McGill University
Faculty of Science

Midterm Examination

Math 317 – Numerical Analysis

Examinar: Tiago Salvador

Date: October 27th, 2016

Time: 8:35 AM - 9:55 AM

Student name (last, first)	**Student number (McGill ID)**

This exam contains a total of 7 **pages** (including this cover page) and **5 questions**.

INSTRUCTIONS

- Print your full name and student number clearly on each page
- Answer all 5 questions direcly on the exam; show your work
- If you need more space, use the back of the pages
- This a is a **closed** book exam
- Calculators, notes, formula sheets are **not** permitted.

Question	Points	Score
1	10	
2	10	
3	10	
4	10	
5	10	
Total:	50	

1. (10 marks) Compute the Lagrange interpolation polynomial for $\{(0,1),(1,2),(2,3),(3,-2)\}$ and express the polynomial in the simplest form.

   We compute the divided differences using the table.

x_i	$f[x_i]$	$f[x_i, x_{i+1}]$	$f[x_i, x_{i+1}, x_{i+2}]$	$f[x_i, x_{i+1}, x_{i+2}, x_{i+3}]$
0	1			
		1		
1	2		0	
		1		-1
2	3		-3	
		-5		
3	-2			

   Taking the upper diagonal entries, Newton's divided formula simplifies to

$$L_2(x) = 1 + x - x(x-1)(x-2) = 1 - x + 3x^2 - x^3.$$

2. (10 marks) A clampled cubic spline S for a function f is defined on $[-1, 1]$ by

$$S(x) = \begin{cases} S_0(x) = 1 - 6(x+1) + 3(x+1)^2 - 2(x+1)^3 & \text{if } x \in [-1, 0), \\ S_1(x) = a + bx + cx^2 + dx^3 & \text{if } x \in [0, 1]. \end{cases}$$

Given $f'(-1) = f'(1)$, find the constants a, b, c and d.

By construction, the spline is twice continuous differentiable at $x = 0$. This means that

i) $S_0(0) = S_1(0)$

ii) $S_0'(0) = S_1'(0)$

iii) $S_0''(0) = S_1''(0)$

Condition i) leads to

$$S_0(0) = S_1(0) \iff 1 - 6 + 3 - 2 = a \iff a = -4.$$

We have that

$$S'(x) = \begin{cases} S_0'(x) = -6 + 6(x+1) - 6(x+1)^2 & \text{if } x \in [-1, 0) \\ S_1'(x) = b + 2cx + 3dx^2 & \text{if } x \in [0, 1] \end{cases}$$

Hence ii) leads to

$$b = -6 + 6 - 6 = -6.$$

We also have

$$S''(x) = \begin{cases} S_0''(x) = 6 - 12(x+1) & \text{if } x \in [-1, 0) \\ S_1''(x) = 2c + 6dx & \text{if } x \in [0, 1] \end{cases}$$

Thus iii) leads to

$$6 - 12 = 2c \iff c = -3.$$

Finally, since it is a clamped cubic spline, we have $S_0'(-1) = f'(-1)$ and $S_1'(1) = f'(1)$. We then have

$$f'(-1) = S_0'(-1) = -6$$

and

$$S_1'(1) = b + 2c + 3d = -12 + 3d.$$

By assumption, $f'(1) = f'(3)$ and therefore

$$d = 2.$$

We then have $S(x) = -4 + 6x - 3x^2 - 3x^3$.

3. (a) (5 marks) Find the constants a, b, c such that the finite difference of the first derivative

$$D_h f(x_0) := af(x_0 - h) + bf(x_0) + cf(x_0 + h)$$

has the highest degree of accuracy possible.

The idea is for the formula to be exact to the highest degree polynomial possible. Hence, taking $f(x) = 1$ we get

$$D_h f(x_0) = Df(x_0) \iff a + b + c = 0.$$

With $f(x) = x$ we obtain

$$\begin{aligned} D_h f(x_0) = Df(x_0) &\iff a\,(x_0 - h) + bx_0 + c(x_0 + h) = 1 \\ &\iff x_0(a + b + c) + h(-a + c) = 1 \\ &\implies -a + c = \frac{1}{h}, \end{aligned}$$

where we used the first equation. Finally, for $f(x) = x^2$, we get

$$\begin{aligned} D_h f(x_0) = Df(x_0) &\iff a\,(x_0 - h)^2 + bx_0^2 + c(x_0 + h)^2 = 2x_0 \\ &\iff (a + b + c)x_0^2 + 2x_0 h(-a + c) + h^2(a + c) = 2x_0 \\ &\implies a + c = 0, \end{aligned}$$

where we used the second equation. We thus have

$$\begin{cases} a + b + c = 0 \\ -a + c = \frac{1}{h} \\ a + c = 0. \end{cases} \iff \begin{cases} a = -\frac{1}{2h} \\ b = 0 \\ c = \frac{1}{2h} \end{cases}$$

Thus the finite difference formula is given by

$$D_h f(x_0) = \frac{f(x_0 + h) - f(x_0 - h)}{2h}.$$

(b) (5 marks) Assume $f \in C^3(\mathbb{R})$. Using the Taylor expansion, find a formula for the error of $D_h f$ and express it in the simplest form.

