Math 317 Assignment 2

Due in class: October 20, 2016

Instructions: Submit a hard copy of your solution with your name and student number. (**No name = zero grade!**) You must include all relevant program code, electronic output and explanations of your results. Write your own codes and comment them. Late assignment will not be graded and will receive a grade of zero.

1. (15 marks) Using Newton's method, find a zero to the system of equations

$$F(x,y) = \begin{pmatrix} x^2+y^2-4 \\ x+\sin(xy)-y \end{pmatrix},$$

with $\vec{x_0} = (1,1)^T$. Use the stopping criterion of $||F(\vec{x_k})||_2 < 10^{-8}$.

Solution: The Jacobian for this system is

$$J_F(x,y) = \begin{pmatrix} 2x & 2y \\ 1+y\cos(xy) & x\cos(xy)-1 \end{pmatrix}.$$

In MATLAB we run the script "newtonSystem.m". The output follows.

```
>> newtonSystem([1;1],1e-8);
(x,y) = (0.8091133546619820,2.1908866453380180) for k=1
(x,y) = (0.8255246464035786,1.8528486397672017) for k=2
(x,y) = (0.8244071825840399,1.8224376192879208) for k=3
(x,y) = (0.8244888454959339,1.8221466008566216) for k=4
(x,y) = (0.8244888339654131,1.8221465810044906) for k=5
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2. Consider the following data set $\{(-1,0),(1,6),(2,9)\}$.

   (a) (12 marks) Compute the Lagrange interpolation polynomial $L_2(x)$:

   i. solving the linear system with the Vandermonde matrix;

Solution: We suppose that $L_2(x) = a_0 + a_1x + a_2x^2$ and impose the interpolation conditions

$$L_2(-1)=0, \quad L_2(1)=6, \quad L_2(2)=9.$$

This leads to the linear system

$$\begin{pmatrix} 1 & -1 & 1 \\ 1 & 1 & 1 \\ 1 & 2 & 4 \end{pmatrix} \begin{pmatrix} a_0 \\ a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 6 \\ 9 \end{pmatrix},$$

which has solution $a_0 = 3$, $a_1 = 3$, $a_2 = 0$. Thus $L_2(x) = 3 + 3x$.

ii. using the Lagrange basis polynomials;

Solution: We have

$$\begin{aligned}
l_0(x) &= \frac{x-1}{-1-1}\frac{x-2}{-1-2} = \frac{1}{6}(x-1)(x-2) = \frac{1}{2}x^2 - \frac{1}{2}x + \frac{1}{3},\\
l_1(x) &= \frac{x-(-1)}{1-(-1)}\frac{x-2}{1-2} = -\frac{1}{2}(x+1)(x-2) = -\frac{1}{2}x^2 + \frac{1}{2}x + 1,\\
l_2(x) &= \frac{x-(-1)}{2-(-1)}\frac{x-1}{2-1} = \frac{1}{3}(x+1)(x-1) = \frac{1}{3}x^2 - \frac{1}{3}.
\end{aligned}$$

Hence

$$\begin{aligned}
L_2(x) &= 0\, l_0(x) + 6\, l_1(x) + 9\, l_2(x)\\
&= 0\left(\frac{1}{2}x^2 - \frac{1}{2}x + \frac{1}{3}\right) + 6\left(-\frac{1}{2}x^2 + \frac{1}{2}x + 1\right) + 9\left(\frac{1}{3}x^2 - \frac{1}{3}\right)\\
&= -3x^2 + 3x + 6 + 3x^2 - 3\\
&= 3 + 3x
\end{aligned}$$

iii. using Newton's divided differences.

Solution: We compute the divided differences using the table.

x_i	$f[x_i]$	$f[x_i, x_{i+1}]$	
-1	0		
		3	
1	6		0
		3	
2	9		

Consequently

$$L_2(x) = 0 + 3(x-(-1)) + 0(x-(-1))(x-1) = 3 + 3x.$$

(b) (3 marks) Add the point $(-2, 3)$ to the data set. Compute the Lagrange interpolation polynomial $L_3(x)$.

