Nonparametric Density Estimation

Sections 17.1-17.8 of Hansen (2022)

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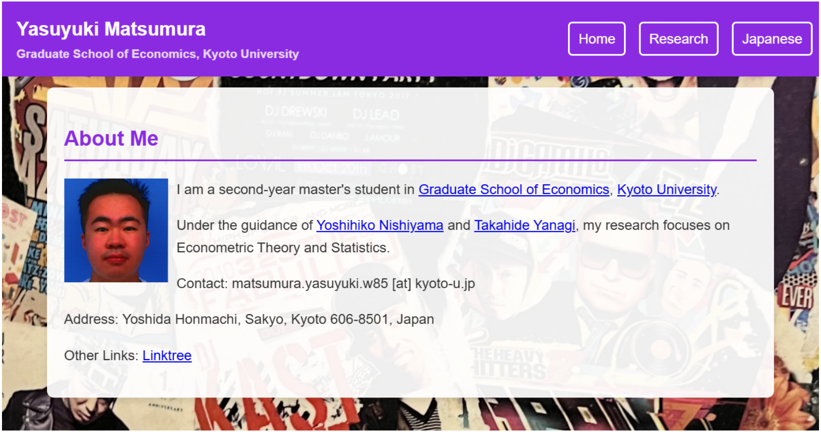
https://yasu0704xx.github.io



This slide is available on

https://github.com/yasu0704xx/Econometrics2025.

Introduction

- As a general rule, density functions can take any shape. They are inherently nonparametric and cannot be described by a finite set of parameters.
- That is, functional and/or distributional specifications relied on when estimating density functions may be incorrect.
- If we assume that such specifications are "true," we might obtain incorrect empirical conclusions.
- Thus, it would be disirable if we develop estimation procedures without requiring functional and/or distributional specifications.
- Nonparametric kernel methods achieve such a goal.

Setup

- Here we review Sections 17.1-17.8 of Hansen (2022) [1].
- We proceed with a discussion of how to estimate the probability density function $f(x)$ of a real-valued random variable X for which we have n IID observations $X_1, \cdots X_n$.
- We assume that $f(x)$ is continuous.
- The goal is to estimate $f(x)$ either at a single point x or a set of points in the interior of the support of X.

References

- Excellent textbooks on nonparametric density estimation include Silverman (1986) [7] and Scott (1992) [6].
- The following textbooks are often refered:
  - Silverman (1986) [7],
  - Scott (1992) [6],
  - van der Vaart (1998, Chapter 24) [8],
  - Pagan and Ullah (1999, Chapter 2) [3], and
  - Li and Racine (2007, Chapter 1) [2].
- 日本語の文献：
  - 西山・人見 (2023，第 1 章) [11]
  - 末石 (2015，第 9 章) [10]
  - 清水 (2023，第 5 章) [9]

Contents

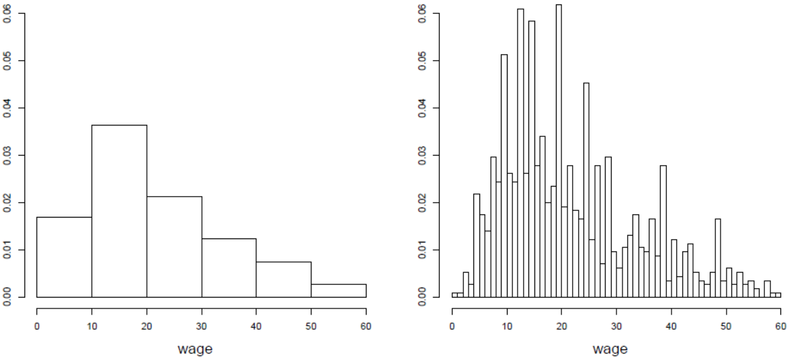
Idea behind Kernel Density Estimation

Histogram

- A simple and familiar estimator of $f(x)$ is a histogram.
- Devide the range of $f(x)$ into B bins of width w.
- Counting the number of observations n_j in each bin j, we obtain the histogram estimator of $f(x)$ for x in the j-th bin:

$$\hat{f}(x) = \frac{n_j}{nw}. \tag{1}$$

- The histogram is the plot of these heights, displayed as rectangles.



