Supplementary Notes 2

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Abstract

In this set of notes we discuss the fixed point methods in general and briefly the Newton's method.

Contents

1 Relation between root finding methods and fixed point methods

In the last set of notes, we discussed the bisection method. We call it a root finding method since given a function f it finds an approximation for the root x^* of f (i.e., $f(x^*) = 0$). In what follows, we will focus mainly on fixed point methods: these are methods that look for the fixed point x^* of a function g (i.e., $g(x^*) = x^*$). The relation between the two will be the following: x^* is a root of f if and only if x^* is a fixed point of g, i.e.,

$$f(x^*) = 0 \iff g(x^*) = x^*.$$

Example 1.1. *Let $f(x) = x^3 - 2x - 5$ and let x^* be its unique real root. A possible function g whose fixed point is also the root of f is $g(x) = (2x+5)^{1/3}$. Indeed,*

$$f(x^*) = 0 \iff (x^*)^3 - 2x^* - 5 = 0 \iff (x^*)^3 = 2x^* + 5 \iff x^* = g(x^*).$$

2 Fixed Point Iteration

Example 2.1. *Consider the equation $x^3 + 4x^2 - 10 = 0$ which has a unique root x^* in $[1, 2]$. Using simple algebraic manipulation, we can rewrite the equation as the fixed point problem $x = g(x)$ for different choices of g. We can define*

a) $g_1(x) = x - x^3 - 4x^2 + 10$

b) $g_2(x) = \left(\frac{10}{x} - 4x\right)^{1/2}$

c) $g_3(x) = \frac{1}{2}(10 - x^3)^{1/2}$

d) $g_4(x) = \left(\frac{10}{4+x}\right)^{1/2}$

e) $g_5(x) = x - \frac{x^3+4x^2-10}{3x^2+8x}$

n	*a)*	*b)*	*c)*	*d)*	*e)*
0	1.5	1.5	1.5	1.5	1.5
1	-0.875	0.8165	1.286953768	1.348399725	1.37333333
2	6.732	2.9969	1.402540804	1.367376372	1.365262015
3	-469.7	$(-8.65)^{1/2}$	1.345458374	1.364957015	1.365230014
4	1.03×10^8		1.375170253	1.365264748	1.365230013
5			1.360094193	1.365225594	
6			1.367846968	1.365230576	
7			1.363887004	1.365229942	
8			1.365916734	1.365230022	
9			1.364878217	1.365230012	
10			1.365410062	1.365230014	
15			1.365223680	1.365230013	
20			1.365230236		
25			1.365230006		
30			1.365230013		

Table 1: Fixed point iterations obtained for Example 2.1 with $x_0 = \frac{3}{2}$. The actual root is $1.365230013\ldots$.

Even though all functions have x^* as a fixed point, not all of them can be used as a fixed point iteration. For $x_0 = \frac{3}{2}$, the choice a) is divergent and b) becomes undefined since it involves the square root of a negative number. All other choices are convergent (see Table 2).

From the example above, the following question arises:

How can we find a fixed-point problem that produces a sequence that reliably and rapidly converges to a solution to a given root-finding problem?

The following theorem addresses this question.

Theorem 2.1 (Fixed Point Theorem). *Let $g \in C[a,b]$ and $a \le g(x) \le b$ for all $x \in [a,b]$. Suppose as well that g' exists on (a,b) and that there is a positive constant L such that $|g'(x)| \le L < 1$ for all $x \in (a,b)$. Then for any number $x_0 \in [a,b]$, the sequence defined by $x_{k+1} = g(x_k)$, $k \ge 0$, converges to the unique fixed point $x^* \in [a,b]$. Additionally, we have the following estimate*

$$|x_k - x^*| \le L^n \max\{x_0 - a, b - x_0\}.$$

Example 2.2. *We revisit Example 2.1.*

a) For $g_1(x) = x - x^3 - 4x^2 + 10$, we have $g_1(1) = 6$ and $g_1(2) = -12$, so g_1 does not map $[1,2]$ into itself. Moreover, $g'(x) = 1 - 3x^2 - 8x$, so $|g'(x)| > 1$ for all $x \in [1,2]$. Although Theorem 2.1 does not guarantee that the method must fail for this choice of g, there is no reason to expect convergence.