Solution: In the best method to use in this case is the Newton's divided difference since we can use $L_2(x)$ in the previous question to compute $L_3(x)$. We compute the Newton's divided difference table with the new node. Notice that we only need to compute 3 new entries.

x_i	$f[x_i]$	$f[x_i, x_{i+1}]$	$f[x_i, x_{i+1}, x_{i+2}]$	$f[x_i, x_{i+1}, x_{i+2}, x_{i+3}]$
-2	3			
		-3		
-1	0		2	
		3		$-\frac{1}{2}$
1	6		0	
		3		
2	9			

Consequently

$$\begin{aligned} L_3(x) &= L_2(x) - \frac{1}{2}(x-(-1))(x-1)(x-2) \\ &= 3+3x-\frac{1}{2}(x^2-1)(x-2) \\ &= 3+3x-\frac{1}{2}(x^3-2x^2-x+2) \\ &= 3+3x-\frac{1}{2}x^3+x^2+\frac{1}{2}x-1 \\ &= 2+\frac{10}{2}x+x^2-\frac{1}{2}x^3. \end{aligned}$$

3. For a general set of points $\{x_0, \ldots, x_n\}$ belonging to $[a,b]$, we derived in class the error bound of Lagrange interpolation,

$$|f(x)-L_n(x)| \leq \frac{M_{n+1}}{(n+1)!} \max_{x\in[a,b]} |(x-x_0)\cdots(x-x_n)|,$$

where $M_i = \max_{x\in[a,b]} |f^{(i)}(x)|$. This exercise is to show an error bound for the case when the points are equally-spaced, i.e. $x_k = a+kh$ with $h = \frac{b-a}{n}$.

(a) (5 marks) For any $x \in [a,b)$, we can write $x = x_0 + (k+t)h$ for some unique $0 \leq t < 1$ and $k = 0, 1, \ldots, n-1$. Show that,

$$|(x-x_0)\cdots(x-x_n)| = h^{n+1}(k+t)(k-1+t)\cdots(1+t)t(1-t)(2-t)\cdots(n-k-t).$$

Solution: Replacing x by $x_0+(k+t)h$ and each x_j by x_0+jh, we get

$$\begin{aligned} & |(x-x_0)\cdots(x-x_n)| \\ =\ & |(x-x_0)(x-x_1)\ldots(x-x_{k-1})(x-x_k)(x-x_{k+1})(x-x_{k+2})\ldots(x-x_n)| \\ =\ & |((k+t)h)((k+t-1)h)\ldots((t+1)h)(th)((t-1)h)((t-2)h)\ldots((k+t-n)h)| \\ =\ & h^{n+1}|(k+t)(k+t-1)\ldots(t+1)t(t-1)(t-2)\ldots(k+t-n)| \\ =\ & h^{n+1}(k+t)(k+t-1)\ldots(t+1)t(1-t)(2-t)\ldots(n-k-t) \\ =\ & h^{n+1}(k+t)(k-1+t)\cdots(1+t)t(1-t)(2-t)\cdots(n-k-t). \end{aligned}$$

(b) (5 marks) Show that for $0 \leq t \leq 1$

i. $t(1-t) \leq \frac{1}{4}$

Solution: Let $g(t) = t(1-t)$. g has a maximum at $t = 1/2$. Hence

$$g(t) \leq g\left(\frac{1}{2}\right) = \frac{1}{4} \Longleftrightarrow t(1-t) \leq \frac{1}{4}.$$

ii. $(k+t)(k-1+t)\cdots(1+t) \leq (k+1)!$

Solution: For $0 \le t \le 1$, we have

$$(k+t)(k-1+t)\cdots(1+t) \le (k+1)k\cdots 2 = (k+1)!.$$

iii. $(2-t)\cdots(n-k-t) \le (n-k)!$.

