CSE202
Design and Analysis of Algorithms

Constantin Enea

Week 1: Overview & Basics

I. Overview of the Course

Summary of Last Year

CSE 101/102: Computer programming.
Data structures and algorithms.
Basic graph/tree algorithms.

CSE 103: Introduction to algorithms.
Divide and conquer. Sorting. Searching
Dynamic programming. Greedy algorithms.
Graph algorithms.

Plan for this Course

Introduction

Divide & Conquer

Randomization

Amortization, balancing

String Algorithms

P vs NP - Approximation algorithms

Basic principles through many examples

data-structures along the way

Organization

Lectures:	Wednesdays 10:15—11:45 amphi Cauchy	
Tutorials in Python	Thursdays 13:15—15:15 rooms 33 & 35 & 36	Antoine Lavignotte Martin Krejka Simon Bliudze
Material:	see Moodle	
Questions:	constantin.enea@polytechnique.edu	

Assessment

	Week	Ratio
Homework assignments	1—14	10 %
Weekly tutorials	1—14	10 %
Midterm	7	40 %
Final	15	40 %

Midterm: programming
Final: written exam

Exam rules:
No collaboration,
no laptop,
no internet.

II. Algorithms

An algorithm is a finite answer to an infinite number of questions.
Stephen Kleene

Example of Algorithm: Binary Powering

1. A well-specified problem:

Input: $(x, n) \in \mathbb{A} \times \mathbb{N}$

Output: $y \in \mathbb{N}$ such that $y = x^n$

2. A method to solve it:

Idea. $x^n = \begin{cases} (x^{n/2})^2, & \text{for even } n \\ (x^{(n-1)/2})^2 x, & \text{otherwise} \end{cases}$

```
Binary-Powering(x,n)
  if n = 0 then return 1
  tmp ← Binary-Powering(x,n/2)
  tmp ← tmp * tmp
  if ( n is even ) return tmp
  return tmp * x
```

Pseudocode.

Example of Algorithm: Binary Powering

Algorithm:

Binary-Powering(x,n)

if n = 0 then return 1
tmp ← **Binary-Powering**(x,n/2)
tmp ← tmp * tmp
if (n is even) return tmp
return tmp * x

Implemented in programs:

```
pow(int, int):
        mov     eax, 1
        test    esi, esi
        jne     .L8
        ret
.L8:
        push    rbp
        push    rbx
        sub     rsp, 8
        mov     ebp, edi
        mov     ebx, esi
        shr     esi, 31
        add     esi, ebx
        sar     esi
        call    pow(int, int)
        imul    eax, eax
        test    bl, 1
        je      .L1
        imul    eax, ebp
.L1:
        add     rsp, 8
        pop     rbx
        pop     rbp
        ret
```

Example of Algorithm: Binary Powering

Algorithm:

Binary-Powering(x,n)

if n = 0 then return 1
tmp ← **Binary-Powering**(x,n/2)
tmp ← tmp * tmp
if (n is even) return tmp
return tmp * x

Implemented in programs:

```
def binpow(x,n):
  if n==0: return 1
  tmp=binpow(x,n//2)
  tmp=tmp*tmp
  if n%2==0: return tmp
  return tmp*x
```

Paying attention to details in the programming language semantics, e.g., integer precision

- C/C++, Java have fixed integer precision => overflow

Correctness

Def. An algorithm is *correct* if

1. it terminates;
2. it computes what its specification claims.

A useful proof technique: look for variants and invariants.

```
# Input:  x that can be multiplied
#         n nonnegative integer
# Output: xⁿ
def binpow(x,n):
    if n==0: return 1
    # n>0
    tmp=binpow(x,n//2) # n//2 < n
    # tmp = x^(n//2)
    tmp=tmp*tmp # tmp = x^(2*(n//2))
    if n%2==0: return tmp
    return tmp*x
```

Specification

$n > 0 \Rightarrow n//2 < n$ proves termination

(Obvious) invariants

Correctness by induction

Correctness: a less obvious example

```
# Input:   x that can be multiplied
#          n nonnegative integer
# Output:  xⁿ
def binpow2(x,n):# let n₀=n, x₀=x
    if n==0: return 1
    y = 1
    while n>1: # y*(xⁿ)=x₀ⁿ⁰
        if n%2==1: y *= x
        x *= x
        n //= 2
    return y*x
```

Termination: same argument

Correctness: invariant

Proof. In one iteration of the loop, $y*(x^n)$ becomes

$$y*(x^2)^{(n//2)}=y*(x^n) \quad \text{for even } n$$

$$y*x*(x^2)^{(n//2)}=y*(x^n) \quad \text{for odd } n$$

Termination is a very hard problem

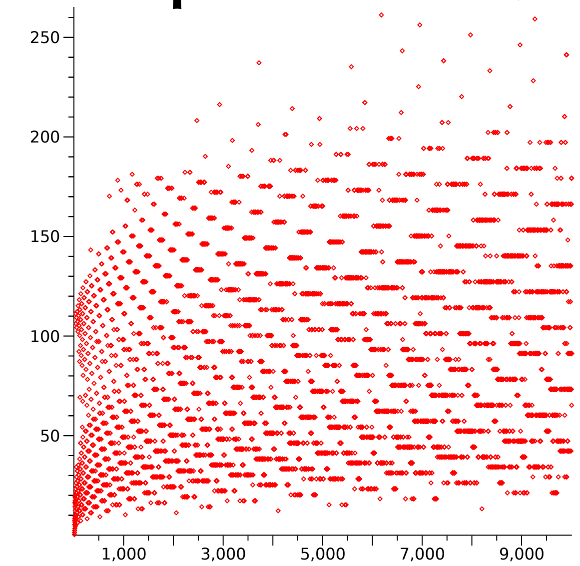
The general problem is undecidable. (See CSE 203)

Already hard for seemingly simple programs:

```
def syracuse(n):
    if n==1: return
    if n%2==0: return syracuse(n//2)
    return syracuse(3*n+1)
```

Conjecture. (3n+1 conjecture, Syracuse problem,…)
This program terminates.

