Math 317 Assignment 3

Due in class: November 17th, 2016

Instructions: Submit a hard copy of your solution with your name and student number. (**No name = zero grade!**) You must include all relevant program code, electronic output and explanations of your results. Write your own codes and comment them. Late assignment will not be graded and will receive a grade of zero.

1. Consider the integral $I(f) = \int_0^1 f(x)dx$.
   (a) Determine the two point Gauss quadrature for I on $[0, 1]$. What is its degree of accuracy?
   (b) Determine c_0, c_1 and x_1 so that the quadrature $I_h(f) = c_0 f(0) + c_1 f(x_1)$ has the highest degree of accuracy possible. State this degree.
   (c) Recall from probability, the Gaussian distribution with mean $\mu = 0$ and standard deviation $\sigma = 1$ is $f(x) = \frac{e^{-x^2/2}}{\sqrt{2\pi}}$. In this case, compare the approximate values for I using the two point Gauss quadrature and your quadrature in part (b). (The exact value of I is $0.3413447\ldots$).

2. Consider the integral $I(f) = \int_0^1 f(x)dx$.
   (a) Derive the formula for the composite trapezoidal rule and its error.
   (b) Using Richard's extrapolation method, we can use the composite trapezoidal rule to derive a more accurate quadrature. Such quadrature is given by

   $$I_h^R(f) = \frac{4I_{\frac{h}{2}}(f) - I_h(f)}{3}$$

   where I_h denotes here the composite trapezoidal rule and has error $O(h^4)$. The goal of this question is to perform a convergence analysis of the composite Trapezoidal rule and the improved quadrature for $I(f) = \int_0^1 e^{-x}\, dx$. In order to do so write a program to approximate using both quadratures for $h = 1, 2^{-1}, \ldots 2^{-8}$, plot log(error) versus log(h) confirm their convergence rate by estimating the slopes of the lines in the loglog plot.

3. This exercise is to derive the order conditions for linear multistep method. Recall a k-step linear multi-step method for first-order initial value problems has the form:

   $$\Phi_h := y_{n+1} + \sum_{i=0}^{k-1} a_i y_{n-k+1+i} - h \sum_{i=0}^{k} b_i f_{n-k+1+i} = 0,$$

   where a_i, b_i are constants and $f_i := f(t_i, y_i)$.

   (a) Denoting $a_k = 1$ and y as the exact solution to the initial value problem, show that the local truncation error is,

   $$\tau_h(t_n) = \sum_{i=0}^{k} a_i y(t_{n-k+1} + ih) - h \sum_{i=0}^{k} b_i y'(t_{n-k+1} + ih).$$

   (b) By Taylor expanding y, y' around t_{n-k+1}, show that for some $\xi_i, \eta_i \in [t_{n-k+1}, t_{n-k+1} + ih]$

$$\tau_h(t_n) = \sum_{i=0}^{k} a_i \left(\sum_{q=0}^{p} \frac{(ih)^q}{q!} y^{(q)}(t_{n-k+1}) + \frac{(ih)^{p+1}}{(p+1)!} y^{(p+1)}(\xi_i) \right) - h \sum_{i=0}^{k} b_i \left(\sum_{q=1}^{p} \frac{(ih)^{q-1}}{(q-1)!} y^{(q)}(t_{n-k+1}) + \frac{(ih)^p}{p!} y^{(p+1)}(\eta_i) \right).$$

(c) Show that the local truncation error can be written in the form,

$$\tau_h(t_n) = \sum_{q=0}^{p} \left(\frac{h^q}{q!} y^{(q)}(t_{n-k+1}) C_q \right) + \frac{h^{p+1}}{(p+1)!} D,$$

where

$$C_q = \sum_{i=0}^{k} i^q a_i - q \sum_{i=0}^{k} i^{q-1} b_i, \quad D = \sum_{i=0}^{k} \left(i^{p+1} a_i y^{(p+1)}(\xi_i) - (p+1) i^p b_i y^{(p+1)}(\eta_i) \right).$$

(d) Conclude that a k-step linear multi-step method is of order p if and only if a_i, b_i satisfies $C_q = 0$ for all $q = 0, \ldots, p$. Or equivalently, a_i, b_i satisfies for all $q = 0, \ldots, p$,

$$q \sum_{i=0}^{k} i^{q-1} b_i = k^q + \sum_{i=0}^{k-1} i^q a_i. \text{ (i.e. order conditions)}$$

4. The implicit 2-step Milne-Simpson method is:

$$y_{n+1} = y_{n-1} + \frac{h}{3}(f_{n+1} + 4f_n + f_{n-1}).$$

(a) Show that the local truncation error is $O(h^5)$.

(b) Show that the method is zero-stable and conclude that it is convergent.

5. Consider the I.V.P. on $t \in [0, T]$:

$$y' = y^2(1 - \epsilon y).$$

(a) For $T = 20, y(0) = 0.1, \epsilon = 0.03$, use the forward Euler and Trapezoidal method to solve the I.V.P. with $N = 500$ and plot both solutions versus t.

(b) For the equilibrium solution $y^* = 1/\epsilon$, show that the I.V.P. is approximately,

$$y' \approx -\frac{1}{\epsilon}(y - y^*) \text{ when } |y - y^*| \text{ is small.}$$

Hint: Taylor expand $f(y) = y^2(1 - \epsilon y)$ *around* $y = y^*$.

(c) Plot both solutions when $N = 250$ and use part (b) to explain what is happening.

6. Consider the non-dimensionalized pendulum problem

$$\begin{cases} \theta''(t) + \sin(\theta(t)) = 0, & t \in [0, T], \\ \theta(0) = a, \\ \theta'(0) = b. \end{cases}$$

Let $\theta(t)$ denote the exact solution.

(a) Write the second order equation as a system of first order equations.

(b) For $N = 1000$, use the forward Euler and improved Euler's method to solve the first order system for $a = \pi/4, b = 0$ up to $T = 30$. Plot the two solutions as $\theta(t)$ versus t and as $\theta'(t)$ versus $\theta(t)$.

(c) Let $E(t) = \frac{(\theta'(t))^2}{2} - \cos(\theta(t))$ denote the energy of the pendulum. Show that the energy is conserved, i.e.,

$$\frac{d}{dt} E(t) = 0.$$

(d) Plot the energy computed using the two methods as a function of t. Which method has the least "energy drift"?