Solution: For $0 \le t \le 1$, we have

$$(2-t)\cdots(n-k-t) \le 2\cdots(n-k) = (n-k)!.$$

(c) (5 marks) Show that $(k+1)!(n-k)! \le n!$ for any $k = 0, 1, \ldots, n-1$ and conclude that for equally-spaced points,

$$|f(x) - L_n(x)| \le M_{n+1} \frac{h^{n+1}}{4(n+1)}.$$

Solution: We have

$$\begin{aligned} n! &= n(n-1)\cdots(n-(k-1))(n-k)! \\ &\ge (k+1)k\cdots 2(n-k)! \\ &= (k+1)!(n-k)! \end{aligned}$$

since for $k = 0, 1, \ldots, n-1$, we always have $n \ge k+1$.
By (a) we have that

$$\begin{aligned} &\max_{x\in[a,b]} |(x-x_0)\cdots(x-x_n)| \\ &= h^{n+1} \max_{t\in[0,1]} (k+t)(k-1+t)\cdots(1+t)t(1-t)(2-t)\cdots(n-k-t) \end{aligned}$$

and by (b) we have

$$\max_{x\in[a,b]} |(x-x_0)\cdots(x-x_n)| \le h^{n+1}(k+1)!(n-k)!\frac{1}{4}$$

and by the inequality just proved above we get

$$\max_{x\in[a,b]} |(x-x_0)\cdots(x-x_n)| \le h^{n+1} n! \frac{1}{4}.$$

Thus

$$\begin{aligned} |f(x) - L_n(x)| &\le \frac{M_{n+1}}{(n+1)!} \max_{x\in[a,b]} |(x-x_0)\cdots(x-x_n)| \\ &\le \frac{M_{n+1}}{(n+1)!} h^{n+1} n! \frac{1}{4} \\ &= M_{n+1} \frac{h^{n+1}}{4(n+1)}. \end{aligned}$$

4. Consider the function $f(x) = x + x^4$. The goal of this question is to compute the Hermite interpolation polynomial using two different approaches.

(a) (5 marks) Set up an appropriate linear system and compute the Hermite interpolation polynomial of f with nodes $x_0 = 0$, $x_1 = 1$.

Solution: Let H denote the Hermite interpolation polynomial. Then H is a polynomial of degree at most $2n+1 = 3$ that satisfies

$$H(0) = f(0), \quad H(1) = f(1), \quad H'(0) = f'(0), \quad H'(1) = f'(1) \tag{1}$$

Since H is a polynomial of degree at most 3, we can write it as

$$H(x) = a + bx + cx^2 + dx^3,$$

where $a, b, c, d \in \mathbb{R}$ are unknowns. Since $H'(x) = b + 2cx + 3dx^2$, the conditions (1) lead to the linear system

$$\begin{cases} a = 0 \\ a+b+c+d = 2 \\ b = 1 \\ b + 2c + 3d = 5 \end{cases} \iff \begin{cases} a = 0 \\ b = 1 \\ c + d = 1 \\ 2c + 3d = 4 \end{cases} \iff \begin{cases} a = 0 \\ b = 1 \\ c = -1 \\ d = 2 \end{cases}$$

Hence $H(x) = x - x^2 + 2x^3$.

(b) (5 marks) Recall the definition of Hermite basis functions.

Definition. *Given the nodes $\{x_0, \ldots, x_n\}$, for $k = 0, \ldots, n$, the Hermite basis functions $h_k(x)$ and $\hat{h}_k(x)$ are polynomials of degree $2n+1$ which satisfy*

$$h_k(x_i) = \begin{cases} 0 & \text{if } i \neq k, \\ 1 & \text{if } i = k, \end{cases} \quad \hat{h}_k(x_i) = 0, \quad h'_k(x_i) = 0, \quad \hat{h}'_k(x_i) = \begin{cases} 0 & \text{if } i \neq k, \\ 1 & \text{if } i = k., \end{cases}$$

for $i = 0, \ldots, n$

Compute the Hermite basis functions associated with the nodes $x_0 = 0$, $x_1 = 1$.

