Practice Problems Exam

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1. A natural cubic spline for a function f is defined on $[-1, 1]$ by

$$S(x) = \begin{cases} S_0(x) = -4 + x + x^3, & x \in [0, 1), \\ S_1(x) = -2 + 4(x-1) + 3(x-1)^2 - (x-1)^3, & x \in [1, 2]. \end{cases}$$

Compute the Lagrange interpolation polynomial of f on the nodes $x_0 = 0$, $x_1 = 1$, $x_2 = 2$.

2. (Final Exam Math 317 Fall 2010) Say we have an analytical function $f(x)$ and its numerical approximation $\hat{f}(x)$ such that $\hat{f}(x) = f(x) + e(x)$ where $e(x)$ is then the round-off error and both the function and the error are sufficiently smooth for our purposes. Now we approximate the derivative with a finite difference scheme. Let's use

$$\hat{D}(x) = \frac{\hat{f}(x+h) - \hat{f}(x-h)}{2h}.$$

Show that the final error is the sum of round-off and truncation errors and that one becomes large at small h, while the other becomes large at large h. *Hint: consider the total error* $f'(x) - \hat{D}(x)$ *and recall that*

$$f'(x) = \frac{f(x+h) - f(x-h)}{2h} - \frac{f^{(3)}(\xi)}{6}h^2.$$

3. (Final Exam Math 317 Fall 2004)

   (a) Use the Taylor expansion

   $$f(x^*) = f(x) + (x^* - x)f'(x) + \frac{(x^* - x)^2}{2}f''(\xi)$$

   for $f \in C^2[a, b]$ to derive Newton's method for approximating a root x^* of the equation $f(x) = 0$.

   (b) Show that Newton's method can be written as a fixed point iteration $x_{n+1} = g(x_n)$ for a suitable choice of $g(x)$.

   (c) Show that $g'(x^*) = 0$ provided that $f'(x^*) \neq 0$. What can we say about the fixed point iteration in such a case?

   (d) Suppose $f'(x^*) = 0$ and $f''(x^*) \neq 0$. Show that we can write $f(x) = (x - x^*)^2 r(x)$ for some function $r(x)$ with $r(x^*) \neq 0$.

   (e) Find $\lim_{x \to x^*} g'(x)$ for Newton's method when $f'(x^*) = 0$ but $f''(x^*) \neq 0$. What can we say about the fixed point iteration in such case?

   (f) The root $x^* = 5$ of $f(x) = x^3 - 9x^2 + 15x + 25$ is approximated using Newton's method method with $x_0 = 3$. What is the order of convergence?

4. Consider the function $f(x) = x^2 - 4x + 3$ and the fixed point iterations $x_{n+1} = g(x_n)$ given by

$$i.\ g(x) = \sqrt{4x - 3}, \qquad ii.\ g(x) = \frac{x^2 + 3}{4}.$$

Analyzing the local convergence of the fixed point iteration, explain which method you would use to approximate each root of f.

5. (Final Exam Math 317 Fall 2010) Suppose we have a computer that can't divide. One way to get around this is to create a routine that finds roots of $f(x) = \frac{1}{x} - a$ where $a \neq 0$. We then input a and get its inverse. (You (human) can use division in formulating the algorithm, but the algorithm itself must not require the computer to divide). Show how you would use Newton's method to do this. Show if/when it converges. What is the order of convergence?.

6. (Final Exam Math 317 2005)

   (a) What is the key difference between Lagrange and Hermite interpolation? What is the difference between a clamped and a natural cubic spline?

   (b) A natural cubic spline S on $[0, 2]$ has the formula

$$S(x) = \begin{cases} S_0(x) = 1 + 2x - x^3 & 0 \leq x < 1 \\ S_1(x) = a + b(x-1) + c(x-1)^2 + d(x-1)^3 & 1 \leq x \leq 2. \end{cases}$$

   Find a, b, c and d.

7. (Final Exam Math 317 Fall 2010) How would you go about calculating quadratic splines)? In other words, let's assume you want to form a piecewise polynomial interpolation using polynomials of degree two. Show how to calculate these for data points x_i and f_i with $i = 0, \ldots, n$.

8. Let f be defined on $[a, b]$ and let the nodes $a = x_0 < x_1 < x_2 = b$ be given. A quadratic spline interpolation function S is given by

$$S(x) = \begin{cases} S_0(x) = a_0 + b_0(x - x_0) + c_0(x - x_0)^2 & x \in [x_0, x_1] \\ S_1(x) = a_1 + b_1(x - x_1) + c_1(x - x_1)^2 & x \in [x_1, x_2] \end{cases}$$

   such that

   i) $S(x_i) = f(x_i)$ for $i = 0, 1, 2$

   ii) $S \in C^1[x_0, x_2]$

   Show that conditions $i)$ and $ii)$ lead to five equations in the six unknowns a_0, b_0, c_0, a_1, b_1 and c_1. Determine those equations. The problem is to decide what additional condition to impose to make the solution unique. Does the condition $S \in C^2[x_0, x_2]$ lead to a meaningful solution?

