MATH 356 - FALL 2016 - HOMEWORK 4 SOLUTION

This homework set is on **Inequalities, Multiple Random Variables, Independence and Conditional Distributions**. It's due on **Nov. 10th in class**. Late homework will **not** be accepted.

Problem 1. (20pts) Given a probability space $(\Omega, \mathcal{S}, \mathbb{P})$ and $R > 0$, let (X, Y) be a continuous bivariate random variable on Ω with the joint PDF given by

$$f(x,y) = \begin{cases} \frac{1}{\pi R^2}, & \text{if } x^2+y^2 \le R^2, \\ 0, & \text{otherwise.} \end{cases}$$

We say that (X, Y) has the uniform distribution on the disk $D := \left\{(x,y) : x^2+y^2 \le R^2\right\}$.

(i) (5pts) Determine the marginal DF F_X and the marginal PDF f_X of X, as well as the marginal DF F_Y and the marginal PDF f_Y of Y.

Solution: For $x \in [-R, R]$,

$$f_X(x) = \int_{-\sqrt{R^2-x^2}}^{\sqrt{R^2-x^2}} \frac{1}{\pi R^2} dy = \frac{2\sqrt{R^2-x^2}}{\pi R^2};$$

for $x \notin [-R, R]$, $f_X(x) = 0$. For symmetry,

$$f_Y(y) = \begin{cases} \frac{2\sqrt{R^2-x^2}}{\pi R^2}, & \text{if } y \in [-R, R], \\ 0, & \text{otherwise.} \end{cases}$$

For $x \in [-R, R]$,

$$\begin{aligned} F_X(x) &= \frac{2}{\pi R^2} \int_{-R}^{x} \sqrt{R^2-t^2} dt \\ &= \frac{2}{\pi R^2} \left[t\sqrt{R^2-t^2} \Big|_{-R}^{x} + \int_{-R}^{x} \frac{t^2 dt}{\sqrt{R^2-t^2}} \right] \\ &= \frac{2}{\pi R^2} \left[x\sqrt{R^2-x^2} - \int_{-R}^{x} \sqrt{R^2-t^2} dt + R^2 \int_{-R}^{x} \frac{dt}{\sqrt{R^2-t^2}} \right] \\ &= \frac{2}{\pi R^2} \left[x\sqrt{R^2-x^2} + R^2 \left(\arcsin \frac{x}{R} + \frac{\pi}{2} \right) \right] - F_X(x). \end{aligned}$$

So, we get that

$$F_X(x) = \begin{cases} 0, & x < -R, \\ \frac{x\sqrt{R^2-x^2}+R^2\left(\arcsin\frac{x}{R}+\frac{\pi}{2}\right)}{\pi R^2}, & x \in [-R, R], \\ 1, & x > R. \end{cases}$$

$F_Y(y)$ takes the same formula as above with x replaced by y.

(ii) (5pts) Show that X and Y are NOT independent by finding two Borel sets $A, B \in \mathcal{B}(\mathbb{R})$ such that

$$\mathbb{P}(X \in A, Y \in B) \neq \mathbb{P}(X \in A)\,\mathbb{P}(Y \in B).$$

Solution: For example, let $A = \left[-\frac{\sqrt{2}}{2}R, \frac{\sqrt{2}}{2}R\right]$ and $B = \left[0, \frac{\sqrt{2}}{2}R\right]$. Then, $\{X \in A, Y \in B\}$ corresponds to a rectangle within the disk, and area of the rectangle is R^2. So

$$\mathbb{P}(X \in A, Y \in B) = \frac{1}{\pi}.$$

However,

$$\mathbb{P}(X \in A) = F_X\left(\frac{\sqrt{2}}{2}R\right) - F_X\left(-\frac{\sqrt{2}}{2}R\right) = \frac{1}{\pi} + \frac{1}{2},$$

and

$$\mathbb{P}(Y \in B) = F_Y\left(\frac{\sqrt{2}}{2}R\right) - F_Y(0) = \frac{1}{2\pi} + \frac{1}{4},$$

so

$$\mathbb{P}(X \in A)\,\mathbb{P}(Y \in B) \neq \mathbb{P}(X \in A, Y \in B).$$

(iii) (5pts) Compute $\mathbb{P}(|Y| \leq |X|)$ and $\mathbb{P}(|X+Y| \leq R)$. Draw in the disk D the region corresponding to each of these events.

