Math 340 Homework 7

Dr. Ebrahimian

The following problems are due 10/18/2024 before class starts.

Problems for Grading

1. (10 pts) Find the derivative and differential of the function $f : \mathbb{R} \to \mathbb{R}^3$ given by

$$f(t) = (1 - t^2, 2 + \sin t, \tan(1 + t^2)).$$

2. (10 pts) Consider the set

$$A = \{(x, y) \in \mathbb{R}^2 \mid x \geq 0, \text{ and } y > 0\}.$$

   (a) Geometrically sketch this set and explain if it is closed, open or neither closed nor open.

   (b) Carefully prove your claim in part (a).

3. (10 pts) Suppose A is a compact subset of $\mathbb{R}^n$ and B is an open subset of $\mathbb{R}^n$. Show that $A - B$ is compact.

4. (10 pts) Suppose $f : \mathbb{R} \to \mathbb{R}^n$ is differentiable. Prove that $||f(t)||$ is constant, if and only if $f(t)$ and $f'(t)$ are orthogonal for every $t \in \mathbb{R}$.

   Hint: $||f(t)||^2 = f(t) \cdot f(t)$.

5. (10 pts) Prove that every non-empty open subset of $\mathbb{R}^n$ is a union of a collection of open balls; all of which have an irrational radius.

6. (10 pts) Let D be a non-empty compact subset of $\mathbb{R}^n$. For every $\mathbf{x} \in \mathbb{R}^n$ let $f(\mathbf{x})$ be the minimum distance between $\mathbf{x}$ and points of D. (See Example 7.17). Prove that $f : \mathbb{R}^n \to \mathbb{R}$ is continuous.

   Hint: Use the definition of limit.

7. (10 pts) Suppose $f, g : \mathbb{R} \to \mathbb{R}^n$ are two differentiable curves and $(s_0, t_0) \in \mathbb{R}^2$ is a point for which the points $f(t_0)$ and $g(s_0)$ are closer than any other points on the two curves. Prove that $f(t_0) - g(s_0)$ is orthogonal to both $f'(t_0)$ and $g'(s_0)$. Use this fact to find the closest distance between lines $f(t) = (t + 1, t, t - 1)$ and $g(s) = (2s, s - 1, 2s + 1)$.

   Hint: Show that t_0 must be a critical point of $||f(t) - g(s_0)||^2$.

8. (10 pts) Find the length of the curve $f : [0, 1] \to \mathbb{R}^4$ given by $f(t) = \left(1, \frac{4}{3}t^{3/2}, \frac{t^2}{2}, 2t\right)$.

Practice Problems

9. Example 7.13.

10. Example 7.14.

11. Example 7.20.

12. Example 7.21.

13. Example 7.22.

14. Exercise 7.1.

15. Exercise 7.10.

16. Exercise 7.20.