

Semiparametric Single Index Models

Li and Racine (2007, Chapter 8)

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- A semiparametric single index model is given by

$$Y = g(X^T \beta_0) + u,$$

where

$Y \in \mathbb{R}$: a dependent variable,

$X \in \mathbb{R}^q$: a $q \times 1$ explanatory vector,

$\beta_0 \in \mathbb{R}^q$: a $q \times 1$ vector of unknown parameters,

$u \in \mathbb{R}$: an error term which satisfies $\mathbb{E}(u \mid X) = 0$,

$g(\cdot)$: an unknown distribution function.

- Even though x is a $q \times 1$ vector, $x^T \beta_0$ is a scalar of a single linear combination, which is called **a single index**.
- By the form of the single index model, we obtain

$$\mathbb{E}(Y \mid X) = g(X^T \beta_0),$$

which means that the conditional expectation of Y only depends on the vector X through a single index $X^T \beta_0$.

- The model is semiparametric when $\beta \in \mathbb{R}^q$ is estimated with the parametric methods and $g(\cdot)$ with the nonparametric methods.

Examples of Parametric Single Index Model

- If $g(\cdot)$ is the identity function, then the model turns out to be a linear regression model:

$$Y = g(X^T \beta_0) + u = X^T \beta_0 + u.$$

- If $g(\cdot)$ is the CDF of Normal(0, 1), then the model turns out to be a probit model.
 - See the textbook for further discussions on a probit model.
- If $g(\cdot)$ is the CDF of logistic distribution, then the model turns out to be a logistic regression model.

Identification Conditions

Ichimura's (1993) Method

Direct Semiparametric Estimators for β

Bandwidth Selection

Klein and Spady's (1993) Estimator

Lewbel's (2000) Estimator

Manski's (1975) Maximum Score Estimator

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Multinomial Discrete Choice Models

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References

Identification Conditions

Identification Conditions

Proposition 8.1 (Identification of a Single Index Model)

For the semiparametric single index model $Y = g(x^T \beta_0) + u$, identification of β_0 and $g(\cdot)$ requires that

- (i) x should not contain a constant/an intercept, and must contain at least one continuous variable. Moreover, $\|\beta_0\|=1$.
- (ii) $g(\cdot)$ is differentiable and is not a constant function on the support of $x^T \beta_0$.
- (iii) For the discrete components of x , varying the values of the discrete variables will not divide the support of $x^T \beta_0$ into disjoint subsets.

Identification Condition (i)

- Note that the location and the scale of β_0 are not identified.
- The vector x cannot include an intercept because the function $g(\cdot)$ (which is to be estimated in nonparametric manners) includes any location and level shift.
 - That is, β_0 cannot contain a location parameter.

Identification Condition (i)

- Some normalization criterion (scale restrictions) for β_0 are needed.
 - One approach is to set $\|\beta_0\| = 1$.
 - The second approach is to set one component of β_0 to equal one. This approach requires that the variable corresponding to the component set to equal one is continuously distributed and has a non-zero coefficient.
 - Then, x must be dimension 2 or larger. If x is one-dimensional, then $\beta_0 \in \mathbb{R}^1$ is simply normalized to 1, and the model is the one-dimensional nonparametric regression $E(Y | x) = g(x)$ with no semiparametric component.

Identification Conditions (ii) and (iii)

- The function $g(\cdot)$ cannot be a constant function and must be differentiable on the support of $x^T \beta_0$.
- x must contain at least one continuously distributed variable and this continuous variable must have non-zero coefficient.
 - If not, $x^T \beta_0$ only takes a discrete set of values and it would be impossible to identify a continuous function $g(\cdot)$ on this discrete support.

Ichimura's (1993) Method

Ichimura's Method

- Textbook: Sections 8.2; 8.4.1; and 8.12.
- Suppose that the functional form of $g(\cdot)$ were known.
- Then we could estimate β_0 by minimizing the least-squares criterion:

$$\sum_{i=1} [Y_i - g(X_i^T \beta)]^2$$

with respect to β .

- We could think about replacing $g(\cdot)$ with a nonparametric estimator $\hat{g}(\cdot)$.
- However, since $g(z)$ is the conditional mean of Y_i given $X_i^T \beta_0 = z$, $g(\cdot)$ depends on unknown β_0 , so we cannot estimate $g(\cdot)$ here.