Solution: $\{|Y| \leq |X|\}$ corresponds to two opposite "slices" of the disk each of which has a central angle of $90°$. So

$$\mathbb{P}(|Y| \leq |X|) = \frac{\frac{1}{2}\pi R^2}{\pi R^2} = \frac{1}{2}.$$

$\{|X+Y| \leq R\}$ corresponds to a square "diamond" centered in the disk with the four corners on the circle. So

$$\mathbb{P}(|X+Y| \leq R) = \frac{2R^2}{\pi R^2} = \frac{2}{\pi}.$$

(iv) (5pts) Set $R = 4$. Compute $\mathbb{P}\left(|XY| \leq \sqrt{3}\right)$. Draw in the disk D the region corresponding to this event.

Solution: Due to symmetry, we only need to find the area of the target region in the first quadrant. Assume that the hyperbola $y = \sqrt{3}/x$ and the circle $y = \sqrt{16-x^2}$ intersect at two points $\left(a, \sqrt{3}/a\right)$ and $\left(b, \sqrt{3}/b\right)$ (assuming that $b > a$) in the first quadrant. Then the area of the region below the circle $y = \sqrt{16-x^2}$ but above the parabola $y = \sqrt{3}/x$, i.e., the region defined by

$$\left\{(x,y) : x>0, y>0, \sqrt{3}/x < y < \sqrt{16-x^2}\right\},$$

is given by

$$\begin{aligned}\int_a^b \left(\sqrt{16-x^2}-\frac{\sqrt{3}}{x}\right)dx &= \int_a^b \sqrt{16-x^2}dx - \int_a^b \frac{\sqrt{3}}{x}dx \\ &= \frac{1}{2}\left[x\sqrt{16-x^2}+16\arcsin\frac{x}{4}\right]\Bigg|_a^b - \sqrt{3}\ln x\Big|_a^b \\ &= 8\left(\arcsin b - \arcsin a\right) - \sqrt{3}\ln\frac{b}{a}.\end{aligned}$$

Therefore, the desired probability is the difference between 16π (the area of the disk) and four times the area computed above, i.e.,

$$\mathbb{P}\left(|XY| \le \sqrt{3}\right) = 16\pi - 32\left(\arcsin b - \arcsin a\right) + 4\sqrt{3}\ln\frac{b}{a}$$

where $a = \sqrt{8-\sqrt{61}}$ and $b = \sqrt{8+\sqrt{61}}$ by direct computation.

Problem 2. (12pts) Consider the random variable (X,Y) as in Problem 1 restricted in the first quadrant with the origin $(0,0)$, the $x-$axis and $y-$axis removed, i.e., assume that the joint PDF of (X,Y) is

$$f(x,y) = \begin{cases} \frac{4}{\pi R^2}, & \text{if } x>0,\ y>0, \text{ and } x^2+y^2 \le R^2, \\ 0, & \text{otherwise.}\end{cases}$$

Set $Q := \left\{(x,y) : x>0, y>0, x^2+y^2 \le R^2\right\}$. Consider the transformations of (X,Y) given by

$$r(X,Y) = \begin{cases} \sqrt{X^2+Y^2}, & \text{if } (X,Y) \in Q, \\ 0, & \text{otherwise}\end{cases} \text{ and } \theta(X,Y) := \begin{cases} \arctan\frac{Y}{X}, & \text{if } (X,Y) \in Q, \\ 0, & \text{otherwise.}\end{cases}.$$

(i) (4pts) Determine the DF of the random variable r, i.e., $F_r(t) = \mathbb{P}(r \le t)$. Is r a continuous random variable? If so, write down the PDF of r.

Solution: For each $t \in \mathbb{R}$,

$$F_r(t) = \mathbb{P}(r \le t) = \mathbb{P}\left(X^2+Y^2 \le t^2\right) = \begin{cases} 0, & \text{if } t \le 0, \\ t^2/R^2, & \text{if } 0 \le t \le R, \\ 1, & \text{if } t \ge R.\end{cases}$$

r is a random variable with continuous distribution, and its PDF is given by

$$f_r(t) = \begin{cases} 0, & \text{if } t < 0 \text{ or } t > R, \\ 2t/R^2, & \text{if } t \in (0, R). \end{cases}$$

(ii) (4pts) Determine the DF of the random variable θ, i.e., $F_\theta(\alpha) = \mathbb{P}(\theta \le \alpha)$. Is θ a continuous random variable? If so, write down the PDF of θ.

Solution: For each $\alpha \in \mathbb{R}$,

$$F_\theta(\alpha) = \mathbb{P}(\theta \le \alpha) = \mathbb{P}\left(\arctan\frac{Y}{X} \le \alpha\right) = \begin{cases} 0, & \text{if } \alpha \le 0, \\ 2\alpha/\pi, & \text{if } 0 \le \alpha \le \frac{\pi}{2}, \\ 1, & \text{if } \alpha \ge \frac{\pi}{2}. \end{cases}$$

θ is a random variable with continuous distribution, and its PDF is given by

$$f_\theta(\alpha) = \begin{cases} 0, & \text{if } \alpha < 0 \text{ or } \alpha > \frac{\pi}{2}, \\ 2/\pi, & \text{if } \alpha \in \left(0, \frac{\pi}{2}\right). \end{cases}$$

(iii) (4pts) Determine the joint DF of (r, θ), i.e., $F_{(r,\theta)}(t, \alpha) = \mathbb{P}(r \le t, \theta \le \alpha)$, and show that r and θ are independent random variables.