(a) Bin Width = 10

(b) Bin Width = 1

Figure 17.1: Histogram Estimate of Wage Density for Asian Women

Empirical Distribution Function

- Let us generalize the concept of histogram estimator.
- The empirical distribution function is defined as

$$F_n(x) = \frac{1}{n}\sum_{i=1}^{n} 1(X_i \leq x).$$

- Let $F(x) = \int_{-\infty}^{x} f(x)dx$ denote the (cumulative) distribution function.
- By L.L.N. and C.L.T.,[1] we obtain

$$F_n(x) \xrightarrow{p} F(x),$$
$$\sqrt{n}(F_n(x) - F(x)) \xrightarrow{d} \text{Normal}(0, F(x)(1 - F(x))).$$

[1] We discuss these convergences in Chapter 18 of Hansen (2022) [1].

Naive Estimator

- Since $F_n(x)$ is discrete, the empirical distribution function is not differentiable.
- Instead, let us consider approximate the "derivative" of $F_n(x)$.
- Note that, for $h \to 0$, it holds that

$$f(x) \approx \frac{F(x+h) - F(x-h)}{2h}.$$

- Replacing f and F with f_n and F_n, respectively, we obtain the naive estimator (the Rosenblatt estimator[2]) of $f(x)$:

$$f_n(x) = \frac{F_n(x+h) - F_n(x-h)}{2h}.$$

- Under certain conditions, it can be shown that $f_n(x) \xrightarrow{p} f(x)$.

[2] Rosenblatt (1956) [5]

- The naive estimator of $\phi(x)$ using IID observations $X_1, \cdots X_{100} \sim \text{Normal}(0,1)$: [3]

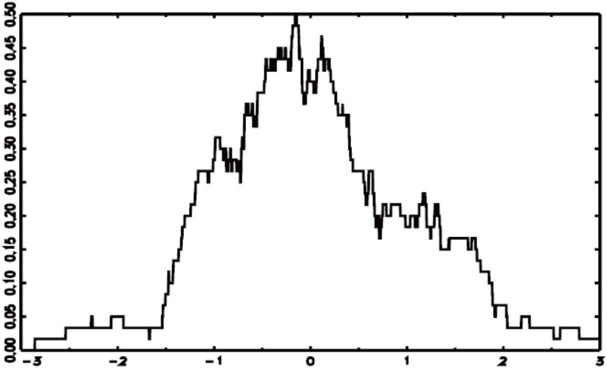


図 1: ナイーブ推定量 $(h = 0.3)$

[3] Cited from the lecture note 01 by N. Sueishi.

Problems arising with Naive Estimator

- Although $f_n(x)$ is an asymptotically "good" estimator, it is still displayed as rectangles, and then looks different from $f(x)$.
- Also, $f_n(x)$ is not differentiable.

Idea behind Kernel Density Estimation

- The naive estimator can be rewritten as

$$
\begin{aligned}
f_n(x) &= \frac{1}{2nh} \sum_{i=1}^{n} 1(x - h \leq X_i \leq x + h) \\
&= \frac{1}{nh} \sum_{i=1}^{n} k_0\left(\frac{X_i - x}{h}\right),
\end{aligned}
$$

  where $k_0(\cdot)$ is given by

$$
k_0(u) = \frac{1}{2} \cdot 1(-1 \leq u \leq 1).
$$

- Replacing $k_0(\cdot)$ with some smooth function, we can obtain a differentiable, smooth estimator of $f(x)$...?

Kernel Density Estimator

Kernel Density Estimator

- The kernel density estimator of $f(x)$ is given by

$$\hat{f}(x) = \frac{1}{nh} \sum_{i=1}^{n} K\left(\frac{X_i - x}{h}\right), \tag{2}$$

  where $K(u)$ is a weighting function known as a kernel function, and $h > 0$ is a scalar known as a bandwidth.
- $\hat{f}(x)$ is also called the Parzen-Rosenblatt estimator.[4]
- $K(u)$ need to satisfy several properties in order to construct the estimator given in (2).

[4] Parzen (1962) [4]

Kernel Function

Def 17.1

- A (second-order) kernel function $K(u)$ satisfies

$$0 \leq K(u) \leq \bar{K} < \infty,$$

Bias

Variance

Variance Estimation and Standard Errors

Integrated Mean Squared Error (IMSE)

Optimal Kernel

Refernces

References i

- Hansen, B. E. (2022). *Probability and Statistics for Economists.* Princeton.
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- Parzen, E. (1962). "On the Estimation of a Probability Density Function and Mode," *Annals of Mathematical Statistics, 33,* 1065-1076.

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- Scott, D. W. (1992). *Multivariate Density Estimation: Theory, Practice, and Visualization.* Wiley.
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