Computer Problem set 2.

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The objective of this computer set is to implement different ways to simulate a discretization of the Brownian motion. The first question uses the forward approach by exploiting the independence of the increments of Brownian motion. The second question uses a backward approach with the conditional distribution. The third question is an application of both methods to compute the quadratic variation of the Brownian motion.

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1.1 Question 1-a

Let $n > 0$ and $1 \leq i \leq 2^n$ be integers.

We define the random variable Z_i as follows:

$$Z_i = \frac{W_{t_i^n} - W_{t_{i-1}^n}}{\sqrt{\Delta T}}$$

Since $(W_t)_{t \geq 0}$ is a Brownian motion, then by definition $W_{t_i^n} - W_{t_{i-1}^n}$ is a $\mathcal{N}(0, \Delta T)$ random varibale and thus Z_i is a $\mathcal{N}(0,1)$ random variable. And again since $(W_t)_{t \geq 0}$ is a Brownian motion the increments are independent which means that the variables $(Z_j)_{1 \leq j \leq 2^n}$ are iid $\mathcal{N}(0,1)$ random variables.

1.2 Questions 1-b , 1-c

Simulation results for T=1 and (N = 1000 number of copies):

As predicted by theroy, we have:

$$Var(W_T) \simeq T$$

and

$$CovMatrix(W_T, W_{\frac{T}{2}}) \simeq \begin{bmatrix} T & \frac{T}{2} \\ \frac{T}{2} & \frac{T}{2} \end{bmatrix}$$

We observe when estimating the quantities $Var(W_T)$ and $CovMatrix(W_T, W_{\frac{T}{2}})$ using the N discrete trajectories the results do not vary much when n becomes bigger. This observation may seem weird at first since one would expect that the estimations will be more accurate. However, it is natural that increasing n does not affect the precision because it's only a parameter of discretization and since we only compute $Var(W_T)$ and $CovMatrix(W_T, W_{\frac{T}{2}})$ it does not matter how large n is because only the number of copies N affects the estimation precision of these two quantities, while n quantifies the extent of detail to which we have information on the random trajectories.

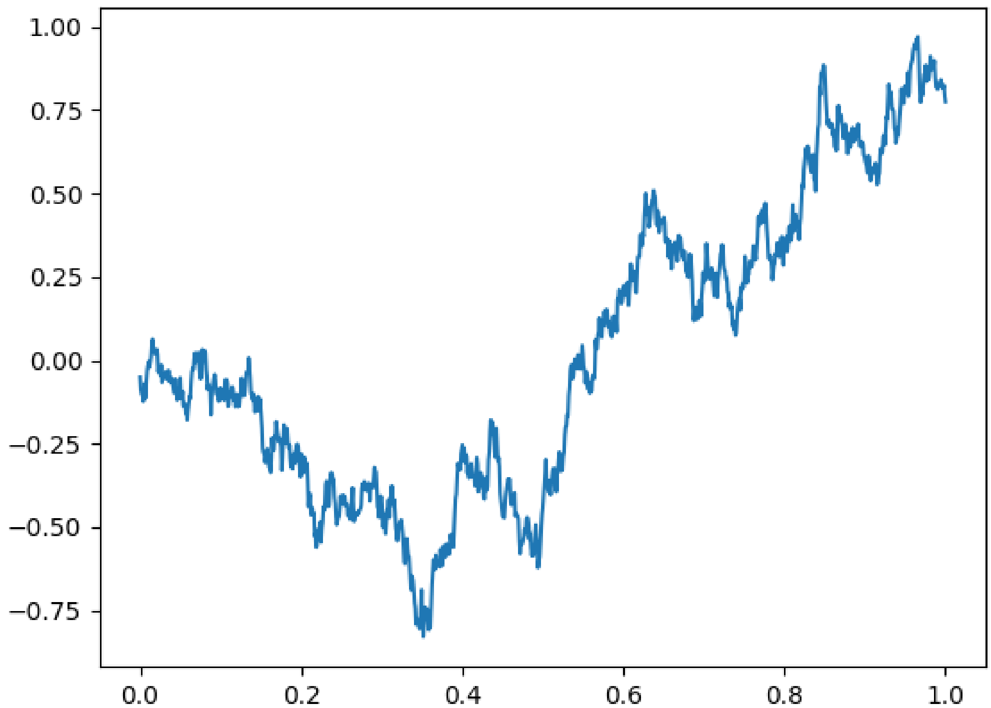

Figure 1: Trajectory simulated by forward approach

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2.1 Question 2.a

We have: $W_{\bar{s}}|(W_{s_1} = x_1, W_{s_2} = x_2) = W_{s_1} + (W_{\bar{s}} - W_{s_1})|(W_{s_1} = x_1, W_{s_2} = x_2)$ Then since W_{s_1} , $(W_{\bar{s}} - W_{s_1})$ are independent and W_{s_1} , $(W_{s_2} - W_{s_1})$ we can write the following:

$$W_{\bar{s}}|(W_{s_1} = x_1, W_{s_2} = x_2) = x_1 + (W_{\bar{s}} - W_{s_1})|(W_{s_2} - W_{s_1} = x_2 - x_1)$$

And since $(W_t - W_{s_1})$ is a Brownian motion, we can deduce from the information given that: $(W_{\bar{s}} - W_{s_1})|(W_{s_2} - W_{s_1} = x_2 - x_1)$ is normally distributed with conditional mean $\frac{\bar{s}-s_1}{s_2-s_1}(x_2 - x_1)$ and conditional variance $\frac{\bar{s}-s_1}{s_2-s_1}(s_2 - \bar{s})$.