b) With $g_2(x) = \left(\frac{10}{x} - 4x\right)^{1/2}$, we can see that g_2 does not map $[1,2]$ into itself and that the sequence $\{x_n\}_{n=0}^{\infty}$ is not defined when $x_0 = \frac{3}{2}$. Moreover, there is no interval containing $x^* \approx 1.365$ such that $|g_2'(x)| < 1$ since $|g_2'(x^*)| \approx 3.4$. There is therefore no reason to expect convergence in this case.

c) For the function $g_3(x) = \frac{1}{2}(10 - x^3)^{1/2}$, we have

$$g_3'(x) = -\frac{3}{4}x^2(10 - x^3)^{-1/2} < 0 \quad \text{on } [1,2]$$

and so g_3 is strictly decreasing. However $|g_3'(2)| \approx 2.12$, so the condition $|g_3'(x)| \le L < 1$ fails on $[1,2]$. However, if we take a closer look at the sequence $\{x_n\}_{n=0}^{\infty}$ starting with $x_0 = \frac{3}{2}$ we conclude *that it is enough to consider the interval* $\left[1, \frac{3}{2}\right]$ *instead of* $[1, 2]$. *On this interval we still have that* g_3 *is strictly decreasing, but in addition*

$$1 < 1.28 \approx g_3\left(\frac{3}{2}\right) \le g_3(x) \le g_3(1) = \frac{3}{2}$$

for all $x \in \left[1, \frac{3}{2}\right]$. *Since it is also true that* $|g_3'(x)| \le \left|g_3'(\frac{3}{2})\right| \approx 0.66$ *for all* $x \in \left[1, \frac{3}{2}\right]$, *Theorem 2.1 confirms the convergence of which we were already aware.*

d) For $g_4(x) = \left(\frac{10}{4+x}\right)^{1/2}$ *we have*

$$|g_4'(x) = \left|\frac{-5}{\sqrt{10}(4+x)^{3/2}}\right| \le \frac{5}{\sqrt{10}\,5^{3/2}} \approx 0.1414 \quad \textit{for all } x \in [1, 2].$$

The bound on the magnitude of $g_4'(x)$ *is much smaller than the bound (found in c)) on the magnitude of* $g_3'(x)$, *which explains the faster convergence using* g_4.

e) The sequence defined by

$$g_5(x) = x - \frac{x^3 + 4x^2 - 10}{3x^2 + 8x}$$

converges much more rapidly than other choices. It corresponds in fact to the Newton's method.

Exercise 2.1. *Consider the fixed point iteration* $x_{k+1} = g(x_k)$ *with* $x_0 \in [0, 2\pi]$ *where* $g(x) = \pi + \frac{1}{2}\sin\left(\frac{x}{2}\right)$. *Show that* g *has a unique fixed point on* $[0, 2\pi]$ *and that the fixed point iteration converges. Estimate the number of iterations required to achieve* 10^{-2} *accuracy.*

Solution: We need to prove the assumptions of Theorem 2.1. For all $x \in [0, 2\pi]$, we have

$$0 \le \sin\left(\frac{x}{2}\right) \le 1 \Rightarrow 0 \le \frac{1}{2}\sin\left(\frac{x}{2}\right) \le \frac{1}{2} \Rightarrow \pi \le \pi + \frac{1}{2}\sin\left(\frac{x}{2}\right) \le \pi + \frac{1}{2} \Leftrightarrow \pi \le g(x) \le \pi + \frac{1}{2}$$

and so for all $x \in [0, 2\pi]$ $0 \le g(x) \le 2\pi$. In addition,

$$|g'(x)| = \frac{1}{4}\left|\cos\left(\frac{x}{2}\right)\right| \le \frac{1}{4} = L.$$

We can then conclude that g satisfies the two conditions of Theorem 2.1 with $L = \frac{1}{4}$. Therefore, g has a unique fixed point on $[0, 2\pi]$ and the fixed point iteration converges.

