# COMP251: Divide-and-Conquer (1)

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Based on (Kleinberg & Tardos, 2005) & slides from (Snoeyink,2004)

# Divide and Conquer

- Recursive in structure
  - ***Divide*** the problem into sub-problems that are similar to the original but smaller in size
  - ***Conquer*** the sub-problems by solving them recursively. If they are small enough, just solve them in a straightforward manner.
  - ***Combine*** the solutions to create a solution to the original problem

# An Example: Merge Sort

***Sorting Problem***: Sort a sequence of *n* elements into non-decreasing order.

- ***Divide***: Divide the *n*-element sequence to be sorted into two subsequences of *n/2* elements each
- ***Conquer:*** Sort the two subsequences recursively using merge sort.
- ***Combine***: Merge the two sorted subsequences to produce the sorted answer.

# Sorting applications

Obvious applications.

- Organize an MP3 library.
- Display Google PageRank results.
- List RSS news items in reverse chronological order.

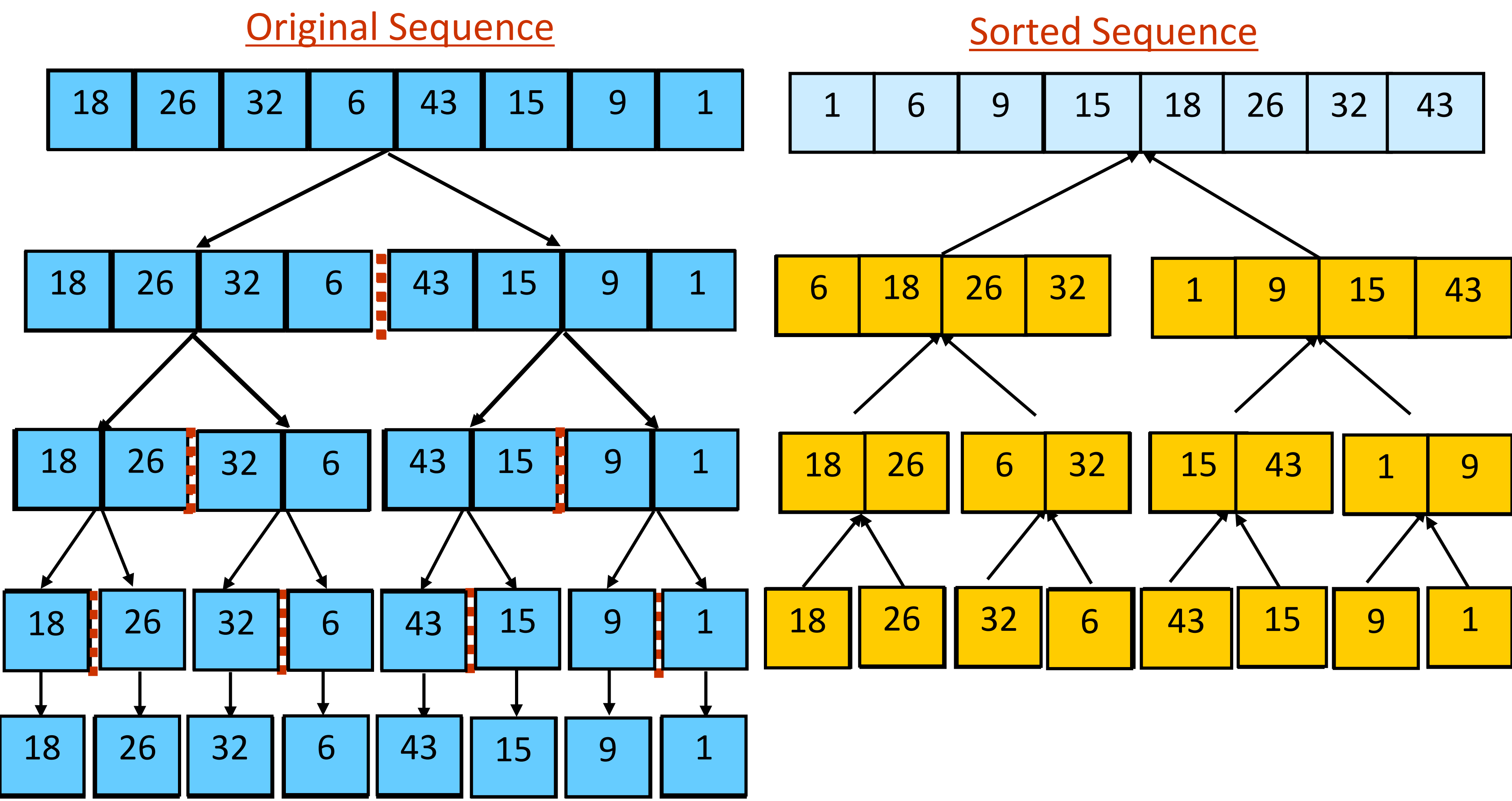
Some problems become easier once elements are sorted.

