COMP251: Amortized Analysis

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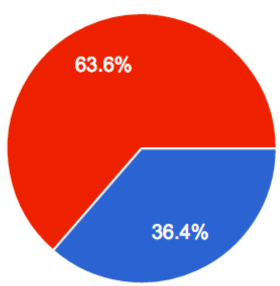
Based on (Cormen *et al.*, 2009)

$$T(n) = 2 \cdot T\left(\frac{n}{5}\right) + n^3$$

What is the height of the recursion tree?

- $\log_3 n$ ✗
- $\log_5 n$ ✓
- $\log_2 n$ ✗



log_3 (n)	**4**	36.4%
log_5 (n)	**7**	63.6%
log_2 (n)	**0**	0%

$$T(n) = 3 \cdot T\left(\frac{n}{4}\right) + n \log n$$

$$a = 3; b = 4$$
$$k = \log_4 3$$
$$f(n) = n \log n$$

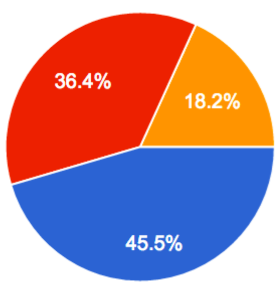
(A)

$$\boldsymbol{f(n) = \Omega\left(n^{\log_4 3 + (1 - \log_4 3)}\right)}$$

(B)

$$\boldsymbol{3 \cdot \left(\frac{n}{4} \cdot \log \frac{n}{4}\right) \le \frac{3}{4} \cdot n \log n}$$

- $\Theta(n(\log n)^2)$ ✗
- $\Theta(n \log n)$ ✓ (case 3)
- $\Theta(n \log_4 3)$ ✗
- Not applicable ✗

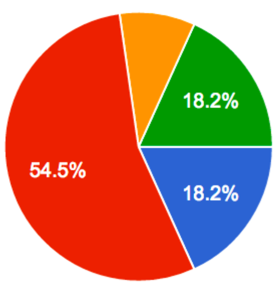


\Tetha (n * log^2 n)	5	45.5%
\Tetha (n * log n)	4	36.4%
\Tetha (n^log_4(3))	2	18.2%
The master theorem cannot be applied	0	0%

$$T(n) = 4 \cdot T\left(\frac{n}{2}\right) + \log n$$

$a = 4; b = 2$
$k = \log_2 4 = 2$
$f(n) = \log n$
$\boldsymbol{f(n) = O(n^{2-1})}$

- $\Theta(\log n)$ ✗
- $\Theta(n^2)$ ✓ (case 1)
- $\Theta((\log n)^2)$ ✗
- Not applicable ✗



\Tetha (log n)	**2**	18.2%
\Tetha (n^2)	**6**	54.5%
\Theta (log^2 n)	**1**	9.1%
The master theorem cannot be applied	**2**	18.2%

Overview

- Analyze a sequence of operations on a data structure.
- We will talk about average cost in the worst case (i.e. not averaging over a distribution of inputs. No probability!)
- **Goal:** Show that although some individual operations may be expensive, on average the cost per operation is small.
- 3 methods:
  1. aggregate analysis
  2. accounting method
  3. potential method

Aggregate analysis

Stack operations

- PUSH(S, x): $O(1)$ each $\Rightarrow$ $O(n)$ for any sequence of n operations.
- POP(S): $O(1)$ each $\Rightarrow$ $O(n)$ for any sequence of n operations.
- MULTIPOP(S,k):

```
while S≠Ø and k>0 do
    POP(S)
    k⟵k−1
```

Running time of MULTIPOP?

Running time of MULTIPOP

- Linear in # of POP operations.
- Let each PUSH/POP cost 1.
- # of iterations of **while** loop is min(s, k), where s = # of objects on stack. Therefore, total cost = min(s, k).

Sequence of n PUSH, POP, MULTIPOP operations:

- Worst-case cost of MULTIPOP is $O(n)$.
- Have n operations.
- Therefore, worst-case cost of sequence is $O(n^2)$.

But:

- Each object can be popped only once per time that it is pushed.
- Have $\le n$ PUSHes $\Rightarrow$ $\le n$ POPs, including those in MULTIPOP.
- Therefore, total cost = $O(n)$.
- Average over the n operations $\Rightarrow$ $O(1)$ per operation on average.

Binary counter

- k-bit binary counter $A[0 \,..\, k-1]$ of bits, where $A[0]$ is the least significant bit and $A[k-1]$ is the most significant bit.
- Counts upward from 0.
- Value of counter is: $$\sum_{i=0}^{k-1} A[i] \cdot 2^i$$
- Initially, counter value is 0, so $A[0 \,..\, k-1] = 0$.
- To increment, add 1 (mod 2k):

```
Increment(A,k):
i←0
while i<k and A[i]=1 do
   A[i]←0
   i←i+1
if i < k then
   A[i] ← 1
```

Example (1)

k=3

Counter Value	A 2 1 0	cost
0	0 0 <u>0</u>	0
1	0 <u>0 1</u>	1
2	0 1 <u>0</u>	3
3	<u>0 1 1</u>	4
4	1 0 <u>0</u>	7
5	1 <u>0 1</u>	8
6	1 1 <u>0</u>	10
7	<u>1 1 1</u>	11
0	0 0 0	14

Cost of INCREMENT = Θ(# of bits flipped)

Analysis: Each call could flip *k* bits,
so *n* INCREMENTs takes $O(nk)$ time.