Solution: We need to compute $h_0(x)$, $h_1(x)$, $\hat{h}_0(x)$, $\hat{h}_1(x)$.

- $h_0(x)$ is a polynomial of degree 3 such that

  $$h_0(0) = 1, \quad h_0(1) = 0, \quad h'_0(0) = 0, \quad h'_0(1) = 0.$$

  Letting $h_0(x) = a + bx + cx^2 + dx^3$, we obtain the linear system

  $$\begin{cases} a = 1 \\ a+b+c+d = 0 \\ b = 0 \\ b + 2c + 3d = 0 \end{cases} \iff \begin{cases} a = 1 \\ b = 0 \\ c + d = -1 \\ 2c + 3d = 0 \end{cases} \iff \begin{cases} a = 1 \\ b = 0 \\ c = -3 \\ d = 2 \end{cases}$$

  Hence $h_0(x) = 1 - 3x^2 + 2x^3$.

- $h_1(x)$ is a polynomial of degree 3 such that

  $$h_1(0) = 0, \quad h_1(1) = 1, \quad h'_1(0) = 0, \quad h'_1(1) = 0.$$

Letting $h_1(x) = a + bx + cx^2 + dx^3$, we obtain the linear system

$$\begin{cases} a = 0 \\ a + b + c + d = 1 \\ b = 0 \\ b + 2c + 3d = 0 \end{cases} \iff \begin{cases} a = 0 \\ b = 0 \\ c + d = 1 \\ 2c + 3d = 0 \end{cases} \iff \begin{cases} a = 0 \\ b = 0 \\ c = 3 \\ d = -2 \end{cases}$$

Hence $h_1(x) = 3x^2 - 2x^3$.

- $\hat{h}_0(x)$ is a polynomial of degree 3 such that

$$\hat{h}_0(0) = 0, \quad \hat{h}_0(1) = 0, \quad \hat{h}_0'(0) = 1, \quad \hat{h}_0'(1) = 0.$$

Letting $h_0(x) = a + bx + cx^2 + dx^3$, we obtain the linear system

$$\begin{cases} a = 0 \\ a + b + c + d = 0 \\ b = 1 \\ b + 2c + 3d = 0 \end{cases} \iff \begin{cases} a = 0 \\ b = 1 \\ c + d = -1 \\ 2c + 3d = -1 \end{cases} \iff \begin{cases} a = 0 \\ b = 1 \\ c = -2 \\ d = 1 \end{cases}$$

Hence $\hat{h}_0(x) = x - 2x^2 + x^3$.

- $\hat{h}_1(x)$ is a polynomial of degree 3 such that

$$\hat{h}_1(0) = 0, \quad \hat{h}_1(1) = 0, \quad \hat{h}_1'(0) = 0, \quad \hat{h}_1'(1) = 1.$$

Letting $h_0(x) = a + bx + cx^2 + dx^3$, we obtain the linear system

$$\begin{cases} a = 0 \\ a + b + c + d = 0 \\ b = 0 \\ b + 2c + 3d = 1 \end{cases} \iff \begin{cases} a = 0 \\ b = 0 \\ c + d = 0 \\ 2c + 3d = 1 \end{cases} \iff \begin{cases} a = 0 \\ b = 0 \\ c = -1 \\ d = 1 \end{cases}$$

Hence $\hat{h}_1(x) = -x^2 + x^3$.

(c) (5 marks) Show that for $k = 0, \ldots, n$, the Hermite basis functions can be expressed as

$$h_k(x) = (1 - 2(x - x_k)l_k'(x_k))\, l_k^2(x) \quad \text{and} \quad \hat{h}_k(x) = (x - x_k)l_k^2(x),$$

where l_k is the k-th Lagrange basis polynomial. *Hint: you simply have to check the definition given in (b).*

Solution: We first observe that $h_k(x)$ and $\hat{h}_k(x)$ are polynomials of degree $2n+1$ since $l_k(x)$ are polynomials of degree n. Then we only have to check the 4 conditions in the definition of Hermite basis functions.

