Math 318: Assignment 4 solutions

Problem 1

Part (1)

The atoms are the following:

$$\begin{array}{rr} p \wedge q \wedge r & p \wedge q \wedge \neg r \\ \neg p \wedge q \wedge r & \neg p \wedge q \wedge \neg r \\ p \wedge \neg q \wedge r & p \wedge \neg q \wedge \neg r \\ \neg p \wedge \neg q \wedge r & \neg p \wedge \neg q \wedge \neg r \end{array}$$

Part (2)

Since there are 8 atoms, there are 2^8 elements in B.

Problem 2

Part (1)

Yes, take $B = \mathcal{P}(\mathbb{N})$ with the usual Boolean algebra structure. The atoms of B are the singleton sets, and every nonempty set in B contains a singleton, so for every nonzero $b \in B$, there is an atom $a \in B$ with $a \leq b$. $\square$

Part (2)

Yes, take $B = C \times D$ where C is a two-element Boolean algebra (such as $\mathcal{P}(\{\varnothing\})$) and D is any atomless Boolean algebra (such as the Lindenbaum-Tarski algebra on an infinite set of variables). B can be given a Boolean algebra structure by defining all operations pairwise. Note that D is atomless so it is infinite, and thus B is infinite. The atoms of $C \times D$ are of the form $(c, 0)$ and $(0, d)$ where c is an atom of C and d is an atom of D. Since C has one atom and d has no atoms, B has exactly one atom $(1, 0)$ corresponding to the one atom of C. $\square$

Problem 3

Part (1)

We claim that $\mid$ is transitive on $\mathbb{N}$. It is reflexive since $n = 1 \cdot n$ so $n \mid n$. Now suppose $n \mid m$ and $m \mid n$. Then $m = kn$ and $n = lm$ for some k, l so $m = kn = klm$. If $m = 0$, then $n = l \cdot 0 = 0$ so $n = m$. Otherwise, $kl = 1$ so $k = l = 1$ and thus $n = 1 \cdot m = m$, so it is antisymmetric. Finally, if $n \mid m$ and $m \mid p$, then $m = kn$ and $p = lm$ for some k, l so $p = lm = lkn$. Thus $n \mid p$ and it is transitive. Thus it is a partial order. $\square$

Part (2)

Note that for any n, we have $n = n \cdot 1$, so $1 \mid n$. Thus 1 is a least element. $\square$

Part (3)

Note that for any n, we have $0 = 0 \cdot n$, so $n \mid 0$. Thus 0 is a greatest element. $\square$

Problem 4

Note that P itself is not a chain since $2 \nmid 3$ and $3 \nmid 2$. Thus we need at least two chains to cover P. If we let $C_1 = \{1, 3\}$ and $C_2 = \{2, 4\}$, then these are both chains and $P = C_1 \cup C_2$. Thus the minimal number of chains needed to cover P is 2. $\square$

Problem 5

The following is one of many solutions which work:

$$\varnothing < \{1\} < \{2\} < \{3\} < \{1, 2\} < \{1, 3\} < \{2, 3\} < \{1, 2, 3\}$$

Problem 6

Suppose $B \in \mathcal{P}(\mathbb{N})$ has the property that $B \subset \{1, 2\}$ and $B \subset \{1, 3\}$. Then $B \subset \{1, 2\} \cap \{1, 3\}$ so $B \subset \{1\}$, so $B \notin P$. Thus since $\{1, 2\}$ and $\{1, 3\}$ are in P and they don't have a lower bound in P, P does not have a lower bound. $\square$

Problem 7

We claim that the least fixed point is $A = \{(01)^n : n \geq 0\}$. First of all, note that A is a fixed point since we have:

$$f(A) = A \cup \{w01 : w \in A\} \cup \{\epsilon\} = A \cup \{(01)^n : n \geq 1\} \cup \{\epsilon\} = A$$

Now let B be a fixed point of f. We show that for $(01)^n \in B$ by induction on $n \geq 0$. For $n = 0$, we have $\epsilon \in f(B) = B$. Now if $(01)^n \in B$, then we have $(01)^{n+1} = 01(01)^n \in f(B) = B$. Thus our claim holds, and thus $A \subset B$. Thus A is a least fixed point of f. $\square$

Problem 8

Consider the map $T : \mathcal{P}(A) \to \mathcal{P}(A)$ defined by $T(E) = g(f(E)^c)^c$. Note that this is monotone since for $E, F \subset A$, we have:

$$E \subset F \implies f(E) \subset f(F) \implies f(E)^c \supset f(F)^c \implies g(f(E)^c) \supset g(f(F)^c) \implies g(f(E)^c)^c \subset g(f(F)^c)^c$$

Thus by the Knaster-Tarski fixed point theorem, T has a fixed point $A_1 \subset \mathcal{P}(A)$. Let $A_2 = A_1^c$, $B_1 = f(A_1)$ and $B_2 = B_1^c$. Then $A_1 \sqcup A_2 = A$ and $B_1 \sqcup B_2 = B$, so it suffices to show that $g(B_2) = A_2$. Well since A_1 is a fixed point of T, we have $A_1 = g(f(A_1)^c)^c = g(B_1^c)^c = g(B_2)^c$, so we have $g(B_2) = A_1^c = A_2$. This concludes the proof.
Now we turn to the proof of the Cantor-Schröder-Bernstein theorem. Let $f : A \to B$ and $g : B \to A$ be injections and let A_1, A_2, B_1, B_2 be as in the statement of the problem. Since f is an injection, it is a bijection onto its image, so since $f(A_1) = B_1$, we have $A_1 \sim B_1$. Similarly $A_2 \sim B_2$. Thus $A = A_1 \sqcup A_2 \sim B_1 \sqcup B_2 = B$ and this proves the CSB theorem. $\square$