Supplementary Notes 12

Tiago Salvador (tiago.saldanhasalvador@mail.mcgill.ca)

Abstract

In this tutorial we will deal with real-valued square systems of order n, that is, systems of the form $Ax = b$ with $A \in \mathbb{R}^{n \times n}$ and $b \in \mathbb{R}^n$. In such case we know from Linear Algebra that the system has a unique solution if and only if A is non-singular. We will focus on iterative methods to solve the linear system.

Contents

1 Numerical methods for solving linear systems

We are interested in solving $Ax = b$ where A is a $n \times n$ matrix and x, b are column vectors with n entries. There are two classes of methods.

- Direct methods (solve the linear systems exactly, ignoring round-off errors)
  - Gaussian elimination
  - LU/QR/Cholesky decomposition
- Iterative methods
  - Richardson
  - Jacobi
  - Gauss-Seidel
  - Successive over relaxation (SOR)
  - Conjugate gradient (seen in Math 327/578)
  - Multigrid (seen in Math 327/578)

Often in applications, A is a very large but sparse matrix. The sparsity of the matrix can be exploited for iterative methods to give accurate approximations efficiently. Moreover, if x is a solution to some discretization of ODEs/PDEs (e.g. $A = A_h$ of the Poisson problem), it is enough to compute x up to some error tolerance.

2 On the convergence of iterative methods

Consider an iterative method of the form

$$x^{(k+1)} = Bx^{(k)} + c.$$

(In fact, we have a fixed point iteration $x^{(k+1)} = G(x^{(k)})$ where $G(x) = Bx + c$.) We assume the method is consistent with the linear system $Ax = b$, i.e.,

$$x = Bx + c \Longleftrightarrow Ax = b.$$

Define the error at the k^{th} iteration as $e^{(k)} = x - x^{(k)}$. Hence

$$e^{(k+1)} = x - x^{(k+1)} = Bx + c - (Bx^{(k)} + c) = B(x - x^{(k)}) = Be^{(k)} = \ldots = B^{k+1}e^{(0)}.$$

Theorem 2.1. *Let $x^{(0)}$ be an initial guess. Then*

$$\lim_{k\to\infty} \left|\left|e^{(k)}\right|\right|_2 = 0 \Longleftrightarrow \rho(B) < 1 \Longleftrightarrow \max_{i=1,\ldots,n} |\lambda_i(B)| < 1$$

Definition 2.2. *The spectral radius of A, $\rho(A)$, is given by*

$$\rho(A) = \max_{i=1,\ldots,n} |\lambda_i(A)|.$$

Assuming B is symmetric,

$$\left|\left|e^{(k+1)}\right|\right|_2 \leq ||B||_2 \left|\left|e^{(k)}\right|\right|_2 = \rho(B) \left|\left|e^{(k)}\right|\right|_2 \Longrightarrow \lim_{k\to\infty} = \frac{||e^{(k+1)}||_2}{||e^{(k)}||_2} \leq \rho(B) < 1.$$

Thus the fixed point iteration converges at least linearly and the rate of convergence (i.e., A.E.C for linear convergence) $r \leq \rho(B)$. The smaller $\rho(B)$, the faster the iterations converge.

3 Classic iterative methods for the solution of linear systems

A general idea to obtain iterative methods is based on an additive splitting of the matrix A of the form $A = M + N$ where M is nonsingular. We call M preconditioner or preconditioning matrix.

We have

$$Ax = b \Longleftrightarrow (M + N)x = b \Longleftrightarrow Mx = b - Nx \Longleftrightarrow x = M^{-1}(b - Nx).$$

From this we can construct the fixed point iteration

$$x^{(k+1)} = Bx^{(k)} + c$$

where $B = -M^{-1}N$ and $c = M^{-1}b$. We have then obtained a fixed point iteration to compute the solution of $Ax = b$. The matrix B is called the iteration matrix. Here we are interested in solving systems of the form $Mx = b - Nx$. Notice that this is not equivalent to invert M and can be done significantly faster. Typically we want M to be simple, say diagonal or triangular.

Example 3.1 (Richardson's Method)**.** *If we take $M = \frac{1}{\omega}I$ with $\omega \neq 0$, we get the iterative method called Richardson's method. In this case*

$$M^{-1} = \omega I, \quad N = A - M = A - \frac{1}{\omega}I$$

and so

$$x^{(k+1)} = M^{-1}\left(B - Nx^{(k)}\right) = (\omega I)\left(b - \left(A - \frac{1}{\omega}I\right)x^{(k)}\right) = w\left(b - Ax^{(k)}\right) + x^{(k)}$$