We start by expanding f in a second order Taylor polynomial about a point x_0 and evaluate at $x_0 + h$ and $x_0 - h$. We get

$$f(x_0 + h) = f(x_0) + hf'(x_0) + \frac{h^2}{2} f''(x_0) + \frac{h^3}{6} f^{(3)}(\xi_+)$$

and

$$f(x_0 - h) = f(x_0) - hf'(x_0) + \frac{h^2}{2} f''(x_0) - \frac{h^3}{6} f^{(3)}(\xi_-)$$

where $x_0 - h < \xi_- < x_0 < \xi_+ < x_0 + h$. Plugging both equation into $D_h f(x_0)$ gives

$$D_h f(x_0) = f'(x_0) + \frac{h^2}{12} \left(f^{(3)}(\xi_+) + f^{(3)}(\xi_-) \right).$$

By the Intermediate Value Theorem there is ξ between ξ_+ and ξ_-, and hence in $(x_0 - h, x_0 + h)$ such that

$$f^{(3)}(\xi) = \frac{\left(f^{(3)}(\xi_+) + f^{(3)}(\xi_-)\right)}{2}$$

since $\frac{\left(f^{(3)}(\xi_1 +) + f^{(3)}(\xi_-)\right)}{2}$ lies between $f^{(3)}(\xi_+)$ and $f^{(3)}(\xi_-)$. This allows us to write:

$$D_h f(x_0) = f'(x_0) + \frac{h^2}{6} f^{(3)}(\xi).$$

where $\xi \in (x_0 - h, x_0 + h)$. The error is then given by

$$error = f'(x_0) - D_h f(x_0) = -\frac{h^2}{6} f^{(3)}(\xi).$$

4. Consider the weighted integral $I(f) := \int_{-1}^{1} f(x)xdx$.

   (a) (7 marks) Find the constants a, b, c such that the 3-point quadrature

   $$I_h(f) := af(-1) + bf(0) + cf(1)$$

   has the highest degree of accuracy with respect to $I(f)$.
   The idea is for the formula to be exact for the highest degree of polynomial possible. We start by taking $f(x) = 1$ which leads to

   $$I_h(f) = I(f) \Longleftrightarrow a + b + c = \int_{-1}^{1} xdx \Longleftrightarrow a + b + c = 0.$$

   With $f(x) = x$, we obtain

   $$I_h(f) = I(f) \Longleftrightarrow -a + c = \int_{-1}^{1} x^2dx \Longleftrightarrow -a + c = \frac{2}{3}.$$

   Finally, with $f(x) =^2$, we get

   $$I_h(f) = I(f) \Longleftrightarrow a + c = \int_{-1}^{1} x^2dx \Longleftrightarrow a + c = 0.$$

   We have a linear system to solve

   $$\begin{cases} a + b + c = 0 \\ -a + c = \frac{2}{3} \\ a + c = 0 \end{cases} \Longleftrightarrow \begin{cases} a = -\frac{1}{3} \\ b = 0 \\ c = \frac{1}{3} \end{cases}$$

   (b) (3 marks) Find the degree of accuracy of $I_h(f)$ from part (a).
   To find the degree of accuracy, we check if the formula is exact for $f(x) = x^3$. We have

   $$I(f) = \int_{-1}^{1} x^4dx = \frac{2}{5} \quad \text{and} \quad I_h(f) = \frac{1}{3} + \frac{1}{3} = \frac{2}{3}$$

   and so $I(x^3) \neq I_h(x^3)$ and the degree of accuracy is 2.

5. Consider the fixed point iteration $x_{n+1} = g(x_n)$ where

$$g(x) = x + \mu(2 - e^x)$$

with $\mu \neq 0$.

(a) (2 marks) Find the fixed point x^* of g.

The fixed point x^* satisfies $g(x^*) = x^*$. Using the given formula for g, we get

$$\mu(2 - e^x) = 0.$$

Since $\mu \neq 0$, we obtain $x^* = \log(2)$.

(b) (5 marks) Show that the fixed point iteration converges locally if $\mu \in (0, 1)$.

We know from class that the fixed point method converges locally if $|g'(x^*)| < 1$. Here, since $g'(x) = 1 - \mu e^x$, we have local convergence if $\left|1 - \mu e^{x^*}\right| < 1$. Plugging in $x^* = \log(2)$, leads to

$$\begin{aligned}
\left|1 - \mu e^{x^*}\right| < 1 &\iff |1 - 2\mu| < 1 \\
&\iff -1 < 1 - 2\mu < 1 \\
&\iff -2 < -2\mu < 0 \\
&\iff 1 > \mu > 0.
\end{aligned}$$

(c) (3 marks) Find the value μ for which the fixed point iteration converges at least quadratically.

To have (at least) quadratic convergence we need $g'(x^*) = 0$. Hence

$$g'(x^*) = 0 \iff 1 - \mu e^{x^*} = 0 \iff 1 - 2\mu = 0 \iff \mu = \frac{1}{2}.$$

Thus $\mu = 1/2$.