NON-HOMOGENEITY OF LIE AND JORDAN PRODUCTS UNDER NON-ABELIAN GRADINGS

YASSINE AIT MOHAMED

Let R be a G-graded associative ring, where G is a group with identity e. For $x, y \in R$, we denote $[x, y] = xy - yx$ (Lie product) and $x \circ y = xy + yx$ (Jordan product).

Proposition 1. *If G is abelian, then $[x, y] \in R_{gh}$ and $x \circ y \in R_{gh}$.*

Proof. Straightforward. $\square$

When G is non-abelian, homogeneity fails. We demonstrate this with an explicit counterexample.

Example 1. *Consider $R = M_4(k)$ graded by $D_{10} = \langle a, b \mid a^5 = b^2 = e, ba = a^{-1}b \rangle$, with grading components:*

$$R_e := \begin{pmatrix} k & 0 & 0 & 0 \\ 0 & k & 0 & 0 \\ 0 & 0 & k & 0 \\ 0 & 0 & 0 & k \end{pmatrix}, \quad R_a := \begin{pmatrix} 0 & k & 0 & 0 \\ 0 & 0 & k & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad R_{a^2} := \begin{pmatrix} 0 & 0 & k & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$R_{a^3} := \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ k & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad R_b := \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & k \\ 0 & 0 & 0 & 0 \\ 0 & k & 0 & 0 \end{pmatrix}, \quad R_{ab} := \begin{pmatrix} 0 & 0 & 0 & k \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ k & 0 & 0 & 0 \end{pmatrix}$$

$$R_{a^4b} := \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & k \\ 0 & 0 & k & 0 \end{pmatrix}, \quad R_{a^4} := \begin{pmatrix} 0 & 0 & 0 & 0 \\ k & 0 & 0 & 0 \\ 0 & k & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad R_{a^2b} = R_{a^3b} = \{0\}$$

Direct computation yields

$$[R_b, R_{a^4}] = R_b \circ R_{a^4} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & k \\ k & 0 & 0 & 0 \end{pmatrix} \notin H(R)$$