$$h_k(x_i) = \begin{cases} 0 & \text{if } i \neq j, \\ 1 & \text{if } i = j, \end{cases} \quad \hat{h}_k(x_i) = 0, \quad h_k'(x_i) = 0, \quad \hat{h}_k'(x_i) = \begin{cases} 0 & \text{if } i \neq k, \\ 1 & \text{if } i = k., \end{cases}$$

- $h_k(x_i) = \begin{cases} 0 & \text{if } i \neq k, \\ 1 & \text{if } i = k. \end{cases}$

We have

$$h_k(x_i) = \begin{cases} (1 - 2(x_i - x_k)l_k'(x_k)) \times 0^2 = 0 & \text{if } i \neq k, \\ (1 - 2(x_k - x_k)l_k'(x_k)) \times 1^2 = 1 & \text{if } i = k. \end{cases}$$

- $h_k'(x_i) = 0$

  We have

  $$h_k'(x) = -2l_k'(x)l_k^2(x_k) + (1 - 2(x - x_k)l_k'(x_k))\, 2l_k'(x)l_k(x)$$

  So if $i \neq k$, $h_k'(x_i) = 0$ since $h_k'(x)$ is a multiple of $l_k(x)$. If $i = k$, then

  $$\begin{aligned} h_k'(x_k) &= -2l_k'(x_k)l_k^2(x_k) + (1 - 2(x_k - x_k)l_k'(x_k))\, 2l_k'(x_k)l_k(x_k) \\ &= -2l_k'(x_k) + 2l_k'(x_k) \\ &= 0, \end{aligned}$$

  since $l_k(x_k) = 1$.

- $\hat{h}_k(x_i) = 0$

  We have

  $$\hat{h}_k(x_i) = \begin{cases} (x_i - x_k) \times 0^2 = 0 & \text{if } i \neq k, \\ (x_k - x_k) \times 1^2 = 0 & \text{if } i = k. \end{cases}$$

- $\hat{h}_k'(x_i) = \begin{cases} 0 & \text{if } i \neq k, \\ 1 & \text{if } i = k. \end{cases}$

  We have

  $$h_k(x_i) = \begin{cases} (1 - 2(x_i - x_k)l_k'(x_k)) \times 0^2 = 0 & \text{if } i \neq k, \\ (1 - 2(x_k - x_k)l_k'(x_k)) \times 1^2 = 1 & \text{if } i = k. \end{cases}$$

(d) (5 marks) Use the formulas in (c) to compute the Hermite basis functions associated with the nodes $x_0 = 0$, $x_1 = 1$. Compare your answer with (b).

Solution: We first compute the Lagrange basis polynomials. We have

$$l_0(x) = \frac{x - x_1}{x_0 - x_1} = 1 - x, \quad l_0(x) = \frac{x - x_0}{x_1 - x_0} = x.$$

Moreover $l_0'(x) = -1$ and $l' - 1(x) = 1$. Then

$$\begin{aligned}
h_0(x) &= (1 - 2(x - x_0)l_0'(x_0))\, l_0^2(x) \\
&= (1 + 2x)(1 - x)^2 \\
&= (1 + 2x)(1 - 2x + x^2) \\
&= 1 - 2x + x^2 + 2x - 4x^2 + 2x^3 \\
&= 1 - 3x^2 + 2x^3 \\
h_1(x) &= (1 - 2(x - x_1)l_1'(x_1))\, l_1^2(x) \\
&= (1 - 2(x - 1))x^2 \\
&= 3x^2 - 2x^3 \\
\hat{h}_0(x) &= (x - x_0)l_0^2(x) \\
&= x(1 - x)^2 \\
&= x(1 - 2x + x^2) \\
&= x - 2x^2 + x^3 \\
\hat{h}_1(x) &= (x - x_1)l_1^2(x) \\
&= (x - 1)x^2 \\
&= -x^2 + x^3.
