Honors Linear Algebra and Multivariable Calculus

Math 340

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Notations

- $\in$, belongs to.
- $\forall$, for all.
- $\exists$, there exists or for some.
- D_f, the domain of function f.
- $\operatorname{Im} f$ or R_f, the image of function f.
- $\mathbb{N}$, the set of nonnegative integers.
- $\mathbb{Z}^+$, the set of positive integers.
- $\mathbb{Q}$, the set of rational numbers.
- $\mathbb{R}$, the set of real numbers.
- $A \subseteq B$, set A is a subset of set B.
- $A \subsetneqq B$, set A is a proper subset of set B.
- $A \cup B$, the union of sets A and B.
- $A \cap B$, the intersection of sets A and B.
- $\bigcup\limits_{i=1}^{n} A_i$, the union of sets $A_1, A_2, \ldots, A_n$.
- $\bigcap\limits_{i=1}^{n} A_i$, the intersection of sets $A_1, A_2, \ldots, A_n$.
- $A_1 \times A_2 \times \cdots \times A_n$, the Cartesian product of sets $A_1, A_2, \ldots, A_n$.
- $\emptyset$, the empty set.
- $f^{-1}(T)$, the inverse image (or pre-image) of set T under function f.
- $f(S)$, the image of set S under function f.
- span $\mathcal{S}$, the subspace spanned by set $\mathcal{S}$.
- $\dim V$, the dimension of vector space V.
- $\langle \mathbf{v}, \mathbf{w} \rangle$, the inner product of vectors $\mathbf{v}$ and $\mathbf{w}$.
- $\mathbf{v} \cdot \mathbf{w}$, the standard inner product of vectors $\mathbf{v}, \mathbf{w} \in \mathbb{R}^n$.
- $||\mathbf{v}||$, the norm of vector $\mathbf{v}$.
- $\det A$, the determinant of a square matrix A.

- $D_{\mathbf{u}} f(\mathbf{x}_0)$, the directional derivative of f at $\mathbf{x}_0$ with respect to the nonzero vector $\mathbf{u}$.
- $f_x, D_1 f, \dfrac{\partial f}{\partial x}$, the partial derivative of f with respect to x.
- $\mathbf{u} \times \mathbf{v}$, the cross product of $\mathbf{u}$ and $\mathbf{v}$.
- ∇f, the gradient of a scalar function f.
- curl $\mathbf{F}$, the curl of a vector field $\mathbf{F}$.
- div $\mathbf{F}$, the divergence of a vector field $\mathbf{F}$.

Contents

These note may contain occasional typos or errors. Feel free to email me at ebrahimi@umd.edu if you notice a typo or an error.

Week 1

1.1 Sets

YouTube Video: https://youtu.be/nU8U1G6BNqk

A **set** is a well-defined collection of unordered **elements**. Each set is usually defined either by listing all of its elements or by a property as below:

$$S = \{s_1, s_2, \ldots, s_n\} \qquad \text{or} \qquad S = \{s \mid s \text{ satisfies property } \mathcal{P}\}$$

Note that the order of elements in a set and repetition do not matter. So, $\{1,1,2\}$, $\{1,2\}$, $\{1,2,1,2,2\}$ and $\{2,1\}$ are all the same set.

Notation: Instead of "x is an element of the set A" or "x belongs to the set A", we write "$x \in A$".

Definition 1.1. Let A and B be two sets for which the following statement is true:

"If $x \in A$, then $x \in B$."

Then, we say A is a **subset** of B, in which case we write $A \subseteq B$.
We say a subset A of a set B is **proper** if $A \neq B$, in which case we write $A \subsetneqq B$ or $A \subsetneq B$.
The **union** of A and B, denoted by $A \cup B$, is the set consisting of all elements that are in A or B (or both).
The **intersection** of A and B, denoted by $A \cap B$, is the set consisting of all elements that are in both A and B. In other words

$$A \cup B = \{x \mid x \in A, \text{ or } x \in B\}, \qquad \text{and} \qquad A \cap B = \{x \mid x \in A, \text{ and } x \in B\}.$$

The union and intersection of n sets is defined similarly:

$$\bigcup_{i=1}^{n} A_i = \{x \mid x \in A_i, \text{ for some } i\}, \qquad \text{and} \qquad \bigcap_{i=1}^{n} A_i = \{x \mid x \in A_i, \text{ for all } i\}.$$

The union and intersection of infinitely many sets $A_1, A_2, \ldots$ are defined similarly and they are denoted by $\bigcup\limits_{n=1}^{\infty} A_n$ and $\bigcap\limits_{n=1}^{\infty} A_n$.

The **empty set** or the **null set** is the set with no elements. It is denoted by $\emptyset$ or $\{\}$.

Remark. The word "or" in mathematics is not exclusive. In other words, for two statements p and q, the statement "p or q" means "p or q or both". For example, "$x \in A$ or $x \in B$" means, "x is an element of A or x is an element of B or x is an element of both A and B."

Remark. Sometimes sets are labeled by elements of another set. For example, instead of $\bigcup_{n=0}^{\infty} A_n$ we may write $\bigcup_{n\in\mathbb{N}} A_n$ and instead of $\bigcup_{n=-\infty}^{\infty} A_n$ we may write $\bigcup_{n\in\mathbb{Z}} A_n$. This is especially useful when there are too many sets to label them using only integers. For example, in the union $\bigcup_{r\in\mathbb{R}} A_r$, there is a set A_r corresponding to every real number r.

Definition 1.2. We say two sets A and B are **equal** if and only if $A \subseteq B$ and $B \subseteq A$, in which case we write $A = B$.

Example 1.1. Prove that for every three sets A, B, and C we have $(A \cup B) \cap C = (A \cap C) \cup (B \cap C)$.

Definition 1.3. An **ordered pair** (a, b) is two objects a and b with a specified order. Two ordered pairs (a, b) and (c, d) are the same if and only if $a = c$ and $b = d$. An n-**tuple** $(a_1, a_2, \ldots, a_n)$ is n objects $a_1, a_2, \ldots, a_n$ with a specified order. Two n-tuples $(a_1, a_2, \ldots, a_n)$ and $(b_1, b_2, \ldots, b_n)$ are equal if and only if $a_i = b_i$ for $i = 1, \ldots, n$.

