# Supplementary Notes 9

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**Abstract**

We discuss linear multistep methods focusing on the properties of consistency, zero-stability and convergence.

## Contents



## 1 Multistep methods for initial value problems

In the last set of notes, it is pointed out that all consistent 1-step methods are convergent. The natural question to ask is if that is also the case for $k$-step methods with $k \geq 2$, i.e., does every consistent $k$-step method converge? The answer is no as we will see here: we need consistency and a form of stability, which we will make precise.

Recall that we are interested in solving

$$\begin{cases} y'(t) = f(t, y(t)) & t \in [a, b], \\ y(a) = y_0. \end{cases} \tag{IVP}$$

We will focus here on linear multistep methods.

**Definition 1.1.** *A $k$-step method is linear multistep method if the discretization $\Phi_h$ can be written in the form*

$$\Phi_h = y_{n+1} + a_{k-1}y_n + \ldots + a_0 y_{n+1-k} = h\left(b_k f(t_{n+1}, y_{n+1}) + \ldots + b_0 f(t_{n+1-k}, y_{n+1-k})\right) \tag{1}$$

*for $n = k-1, \ldots, N-1$, where $a_i$, $b_i$ are constants and $f_i = f(t_i, y_i)$*

**Remark 1.2.** Note that for a $k$-step linear multistep method:

- the discretization $\Phi_h$ is linear in $y_{n+1}, y_n, \ldots, y_{n-k+1}$ and $f_{n+1}, f_n, \ldots, f_{n-k+1}$,
- if $b_k = 0$ the method is called explicit, otherwise, it is implicit.
- the $k-1$ initial $y_n$ are usually obtained using a 1-step method.

Not all arbitrary constants $a_i$, $b_i$ define a consistent $k$-step linear multistep method.

**Theorem 1.3.** *A $k$-step linear multistep method is consistent if and only if*

$$\sum_{i=0}^{k-1} a_i = -1, \quad \sum_{i=0}^{k} b_i = k + \sum_{i=0}^{k-1} i a_i. \tag{consistency conditions}$$

*A $k$-step linear multistep method is of order $p$ if and only if for all $q = 0, \ldots, p$*

$$q\sum_{i=0}^{k} i^{q-1} b_i = k^q + \sum_{i=0}^{k-1} i^s a_i. \tag{order conditions}$$

**Remark 1.4.** Notice that we recover the consistency conditions in the order conditions when we take $q = 0, 1$.

Unfortunately not all consistent multistep method are convergent.

**Definition 1.5.** *Let $\{a_i\}_{i=0}^{k-1}$ be the coefficients of a $k$-step linear multistep method. The characteristic polynomial is*

$$p(\lambda) = \lambda^k + a_{k-1}\lambda^{k-1} + \ldots + a_0.$$

**Definition 1.6.** *A $k$-step linear multistep method is called zero-stable if the roots $\{\lambda_1, \ldots, \lambda_k\}$ of its characteristic polynomial satisfy*

- $|\lambda_i| \leq 1$ *(i.e. all roots have modulus less than or equal to 1).*
- *if $\lambda_i = 1$, the $\lambda_i \neq \lambda_j$ for all $j \neq i$ (i.e., all roots with modulus 1 are distinct).*

**Remark 1.7.** Note that the roots of the characteristic polynomial may be complex. By absolute value of a complex number $z = x + iy$ we mean $|z| = \sqrt{x^2 + y^2}$.

**Remark 1.8.** Zero-stability means that a perturbation of size $\epsilon$ in the starting values causes the numerical solution to change by no more than $K\epsilon$ for some value $K$ which does not depend on the step size $h$ for a fixed time interval. It is called zero-stability because it is enough to check the condition for the differential equation $y' = 0$

**Remark 1.9.** Notice that $\lambda = 1$ is always a root of the characteristic polynomial of a consistent linear multistep method.

**Theorem 1.10.** *A linear multistep method is convergent if and only if it is consistent and zero-stable.*

The zero-stability condition helps us conclude that a consistent multistep method converges in the limit as $h \to 0$. However, in practice we will work with a fixed finite $h$. For efficiency reasons we want to take as large time step as possible, but this won't be possible in general since there are restrictions on how large $h$ can be. We will address this in the next section.

**Exercise 1.1.** Use the three-point endpoint formula to approximate $y'$ and derive two methods to numerically solve (IVP). Characterize both methods in terms of their consistency, order, zero-stability and convergence.