Open since 1937!



stopping times

III. Complexity

Complexity

How long will my program take?
Do I have enough memory?

The scientific approach:

1. Experiment for various sizes;
2. Model;
3. Analyse the model;
4. Validate with experiments;
5. If necessary, go to 2.

Experimental Determination of (Polynomial) Complexity

If the time for a computation grows like $C(n) \sim K n^{\alpha} \log^{p} n$

then doubling n should take time $C(2n) \sim K 2^{\alpha} n^{\alpha} \log^{p} n$

so that
$$\alpha \approx \log_2 \frac{C(2n)}{C(n)}\,.$$

Example: matrix product

n	10	20	40	80
time (s)	0,023	0,158	1,159	9,075
ln2(ratio)		2.78	2.88	2.97

```
from sympy.matrices import randMatrix
import timeit

def testMatrixMul(size,nbtests):
    total = 0
    for i in range(nbtests):
      A = randMatrix(size)*1.
      B = randMatrix(size)*1.
      def doit():
        return A*B
      total += timeit.timeit(doit,number=1)
    return total/nbtests
```

suggests **cubic** complexity.

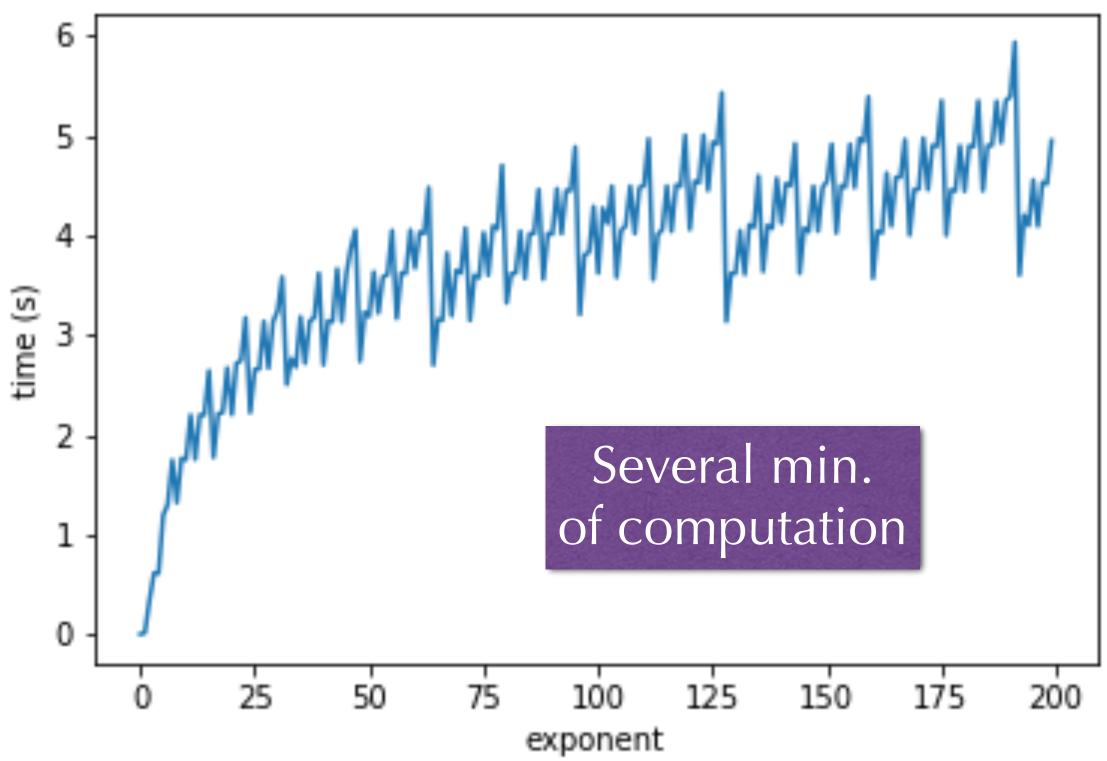
Blackboard:
3 is expected

Binary Powering 1. Model

1. Experiment

```
from sympy.matrices import randMatrix
import timeit

def test(size,maxpow):
    A = randMatrix(size)*1.
    val = [0 for i in range(maxpow)]
    for i in range(maxpow):
        def doit():
            return binpow(A,i)
        val[i] = timeit.timeit(doit,number=3)
    return val
```