- Identify statistical outliers.
- Binary search in a database.
- Remove duplicates in a mailing list.

Non-obvious applications.

- Convex hull.
- Closest pair of points.
- Interval scheduling / interval partitioning.
- Minimum spanning trees (Kruskal's algorithm).
- Scheduling to minimize maximum lateness or average completion time.
- ...

# Merge Sort – Example



# Merge-Sort (A, p, q, r)

**INPUT:** **a sequence of *n* numbers stored in array A**

**OUTPUT:** **an ordered sequence of *n* numbers**

```
MergeSort (A, p, r)   // sort A[p..r] by divide & conquer
  if p < r
     then q ← ⌊(p+r)/2⌋
          MergeSort (A, p, q)
          MergeSort (A, q+1, r)
          Merge (A, p, q, r) // merges A[p..q] with A[q+1..r]
```

Initial Call: MergeSort($A$, 1, $n$)

# Procedure Merge

**Merge($A, p, q, r$)**

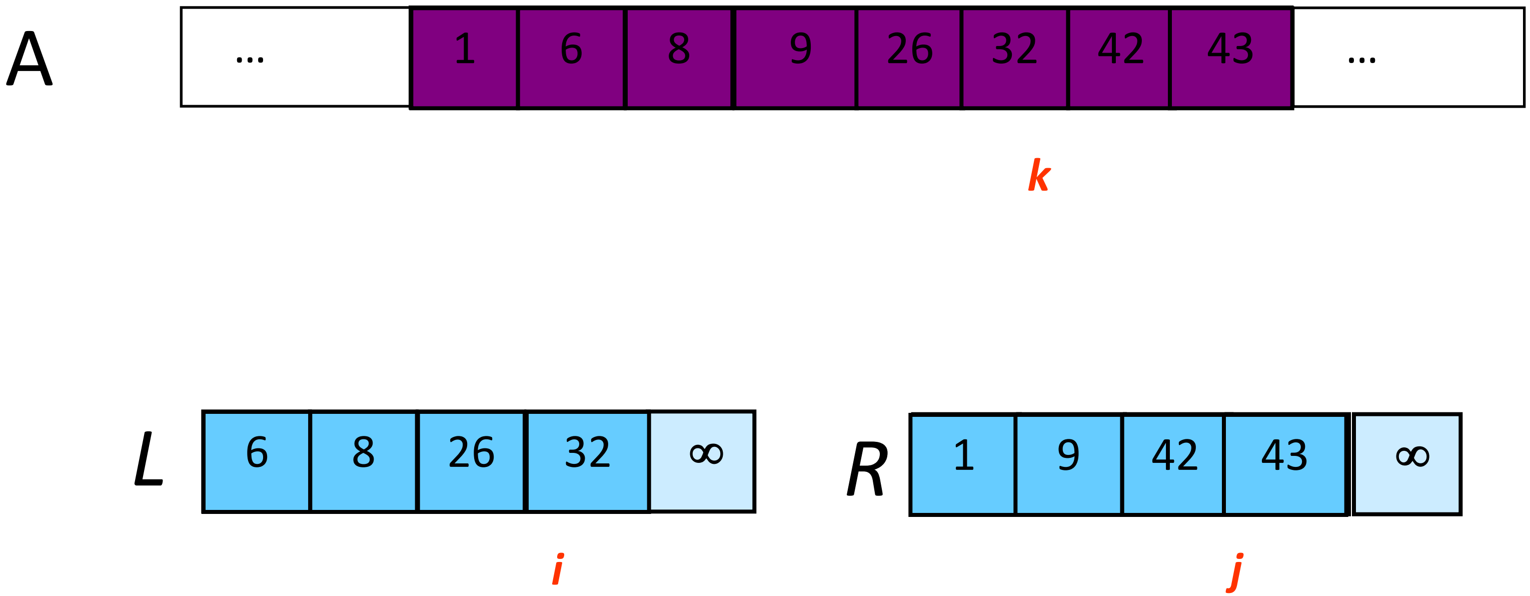
```
n1 ← q − p + 1
n2 ← r − q
for i ← 1 to n1
    do L[i] ← A[p + i − 1]
for j ← 1 to n2
    do R[j] ← A[q + j]
L[n1+1] ← ∞
R[n2+1] ← ∞
i ← 1
j ← 1
for k ← p to r
    do if L[i] ≤ R[j]
        then A[k] ← L[i]
                i ← i + 1
        else A[k] ← R[j]
                j ← j + 1
```