9. Determine a quadratic spline Q that interpolates the data $f(0) = 0$, $f(1) = 1$, $f(2) = 2$ and satisfies $Q'(0) = 2$.

10. (a) Define the degree of accuracy (also known as the degree of precision) of a quadrature formula $I_h(f)$ for approximating the integral

$$I(f) = \int_a^b f(x)\, dx.$$

   (b) Find the degree of accuracy p of the quadrature formula

$$I_h(f) = \frac{3}{2} h \left[f(x_1) + f(x_2) \right]$$

   where $a = x_0$, $b = x_3$ and $h = x_{i+1} - x_i$.

   (c) (Final Exam Math 317 Fall 2005) Find constants α, β and γ such that the degree of accuracy of the quadrature formula

$$I_h(f) = h \left[\alpha f(a) + \beta f(a + \gamma h) \right]$$

   is as large as possible, where $h = b - a$.

(d) Determine constants a, b, c, d and e that will produce a quadrature formula

$$\int_{-1}^{1} f(x)\,dx \approx af(-1) + bf(0) + cf(1) + df'(-1) + ef'(1)$$

that has degree of precision at least 4.

11. (Final Exam Math 317 Fall 2004) Consider the initial value problem

$$\begin{cases} y'(t) = f(t, y(t)) = \lambda y(t), & 0 \le t \le T, \\ y(0) = y_0 > 0, \end{cases}$$

where $\lambda < 0$. Suppose you approximate the solution $y(\cdot)$ using the Runge-Kutta method

$$y_{n+1} = y_n + \frac{1}{4} h f(t_n, y_n) + \frac{3}{4} h f\left(t_n + \frac{2}{3}h, y_n + \frac{2}{3} h f(t_n, y_n)\right).$$

(a) Show that $y(t_{n+1}) = e^{h\lambda} y(t_n)$.

(b) Show that $y_{n+1} = \left(1 + h\lambda + \frac{(h\lambda)^2}{2}\right) y_n$.

(c) Under what conditions on h does $\lim_{n\to\infty} y_n = 0$?

(d) Define local truncation error $\tau_h(t_n)$ and show that for this problem

$$\tau_h(t_n) = \frac{h^3 \lambda^3}{6} y(\xi),$$

where $\xi \in (t_n, t_{n+1})$.

12. Show that the Newton-Cotes quadrature with $n+1$ points has degree of accuracy at least n.

13. The k-step Adam-Bashforth method is given by

$$y_{n+1} = y_n + \int_{t_n}^{t_{n+1}} p(t)\,dt$$

where $p(t)$ is the polynomial of degree $k-1$ such that $p(t_{n-i}) = f_{n-i}$, for $i = 0, \ldots, k-1$.

(a) Derive the 2-step Adam-Bashforth method.

(b) Using the Taylor expansion, show that the local truncation error of the 2-step Adam-Bashforth method is $\mathcal{O}(h^3)$

14. The k-step Adam-Moulton method is given by

$$y_{n+1} = y_n + \int_{t_n}^{t_{n+1}} p(t)\,dt$$

where $p(t)$ is the polynomial of degree k such that $p(t_{n-i}) = f_{n-i}$, for $i = -1, \ldots, k-1$.

(a) Derive the 2-step Adam-Moulton method.

(b) Using the Taylor expansion, show that the local truncation error of the 2-step Adam-Moulton method is $\mathcal{O}(h^4)$.

15. The improved Euler's method is given by

$$\begin{aligned} \tilde{y}_{n+1} &= y_n + h f(t_n, y_n) \\ y_{n+1} &= y_n + \frac{h}{2}\left(f(t_n, y_n) + f(t_{n+1}, \tilde{y}_{n+1})\right). \end{aligned}$$

Show that the method has order 2. *Hint: Show first that*

$$f(t+h, y+ch) = f(t, y) + f_t(t, y) + c f_y(t, y) + \mathcal{O}(h^2).$$

by Taylor expanding $g(x) = f(t+x, y+cx)$ *at* $x = h$ *about* $x_0 = 0$.