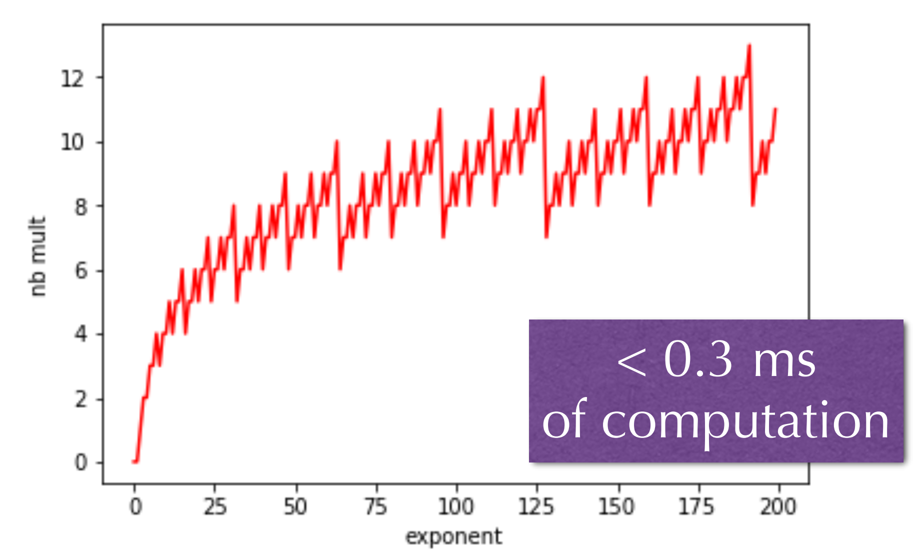
x is a 20x20 matrix of floats

```
def binpow(x,n):
    if n==0: return 1
    if n==1: return x
    tmp=binpow(x,n//2)
    tmp=tmp*tmp
    if n%2==0: return tmp
    return tmp*x
```

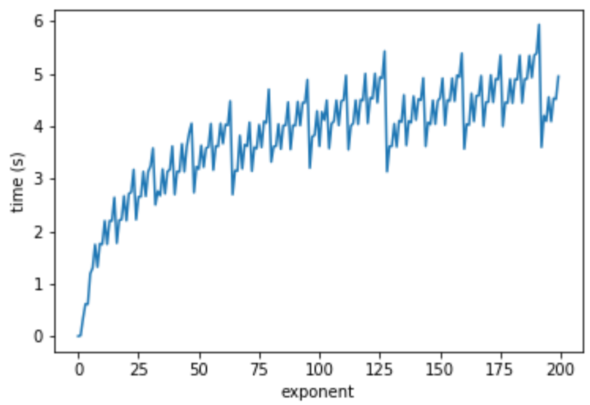
2. Model: count multiplications only

$$C(n) = \begin{cases} C(n/2) + 1, & \text{for even } n > 0 \\ C((n-1)/2) + 2, & \text{for odd } n > 1 \end{cases}$$

