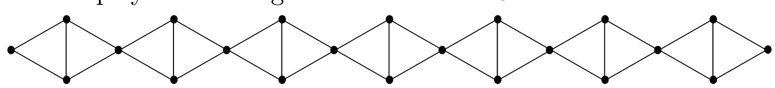
EXERCISE FOR CSE202 – WEEK 14

The NearestNeighbour heuristic for the metric Traveling Salesman Problem is very natural: start from an arbitrary vertex and at each step, take the edge to the closest unvisited vertex (breaking ties arbitrarily). Conclude by adding the edge connecting the final vertex to the starting one. The aim of this exercise is to show that this does not give an approximation of the optimal circuit by a constant factor.

For $k \geq 1$, consider the set of $3 \times 2^k - 2$ points in $\mathbb{R}^2$ arranged in a chain of $2^k - 1$ diamonds as displayed in the figure below for $k = 3$.



The points are arranged on 3 parallel lines, all the edges shown have length 1. The graph is completed by edges (not shown) between all pairs of vertices, with length obtained using the Euclidean distance.

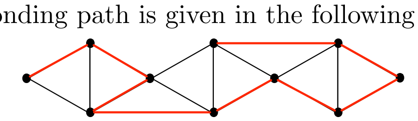
Call ℓ_k the left-most vertex, r_k the rightmost one, u_k the upper-middle one and d_k the lower-middle one.

Question 1. *In the case $k = 1$ observe that there is a path of length 3 starting from ℓ_1, ending at u_1 (or d_1) and using only nearest neighbours.*

Solution. The paths $\ell_1 \to d_1 \to r_1 \to u_1$ and its mirror image satisfy the requirements. $\square$

Question 2. *By "gluing" two such paths, deduce a path of length $7 + 2\sqrt{3}$ for the case $k = 2$ starting from ℓ_2, ending at u_2 (or d_2) and using only nearest neighbours.*

Solution. The corresponding path is given in the following picture:



It starts at ℓ_2, follows a path as in the previous question, goes to the nearest unvisited neighbour d_2 at distance $\sqrt{3}$, from where it goes to the left-most vertex of the next diamond, follows a path as in the previous question and reaches the last unvisited vertex u_2 after a final step of length $\sqrt{3}$. $\square$

Question 3. *Deduce that in the general case, there exists a path of length $2^{k-1}(4 + (k-1)\sqrt{3}) - 1$, starting from ℓ_k, ending at u_k (or d_k) and using only nearest neighbours.*

Solution. The general case is constructed in the same way: the path starts at ℓ_k, follows the path of the case $k-1$ (of length $2^{k-2}(4+(k-2)\sqrt{3})-1$), then jumps to d_k with a step of length $2^{k-2}\sqrt{3}$, goes to the left-most vertex of the next diamond, follows a path of the case $k-1$ and reaches the last unvisited vertex u_k after another final step of length $2^{k-2}\sqrt{3}$. The total length is therefore

$$2 \times (2^{k-2}(4 + (k-2)\sqrt{3}) - 1) + 2 \times 2^{k-2}\sqrt{3} + 1 = 2^{k-1}(4 + (k-1)\sqrt{3}) - 1$$

as was to be proved. $\square$

Question 4. *By adding the missing edge, complete this path into a traveling salesman circuit of these vertices using only nearest neighbours.*

Solution. Setting the origin in the center of the graph, with orientation and norm as in the picture gives the coordinates of ℓ_k and u_k as $(-(2^{k-1}-1/2)\sqrt{3},0)$ and $(0,1/2)$ so that the last edge has length $\sqrt{1/4+3(2^{k-1}+1/2)^2}$. ☐

Question 5. *Compare the length of that circuit with that obtained by starting at ℓ_k, zigzagging along the upper and middle vertices and coming back via the lower ones. Conclude that the ratio between the length of the circuit computed with the nearest-neighbour heuristic on a graph with n vertices and the optimal one can be as large as $\Omega(\log n)$.*

Solution. The total length of that nearest-neighbour circuit is therefore

$$2^{k-1}((k-1)\sqrt{3}-1)+\sqrt{1/4+3(2^{k-1}+1/2)^2}\sim\frac{\sqrt{3}}{2}k2^k,\quad k\to\infty.$$

The circuit described in the question, which is necessarily at least as long as the optimal, has length

$$2^{k+1}+(2^k-2)\sqrt{3}\sim(2\sqrt{3}+1)2^k,\quad k\to\infty.$$

It follows that the ratio between the result of the nearest-neighbour heuristic and the optimal circuit behaves asymptotically at least like

$$\frac{\sqrt{3}}{2(2\sqrt{3}+1)}k=\Omega(\log n),\quad n\to\infty.$$

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