**Solution:** The three-point endpoint formula tells us that

$$y'(t_n) \approx \frac{-3y(t_n) + 4y(t_n + \Delta t) - y(t_n + 2\Delta t)}{2\Delta t}.$$

On one hand, if we take $\Delta t = h$, we get

$$y'(t_n) \approx \frac{-3y(t_n) + 4y(t_{n+1}) - y(t_{n+2})}{2h}.$$

Hence, since $y'(t_n) = f(t_n, y(t_n))$ we have

$$\frac{-3y(t_n) + 4y(t_{n+1}) - y(t_{n+2})}{2h} \approx f(t_n, y(t_n)).$$

Finally, replacing the exact solution $y(t_n)$ by the approximation $y_n$ leads to

$$\frac{-3y_n + 4y_{n+1} - y_{n+2}}{2h} = f(t_n, y_n).$$

Solving for $y_{n+2}$ and relabelling the indices leads to the explicit method

$$y_{n+1} - 4y_n + 3y_{n-1} = -2hf(t_{n-1}, y_{n-1}). \tag{2}$$

On the other hand, if we take $\Delta t = -h$, we get

$$y'(t_n) \approx \frac{3y(t_n) - 4y(t_{n-1}) + y(t_{n-2})}{2h},$$

leading to the implicit method given by

$$y_{n+1} - \frac{4}{3}y_n + \frac{1}{3}y_{n-1} = \frac{2h}{3}f(t_{n+1}, y_{n+1}). \tag{3}$$

In method (2), we have $a_0 = 3$, $a_1 = -4$, $b_0 = -2$, $b_1 = 0$ and $b_2 = 0$. Using Theorem 1.3, we conclude that the method is consistent of order 2. However, it's not zero-stable since the characteristic polynomial given by $p(\lambda) = \lambda^2 - 4\lambda + 3$ has roots 1 and 3. Therefore by Theorem 1.10, the method is not convergent.

In method (3), we have $a_0 = 1/3$, $a_1 = -4/3$, $b_0 = 0$, $b_1 = 0$ and $b_2 = 2/3$. Using Theorem 1.3, we conclude that the method is consistent of order 2. Moreover, it's zero-stable since the characteristic polynomial given by $p(\lambda) = \lambda^2 - 4/3\lambda + 1/3$ has roots 1/3 and 1. Therefore by Theorem 1.10, the method is convergent.

**Exercise 1.2.** The midpoint method is given by

$$y_{n+1} = y_{n-1} + 2hf(t_n, y_n).$$

(a) Show that the method is second order using Theorem 1.3.

**Solution:** The midpoint method is a 2-step linear multistep method with $a_0 = -1$, $a_1 = 0$, $b_0 = 0$, $b_1 = 2$ and $b_2 = 0$. We have

$$\sum_{j=0}^{1} a_j = 0 + (-1) = -1$$

$$\sum_{j=0}^{2} b_j = 0 + 2 + 0$$

$$2 + \sum_{j=0}^{1} ja_j = 2 + 0 \times (-1) + 1 \times 0 = 0$$

$$2\sum_{j=0}^{2} jb_j = 2(0 \times 0 + 1 \times 2 + 3 \times 0 = 4$$

$$2^2 + \sum_{j=0}^{1} j^2 a_j = 4 + 0 \times (-1) + 1 \times 0 = 4$$

and so the order conditions of order 2 are satisfied and we are done.

(b) Show that the local truncation error is $\mathcal{O}(h^3)$ using the Taylor approximation.

**Solution:** The midpoint method has a local truncation error at the $n^{th}$ step given by

$$\tau_h(t_n) = y(t_{n+1}) - y(t_{n-1}) - 2hf(t_n, y(t_n))$$

for each $i = 0, \ldots, N-1$. If we Taylor expand $y(t)$ about $t = t_n$ and evaluate at $t = t_{n+1}$ and $t_{n-1}$, we get

$$\begin{aligned} y(t_{n+1}) &= y(t_n) + hy'(t_n) + \frac{h^2}{2}y''(t_n) + \frac{h^3}{6}y^{(3)}(\xi_n^+) \\ y(t_{n-1}) &= y(t_n) - hy'(t_n) + \frac{h^2}{2}y''(t_n) - \frac{h^3}{6}y^{(3)}(\xi_n^-) \end{aligned}$$

where $\xi_n^\pm \in (t_n; t_{n\pm1})$. Here we stop at the third derivative since this term is multiplied by $h^3$ and we want to show that the local truncation error is $\mathcal{O}(h^3)$.

Hence, since $f(t_n, y(t_n)) = y'(t_n)$, we get subtracting the two equalities above and using the intermediate value theorem

$$\begin{aligned} \tau_h(t_n) &= \frac{h^3}{6}\left(y^{(3)}(\xi_n^+) + y^{(3)}(\xi_n^-)\right) \\ &= \frac{h^3}{3}y^{(3)}(\xi_n) \end{aligned}$$

where $\xi_n \in (t_{n-1}, t_{n+1})$. When $y^{(3)}(\cdot)$ is known to be bounded by a constant $M_3$ on $[0, T]$, we have

$$|\tau_h(t_n)| \leq \frac{M_3}{3}h^3$$

and so the local truncation error in the midpoint method is $\mathcal{O}(h^3)$.

(c) Show that the method is zero-stable. Can you conclude that it is convergent?

**Solution:** The characteristic polynomial is given by $p(\lambda) = \lambda^2 - 1$ and so the roots are $\pm 1$. Hence the midpoint method is zero-stable. The method is thus consistent and zero-stable and so we can conclude that it is convergent.

# References


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