Math 340 Homework 3

Dr. Ebrahimian

Due 9/20/2024 before the class starts

- You are expected to solve all of the following problems, but only problems under "Problems for Grading" must be submitted for grading. You will have a quiz on Friday 9/20/2024 based on these problems. **Late submission will not be accepted.**
- If you are not typing your work (which is fine) please make sure your work is legible.
- Prove **all** of your answers.

Problems for Grading

Instructions for submission: Same as before!

1. (10 pts) Determine the dimension of the subspace of $\mathbb{R}^3$ generated by vectors $(1, 2, -1), (2, 3, 4)$, and $(4, 10, 2)$.

2. (10 pts) Suppose the homogeneous system

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = 0 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = 0 \\ \qquad\qquad\vdots \\ a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n = 0 \end{cases}$$

has only the trivial solution. Prove that for every $b_1, b_2, \dots, b_n \in \mathbb{R}$, the system

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2 \\ \qquad\qquad\vdots \\ a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n = b_n \end{cases}$$

has a unique solution.

Hint: Use $\dim \mathbb{R}^n = n$, and consider the vectors $(a_{11}, a_{21}, \dots, a_{n1}), \dots, (a_{1n}, a_{2n}, \dots, a_{nn})$.

3. (10 pts) Let V be a subspace of $\mathbb{R}^n$. Prove that if $\mathcal{A} = \{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k\}$ is a linearly independent set of vectors in V, then there is a basis for V that contains $\mathcal{A}$.

Hint: Consider the subspace generated by $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k$. If this subspace is not V and then add an element $\mathbf{v}_{k+1}$ from V but outside of span $\mathcal{A}$ to the set $\mathcal{A}$. Show this new larger set is linearly independent. Repeat this until you get a basis. You must show this process ends. (The solution is essentially presented in one of the YouTube videos!)

4. (10 pts) Suppose W and V are subspaces of $\mathbb{R}^n$ for which $W \subseteq V$. Prove that if $\dim W = \dim V$, then $W = V$.

5. (10 pts) Find the angle between $(1, 2, -1)$ and $(0, 2, -1)$ in $\mathbb{R}^3$:

   (a) with the standard inner product.

   (b) with the inner product $\langle (x_1, x_2, x_3), (y_1, y_2, y_3) \rangle = x_1 y_1 + 2x_2 y_2 + 3x_3 y_3$.

6. (10 pts) Find an orthogonal basis for $\mathbb{R}^3$ for which one of the elements of this basis is $(1, 2, -1)$.

   Hint: Use the idea of echelon form to extend this vector to a basis. Then apply Gram-Schmidt. See Example 3.20.

7. (10 pts) Let A be an $m \times n$ matrix with real entries. We have shown that Row (A) and Ker A are both subspaces of $\mathbb{R}^n$. What is the relationship between Ker A and (Row $(A))^{\perp}$? Justify your answer.

   Hint: Show that a vector is in (Row $(A))^{\perp}$ if and only if it is orthogonal to all rows of A.

8. (10 pts) Let $\mathbf{v}, \mathbf{w} \in \mathbb{R}^n$, and $\langle , \rangle$ be an inner product of $\mathbb{R}^n$. Prove that $|\langle \mathbf{v}, \mathbf{w} \rangle| = ||\mathbf{v}||\ ||\mathbf{w}||$ if and only if $\mathbf{w}$ is a scalar multiple of $\mathbf{v}$ or $\mathbf{v} = \mathbf{0}$.

   Hint: Follow the proof of Cauchy-Schwarz inequality and see when the equality holds.

Practice Problems

The following examples and exercises are from the "Honors Linear Algebra and Multivariable Calculus" PDF file posted on ELMS under "Files".

9. Example 3.14.
10. Example 3.15.
11. Example 3.16.
12. Example 3.19.
13. Example 3.20.
14. Exercise 3.11.

15. Exercise 3.12.

16. Exercise 3.13.

17. Exercise 3.17.

18. Exercise 3.23.

Challenge Problems

Exercises 3.34, 3.35 and 3.38.