COMP251: Randomized Algorithms

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Based on (Kleinberg & *Tardos*, 2006)

Algorithm Design Techniques

- Greedy Algorithms
- Dynamic Programming
- Divide-and-Conquer
- Network Flows
- Randomization

Randomization

Principle: Allow fait coin flip in unit time.

Why? Can lead to simplest, fastest, or only known algorithm for a particular problem.

Examples:
- Quicksort
- Graph Algorithms
- Hashing
- Monte-Carlo integration
- Cryptography

Global Min Cut

Definition: Given a connected, undirected graph G=(V,E), find a cut with minimum cardinality.

Applications:
- Partionning items in database
- Identify clusters of related documents
- Network reliability
- TSP solver

Network solution:
- Replace every edge (u,v) with 2 antiparallel edges (u,v) & (v,u)
- Pick some vertex s, and compute min s-v cut for each other vertex v.

False Intuition: Global min-cut is harder that min s-t cut!

Contraction algorithm

Contraction algorithm. [Karger 1995]

- Pick an edge $e = (u, v)$ uniformly at random.
- Contract edge e.
  - replace u and v by single new super-node w
  - preserve edges, updating endpoints of u and v to w
  - keep parallel edges, but delete self-loops
- Repeat until graph has just two nodes u_1 and v_1.
- Return the cut (all nodes that were contracted to form v_1).

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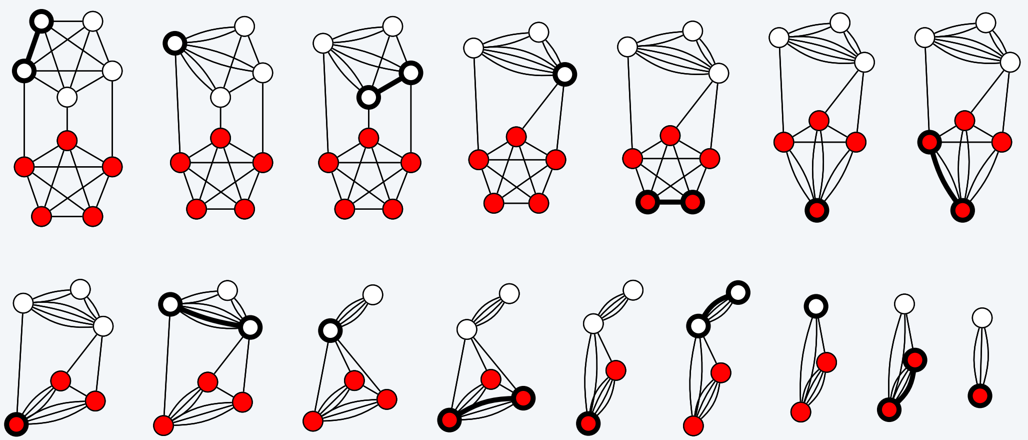


Reference: Thore Husfeldt

Contraction Algorithm

Randomization

Contraction(V,E):

While $|V| > 2$ do

Choose $e \in E$ uniformly at random

$G \leftarrow G - \{e\}$ // contract G

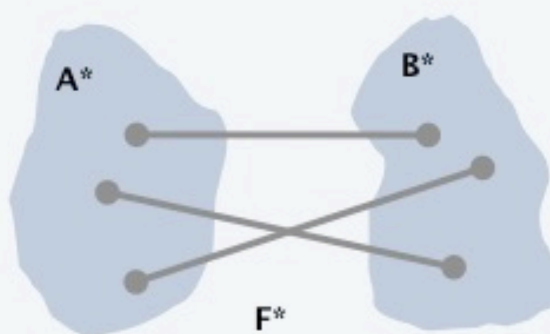
return { the only cut in G }

Contraction algorithm

Claim. The contraction algorithm returns a min cut with prob $\geq 2 / n^2$.
$(n = |V|)$

Pf. Consider a global min-cut (A^*, B^*) of G.

- Let F^* be edges with one endpoint in A^* and the other in B^*.
- Let $k = |F^*|$ = size of min cut.
- In first step, algorithm contracts an edge in F^* probability $k / |E|$.
- Every node has degree $\geq k$ since otherwise (A^*, B^*) would not be a min-cut $\Rightarrow |E| \geq \frac{1}{2} k n$. $\Leftrightarrow \frac{k}{|E|} \leq \frac{2}{n}$
- Thus, algorithm contracts an edge in F^* with probability $\leq 2 / n$.



