Semiparametric Single Index Models

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December 2, 2024

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Introduction

- A semiparametric single index model is given by

$$Y = g(X^T \beta_0) + u,$$

where

$Y \in \mathbb{R}$: a dependent variable,

$X \in \mathbb{R}^q$: a $q \times 1$ explanatory vector,

$\beta_0 \in \mathbb{R}^q$: a $q \times 1$ vector of unknown parameters,

$u \in \mathbb{R}$: an error term which satisfies $\mathbb{E}(u \mid X) = 0$,

$g(\cdot)$: an unknown distribution function.

- Even though x is a $q \times 1$ vector, $x^T\beta_0$ is a scalar of a single linear combination, which is called a single index.
- By the form of the single index model, we obtain

$$\mathbb{E}(Y \mid X) = g(X^T\beta_0),$$

which means that the conditional expectation of Y only depends on the vector X through a single index $X^T\beta_0$.

- The model is semiparametric when $\beta \in \mathbb{R}^q$ is estimated with the parametric methods and $g(\cdot)$ with the nonparametric methods.

Examples of Parametric Single Index Model

- If $g(\cdot)$ is the identity function, then the model turns out to be a linear regression model:

$$Y = g(X^T\beta_0) + u = X^T\beta_0 + u.$$

- If $g(\cdot)$ is the CDF of Normal$(0, 1)$, then the model turns out to be a probit model.
  - See the textbook for further discussions on a probit model.
- If $g(\cdot)$ is the CDF of logistic distribution, then the model turns out to be a logistic regression model.

TOC

Identification Conditions

Identification Conditions

Proposition 8.1 (Identification of a Single Index Model)

For the semiparametric single index model $Y = g(x^T\beta_0) + u$, identification of β_0 and $g(\cdot)$ requires that

- (i) x should not contain a constant/an intercept, and must contain at least one continuous variable. Moreover, $\|\beta_0\|=1$.
- (ii) $g(\cdot)$ is differentiable and is not a constant function on the support of $x^T\beta_0$.
- (iii) For the discrete components of x, varying the values of the discrete variables will not divide the support of $x^T\beta_0$ into disjoint subsets.

Identification Condition (i)

- Note that the location and the scale of β_0 are not identified.
- The vector x cannot include an intercept because the function $g(\cdot)$ (which is to be estimated in nonparametric manners) includes any location and level shift.
  - That is, β_0 cannot contain a location parameter.

Identification Condition (i)

- Some normalization criterion (scale restrictions) for β_0 are needed.
  - One approach is to set $\|\beta_0\|$.
  - The second approach is to set one component of β_0 to equal one. This approach requires that the variable corresponding to the component set to equal one is continuously distributed and has a non-zero coefficient.
  - Then, x must be dimension 2 or larger. If x is one-dimensional, then $\beta_0 \in \mathbb{R}^1$ is simply normalized to 1, and the model is the one-dimensional nonparametric regression $E(Y \mid x) = g(x)$ with no semiparametric component.

Identification Conditions (ii) and (iii)

- The function $g(\cdot)$ cannot be a constant function and must be differentiable on the support of $x^T\beta_0$.
- x must contain at least one continuously distributed variable and this continuous variable must have non-zero coefficient.
    - If not, $x^T\beta_0$ only takes a discrete set of values and it would be impossible to identify a continuous function $g(\cdot)$ on this discrete support.

Ichimura's (1993) Method

Ichimura's Method

- Textbook: Sections 8.2; 8.4.1; and 8.12.
- Suppose that the functional form of $g(\cdot)$ were known.
- Then we could estimate β_0 by minimizing the least-squares criterion:

$$\sum_{i=1} \left[Y_i - g(X_i^T \beta)\right]^2$$

with respect to β.

- We could think about replacing $g(\cdot)$ with a nonparametric estimator $\hat{g}(\cdot)$.
- However, since $g(z)$ is the conditional mean of Y_i given $X_i^T \beta_0 = z$, $g(\cdot)$ depends on unknown β_0, so we cannot estimate $g(\cdot)$ here.

Ichimura's Method

- Nevertheless, for a fixed value of β, we can estimate

$$G(X_i^T\beta) := \mathbb{E}(Y_i \mid X_i^T\beta) = \mathbb{E}(g(X_i^T\beta) \mid X_i^T\beta).$$

- In general $G(X_i^T\beta) \neq g(X_i^T\beta)$.
- When $\beta = \beta_0$, it holds that $G(X_i^T\beta_0) = g(X_i^T\beta_0)$.