**Input:** Array containing sorted subarrays $A[p..q]$ and $A[q+1..r]$.

**Output:** Merged sorted subarray in $A[p..r]$.

**Sentinels**, to avoid having to check if either subarray is fully copied at each step.

# Merge – Example



# Correctness of Merge

**Merge(*A*, *p*, *q*, *r*)**

```
n1 ← q − p + 1
n2 ← r − q
for i ← 1 to n1
    do L[i] ← A[p + i − 1]
for j ← 1 to n2
    do R[j] ← A[q + j]
L[n1+1] ← ∞
R[n2+1] ← ∞
i ← 1
j ← 1
for k ← p to r
    do if L[i] ≤ R[j]
        then A[k] ← L[i]
                 i ← i + 1
        else A[k] ← R[j]
                 j ← j + 1
```

**Loop Invariant property (main *for* loop)**

- At the start of each iteration of the for loop, Subarray $A[p..k-1]$ contains the $k-p$ smallest elements of $L$ and $R$ in sorted order.
- $L[i]$ and $R[j]$ are the smallest elements of $L$ and $R$ that have not been copied back into $A$.

**Initialization:**

Before the first iteration:

- $A[p..k-1]$ is empty.
- $i = j = 1$.
- $L[1]$ and $R[1]$ are the smallest elements of $L$ and $R$ not copied to $A$.

# Correctness of Merge

**Merge(*A*, *p*, *q*, *r*)**

```
1.   n1 ← q – p + 1
2.   n2 ← r – q
3.   for i ← 1 to n1
4.      do L[i] ← A[p + i – 1]
5.   for j ← 1 to n2
6.      do R[j] ← A[q + j]
7.   L[n1+1] ← ∞
8.   R[n2+1] ← ∞
9.   i ← 1
10.  j ← 1
11.  for k ←p to r
12.     do if L[i] ≤ R[j]
13.        then A[k] ← L[i]
14.                 i ← i + 1
15.        else A[k] ← R[j]
16.                 j ← j + 1
```

## Maintenance:

**Case 1:** $L[i] \le R[j]$

- By LI, $A$ contains $p - k$ smallest elements of $L$ and $R$ in sorted order.
- By LI, $L[i]$ and $R[j]$ are the smallest elements of $L$ and $R$ not yet copied into $A$.
- Line 13 results in $A$ containing $p - k + 1$ smallest elements (again in sorted order). Incrementing $i$ and $k$ reestablishes the LI for the next iteration.