- Nevertheless, for a fixed value of β , we can estimate

$$G(X_i^T \beta) := \mathbb{E}(Y_i \mid X_i^T \beta) = \mathbb{E}(g(X_i^T \beta_0) \mid X_i^T \beta).$$

- In general $G(X_i^T \beta) \neq g(X_i^T \beta)$.
- When $\beta = \beta_0$, it holds that $G(X_i^T \beta_0) = g(X_i^T \beta_0)$.¹

¹一般の β を用いて条件付けると、 G と g は通常は一致しないが、正しいインデックス $\beta = \beta_0$ のときだけ一致するということ。

- First, we estimate $G(X_i^T \beta)$ with the leave-one-out NW estimator:

$$\begin{aligned}\hat{G}_{-i}(X_i^T \beta) &:= \hat{\mathbb{E}}_{-i}(Y_i \mid X_i^T \beta) \\ &= \frac{\sum_{j \neq i} Y_j K\left(\frac{X_j^T \beta - X_i^T \beta}{h}\right)}{\sum_{j \neq i} K\left(\frac{X_j^T \beta - X_i^T \beta}{h}\right)}.\end{aligned}$$

Ichimura's Method

- Second, using the leave-one-out NW estimator $\hat{G}_{-i}(X_i^T \beta)$, we estimate β with

$$\begin{aligned}\hat{\beta} &:= \arg \min_{\beta} \sum_{i=1}^n \left[Y_i - \hat{G}_{-i}(X_i^T \beta) \right]^2 w(X_i) \mathbf{1}(X_i \in A_n) \\ &:= \arg \min_{\beta} S_n(\beta),\end{aligned}$$

which is called **Ichimura's estimator (the WSLs estimator)**.

- $w(X_i)$ is a nonnegative weight function.
- $\mathbf{1}(X_i \in A_n)$ is a trimming function to trim out small values of $\hat{p}(X_i^T \beta) = \frac{1}{nh} \sum_{j \neq i} K \left(\frac{X_j^T \beta - X_i^T \beta}{h} \right)$, so that we do not suffer the random denominator problem.
 - $A_\delta = \{x : p(x^T \beta) \geq \delta, \text{ for } \forall \beta \in \mathcal{B}\}$.
 - $A_n = \{x : \|x - x^*\| \leq 2h, \text{ for } \exists x^* \in A_\delta\}$, which shrinks to A_δ as $n \rightarrow \infty$ and $h \rightarrow 0$.

- Let $\hat{\beta}$ denote the semiparametric estimator of β_0 obtained from minimizing $S_n(\beta)$.
- To derive the asymptotic distribution of $\hat{\beta}$, the following conditions are needed:

Asymptotic Distribution of Ichimura's Estimator

Assumption 8.1

The set A_δ is compact, and the weight function $w(\cdot)$ is bounded and positive on A_δ . Define the set

$$D_z = \{z : z = x^T \beta, \beta \in \mathcal{B}, x \in A_\delta\}.$$

Letting $p(\cdot)$ denote the PDF of $z \in D_z$, $p(\cdot)$ is bounded below by a positive constant for $\forall z \in D_z$

Assumption 8.2

$g(\cdot)$ and $p(\cdot)$ are 3 times differentiable w.r.t. $z = x^\beta$. The third derivatives are Lipschitz continuous uniformly over \mathcal{B} for $\forall z \in D_z$.

Asymptotic Distribution of Ichimura's Estimator

Assumption 8.3

The kernel function is a bounded second order kernel, which has bounded support; is twice differentiable; and its second derivative is Lipschitz continuous.

Assumption 8.4

$\mathbb{E}(|Y^m|) < \infty$ for $\exists m \geq 3$. $\text{var}(Y | x)$ is bounded and bounded away from zero for $\forall x \in A_\delta$. $\frac{q \ln(h)}{nh^{3+\frac{3}{m-1}}} \rightarrow 0$ and $nh^8 \rightarrow 0$ as $n \rightarrow \infty$.

Asymptotic Distribution of Ichimura's Estimator

Theorem 8.1. Under assumptions 8.1 through 8.4,

$$\sqrt{n}(\hat{\beta} - \beta_0) \xrightarrow{d} \text{Normal}(0, \Omega_I),$$

with

$$\Omega_I = V^{-1} \Sigma V^{-1},$$

$$V = \mathbb{E}\{w(X_i)(g_i^{(1)})^2 \\ \times (X_i - E_A(X_i \mid X_i^T \beta_0))(X_i - E_A(X_i \mid X_i^T \beta_0))^T\},$$

$$\Sigma = \mathbb{E}\{w(X_i)\sigma^2(X_i)(g_i^{(1)})^2 \\ \times (X_i - E_A(X_i \mid X_i^T \beta_0))(X_i - E_A(X_i \mid X_i^T \beta_0))^T\},$$

where

- $(g_i^{(1)}) = \frac{\partial g(v)}{\partial v} \big|_{v=X_i^T \beta_0},$
- $\mathbb{E}_A(X_i \mid v) = \mathbb{E}(X_i \mid x_A^T \beta_0 = v),$
- x_A has the distribution of X_i conditional on $X_i \in A_\delta.$

- See Ichimura (1993); and Hardle, Hall and Ichimura (1993) for the proof of **Theorem 8.1**.
- Horowitz (2009) provides an excellent heuristic outline for proving **Theorem 8.1**, using only the familiar Taylor series methods, the standard LLN, and the Lindeberg-Levy CLT.

