Semiparametric Single Index Models

Li and Racine (2007, Chapter 8)

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Introduction

- A semiparametric single index model is given by

$$Y = g(X^T\beta_0) + u,$$

where

$Y \in \mathbb{R}$: a dependent variable,

$X \in \mathbb{R}^q$: a $q \times 1$ explanatory vector,

$\beta_0 \in \mathbb{R}^q$: a $q \times 1$ vector of unknown parameters,

$u \in \mathbb{R}$: an error term which satisfies $\mathbb{E}(u \mid X) = 0$,

$g(\cdot)$: an unknown distribution function.

Introduction

- Even though x is a $q \times 1$ vector, $x^T \beta_0$ is a scalar of a single linear combination, which is called a single index.
- By the form of the single index model, we obtain

$$\mathbb{E}(Y \mid X) = g(X^T \beta_0),$$

  which means that the conditional expectation of Y only depends on the vector X through a single index $X^T \beta_0$.
- The model is semiparametric when $\beta \in \mathbb{R}^q$ is estimated with the parametric methods and $g(\cdot)$ with the nonparametric methods.

Examples of Parametric Single Index Model

- If $g(\cdot)$ is the identity function, then the model turns out to be a linear regression model:

$$Y = g(X^T \beta_0) + u = X^T \beta_0 + u.$$

- If $g(\cdot)$ is the CDF of Normal$(0, 1)$, then the model turns out to be a probit model.
    - See the textbook for further discussions on a probit model.
- If $g(\cdot)$ is the CDF of logistic distribution, then the model turns out to be a logistic regression model.

TOC

Identification Conditions

Ichimura's (1993) Method

Direct Semiparametric Estimators for β

Bandwidth Selection

Klein and Spady's (1993) Estimator

Lewbel's (2000) Estimator

Manski's (1975) Maximum Score Estimator

Horowitz's (1992) Smoothed Maximum Score Estimator

Han's (1987) Maximum Rank Estimator

Multinomial Discrete Choice Models

Ai's (1997) Semiparametric Maximum Likelihood Approach

References

Identification Conditions

Identification Conditions

Proposition 8.1 (Identification of a Single Index Model)

For the semiparametric single index model $Y = g(x^T\beta_0) + u$, identification of β_0 and $g(\cdot)$ requires that

- (i) x should not contain a constant/an intercept, and must contain at least one continuous variable. Moreover, $\|\beta_0\|=1$.
- (ii) $g(\cdot)$ is differentiable and is not a constant function on the support of $x^T\beta_0$.
- (iii) For the discrete components of x, varying the values of the discrete variables will not divide the support of $x^T\beta_0$ into disjoint subsets.

Identification Condition (i)

- Note that the location and the scale of β_0 are not identified.
- The vector x cannot include an intercept because the function $g(\cdot)$ (which is to be estimated in nonparametric manners) includes any location and level shift.
    - That is, β_0 cannot contain a location parameter.

Identification Condition (i)

- Some normalization criterion (scale restrictions) for β_0 are needed.
  - One approach is to set $\|\beta_0\| = 1$.
  - The second approach is to set one component of β_0 to equal one. This approach requires that the variable corresponding to the component set to equal one is continuously distributed and has a non-zero coefficient.
  - Then, x must be dimension 2 or larger. If x is one-dimensional, then $\beta_0 \in \mathbb{R}^1$ is simply normalized to 1, and the model is the one-dimensional nonparametric regression $E(Y \mid x) = g(x)$ with no semiparametric component.

Identification Conditions (ii) and (iii)

- The function $g(\cdot)$ cannot be a constant function and must be differentiable on the support of $x^T\beta_0$.
- x must contain at least one continuously distributed variable and this continuous variable must have non-zero coefficient.
    - If not, $x^T\beta_0$ only takes a discrete set of values and it would be impossible to identify a continuous function $g(\cdot)$ on this discrete support.

Ichimura's (1993) Method

Ichimura's Method

- Textbook: Sections 8.2; 8.4.1; and 8.12.
- Suppose that the functional form of $g(\cdot)$ were known.
- Then we could estimate β_0 by minimizing the least-squares criterion:

$$\sum_{i=1} \left[Y_i - g(X_i^T \beta)\right]^2$$

  with respect to β.
- We could think about replacing $g(\cdot)$ with a nonparametric estimator $\hat{g}(\cdot)$.
- However, since $g(z)$ is the conditional mean of Y_i given $X_i^T \beta_0 = z$, $g(\cdot)$ depends on unknown β_0, so we cannot estimate $g(\cdot)$ here.