**Case 2:** Similar arguments with $L[i] > R[j]$

## Termination:

- On termination, $k = r + 1$.
- By LI, $A$ contains $r - p + 1$ smallest elements of $L$ and $R$ in sorted order.
- $L$ and $R$ together contain $r - p + 3$ elements including the two sentinels. So all elements are sorted.

# Analysis of Merge Sort

- Running time ***T*(*n*)** of Merge Sort:
- Divide: computing the middle takes $\Theta(1)$
- Conquer: solving 2 subproblems takes $2T(n/2)$
- Combine: merging $n$ elements takes $\Theta(n)$
- Total:

$$T(n) = \Theta(1) \qquad \text{if } n = 1$$
$$T(n) = 2T(n/2) + \Theta(n) \quad \text{if } n > 1$$

$$\Rightarrow T(n) = \Theta(n \lg n)$$

# A useful recurrence relation

Def. $T(n)$ = max number of compares to mergesort a list of size $\leq n$.

Note. $T(n)$ is monotone nondecreasing.

Mergesort recurrence.

$$T(n) \leq \begin{cases} 0 & \text{if } n = 1 \\ T(\lceil n/2 \rceil) + T(\lfloor n/2 \rfloor) + n & \text{otherwise} \end{cases}$$

Solution. $T(n)$ is $O(n \log_2 n)$.

Assorted proofs. We describe several ways to prove this recurrence. Initially we assume $n$ is a power of $2$ and replace $\leq$ with $=$.

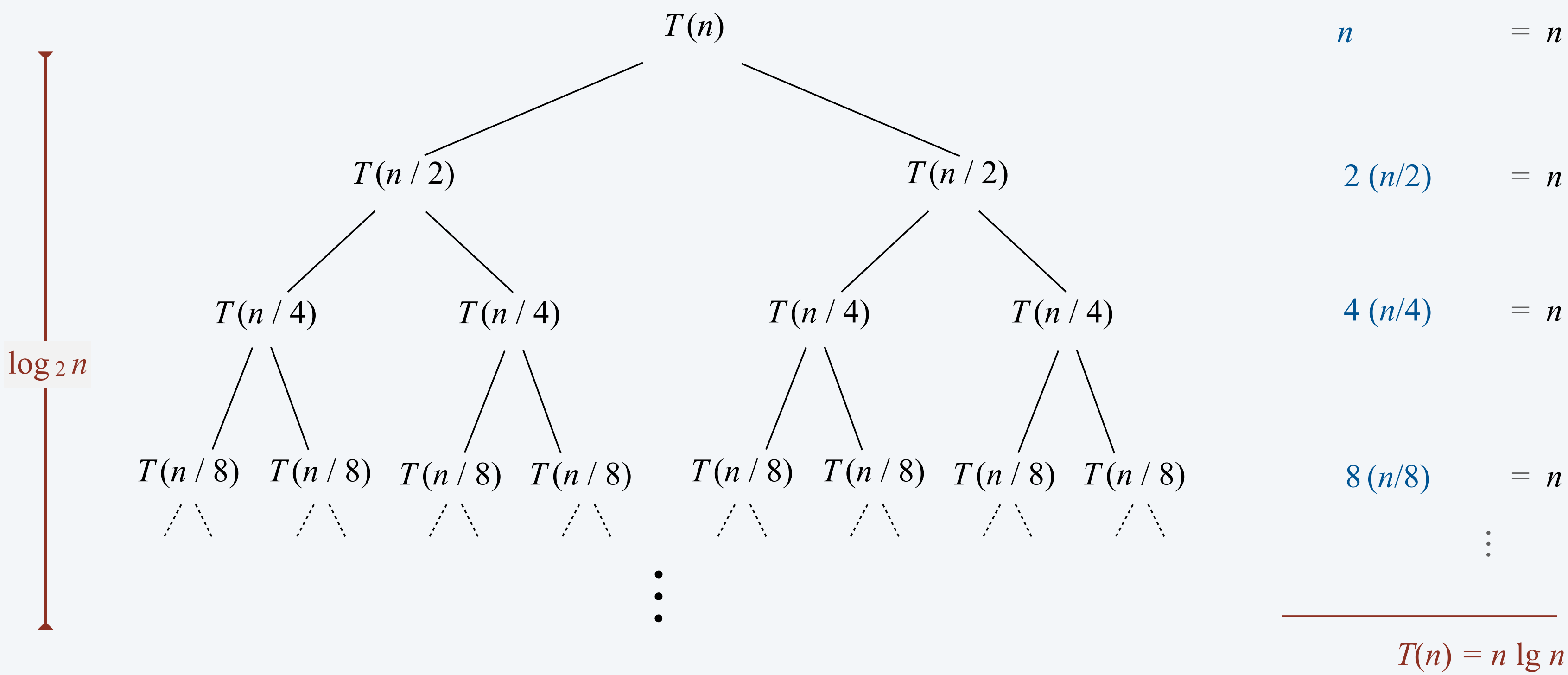
# Divide-and-conquer recurrence: proof by recursion tree