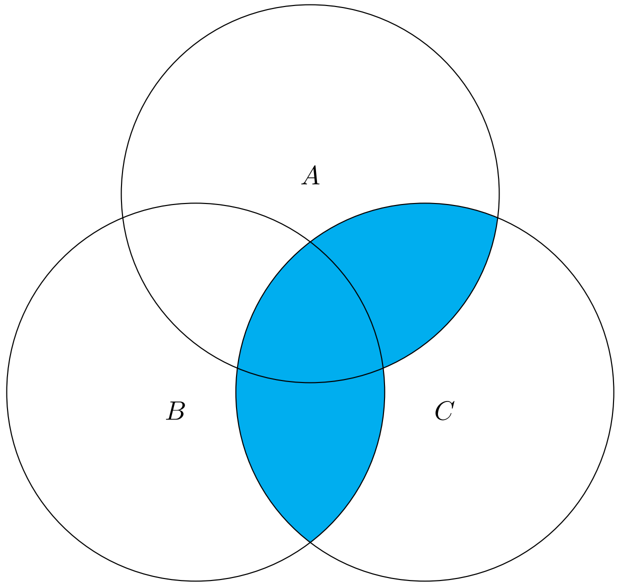
Definition 1.4. The **Cartesian product** of n sets $A_1, A_2, \ldots, A_n$, denoted by $A_1 \times A_2 \times \cdots \times A_n$, is the set of all n-tuples $(a_1, a_2, \ldots, a_n)$ for which $a_i \in A_i$ for all i. The Cartesian product of n copies of a set A is denoted by A^n.

Example 1.2. Every point on the plane can be represented by an element of the set $\mathbb{R}^2 = \{(x, y) \mid x, y \in \mathbb{R}\}$. Every point on the n-dimensional space can be represented by an element of the set

$$\mathbb{R}^n = \{(x_1, \ldots, x_n) \mid x_1, \ldots, x_n \in \mathbb{R}\}.$$

Definition 1.5. We say two sets A and B are **disjoint** if $A \cap B = \emptyset$. We say sets $A_1, A_2, \ldots, A_n$ are **disjoint** if $\bigcap_{i=1}^{n} A_i = \emptyset$. We say $A_1, A_2, \ldots, A_n$ are **pairwise disjoint** if for every $i \neq j$, A_i and A_j are disjoint. Similarly, these notions are defined for an infinite collection of sets A_i with $i \in I$.

To understand sets we often picture them as ovals or circles. The following shows the **Venn diagram** of $(A \cup B) \cap C$.



Definition 1.6. For two sets A, B, the **difference** $A - B$ consists of all elements of A that are not in B.

$$A - B = \{x \in A \mid x \notin B\}.$$

When dealing with sets, we often assume all of our sets are subsets of a given larger set U. This set is called a **universal set**. For example in number theory the universal set is usually $\mathbb{Z}$, the set of integers. In calculus we deal with real numbers and thus our universal set is typically $\mathbb{R}$.

Assume A is a subset of the universal set U. The **complement** of A in U is the set consisting of all elements of U that are not in A. The complement of A is denoted by A^c.

Theorem 1.1 (De Morgan's Laws)**.** *Given n subsets $A_1, \ldots, A_n$ of a universal set U we have:*

(a) $\left(\bigcap_{j=1}^{n} A_j\right)^c = \bigcup_{j=1}^{n} A_j^c$.

(b) $\left(\bigcup_{j=1}^{n} A_j\right)^c = \bigcap_{j=1}^{n} A_j^c$.

Remark. Similar to above, given a nonempty set of indices I and a collection of subsets A_i of a universal set U, for every $i \in I$, we have:

(a) $\left(\bigcap_{i\in I} A_i\right)^c = \bigcup_{i\in I} A_i^c$.

(b) $\left(\bigcup_{i\in I} A_i\right)^c = \bigcap_{i\in I} A_i^c$.

1.2 Functions

YouTube Video: https://youtu.be/lwQG_dMXQOA

Definition 1.7. Given two nonempty sets A and B, a **function** or a **mapping** $f : A \to B$ is a rule that assigns to every element $a \in A$ an element $f(a) \in B$. The set A is called the **domain** of f and is denoted by D_f. The set B is called the **co-domain** of f. The **range** or **image** of f, denoted by R_f or $\operatorname{Im} f$, is the set $\operatorname{Im} f = \{f(a) \mid a \in A\}$.

Two functions f and g are called **equal** if they have the same domain, the same co-domain, and $f(x) = g(x)$ for all x in their common domain.

f is called **surjective** or **onto** if for every $b \in B$ there is $a \in A$ for which $f(a) = b$.

f is called **injective** or **one-to-one** if whenever $f(a_1) = f(a_2)$ we also have $a_1 = a_2$.

f is called **bijective** if it is injective and surjective.

The **composition** $f \circ g$ of two functions f, g with $R_g \subseteq D_f$, is a function from D_g to the co-domain of f given by $f \circ g(x) = f(g(x))$, for all $x \in D_g$.

The function $\mathrm{id}_A : A \to A$ defined by $\mathrm{id}_A(a) = a$, for all $a \in A$ is called the **identity** function of A.

A function $f : A \to B$ is called **invertible** if and only if there is a function $g : B \to A$ for which $f \circ g = \mathrm{id}_B$ and $g \circ f = \mathrm{id}_A$. The function g is called the **inverse** of f and is denoted by f^{-1}.

Example 1.3 (Projection)**.** The function $\pi_1 : A \times B \to A$ defined by $\pi_1(a, b) = a$ is called the projection onto the first component. Similarly, the function $\pi_i : A_1 \times \cdots \times A_n \to A_i$ defined by $\pi_i(a_1, \ldots, a_n) = a_i$ is called the **projection** onto the i-th component.

Definition 1.8. Given a function $f : A \to B$, and a subset S of A, the **image** of S under f is the set $f(S) = \{f(s) \mid s \in S\}$. If T is a subset of B, then the **pre-image** or **inverse image** of T under f is the set $f^{-1}(T) = \{a \in A \mid f(a) \in T\}$.

Note that the pre-image and image are both sets.

Example 1.4. Let $f : A \times B \to B$ be the projection onto the second component. For every $b \in B$ find the pre-image of $\{b\}$ under f.

Example 1.5. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a function defined by $f(x, y) = 2x + 3y$. For every real number b, evaluate and describe $f^{-1}(\{b\})$. How do these pre-images change as b changes?