16. Consider the boundary value problem on $[0, 1]$,

$$\begin{cases} -u''(x) = f(x), \\ u(0) = 0 \\ u(1) = 0 \end{cases}$$

(a) Using the approximation $u''(x_k) = \frac{u_{k+1} - 2u_k + u_{k-1}}{h^2} + \mathcal{O}(h^2)$ using finite differences, write down the linear system of the form $A_h \vec{u_h} = \vec{f_h}$ using equally-spaced points $x_k = kh$ for $k = 0, \ldots, N$ with $h = \frac{1}{N}$.

(b) For the discretization of part (a), show that the consistency error measured in the l_2 norm is $\mathcal{O}(h^p)$ for some $p > 0$. Find the exponent p in this case.

(c) Show that the discretization of part (a) is stable. *Hint: Recall that for constants* $a \in \mathbb{R}$, $b < 0$, *an* $n \times n$ *matrix of the form*

$$A = \begin{pmatrix} a & b & & & \\ b & a & b & & \\ & \ddots & \ddots & \ddots & \\ & & b & a & b \\ & & & b & a \end{pmatrix}$$

has eigenvalues $\lambda_k = a + 2b\cos(\frac{k\pi}{n+1})$ *for* $k = 1, \ldots, n$. *Also note that for* $\frac{\sin(x)}{x} \geq \frac{2}{\pi}$ *for* $0 < x < \frac{\pi}{2}$.

(d) Conclude that $\vec{u_h}$ from part (a) converges to the exact solution $\vec{u}$ as $h \to 0$.

17. Consider the B.V.P on $[a, b]$ given by

$$\begin{cases} u''(t) = f(x, u, u') \\ u(a) = \alpha \\ u(b) = \beta \end{cases} \tag{6}$$

and the I.V.P.

$$\begin{cases} y''(t) = f(x, y, y') \\ y(a) = \alpha \\ y'(a) = s \end{cases} \tag{7}$$

where s is an (unknown) parameter. For a fixed $s \in \mathbb{R}$, denote by $y(x; s)$ the exact solution of (7). Denote by $\left\{y_0^{(k)}, \ldots, y_N^{(k)}\right\}$ the numerical solution of (7) with $s = s_k$.

Algorithm Shooting method with secant method

Pick two guesses s_0, s_1 and N, *tol*.
Compute $\left\{y_0^{(0)}, \ldots, y_N^{(0)}\right\}$ and $\left\{y_0^{(1)}, \ldots, y_N^{(1)}\right\}$.
Set $k = 1$.
while $\left|y_N^{(k)} - \beta\right| < tol$ **do**
  Set $s_{k+1} = s_k - \frac{y_N^{(k)} - \beta}{y_N^{(k)} - y_N^{(k-1)}}(s_k - s_{k-1})$.
  Compute $\left\{y_0^{(k)}, \ldots, y_N^{(k)}\right\}$.
  Set $k = k + 1$.
end while
Output $\left\{y_0^{(k)}, \ldots, y_N^{(k)}\right\}$.

(a) Consider the function $z(x; s) := \frac{\partial y(x;s)}{\partial s}$. Determine the augmented second order I.V.P for y and z. *Hint: What ODE does z'' satisfy? Do not forget to include the initial conditions.*

(b) Write the algorithm for the shooting method with Newton's method. *Modify the above algorithm as needed.*

(a) Compute the first iteration of the Gauss-Seidel method for the linear system

$$\begin{aligned} 4x_1 - 2x_2 \qquad\quad &= 2 \\ -2x_1 + 4x_2 - 2x_3 &= 2 \\ -2x_2 + 4x_3 &= -1 \end{aligned} \qquad \text{with } \boldsymbol{x}_0 = \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}.$$

(b) Does the Gauss-Seidel method converge in this case? Justify your answer brifly.

18. Recall that the S.O.R. method can be written as

$$(D + \theta L)x^{(k+1)} = -\left(\theta U + (\theta - 1)D\right) x^{(k)} + \theta b.$$

Show that $0 < \theta < 2$ is a necessary condition for convergence. *Hint: show first that for any matrix B, if $|\det(B)| \geq 1$ then $\rho(B) \geq 1$.*

19. Let $A \in \mathbb{R}^{n \times n}$ be such that $A = (1 + \omega)M - (N + \omega M)$, with $M^{-1}N$ nonsingular and with real eigenvalues $1 > \lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_n$ and $\omega \neq -1$. Find the values of $\omega \in \mathbb{R}$ for which the following iterative method

$$(1 + \omega)Mx^{(k+1)} = (N + \omega M)x^{(k)} + b, \quad k \geq 0,$$

converges for any $x^{(0)}$ to the solution of the linear system $Ax = b$. Determine also the value of ω for which the convergence rate is optimal.