Solution: By the result derived in class on the circular coordinate transformation, we have that (r, θ) is a continuous type random variable with the joint PDF given by

$$f_{(r,\theta)}(t, \alpha) = f_{(X,Y)}(t\cos\alpha, t\sin\alpha) \cdot t = \begin{cases} \frac{4t}{\pi R^2}, & \text{if } t \in (0, R) \text{ and } \alpha \in \left(0, \frac{\pi}{2}\right), \\ 0, & \text{otherwise.} \end{cases}$$

Clearly that

$$f_{(r,\theta)}(t, \alpha) = f_r(t) \cdot f_\theta(\alpha),$$

which implies that r and θ are independent random variables. Finally,

$$F_{(r,\theta)}(t, \alpha) = F_r(t) \cdot F_\theta(\alpha)$$

where F_r and F_θ are as derived above in (i) and (ii).

Problem 3. (12pts) Given a probability space $(\Omega, \mathcal{S}, \mathbb{P})$, for each $n \in \mathbb{N}$, assume that $X_1, \cdots, X_n$ are independent random variables on Ω, and for each $i = 1, \cdots, n$, X_i has the uniform distribution on $[0, 1]$, i.e., the PDF of X_i is

$$f_{X_i}(x) = \begin{cases} 1, & 0 \le x \le 1, \\ 0, & \text{otherwise.} \end{cases}$$

Set

$$Z_n := X_1 + X_2 + \cdots + X_n.$$

(i) (5pts) Show, by an induction in n, that the DF F_{Z_n} of Z_n satisfies that, if $0 \leq x \leq 1$, then

$$F_{Z_n}(x) = \frac{x^n}{n!}.$$

Note that this formula only applies to x that is in $[0,1]$. For general x, the formula of $F_{Z_n}(x)$ is much more complicated, and you are not required to derive it.

Proof: The statement is clearly true for $n = 1$. Assume that it is true for n. Let $f_{(Z_n, X_{n+1})}$ be the joint PDF of (Z_n, X_{n+1}), and since Z_n and X_{n+1} are independent,

$$f_{(Z_n, X_{n+1})}(z, x) = f_{Z_n}(z) \cdot f_{X_{n+1}}(x).$$

By the inductive assumption, for $z \in [0,1]$,

$$f_{Z_n}(z) = \frac{z^{n-1}}{(n-1)!}.$$

So, for each $c \in [0,1]$,

$$\begin{aligned} F_{Z_{n+1}}(c) = \mathbb{P}(Z_n + X_{n+1} \leq c) &= \iint_{\{(z,x) \in [0,1]^2 : z+x \leq c\}} \frac{z^{n-1}}{(n-1)!} dz dx \\ &= \int_0^c \int_0^{c-x} \frac{z^{n-1}}{(n-1)!} dz dx \\ &= \frac{c^{n+1}}{(n+1)!}. \end{aligned}$$

(ii) (4pts) For each sample point $\omega \in \Omega$, define

$$N(\omega) := \min\{n \in \mathbb{N} : Z_n(\omega) > 1\}.$$

That is, $N(\omega)$ is the first time that the sum of $X_i(\omega)$'s exceeds 1, so $N(\omega) = k$ if and only if

$$X_1(\omega) + \cdots + X_{k-1}(\omega) \leq 1 \text{ and } X_1(\omega) + \cdots + X_k(\omega) > 1.$$

$N : \Omega \to \mathbb{N}$ is a discrete RV on Ω (you don't have to prove this statement). Show that, for $k \in \mathbb{N}$,

$$\mathbb{P}(N \geq k) = \frac{1}{(k-1)!}.$$

Proof: Note that, the event $\{N \geq k\}$ is equivalent to the event $\{Z_{k-1} \leq 1\}$. Therefore,

$$\mathbb{P}(N \geq k) = F_{Z_{k-1}}(1) = \frac{1}{(k-1)!}.$$

(iii) (3pts) Show that $\mathbb{E}[N] = e$.

