MAT250 Calculus Final

Problem 1

Chain rule

$$\frac{d}{dx}(f_1 \circ f_2)(x) = \frac{d}{dx} f_1\left(f_2(x)\right) = \frac{du_1}{du_2}\frac{du_2}{dx} \quad [u_1 = f_1(u_2) \text{ and } u_2 = f_2(x)]$$

Derivatives of integer power functions

$$\frac{d}{dx}x^k = kx^{k-1} \quad \text{for } k \in \mathbb{Z} \quad [\underline{\mathbb{Z} \text{ is the set of integers (整数)}}]$$

(1) Using the chain rule and the formula for derivatives of integer power functions above, *prove* the equation below.

$$\frac{d}{dx}x^{\frac{1}{m}} = \frac{1}{m}x^{\left(\frac{1}{m}-1\right)} \quad \text{for } m \in \mathbb{N} \quad [\mathbb{N} \text{ is the set of natural numbers (自然数)}].$$

(2) Using the chain rule, the formula for derivatives of integer power functions and the conclusion in the problem above, *prove* the equation below.

$$\frac{d}{dx}x^q = qx^{(q-1)} \quad \text{for } q \in \mathbb{Q} \quad [\mathbb{Q} \text{ is the set of rational numbers (有理数)}].$$

Here, $\mathbb{Q}$ is defined as $\mathbb{Q} = \left\{\frac{k}{m} \middle| k \in \mathbb{Z}, m \in \mathbb{N}\right\}$.

Problem 2

Perform the following differentiations. Show the deriving processes as far as possible. Here you can use any theorems and formulas.

(1) $\dfrac{d}{dx}\left[\dfrac{1}{(\sin x)^2}\right]$ where $x \notin \{k\pi | k \in \mathbb{Z}\}$

(2) $\dfrac{d}{dx}\left[5^{\frac{1}{x^2}}\right], \quad x \neq 0$

(3) $\dfrac{d}{dx}\left\{\left[\log\left(\dfrac{1}{x^2}\right)\right]^2\right\}, \quad x \neq 0$

(4) $\dfrac{d}{dx}x^{(\cos x)^2}, \quad x > 0$

Problem 3

(a) Consider a function $f(x) = e^{-x}$.

(a-1) Find $\frac{d^n f}{dx^n}(x)$. Here $n \geq 0$ is an integer (整数).

(a-2) Derive the Taylor's series of $f(x)$ at $x = 0$. Please use $\sum$-notation.

(b) Consider a function $f(x) = \frac{1}{1+x}$ $(-1 < x < 1)$.

(b-1) Find $\frac{d^n f}{dx^n}(x)$. Here $n \geq 0$ is an integer (整数).

(b-2) Derive the Taylor's series of $f(x)$ at $x = 0$. Please use $\sum$-notation.

Problem 4

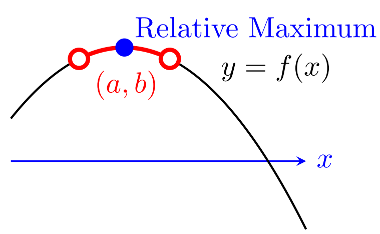
Prove $\frac{df}{dx}(x_0) = 0$ when the function f is differentiable at x_0 and has a relative maximum at x_0. Note that the derivative of f at x_0 is defined as

$$\frac{df}{dx}(x_0) = \lim_{\Delta x \to 0} \frac{f(\boldsymbol{x_0 + \Delta x}) - f(\boldsymbol{x_0})}{\Delta x}.$$

Relative Maximum (極大値)

f is said to has a relative maximum (極大値) at x_0 if

there exists an *open* interval (a, b) such that $a < x_0 < b$, and $f(x_0) \geq f(x)$ for any $x \in (a, b)$.