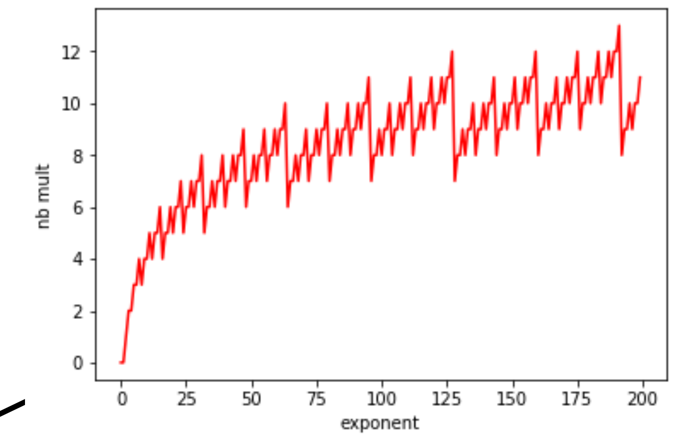
$$C(0) = C(1) = 0.$$



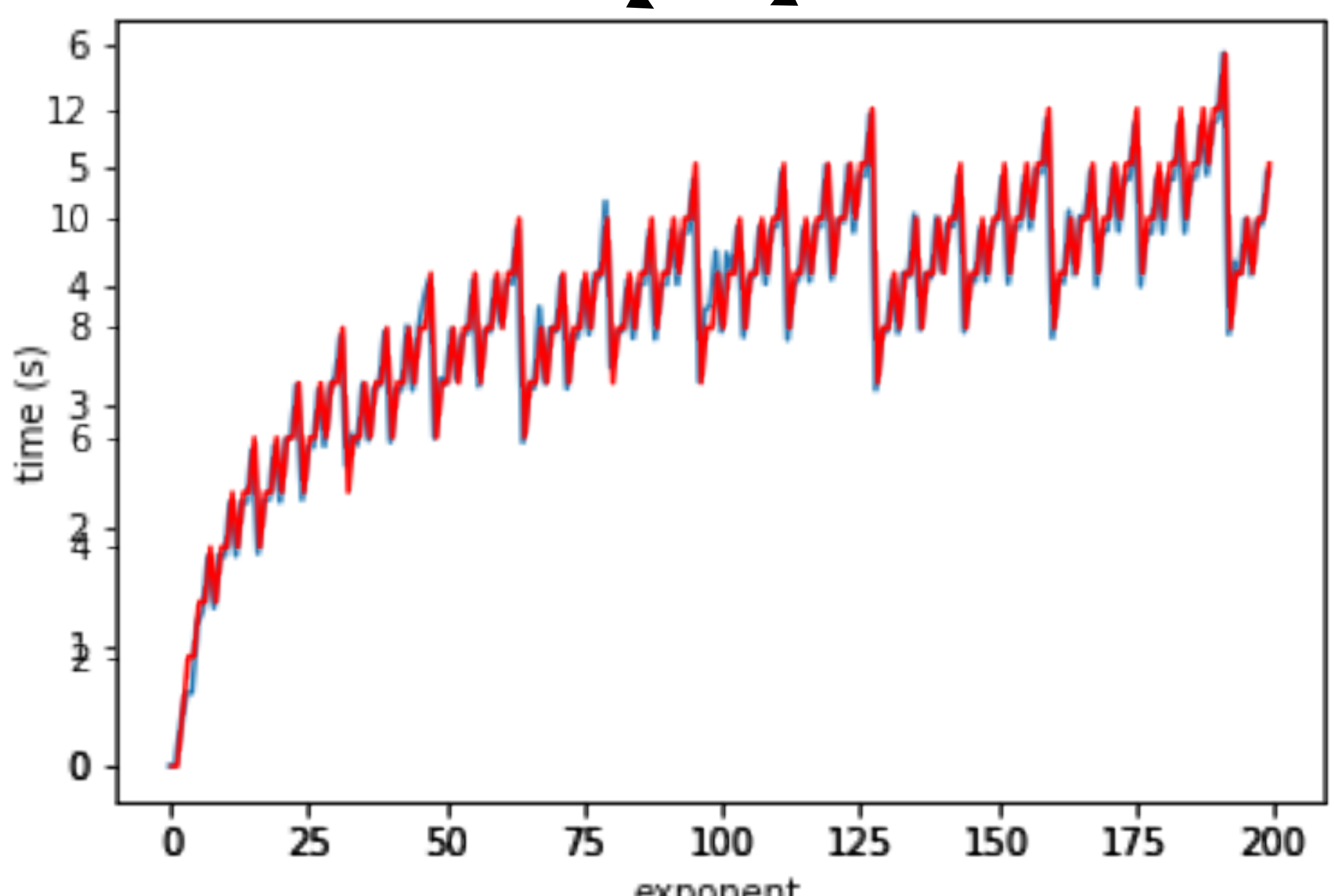
Binary Powering 2. Comparison



time



nb mult



Asymptotically, the cost of multiplications dominates

Binary Powering 3. Analysis

$$C(n) = 1 + \begin{cases} C(n/2), & \text{for even } n > 0 \\ C((n-1)/2) + 1, & \text{for odd } n > 1 \end{cases} \quad \text{with } C(0) = C(1) = 0$$

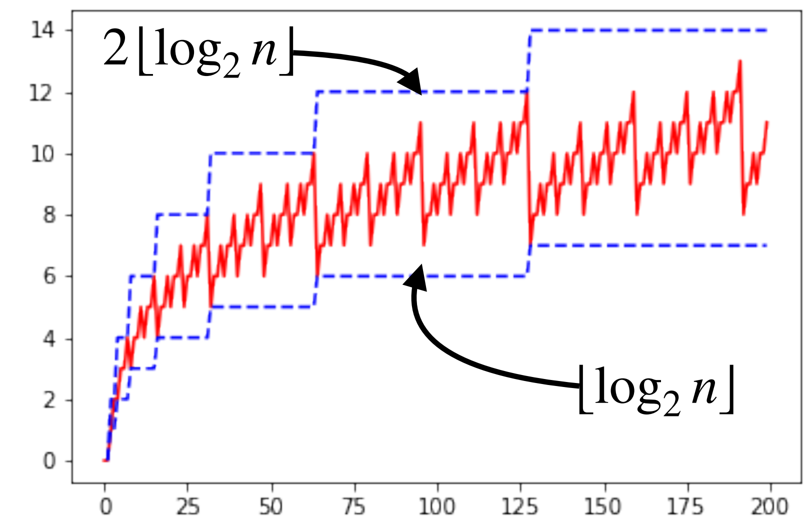
Lemma. For $n \geq 1$, $C(n) = \lfloor \log_2 n \rfloor - 1 + \lambda(n)$,
where $\lambda(n)$ is the number of 1's in the binary expansion of n.

Blackboard proof

Ex. $82 = 64 + 16 + 2 = \overline{1010010}^2$ → 6-1+3=8 mult.

Consequence:

$$\underbrace{\boxed{\lfloor \log_2 n \rfloor \leq C(n) \leq 2\lfloor \log_2 n \rfloor}}_{C(n)=O(\log n)}$$



Notation

$$f(n) \sim g(n) \quad \text{means} \quad \lim_{n\to\infty} f(n)/g(n) = 1$$

Recall: $$f(n) = O(g(n)) \quad \text{means} \quad \exists K \, \exists M \, \forall n \geq M, |f(n)| \leq Kg(n)$$

$$f(n) = \Theta(g(n)) \quad \text{means} \quad f(n) = O(g(n)) \text{ and } g(n) = O(f(n))$$