Theorem 1.2 (Properties of Pre-image)**.** *Suppose $f : A \to B$ is a function, $S \subseteq A$, $T \subseteq B$, and $T_i \subseteq B$ for every $i \in I$, where I is a nonempty set of indices. Then*

(a) $S \subseteq f^{-1}(f(S))$, *and* $f(f^{-1}(T)) \subseteq T$.

(b) $f^{-1}(\bigcup\limits_{i \in I} T_i) = \bigcup\limits_{i \in I} f^{-1}(T_i)$.

(c) $f^{-1}(\bigcap\limits_{i \in I} T_i) = \bigcap\limits_{i \in I} f^{-1}(T_i)$.

Proof. (a) Suppose $s \in S$. By definition of image, $f(s) \in f(S)$. Therefore, by definition of pre-image $s \in f^{-1}(f(S))$. This completes the proof of $S \subseteq f^{-1}(f(S))$.

Suppose $x \in f(f^{-1}(T_i))$. By definition $x = f(y)$, for some $y \in f^{-1}(T_i)$. Therefore, $f(y) \in T_i$, which means $x \in T_i$. This means $f(f^{-1}(T_i)) \subseteq T_i$.

Parts (b) and (c) are left as exercises. □

1.3 Proofs

YouTube Video: https://youtu.be/sW823lmew64

In writing proofs you should note the following:

- You cannot prove a *universal statement* (statements involving *for every* or *for all*) by examples. For example if you are asked to prove "The sum of every two odd integers is even." your proof may not be "3 is odd, 5 is odd, 3+5=8 is even. Therefore, the sum of every two odd integers is even."

  On the other hand, for *existential statements* (when a statement is asking you to show something exists), giving an example and showing that the example satisfies all the required conditions is enough.

- Do not use the same variable for two different things.

- You may not assume anything but what is given in the assumptions.

- All steps must be justified and the justifications must all be clearly stated.

- You may only use known facts. These are typically things that have been previously proved as theorems or are facts stated in definitions.

- To prove a statement of the form "p if and only if q" we will need to prove both "If p, then q" and "If q, then p".

To prove a *statement* (usually of the form "If p then q"), there are three main methods of proof. We will look at each one via examples.

1.3.1 Direct Proof

In this method we start from the assumption and by taking logical steps we end up with the conclusion.

Example 1.6. Prove that the function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = x^3 + 2x$ is one-to-one.

Solution. By definition of one-to-one, we need to prove if $f(x) = f(y)$, then $x = y$.

Suppose $f(x) = f(y)$. Then $x^3 + 2x = y^3 + 2y$. Therefore, $x^3 - y^3 + 2(x - y) = 0$, which implies $(x - y)(x^2 + xy + y^2 + 2) = 0$. This means either $x = y$ or $x^2 + xy + y^2 + 2 = 0$. If the second equality holds, by the quadratic formula we obtain $x = \dfrac{-y \pm \sqrt{y^2 - 4(y^2 + 2)}}{2}$. The discriminant is $-3y^2 - 8$ which is negative. Therefore, this equality is impossible, and hence $x = y$. This means f is one-to-one. $\square$

1.3.2 Proof by Contradiction

In this method, we assume the conclusion is false while the assumption is true. After taking logical steps we obtain a contradiction. A contradiction is a statement that is false: Either it violates a fact in math such as a theorem or a definition or it violates the assumptions. When using proof by contradiction make sure to clearly specify you are using this method. You could say " On the contrary assume..." or "By way of contradiction assume..." or simply state "We will use proof by contradiction."

Example 1.7. Prove that there are infinitely many primes.

Solution. On the contrary assume there are only a finite number of primes, and let $p_1, p_2, \ldots, p_n$ be the list of all primes. Since the integer $d = p_1 \cdots p_n + 1$ is more than one, d has a prime factor. Since $p_1, p_2, \ldots, p_n$ is the list of all primes, one of the p_i's must divide d. On the other hand p_i divides $p_1 p_2 \cdots p_n$. Therefore, p_i must divide $d - p_1 p_2 \cdots p_n = 1$. This is a contradiction. Therefore, the initial assumption must be false, and thus there must exist infinitely many primes. $\square$

1.3.3 Proof by Induction

To prove a statement $P(n)$ (i.e. a statement that depends on a positive integer n) we will:

- Prove $P(1)$ (basis step); and
- Assume $P(n)$ holds for some $n \geq 1$, and then prove $P(n+1)$ (inductive step).

If you need to use $P(n-1)$ in your proof of $P(n+1)$, then the basis step must involve two consecutive integers, e.g. $P(1)$ and $P(2)$.

Often times we use what is called **strong induction** which involves assuming $P(1), \ldots, P(n)$ and then proving $P(n+1)$ in addition to proving the basis step.

When employing the method of mathematical induction keep in mind to always start your proof by "We will prove *the statement* by induction on *the variable*". Replace "the statement" and "the variable" accordingly. Also, clearly separate the basis step and the inductive step.

Example 1.8. Prove that the sum of the first n positive odd integers is n^2.

1.4 $\mathbb{R}^n$ as a Vector Space

YouTube Video: https://youtu.be/Bwpk4fPJmoU

As we saw earlier, elements of $\mathbb{R}^n$ are n-tuples of the form $(x_1, x_2, \ldots, x_n)$, where x_j's are real numbers. Each one of these elements is called a **vector** and these vectors can be added componentwise as follows:

$$(x_1, x_2, \ldots, x_n) + (y_1, y_2, \ldots, y_n) = (x_1 + y_1, x_2 + y_2, \ldots, x_n + y_n).$$

Each vector can also be scaled by any real number c (also called a **scalar**) as follows:

$$c(x_1, x_2, \ldots, x_n) = (cx_1, cx_2, \ldots, cx_n)$$

This vector addition and scalar multiplication satisfy the following properties.

(I) (Closure) For every two vectors $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$, and every scalar $c \in \mathbb{R}$, both $\mathbf{x} + \mathbf{y}$ and $c\mathbf{x}$ are in $\mathbb{R}^n$.

(II) (Associativity) For every $\mathbf{x}, \mathbf{y}, \mathbf{z} \in \mathbb{R}^n$, and every $a, b \in \mathbb{R}$, we have $\mathbf{x} + (\mathbf{y} + \mathbf{z}) = (\mathbf{x} + \mathbf{y}) + \mathbf{z}$, and $a(b\mathbf{x}) = (ab)\mathbf{x}$.

(III) (Commutativity) For every $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$, we have $\mathbf{x} + \mathbf{y} = \mathbf{y} + \mathbf{x}$.

