Math 317 Assignment 1

Due in class: September 29, 2016

Instructions: Submit a hard copy of your solution with your name and student number. (**No name = zero grade!**) You must include all relevant program code, electronic output and explanations of your results. Write your own codes and comment them. Late assignment will not be graded and will receive a grade of zero.

1. Let a be a nonzero real number and $fl(a)$ be a k-digit rounding approximation to a is base 10. Show that
$$\left|\frac{a - fl(a)}{a}\right| \leq \frac{1}{2}10^{1-k}.$$

2. (a) Find $P_9(x)$ about $x_0 = 0$ of
$$f(x) = \frac{\sin(x^3) + e^{x^2} - 1}{x^2}.$$
   (b) Let $p \geq 0$. For $n = 1, 2, \ldots$ find $P_n(x)$ about $x_0 = 0$ of $f(x) = (1 + x)^p$ and find an upper bound depending on n, p for the error on $x \in [0, 1]$.

3. Consider the following function $f(x) = x^3 - 2x - 5$.
   (a) Show that f has one unique root x^* with $x^* \in [2, 3]$.
   (b) Estimate the number of iterations for the bisection method to reach an accuracy of 10^{-10} and compare with the actual number of iterations.
   (c) Consider the fixed point iteration $x_{n+1} = g(x_n)$ with $x_0 \in [2, 3]$, where $g(x) = (5 + 2x)^{\frac{1}{3}}$. Prove that x_n converges to x^*.
   (d) For $x_0 = 2.5$, compute the number of iterations needed to reach an accuracy of 10^{-10} for
      i. the fixed point method in part (c),
      ii. Newton's method,
      iii. secant method (use $x_1 = 3$).
   (e) Rank all four methods by how fast they converge.

   Hint: to compute the actual number of iterations for each method, you may use the fact that $x^* = 2.09455148154233\ldots$

4. Consider the iteration $x_{n+1} = g(x_n)$ where
$$g(x) = \frac{\lambda x - \log(x) - 2(x + 1)}{\lambda - 2}$$
   with $\lambda \neq 2$.
   (a) Compute the fixed point x^* of g.
   (b) Determine the values of λ for which $|g'(x^*)| < 1$. Why are these values important?
   (c) Determine the value of λ such that x_n converges as fast as possible to x^*.

5. Assume that $g \in C^p[a, b]$ has a fixed point x^* and that the fixed point iteration of g converges. Show that if $g^{(i)}(x^*) = 0$ for all $i = 1, \ldots, p - 1$ and $g^{(p)}(x^*) \neq 0$, then the fixed point iteration converges at order p. State the asymptotic error constant in this case. *Hint: Expand g using Taylor's polynomial around $x_0 = x^*$.)*