Semiparametric Single Index Models

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Introduction

- A semiparametric single index model is given by

$$Y = g(X^T \beta_0) + u,$$

where

$$\begin{aligned} & Y \in \mathbb{R} : \text{a dependent variable,} \\ & X \in \mathbb{R}^q : \text{a } q \times 1 \text{ explanatory vector,} \\ & \beta_0 \in \mathbb{R}^q : \text{a } q \times 1 \text{ vector of unknown parameters,} \\ & u \in \mathbb{R} : \text{an error term which satisfies } \mathbb{E}(u \mid X) = 0, \\ & g(\cdot) : \text{an unknown distribution function.} \end{aligned}$$

Introduction

- Even though x is a $q \times 1$ vector, $x^T\beta_0$ is a scalar of a single linear combination, which is called a single index.
- By the form of the single index model, we obtain

$$\mathbb{E}(Y \mid X) = g(X^T\beta_0),$$

  which means that the conditional expectation of Y only depends on the vector X through a single index $X^T\beta_0$.
- The model is SEMIPARAMETRIC when $\beta \in \mathbb{R}^q$ is estimated with the parametric methods and $g(\cdot)$ with the nonparametric methods.
- Some of the PARAMETRIC single index models are really familiar with us.

Examples of Parametric Single Index Model

- If $g(\cdot)$ is the identity function, then the model turns out to be a linear regression model:

$$Y = g(X^T\beta_0) + u = X^T\beta_0 + u.$$

- If $g(\cdot)$ is the CDF of Normal$(0, 1)$, then the model turns out to be a probit model.
  - See the textbook for further discussions on a probit model.
- If $g(\cdot)$ is the CDF of logistic distribution, then the model turns out to be a logistic regression model.

1. Identification Conditions
2. Estimation: Ichimura (1993)'s Method
3. Direct Semiparametric Estimators for β
4. Bandwidth Selection
5. Klein and Spady (1993)
6. Lewbel (2000)
7. Manski's (1975) Maximum Score Estimator
8. Horowitz's (1992) Smoothed Maximum Score Estimator
9. Han's (1987) Maximum Rank Estimator
10. Multinomial Discrete Choice Models
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Identification Conditions

Proposition 8.1 (Identification of a Single Index Model)

For the semiparametric single index model $Y = g(x^T\beta_0) + u$, identification of β_0 and $g(\cdot)$ requires that

- (i) x should not contain a constant/intercept, and must contain at least one continuous variable. Moreover, $\|\beta_0\|=1$.
- (ii) $g(\cdot)$ is differentiable and is not a constant function on the support of $x^T\beta_0$.
- (iii) For the discrete components of x, varying the values of the discrete variables will not divide the support of $x^T\beta_0$ into disjoint subsets.

Identification Condition (i)

- Note that the location and the scale of β_0 are not identified.
- The vector x cannot include an intercept because the function $g(\cdot)$ (which is to be estimated in nonparametric manners) includes any location and level shift.
  - That is, β_0 cannot contain a location parameter.
- Some normalization criterion (scale restrictions) for β_0 are needed.
  - One approach is to set $\|\beta_0\|$.
  - The second approach is to set one component of β_0 to equal one. This approach requires that the variable corresponding to the component set to equal one is continuously distributed and has a non-zero coefficient.
  - Then, x must be dimension 2 or larger. If x is one-dimensional, then $\beta_0 \in \mathbb{R}^1$ is simply normalized to 1, and the model is the one-dimensional nonparametric regression $E(Y \mid x) = g(x)$ with no semiparametric component.

Identification Conditions (ii) and (iii)

- The function $g(\cdot)$ cannot be a constant function and must be differentiable on the support of $x^T\beta_0$.
- x must contain at least one continuously distributed variable and this continuous variable must have non-zero coefficient.
    - If not, $x^T\beta_0$ only takes a discrete set of values and it would be impossible to identify a continuous function $g(\cdot)$ on this discrete support.

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Ichimura (1993)'s Method: Semiparametric Least Squares

- Suppose that the functional form of $g(\cdot)$ were known.
- Then we could estimate β_0 by minimizing the least-squares criterion:

$$\sum_{i=1} \left[Y_i - g(X_i^T \beta) \right]^2$$

with respect to β.

- We could think about replacing $g(\cdot)$ with a nonparametric estimator $\hat{g}(\cdot)$.
- However, since $g(z)$ is the conditional mean of Y_i given $X_i^T \beta_0 = z$, $g(\cdot)$ depends on unknown β_0, so we cannot estimate $g(\cdot)$ here.

- Nevertheless, for a fixed value of β, we can estimate

$$G(X_i^T\beta) := \mathbb{E}(Y_i \mid X_i^T\beta) = \mathbb{E}(g(X_i^T\beta) \mid X_i^T\beta).$$

- In general $G(X_i^T\beta) \neq g(X_i^T\beta)$.
- When $\beta = \beta_0$ [1], it hold that $G(X_i^T\beta_0) = g(X_i^T\beta_0)$.

[1] Recall that β_0 is the true value of β.

Ichimura (1993)'s Method: Semiparametric Least Squares

- First, we estimate $G(X_i^T\beta)$ with the leave-one-out NW estimator:

$$\hat{G}_{-i}(X_i^T\beta) := \hat{\mathbb{E}}_{-i}(Y_i \mid X_i^T\beta) = \frac{\sum_{j\neq i} Y_j K\left(\frac{X_j^T\beta - X_i^T\beta}{h}\right)}{\sum_{j\neq i} K\left(\frac{X_j^T\beta - X_i^T\beta}{h}\right)}.$$

- Second, using the leave-one-out NW estimator $\hat{G}_{-i}(X_i^T\beta)$, we estimate β with

$$\hat{\beta} := \arg\min_{\beta} \sum_{i=1}^{n} \left[Y_i - \hat{G}_{-i}(X_i^T\beta)\right]^2 w(X_i)\mathbf{1}(X_i \in A_n) := S_n(\beta),$$

where

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- 西山慶彦，人見光太郎 (2023) 『ノン・セミパラメトリック統計解析（理論統計学教程：数理統計の枠組み）』共立出版.

References (2)

Useful references also include some lecture notes of the following topic courses:

- ECON 718 NonParametric Econometrics (Bruce Hansen, Spring 2009, University of Wisconsin-Madison),
- セミノンパラメトリック計量分析（末石直也，2014 年度後期，京都大学）.