From Theorem 2.1, we have the error estimate

$$|x_k - x^*| \le L^k \max\{x_0 - a, b - x_0\}$$

with $L = \frac{1}{4}$. We first observe that

$$\max\{x_0 - 0, 2\pi - x_0\} \le 2\pi$$

for all $x_0 \in [0, 2\pi]$. Then we can rewrite the inequality as

$$|x_k - x^*| \le \left(\frac{1}{4}\right)^k 2\pi.$$

For 10^{-2} accuracy we need

$$\left(\frac{1}{4}\right)^k 2\pi < 10^{-2}$$
$$\left(\frac{1}{4}\right)^k < \frac{10^{-2}}{2\pi}$$
$$k \log\left(\frac{1}{4}\right) < \log\left(\frac{10^{-2}}{2\pi}\right)$$
$$k > \frac{\log\left(\frac{10^{-2}}{2\pi}\right)}{\log\left(\frac{1}{4}\right)} \approx 4.65.$$

So $k = 5$ is enough to guarantee 10^{-2} accuracy.

Exercise 2.2. *Show that $g(x) = 2^{-x}$ has a unique fixed point on $\left[\frac{1}{3}, 1\right]$. Use a fixed-point iteration to find an approximation to the fixed point accurate to within 10^{-4}. Estimate the number of iterations required to achieve such accuracy and compare this theoretical estimate to the number actually needed.*

Solution: Since g is a decreasing function

$$\frac{1}{2} = g(1) \leq g(x) \leq g\left(\frac{1}{3}\right) < 1.$$

Furthermore, $g'(x) = -\log(2)2^{-x}$. For $x \in \left[\frac{1}{3}, 1\right]$, we have

$$|g'(x)| = 2^{-x}\log(2) \leq 2^{-\frac{1}{3}}\log(2) := L < 1$$

for all $x \in \left[\frac{1}{3}, 1\right]$. We can then conclude that g has a fixed point in $\left[\frac{1}{3}, 1\right]$.

We consider the fixed point iteration $x_{k+1} = g(x_k)$ with $x_0 \in \left[\frac{1}{3}, 1\right]$.

We will use the estimate

$$|x_k - x^*| \leq L^k \max\{x_0 - a, b - x_0\}$$

to find an upper bound on the number of iterations required to achieve an accuracy of 10^{-4}. For all $x_0 \in [\frac{1}{3}, 1]$,

$$\max\left\{x_0 - \frac{1}{3}, 1 - x_0\right\} \leq \frac{2}{3}.$$

The inequality then becomes

$$|x_k - x^*| \leq \frac{2}{3}L^n.$$

To get the 10^{-4} accuracy we need

$$\frac{2}{3}L^k < 10^{-4} \iff k > \frac{\log\left(\frac{3}{2}10^{-4}\right)}{\log(L)} \approx 14.73.$$

So $k = 15$ is enough to theoretically guarantee 10^{-4} accuracy, but in practice in this case we only need 10 if $x_0 = \frac{1}{3}$ (check this!).

3 Error analysis for iterative methods

Definition 3.1. *Let $\{x_0, x_1, \ldots\}$ be a sequence. We say the sequence converges to x^* if*

$$\lim_{k\to\infty} x_k = x^*.$$

If in addition, $x_k \neq x^*$ for all $k \in \mathbb{N}$, then we say the sequence converges to x^* at order p with an asymptotic error constant $C > 0$ if,

$$\lim_{n\to\infty} \frac{|x_{k+1} - x^*|}{|x_k - x^*|^p} = C.$$

Remark 3.2. *When $p = 1$, the convergence is called linear. In this case, C is also called the rate of convergence and C must be less than 1 for the convergence. The smaller the C is, the faster the sequence converges. When $p = 2$, the convergence is called quadratic.*

In general a sequence with a high order of convergence converges more rapidly than a sequence with a lower order. The asymptotic constant also affects the speed of convergence but is not as important as the order.

Theorem 3.3. *Let x^* be a fixed point of a function g, which is continuous and differentiable in an open interval I containing x^*. If $|g'(x^*)| < 1$, then there exists $\delta > 0$ such that the sequence $x_{k+1} = g(x_k)$ for $k \geq 0$ converges for any $x_0 \in [x^* - \delta, x^* + \delta]$. If in addition $g'(x^*) \neq 0$, then the convergence is linear with*

$$\lim_{k\to\infty} \frac{|x_{k+1} - x^*|}{|x_k - x^*|} = |g'(x^*)|.$$

Remark 3.4. *If $|g'(x^*)| > 1$, one can show that the fixed point iteration is not convergent. In the case $|g'(x^*)| = 1$ no conclusion can de drawn since both situations (convergence and nonconvergence) are possible depending on the problem.*

Theorem 3.3 tells us that if $g'(x^*) \neq 0$, the fixed point iteration exhibits linear convergence with asymptotic error constant $|g'(x^*)|$. It also implies that high-order convergence for fixed point methods can occur only when $g'(x^*) = 0$.

