Math 340 - Exam 2 Study Guide

Exam 2 will cover weeks 5-9 of the class.

Sample Problems

1. Evaluate the following determinant by each of the following methods.

   (a) Using row operations, i.e. Theorem 5.1.

   (b) Using co-factor expansion.

$$\det \begin{pmatrix} 1 & -1 & 2 \\ 0 & 2 & -1 \\ 3 & 4 & 1 \end{pmatrix}$$

2. Show that if the entries of an invertible matrix are all rational, then all entries of its inverse are also rational. (Example 5.11)

3. Suppose $\mathbf{a}$ is a limit point of D, where D is a subset of $\mathbb{R}^n$. Prove that every open ball centered at $\mathbf{a}$ contains infinitely many points of D. (Example 6.9.)

4. Find each limit or show it does not exist. Do it once using the definition of limit, and once using theorems if possible.

   (a) $\displaystyle\lim_{(x,y)\to(1,0)} \frac{x^2+2y}{x+y}$.

   (b) $\displaystyle\lim_{(x,y)\to(0,0)} (x+y)\sin\left(\frac{1}{x^2+y^2}\right)$.

   (c) $\displaystyle\lim_{(x,y)\to(0,0)} \frac{y^2}{x^2+y^2}$.

   (d) $\displaystyle\lim_{(x,y)\to(2,1)} xy-x^2+y$.

   (Example 6.11.)

5. Show that the following function is not continuous at $(0,0)$.

$$f(x,y) = \begin{cases} \dfrac{x^2-y^2}{x^2+y^2} & \text{if } (x,y)\neq(0,0) \\ 0 & \text{if } x=y=0 \end{cases}$$

   (Exercise 6.8.)

6. Suppose $\mathbf{x}_0$ is a point in $\mathbb{R}^n$ and D is a nonempty compact subset of $\mathbb{R}^n$. Prove that there exists a closest point $\mathbf{y}_0 \in D$ to x_0. In other words $||\mathbf{x}_0 - \mathbf{y}_0|| \le ||\mathbf{x}_0 - \mathbf{y}||$ for all $\mathbf{y} \in D$. (Example 7.17.)

7. Prove that the intersection of a closed subset of $\mathbb{R}^n$ and a compact subset of $\mathbb{R}^n$ is compact. (Example 7.19.)

8. Prove that the function

$$f(x, y, z) = \sin(x + 2y + 3z) + \cos(z) + \sin(x - y) + \cos(x + y)$$

attains its maximum and minimum values over $\mathbb{R}^3$. (Example 7.21.)

9. Given a real number a, find the derivative and the differential of each of the following functions at a:

   (a) $f(x) = (1 + x, e^x, \sin(2x))$.

   (b) $g(x) = (x^2, 3, x)$.

   (c) $h(t) = (1 + t^2, 2t - \cos t, \sqrt{1 + t^2})$.

   (Example 7.17.)

10. Consider the function given by

$$f(x, y) = \begin{cases} \dfrac{x^2y - y^3}{x^2 + y^2} & \text{if } (x, y) \neq (0, 0) \\ 0 & \text{otherwise} \end{cases}$$

   (a) Show that $f(x, y)$ is continuous on $\mathbb{R}^2$.

   (b) Find $D_{\mathbf{u}}f(0, 0)$ for every nonzero vector $\mathbf{u} = (a, b)$.

   (c) Show that f is not differentiable at $(0, 0)$.

   (Example 8.14.)

11. Find the maximum and minimum directional derivatives of the function $f(x, y) = x^3 \sin y + xe^y$ at the origin. (Example 8.15)

12. Suppose $L : \mathbb{R}^n \to \mathbb{R}^m$ is a linear mapping with matrix A.

   (a) Using the definition of derivatives, show that the differential of L is itself. Deduce the derivative of L is A.

   (b) Conversely, prove that if the derivative of a function $F : \mathbb{R}^n \to \mathbb{R}^m$, satisfying $F(\mathbf{0}) = \mathbf{0}$, is a constant matrix A, then F is linear.

   (Exercise 8.4.)

13. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a function with continuous second partials. Define a function g by $g(r, \theta) = f(r\cos\theta, r\sin\theta)$. Prove that

$$||\nabla f||^2 = \left(\frac{\partial g}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial g}{\partial \theta}\right)^2.$$

   (Exercise 8.8.)

14. Find and classify all critical points of each function:

    (a) $f(x, y) = x^2 + y^2 + xy + 2x - 2y$.

    (b) $f(x, y) = x^4 + x^2 + y^4$.

    (Example 9.10.)

15. Find the plane or hyper-plane tangent to each manifold at the given point. Assume the given set is a manifold.

    (a) $x_1^2 + 3x_2^2 + x_3^2 = 2$ at $(1, 0, -1)$.

    (b) $x_1^4 + 4x_2 \sin(x_1 x_3) + x_3^2 + 3x_4^2 = 4$ at $(0, 0, 1, -1)$.

    (Example 9.11.)

16. In each case below, find the maximum and minimum values of the given function subject to the given constraint or show they do not exist:

    (a) $f(x, y) = x^3 + 2y^2$ given that $x^2 + 3y^2 = 1$.

    (b) $f(x, y) = 3x^4 + 4y^4$ with the constraint $x^2 + y^2 = 1$.

    (c) $f(x, y, z) = \sin x + \sin y + \sin z$ subject to $x + y + z = \pi$.

    (d) $f(x, y, z) = x^2 + 2y^2 + z^2$ given $3x + 2y + z = 1$.

    (Example 9.12.)

For more practice problems check the examples in the textbook. Solutions to homework problems are posted on ELMS under “Files”.