Ichimura's Method

- Nevertheless, for a fixed value of β, we can estimate

$$G(X_i^T\beta) := \mathbb{E}(Y_i \mid X_i^T\beta) = \mathbb{E}(g(X_i^T\beta_0) \mid X_i^T\beta).$$

- In general $G(X_i^T\beta) \neq g(X_i^T\beta)$.
- When $\beta = \beta_0$, it holds that $G(X_i^T\beta_0) = g(X_i^T\beta_0)$. [1]

[1] 一般の $X_i^T\beta$ を用いて条件付けると，G と g は通常は一致しないが，正しいインデックス $X_i^T\beta = X_i^T\beta_0$ のときだけ一致するということ．

Ichimura's Method

- First, we estimate $G(X_i^T\beta)$ with the leave-one-out NW estimator:

$$\begin{aligned}\hat{G}_{-i}(X_i^T\beta) &:= \hat{\mathbb{E}}_{-i}(Y_i \mid X_i^T\beta) \\ &= \frac{\sum_{j\neq i} Y_j K\left(\frac{X_j^T\beta - X_i^T\beta}{h}\right)}{\sum_{j\neq i} K\left(\frac{X_j^T\beta - X_i^T\beta}{h}\right)}.\end{aligned}$$

Ichimura's Method

- Second, using the leave-one-out NW estimator $\hat{G}_{-i}(X_i^T\beta)$, we estimate β with

$$\hat{\beta} := \arg\min_{\beta} \sum_{i=1}^{n} \left[Y_i - \hat{G}_{-i}(X_i^T\beta)\right]^2 w(X_i)\mathbf{1}(X_i \in A_n)$$
$$:= \arg\min_{\beta} S_n(\beta),$$

  which is called Ichimura's estimator (the WSLS estimator).
- $w(X_i)$ is a nonnegative weight function.
- $\mathbf{1}(X_i \in A_n)$ is a trimming function to trim out small values of $\hat{p}(X_i^T\beta) = \dfrac{1}{nh}\sum_{j\neq i} K\left(\frac{X_j^T\beta - X_i^T\beta}{h}\right)$, so that we do not suffer the random denominator problem.
  - $A_\delta = \{x : p(x^T\beta) \geq \delta,$ for $^\forall\beta \in \mathcal{B}\}$.
  - $A_n = \{x : ||x - x^\star|| \leq 2h,$ for $^\exists x^\star \in A_\delta\}$, which shrinks to A_δ as $n \to \infty$ and $h \to 0$.

Asymptotic Distribution of Ichimura's Estimator

- Let $\hat{\beta}$ denote the semiparametric estimator of β_0 obtained from minimizing $S_n(\beta)$.
- To derive the asymptotic distribution of $\hat{\beta}$, the following conditions are neeeded:

Asymptotic Distribution of Ichimura's Estimator

Assumption 8.1

The set A_δ is compact, and the weight function $w(\cdot)$ is bounded and posotive on A_δ. Define the set

$$D_z = \{z : z = x^T\beta, \beta \in \mathcal{B}, x \in A_\delta\}.$$

Letting $p(\cdot)$ denote the PDF of $z \in D_z$, $p(\cdot)$ is bounded below by a positive constant for ${}^\forall z \in D_z$

Assumption 8.2

$g(\cdot)$ and $p(\cdot)$ are 3 times differentiable w.r.t. $z = x^\beta$. The third derivatives are Lipschitz continuous uniformly over $\mathcal{B}$ for ${}^\forall z \in D_z$.

Asymptotic Distribution of Ichimura's Estimator

Assumption 8.3

The kernel function is a bounded second order kernel, which has bounded support; is twice differentiable; and its second derivative is Lipschitz continuous.

Assumption 8.4

$\mathbb{E}(|Y^m|) < \infty$ for ${}^{\exists}m \geq 3$. var$(Y \mid x)$ is bounded and bounded away from zero for ${}^{\forall}x \in A_\delta$. $\frac{q\ln(h)}{nh^{3+\frac{3}{m-1}}} \to 0$ and $nh^8 \to 0$ as $n \to \infty$.