Theorem 3.5. *Let $g \in C^{p+1}(I)$ where I in an open interval containing x^*, a fixed point of g, and $p \geq 1$ is an integer. If $g^{(i)}(x^*) = 0$ for $1 \leq i \leq p$ and $g^{(p+1)}(x^*) \neq 0$, then then there exists $\delta > 0$ such that the sequence $x_{n+1} = g(x_n)$ for $n \geq 0$ converges for any $x_0 \in [x^* - \delta, x^* + \delta]$ with*

$$\lim_{n\to\infty} \frac{|x_{n+1} - x^*|}{|x_n - x^*|^{p+1}} = \left| \frac{g^{(p+1)}(x^*)}{(p+1)!} \right|.$$

Example 3.1. *We revisit Example 2.1 once again. We have*

a) $g_1'(x^*) \approx 15.5134$

b) $g_2'(x^*) \approx 3.42991$

c) $g_3'(x^*) \approx 0.511961$

d) $g_4'(x^*) \approx 0.127229$

e) $g_5'(x^*) = 0$

These are in accordance with what we just saw. For examples a) and b), $|g'(x^*)| > 1$ and so there is no convergence. For the remaining three examples $|g'(x^*)| < 1$ and so we have convergence (provided x_0 is close enough to x^*). Example e) is the one with the fastest convergence which agrees with the theory since it has convergence (at least) quadratically. Between Examples c) and d), Example d) is the faster since it has a smaller asymptotic error constant.

Exercise 3.1. *Consider the iteration* $x_{n+1} = g(x_n)$ *where*

$$g(x) = x + \mu(3 - e^x)$$

with $\mu \neq 0$. *What is the fixed point* x^* *of this method? For what values of the parameter* μ *is* $|g'(x^*)| < 1$*? Why is this important? For what value of* μ *is* $|g'(x^*)| = 0$*? Why do we care about this?*

Solution: The fixed point x^* satisfies $g(x^*) = x^*$. Using the given formula for g, we get

$$\mu(3 - e^x) = 0.$$

Since $\mu \neq 0$, we obtain $x^* = \log(3)$. So the real goal of this iteration formula is to compute $\log(3)$.

It's easy to see that $g'(x) = 1 - \mu e^x$. Thus to have $|g'(x^*)| < 1$ we need $|1 - \mu e^{x^*}| < 1$. Plugging in the value for x^* and working out the inequality leads to

$$0 < \mu < \frac{2}{3}.$$

This values of μ are important since by Theorem 3.3 we are guaranteed local convergence of the fixed point iteration.

Furthermore, to have more than linear convergence we need $g'(x^*) = 0$. This is equivalent to demanding

$$1 - \mu e^{x^*} = 1 - \mu e^{\log(3)} = 1 - 3\mu = 0.$$

Thus $\mu = 1/3$ gives us at least quadratic convergence. This means the iterates will move to x^* faster.

4 Newton's method

Newton's method is given by

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}.$$

Theorem 4.1. *Let* $f \in C^2[a,b]$ *and* $x^* \in [a,b]$ *be a root of* f *with* $f'(x^*) \neq 0$. *Then there is a neighbourhood* $[x^* - \delta, x^* + \delta]$ *near* x^* *for some* $\delta > 0$ *such that for any* $x_0 \in [x^* - \delta, x^* + \delta]$, *Newton's method converges (at least) quadratically.*

Proof. The idea here is to look at Newton's method as a fixed point method. In fact, Newtow's method finds the fixed point of the function g given by

$$g(x) = x - \frac{f(x)}{f'(x)}$$

According to Theorem 3.5 our result is proved provided $g'(x^*) = 0$. Indeed, we have

$$g'(x) = 1 - \frac{f'(x)f'(x) - f(x)f''(x)}{f'(x)^2} = \frac{f(x)f''(x)}{f'(x)^2}$$

Since $f(x^*) = 0$ and $f'(x^*) \neq 0$, we conclude $g'(x^*) = 0$ as desired. □

References

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