Proof: By the previous result,

$$\mathbb{E}[N] = \sum_{k=1}^{\infty} \mathbb{P}(N \geq k) = \sum_{l=0}^{\infty} \frac{1^l}{l!} = e.$$

Problem 4. (20pts) Given constants $m_1, m_2 \in \mathbb{R}$, $\sigma_1^2 > 0$, $\sigma_2^2 > 0$, $\sigma_1 = \sqrt{\sigma_1^2}$, $\sigma_2 = \sqrt{\sigma_2^2}$ and $c \in \mathbb{R}$ such that $|c| < \sigma_1\sigma_2$, let function $f\colon \mathbb{R}^2 \to [0,\infty)$ be given by, for $(x,y) \in \mathbb{R}^2$,

$$f(x,y) = \frac{1}{2\pi}\frac{1}{\sqrt{\sigma_1^2\sigma_2^2 - c^2}}\exp\left[-\frac{\sigma_2^2(x-m_1)^2 - 2c(x-m_1)(y-m_2) + \sigma_1^2(y-m_2)^2}{2(\sigma_1^2\sigma_2^2 - c^2)}\right]. \quad (\star)$$

(i) (4pts) Show that $f(x,y)$ is a joint PDF, i.e.,

$$\iint_{\mathbb{R}^2} f(x,y)\,dxdy = 1.$$

Proof: By direct computation, one can verify that

$$f(x,y) = \frac{1}{\sqrt{2\pi}\cdot\sqrt{\left(\sigma_1^2 - \frac{c^2}{\sigma_2^2}\right)}}\exp\left\{-\frac{\left[x - \left(m_1 + \frac{c(y-m_2)}{\sigma_2^2}\right)\right]^2}{2\left(\sigma_1^2 - \frac{c^2}{\sigma_2^2}\right)}\right\}\cdot\frac{1}{\sqrt{2\pi}\cdot\sigma_2^2}\exp\left[-\frac{(y-m_2)^2}{2\sigma_2^2}\right]. \quad (A)$$

In fact, we also have that

$$f(x,y) = \frac{1}{\sqrt{2\pi}\cdot\sqrt{\left(\sigma_2^2 - \frac{c^2}{\sigma_1^2}\right)}}\exp\left\{-\frac{\left[y - \left(m_2 + \frac{c(x-m_1)}{\sigma_1^2}\right)\right]^2}{2\left(\sigma_2^2 - \frac{c^2}{\sigma_1^2}\right)}\right\}\cdot\frac{1}{\sqrt{2\pi}\cdot\sigma_1^2}\exp\left[-\frac{(x-m_1)^2}{2\sigma_1^2}\right]. \quad (B)$$

Therefore, following either of the expressions above, we can see that

$$\iint f(x,y)\,dxdy = 1.$$

(ii) (5pts) Let (X,Y) be a bivariate random variable on some probability space $(\Omega,\mathcal{S},\mathbb{P})$ that has continuous distribution with $f(x,y)$ in $(\star)$ being the joint PDF. Determine the marginal PDF f_X of X and the marginal PDF f_Y of Y. What kind of the distributions are the marginal distributions of X and Y?

Solution: Using (B), we have that, for every $x \in \mathbb{R}$,

$$f_X(x) = \int_{\mathbb{R}} f(x,y)\,dy = \frac{1}{\sqrt{2\pi}\cdot\sigma_1^2}\exp\left[-\frac{(x-m_1)^2}{2\sigma_1^2}\right]$$

so X is an $N(m_1,\sigma_1^2)$ random variable. Similarly, using (A), for every $y \in \mathbb{R}$,

$$f_Y(y) = \int_{\mathbb{R}} f(x,y)\,dx = \frac{1}{\sqrt{2\pi}\cdot\sigma_2^2}\exp\left[-\frac{(y-m_2)^2}{2\sigma_2^2}\right],$$

so Y is an $N(m_2,\sigma_2^2)$ random variable.

(iii) (5pts) Show that

$$\mathrm{Cov}(X,Y) = c,$$

$$\operatorname{Corr}(X,Y)=\frac{c}{\sigma_1\sigma_2},$$

and the joint MGF of (X,Y), i.e.,

$$M(s_1,s_2)=\mathbb{E}\left[e^{s_1X_1+s_2X_2}\right]$$

exists for all $s_1,s_2\in\mathbb{R}$ and

$$M(s_1,s_2)=\exp\left[s_1m_1+s_2m_2+\frac{1}{2}\left(\sigma_1^2s_1^2+2cs_1s_2+\sigma_2^2s_2^2\right)\right].$$