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- Let $k = |F^*|$ = size of min cut.
- Let G' be graph after j iterations. There are $n' = n - j$ supernodes.
- Suppose no edge in F^* has been contracted. The min-cut in G' is still k.
- Since value of min-cut is k, $|E'| \geq \frac{1}{2} k n'$.
- Thus, algorithm contracts an edge in F^* with probability $\leq 2 / n'$.
- Let E_j = event that an edge in F^* is not contracted in iteration j.

$$
\begin{aligned}
\Pr[E_1 \cap E_2 \cdots \cap E_{n-2}] &= \Pr[E_1] \times \Pr[E_2 \mid E_1] \times \cdots \times \Pr[E_{n-2} \mid E_1 \cap E_2 \cdots \cap E_{n-3}] \\
&\geq \left(1 - \tfrac{2}{n}\right)\left(1 - \tfrac{2}{n-1}\right) \cdots \left(1 - \tfrac{2}{4}\right)\left(1 - \tfrac{2}{3}\right) \\
&= \left(\frac{n-2}{n}\right)\left(\frac{n-3}{n-1}\right) \cdots \left(\frac{2}{4}\right)\left(\frac{1}{3}\right) \\
&= \frac{2}{n(n-1)} \\
&\geq \frac{2}{n^2}
\end{aligned}
$$

Contraction algorithm

Amplification. To amplify the probability of success, run the contraction algorithm many times.

with independent random choices,

Claim. If we repeat the contraction algorithm $n^2 \ln n$ times, then the probability of failing to find the global min-cut is $\leq 1 / n^2$.

Pf. By independence, the probability of failure is at most

$$\left(1 - \frac{2}{n^2}\right)^{n^2 \ln n} = \left[\left(1 - \frac{2}{n^2}\right)^{\frac{1}{2}n^2}\right]^{2 \ln n} \leq \left(e^{-1}\right)^{2 \ln n} = \frac{1}{n^2}$$

$(1 - 1/x)^x \leq 1/e$

Contraction algorithm: example execution

trial 1

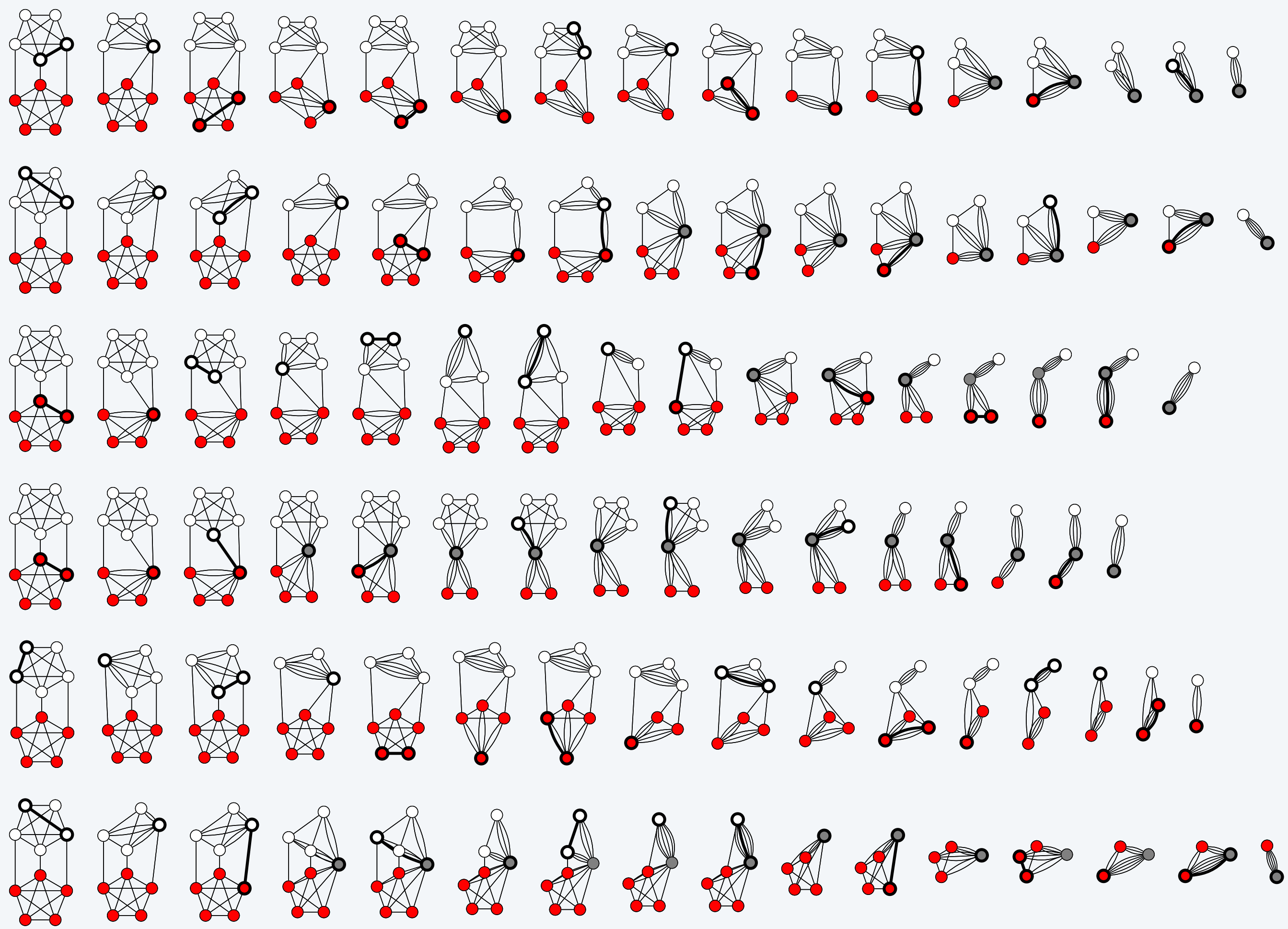
trial 2

trial 3

trial 4

trial 5
(finds min cut)

trial 6



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Global min cut: context

Remark. Overall running time is slow since we perform $\Theta(n^2 \log n)$ iterations and each takes $\Omega(m)$ time. Where $m = |E|$. Overall complexity $O(n^2 m \log n)$

Improvement. [Karger-Stein 1996] $O(n^2 \log^3 n)$.