Asymptotic Distribution of Ichimura's Estimator

Theorem 8.1. Under assumptions 8.1 through 8.4,

$$\sqrt{n}(\hat{\beta} - \beta_0) \xrightarrow{d} \text{Normal}(0, \Omega_I),$$

with

$$\begin{aligned}
\Omega_I &= V^{-1}\Sigma V^{-1}, \\
V &= \mathbb{E}\{w(X_i)(g_i^{(1)})^2 \\
&\quad \times (X_i - E_A(X_i \mid X_i^T\beta_0))(X_i - E_A(X_i \mid X_i^T\beta_0))^T\}, \\
\Sigma &= \mathbb{E}\{w(X_i)\sigma^2(X_i)(g_i^{(1)})^2 \\
&\quad \times (X_i - E_A(X_i \mid X_i^T\beta_0))(X_i - E_A(X_i \mid X_i^T\beta_0))^T\},
\end{aligned}$$

where

- $(g_i^{(1)}) = \frac{\partial g(v)}{\partial v} \mid_{v = X_i^T\beta_0}$,
- $\mathbb{E}_A(X_i \mid v) = \mathbb{E}(X_i \mid x_A^T\beta_0 = v)$,
- x_A has the distribution of X_i conditional on $X_i \in A_\delta$.

Asymptotic Distribution of Ichimura's Estimator

- See Ichimura (1993); and Hardle, Hall and Ichimura (1993) for the proof of **Theorem 8.1**.
- Horowitz (2009) provides an excellent heuristic outline for proving **Theorem 8.1**, using only the familiar Taylor series methods, the standard LLN, and the Lindeberg-Levy CLT.

Optimal Weight under the Homoscedasticity Assumption

- We introduce the following homoscedasticity assumption:

$$\mathbb{E}(u_i^2 \mid X_i) = \sigma^2.$$

- Under this assumption, the optimal choice of $w(\cdot)$ is $w(X_i) = 1$.
- In this case,

$$\hat{\beta} = \arg\min_{\beta} \sum_{i=1}^{n} (Y_i - \hat{G}_{-i}(X_i^T \beta)^2) \mathbf{1}(X_i \in A_n)$$

is semiparametrically efficient in the sense that Ω_I is the semiparametric variance lower bound (conditional on $X \in A_\delta$).

Optimal Weight under Heteroscedasticity

- In general, $\mathbb{E}(u_i^2 \mid X_i) = \sigma^2(X_i)$.
- An infeasible case: If one assues that $\mathbb{E}(u_i^2 \mid X_i) = \sigma^2(X_i^T\beta_0)$, that is, the conditional variance depends only on the single index $X_i^T\beta_0$, the choice of $w(X_i) = \frac{1}{\sigma^2(X_i^T\beta_0)}$ can lead to a semiparametrically efficient estimation.
- We could adopt a two-step procedure as follows.

Two-Step Procedure to Choose Optimal Weight

- Suppose that the conditional variance is a function of $X_i^T \beta_0$ (Let $\sigma^2(X_i^T \beta_0)$ denote it).
- The first step: Use $w(X_i) = 1$ to obtain a $\sqrt{n}$-consistent estimator of β_0.
- Let $\tilde{\beta}_0$ denote the estimator of β_0, and $\tilde{u}_i = Y_i - \hat{g}(X_i^T \tilde{\beta}_0)$ denote the residual obtained from $\tilde{\beta}_0$.
- We can obtain a consistent nonparametric estimator of the conditional variance: $\hat{\sigma}^2(X_i^T \tilde{\beta}_0)$.

Two-Step Procedure to Choose Optimal Weight

- The second step: Estimate β_0 again using $w(X_i) = \frac{1}{\hat{\sigma}^2(X_i^T \tilde{\beta}_0)}$:

$$\hat{\beta}_0 = \arg\min_{\beta} \sum_{i=1}^{n} \left[Y_i - \hat{G}_{-i}(X_i^T \beta) \right]^2 \frac{1}{\hat{\sigma}^2(X_i^T \tilde{\beta}_0)} \mathbf{1}(X_i \in A_n).$$

- The estimator $\hat{\beta}_0$ is semiparametrically efficient because $\hat{\sigma}^2(v) - \sigma^2(v)$ converges to zero at a particular rate uniformly over $v \in D_v$ (D_v is the support of $X_i^T \beta_0$). [2]

[2] $\hat{\sigma}^2(X_i^T \beta)$ を用いるケースもある.