Example (2)

Bit	Flips how often	Time in n INCREMENTs
0	Every time	n
1	½ of the time	floor(n/2)
2	¼ of the time	floor(n/4)
…	…	
i	$1/2^i$ of the time	floor($n/2^i$)
…	…	
i≥k	Never	0

Thus, total # flips = $$\sum_{i=0}^{k-1} \lfloor n/2^i \rfloor < n \cdot \sum_{i=0}^{\infty} 1/2^i = n\left(\frac{1}{1-1/2}\right) = 2 \cdot n$$

Therefore, n INCREMENTs costs $O(n)$.
Average cost per operation = $O(1)$.

Accounting method

Assign different charges to different operations.

- Some are charged more than actual cost.
- Some are charged less.

Amortized cost = amount we charge.

When amortized cost > actual cost, store the difference *on specific objects* in the data structure as ***credit***.
Use credit later to pay for operations whose actual cost > amortized cost.

Differs from aggregate analysis:

- In the accounting method, different operations can have different costs.
- In aggregate analysis, all operations have same cost.

But we need to guarantee that the credit never goes negative.

Definition

Let c_i = cost of actual i^{th} operation.

$\hat{c}_i$ = amortized cost of i^{th} operation.

Then require $\sum_{i=1}^{n} \hat{c}_i \geq \sum_{i=1}^{n} c_i$ for all sequences of n operations.

Total credit stored = $\sum_{i=1}^{n} \hat{c}_i - \sum_{i=1}^{n} c_i \geq 0$

Stack

Operation	Actual cost	Amortized cost
PUSH	1	2
POP	1	0
MULTIPOP	min(k,s)	0

Intuition: When pushing an object, pay $2.

- $1 pays for the PUSH.
- $1 is prepayment for it being popped by either POP or MULTIPOP.
- Since each object has $1, which is credit, the credit can never go negative.
- Total amortized cost (= $O(n)$) is an upper bound on total actual cost.

Binary counter

Charge \$2 to set a bit to 1.

- \$1 pays for setting a bit to 1.
- \$1 is prepayment for flipping it back to 0.
- Have \$1 of credit for every 1 in the counter.
- Therefore, credit ≥ 0.

Amortized cost of INCREMENT:

- Cost of resetting bits to 0 is paid by credit.
- At most 1 bit is set to 1.
- Therefore, amortized cost ≤ \$2.
- For n operations, amortized cost = $O(n)$.

Dynamic tables

Scenario

- Have a table - maybe a hash table.
- Don't know in advance how many objects will be stored in it.
- When it fills, must reallocate with a larger size, copying all objects into the new, larger table.
- When it gets sufficiently small, *might* want to reallocate with a smaller size.

Goals

1. $O(1)$ amortized time per operation.
2. Unused space always $\leq$ constant fraction of allocated space.

Load factor α = (# items stored) / (allocated size)

Never allow $\alpha > 1$; Keep $\alpha >$ a constant fraction $\Rightarrow$ Goal 2.

Table expansion

Consider only insertion.

- When the table becomes full, double its size and reinsert all existing items.
- Guarantees that $\alpha \geq ½$.
- Each time we insert an item into the table, it is an ***elementary insertion***.

```
TABLE-INSERT(T,x)
if size[T ]=0
   then allocate table[T] with 1 slot
      size[T]←1
if num[T]=size[T] then
   allocate new-table with 2 · size[T] slots
   insert all items in table[T] into new-table
   free table[T]
   table[T]←new-table
   size[T]←2·size[T]
insert x into table[T]
num[T]←num[T] + 1
```

(Initially, $num[T]=size[T]=0$)

Aggregate analysis

- Charge 1 per elementary insertion.
- Count only elementary insertions (other costs = constant).

c_i = actual cost of i^{th} operation

- If not full, $c_i = 1$.
- If full, have $i-1$ items in the table at the start of the i^{th} operation. Have to copy all $i-1$ existing items, then insert i^{th} item $\Rightarrow c_i = i$.

n operations $\Rightarrow c_i = O(n) \Rightarrow O(n^2)$ time for n operations

$$c_i = \begin{cases} i & \text{if } i-1 \text{ is power of } 2 \\ 1 & \text{Otherwise} \end{cases}$$

Total cost = $\displaystyle\sum_{i=1}^{n} c_i \le n + \sum_{j=0}^{\lfloor \log n \rfloor} 2^j = n + \frac{2^{\lfloor \log n \rfloor + 1} - 1}{2-1} < n + 2n = 3n$

Amortized cost per operation = 3.

Accounting method

Charge \$3 per insertion of *x*.

- \$1 pays for *x*'s insertion.
- \$1 pays for *x* to be moved in the future.
- \$1 pays for some other item to be moved.

Suppose we've just expanded, *size*=*m* before next expansion, *size*=2*m* after next expansion.

- Assume that the expansion used up all the credit, so that there's no credit stored after the expansion.
- Will expand again after another *m* insertions.
- Each insertion will put \$1 on one of the *m* items that were in the table just after expansion and will put \$1 on the item inserted.
- Have \$2*m* of credit by next expansion, when there are 2*m* items to move. Just enough to pay for the expansion…