(IV) (Additive Identity) For every $\mathbf{x} \in \mathbb{R}^n$ we have $\mathbf{x} + (0, 0, \ldots, 0) = \mathbf{x}$. (The vector $(0, 0, \ldots, 0)$ is called the **zero vector** and is denoted by $\mathbf{0}$.)

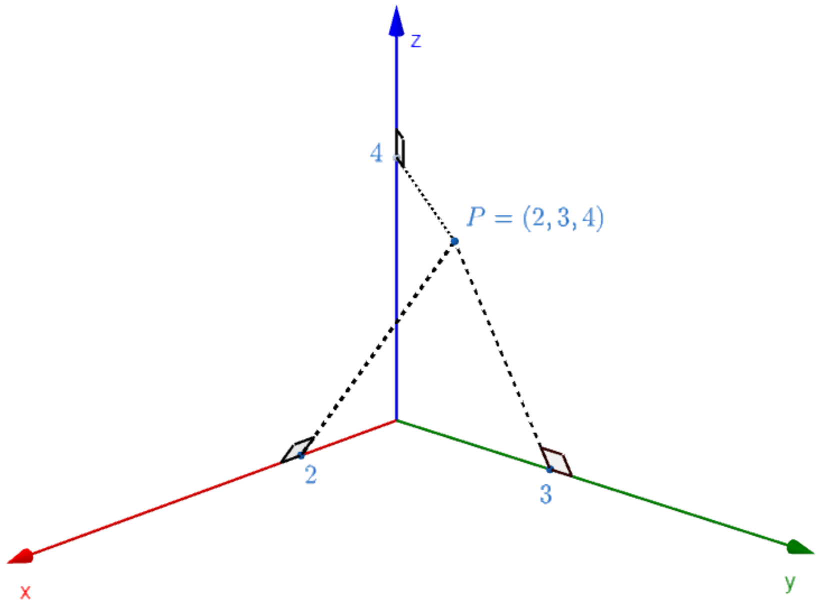
(V) (Additive Inverse) For every $\mathbf{x} \in \mathbb{R}^n$, there is a vector $\mathbf{y} \in \mathbb{R}^n$ for which $\mathbf{x} + \mathbf{y} = \mathbf{0}$. (This vector $\mathbf{y}$ is called the **additive inverse** of $\mathbf{x}$ and is denoted by $-\mathbf{x}$. It is given by $-(x_1, x_2, \ldots, x_n) = (-x_1, -x_2, \ldots, -x_n)$.)

(VI) (Distributivity) For every $a, b \in \mathbb{R}$ and $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$, we have $(a+b)\mathbf{x} = a\mathbf{x} + b\mathbf{x}$, and $a(\mathbf{x}+\mathbf{y}) = a\mathbf{x} + a\mathbf{y}$.

(VII) (Multiplicative Identity) For every $\mathbf{x} \in \mathbb{R}^n$ we have $1\mathbf{x} = \mathbf{x}$.

The seven properties I-VII listed above are called **vector space** properties of $\mathbb{R}^n$. This is often phrased as "$\mathbb{R}^n$ is a vector space." Note that sometimes we refer to elements of $\mathbb{R}^n$ as **points**. This is only for conceptualizing these objects. The math does not change. When elements of $\mathbb{R}^n$ are seen as points, the zero vector is referred to as the **origin**.

Geometrically, elements of $\mathbb{R}^2$ can be represented by points on a plane. Elements of $\mathbb{R}^3$ can be represented by points in a 3D space. To do that, we need three axes, x-, y-, and z-**axes**. These three axes must satisfy the right-hand rule. The coordinates of each point can be found by dropping perpendiculars to the axes.



The set of all points with positive coordinates, is called the **first octant**.

There are three planes each containing two of the x-, y-, and z- axes. Each of the three xy-, xz- and yz-planes is called a **coordinate plane**.

Theorem 1.3. *The distance between two points* (x_1, y_1, z_1) *and* (x_2, y_2, z_2) *in* $\mathbb{R}^3$ *is given by*

$$\sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2}.$$

1.5 Warm-ups

Example 1.9. How many elements does the set $\{2, 1, 3, 2\}$ have? How about the set $\{3, 2, 1\}$? How are these two sets related?

Solution. Since repetition and order does not matter in a set, these two sets are the same sets:

$$\{2, 1, 3, 2\} = \{3, 2, 1\}.$$

So, these sets both have three elements. $\square$

Example 1.10. Let E be the set of all even integers and O be the set of all odd integers. Describe $E \cup O$ and $E \cap O$.

Solution. $E \cup O$ is the set of all integers that are odd or even. Since every integer is either odd or even, $E \cup O = \mathbb{Z}$.

By definition of intersection, $E \cap O$ is the set of all integers that are both even and odd. Since no integer is both even and odd, $E \cap O = \emptyset$. $\square$

Example 1.11. Consider the function $f : \{1, 2, 3\} \to \{1, 2, 3, 4\}$ defined by $f(1) = 2, f(2) = 2$, and $f(3) = 4$. Find the domain of f, the co-domain of f, the image of f, $f(\{1, 2\})$, and $f^{-1}(\{2, 3\})$.

Solution. The domain of f is $\{1, 2, 3\}$. The co-domain of f is $\{1, 2, 3, 4\}$. The image of f is $\{f(1), f(2), f(3)\} = \{2, 4\}$.

$$f(\{1, 2\}) = \{f(1), f(2)\} = \{2, 2\} = \{2\},$$

and

$$f^{-1}(\{2, 3\}) = \{x \in \{1, 2, 3\} \mid f(x) \in \{2, 3\}\}.$$

Thus, $f^{-1}(\{2, 3\}) = \{1, 2\}$. $\square$

1.6 More Examples

Example 1.12. Given sets $A = \{1, 2\}, B = \{0, 1, -1\}$, write each of the following sets by listing all of its elements:

(a) $A \cup B$

(b) $A \cap B$

(c) $A \times B$

Solution. (a) $A \cup B$ consists of all elements that are in A or B. Thus, $A \cup B = \{1, 2, 0, -1\}$.

(b) $A \cap B$ consists of all elements that are in both A and B. Thus, $A \cap B = \{1\}$.

(c) $A \times B$ consists of all elements of the form (a, b), where $a \in A$ and $b \in B$. Thus,

$$A \times B = \{(1, 0), (1, 1), (1, -1), (2, 0), (2, 1), (2, -1)\}$$

$\square$

Example 1.13. Prove that for all sets $A, B_1, B_2, \ldots$, we have $A \bigcap (\bigcup\limits_{n=1}^{\infty} B_n) = \bigcup\limits_{n=1}^{\infty} (A \cap B_n)$.