Bandwidth Selection for Ichimura's Estimator

- Recall that we assume in Assumption 8.4 that $\frac{q\ln(h)}{nh^{3+\frac{3}{m-1}}} \to 0$ and $nh^8 \to 0$ as $n \to \infty$, where $m \geq 3$ is a positive integer whose specific value depends on the existence of the number of finite moments of Y along with the smoothness of the unknown function $g(\cdot)$. [3]
- The range of permissive smoothing parameters allows for optimal smoothing, i.e., $h = O(n^{-\frac{1}{5}})$. [4]

[3] Assumption 8.4 は，g をノンパラメトリックに推定することがパラメトリックパートの収束レートに影響を与えないための十分条件になっている．

[4] このオーダーで選んだ h は，Assumption 8.4 を満たしている．

Bandwidth Selection for Ichimura's Estimator

- Our aim is to choose $\hat{\beta}$ close to β_0, and h close to the value h_0, which minimize the average of

$$\mathbb{E}\{\hat{g}(X_i^T\beta_0 \mid X_i^T\beta_0) - g(X_i^T\beta_0)\}^2.$$

- Hardle, Hall and Ichimura (1993) suggest picking β and the bandwidth h jointly by minimization of $S_n(\beta)$.
- Further discussions follow in Section 8.4.

Direct Semiparametric Estimators for β

Direct Semiparametric Estimators for β

- Textbook: Sections 8.3; and 8.4.2.
- Here we review:
  - Hardle and Stoker's (1989) Average Derivative Estimator,
  - Powell, Stock and Stoker's (1989) Density-Weighted Average Derivative Estimator, and
  - Li, Lu and Ullah's (2003) Estimator.
- The advantage of the direct estimation method is that we can estimate β_0 and $g(x^T\beta_0)$ directly without running the nonlinear least squares, which leads to the computational simplicity.
- We still suffer from a finite-sample problem.

Hardle and Stoker's (1989) Average Derivative Estimator

- Suppose that x is a $q \times 1$ vector of continuous variables.
- Then we obtain the average derivative of $\mathbb{E}(Y \mid X = x)$:

$$\mathbb{E}\left[\frac{\partial \mathbb{E}(Y \mid X = x)}{\partial x}\right] = \mathbb{E}\left[g^{(1)}(x^T \beta_0)\right] \beta_0$$

- Recall that the scale of β_0 is not identified, which means that the constant $\mathbb{E}\left[g^{(1)}(x^T \beta_0)\right]$ does not matter. That is, a normalized estimation of $\mathbb{E}\left[\frac{\partial \mathbb{E}(Y|X=x)}{\partial x}\right]$ is an estimation of normalized β_0.

Hardle and Stoker's (1989) Average Derivative Estimator

- Let $\hat{\mathbb{E}}(Y_i \mid X_i)$ denote the NW estimator of $\mathbb{E}(Y_i \mid X_i)$:

$$\hat{\mathbb{E}}(Y_i \mid X_i) = \frac{\sum_{j=1}^{n} Y_j K\left(\frac{X_i - X_j}{a}\right)}{\sum_{j=1}^{n} K\left(\frac{X_i - X_j}{a}\right)}.$$

- Assuming that the kernel function is differentiable, we can estimate β_0, estimating $\mathbb{E}\left[\frac{\partial \mathbb{E}(Y|X=x)}{\partial x}\right]$ with its sample analogue:

$$\tilde{\beta}_{ave} = \frac{1}{n}\sum_{i=1}^{n} \frac{\partial \hat{\mathbb{E}}(Y_i \mid X_i)}{\partial X_i}.$$

- The scale normalization can also be implemented by $\frac{\tilde{\beta}_{ave}}{|\tilde{\beta}_{ave}|}$.

Hardle and Stoker's (1989) Average Derivative Estimator

- An issue raised with this estimator is the random denominator problem, which leads to a difficulty in the derivation of the asymptotic properties.
- Rilstone (1991) establishes the $\sqrt{n}$-normality using a trimming function.

