Math 340 - Exam 1 Study Guide

Exam 1 will cover the first four weeks of the class. It will not cover determinants. The problems on the exam will be similar to the ones listed below.

Sample Problems

1. Prove that for all sets $A, B_1, B_2, \ldots$, we have $A \bigcap (\bigcup_{n=1}^{\infty} B_n) = \bigcup_{n=1}^{\infty} (A \cap B_n)$. (Example 1.13)

2. Define the Fibonacci sequence F_n by $F_0 = 0, F_1 = 1$, and $F_{n+2} = F_{n+1} + F_n$ for all $n \geq 0$. Prove that $F_n < 2^n$ for all $n \geq 0$. (Example 1.17)

3. Prove that for every positive integer n, there is a polynomial $p_n(x)$ for which the n-th derivative of e^{x^2} at x is equal to $p_n(x)e^{x^2}$. (Example 1.20)

4. Prove that
$$\bigcup_{x \in [0,1]} ([x,1] \times [0,x^2]) = \{(x,y) \mid 0 \leq x \leq 1 \text{ and } 0 \leq y \leq x^2\}$$
(Exercise 1.8)

5. Prove parts (b) and (c) of Theorem 1.2: Suppose $f : A \to B$ is a function, and $T_i \subseteq B$ for $i \in I$. Then
   (a) $f^{-1}(\bigcup_{i \in I} T_i) = \bigcup_{i \in I} f^{-1}(T_i)$.
   (b) $f^{-1}(\bigcap_{i \in I} T_i) = \bigcap_{i \in I} f^{-1}(T_i)$.

   (Exercise 1.15)

6. Determine if each of the following is a subspace of $\mathbb{R}^2$.
   (a) The set of points on the line $3x + 2y = 1$.
   (b) The set of points on the line $4x - 3y = 0$.
   (c) The set of points on the unit circle $x^2 + y^2 = 1$.

   (Example 2.9)

7. Let S and T be two subsets of $\mathbb{R}^n$. Then span S = span T if and only if $S \subseteq$ span T and $T \subseteq$ span S (Example 2.12)

8. Suppose $\mathbf{v}_1, \ldots, \mathbf{v}_n$ are linearly independent vectors of $\mathbb{R}^k$ and $\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_m$ are also linearly independent vectors of $\mathbb{R}^k$. Prove that $\mathbf{v}_1, \ldots, \mathbf{v}_n, \mathbf{w}_1, \ldots, \mathbf{w}_m$ are linearly independent if and only if

$$\operatorname{span}\{\mathbf{v}_1, \ldots, \mathbf{v}_n\} \cap \operatorname{span}\{\mathbf{w}_1, \ldots, \mathbf{w}_m\} = \{\mathbf{0}\}.$$

(Example 2.14)

9. Determine if each of the following matrices is in echelon form, reduced echelon form or neither. If the matrix is not in reduced echelon form, turn it into reduced echelon form by appropriate elementary row operations. In each step make sure you specify which row operation is used.

(a) $\begin{pmatrix} 1 & 0 & -1 \\ 0 & -1 & 0 \\ 0 & 2 & 3 \end{pmatrix}$

(b) $\begin{pmatrix} -1 & 0 & 2 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 5 \end{pmatrix}$

(Example 2.15)

10. Suppose U and W are subspaces of $\mathbb{R}^n$ for which $U \cup W$ is also a subspace. Prove that $U \subseteq W$ or $W \subseteq U$. (Exercise 2.3)

11. Consider the homogeneous system

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1k}x_k = 0 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2k}x_k = 0 \\ \qquad\qquad\vdots \\ a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nk}x_k = 0 \end{cases}$$

Prove that the set of vectors $(x_1, x_2, \ldots, x_k) \in \mathbb{R}^k$ satisfying the system above is a subspace of $\mathbb{R}^k$. (Exercise 2.4)

12. Prove that if $n > 1$, then $\mathbb{R}^n$ can be written as the union of all of its proper subspaces. (Exercise 2.11)

13. Prove that if $||\cdot||$ is a norm relative to an inner product of $\mathbb{R}^n$ and $\mathbf{v}, \mathbf{w} \in \mathbb{R}^n$, then

$$||\mathbf{v} + \mathbf{w}||^2 + ||\mathbf{v} - \mathbf{w}||^2 = 2(||\mathbf{v}||^2 + ||\mathbf{w}||^2).$$

(Example 3.15)

14. Consider the subspace V of $\mathbb{R}^4$ spanned by $\mathbf{v} = (1, 2, 0, 1)$ and $\mathbf{w} = (1, -1, 1, 2)$. Find a basis for the orthogonal complement of V relative to the dot product. (Example 3.16)

15. Determine the dimension of each vector space.

(a) The subspace of $\mathbb{R}^3$ generated by vectors $(1, 2, -1), (2, 3, 4)$, and $(4, 10, 2)$.

