EXERCISE FOR CSE202 – WEEK 4

Given a truncated power series

$$A = a_0 + a_1X + a_2X^2 + \cdots + O(X^n),$$

its derivative and its integral are defined by

$$A' = a_1 + 2a_2X + \cdots + O(X^{n-1}), \qquad \int A = a_0X + a_1X^2/2 + \cdots + O(X^{n+1}).$$

From these definitions, given a power series

$$A = 1 + a_1X + a_2X^2 + \cdots + O(X^n),$$

one can compute its logarithm via the formula

$$\log(A) = \int \frac{A'}{A}.$$

Conversely, given a truncated power series

$$S = s_1X + s_2X^2 + \cdots + O(X^n),$$

one can define its exponential as

$$\exp(S) = 1 + S + \frac{1}{2!}S^2 + \cdots + \frac{1}{(n-1)!}S^{n-1} + O(X^n).$$

It turns out that this is sufficient to ensure that $\exp(\log A) = A$ for any A with $A(0) = 1$.

Question 1. *Show that the complexity of the computation of* $\log(A)$ *given* A *is* $O(\mathrm{Mul}(n))$ *operations on the coefficients.*

Solution : Computing the derivative or the integral of a truncated series uses only $O(n)$ operations: one per coefficient. Apart from this, the formula uses one division, which has complexity $O(\mathrm{Mul}(n))$ as seen in the course. □

Question 2. *Using the equation* $S - \log y = 0$*, design an algorithm based on Newton iteration for the computation of* $\exp(S)$*. Assuming this algorithm has quadratic convergence, show that its complexity is* $O(\mathrm{Mul}(n))$ *operations again.*

Solution : Starting from $\phi(y) = S - \log y$, Newton's iteration takes the form

$$y_{n+1} = y_n + y_n(S - \log y_n).$$

Provided the convergence is quadratic, the algorithm that follows is the following

If $n = 1$ return $1 + O(X)$
Let $k = \lceil n/2 \rceil$
Compute recursively $\exp(S + O(X^k)) = G_k + O(X^k)$ at precision k;
Compute $H_k = \log G_k + O(X^k)$ by the method of part (1).
Return $G_k + G_k(S - H_k) + O(X^n)$

The complexity satisfies $C(n) \leq C(\lceil n/2 \rceil) + O(\mathrm{Mul}(n))$ and thus, by the same techniques as usual $C(n) = O(\mathrm{Mul}(n))$. □

Question 3. *(More difficult). Prove that quadratic convergence. Indication: use the Taylor expansion*

$$\phi(\exp(S)) = 0 = \phi(y_n) + \phi'(y_n)(e^S - y_n) + O((e^S - y_n)^2).$$

Solution : Replacing $\phi(y_n)$ in the given Taylor expansion by its value in terms of y_{n+1}, namely

$$\phi(y_n) = \phi'(y_n)(y_n - y_{n+1})$$

gives

$$0 = \phi'(y_n)(e^S - y_{n+1}) + O((e^S - y_n)^2)$$

which concludes once we notice that $1/\phi'(y_n) = -y_n = -1 + O(X)$. □