Proof: Again, we can use (A) or (B) to compute these integrals. For example, using (A),

$$\begin{aligned}
&\operatorname{Cov}(X,Y)\\
=\;&\iint_{\mathbb{R}^2}(x-m_1)(y-m_2)f(x,y)\,dxdy\\
=\;&\int_{\mathbb{R}}\left(\int_{\mathbb{R}}\frac{(x-m_1)}{\sqrt{2\pi}\cdot\sqrt{\left(\sigma_1^2-\frac{c^2}{\sigma_2^2}\right)}}\exp\left\{-\frac{\left[x-\left(m_1+\frac{c(y-m_2)}{\sigma_2^2}\right)\right]^2}{2\left(\sigma_1^2-\frac{c^2}{\sigma_2^2}\right)}\right\}dx\right)\frac{(y-m_2)}{\sqrt{2\pi}\cdot\sigma_2^2}\exp\left[-\frac{(y-m_2)^2}{2\sigma_2^2}\right]dy\\
=\;&\int_{\mathbb{R}}\frac{c(y-m_2)}{\sigma_2^2}\cdot\frac{(y-m_2)}{\sqrt{2\pi}\cdot\sigma_2^2}\exp\left[-\frac{(y-m_2)^2}{2\sigma_2^2}\right]dy\\
=\;&\frac{c}{\sigma_2^2}\cdot\sigma_2^2=c.
\end{aligned}$$

It follows immediately that $\operatorname{Corr}(X,Y)=\frac{c}{\sigma_1\sigma_2}$. Using (B),

$$\begin{aligned}
&M(s_1,s_2)\\
=\;&\iint_{\mathbb{R}^2}e^{s_1x}\cdot e^{s_2y}f(x,y)\,dxdy\\
=\;&\int_{\mathbb{R}}\left(\int_{\mathbb{R}}\frac{e^{s_2y}}{\sqrt{2\pi}\cdot\sqrt{\left(\sigma_2^2-\frac{c^2}{\sigma_1^2}\right)}}\exp\left\{-\frac{\left[y-\left(m_2+\frac{c(x-m_1)}{\sigma_1^2}\right)\right]^2}{2\left(\sigma_2^2-\frac{c^2}{\sigma_1^2}\right)}\right\}dy\right)\frac{e^{s_1x}}{\sqrt{2\pi}\cdot\sigma_1^2}\exp\left[-\frac{(x-m_1)^2}{2\sigma_1^2}\right]dx\\
=\;&\int_{\mathbb{R}}e^{s_2m_2+s_2\frac{c(x-m_1)}{\sigma_1^2}+\frac{s_2^2}{2}\left(\sigma_2^2-\frac{c^2}{\sigma_1^2}\right)}\cdot\frac{e^{s_1x}}{\sqrt{2\pi}\cdot\sigma_1^2}\exp\left[-\frac{(x-m_1)^2}{2\sigma_1^2}\right]dx\\
=\;&e^{s_2m_2-s_2\frac{cm_1}{\sigma_1^2}+\frac{s_2^2}{2}\left(\sigma_2^2-\frac{c^2}{\sigma_1^2}\right)}\int_{\mathbb{R}}\frac{e^{\left(s_1+\frac{s_2c}{\sigma_1^2}\right)x}}{\sqrt{2\pi}\cdot\sigma_1^2}\exp\left[-\frac{(x-m_1)^2}{2\sigma_1^2}\right]dx\\
=\;&e^{s_2m_2-s_2\frac{cm_1}{\sigma_1^2}+\frac{s_2^2}{2}\left(\sigma_2^2-\frac{c^2}{\sigma_1^2}\right)}\cdot e^{\left(s_1+\frac{s_2c}{\sigma_1^2}\right)m_1+\frac{\left(s_1+\frac{s_2c}{\sigma_1^2}\right)^2\sigma_1^2}{2}}\\
=\;&e^{s_2m_2+s_1m_1+\frac{s_2^2\sigma_2^2}{2}+s_1s_2c+\frac{s_1^2\sigma_1^2}{2}}.
\end{aligned}$$

(iv) (3pts) Further determine the conditional PDF $f_{X|Y}$ of X given Y. What kind of the distribution is this conditional distribution? Let Z be another continuous random variable on Ω and Z has $f_{X|Y}$ as its PDF. Determine $\mathbb{E}[Z]$ and $\mathrm{Var}(Z)$.

Solution: It's obvious from (A) that

$$f_{X|Y}(x|y) = \frac{f_{(X,Y)}(x,y)}{f_Y(y)} = \frac{1}{\sqrt{2\pi}\cdot\sqrt{\left(\sigma_1^2 - \frac{c^2}{\sigma_2^2}\right)}} \exp\left\{-\frac{\left[x - \left(m_1 + \frac{c(y-m_2)}{\sigma_2^2}\right)\right]^2}{2\left(\sigma_1^2 - \frac{c^2}{\sigma_2^2}\right)}\right\}.$$

If Z has the above PDF, then Z is a Gaussian random variable with expectation

$$\mathbb{E}[Z] = m_1 + \frac{c(y-m_2)}{\sigma_2^2}$$

and variance

$$\mathrm{Var}(Z) = \sigma_1^2 - \frac{c^2}{\sigma_2^2}.$$

(v) (3pts) Based on the statements in (i)-(iv), determine an "if and only if" condition, i.e., a both necessary and sufficient condition, for X and Y to be independent.