Solution. Suppose $x \in A \bigcap (\bigcup\limits_{n=1}^{\infty} B_n)$. By definition of intersection, $x \in A$ and $x \in \bigcup\limits_{n=1}^{\infty} B_n$. By definition of union, $x \in B_n$ for some n. This means $x \in A \cap B_n$ and thus $x \in \bigcup\limits_{n=1}^{\infty} (A \cap B_n)$. Therefore,

$$A \bigcap (\bigcup_{n=1}^{\infty} B_n) \subseteq \bigcup_{n=1}^{\infty} (A \cap B_n). \qquad (*)$$

Suppose $x \in \bigcup_{n=1}^{\infty}(A \cap B_n)$. By definition of union, $x \in A \cap B_n$ for some n. Thus, by definition of intersection, $x \in A$ and $x \in B_n$. Therefore, $x \in A$ and $x \in \bigcup_{n=1}^{\infty} B_n$. This implies $x \in A \bigcap (\bigcup_{n=1}^{\infty} B_n)$. Therefore,

$$\bigcup_{n=1}^{\infty}(A \cap B_n) \subseteq A \bigcap (\bigcup_{n=1}^{\infty} B_n). \qquad (**)$$

Combining $(*)$ and $(**)$ we obtain the result. □

Example 1.14. Describe each set as a subset of $\mathbb{R}^2$.

(a) $[0, 1] \times \{1\}$.

(b) $[1, 2] \times [0, 1]$.

Solution. (a) This is the set of all (x, y), where $x \in [0, 1]$ and $y = 1$. This is a horizontal segment connecting $(0, 1)$ and $(1, 1)$.

(b) This set consists of all points (x, y) for which $x \in [1, 2]$ and $y \in [0, 1]$. This is a filled square with vertices $(1, 0), (2, 0), (1, 1)$, and $(2, 1)$. □

Example 1.15. Let C be the unit circle $x^2 + y^2 = 1$ in the xy-plane. Geometrically describe the set $C \times \mathbb{R}$.

Solution. $C \times \mathbb{R}$ is the set of all (x, y, z) for which $x^2 + y^2 = 1$. This means $C \times \mathbb{R}$ is the union of the translation of the unit circle C in the direction of the z-axis. This is a right circular cylinder. □

Example 1.16. Suppose X and Y are finite nonempty sets of sizes m and n respectively. Let Y^X be the set of all function $f : X \to Y$. What is the size of Y^X? (This should tell you why we use the notation "Y^X".)

Solution. Let $f : X \to Y$ be a function. For each $x \in X$, $f(x)$ could be any element of Y. Thus, there are n possible values for $f(x)$. Since this is true for each element of X, there are n^m functions $f : X \to Y$. □

Example 1.17. Define the Fibonacci sequence F_n by $F_0 = 0, F_1 = 1$, and $F_{n+2} = F_{n+1} + F_n$ for all $n \geq 0$. Prove that $F_n < 2^n$ for all $n \geq 0$.

Sketch. The fact that each term of the sequence depends on the previous terms reminds us of the method of Mathematical Induction. So, we will employ this method. However since each term depends on the previous two terms, we will have to start with proving the given statement for two values of n.

Solution. We will prove $F_n < 2^n$ by induction on n.

Basis step: $F_0 = 0 < 2^0 = 1$, and $F_1 = 1 < 2^1$.

Inductive step: Suppose for some $n \geq 1$, $F_k < 2^k$ for $k = 0, \ldots, n$. By assumption $F_{n+1} = F_n + F_{n-1} < 2^n + 2^{n-1} = 2^{n-1}(2+1) < 2^{n+1}$, as desired. This completes the solution. □

Example 1.18. Prove that if a real number x satisfies $|x| + x > 0$, then x is positive.

Solution. On the contrary assume x is not positive. Therefore, we have two cases:

Case I: $x = 0$. This means $|x| + x = 0$, which is a contradiction.

Case II: $x < 0$. This implies $|x| = -x$ and thus, $|x| + x = 0$, which is a contradiction.
Therefore x must be positive. □

Definition 1.9. Suppose D is a subset of $\mathbb{R}$. A function $f : D \to \mathbb{R}$ is said to be **periodic** if there is a positive real number T for which:

- For every $x \in \mathbb{R}$, the number x is in D if and only if $x + T \in D$, and
- $f(x + T) = f(x)$ for every $x \in D$.

Example 1.19. Prove $\sin x$ and $\sin(\pi x)$ are both periodic, but their sum is not.

Solution. First note that the domain of both functions is $\mathbb{R}$. Thus, $x \in \mathbb{R}$ iff $x + T \in \mathbb{R}$ is valid for every $x, T \in \mathbb{R}$.

By properties of $\sin x$ we know $\sin(x + 2\pi) = \sin x$ and $\sin(\pi(x + 2)) = \sin(\pi x)$. Thus, both functions $\sin x$ and $\sin(\pi x)$ are periodic.

To prove $\sin x + \sin(\pi x)$ is not periodic we will use proof by contradiction. Assume on the contrary there is a positive real number T for which for every $x \in \mathbb{R}$ we have

$$\sin(x + T) + \sin(\pi(x + T)) = \sin x + \sin(\pi x) \qquad (*)$$

Differentiating $(*)$ with respect to x twice, we obtain

$$-\sin(x + T) - \pi^2 \sin(\pi(x + T)) = -\sin x - \pi^2 \sin(\pi x) \qquad (**)$$

Adding $(*)$ and $(**)$ and dividing both sides by $1 - \pi^2$ we conclude $\sin(\pi(x+T)) = \sin(\pi x)$ for every $x \in \mathbb{R}$. Combining this with $(*)$ we conclude that $\sin(x + T) = \sin x$ for all $x \in \mathbb{R}$. This implies $T = 2k\pi$ for some positive integer k. Substituting x by x/π in $\sin(\pi(x + T)) = \sin(\pi x)$ we conclude $\sin(x + \pi T) = \sin(x)$. Therefore, $\pi T = 2n\pi$ for some positive integer n. Therefore, $\pi(2k\pi) = 2n\pi$. Hence $\pi = n/k$ is rational, which is a contradiction. □

Example 1.20. Prove that for every positive integer n, there is a polynomial $p_n(x)$ for which the n-th derivative of e^{x^2} at x is equal to $p_n(x)e^{x^2}$.