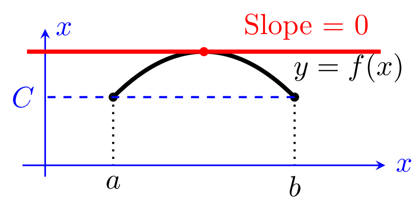


Problem 5

Using the Rolle's theorem,

Rolle's theorem (ロルの定理)

The function g is *continuous* on the closed interval $[a, b]$, is differentiable on the open interval (a, b), where $a < b$, and satisfies $g(a) = g(b)\,(= C)$. Then, there exists at least one c such that $\frac{dg}{dx}(c) = 0$ and $c \in (a, b)$.



we would like to prove the Taylor's theorem below.

Taylor's theorem (テイラーの定理)

The function f is continuous on $[a, b]$ and n times differentiable ($n \in \mathbb{N}$) on the open interval (a, b), where $a < b$. Then there exists $c \in (a, b)$ such that

$$f(b) = \sum_{k=0}^{n-1} \frac{1}{k!} \frac{d^k f}{dx^k}(a)\ (b-a)^k + \frac{1}{n!} \frac{d^n f}{dx^n}(c)\ (b-a)^n.$$

Here we assume that f is continuous on $[a, b]$ and n times differentiable ($n \in \mathbb{N}$) on the open interval (a, b), where $a < b$. Then, using f, define the function g as

$$g(x) = f(b) - \sum_{k=0}^{n-1} \frac{1}{k!} \frac{d^k f}{dx^k}(x)\,(b-x)^k - R\,(b-x)^n,$$

where R is the constant defined by

$$R = \frac{1}{(b-a)^n} \left[f(b) - \sum_{k=0}^{n-1} \frac{1}{k!} \frac{d^k f}{dx^k}(a)\,(b-a)^k \right].$$

(1) Please confirm $g(a) = g(b)$. (Here you just need to show the calculation process to get $g(a) = g(b)$.)

(2) By performing the differentiation for $g(x)$, find $\frac{dg}{dx}(x)$. In the final expression, keep R and do *not* have $\sum$-notation. Please provide the calculation process here.

(3) Fill $\boxed{\text{(3-A)}}$ and $\boxed{\text{(3-C)}}$ to make the following sentences correct.

Function $g(x)$ is continuous on the closed interval $[a, b]$ and differentiable on the open interval (a, b) [†]. Furthermore, one finds $g(a) = g(b)$. Therefore, the Rolle's theorem can be applied and one finds that there exists $c \in (a, b)$ such that

$$\frac{dg}{dx}(c) \quad \boxed{\text{(3-A [put >, <, or =])}} \quad \boxed{\text{(3-B)}}.$$

Hence, one has

$$R = \boxed{\text{(3-C)}}.$$

Inserting this to

$$R = \frac{1}{(b-a)^n}\left[f(b) - \sum_{k=0}^{n-1} \frac{1}{k!}\frac{d^k f}{dx^k}(a)\,(b-a)^k\right],$$

then finally, one finds

$$f(b) = \sum_{k=0}^{n-1} \frac{1}{k!}\frac{d^k f}{dx^k}(a)\ (b-a)^k + \frac{1}{n!}\frac{d^n f}{dx^n}(c)\ (b-a)^n.$$

[†] Since f is n times differentiable on (a, b), $\frac{d^k f}{dx^x}(x)$ is continuous on $[a, b]$ and differentiable on (a, b) for $k \in \mathbb{Z}$, $0 \le k \le n-1$. Besides, $(b-x)^k$ is continuous and differentiable on $\mathbb{R}$ for $k \in \mathbb{Z}$, $k \ge 0$. Thus, it can be shown that $g(x) = f(b) - \sum_{k=0}^{n-1} \frac{1}{k!}\frac{d^k f}{dx^k}(x)\,(b-x)^k - R\,(b-x)^n$, is continuous on $[a, b]$ and differentiable on (a, b) by using the sum rule and product rule of limits (for the continuity) and of derivatives (for differentiability).