Exs.:

$$\log(2n) = O(\log n)$$

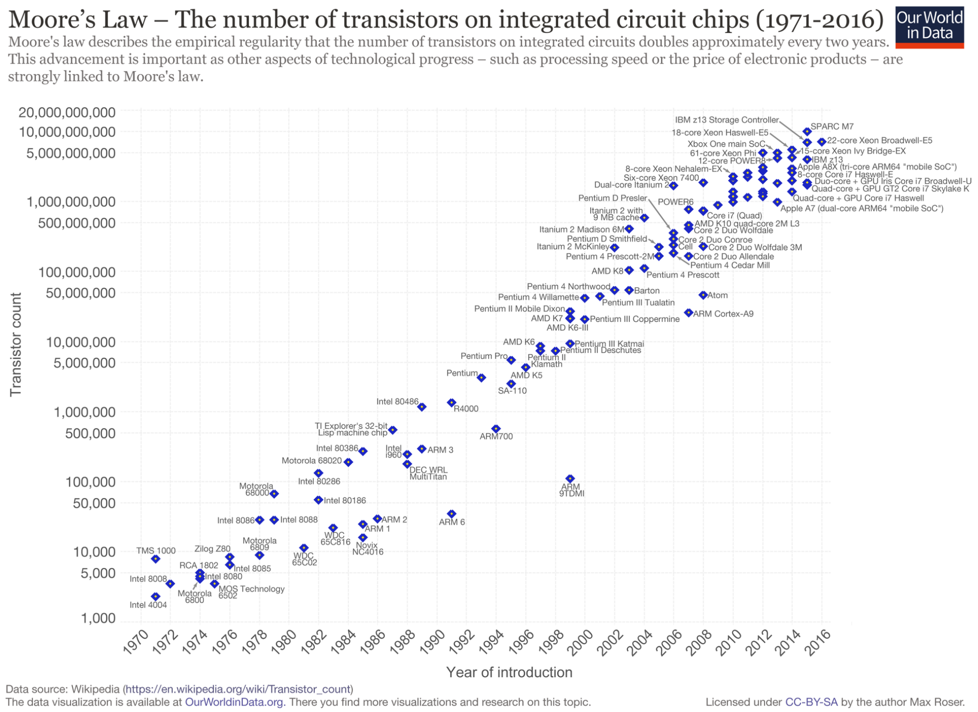
$$10^{10^{10}} n = O(n)$$

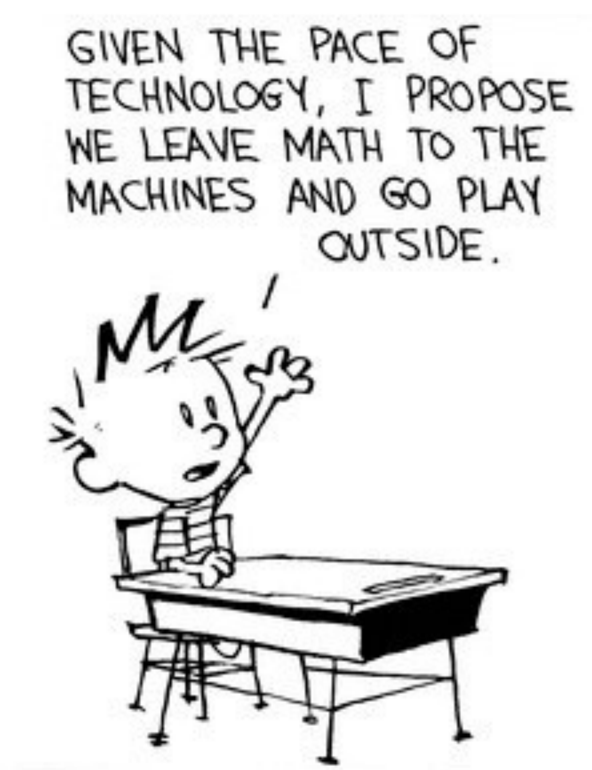
$$10^{10^{10}} n + n^2 = O(n^2)$$

$$n + n^2 = O(n^{20})$$

Moore's "law"

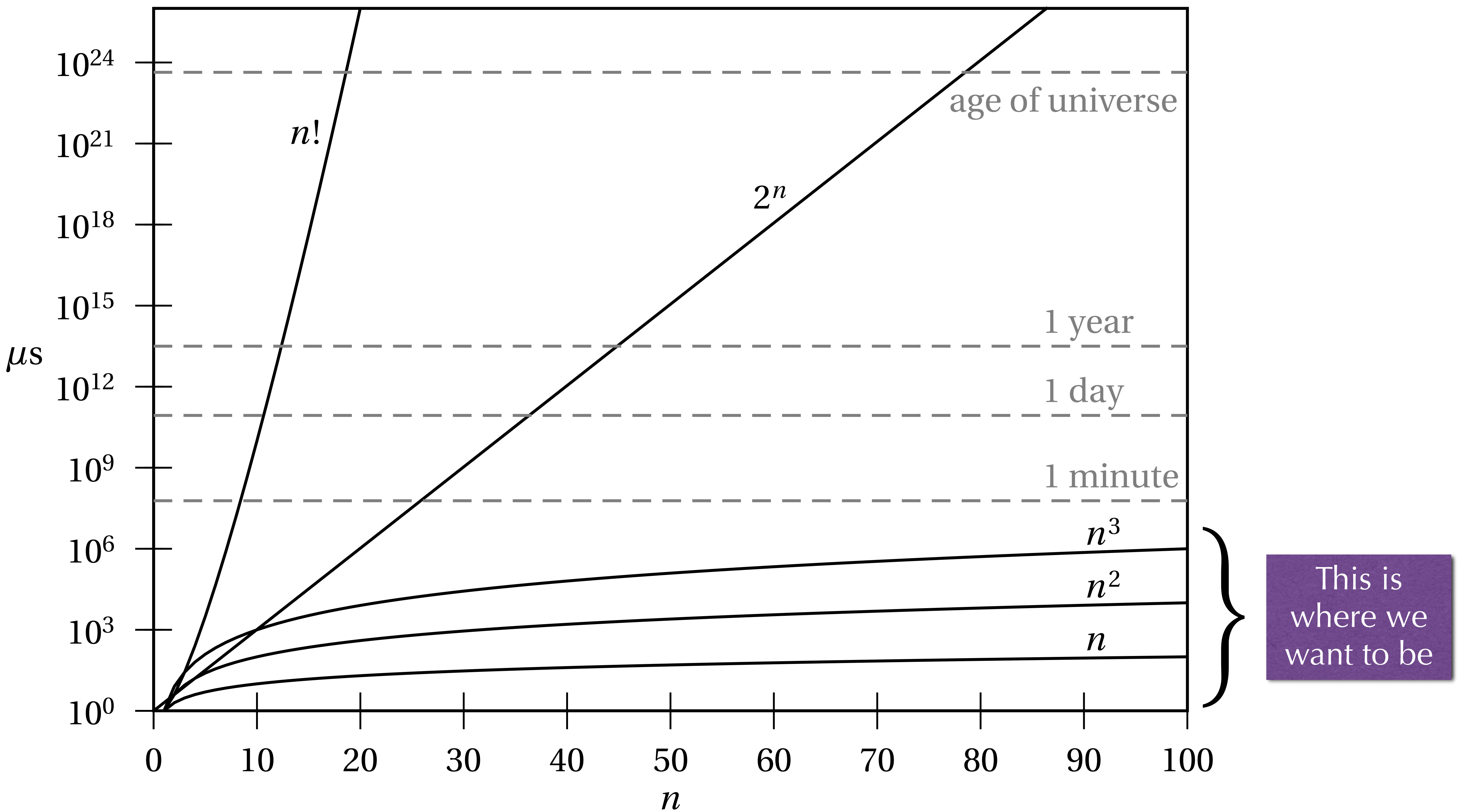
Gordon Moore, co-founder of Intel, predicted in 1965 that the number of transistors on integrated circuits would double every year for 10 years.