Solution. We will prove this by induction on n.

Basis step. By the Chain Rule, the first derivative of e^{x^2} is $2xe^{x^2}$. The fact that $p_1(x) = 2x$ is a polynomial proves the claim for $n = 1$.

Inductive step. Suppose the n-th derivative of e^{x^2} is $p_n(x)e^{x^2}$ for a polynomial p_n. Differentiating this using the Product Rule and the Chain Rule we conclude that the $(n+1)$-th derivative of e^{x^2} is equal to $p'_n(x)e^{x^2} + p_n(x)2xe^{x^2} = (p'_n(x) + 2xp_n(x))e^{x^2}$. Since the derivative of a polynomial is a polynomial and the product and sum of polynomials are polynomials, $p'_n(x) + 2xp_n(x)$ is a polynomial. So, setting $p_{n+1}(x) = p'_n(x) + 2xp_n(x)$ we conclude the $(n+1)$-th derivative of e^{x^2} is equal to $p_{n+1}(x)e^{x^2}$ for some polynomial p_{n+1}, as desired. □

Example 1.21. Let $f : A \to B$ be a function, $S \subseteq A$, and $T \subseteq B$. Prove that:

(a) If f is one-to-one, then $S = f^{-1}(f(S))$.

(b) If f is onto, then $T = f(f^{-1}(T))$.

Solution. (a) By Theorem 1.2, $S \subseteq f^{-1}(f(S))$. It is enough to show $f^{-1}(f(S)) \subseteq S$. Suppose $x \in f^{-1}(f(S))$. By definition of pre-image, $f(x) \in f(S)$. By definition of $f(S)$ we conclude $f(x) = f(s)$ for some $s \in S$. Since f is one-to-one, $x = s$ and thus $x \in S$. This shows $f^{-1}(f(S)) \subseteq S$, as desired.

(b) By Theorem 1.2, $f(f^{-1}(T)) \subseteq T$. Thus, it is enough to prove $T \subseteq f(f^{-1}(T))$. Let $x \in T$. Since f is onto, there is $a \in A$ such that $f(a) = x$. Thus, by definition of pre-image $a \in f^{-1}(T)$. Therefore, by definition of image $f(a) \in f(f^{-1}(T))$. Since $f(a) = x$, we obtain $x \in f(f^{-1}(T))$. Therefore, $T \subseteq f(f^{-1}(T))$, as desired. □

Example 1.22. Let $f : \mathbb{R} \to \mathbb{R}$ be defined as $f(x) = x^2$. Find each of the following:

(a) $f([0, 1))$.

(b) $f^{-1}([-1, 0))$.

(c) $f^{-1}((0, 2))$.

Solution. (a) Note that if $x \in [0, 1)$, then $0 \le x^2 < 1$ and thus $f([0, 1)) \subseteq [0, 1)$. Furthermore, if $y \in [0, 1)$, then $\sqrt{y} \in [0, 1)$ and $f(\sqrt{y}) = y$. Therefore, $[0, 1) \subseteq f([0, 1))$. This shows $f([0, 1)) = [0, 1)$.

(b) By definition of pre-image, $x \in f^{-1}([-1, 0))$ if and only if $f(x) \in [-1, 0)$ if and only if $-1 \le x^2 < 0$, which is impossible. Therefore, $f^{-1}([-1, 0)) = \emptyset$.

(c) By definition of pre-image, $x \in f^{-1}(0,2)$ if and only if $f(x) \in (0,2)$, i.e. $0 < x^2 < 2$. This holds if and only if $0 < x < \sqrt{2}$ or $-\sqrt{2} < x < 0$. Therefore, $f^{-1}((0,2)) = (0, \sqrt{2}) \cup (-\sqrt{2}, 0)$. $\square$

Example 1.23. Let $f : A \to B$ be a function. Find each of the following:

(a) $f(\emptyset)$.

(b) $f^{-1}(\emptyset)$.

(c) $f^{-1}(B)$.

Solution. (a) $f(\emptyset)$ consists of all elements of the form $f(x)$, where $x \in \emptyset$, but since $\emptyset$ contains no elements, $f(\emptyset) = \emptyset$.

(b) $f^{-1}(\emptyset)$ consists of all elements $a \in A$ for which $f(a) \in \emptyset$. Since $\emptyset$ contains no elements $f^{-1}(\emptyset) = \emptyset$.

(c) $f^{-1}(B)$ consists of all elements $a \in A$ for which $f(a) \in B$, but since B is the co-domain, $f(a)$ is always in B, and thus $f^{-1}(B) = A$. $\square$

Example 1.24. Let $f : A \to B$ be a function, and S_i with $i \in I$ be a collection of subsets of A. Prove that

(a) $f\left(\bigcup\limits_{i\in I} S_i\right) = \bigcup\limits_{i\in I} f\left(S_i\right)$.

(b) $f\left(\bigcap\limits_{i\in I} S_i\right) \subseteq \bigcap\limits_{i\in I} f\left(S_i\right)$. By an example show that the equality does not always hold.

Solution. (a) Let $x \in f\left(\bigcup\limits_{i\in I} S_i\right)$. By definition of image, $x = f(s)$ for some $s \in \bigcup\limits_{i\in I} S_i$. By definition of union, $s \in S_j$ for some $j \in I$. Therefore, $x = f(s) \in f(S_j)$, which implies $x \in \bigcup\limits_{i\in I} f(S_i)$, by definition of union. The other inclusion is similar and is left as an exercise.

(b) Let $x \in f\left(\bigcap\limits_{i\in I} S_i\right)$. By definition of image, $x = f(s)$ for some $s \in \bigcap\limits_{i=1}^{n} S_i$. By definition of intersection, $s \in S_i$ for all $i \in I$, and thus $x = f(s) \in f(S_i)$ for all $i \in I$, by definition of image. Therefore, $x \in \bigcap\limits_{i\in I} f\left(S_i\right)$, by definition of intersection. This completes the proof.

Consider $f : \{1,2\} \to \{1\}$ given by $f(1) = f(2) = 1$. Let $S_1 = \{1\}$ and $S_2 = \{2\}$. Then, $S_1 \cap S_2 = \emptyset$ and thus $f(S_1 \cap S_2) = \emptyset$. On the other hand $f(S_1) = f(S_2) = \{1\}$ and thus $f(S_1) \cap f(S_2) \neq \emptyset$. $\square$

Example 1.25. Determine if each function below is one-to-one, onto, both or neither.