Example Solutions

Problem 1

(1) Put $f(x) = x^{\frac{1}{m}}$. Then,

$$
\begin{aligned}
f(x) &= x^{\frac{1}{m}} \\
[f(x)]^m &= x \\
\frac{d}{dx}[f(x)]^m &= \frac{d}{dx}x \\
m[f(x)]^{m-1}\frac{df}{dx}(x) &= 1 \\
\frac{d}{dx}x^{\frac{1}{m}} = \frac{df}{dx}(x) &= \frac{1}{m[f(x)]^{m-1}} = \frac{1}{m}x^{-\frac{m-1}{m}} = \frac{1}{m}x^{\left(\frac{1}{m}-1\right)}.
\end{aligned}
$$

(2) Since $q \in \mathbb{Q}$, one can have the expression of q in the form of $q = \frac{k}{m}$, where $k \in \mathbb{Z}$ and $m \in \mathbb{N}$. Then,

$$
\begin{aligned}
\frac{d}{dx}x^q &= \frac{d}{dx}\left(x^{\frac{1}{m}}\right)^k \\
&= k\left(x^{\frac{1}{m}}\right)^{k-1} \times \frac{d}{dx}x^{\frac{1}{m}} \\
&= k\left(x^{\frac{1}{m}}\right)^{k-1} \times \frac{1}{m}x^{\left(\frac{1}{m}-1\right)} \\
&= \frac{k}{m}x^{\frac{k}{m}-1} = qx^{q-1}.
\end{aligned}
$$

Problem 2

(1) $\displaystyle \frac{d}{dx}\left[\frac{1}{(\sin x)^2}\right] = \frac{d}{dx}\left[(\sin x)^{-2}\right] = -2(\sin x)^{-2-1}\left[\frac{d}{dx}\sin x\right] = -\frac{2\cos x}{(\sin x)^3}$

(2) $\displaystyle \frac{d}{dx}\left[5^{(x^2-2x)}\right] = \frac{d}{dx}\left[e^{(x^2-2x)\log 5}\right] = \left[e^{(x^2-2x)\log 5}\right]\frac{d}{dx}\left[(x^2-2x)\log 5\right]$

$\displaystyle = \left[e^{(x^2-2x)\log 5}\right][(2x-2)\log 5] = [2(x-1)\log 5]\left[5^{(x^2-2x)}\right]$

(3) $\displaystyle \frac{d}{dx}\left\{\left[\log\left(\frac{1}{x^2}\right)\right]^2\right\} = \frac{d}{dx}\left\{\left[\log x^{-2}\right]^2\right\} = \frac{d}{dx}\left\{[-2\log x]^2\right\}$

$\displaystyle = 2[-2\log x]\frac{d}{dx}[-2\log x] = 2[-2\log x]\frac{-2}{x} = \frac{8\log x}{x}$

(4) $\displaystyle \frac{d}{dx}x^{(\cos x)^2} = x^{(\cos x)^2}\frac{d}{dx}\log\left[x^{(\cos x)^2}\right] = x^{(\cos x)^2}\frac{d}{dx}(\cos^2 x)\log x$

$\displaystyle = x^{(\cos x)^2}\left[2\cos x(-\sin x)\log x + \frac{\cos^2 x}{x}\right] = x^{(\cos x)^2}\left[\frac{\cos^2 x}{x} - 2\cos x\sin x\log x\right]$

Problem 3

(a) Consider a function $f(x) = e^{-x}$.

(a-1)

$$\frac{d^n f}{dx^n}(x) = (-1)^n e^{-x}$$

(a-2)

$$\frac{d^n f}{dx^n}(0) = (-1)^n$$

Thus, the Taylor's series of $f(x) = \log(1+x)$ is

$$f(x) = e^{-x} = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} x^n.$$

(b) Consider a function $f(x) = \frac{1}{1+x}$ $(-1 < x < 1)$.

