Computer Problem set 3.

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The objective of this CPS is to simulate various ways of defining stochastic integrals and to detect the differences between the stochastic integral of a random process (Ito's integral) with the stochastic integral of a continuous function (Weiner's Integral). The questions 1 and 2 do illustrate the difference between the two and the importance of a precise definition of stochastic integrals.

1

In this question we simulate different series that can define the stochastic integral of a random process the and compare the results

1.1 Questions 1-a , 1-b, 1-c

The variables I_n , J_n can be interpreted as different Reimann series candidates to define the stochastic integral:

$$\int_0^T W_s dWs$$

And K_n is a the mean of the two series. When integrating a continuous function in the classic Reimann integral the three series converge to the same value that is the integral of the function. In the case of random processes however, we can see already that the expectations of the three series are completely different. A simple calculation shows that:

$$\mathbb{E}(I_n) = 0$$
$$\mathbb{E}(J_n) = T$$
$$\mathbb{E}(K_n) = \frac{T}{2}$$

Thus the three series cannot define a single quantity when n tends to $+\infty$.

We observe in the following graphic the means of $\frac{1}{2}W_T - I_n$, $\frac{1}{2}W_T - I_n$ and $\frac{1}{2}W_T - I_n$ and confidence intervals for different values of n. The figure clearly shows the difference in the means of the three variables. Ito's integral $\int_0^T W_s dWs$ is defined by the serie I_n, and it's value is $\frac{1}{2}(W_T^2 - T)$

And the reason why the limits of the three series are different is the irregularity of the Brownian motion and in particular it's non null quadratic variation.

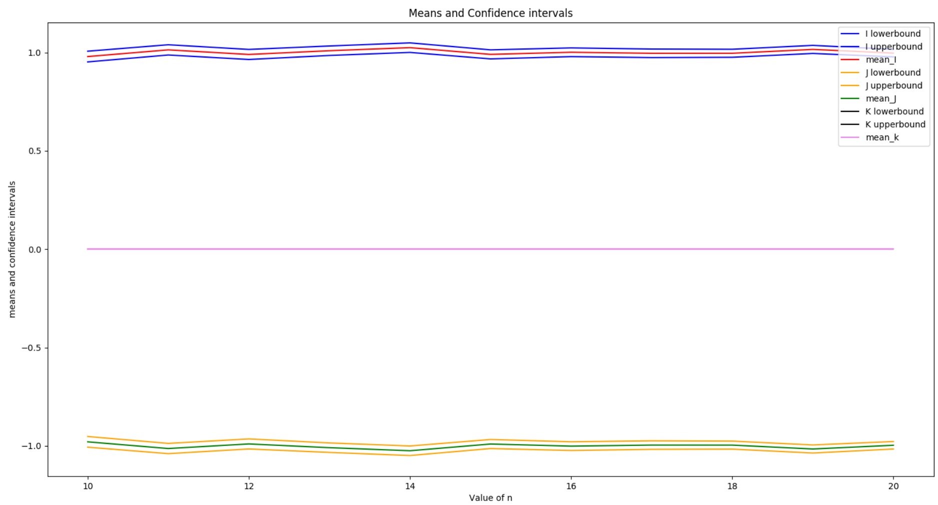

Figure 1: Quadratic Variation estimation error

2

In this question we simulate different series that can define the stochastic integral of a deterministic continuous function and compare the results.

The variables A_n , B_n and C_n can be interpreted as different Reimann series candidates to define the stochastic integral:

$$\int_0^T exp(s)dWs$$

The remarkable difference between this integral and the integral of question 1 is that in this case the three variables have the same mean and converge to the same quantity. It is also important not to forget that the function exp being deterministic is $\mathcal{F}_t$-measurale for every $t \geq 0$. Therefore choosing the point $exp(W_{t_{i-1}})$ or $exp(W_{t_i})$ in the Reimann sum yield the same results.

3

The mean of the variable $A_n = sin(W_T) - \frac{1}{2n}\sum_{i=1}^{n} sin(W_{t_{i-1}})$ is 0 because the distribution of W_t is symmetric with respect to 0 for every $t \geq 0$ and sin is an odd function. The confidence intervalles however are not affected by the bariable n but rather by the variable M which represents the Monte-Carlo simulation effort.

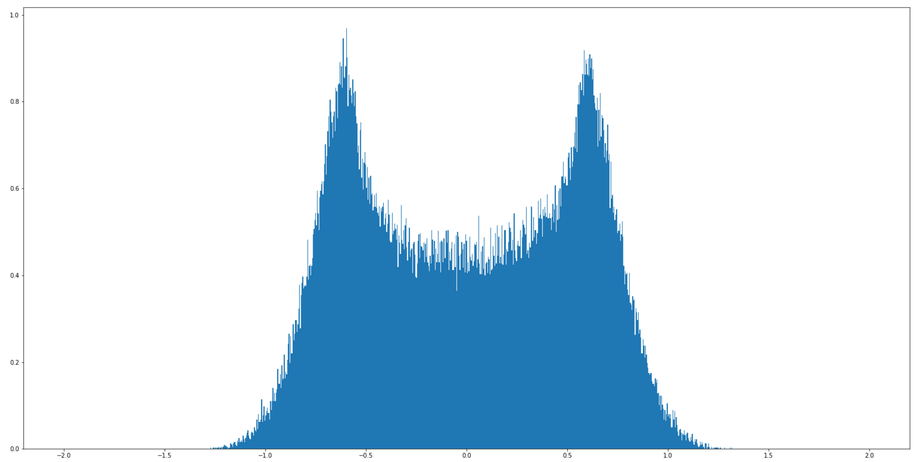

Figure 2: Distribution of A_n for n=100