(a) $f : \mathbb{R}^2 \to \mathbb{R}^2$ given by $f(x,y) = (x + 2y, x - y)$.

(b) $f : \mathbb{Z}^2 \to \mathbb{Q}^+$ given by $f(m,n) = 2^m \cdot 3^n$.

Solution. (a) Suppose $f(x,y) = f(a,b)$. This implies

$$\begin{cases} x + 2y = a + 2b \\ x - y = a - b \end{cases}$$

Subtracting the two equations above we obtain $3y = 3b$ and thus $y = b$. Substituting into the first equation we obtain $x = a$. Therefore, f is one-to-one.

Given $(a,b) \in \mathbb{R}^2$, we will need to determine if there is $(x,y) \in \mathbb{R}^2$ for which $f(x,y) = (a,b)$. Solving the system

$$\begin{cases} x + 2y = a \\ x - y = b \end{cases}$$

we obtain $x = (a+2b)/3$ and $y = (a-b)/3$. Therefore, this function is also onto.

(b) Suppose $f(m,n) = f(r,s)$ for some integers m,n,r,s. Therefore, $2^m \cdot 3^n = 2^r \cdot 3^s$. Without loss of generality assume $m \geq r$. We see that $2^{m-r} = 3^{s-n}$. If the exponent $m-r$ is positive, then the left side is even, while the right side is not. This contradiction shows $m = r$ and thus $n = s$. Therefore, f is one-to-one.

This function is not onto. For example $f(m,n) = 5$ has no solutions, because $2^m \cdot 3^n = 5$ is impossible by the uniqueness of prime factorization. $\square$

Example 1.26. For a function $f : A \to B$ prove that the equality $f(S_1 \cap S_2) = f(S_1) \cap f(S_2)$ holds for all subsets S_1, S_2 of A if and only if f is one-to-one.

Solution. First, note that by Example 1.24, we know

$$f(S_1 \cap S_2) \subseteq f(S_1) \cap f(S_2).$$

($\Leftarrow$) Suppose f is one-to-one. Let $x \in f(S_1) \cap f(S_2)$. By definition, $x = f(s_1) = f(s_2)$ for some $s_1 \in S_1$, and some $s_2 \in S_2$. Since f is one-to-one, we have $s_1 = s_2$. Therefore, $s_1 \in S_1 \cap S_2$. This means $x \in f(S_1 \cap S_2)$. This shows $f(S_1) \cap f(S_2) = f(S_1 \cap S_2)$.

($\Rightarrow$) Assume $f(S_1 \cap S_2) = f(S_1) \cap f(S_2)$ for every two subsets S_1, S_2 of A. Suppose $f(a) = f(b)$, and let $S_1 = \{a\}, S_2 = \{b\}$. We know $f(S_1) = \{f(a)\}$ and $f(S_2) = \{f(b)\} = \{f(a)\}$. Therefore, $f(S_1) \cap f(S_2) = \{f(a)\}$. If $a \neq b$, then $S_1 \cap S_2 = \emptyset$, which means $f(S_1 \cap S_2) = \emptyset \neq \{f(a)\}$. Therefore, $a = b$. This shows f is one-to-one. $\square$

Example 1.27. Suppose c is a real number and $\mathbf{v}$ is a vector in $\mathbb{R}^n$. Prove that if $c\mathbf{v} = \mathbf{0}$, then $c = 0$ or $\mathbf{v} = \mathbf{0}$.

Solution. Let $\mathbf{v} = (x_1, \ldots, x_n)$. On the contrary, assume neither c is zero, nor $\mathbf{v}$ is the zero vector. Therefore, x_i is not zero for some i. Since $c \neq 0$, we have $cx_i \neq 0$. Thus,

$$c\mathbf{v} = (cx_1, \ldots, cx_i, \ldots, cx_n) \neq \mathbf{0}.$$

This contraction shows $c = 0$ or $\mathbf{v} = \mathbf{0}$. $\square$

Further Reading: Click here for further reading on Sets, Maps, and Vector Spaces.

1.7 Exercises

Exercise 1.1. *Given the following sets A, B, C write down each of the sets $(A \times B) \cap C$, $A \cup (B \cap C)$ and $A \times B \times C$ by listing all of their elements in braces.*

$$A = \{1, -1\}, B = \{1, 0\}, C = \{(1, 1), (1, 0)\}.$$

Exercise 1.2. *For n sets $A_1, A_2, \ldots, A_n$, prove that $A_1 \times A_2 \times \cdots \times A_n = \emptyset$ if and only if $A_i = \emptyset$ for some i.*

Hint: Proof by contradiction might be useful.

Exercise 1.3. *Suppose for two nonempty sets A, B we know $A \times B = B \times A$. Prove that $A = B$.*

Exercise 1.4. *Prove or disprove:*

(a) For every three sets A, B, C we have $A - (B \cup C) = (A - B) \cap (A - C)$.

(b) For every three sets A, B, C we have $A - (B \cap C) = (A - B) \cup (A - C)$.

Exercise 1.5. *Determine which of the following statements are true.*

(a) $(\mathbb{Z} \times \mathbb{R}) \cup (\mathbb{R} \times \mathbb{Z}) = \mathbb{R} \times \mathbb{R}$.

(b) $(\mathbb{Z} \times \mathbb{R}) \cap (\mathbb{R} \times \mathbb{Z}) = \mathbb{Z} \times \mathbb{Z}$.

(c) $(\mathbb{R} - \mathbb{Z}) \times \mathbb{Z} = (\mathbb{R} \times \mathbb{Z}) - (\mathbb{Z} \times \mathbb{Z})$

Definition 1.10. The **power set** of a set A, denoted by $\mathcal{P}(A)$ is the set consisting of all subsets of A.

Exercise 1.6. *Prove or disprove each of the following:*

(a) For every two sets A, B, we have $\mathcal{P}(A \cap B) = \mathcal{P}(A) \cap \mathcal{P}(B)$.

(b) For every two sets A, B, we have $\mathcal{P}(A \cup B) = \mathcal{P}(A) \cup \mathcal{P}(B)$.

(c) For every two sets A, B, we have $\mathcal{P}(A \times B) = \mathcal{P}(A) \times \mathcal{P}(B)$.