**Proposition.** If $T(n)$ satisfies the following recurrence, then $T(n) = n \log_2 n$.

assuming n
is a power of 2

$$T(n) = \begin{cases} 0 & \text{if } n = 1 \\ 2\,T(n/2) + n & \text{otherwise} \end{cases}$$

**Pf 1.**



# Proof by induction

Proposition. If $T(n)$ satisfies the following recurrence, then $T(n) = n \log_2 n$.

assuming n is a power of 2

$$T(n) = \begin{cases} 0 & \text{if } n = 1 \\ 2\,T(n/2) + n & \text{otherwise} \end{cases}$$

Pf 2. [by induction on $n$]

- Base case: when $n = 1$, $T(1) = 0$.
- Inductive hypothesis: assume $T(n) = n \log_2 n$.
- Goal: show that $T(2n) = 2n \log_2 (2n)$.

$$\begin{aligned} T(2n) &= 2\,T(n) + 2n \\ &= 2\,n \log_2 n + 2n \\ &= 2\,n\,(\log_2 (2n) - 1) + 2n \\ &= 2\,n \log_2 (2n). \quad ▪ \end{aligned}$$

# Analysis of mergesort recurrence

Claim. If $T(n)$ satisfies the following recurrence, then $T(n) \le n \lceil \log_2 n \rceil$.

$$T(n) \le \begin{cases} 0 & \text{if } n = 1 \\ T(\lceil n/2 \rceil) + T(\lfloor n/2 \rfloor) + n & \text{otherwise} \end{cases}$$

Pf. [by strong induction on $n$]

- Base case: $n = 1$.
- Define $n_1 = \lfloor n/2 \rfloor$ and $n_2 = \lceil n/2 \rceil$.
- Induction step: assume true for $1, 2, \ldots, n-1$.

$$\begin{aligned} n_2 &= \lceil n/2 \rceil \\ &\le \left\lceil 2^{\lceil \log_2 n \rceil} / 2 \right\rceil \\ &= 2^{\lceil \log_2 n \rceil} / 2 \end{aligned}$$

$$\begin{aligned} T(n) &\le T(n_1) + T(n_2) + n \\ &\le n_1 \lceil \log_2 n_1 \rceil + n_2 \lceil \log_2 n_2 \rceil + n \\ &\le n_1 \lceil \log_2 n_2 \rceil + n_2 \lceil \log_2 n_2 \rceil + n \\ &= n \lceil \log_2 n_2 \rceil + n \qquad \longleftarrow \qquad \log_2 n_2 \le \lceil \log_2 n \rceil - 1 \\ &\le n (\lceil \log_2 n \rceil - 1) + n \\ &= n \lceil \log_2 n \rceil. \quad \blacksquare \end{aligned}$$

# Arithmetic operations

Given 2 (binary) numbers, we want efficient algorithms to:

- Add 2 numbers
- **Multiply 2 numbers** (here, we will use a divide-and-conquer method!)

