Math 318: Assignment 5 Solutions

Problem 1

Let s be the truth assignment $s(p) = 1$, $s(q) = 1$, $s(r) = 0$. Then we have $\tilde{s}(p \wedge q) = 1$, $\tilde{s}((\neg p) \vee q) = 1$ and $\tilde{s}(p \vee r) = 1$. However, since $\tilde{s}(r) = 0$ as well, we have $\Gamma \not\models r$. Thus by the soundness theorem, we have $\Gamma \nvdash r$. $\square$

Problem 2

Since $(\neg p) \wedge q = \neg(p \vee \neg q) = \neg(q \to p)$, we have that for any truth assignment s, $\tilde{s}(q \to p) \neq \tilde{s}((\neg p) \wedge q)$. Thus Γ is not satisfiable, so by the completeness theorem, Γ is inconsistent. Thus Γ proves anything, and in particular $\Gamma \vdash p$. $\square$

Formal deduction preamble

Recall from class that we have the following lemmas:

Lemma 1. *$\vdash \phi \to \phi$ for every ϕ.*

Lemma 2. *$\{\phi\} \vdash \psi \to \phi$ for every ϕ and ψ.*

We will also use the following:

Lemma 3. *If $\Gamma \vdash \phi \to (\psi \to \chi)$, then $\Gamma \vdash (\phi \to \psi) \to (\phi \to \chi)$.*

Proof. A formal proof from Γ is as follows:

(1)	$(\phi \to (\psi \to \chi)) \to ((\phi \to \psi) \to (\phi \to \chi))$	(A2)
(2)	$\phi \to (\psi \to \chi)$	(given)
(3)	$(\phi \to \psi) \to (\phi \to \chi)$	(MP with (1) and (2))

$\square$

Below, when we reference a lemma or a previous problem, we really are inserting the formal proof.

Problem 3

A formal proof is as follows:

(1)	$((\neg p) \to (\neg q)) \to (q \to p)$	(A3)
(2)	$(((\neg p) \to (\neg q)) \to q) \to (((\neg p) \to (\neg q)) \to p)$	(Lemma 3 with (1))

Problem 4

Part (a)

Let $\Gamma = \{\alpha \to \beta, \beta \to \gamma\}$. A formal proof from Γ is as follows:

(1)	$\alpha \to (\beta \to \gamma)$	(Lemma 2 with $\beta \to \gamma \in \Gamma$)
(2)	$(\alpha \to \beta) \to (\alpha \to \gamma)$	(Lemma 3 with (1))
(3)	$\alpha \to \beta$	($\alpha \to \beta \in \Gamma$)
(4)	$\alpha \to \gamma$	(MP with (2) and (3))

Part (b)

Let $\Gamma = \{\alpha \to \beta\}$. A formal proof from Γ is as follows:

(1)	$(\beta \to \gamma) \to (\alpha \to (\beta \to \gamma))$	(A1)
(2)	$(\alpha \to (\beta \to \gamma)) \to ((\alpha \to \beta) \to (\alpha \to \gamma))$	(A2)
(3)	$(\beta \to \gamma) \to ((\alpha \to \beta) \to (\alpha \to \gamma))$	(Problem 4a with (1) and (2))
(4)	$((\beta \to \gamma) \to (\alpha \to \beta)) \to ((\beta \to \gamma) \to (\alpha \to \gamma))$	(Lemma 3 with (3))
(5)	$(\beta \to \gamma) \to (\alpha \to \beta)$	(Lemma 2 with $\alpha \to \beta \in \Gamma$)
(6)	$(\beta \to \gamma) \to (\alpha \to \gamma)$	(MP with (4) and (5))

Part (c)

This one is similar to the previous one. A formal proof is as follows:

(1)	$(\beta \to \gamma) \to (\alpha \to (\beta \to \gamma))$	(A1)
(2)	$(\alpha \to (\beta \to \gamma)) \to ((\alpha \to \beta) \to (\alpha \to \gamma))$	(A2)
(3)	$(\beta \to \gamma) \to ((\alpha \to \beta) \to (\alpha \to \gamma))$	(Problem 4a with (1) and (2))
(4)	$((\beta \to \gamma) \to (\alpha \to \beta)) \to ((\beta \to \gamma) \to (\alpha \to \gamma))$	(Lemma 3 with (3))
(5)	$(\alpha \to \beta) \to ((\beta \to \gamma) \to (\alpha \to \beta))$	(A1)
(6)	$(\alpha \to \beta) \to ((\beta \to \gamma) \to (\alpha \to \gamma))$	(Problem 4a with (4) and (5))

Problem 5

A formal proof is as follows:

(1)	$\neg\neg\alpha \to (\neg\neg\neg\neg\alpha \to \neg\neg\alpha)$	(A1)
(2)	$(\neg\neg\neg\neg\alpha \to \neg\neg\alpha) \to (\neg\alpha \to \neg\neg\neg\alpha)$	(A3)
(3)	$\neg\neg\alpha \to (\neg\alpha \to \neg\neg\neg\alpha)$	(Problem 4a with (1) and (2))
(4)	$(\neg\alpha \to \neg\neg\neg\alpha) \to (\neg\neg\alpha \to \alpha)$	(A3)
(5)	$\neg\neg\alpha \to (\neg\neg\alpha \to \alpha)$	(Problem 4a with (3) and (4))
(6)	$(\neg\neg\alpha \to \neg\neg\alpha) \to (\neg\neg\alpha \to \alpha)$	(Lemma 3 with (5))
(7)	$\neg\neg\alpha \to \neg\neg\alpha$	(Lemma 1)
(8)	$\neg\neg\alpha \to \alpha$	(MP with (6) and (7))

Problem 6

A formal proof is as follows:

(1)	$\neg\neg\neg\alpha \to \neg\alpha$	(Problem 5 with $\neg\alpha$)
(2)	$(\neg\neg\neg\alpha \to \neg\alpha) \to (\alpha \to \neg\neg\alpha)$	(A3)
(3)	$\alpha \to \neg\neg\alpha$	(MP with (1) and (2))