\end{aligned}$$

(e) (5 marks) Compute the Hermite interpolation polynomial of f with nodes $x_0 = 0$, $x_1 = 1$ using the Hermite basis functions computed in (b) and (d).

Solution: The Hermite polynomial can be expressed as a linear combination of $h_k(x)$, $\hat{h}_k(x)$ as

$$H_{2n+1}(x) = \sum_{k=0}^{n} f(x_k)h_k(x) + \sum_{k=0}^{n} f'(x_k)\hat{h}_k(x)$$

In our case this leads to

$$H_3(x) = 0(1-3x^2+2x^3)+2(3x^2-2x^3)+1(x-2x^2+x^3)+5(-x^2+x^3) = x-x^2+2x^3.$$

5. (5 marks) Determine the natural cubic spline S that interpolates the data $f(0) = 3$, $f(1) = 2$ and $f(2) = 9$.

Solution: We are given three points $x_0 = 0$, $x_1 = 1$ and $x_2 = 2$ and so $n = 2$. Therefore there will be two pieces S_0 and S_1 to the spline. The goal is then to determine $a_0, b_0, c_0, d_0, a_1, b_1, c_1$ and d_1 where

$$S_0(x) = a_0 + b_0(x - x_0) + c_0(x - x_0)^2 + d_0(x - x_0)^3$$

and

$$S_1(x) = a_1 + b_1(x - x_1) + c_1(x - x_1)^2 + d_1(x - x_1)^3.$$

This is done by imposing the eight conditions from the definition of free cubic spline.

- Three interpolation conditions: $S_0(x_0) = f(x_0)$, $S_1(x_1) = f(x_1)$ and $S_1(x_2) = f(x_2)$.

This leads to

$$\begin{cases} a_0 = f(x_0) = 3 \\ a_1 = f(x_1) = 2 \\ 9 = S_1(x_2) = a_1 + b_1(x_2 - x_1) + c_1(x_2 - x_1)^2 + d_1(x_2 - x_1)^3 = 2 + b_1 + c_1 + d_1 \end{cases}$$

- One continuity condition: $S_0(x_1) = S_1(x_1)$.

Thus

$$2 = S_1(x_1) = S_0(x_1) = a_0 + b_0(x_1 - x_0) + c_1(x_1 - x_0)^2 + d_1(x_1 - x_0)^3 = 3 + b_0 + c_0 + d_0.$$

- Two smoothness conditions: $S_0'(x_1) = S_1'(x_1)$ and $S_0''(x_1) = S_1''(x_1)$

We then have

$$\begin{cases} b_0 + 2c_0(x_1 - x_0) + 3d_0(x_1 - x_0)^2 = b_1 + 2c_1(x_1 - x_1) + 3d_1(x_1 - x_1)^2 \\ 2c_0 + 6d_0(x_1 - x_0) = 2c_1 + 6d_1(x_1 - x_1) \end{cases}$$

By plugging in all the values we know, we get

$$\begin{cases} b_0 + 2c_0 + 3d_0 = b_1 \\ 2c_0 + 6d_0 = 2c_1 \end{cases}$$

- Free boundary conditions: $S''(x_0) = 0$ and $S''(x_2) = 0$.

We then have

$$\begin{cases} S''(x_0) = 2c_0 = 0 \\ S''(x_2) = 2c_1 + 6d_1(x_1 - x_0) = 2c_1 + 6d_1 = 0 \end{cases}$$

Let us summarize all the equations.