The expression Moore's "law" is commonly used to mean that the speed and memory of computers is expected to double every 18 months.

Orders of Growth



Moore's "law" means a small vertical shift from one machine to the next

Picture due to Moore & Mertens (2011).

IV. Lower Bounds

Complexity of a Problem

Def. The *complexity of a problem* is that of the most efficient (possibly unknown) algorithm that solves it.

Ex. Sorting n elements has complexity $O(n \log n)$ comparisons.

Proof. Mergesort (CSE103) reaches the bound.

```
def mergesort(l):
  if len(l) <= 1:
    return l
  else:
    l1 = first half of l
    l2 = second half of l
    return merge(mergesort(l1),mergesort(l2))
```

Ex. Sorting n elements has complexity $\Theta(n \log n)$ comparisons.

Proof. k comparisons cannot distinguish more than 2^k permutations and $\log_2 n! \sim n \log_2 n$.

Complexity of Powering

$$(x, n) \in \mathbb{A} \times \mathbb{N} \mapsto x^n \in \mathbb{A}$$

We already know it is $O(\log n)$ multiplications in $\mathbb{A}$.

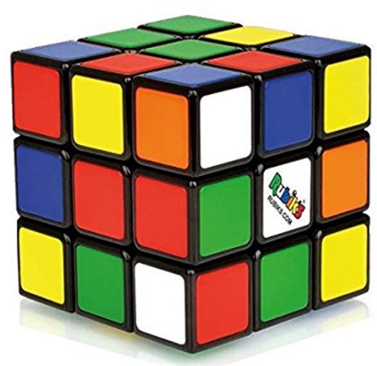
Can this be improved?

Lower bounds on the complexity require a precise definition (a model) of what operations the "most efficient" algorithm can perform.

Ex. If the only available operation in $\mathbb{A}$ is multiplication, x^{2^k} requires k multiplications, so that $\log_2 n$ is a lower bound.

Ex. In floating point arithmetic, $x^n = \exp(n \log x)$ and the complexity hardly depends on n.

Complexity of a Problem



Ex. Rubik's 3x3x3 cube has complexity $O(1)$.
Proof. Store the solutions of each of the finitely many configurations, and look them up.

… would be a better problem.

Simple Lower Bounds

In most useful models, reading the input and writing the output take time. Then,

$$\boxed{\text{size(Input)+size(Output)} \leq \text{complexity.}}$$

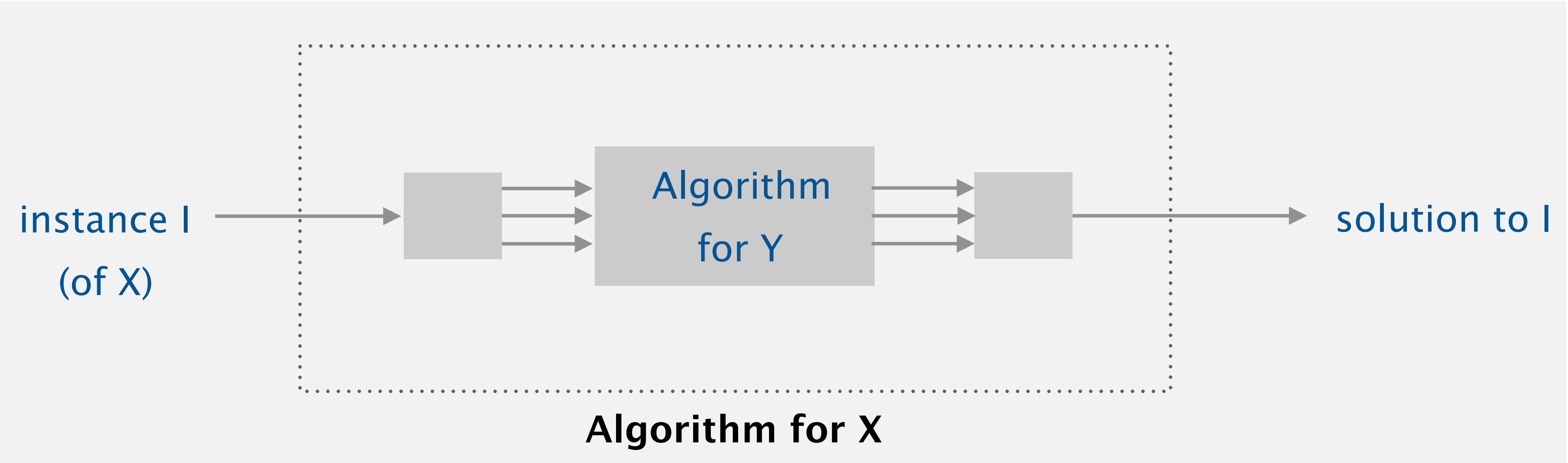
Examples:

Problem	Input	Simple Lower Bound	Best known algorithm	Measure
Sorting	n elts	n	O(n log n)	comparisons
Polynomial multiplication	degree n	n	O(n log n)	ops on coeffs
Matrix multiplication	size n x n	n^2	$O(n^{2.373})$	ops on coeffs
Subset sum	n integers	n	$2^{O(n)}$	time

V. Reductions

Reduction

Problem X reduces to problem Y if you can use an algorithm that solves Y to help solve X



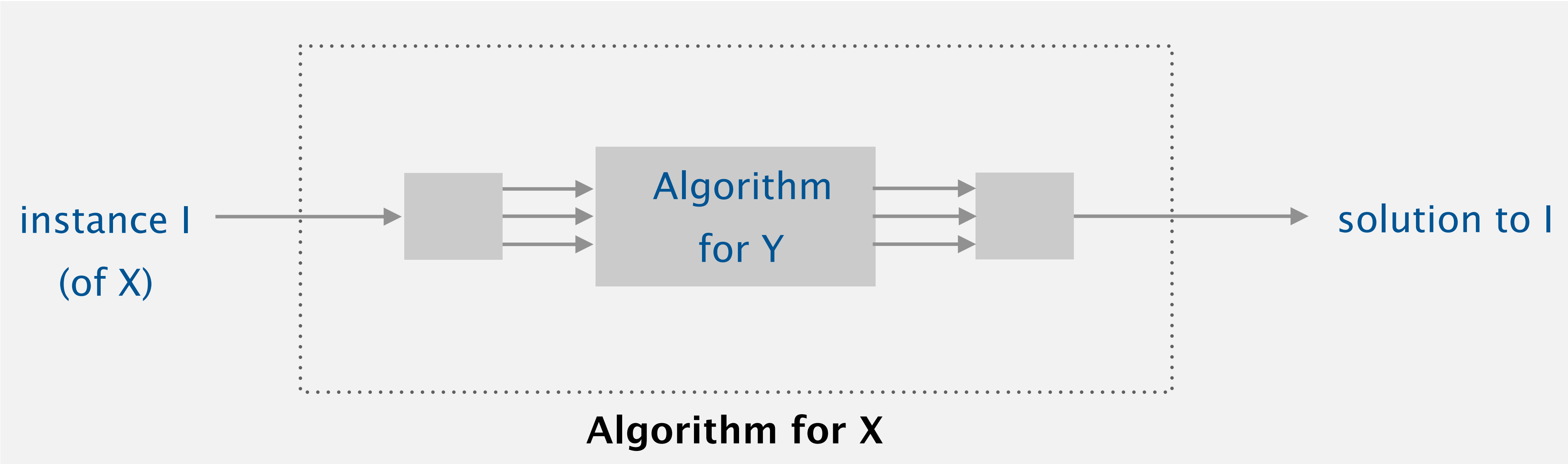
Complexity of solving X = complexity of solving Y + cost of the reduction

perhaps many calls to Y on instances of different sizes (typically, only one call)

preprocessing and postprocessing (typically, less than the cost of solving Y)

Reduction

Problem X reduces to problem Y if you can use an algorithm that solves Y to help solve X



Ex. [powering x^n reduces to multiplication]