Solution: It is clear from above that, X and Y are independent if and only if $c = 0$, i.e., $\mathrm{Cov}(X,Y) = 0$.

Problem 5. (6pts) Given a probability space $(\Omega, \mathcal{S}, \mathbb{P})$, let X and Y be two independent random variables on Ω and both X and Y have the standard Gaussian distribution $N(0,1)$. Denote $U = \frac{X+Y}{\sqrt{2}}$ and $V = \frac{X-Y}{\sqrt{2}}$. Show that U and V are independent and both U and V have the standard Gaussian distribution $N(0,1)$.

Proof: Consider the joint MGF of (U,V). For every $s_1, s_2 \in \mathbb{R}$,

$$\begin{aligned} M_{(U,V)}(s_1,s_2) &= \mathbb{E}\left[e^{s_1 U + s_2 V}\right] = \mathbb{E}\left[e^{\frac{(s_1+s_2)}{\sqrt{2}}X + \frac{(s_1-s_2)}{\sqrt{2}}Y}\right] \\ &= M_X\left(\frac{s_1+s_2}{\sqrt{2}}\right)\cdot M_Y\left(\frac{s_1-s_2}{\sqrt{2}}\right) \\ &= e^{\frac{(s_1+s_2)^2}{4}}\cdot e^{\frac{(s_1-s_2)^2}{4}} = e^{\frac{s_1^2}{2}}\cdot e^{\frac{s_2^2}{2}}. \end{aligned}$$

The right hand side of above is exactly the joint MGF of two independent standard normal random variables.

Problem 6. (15pts) (The converse of Problem 5.) Given a probability space $(\Omega, \mathcal{S}, \mathbb{P})$, let X and Y be two independent random variables on Ω. Assume that X and Y have the same continuous distribution with a common PDF ϕ, i.e.,

$$f_X(t) = f_Y(t) = \phi(t) \;\; \forall t \in \mathbb{R}$$

where f_X and f_Y are the PDFs of X and Y respectively. Further assume that ϕ is smooth (having derivatives of all orders) on $\mathbb{R}$ and

$$\phi(0) = \max_{t \in \mathbb{R}} \phi(t) > 0,$$

$\mathrm{Var}(X)$ exists and equals 1 (so does $\mathrm{Var}(Y)$), and the MGF $M(s)$ of X exists for all $s \in \mathbb{R}$ (same for Y). Denote $U = \frac{X+Y}{\sqrt{2}}$ and $V = \frac{X-Y}{\sqrt{2}}$. Suppose that U and V are also independent random variables, and both U and V have the same distribution as that of X and Y, which means that U and V have the common continuous distribution, and

$$f_U(t) = f_V(t) = \phi(t) \;\; \forall t \in \mathbb{R}$$

where f_U and f_V are the PDFs of U and V respectively.

(i) (3pts) First show that,

$$\mathbb{E}[X] = \mathbb{E}[Y] = \mathbb{E}[U] = \mathbb{E}[V] = 0$$

and

$$\mathrm{Var}(X) = \mathrm{Var}(Y) = \mathrm{Var}(U) = \mathrm{Var}(V) = 1.$$

Proof: By the hypothesis, we have that

$$\mathbb{E}[X] = \mathbb{E}[U] = \mathbb{E}\left[\frac{X+Y}{\sqrt{2}}\right] = \frac{\mathbb{E}[X] + \mathbb{E}[Y]}{\sqrt{2}} = \sqrt{2}\mathbb{E}[X],$$

which implies that $\mathbb{E}[X] = 0$ and hence the expectations of Y, U and V are all zero. Since U, V, X and Y all have the same distribution, they all have variance 1.

Next, show, in two different ways as in (ii) and (iii), that $\phi(t) = \frac{e^{-t^2/2}}{\sqrt{2\pi}}$, i.e., X, Y, U and V all have the standard Gaussian distribution $N(0,1)$.

