# EXERCISE FOR CSE202 – WEEK 10

**Question 1.** *When constructing a BST on $N = 2^n - 1$ keys, if all $N!$ key insertion sequences are equally likely show that the probability that a perfectly balanced tree structure (all $2^n$ leaves on the same level) will be built is*

$$\frac{1}{\prod_{k=1}^{n}(2^k-1)^{2^{n-k}}}.$$

*[Indication: consider what is going on with $n = 2, 3$.]*

*Solution.* For $n = 2$, $N = 3$ and there are two permutations leading to such a balanced BST: 213 and 231, whence a probability $2/6 = 1/3 = 1/(2^1-1)^{2^1}/(2^2-1)^{2^0}$. Thus the formula holds for $n = 2$.

For $n = 3$, $N = 7$, 4 has to be at the root of the tree and then both subtrees will have 3 nodes and must be balanced. Since these events are independent, we obtain the probability

$$\frac{1}{7}\left(\frac{1}{3}\right)^2 = \frac{1}{63} = \frac{1}{(2-1)^4(4-1)^2(8-1)^1}.$$

More generally, the median element has to be at the root of the tree, which occurs with probability $1/N$ and leads to both subtrees having the same size and independent shapes and thus the probability $p_{n+1}$ at size $2^{n+1}-1$ is defined by the recurrence

$$p_{n+1} = \frac{p_n^2}{2^{n+1}-1}, \quad p_0 = 1.$$

The given formula satisfies $p_0 = 1$. It equals $p_n$ since it also satisfies the recurrence:

$$\begin{aligned}
&\frac{1}{2^{n+1}-1}\frac{1}{\left(\prod_{k=1}^{n}(2^k-1)^{2^{n-k}}\right)^2} \\
&\qquad = \frac{1}{2^{n+1}-1}\frac{1}{\prod_{k=1}^{n}(2^k-1)^{2^{n+1-k}}} = \frac{1}{\prod_{k=1}^{n+1}(2^k-1)^{2^{n+1-k}}}. \quad \square
\end{aligned}$$