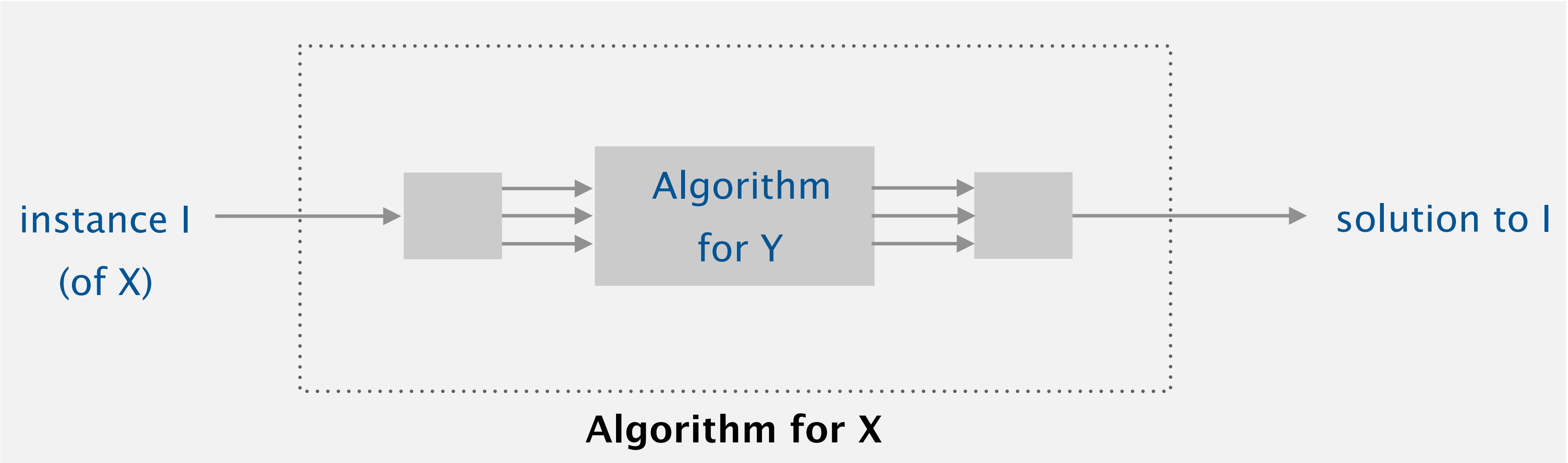
Binary Powering algorithm

Complexity: O(log n) * complexity of multiplication

discussed next week

Reduction

Problem X reduces to problem Y if you can use an algorithm that solves Y to help solve X



Ex. [finding the median of N items (the value separating the higher half from the lower half) reduces to sorting]

To find the median of N items:

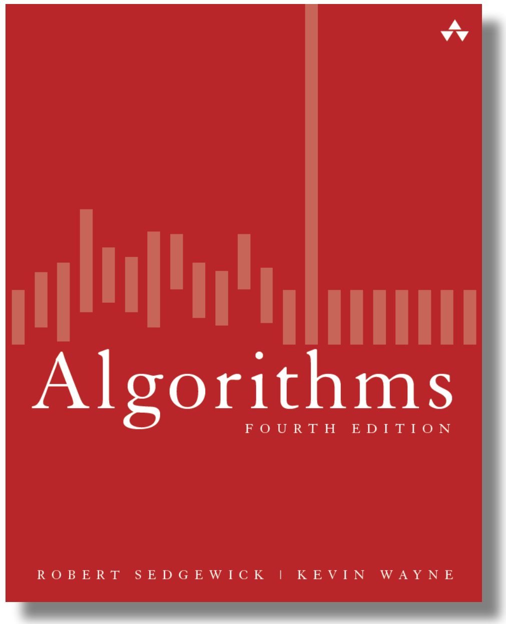
- Sort N items
- Return item in the middle

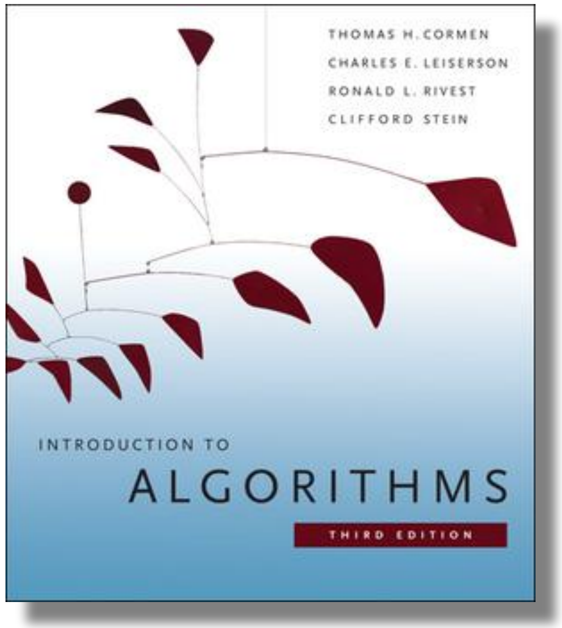
Complexity: N log N + 1 (cost of sorting + cost of reduction)

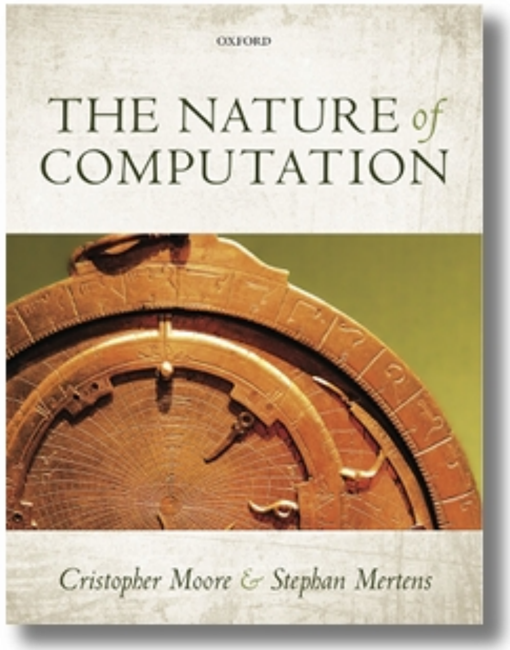
References

The slides are designed to be self-contained.

They were prepared using the following books that I recommend if you want to learn more:







Next

Assignment this week: optimal powering

Next tutorial: fast powering via addition chains

Next week: fast multiplication

Feedback

Moodle

Questions: constantin.enea@polytechnique.edu