Computer Problem Set 5

Girsanov Theorem

The present problem set is attached to Chapter 7, Section 7.2 of the lectures notes. All implementations should be run with the value $T = 1$. Given a scalar Brownian motion B, we consider the Black-Scholes model

$$dS_t \quad = \quad S_t(rdt + \sigma dB_t),$$

with positive parameters r, σ. Our objective is the numerical approximation of

$$\Delta_0 \ := \ \mathbb{P}\big[S_T \geq K\big], \quad \textit{for large} \quad K > 0. \tag{1}$$

By the Girsanov theorem, we may introduce for all $\theta \in \mathbb{R}$ a probability measure $\mathbb{Q}^\theta$, under which $\{B_t^\theta := B_t - \theta t, t \geq 0\}$ is a Brownian motion, and so that:

$$\Delta_0 = \mathbb{E}^{\mathbb{Q}^\theta}\left[\left(Z^\theta\right)^{-1}\mathbf{1}_{\{S_T \geq K\}}\right] \quad \textit{where} \quad \frac{d\mathbb{Q}^\theta}{d\mathbb{P}} = Z^\theta := e^{\theta B_T - \frac{1}{2}\theta^2 T}. \tag{2}$$

Although Δ_0 is independent of θ, various values of θ induce different Monte Carlo approximations of Δ_0. We then introduce the variance of the Monte Carlo estimator based on the last representation

$$V^\theta \quad := \quad \mathbb{E}^{\mathbb{Q}^\theta}\left[\left(Z^\theta\right)^{-2}\mathbf{1}_{\{S_T \geq K\}}\right] - \Delta_0^2.$$

1. Build a program which produces a Monte Carlo approximation of Δ_0 based on the original representation (1) and a sample of M independent copies of S_T.

2. Build a program which produces an $M-$sample Monte Carlo approximation of Δ_0 based on the representation (2) for some value of the parameter θ.

3. Consider the parameters values $S_0 = 100$, $K = 150$, $r = 0.02$, $\sigma = 0.4$, and let θ be ranging in the set $[-3, 3]$. Plot a graph with the difference between the Monte Carlo estimator and the true value of Δ_0 (which can be computed explicitly in terms of the cumulative distribution of the $\mathcal{N}(0, 1)$ distribution).

4. Build a program which produces a Monte Carlo approximation of the variance V^θ for each given value of the parameter θ.

5. Using the values of the parameters of Question 3, plot the Monte Carlo approximation of the variance V^θ in terms of θ. Investigate the stability of the results in terms of the sample size M. Comment the findings.