \\ d_0 + d_1 = 0 \\ c_1 = -3d_1 \end{cases}$$

In the first step we simplified the last two equations and then use them to simplify the first three. We now have

$$\begin{cases} b_0 = 1 + d_1 \\ b_1 = 1 + 2d_1 \\ d_1 = 0 \\ d_0 + d_1 = 0 \\ c_1 = -3d_1 \end{cases} \iff \begin{cases} b_0 = 1 \\ b_1 = 1 \\ d_0 = 0 \\ d_1 = 0 \\ c_1 = 0 \end{cases}$$

The cubic spline is then given by

$$S(x) = \begin{cases} S_0(x) = x & \text{if } x \in [0,1) \\ S_1(x) = 1 + (x-1) = x & \text{if } x \in [1,2) \end{cases}$$

This is expected since the function could have just been $f(x) = x$.

Exercise 3.2. *A clamped cubic spline S for a function f is defined on $[1,3]$ by*

$$S(x) = \begin{cases} S_0(x) = 3(x-1) + 2(x-1)^2 - (x-1)^3 & \textit{if } x \in [1,2), \\ S_1(x) = a + b(x-2) + c(x-2)^2 + d(x-2)^3 & \textit{if } x \in [2,3]. \end{cases}$$

Given $f'(1) = f'(3)$, find a, b, c and d.

Solution: By construction, the spline is twice continuous differentiable at $x = 2$. This means that

(i) $S_0(2) = S_1(2)$

(ii) $S_0'(2) = S_1'(2)$

(iii) $S_0''(2) = S_1''(2)$

Condition (i) leads to

$$S_0(2) = S_1(2) \iff a = 4.$$

We have that

$$S'(x) = \begin{cases} S_0'(x) = 3 + 4(x-1) - 3(x-1)^2 & \text{if } x \in [1,2), \\ S_1'(x) = b + 2c(x-2) + 3d(x-2)^2 & \text{if } x \in [2,3]. \end{cases}$$

Hence (ii) leads to

$$b = 3 + 4 - 3 = 4.$$

We also have

$$S''(x) = \begin{cases} S_0''(x) = 4 - 6(x-1) & \text{if } x \in [1,2), \\ S_1''(x) = 2c + 6d(x-2) & \text{if } x \in [2,3]. \end{cases}$$

Thus (iii) leads to

$$4 - 6 = 2c \iff c = -1.$$

Finally, since it is a clamped cubic spline, we have $S_0'(1) = f'(1)$ and $S_1'(3) = f'(3)$. We then have

$$f'(1) = S_0'(1) = 3$$

and

$$S_1'(3) = b + 2c + 3d.$$

By assumption, $f'(1) = f'(3)$ and therefore

$$d = \frac{3 - b - 2c}{3} = \frac{1}{3}.$$

Exercise 3.3. *A natural cubic spline on* $[0, 2]$ *is defined by*

$$S(x) = \begin{cases} S_0(x) = 1 + 2x - x^3 & \textit{if } x \in [0, 1), \\ S_1(x) = a + b(x-1) + c(x-1)^2 + d(x-1)^3 & \textit{if } x \in [1, 2]. \end{cases}$$

Find a, b, c *and* d.

Solution: As in the previous exercise the spline needs to be twice continuous differentiable at $x = 1$. This means that

(i) $S_0(1) = S_1(1)$

(ii) $S_0'(1) = S_1'(1)$

(iii) $S_0''(1) = S_1''(1)$

Condition (i) leads to

$$S_0(1) = S_1(1) \iff a = 2.$$

We have that

$$S'(x) = \begin{cases} S_0'(x) = 2 - 3x^2 & \text{if } x \in [0, 1), \\ S_1'(x) = b + 2c(x-1) + 3d(x-1)^2 & \text{if } x \in [1, 2]. \end{cases}$$

Hence (ii) leads to

$$b = 2 - 3 = -1.$$

We also have

$$S''(x) = \begin{cases} S_0''(x) = -6x & \text{if } x \in [0, 1), \\ S_1''(x) = 2c + 6d(x-1) & \text{if } x \in [1, 2]. \end{cases}$$

Thus (iii) leads to

$$-6 = 2c \iff c = -3.$$

Finally, since it is a natural cubic spline, we have $S_1''(2) = 0$. We then have

$$2c + 6d = 0 \iff d = -\frac{c}{3} = 1.$$

References

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[3] Spline interpolation Wikipedia

[4] Spline (mathematics) Wikipedia

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