$$\begin{cases} a_0 = 3 \\ a_1 = 2 \\ 2 + b_1 + c_1 + d_1 = 9 \\ 3 + b_0 + c_0 + d_0 = 2 \\ b_0 + 2c_0 + 3d_0 = b_1 \\ 2c_0 + 6d_0 = 2c_1 \\ 2c_0 = 0 \\ 2c_1 + 6d_1 = 0 \end{cases} \iff \begin{cases} a_0 = 0 \\ a_1 = 1 \\ b_0 + d_0 = -1 \\ b_1 + c_1 + d_1 = 7 \\ b_0 + 3d_0 = b_1 \\ 6d_0 = 2c_1 \\ c_0 = 0 \\ 2c_1 + 6d_1 = 0 \end{cases}$$

We have

$$\begin{cases} b_0 + d_0 = -1 \\ b_1 + c_1 + d_1 = 7 \\ b_0 + 3d_0 = b_1 \\ 6d_0 = 2c_1 \\ 2c_1 + 6d_1 = 0 \end{cases} \iff \begin{cases} b_0 - d_1 = -1 \\ b_1 - 2d_1 = 7 \\ b_0 - 3d_1 = b_1 \\ d_0 + d_1 = 0 \\ c_1 = -3d_1 \end{cases} \iff \begin{cases} b_0 = -1 + d_1 \\ b_1 = 7 + 2d_1 \\ -1 + d_1 - 3d_1 = 7 + 2d_1 \\ d_0 + d_1 = 0 \\ c_1 = -3d_1 \end{cases}$$

In the first step we simplified the last two equations and then use them to simplify the first three. We now have

$$\begin{cases} b_0 = -1 + d_1 \\ b_1 = 7 + 2d_1 \\ d_1 = -2 \\ d_0 + d_1 = 0 \\ c_1 = -3d_1 \end{cases} \iff \begin{cases} b_0 = -3 \\ b_1 = 3 \\ d_0 = 2 \\ d_1 = -2 \\ c_1 = 6 \end{cases}$$

The cubic spline is then given by

$$S(x) = \begin{cases} S_0(x) = 3 - 3x + 2x^3 & \text{if } x \in [0, 1) \\ S_1(x) = 2 + 3(x-1) + 6(x-1)^2 - 2(x-1)^3 & \text{if } x \in [1, 2] \end{cases}$$

6. (15 marks) Find a, b, c so that the finite difference of the first derivative

$$D_h f(x_0) = af(x_0) + bf(x_0 + h) + cf(x_0 + 2h)$$

has the highest degree of accuracy possible. State this degree.

Solution: The idea is for the formula to be exact to the highest degree polynomial possible. Hence, taking $f(x) = 1$ we get

$$D_h f(x_0) = Df(x_0) \iff a + b + c = 0.$$

With $f(x) = x$ we obtain

$$\begin{aligned} D_h f(x_0) = Df(x_0) &\iff a\,x_0 + b(x_0 + h) + c(x_0 + 2h) = 1 \\ &\iff x_0(a + b + c) + h(b + 2c) = 1 \\ &\implies b + 2c = \frac{1}{h}, \end{aligned}$$

where we used the first equation. Finally, for $f(x) = x^2$, we get

$$\begin{aligned} D_h f(x_0) = Df(x_0) &\iff a\,x_0 + b(x_0 + h)^2 + c(x_0 + 2h)^2 = 2x_0 \\ &\iff (a + b + c)x_0^2 + 2x_0 h(b + 2c) + h^2(b + 4c) = 2x_0 \\ &\implies b + 4c = 0, \end{aligned}$$

where we used the second equation. We thus have

$$\begin{cases} a + b + c = 0 \\ b + 2c = \frac{1}{h} \\ b + 4c = 0. \end{cases} \iff \begin{cases} a = -\frac{3}{2h} \\ b = \frac{4}{2h} \\ c = -\frac{1}{2h} \end{cases}$$

Thus the finite difference formula is given by

$$D_h f(x_0) = \frac{-3f(x_0) + 4f(x_0 + h) - f(x_0 + 2h)}{2h}.$$

Now, we compute the degree of accuracy, which must be at least two given the conditions imposed above. We take $f(x) = x^3$ and get

$$Dhf(x_0) = \frac{-3x_0^3 + 4(x_0 + h)^2 - (x_0 + 2h)^3}{2h} = -2h^2 + 3x_0^2$$

and

$$Df(x_0) = 3x_0^2.$$

Therefore the degree of accuracy is 2.