(ii) (6pts) Method 1: Show that for every $u, v \in \mathbb{R}$,

$$\phi\left(\frac{u+v}{\sqrt{2}}\right)\phi\left(\frac{u-v}{\sqrt{2}}\right) = \phi(u)\,\phi(v).$$

Then argue that, if $\psi(t) := \frac{\phi(t)}{\phi(0)}$ for $t \in \mathbb{R}$, then

$$\psi\left(\sqrt{2}t\right) = (\psi(t))^2 \quad \forall t \in \mathbb{R}.$$

Finally, establish that $\psi(t) = \exp\left(\frac{\psi''(0)t^2}{2}\right)$, and combine the hypotheses that ϕ is a PDF and $\mathrm{Var}(X) = 1$ to show that

$$\phi(t) = \frac{1}{\sqrt{2\pi}} e^{-\frac{t^2}{2}} \quad \forall t \in \mathbb{R}.$$

Proof: By the result on transformation of bivariate random variables given in class, we know that (U,V) is has the joint density given by

$$f_{(U,V)}(u,v)=f_{(X,Y)}\left(\frac{u+v}{\sqrt{2}},\frac{u-v}{\sqrt{2}}\right)\cdot 1.$$

Further, because U and V are i.i.d. with common PDF ϕ, and meanwhile, X and Y are also i.i.d. with PDF ϕ, the relation above leads to

$$\phi(u)\phi(v)=\phi\left(\frac{u+v}{\sqrt{2}}\right)\phi\left(\frac{u-v}{\sqrt{2}}\right).$$

Define ψ as in the statement. $\psi(0)=1$. Setting $u=v=t$, we get $\psi\left(\sqrt{2}t\right)=(\psi(t))^2$ and by repeating this formula, we have

$$\psi(t)=\left[\psi\left(\frac{t}{2^n}\right)\right]^{2^{2n}}\qquad \forall t\in\mathbb{R}, n\geq 1.$$

Let t be fixed for now. Then we have

$$\psi'(0)=\lim_{n\to\infty}\frac{\psi(t/2^n)-1}{t/2^n}=\frac{1}{t}\lim_{n\to\infty}\frac{(\psi(t))^{2^{-2n}}-1}{2^{-n}}=0$$

and

$$\begin{aligned}\psi''(0) &= \lim_{n\to\infty}\frac{\psi(t/2^n)+\psi(-t/2^n)-2}{t^2/2^{2n}}\\ &= \frac{1}{t^2}\lim_{n\to\infty}\frac{(\psi(t))^{2^{-2n}}+(\psi(-t))^{2^{-2n}}-2}{2^{-2n}}\\ &= \frac{\ln(\psi(t))+\ln(\psi(-t))}{t^2}.\end{aligned}$$

Finally, with Taylor's expansion of ψ near 0, we have that

$$\psi(t)=\left[1+\frac{\psi''(0)}{2}\frac{t^2}{2^{2n}}+o\left(\frac{1}{2^{2n}}\right)\right]^{2^{2n}}\to\exp\left(\frac{\psi''(0)\,t^2}{2}\right)\text{ as }n\to\infty.$$

This means that

$$\phi(t)=\phi(0)\exp\left(\frac{\psi''(0)\,t^2}{2}\right).$$

Since ϕ is a PDF, it must a Gaussian PDF. Combining with the fact that the expectation of X is 0 and the variance of X is 1, we know that ϕ must be the PDF of the standard Gaussian distribution.

(iii) (6pts) Method 2: Consider the MGF $M(s)$ of X (same MGF for Y, U and V). Show that for every $s_1,s_2\in\mathbb{R}$,

$$M(s_1)M(s_2)=M\left(\frac{s_1+s_2}{\sqrt{2}}\right)M\left(\frac{s_1-s_2}{\sqrt{2}}\right).$$

Further deduce that for every $s \in \mathbb{R}$,

$$M\left(\sqrt{2}s\right) = \left(M\left(s\right)\right)^2.$$

From here, establish that $M(s) = e^{s^2/2}$, and hence X, Y, U and V are all standard Gaussian random variables.

Proof: A similar argument as in the proof of Problem 5 leads to

$$M\left(s_1\right) M\left(s_2\right) = M\left(\frac{s_1 + s_2}{\sqrt{2}}\right) M\left(\frac{s_1 - s_2}{\sqrt{2}}\right)$$

which, by setting $s_1 = s_2 = s$, implies that

$$M\left(\sqrt{2}s\right) = \left(M\left(s\right)\right)^2$$

and further

$$M\left(s\right) = \left[M\left(\frac{s}{2^n}\right)\right]^{2^{2n}}.$$

Since $M(0) = 1$, $M'(0) = \mathbb{E}[X] = 0$ and $M''(0) = \mathbb{E}\left[X^2\right] = 1$, we have that

$$M\left(s\right) = \left[1 + \frac{1}{2}\frac{s^2}{2^{2n}} + o\left(\frac{1}{2^{2n}}\right)\right]^{2^{2n}} \to \exp\left(\frac{s^2}{2}\right) \text{ as } n \to \infty.$$