In the next examples we will consider the decomposition of A into $L+D+U$ where L is the strictly lower triangular part of A, D is the diagonal part of A and U is the strictly upper triangular part of A. For instance

$$A = \begin{bmatrix} 3 & \frac{1}{6} & -\frac{1}{4} \\ \frac{5}{6} & 0 & 1 \\ -9 & 2 & \frac{1}{6} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ \frac{5}{6} & 0 & 0 \\ -9 & 2 & 0 \end{bmatrix} + \begin{bmatrix} 3 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \frac{1}{6} \end{bmatrix} + \begin{bmatrix} 0 & \frac{1}{6} & -\frac{1}{4} \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} = L + D + U.$$

Example 3.2 (Jacobi Method). *If we take $M = D$, we get the iterative method called Jacobi's method:*

$$x^{(k+1)} = D^{-1}(-L-U)x^{(k)} + D^{-1}b = D^{-1}\left(b - (L+U)x^{(k)}\right).$$

The element-based formula is thus

$$x_i^{(k+1)} = \frac{1}{a_{ii}}\left(b_i - \sum_{\substack{j=1 \\ j\neq i}}^{n} a_{ij}x_j^{(k)}\right) \quad i = 1, \ldots, n.$$

Remark 3.1. *The Jacobi method is parallelizable since each component in $x^{(k+1)}$ is independent of the others.*

Example 3.3 (Gauss-Seidel Method). *If we take $M = L + D$, we get the iterative scheme called Gauss-Seidel method:*

$$x^{(k+1)} = (L+D)^{-1}(-U)x^{(k)} + (L+D)^{-1}b = (L+D)^{-1}(b - Ux^{(k)}).$$

By taking advantage of the triangular form of P, the entries of $x^{(k+1)}$ can be computed using forward substitution:

$$x_i^{(k+1)} = \frac{1}{a_{ii}}\left(b_i - \sum_{j=1}^{i-1} a_{ij}x_j^{(k+1)} - \sum_{j=i+1}^{n} a_{ij}x_j^{(k)}\right) i = 1, \ldots, n.$$

Example 3.4 (SOR - Successive over-relaxation). *If we take $M = D + \omega L$ with $\omega \in \mathbb{R}$, we get the iterative scheme called the method of successive over-relaxation or SOR. For $\omega = 1$, it reduces to the Gauss-Seidel method. For well-chosen values of ω, SOR can be very fast. The key here is to be able to find near-optimal values of ω, which might not be easy.*

For implementation purposes, we notice that

$$\begin{aligned} Ax = b &\Longleftrightarrow \omega Ax = \omega b \\ &\Longleftrightarrow \omega(D+L)x = \omega b - \omega Ux \\ &\Longleftrightarrow (D+\omega L)x = \omega b - (\omega U + (\omega - 1)D)\, x. \end{aligned}$$

We then write the iterative scheme as

$$x^{(k+1)} = (D+\omega L)^{-1}\left(\omega b - [\omega U + (\omega-1)D]\, x^{(k)}\right)$$

with an element-based formula given by

$$x_i^{(k+1)} = (1-\omega)x_i^{(k)} + \frac{\omega}{a_{ii}}\left(b_i - \sum_{j=1}^{i-1} a_{ij}x_j^{(k+1)} - \sum_{j=i+1}^{n} a_{ij}x_j^{(k)}\right) \quad i = 1, \ldots, n.$$

Exercise 3.1. *Find the first two iterations of the Jacobi and Gauss-Seidel methods for the following linear system*

$$\begin{cases} 3x_1 - x_2 + x_3 = 1 \\ 3x_1 + 6x_2 + 2x_3 = 0 \\ 3x_1 + 3x_2 + 7x_3 = 4 \end{cases}$$

using $x^{(0)} = (0, 0, 0)$.

Solution: We have

$$A = \begin{bmatrix} 3 & -1 & 1 \\ 3 & 6 & 2 \\ 3 & 3 & 7 \end{bmatrix} \text{ and } b = \begin{bmatrix} 1 \\ 0 \\ 4 \end{bmatrix}.$$

We have

$$x_J^{(1)} = D^{-1}\left(b - (L+U)x_J^{(0)}\right) = D^{-1}b = \begin{bmatrix} \frac{1}{3} \\ 0 \\ \frac{4}{7} \end{bmatrix}$$

and

$$x_J^{(2)} = D^{-1}\left(b - (L+U)x_J^{(1)}\right) = \begin{bmatrix} \frac{1}{7} \\ -\frac{5}{14} \\ \frac{3}{7} \end{bmatrix}.$$

We have

$$x_{GS}^{(1)} = \begin{bmatrix} \frac{1}{3} \\ -\frac{1}{6} \\ \frac{1}{2} \end{bmatrix}$$

and

$$x_{GS}^{(2)} = \begin{bmatrix} \frac{1}{9} \\ -\frac{2}{9} \\ \frac{13}{21} \end{bmatrix}.$$

References

[1] R. L. Burden and J. D. Faires. *Numerical Analysis.* 9^{th} edition. Brookes/Cole, 2004.

[2] A. Quarteroni, R. Sacco and F. Saleri. *Numerical Mathematics.* 2^{nd} edition. Springer, 2006.