(b) The subspace of $\mathbb{R}^3$ generated by $(1, 2, 0), (-1, 1, 1)$, and $(1, 5, 1)$.

(Exercise 3.3)

16. Let V be a subspace of $\mathbb{R}^n$. Prove that if $\mathcal{A} = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$ is a linearly independent set of vectors in V, then there is a basis for V that contains $\mathcal{A}$.
(Exercise 3.4)

17. Suppose $c_1, \ldots, c_n$ are real constants. Define a function $\langle , \rangle$ by

$$\langle (x_1, \ldots, x_n), (y_1, \ldots, y_n) \rangle = \sum_{j=1}^{n} c_j x_j y_j.$$

(a) Show $\langle , \rangle$ is linear and symmertic.

(b) Show that $\langle , \rangle$ is an inner product iff $c_1, \ldots, c_n$ are all positive.

(Exercise 3.13)

18. Let $\mathcal{B} = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ be a basis for $\mathbb{R}^n$. For every two vectors

$$\mathbf{v} = a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + \cdots + a_n\mathbf{v}_n, \text{ and } \mathbf{w} = b_1\mathbf{v}_1 + b_2\mathbf{v}_2 + \cdots + b_n\mathbf{v}_n \text{ in } \mathbb{R}^n,$$

define $\langle \mathbf{v}, \mathbf{w} \rangle = a_1b_1 + a_2b_2 + \cdots + a_nb_n$. Prove that this defines an inner product on $\mathbb{R}^n$.
(Exercise 3.17)

19. Prove that if $\mathbf{v}_1, \ldots, \mathbf{v}_n$ are vectors in $\mathbb{R}^m$ with a norm $|| \cdot ||$, then

$$||\mathbf{v}_1 + \cdots + \mathbf{v}_n|| \leq ||\mathbf{v}_1|| + \cdots + ||\mathbf{v}_n||.$$

(Example 4.12)

20. Find Ker$L$, and im$L$, where $L : \mathbb{R}^3 \to \mathbb{R}^2$ is defined by

$$L(\mathbf{v}) = \begin{pmatrix} 1 & 2 & 3 \\ -1 & -1 & 0 \end{pmatrix} \mathbf{v}.$$

(Example 4.13)

21. Determine if each of the following functions is linear:

(a) $L : \mathbb{R}^2 \to \mathbb{R}$, given by $L(x, y) = xy$.

(b) $L : \mathbb{R}^2 \to \mathbb{R}$, given by $L(x, y) = x + 3y$.

(c) $L : \mathbb{R}^n \to \mathbb{R}$, given by $L(x_1, \ldots, x_n) = x_1$.

(Example 4.19)

22. Let V, W be two vector spaces, and let $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ be a basis for V. Assume $S, T : V \to W$ are linear transformations. Prove that $S = T$ if and only if $S(\mathbf{v}_j) = T(\mathbf{v}_j)$ for $j = 1, \ldots, n$.
(Example 4.20)

23. Find all linear transformations $T : \mathbb{R}^3 \to \mathbb{R}^2$ satisfying all of the following:

$$T(1,2,0) = (0,2), T(-1,1,1) = (-2,3), \text{ and } T(1,-2,-1) = (1,-3).$$

(Exercise 4.2)

24. Suppose $T : V \to W$ is a linear transformation between vector spaces. Using induction, prove that for every $c_1, \ldots, c_n \in \mathbb{R}$ and every $\mathbf{v}_1, \ldots, \mathbf{v}_n \in V$, we have

$$T(c_1\mathbf{v}_1 + \cdots + c_n\mathbf{v}_n) = c_1T(\mathbf{v}_1) + \cdots + c_nT(\mathbf{v}_n).$$

(Exercise 4.10)

25. Suppose $L : V \to W$ is a bijective linear transformation. Prove that $L^{-1} : W \to V$ is linear. (Exercise 4.16)

For more practice problems check the examples in the textbook. Solutions to homework problems are posted on ELMS under “Files”.