# Math 318: Assignment 6 Solutions

## Problem 1

One possible model for $\sigma$ is as follows. Let $M = \{0, 1, 2\}$ and let $R^M = M^2$, so in other words, $xR^M y$ for any $x, y \in M$. Then given $x \in M$, we can choose $y$ and $z$ with $y \neq x$, $z \neq x$ and $y \neq z$, and we will have $xR^M y$ and $zR^M x$. Thus $M \models \sigma$. $\square$

## Problem 2

There are many possibilities here. We list a few:

- Let $L = \{f\}$ where $f$ is a unary function symbol, and let $\sigma$ be the conjunction of the following sentences:
  - $\forall x \forall y [f(x) = f(y) \to x = y]$ (injective)
  - $\exists x \forall y [x \neq f(y)]$ (not surjective)

  Note $(\mathbb{N}, S)$ is an infinite model of $\sigma$, where $S$ is the successor function. On the other hand, note that if $\mathcal{M} \models \sigma$, then since $f$ is an injection on $M$ which is not a surjection, we must have that $M$ must be infinite. $\square$

- Let $L = \{<\}$ where $<$ is a binary relation symbol, and let $\sigma$ be the conjunction of the following sentences:
  - $\forall x [\neg x < x]$ (irreflexive)
  - $\forall x \forall y \forall z [x < y \wedge y < z \to x < z]$ (transitive)
  - $\forall x \exists y [x < y]$ (unbounded)

  Note that $(\mathbb{N}, <)$ is an infinite model of $\sigma$. On the other hand, if $\mathcal{M} \models \sigma$, then given $x_0 \in M$, we can use unboundedness to construct a sequence:

  $$x_0 < x_1 < x_2 < \cdots$$

  Now if $i < j$, then $x_i < x_j$ by transitivity and thus $x_i \neq x_j$ by irreflexivity. Thus the $x_n$ are distinct, and thus $M$ is infinite. $\square$

## Problem 3

There are many possibilities here. For example, let $L = \varnothing$ and let $\sigma$ being the sentence $\exists x \forall y [x = y]$. Then models of $\sigma$ are exactly the one-element sets. $\square$

## Problem 4

There are many possibilities here.

### Part (a)

Let $L = \varnothing$ and let $\sigma$ be the sentence $\forall x [x = x]$. Let $n \geq 1$ and let $M$ be a set with $|M| = n$ (in particular, we could choose $M = n$). Then $M \models \sigma$, so $n \in \operatorname{spec} \sigma$. Thus $\mathbb{N}_+ \subset \operatorname{spec} \sigma$, and thus $\operatorname{spec} \sigma = \mathbb{N}_+$. $\square$

## Part (b)

Let $L = \{f\}$ where $f$ is a unary function symbol, and let $\sigma$ be the conjunction of the following sentences:

- $\forall x[f(f(x)) = x]$ (involution)
- $\forall x[x \neq f(x)]$ (no fixed points)

Let $n \geq 1$. Define $\mathcal{M}$ by $M = \{1, 2, \ldots, n\} \sqcup \{-1, -2, \ldots, -n\}$ and $f(x) = -x$. Then $\mathcal{M} \models \sigma$, so $2n \in \operatorname{spec} \sigma$. Thus $X \subset \operatorname{spec} \sigma$. Conversely, let $\mathcal{M}$ be a finite model of $\sigma$. $f$ partitions $M$ into sets of the form $\{x, f(x)\}$ which are of size two since $f$ has no fixed points. Thus $M$ has even order, so $|M| \subset X$. Thus $\operatorname{spec} \sigma \subset X$ and thus $\operatorname{spec} \sigma = X$. $\square$

## Part (c)

Let $L = \{\vee, \wedge, \neg, 0, 1\}$ be the language of Boolean algebras, and let $\sigma$ be the conjunction of the sentences in the theory of Boolean algebras. Let $n \in \mathbb{N}$ and let $M$ be a set with $|M| = n$ (in particular, we could choose $M = n$). Then $\mathcal{P}(M) \models \sigma$, so $2^n \in \operatorname{spec} \sigma$. Thus $X \subset \operatorname{spec} \sigma$. Conversely, let $\mathcal{M}$ be a finite model of $\sigma$. Since $\mathcal{M}$ is a finite Boolean algebra, its order must be a power of two, so we have $|M| \subset X$. Thus $\operatorname{spec} \sigma \subset X$ and thus $\operatorname{spec} \sigma = X$. $\square$

