EXERCISE FOR CSE202 – WEEK 13

In the problem 2-Partition, you are given n positive integers $a_1, \dots, a_n$ and the question is to decide whether there exists a subset $I \subset \{1, \dots, n\}$ such that

$$\sum_{i \in I} a_i = \sum_{i \in \{1,\dots,n\} \setminus I} a_i.$$

Show that 2-Partition is NP-complete.
[Hint: a non-trivial reduction of the SubsetSum problem can be found.]

Solution. 1. It is easy to check in polynomial time that a given I (the certificate) is a subset of $\{1, \dots, n\}$ and is such that the desired equality holds. Thus the problem is in NP.

2. For the reduction we need to show that the SubsetSum problem can be reduced to the 2-Partition problem (not the other way round!).

Given an instance $(a_1, \dots, a_\ell) \in \mathbb{N}^\ell$ and $k \in \mathbb{N}$ of the SubsetSum problem, the idea is to add an element $a_{\ell+1}$ so that the 2-Partition necessarily contains a solution to the SubsetSum instance. Thus, letting $S = a_1 + \cdots + a_\ell$, we distinguish two cases:

— If $2k \geq S$, take $a_{\ell+1} = 2k - S$. The sum of all elements is $2k$. There is a solution to the SubsetSum problem if and only if there is one for the 2-partition problem: in that case one of the parts of a 2-Partition sums to k, does not contain $a_{\ell+1}$ and gives the answer.

— Otherwise, solve the subset sum problem for $k' = S - k$ by the same method, which applies since $2k' = S + (S - 2k) \geq S$. Indeed, the problem for k has a solution if and only if the problem for k' does.

Alternatively, take $a_{\ell+1} = S - 2k$; the sum of all elements is $2S - 2k$; a 2-Partition has two parts each of sum $S - k$; one of them contains $a_{\ell+1}$ and removing it from that part leads to a total of $S - k - (S - 2k) = k$, a solution to the SubsetSum problem.

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