Powell, Stock and Stoker's (1989) Density-Weighted Average Derivative Estimator

- As we obtain the average derivative above, we also obtain the weighted average derivative of $\mathbb{E}(Y \mid X = x)$:

$$\mathbb{E}\left[w(x)\frac{\partial \mathbb{E}(Y \mid X = x)}{\partial x}\right] = \mathbb{E}\left[w(x)g^{(1)}(x^T\beta_0)\right]\beta_0.$$

Powell, Stock and Stoker's (1989) Density-Weighted Average Derivative Estimator

- Let $w(x)$ be the density function $f(x)$, and δ denote the density-weighted average derivative of $\mathbb{E}(Y \mid X = x)$.
- Then we obtain

$$
\begin{aligned}
\delta &= \mathbb{E}\left[f(X)\frac{\partial \mathbb{E}(Y \mid X = x)}{\partial x}\right] \\
&= \mathbb{E}\left[f(X)g^{(1)}(X^T\beta_0)\right] \\
&= \int g^{(1)}(x^T\beta_0)f^2(x)dx \\
&= g(x^T\beta_0)f^2(x) - 2\int g(x^T\beta_0)f^{(1)}(x)f(x)dx.
\end{aligned}
$$

Powell, Stock and Stoker (1989) Density-Weighted Average Derivative Estimator

- Assume that $f(x) = 0$ at the boundary of the support of X. Then we observe that $g(x^T\beta_0)f^2(x) = 0$, that is,

$$\begin{aligned}\delta &= -2\int g(x^T\beta_0)f^{(1)}(x)f(x)dx \\ &= -2\mathbb{E}[g(X^T\beta_0)f^{(1)}(X)] \\ &= -2\mathbb{E}[Yf^{(1)}(X)].\end{aligned}$$

- We can estimate δ by its sample analogue:

$$\hat{\delta} = -\frac{2}{n}\sum_{i=1}^{n} Y_i \hat{f}_{-i}^{(1)}(X_i),$$

where $\hat{f}_{-i}(X_i)$ is the leave-one-out NW estimator of $f(X)$:

$$\hat{f}_{-i}(X_i) = \frac{1}{n-1}\sum_{j\neq i}\left(\frac{1}{h}\right)^q K\left(\frac{X_i - X_j}{h}\right).$$

Powell, Stock and Stoker's (1989) Density-Weighted Average Derivative Estimator

- The textbook uses the NW estimator $\hat{f}^{(1)}(X_i)$ in (8.17).
- However, Powell, Stock and Stoker (1989) define their estimator using the leave-one-out NW estimator $\hat{f}_{-i}^{(1)}(X_i)$.
- Here we proceed with Powell, Stock and Stoker (1989).

Powell, Stock and Stoker's (1989) Density-Weighted Average Derivative Estimator

- A useful representation of $\hat{\delta}$ is given by

$$\hat{\delta} = \frac{-2}{n(n-1)} \sum_{i=1}^{n} \sum_{j \neq i} \left(\frac{1}{h} \right)^{q+1} Y_i K^{(1)} \left(\frac{X_i - X_j}{h} \right).$$

- Under some assumptions, if $h \to 0$ and $nh^{q+2} \to \infty$ hold, then the density-weighted average derivative estimator $\hat{\delta}$ satisfies that

$$\sqrt{n}(\hat{\delta} - \mathbb{E}[\hat{\delta}]) \xrightarrow{d} \text{Normal}(0, \Sigma_\delta),$$

where
$\Sigma_\delta = 4\mathbb{E}[\sigma^2(X) f^{(1)}(X) f^{(1)}(X)^T] + 4\text{Var}(f(X) g^{(1)}(X)).$

U-Statistics Form of $\hat{\delta}$

- Recall that

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Han's (1987) Maximum Rank Estimator

Multinomial Discrete Choice Models

Ai's (1997) Semiparametric Maximum Likelihood Approach

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References (2)

Useful references also include some lecture notes of the following topic courses:

- ECON 718 NonParametric Econometrics (Bruce Hansen, Spring 2009, University of Wisconsin-Madison),
- セミノンパラメトリック計量分析（末石直也，2014 年度後期，京都大学）.