Ichimura's Method

- First, we estimate $G(X_i^T\beta)$ with the leave-one-out NW estimator:

$$\begin{aligned}\hat{G}_{-i}(X_i^T\beta) &:= \hat{\mathbb{E}}_{-i}(Y_i \mid X_i^T\beta) \\ &= \frac{\sum_{j\neq i} Y_j K\left(\frac{X_j^T\beta - X_i^T\beta}{h}\right)}{\sum_{j\neq i} K\left(\frac{X_j^T\beta - X_i^T\beta}{h}\right)}.\end{aligned}$$

Ichimura's Method

- Second, using the leave-one-out NW estimator $\hat{G}_{-i}(X_i^T\beta)$, we estimate β with

$$\begin{aligned}\hat{\beta} &:= \arg\min_{\beta} \sum_{i=1}^{n} \left[Y_i - \hat{G}_{-i}(X_i^T\beta)\right]^2 w(X_i)\mathbf{1}(X_i \in A_n) \\ &:= \arg\min_{\beta} S_n(\beta),\end{aligned}$$

  which is called Ichimura's estimator (the WSLS estimator).
- $w(X_i)$ is a nonnegative weight function.
- $\mathbf{1}(X_i \in A_n)$ is a trimming function to trim out small values of $\hat{p}(X_i^T\beta) = \dfrac{1}{nh}\sum_{j\neq i} K\left(\frac{X_j^T\beta - X_i^T\beta}{h}\right)$, so that we do not suffer the random denominator problem.
  - $A_\delta = \{x : p(x^T\beta) \geq \delta,\ \text{for } {}^\forall\beta \in \mathcal{B}\}$.
  - $A_n = \{x : ||x - x^\star|| \leq 2h,\ \text{for } {}^\exists x^\star \in A_\delta\}$, which shrinks to A_δ as $n \to \infty$ and $h \to 0$.

Asymptotic Distribution of Ichimura's Estimator

- Let $\hat{\beta}$ denote the semiparametric estimator of β_0 obtained from minimizing $S_n(\beta)$.
- To derive the asymptotic distribution of $\hat{\beta}$, the following conditions are neeeded:

Asymptotic Distribution of Ichimura's Estimator

Assumption 8.1

The set A_δ is compact, and the weight function $w(\cdot)$ is bounded and posotive on A_δ. Define the set

$$D_z = \{z : z = x^T\beta, \beta \in \mathcal{B}, x \in A_\delta\}.$$

Letting $p(\cdot)$ denote the PDF of $z \in D_z$, $p(\cdot)$ is bounded below by a positive constant for ${}^\forall z \in D_z$

Assumption 8.2

$g(\cdot)$ and $p(\cdot)$ are 3 times differentiable w.r.t. $z = x^\beta$. The third derivatives are Lipschitz continuous uniformly over $\mathcal{B}$ for ${}^\forall z \in D_z$.

Asymptotic Distribution of Ichimura's Estimator

Assumption 8.3

The kernel function is a bounded second order kernel, which has bounded support; is twice differentiable; and its second derivative is Lipschitz continuous.

Assumption 8.4

$\mathbb{E}(|Y^m|) < \infty$ for $^\exists m \geq 3$. var$(Y \mid x)$ is bounded and bounded away from zero for $^\forall x \in A_\delta$. $\frac{q \ln(h)}{nh^{3+\frac{3}{m-1}}} \to 0$ and $nh^8 \to 0$ as $n \to \infty$.

Asymptotic Distribution of Ichimura's Estimator

Theorem 8.1. Under assumptions 8.1 through 8.4,

$$\sqrt{n}(\hat{\beta} - \beta_0) \xrightarrow{d} \text{Normal}(0, \Omega_I),$$

with

$$\begin{aligned}
\Omega_I &= V^{-1} \Sigma V^{-1}, \\
V &= \mathbb{E}\{w(X_i)(g_i^{(1)})^2 \\
&\quad \times (X_i - E_A(X_i \mid X_i^T \beta_0))(X_i - E_A(X_i \mid X_i^T \beta_0))^T\}, \\
\Sigma &= \mathbb{E}\{w(X_i)\sigma^2(X_i)(g_i^{(1)})^2 \\
&\quad \times (X_i - E_A(X_i \mid X_i^T \beta_0))(X_i - E_A(X_i \mid X_i^T \beta_0))^T\},
\end{aligned}$$

where

- $(g_i^{(1)}) = \frac{\partial g(v)}{\partial v} \mid_{v = X_i^T \beta_0}$,
- $\mathbb{E}_A(X_i \mid v) = \mathbb{E}(X_i \mid x_A^T \beta_0 = v)$,
- x_A has the distribution of X_i conditional on $X_i \in A_\delta$.