Then the distribution of $(W_{\bar{s}} - W_{s_1})|(W_{s_2} - W_{s_1} = x_2 - x_1)$ is $\mathcal{N}(\frac{x_2 - x_1}{2}, \frac{s_2 - s_1}{4})$
And we can deduce that:

$$W_{\bar{s}}|(W_{s_1} = x_1, W_{s_2} = x_2) \overset{D}{\sim} \mathcal{N}(\frac{x_2 + x_1}{2}, \frac{s_2 - s_1}{4})$$

2.2 Question 2.b

We have:

$$W_{\bar{s}}|(W_{s_1} = x_1, W_{s_2} = x_2, (W_u)_{u \notin [s_1, s_2]}) = W_{\bar{s}}|((W_u)_{u \in [0, s_1[}, W_{s_1} = x_1, W_{s_2} = x_2, (W_u)_{u \in [s_2, +\infty[})$$

And since (W_t) is a Markovian process we then have :

$$\begin{aligned}
W_{\bar{s}}|(W_{s_1} = x_1, W_{s_2} = x_2, (W_u)_{u \notin [s_1, s_2]}) &= W_{\bar{s}}|(W_{s_1} = x_1, W_{s_2} = x_2, (W_u)_{u \in [s_2, +\infty[}) \\
&= W_{\bar{s}}|(W_{s_1} = x_1, W_{s_2} = x_2, (W_u - W_{s_2})_{u \in [s_2, +\infty[}) \\
&= W_{\bar{s}}|(W_{s_1} = x_1, W_{s_2} = x_2)
\end{aligned}$$

The last equality is due to the fact that $((W_u - W_{s_2})_{u \in [s_2, +\infty[})$ is independent of $W_{\bar{s}}$
Then, using the result of question 2.b we get that

$$W_{\bar{s}}|(W_{s_1} = x_1, W_{s_2} = x_2, (W_u)_{u \notin [s_1, s_2]}) \overset{D}{\sim} \mathcal{N}(\frac{x_2 + x_1}{2}, \frac{s_2 - s_1}{4})$$

2.3 Questions 2.c and 2.d

When computing $Var(W_T)$ and $CovMatrix(W_T, W_{\frac{T}{2}})$ we get the same results as we did in the forward approach:
As predicted by theroy, we have:

$$Var(W_T) \simeq T$$

and

$$CovMatrix(W_T, W_{\frac{T}{2}}) \simeq \begin{bmatrix} T & \frac{T}{2} \\ \frac{T}{2} & \frac{T}{2} \end{bmatrix}$$

The remark on the effect of n on the estimations remains valid for this approach as well.

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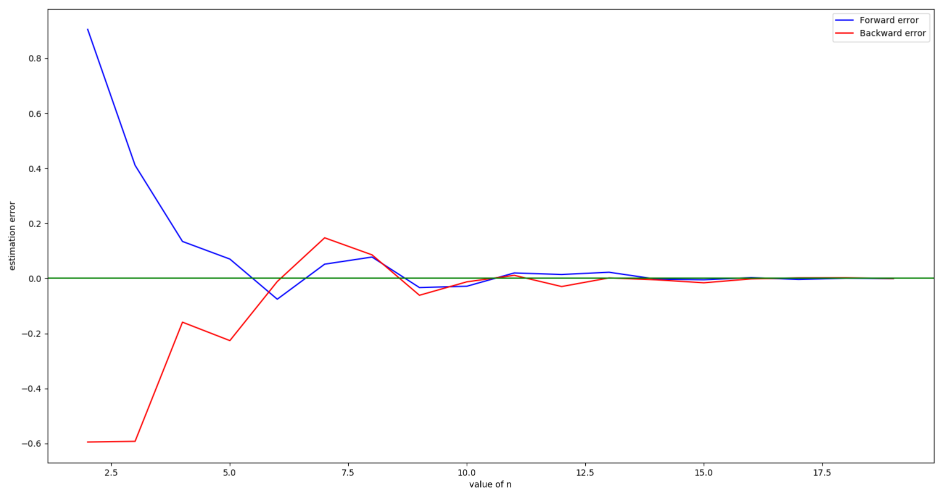

Figure 2: Quadratic Variation estimation error