Math 318: Assignment 2 Solutions

Problem 1

Part (1)

Let $y \in f(f^{-1}(B))$. Then there is some $x \in f^{-1}(B)$ with $y = f(x)$. Since $x \in f^{-1}(B)$, we have $y = f(x) \in B$. Thus $f(f^{-1}(B)) \subset B$. *
On the other hand, let $y \in B$. Since f is surjective, there is some $x \in X$ with $f(x) = y$. Now $f(x) = y \in B$, so $x \in f^{-1}(B)$, and thus $y = f(x) \in f(f^{-1}(B))$. Thus $B \subset f(f^{-1}(B))$.
Thus $B = f(f^{-1}(B))$. $\square$

Part (2)

This is not always true.
For example, take $X = \{0, 1\}$, $Y = \{0\}$, $f : X \to Y$ defined by $f(x) = 0$, and $A = \{0\}$. Then $f(A) = \{0\}$, and $f^{-1}(f(A)) = f^{-1}(\{0\}) = \{0, 1\} \neq A$. † $\square$

Part (3)

Let $y \in f(A \cap f{-}1(B))$. Then there is some $x \in A \cap f^{-1}(B)$ with $y = f(x)$. Since $x \in A$, we have $y = f(x) \in f(A)$. Also since $x \in f^{-1}(B)$, we have $y = f(x) \in B$. Thus $y \in f(A) \cap B$. Thus $f(A \cap f^{-1}(B)) \subset f(A) \cap B$.
On the other hand, let $y \in f(A) \cap B$. Since $y \in f(A)$, there is some $x \in A$ with $f(x) = y$. Now $f(x) = y \in B$, so $x \in f^{-1}(B)$. Thus $x \in A \cap f^{-1}(B)$, and so $f(x) \in f(A \cap f^{-1}(B))$. Thus $f(A) \cap B \subset f(A \cap f^{-1}(B))$.
Thus $f(A \cap f^{-1}(B)) = f(A) \cap B$. $\square$

Problem 2

Note that a function from X to Y is determined by choosing an element of Y for each element of X.

Part (1)

$3^2 = 9$.

Part (2)

$1^5 = 1$.

Part (3)

$0^5 = 0$.

Part (4)

$5^0 = 1$.

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*Note that this does not depend on surjectivity.
†In general, $f^{-1}(f(A)) = A$ iff A is the preimage under f of a subset of Y.

Problem 3

Let $A \preceq B$ denote that there exists an injection from A to B. Note that if $A \subset B$, then $A \preceq B$.

Part (a)

No, these sets are not equinumerous.
We know $[0,1) \sim \mathbb{R}$ and $\mathbb{Q} \sim \mathbb{N}$, but we know that $\mathbb{R} \nsim \mathbb{N}$, so we must have $[0,1) \nsim \mathbb{Q}$. $\square$

Part (b)

Yes, these sets are equinumerous.
We have:

$$[0,\infty) \preceq \mathbb{R} \sim [0,1] \preceq [0,\infty)$$

Thus by Cantor-Schröder-Bernstein, we have $\mathbb{R} \sim [0,\infty)$. Thus:

$$[0,1]^{\mathbb{N}} \sim \mathbb{R}^{\mathbb{N}} \sim \mathbb{R} \sim [0,\infty)$$

$\square$

Part (c)

By our work in Part (b), we have:

$$[0,1]^{\mathbb{N}} \sim \mathbb{R} \sim [0,1] \sim 2^{\mathbb{N}} \preceq \mathbb{Q}^{\mathbb{N}} \preceq [0,1]^{\mathbb{N}}$$

Thus by Cantor-Schröder-Bernstein, we have $[0,1]^{\mathbb{N}} \sim \mathbb{Q}^{\mathbb{N}}$. $\square$

Problem 4

Part (A)

No, it is not countable.
We have $\mathbb{Z}^{\mathbb{N}} \succ 2^{\mathbb{N}}$ and $2^{\mathbb{N}}$ is uncountable, so $\mathbb{Z}^{\mathbb{N}}$ is uncountable. $\square$

Part (B)

Yes, it is countable.
$\mathbb{Z}^3$ and $\mathbb{Z}^7$ are countable since they are finite products of countable sets. Thus $\mathbb{Z}^3 \cup \mathbb{Z}^7$ is countable since it is a union of two countable sets. $\square$

Part (C)

Yes, it is countable.
As in Part (B), for any $n \in \mathbb{N}$, $\mathbb{Z}^n$ is countable since it is a finite product of countable sets. Thus $\bigcup_{n\in\mathbb{N}} \mathbb{Z}^n$ is countable since it is a countable union of countable sets. $\square$

Part (D)

No, it is not countable.
We have $\mathbb{R} \times \mathbb{Q} \succ \mathbb{R} \times \{0\} \sim \mathbb{R}$ which is uncountable, so $\mathbb{R} \times \mathbb{Q}$ is uncountable. $\square$

Problem 5

We prove by induction on $n \geq 0$ that a bijection on an n-element set is a composition of disjoint cycles.
The $n = 0$ case is true since the only bijection on the empty set is vacuously a cycle.
Now let $n > 0$ and suppose that the statement is true for $k < n$. Let $x_0 \in X$. Since X is finite, there must be some $p \leq q$ with $f^p(x_0) = f^q(x_0)$. Let $N \geq 1$ be minimal such that there exists $p \geq 0$ with $f^p(x_0) = f^{p+N}(x_0)$. Since f is a bijection, we have that $x_0 = f^N(x_0)$. Also by our choice of N, for $0 \leq k < l < N$, we have $f^k(x_0) \neq f^l(x_0)$. Let $Y = \{x_0, f(x_0), \dots, f^{N-1}(x_0)\}$. Then f restricts to a cycle on Y and f restricts to a function on $X \setminus Y$, which is a composition of disjoint cycles by the inductive hypothesis. Thus f is also a composition of disjoint cycles. $\square$

Problem 6

First of all, consider the special case where $X = \{f^n(x_0) : n \in \mathbb{Z}\}$ for some $x_0 \in X$. We will show that f is the composition of two involutions. Define g on X by $g(f^n(x_0)) = f^{-n}(x_0)$. Note that g well-defined since if $f^n(x_0) = f^m(x_0)$, then applying f^{-n-m} to both sides gives $f^{-m}(x_0) = f^{-n}(x_0)$. Also g is an involution since $g^2(f^n(x_0)) = g(f^{-n}(x_0)) = f^n(x_0)$. Similarly, if we define h on X by $h(f^n(x_0)) = f^{-n-1}(x_0)$, then h is a well-defined involution on X. Now $(g \circ h)(f^n(x_0)) = g(f^{-n-1}(x_0)) = f^{n+1}(x_0) = f(f^n(x_0))$, so $g \circ h = f$.
Now for the general case, consider the equivalence relation $\sim$ on X defined by $x \sim y$ iff for some $n \in \mathbb{Z}$, we have $f^n(x) = y$ (this is an equivalence relation since f is a bijection).
For each equivalence class C of $\sim$, note that we have $C = \{f^n(x_0) : n \in \mathbb{Z}\}$ for some $x_0 \in C$, so by above, we have that $f|_C = g_C \circ h_C$ for some involutions g_C and h_C on C. [‡] Define g and h on X by $g(x) = g_C(x)$ and $h(x) = h_C(x)$ when $x \in C$. Then g and h are involutions with $f = g \circ h$. $\square$

[‡]Note that this step requires the Axiom of Choice.