Exercise 1.7. *Suppose $J \subseteq I$ are two nonempty sets of indices, and each A_i with $i \in I$ is a set. Prove each of the following:*

(a) $\bigcup_{j\in J} A_i \subseteq \bigcup_{i\in I} A_i$.

(b) $\bigcap_{i\in I} A_i \subseteq \bigcap_{j\in J} A_i$.

Exercise 1.8. *Prove that*

$$\bigcup_{x\in[0,1]} ([x,1]\times[0,x^2]) = \{(x,y) \mid 0\le x\le 1 \text{ and } 0\le y\le x^2\}$$

Exercise 1.9. *Prove that*

$$\bigcap_{x\in[0,1]} ([x,1]\times[0,x^2]) = \{(1,0)\}$$

Exercise 1.10. *Given a nonempty set A, what is the set $\bigcup_{a\in A}\{a\}$? How about $\bigcup_{a\in A}\{(a,1)\}$? Prove your claims.*

Exercise 1.11. *Let X be a nonempty set with n elements. How many one-to-one functions $f : X \to X$ are there? How many onto functions $f : X \to X$ are there?*

Exercise 1.12. *The graph of a function $f : X \to Y$ is defined by $\Gamma_f = \{(x, f(x)) \mid x \in X\}$. Prove that two functions $f, g : X \to Y$ are equal if and only if $\Gamma_f = \Gamma_g$.*

Exercise 1.13. *Suppose f, g are two functions for which $R_g \subseteq D_f$. Prove or disprove each statement.*

(a) If both f and g are injective, then so is $f \circ g$.

(b) If both f and g are surjective, then so is $f \circ g$.

Exercise 1.14. *Determine if each function is injective, surjective or neither.*

(a) $f : \mathbb{R}^2 \to \mathbb{R}^2$ *given by* $f(x,y) = (x+y, xy)$.

(b) $f : \mathbb{R}^n \to \mathbb{R}$ *given by* $f(x_1,\dots,x_n) = x_1+\cdots+x_n$.

(c) $f : \mathbb{R} \to \mathbb{R}$ *given by* $f(x) = x^3 - 5x$.

Exercise 1.15. *Prove parts (b) and (c) of Theorem 1.2.*

Exercise 1.16. *Prove each of the following:*

(a) $\sum_{i=1}^{n} \frac{1}{i^2+i} = \frac{n}{n+1}$ *for every* $n \in \mathbb{Z}^+$.

(b) $2^n > n$ *for every* $n \in \mathbb{Z}^+$.

(c) $2^n \ge n^2$ *for every* $n \in \mathbb{N}$ *with* $n \ge 4$.

Exercise 1.17. *Prove that if for a real number x, the number x^2 is irrational, then so is x.*

Exercise 1.18. *Let E, D be the set of all even and odd integers, respectively. Find a bijection $f : \mathbb{N} \to E$ and another bijection $g : D \to \mathbb{N}$.*

Exercise 1.19. *Suppose $p(x)$ is a polynomial. Prove that for every positive integer n, there is a polynomial $q(x)$ for which the n-th derivative of $e^{p(x)}$ is equal to $q(x)e^{p(x)}$ for every $x \in \mathbb{R}$.*

Exercise 1.20. *Prove that for every positive integer n, there is a polynomial p_n for which the n-th derivative of e^{-1/x^2} at $x \neq 0$ is equal to $p_n(1/x)e^{-1/x^2}$.*

Exercise 1.21. *Carefully prove all vector space properties I-VII of $\mathbb{R}^n$.*

Exercise 1.22. *Let $f : A \to B$ be a function, T be a subset of B. Prove that $f^{-1}(T^c) = (f^{-1}(T))^c$. (Note: For a subset S of A and a subset T of B we have $T^c = B - T$ and $S^c = A - S$.)*

Definition 1.11. A function $f : D \to \mathbb{R}$ is said to be **even** (resp. **odd**) if:

- D is a subset of $\mathbb{R}$ that satisfies $x \in D$ if and only if $-x \in D$, and
- $f(-x) = f(x)$ (resp. $f(-x) = -f(x)$) for every $x \in D$.

Exercise 1.23. *Prove that every function $f : \mathbb{R} \to \mathbb{R}$ can be written as sum of two functions $g, h : \mathbb{R} \to \mathbb{R}$, where g is even and h is odd. Prove the representation $f = g + h$ into sum of an even and an odd function is unique.*

Exercise 1.24. *Suppose functions $f, g : \mathbb{R} \to \mathbb{R}$ are n-times differentiable at some $x_0 \in \mathbb{R}$. Prove*

$$(fg)^{(n)}(x_0) = \sum \binom{n}{k} f^{(k)}(x_0)g^{(n-k)}(x_0).$$

1.8 Challenge Problems

Exercise 1.25. *Let $r \geq 2$ be a fixed positive integer, and let $\mathcal{F}$ be an infinite family of distinct sets, each of size r, no two of which are disjoint. Prove that there exists a set of size $r - 1$ that intersects each set in $\mathcal{F}$.*

Exercise 1.26. *Let A be a nonempty set. Suppose $f : \mathcal{P}(A) \to \mathcal{P}(A)$ is a bijection for which for every subsets X and Y of A:*

$$\textit{If } X \subseteq Y, \textit{ then } f(X) \subseteq f(Y).$$

(a) If A is finite, show that if $f(X) \subseteq f(Y)$, then $X \subseteq Y$.

(b) Show part (a) does not necessarily hold when A is infinite.

1.9 Summary

- To prove $A \subseteq B$, start with $x \in A$ and prove $x \in B$.
- To prove two sets A and B are equal we need to show if $x \in A$, then $x \in B$ and vice-versa.
- For a function $f : A \to B$, a subset S of A, and a subset T of B, we have the following:

$$x \in f(S) \text{ iff } x = f(s) \text{ for some } s \in S, \text{ and } y \in f^{-1}(T) \text{ iff } f(y) \in T.$$

- $f^{-1}(\bigcup_{i\in I} T_i) = \bigcup_{i\in I} f^{-1}(T_i)$ and $f^{-1}(\bigcap_{i\in I} T_i) = \bigcap_{i\in I} f^{-1}(T_i)$.
- $f(f^{-1}(T)) \subseteq T$ and $S \subseteq f^{-1}(f(S))$.
- To prove a statement by contradiction, assume the conclusion is false and after taking logical steps obtain a contradiction.
- To prove a statement depending on a positive integer n, first prove the statement for $n = 1$ (basis step), then prove that if the statement is true for n it must be true for $n + 1$ (inductive step).