# Problem 5

Let $L = \{E\}$ be the language of graphs and let $T$ be the theory of graphs. Suppose we have $T' \supset T$ such that every connected graph satisfies $T'$. Let $L' = L \cup \{a, b\}$ where $a$ and $b$ are constant symbols. For $n \geq 0$, let $\sigma_n$ be the $L'$-sentence saying that there is no path of length $n$ from $a$ to $b$. More precisely, define $\sigma_n$ as follows:

- Let $\sigma_0$ be the sentence $a \neq b$.
- Let $\sigma_1$ be the sentence $\neg(aEb)$.
- For $n \geq 2$, let $\sigma_n$ be the following sentence:

$$\forall x_1, \ldots, x_{n-1} \neg(aEx_1 \wedge x_1Ex_2 \wedge \cdots \wedge x_{n-2}Ex_{n-1} \wedge x_{n-1}Eb)$$

Define the set of $L'$-sentences $\tilde{T} = T' \cup \{\sigma_n : n \geq 0\}$. We claim that $\tilde{T}$ is satisfiable. Let $F \subset \tilde{T}$ be finite. Then $F \subset T' \cup \{\sigma_n : n < N\}$ for some $N$. Consider the $L'$-structure $\mathcal{M}$ where $M = \{0, 1, \ldots, N\}$ and $xEy$ iff $|x - y| = 1$, and $a = 0$ and $b = N$. Now $\mathcal{M}$ is a connected graph, so $\mathcal{M} \models T'$. Also, note that $a$ and $b$ are of distance $N > n$ in $\mathcal{M}$, so $\mathcal{M} \models \sigma_n$ for $n < N$. Thus $\mathcal{M} \models T' \cup \{\sigma_n : n < N\}$, and thus $\mathcal{M} \models F$. Thus every finite subset of $\tilde{T}$ is satisfiable so by the compactness theorem, $\tilde{T}$ is satisfiable. Let $\mathcal{M}$ be a model of $\tilde{T}$. Then $\mathcal{M} \models \{\sigma_n : n \geq 0\}$, so there is no path of any length from $a$ to $b$, so $\mathcal{M}$ is disconnected. But $\mathcal{M} \models T'$ as well. Thus any theory satisfied by every connected graph is also satisfied by a disconnected graph, and thus there cannot be a theory of connected graphs. $\square$

# Problem 6

Let $L = \{<\}$ be the language of orders and let $T$ be the theory of linear orders. Suppose we have $T' \supset T$ such that every well-order models $T'$. Let $L' = L \cup \{c_n : n \geq 0\}$ where each $c_n$ is a constant symbol. Let $\sigma_n$ be the $L'$-sentence $c_{n+1} < c_n$. Define the set of $L'$-sentences $\tilde{T} = T' \cup \{\sigma_n : n \geq 0\}$. Let $F \subset \tilde{T}$ be finite. Then $F \subset T' \cup \{\sigma_n : n < N\}$ for some $N$. Consider the $L'$-structure $\mathcal{M}$ where $M = \{-N, -N+1, \ldots, 1, 0\}$ with the usual order, and $c_n = -n$ for $n \leq N$, and $c_n = 0$ for $n > N$. Now $\mathcal{M}$ is a well-order, so $\mathcal{M} \models T'$. Also, if $n < N$, then $c_{n+1} = -n - 1 < -n = c_n$, so $\mathcal{M} \models \sigma_n$. Thus $\mathcal{M} \models T' \cup \{\sigma_n : n < N\}$, and thus $\mathcal{M} \models F$. Thus every finite subset of $\tilde{T}$ is satisfiable so by the compactness theorem, $\tilde{T}$ is satisfiable. Let $\mathcal{M}$ be a model of $\tilde{T}$. Note that $\{c_n : n \geq 0\} \subset M$ has no lower bound, since for any $c_n$, we have $c_{n+1} < c_n$. Thus $\mathcal{M}$ is not a well-order. But $\mathcal{M} \models T'$ as well. Thus any theory satisfied by every well-order is also satisfied by a non-well-order, and thus there cannot be a theory of well-orders. $\square$