Math 317 - Numerical Analysis

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Theorem

For a fixed $n = 1, 2, \ldots$, *let* $f \in f^{n+1}[a, b]$ *and* $\{x_0, \ldots, x_n\}$ *be distinct on* $[a, b]$. *Then for any* $x \in [a, b]$, *there exists* $\xi(x) \in (a, b)$ *such that*

$$f(x) = L_n(x) + \frac{f^{(n+1)}(\xi(x))}{(n+1)!}(x - x_0) \cdots (x - x_n).$$

Moreover,

$$|f(x) - L_n(x)| \leq \frac{M_{n+1}}{(n+1)!} \max_{x \in [a,b]} |(x - x_0) \cdots (x - x_n)|,$$

where $M_{n+1} = \max_{x \in [a,b]} \left| f^{(n+1)}(x) \right|$.

Equally-spaced nodes

In practice, the nodes are often chosen with equal spacing:

$$x_k = a + kh \quad \text{where} \quad \text{for} \quad k = 0, \ldots, n.$$

where $h = \frac{b-a}{n}$.

Theorem (error bound equally-spaced nodes)

For equally-spaced nodes, we have the following error estimate for Lagrange interpolation

$$|f(x) - L_n(x)| \leq M_{n+1} \frac{h^{n+1}}{4(n+1)}.$$

Proof: See Assignment 2.

Runge's phenomenon

Q: As $n \to \infty$ ($h \to 0$), does the error $|f(x) - L_n(x)| \to 0$ for all $x \in [a, b]$?
A: No... consider the function $f(x) = \frac{1}{1+25x^2}$.

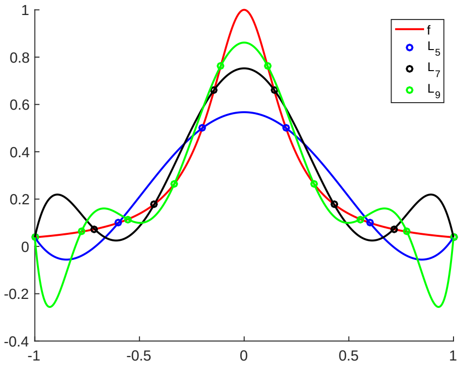

Figure: Lagrange interpolation on equally-spaced nodes.

Runge's phenomenon

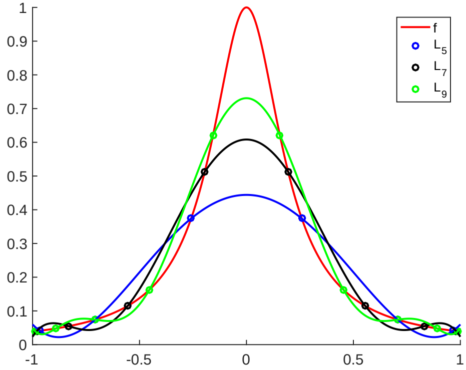

Figure: Chebyshev interpolation.

Runge's phenomenon

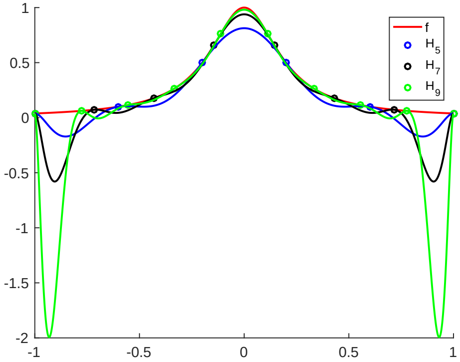

Figure: Hermite interpolation on equally-spaced nodes.

Hermite vs Lagrange

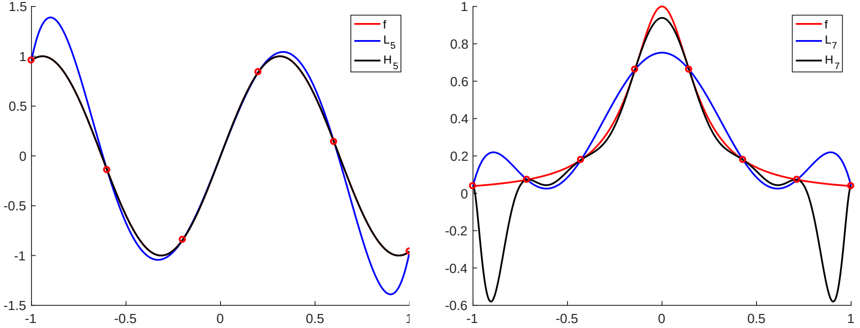

Figure: Comparison between Hermite and Lagrange interpolation on equally space nodes: $f(x) = \sin(x)$ (left); Runge function (right).

Splines vs Lagrange

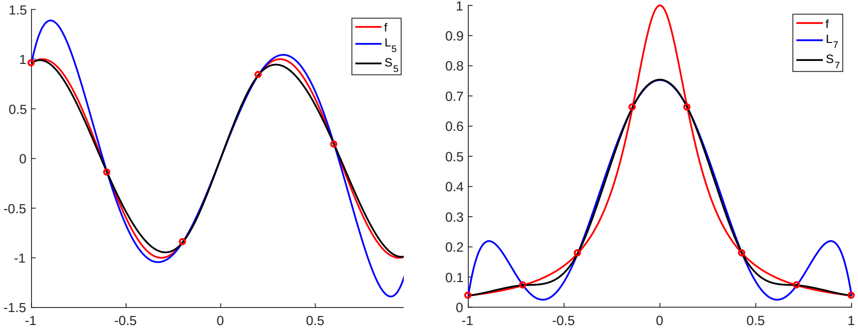

Figure: Comparison between clampled cubic spline and Lagrange interpolation on equally space nodes: $f(x) = \sin(5x)$ (left); Runge function (right).

Natural vs Clamped

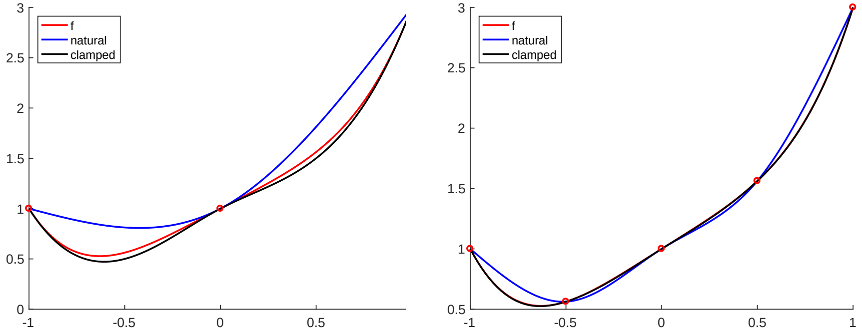

Figure: Comparison between clampled and natural cubic spline for $f(x) = x^4 + x + 1$.