Asymptotic Distribution of Ichimura's Estimator

- See Ichimura (1993); and Hardle, Hall and Ichimura (1993) for the proof of **Theorem 8.1**.
  - Horowitz (2009) provides an excellent heuristic outline for proving **Theorem 8.1**, using only the familiar Taylor series methods, the standard LLN, and the Lindeberg-Levy CLT.
- We can consistently estimate Avar $\left(\sqrt{n}(\hat{\beta} - \beta)\right) = \Omega_I$ with its sample analogue.

Optimal Weight under the Homoscedasticity Assumption

- We introduce the following homoscedasticity assumption:

$$\mathbb{E}(u_i^2 \mid X_i) = \sigma^2.$$

- Under this assumption, the optimal choice of $w(\cdot)$ is $w(X_i) = 1$.
- In this case,

$$\hat{\beta} = \arg\min_{\beta} \sum_{i=1}^{n} (Y_i - \hat{G}_{-i}(X_i^T\beta)^2)\mathbf{1}(X_i \in A_n)$$

is semiparametrically efficient in the sense that Ω_I is the semiparametric variance lower bound (conditional on $X \in A_\delta$).

Optimal Weight under Heteroscedasticity

- In general, $\mathbb{E}(u_i^2 \mid X_i) = \sigma^2(X_i)$.
  - The semiparametrically efficient estimator $\hat{\beta}_{opt}$ has a complicated structure.
- An infeasible case: If one assues that $\mathbb{E}(u_i^2 \mid X_i) = \sigma^2(X_i^T \beta_0)$, that is, the conditional variance depends only on the single index $X_i^T \beta_0$, the choice of $w(X_i) = \frac{1}{\sigma^2(X_i^T \beta_0)}$ can lead to a semiparametrically efficient estimation.
- We could adopt a two-step procedure as follows.

A two-step procedure to Choose Optimal Weight

- Suppose that the conditional variance is a function of $X_i^T \beta_0 : \sigma^2(X_i^T \beta_0)$.
- The first step: Use $w(X_i) = 1$ to obtain a $\sqrt{n}$ consistent estimator of β_0.
- Let $\tilde{\beta}_0$ denote the estimator of β_0, and $\tilde{u}_i = Y_i - \hat{g}$ denote the residual obtained from $\tilde{\beta}_0$
- The second step:

Proof of Theorem 8.1

- The slides for the proof of Theorem 8.1 are associated with Horowitz (2009, Chapter 2).
- Recall that the least squares criterion is that

$$\sum_{i=1}^{n} \left[Y_i - \hat{G}_{-i}(X_i^T \beta) \right]^2 w(X_i) \mathbf{1}(X_i \in A_n).$$

- Hardle, Hall and Ichimura (1993) show that this criterion is asymptotically equivalent to the following criterion:

$$\sum_{i=1}^{n} \left[Y_i - \hat{G}_{-i}(X_i^T \beta) \right]^2 w(X_i),$$

which does not have the trimming function $\mathbf{1}(\cdot)$.

Proof of Theorem 8.1

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Bandwidth Selection for Ichimura's Method

- Hardle, Hall and Ichimura (1993) suggest picking β and the bandwidth h jointly by minimization of $S_n(\beta)$.

Direct Semiparametric Estimators for β

Klein and Spady's (1993) Estimator

Lewbel's (2000) Estimator

Manski's (1975) Maximum Score Estimator

Horowitz's (1992) Smoothed Maximum Score Estimator

Han's (1987) Maximum Rank Estimator

Multinomial Discrete Choice Models

Ai's (1997) Semiparametric Maximum Likelihood Approach

References

References (1)

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- Horowitz, J. L. (2009) *Semiparametric and Nonparametric Methods in Econometrics*, Springer.
- Ichimura, H. (1993) “Semiparametric Least Squares (SLS) and Weighted SLS Estimation of Single-Index Models,” *Journal of Econometrics*, 3, 205-228.
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- 西山慶彦，人見光太郎 (2023) 『ノン・セミパラメトリック統計解析（理論統計学教程：数理統計の枠組み)』共立出版.

References (2)

Useful references also include some lecture notes of the following topic courses:

- ECON 718 NonParametric Econometrics (Bruce Hansen, Spring 2009, University of Wisconsin-Madison),
- セミノンパラメトリック計量分析（末石直也，2014 年度後期，京都大学）.