# Integer addition

Addition. Given two $n$-bit integers $a$ and $b$, compute $a + b$.

Subtraction. Given two $n$-bit integers $a$ and $b$, compute $a - b$.

Grade-school algorithm. $\Theta(n)$ bit operations.

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| 1 | 1 | 1 | 1 | 1 | 1 | 0 | 1 | |
| | 1 | 1 | 0 | 1 | 0 | 1 | 0 | 1 |
| + | 0 | 1 | 1 | 1 | 1 | 1 | 0 | 1 |
| 1 | 0 | 1 | 0 | 1 | 0 | 0 | 1 | 0 |

Remark. Grade-school addition and subtraction algorithms are asymptotically optimal.

# Integer multiplication

Multiplication. Given two $n$-bit integers $a$ and $b$, compute $a \times b$.

Grade-school algorithm. $\Theta(n^2)$ bit operations.

| | | | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | | | | | | | | 1 | 1 | 0 | 1 | 0 | 1 | 0 | 1 |
| | | | | | | | × | 0 | 1 | 1 | 1 | 1 | 1 | 0 | 1 |
| | | | | | | | | 1 | 1 | 0 | 1 | 0 | 1 | 0 | 1 |
| | | | | | | | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | |
| | | | | | | 1 | 1 | 0 | 1 | 0 | 1 | 0 | 1 | | |
| | | | | | 1 | 1 | 0 | 1 | 0 | 1 | 0 | 1 | | | |
| | | | | 1 | 1 | 0 | 1 | 0 | 1 | 0 | 1 | | | | |
| | | | 1 | 1 | 0 | 1 | 0 | 1 | 0 | 1 | | | | | |
| | | 1 | 1 | 0 | 1 | 0 | 1 | 0 | 1 | | | | | | |
| | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | | | | | | | |
| 0 | 1 | 1 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 |

Conjecture. [Kolmogorov 1952] Grade-school algorithm is optimal.

Theorem. [Karatsuba 1960] Conjecture is wrong.

# Divide-and-conquer multiplication

To multiply two $n$-bit integers $x$ and $y$:

- Divide $x$ and $y$ into low- and high-order bits.
- Multiply four ½$n$-bit integers, recursively.
- Add and shift to obtain result.

$$m = \lceil n / 2 \rceil$$

$$a = \lfloor x / 2^m \rfloor \quad b = x \bmod 2^m$$

$$c = \lfloor y / 2^m \rfloor \quad d = y \bmod 2^m$$

use bit shifting
to compute 4 terms

$$\underbrace{(2^m a + b)}_{x} \underbrace{(2^m c + d)}_{y} = 2^{2m} \underbrace{ac}_{1} + 2^m (\underbrace{bc}_{2} + \underbrace{ad}_{3}) + \underbrace{bd}_{4}$$

Ex. $x = \underbrace{1\,0\,0\,0}_{a}\,\underbrace{1\,1\,0\,1}_{b} \quad y = \underbrace{1\,1\,1\,0}_{c}\,\underbrace{0\,0\,0\,1}_{d}$

# Divide-and-conquer multiplication

MULTIPLY$(x, y, n)$

IF $(n = 1)$

RETURN $x \times y$.

ELSE

$m \leftarrow \lceil n / 2 \rceil$.

$a \leftarrow \lfloor x / 2^m \rfloor$; $b \leftarrow x \bmod 2^m$.

$c \leftarrow \lfloor y / 2^m \rfloor$; $d \leftarrow y \bmod 2^m$.

$e \leftarrow$ MULTIPLY$(a, c, m)$.

$f \leftarrow$ MULTIPLY$(b, d, m)$.

$g \leftarrow$ MULTIPLY$(b, c, m)$.

$h \leftarrow$ MULTIPLY$(a, d, m)$.

RETURN $2^{2m}\, e + 2^m\, (g + h) + f$.