Problem 7. (10pts) Given a probability space $(\Omega, \mathcal{S}, \mathbb{P})$, $m \in \mathbb{R}$, $\sigma^2 > 0$ and $\sigma = \sqrt{\sigma^2}$, let $\{X_i : i \in \mathbb{N}\}$ be a sequence of random variables on Ω such that for each $i \in \mathbb{N}$, $\mathbb{E}[X_i] = m$ and $\mathrm{Var}(X_i) = \sigma^2$. Further assume that $\{X_i : i \in \mathbb{N}\}$ is *uncorrelated*, i.e., $\mathrm{Corr}(X_i, X_j) = 0$ whenever $i \neq j$. For each $n \geq 1$, define

$$S_n = \sum_{i=1}^{n} X_i \text{ and } \overline{S}_n = \frac{S_n}{n}.$$

(i) (3pts) Show that for every $n \in \mathbb{N}$,$\mathbb{E}\left[\overline{S}_n\right] = m$ and $\mathrm{Var}\left(\overline{S}_n\right) = \frac{\sigma^2}{n}$.

Proof:

$$\mathbb{E}\left[\overline{S}_n\right] = \frac{\mathbb{E}\left[S_n\right]}{n} = \frac{\sum_{i=1}^{n}\mathbb{E}\left[X_i\right]}{n} = m.$$

$$\begin{aligned}
\operatorname{Var}\left(\overline{S}_n\right) &= \mathbb{E}\left[\left(\overline{S}_n - m\right)^2\right] \\
&= \mathbb{E}\left[\left(\frac{\sum_{i=1}^n (X_i - m)}{n}\right)^2\right] \\
&= \frac{1}{n^2}\left\{\sum_{i=1}^n \mathbb{E}\left[(X_i - m)^2\right] + \sum_{i\neq j} \mathbb{E}\left[(X_i - m)(X_j - m)\right]\right\} \\
&= \frac{1}{n^2}\left\{\sum_{i=1}^n \operatorname{Var}(X) + \sum_{i\neq j} \operatorname{Cov}(X_i, X_j)\right\} \\
&= \frac{1}{n^2}\cdot n\sigma^2 = \sigma^2/n.
\end{aligned}$$

(ii) (4pts) Show that, $\forall \epsilon > 0$ and $n \in \mathbb{N}$,

$$\mathbb{P}\left(\left|\overline{S}_n - m\right| > \epsilon\right) \leq \frac{\sigma^2}{n\epsilon^2}.$$

Proof: By Chebyshev's inequality,

$$\mathbb{P}\left(\left|\overline{S}_n - m\right| > \epsilon\right) \leq \frac{\operatorname{Var}\left(\overline{S}_n\right)}{\epsilon^2} = \frac{\sigma^2}{n\epsilon^2}.$$

(iii) (3pts) In particular, when $\{X_i : i \in \mathbb{N}\}$ is a sequence of independent and identically distributed (i.i.d.) random variables, i.e., X_i's are independent and X_i's share a common distribution, and assume that the common distribution of X_i's is $N\left(m, \sigma^2\right)$. Show that, for every $n \in \mathbb{N}$, if

$$T_n = \frac{S_n - nm}{\sqrt{n}\sigma},$$

then T_n has the distribution $N(0,1)$.

Proof: As seen in class, S_n is an $N\left(nm, n\sigma^2\right)$ random variable, then $T_n = \frac{1}{\sqrt{n}\sigma}S_n - \sqrt{n}\frac{m}{\sigma}$ is also a Gaussian random variable, with expectation $\frac{1}{\sqrt{n}\sigma}nm - \sqrt{n}\frac{m}{\sigma} = 0$ and variance $\frac{n\sigma^2}{n\sigma^2} = 1$. In other words, T_n is an $N(0,1)$ random variable for each $n \geq 1$.

Problem 8. (5pts) Given a probability space $(\Omega, \mathcal{S}, \mathbb{P})$ and $n \in \mathbb{N}$, let $X_1, \cdots, X_n$ be i.i.d. random variables and the common distribution of X_i's is the standard Gaussian distribution $N(0,1)$. Set $Z_n := X_1^2 + \cdots + X_n^2$. Show that, by studying the MGF of Z_n, that Z_n has the $\chi-$square distribution with n degrees of freedom, i.e., the Gamma distribution with parameters $\alpha = \frac{n}{2}, \lambda = \frac{1}{2}$ (as introduced in HW 3 Problem 6).

Proof: Considering the MGF of Z_n. By the result of HW 3 Problem 9, we have that

$$M_{Z_n}(s) = \prod_{i=1}^n \mathbb{E}\left[e^{sX_i^2}\right] = \left(\frac{1}{\sqrt{1-2s}}\right)^n = \left(\frac{1/2}{(1/2) - s}\right)^{n/2}.$$

By HW 3 Problem 6, this is exactly the MGF of the $\chi-$square distribution with n degrees of freedom.