Optimal Weight under the Homoscedasticity Assumption

- We introduce the following homoscedasticity assumption:

$$\mathbb{E}(u_i^2 \mid X_i) = \sigma^2.$$

- Under this assumption, the optimal choice of $w(\cdot)$ is $w(X_i) = 1$.
- In this case,

$$\hat{\beta} = \arg \min_{\beta} \sum_{i=1}^n (Y_i - \hat{G}_{-i}(X_i^T \beta)^2) \mathbf{1}(X_i \in A_n)$$

is **semiparametrically efficient** in the sense that Ω_I is **the semiparametric variance lower bound** (conditional on $X \in A_\delta$).

Optimal Weight under Heteroscedasticity

- In general, $\mathbb{E}(u_i^2 \mid X_i) = \sigma^2(X_i)$.
- **An infeasible case:** If one assumes that $\mathbb{E}(u_i^2 \mid X_i) = \sigma^2(X_i^T \beta_0)$, that is, the conditional variance depends only on the single index $X_i^T \beta_0$, the choice of $w(X_i) = \frac{1}{\sigma^2(X_i^T \beta_0)}$ can lead to a semiparametrically efficient estimation.
- We could adopt a two-step procedure as follows.

A two-step procedure to Choose Optimal Weight

- Suppose that the conditional variance is a function of $X_i^T \beta_0$ (Let $\sigma^2(X_i^T \beta_0)$ denote it).
- **The first step:** Use $w(X_i) = 1$ to obtain a \sqrt{n} -consistent estimator of β_0 .
- Let $\tilde{\beta}_0$ denote the estimator of β_0 , and $\tilde{u}_i = Y_i - \hat{g}(X_i^T \tilde{\beta}_0)$ denote the residual obtained from $\tilde{\beta}_0$.
- We can obtain a consistent nonparametric estimator of the conditional variance: $\hat{\sigma}^2(X_i^T \tilde{\beta}_0)$.

A two-step procedure to Choose Optimal Weight

- **The second step:** Estimate β_0 again using $w(X_i) = \frac{1}{\hat{\sigma}^2(X_i^T \tilde{\beta}_0)}$:

$$\hat{\beta}_0 = \arg \min_{\beta} \sum_{i=1}^n \left[Y_i - \hat{G}_{-i}(X_i^T \beta) \right]^2 \frac{1}{\hat{\sigma}^2(X_i^T \tilde{\beta}_0)} \mathbf{1}(X_i \in A_n).$$

- The estimator $\hat{\beta}_0$ is semiparametrically efficient because $\hat{\sigma}^2(v) - \sigma^2(v)$ converges to zero at a particular rate uniformly over $v \in D_v$ (D_v is the support of $X_i^T \beta_0$).²

² $\hat{\sigma}^2(X_i^T \beta)$ を用いるケースもある.

Bandwidth Selection for Ichimura's Method

- Recall that we assume in Assumption 8.4 that $\frac{q \ln(h)}{nh^{3+\frac{3}{m-1}}} \rightarrow 0$ and $nh^8 \rightarrow 0$ as $n \rightarrow \infty$, where $m \geq 3$ is a positive integer whose specific value depends on the existence of the number of finite moments of Y along with the smoothness of the unknown function $g(\cdot)$.³
- The range of permissive smoothing parameters allows for optimal smoothing, i.e., $h = O(n^{-\frac{1}{5}})$.⁴

³Assumption 8.4 は, g をノンパラメトリックに推定することがパラメトリックパートの収束レートに影響を与えないための十分条件になっている.

⁴このオーダーで選んだ h は, Assumption 8.4 を満たしている.

- Our aim is to choose $\hat{\beta}$ close to β_0 , and h close to the value h_0 , which minimize the average of

$$\mathbb{E}\{\hat{g}(X_i^T \beta_0 | X_i^T \beta_0) - g(X_i^T \beta_0)\}^2.$$

- Hardle, Hall and Ichimura (1993) suggest picking β and the bandwidth h jointly by minimization of $S_n(\beta)$.
- Further discussions follow in Section 8.4.

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- 西山慶彦, 人見光太郎 (2023) 『ノン・セミパラメトリック統計解析 (理論統計学教程：数理統計の枠組み)』共立出版.

References (2)

Useful references also include some lecture notes of the following topic courses:

- ECON 718 NonParametric Econometrics (Bruce Hansen, Spring 2009, University of Wisconsin-Madison),
- セミノンパラメトリック計量分析（末石直也，2014 年度後期，京都大学）.