# Divide-and-conquer multiplication analysis

**Proposition.** The divide-and-conquer multiplication algorithm requires $\Theta(n^2)$ bit operations to multiply two $n$-bit integers.

**Pf.** Apply case 1 of the master theorem to the recurrence:

$$T(n) = \underbrace{4T(n/2)}_{\text{recursive calls}} + \underbrace{\Theta(n)}_{\text{add, shift}} \Rightarrow T(n) = \Theta(n^2)$$

Next class!

# Karatsuba trick

To compute middle term $bc + ad$, use identity:

$$bc + ad = ac + bd - (a - b)(c - d)$$

$$m = \lceil n / 2 \rceil$$

$$a = \lfloor x / 2^m \rfloor \quad b = x \bmod 2^m$$

$$c = \lfloor y / 2^m \rfloor \quad d = y \bmod 2^m$$

middle term

$$(2^m a + b)(2^m c + d) = 2^{2m} ac + 2^m \overline{(bc + ad)} + bd$$

$$= 2^{2m} ac + 2^m (ac + bd - (a - b)(c - d)) + bd$$

1 1 3 2 3

**Bottom line.** Only three multiplication of $n / 2$-bit integers.

# Karatsuba multiplication

KARATSUBA-MULTIPLY($x$, $y$, $n$)

---

IF $(n = 1)$

  RETURN $x \times y$.

ELSE

  $m \leftarrow \lceil n / 2 \rceil$.

  $a \leftarrow \lfloor x / 2^m \rfloor$;  $b \leftarrow x \bmod 2^m$.

  $c \leftarrow \lfloor y / 2^m \rfloor$;  $d \leftarrow y \bmod 2^m$.

  $e \leftarrow$ KARATSUBA-MULTIPLY($a$, $c$, $m$).

  $f \leftarrow$ KARATSUBA-MULTIPLY($b$, $d$, $m$).

  $g \leftarrow$ KARATSUBA-MULTIPLY($a - b$, $c - d$, $m$).

  RETURN $2^{2m}\, e + 2^m\, (e + f - g) + f$.

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## Karatsuba analysis

Proposition. Karatsuba's algorithm requires $O(n^{1.585})$ bit operations to multiply two $n$-bit integers.

Pf. Apply case 1 of the master theorem to the recurrence:

Next class!

$$T(n) = 3\,T(n/2) + \Theta(n) \quad \Rightarrow \quad T(n) = \Theta(n^{\lg 3}) = O(n^{1.585}).$$

Practice. Faster than grade-school algorithm for about 320-640 bits.

# Integer arithmetic reductions

Integer multiplication. Given two $n$-bit integers, compute their product.

| problem | arithmetic | running time |
|---|---|---|
| integer multiplication | $a \times b$ | $\Theta(M(n))$ |
| integer division | $a / b, \; a \bmod b$ | $\Theta(M(n))$ |
| integer square | $a^2$ | $\Theta(M(n))$ |
| integer square root | $\lfloor \sqrt{a} \rfloor$ | $\Theta(M(n))$ |

**integer arithmetic problems with the same complexity as integer multiplication**

# History of asymptotic complexity of integer multiplication

| year | algorithm | order of growth |
|---|---|---|
| ? | brute force | $\Theta(n^2)$ |
| 1962 | Karatsuba-Ofman | $\Theta(n^{1.585})$ |
| 1963 | Toom-3, Toom-4 | $\Theta(n^{1.465})$, $\Theta(n^{1.404})$ |
| 1966 | Toom-Cook | $\Theta(n^{1+\varepsilon})$ |
| 1971 | Schönhage–Strassen | $\Theta(n \log n \log \log n)$ |
| 2007 | Fürer | $n \log n \; 2^{O(\log^* n)}$ |
| ? | ? | $\Theta(n)$ |

**number of bit operations to multiply two n-bit integers**

used in Maple, Mathematica, gcc, cryptography, ...

Remark. GNU Multiple Precision Library uses one of five different algorithm depending on size of operands.

GMP
«Arithmetic without limitations»