- Early iterations are less risky than later ones: probability of contracting an edge in min cut hits 50% when $n / \sqrt{2}$ nodes remain.
- Run contraction algorithm until $n / \sqrt{2}$ nodes remain.
- Run contraction algorithm twice on resulting graph and return best of two cuts.

Extensions. Naturally generalizes to handle positive weights.

Best known. [Karger 2000] $O(m \log^3 n)$.

faster than best known max flow algorithm or deterministic global min cut algorithm

Maximum 3-satisfiability

exactly 3 distinct literals per clause

Maximum 3-satisfiability. Given a 3-SAT formula, find a truth assignment that satisfies as many clauses as possible.

$$
\begin{array}{lcl}
C_1 & = & x_2 \vee \overline{x_3} \vee \overline{x_4} \\
C_2 & = & x_2 \vee x_3 \vee \overline{x_4} \\
C_3 & = & \overline{x_1} \vee x_2 \vee x_4 \\
C_4 & = & \overline{x_1} \vee \overline{x_2} \vee x_3 \\
C_5 & = & x_1 \vee \overline{x_2} \vee \overline{x_4}
\end{array}
$$

Remark. **NP**-hard search problem.

Simple idea. Flip a coin, and set each variable true with probability ½, independently for each variable.

Maximum 3-satisfiability: analysis

Claim. Given a 3-SAT formula with k clauses, the expected number of clauses satisfied by a random assignment is $7k / 8$.

Pf. Consider random variable $Z_j = \begin{cases} 1 & \text{if clause } C_j \text{ is satisfied} \\ 0 & \text{otherwise.} \end{cases}$

- Let Z = weight of clauses satisfied by assignment Z_j.

$$
\begin{aligned}
E[Z] &= \sum_{j=1}^{k} E[Z_j] \\
&= \sum_{j=1}^{k} \Pr[\text{clause } C_j \text{ is satisfied}] \\
&= \tfrac{7}{8} k
\end{aligned}
$$

linearity of expectation

The Probabilistic Method

Corollary. For any instance of 3-SAT, there exists a truth assignment that satisfies at least a $7/8$ fraction of all clauses.

Pf. Random variable is at least its expectation some of the time. ▪

Probabilistic method. [Paul Erdös] Prove the existence of a non-obvious property by showing that a random construction produces it with positive probability!

Maximum 3-satisfiability: analysis

Q. Can we turn this idea into a $7/8$-approximation algorithm?

A. Yes (but a random variable can almost always be below its mean).

Lemma. The probability that a random assignment satisfies $\geq 7k / 8$ clauses is at least $1 / (8k)$.

Pf. Let p_j be probability that exactly j clauses are satisfied; let p be probability that $\geq 7k / 8$ clauses are satisfied.

$$\begin{aligned}
\tfrac{7}{8}k \;=\; E[Z] &= \sum_{j \geq 0} j\, p_j \\
&= \sum_{j < 7k/8} j\, p_j \;+\; \sum_{j \geq 7k/8} j\, p_j \\
&\leq \left(\tfrac{7k}{8} - \tfrac{1}{8}\right) \sum_{j < 7k/8} p_j \;+\; k \sum_{j \geq 7k/8} p_j \\
&\leq \left(\tfrac{7}{8}k - \tfrac{1}{8}\right) \cdot 1 \;+\; k\, p
\end{aligned}$$

Rearranging terms yields $p \geq 1 / (8k)$. ▪

Maximum 3-satisfiability: analysis

Johnson's algorithm. Repeatedly generate random truth assignments until one of them satisfies $\geq 7k / 8$ clauses.

Theorem. Johnson's algorithm is a $7/8$-approximation algorithm.

Pf. By previous lemma, each iteration succeeds with probability $\geq 1 / (8k)$. By the waiting-time bound, the expected number of trials to find the satisfying assignment is at most $8k$. ▪

Monte Carlo vs. Las Vegas algorithms

Monte Carlo. Guaranteed to run in poly-time, likely to find correct answer.
Ex: Contraction algorithm for global min cut.

Las Vegas. Guaranteed to find correct answer, likely to run in poly-time.
Ex: Randomized quicksort, Johnson's MAX-3-SAT algorithm.

stop algorithm
after a certain point

Remark. Can always convert a Las Vegas algorithm into Monte Carlo,
but no known method (in general) to convert the other way.