EXERCISE FOR CSE202 – WEEK 5

In this exercise, n is a power of 2. The complexity of Karatsuba's algorithm obeys

$$C(n) \le 3C(n/2) + 4n.$$

From there, the Master Theorem allows one to conclude that $C(n) = O(n^{\log_2 3})$.

Question 1. *Obtain an explicit bound on the constant of this $O()$ estimate.*

Solution: It is just a matter of following the steps of the proof of the Master theorem in that case and replacing $O()$ estimates by simple bounds. From

$$C(n) \le 4n(1 + 3/2 + \cdots + (3/2)^{k-1}) + 3^k C(1),$$

using $k = \log_2 n$ and $C(1) = 1$ and bounding the geometric series gives

$$\begin{aligned} C(n) &\le 4n(3/2)^{k-1}(1 + 2/3 + \dots) + 3^k \\ &\le 4\,2^k(3/2)^{k-1}3 + 3^k = 3^k(8+1) \\ &\le 9n^{\log_2 3}. \end{aligned}$$

□

Question 2. *Assuming that for a given s, power of 2, the recursion stops when $n \le s$ and the naive multiplication algorithm in $\le 2n^2$ operations is used, show that the complexity is bounded by $f(s)n^{\log_2(3)}$ for an explicit function $f(s)$ that you have to determine.*

Solution: The beginning of the derivation is the same, leading to the inequality

$$C(n) \le 4n(1 + 3/2 + \cdots + (3/2)^{k-1}) + 3^k C(n/2^k).$$

Now, k is chosen in such a way that $n/2^k = s$, i.e., when $2^k = n/s$, which leads to $k = \log_2(n/s)$. With that value of k, and $C(s) = 2s^2$ given by the question, bounding the geometric series gives

$$\begin{aligned} C(n) &\le 4n(3/2)^{k-1}(1 + 2/3 + \dots) + 3^k\,2s^2 \\ &\le 4\,2^k s(3/2)^{k-1}3 + 3^k\,3s^2 = 3^k(8s + 2s^2) \\ &\le 3^{\log_2(n/s)}(8s + 2s^2) \\ &\le n^{\log_2 3}\frac{8s + 2s^2}{s^{\log_2 3}}. \end{aligned}$$

□

Question 3. *Optimizing the choice of s, how low can you get the constant?*

Solution: Differentiating that constant factor with respect to s gives

$$\frac{4(s+2) - 2(s+4)\log_2 3}{s^{\log_2 3}}$$

which vanishes at
$$s = \frac{4(\ln 3 - \ln 2)}{\ln 4 - \ln 3} \approx 5.64.$$
Thus the smallest value at integer powers of 2 is either at $s = 4$ or at $s = 8$. It turns out that they both give the same constant $64/9 \approx 7.11$, which is to be compared with the constant 9 in part 1. of the exercise. Thus this method gives a saving of about 21% compared to the original version of Karatsuba's algorithm. $\square$