(b-1)

$$\frac{d^n f}{dx^n}(x) = \frac{(-1)^n n!}{(1+x)^{n+1}}$$

(b-2)

$$\frac{d^n f}{dx^n}(0) = (-1)^n n!$$

Thus, the Taylor's series of $f(x) = \frac{1}{1+x}$ is

$$f(x) = \frac{1}{1+x} = \sum_{n=0}^{\infty} (-1)^n x^n.$$

Problem 4

Since f has a relative maximum at x_0, there exist an open interval (a, b) such that

$$a < x_0 < b \quad \text{and} \quad f(x_0) \geq f(x) \quad \text{for any } x \in (a, b).$$

Let's say Δx is a non-zero real number satisfying $x_0 + \Delta x \in (a, b)$, then

$$f(x_0) \geq f(x_0 + \Delta x).$$

Therefore,

$$f(x_0 + \Delta x) - f(x_0) \leq 0.$$

(i) When $\Delta x > 0$,

$$\frac{f(x_0+\Delta x)-f(x_0)}{\Delta x} \leq 0.$$

Therefore,

$$\lim_{\Delta x \to 0^+} \frac{f(x_0+\Delta x)-f(x_0)}{\Delta x} \leq 0.$$

(ii) When $\Delta x < 0$,

$$\frac{f(x_0+\Delta x)-f(x_0)}{\Delta x} \geq 0.$$

Therefore,

$$\lim_{\Delta x \to 0^-} \frac{f(x_0+\Delta x)-f(x_0)}{\Delta x} \geq 0.$$

Since Function f is differentiable at x_0, the limit of $\frac{df}{dx}(x_0)$ exits and

$$\begin{aligned}\frac{df}{dx}(x_0) &= \lim_{\Delta x \to 0} \frac{f(x_0+\Delta x)-f(x_0)}{\Delta x} \\ &= \lim_{\Delta x \to 0^+} \frac{f(x_0+\Delta x)-f(x_0)}{\Delta x} = \lim_{\Delta x \to 0^-} \frac{f(x_0+\Delta x)-f(x_0)}{\Delta x}.\end{aligned}$$

Then,

$$0 \leq \lim_{\Delta x \to 0^-} \frac{f(x_0+\Delta x)-f(x_0)}{\Delta x} = \frac{df}{dx}(x_0) = \lim_{\Delta x \to 0^+} \frac{f(x_0+\Delta x)-f(x_0)}{\Delta x} \leq 0.$$

Hence,

$$\frac{df}{dx}(x_0) = 0.$$

Problem 5

Here we assume that f is continuous on $[a,b]$ and n times differentiable (n is a natural number) on the open interval (a,b), where $a<b$. Then, using f, define the function g as

$$g(x) = f(b) - \sum_{k=0}^{n-1} \frac{1}{k!}\frac{d^k f}{dx^k}(x)\,(b-x)^k - R\,(b-x)^n,$$

where R is the constant defined by

$$R = \frac{1}{(b-a)^n}\left[f(b) - \sum_{k=0}^{n-1} \frac{1}{k!}\frac{d^k f}{dx^k}(a)\,(b-a)^k\right].$$

(1) Since

$$\begin{aligned}
g(a) &= f(b) - \sum_{k=0}^{n-1} \frac{1}{k!} \frac{d^k f}{dx^k}(a)\,(b-a)^k - R\,(b-a)^n \\
&= \\
&= f(b) - \sum_{k=0}^{n-1} \frac{1}{k!} \frac{d^k f}{dx^k}(a)\,(b-a)^k - \ (b-a)^n \frac{1}{(b-a)^n} \left[f(b) - \sum_{k=0}^{n-1} \frac{1}{k!} \frac{d^k f}{dx^k}(a)\,(b-a)^k \right] \\
&= \cancel{f(b)} - \cancel{\sum_{k=0}^{n-1} \frac{1}{k!} \frac{d^k f}{dx^k}(a)\,(b-a)^k} - \left[\cancel{f(b)} - \cancel{\sum_{k=0}^{n-1} \frac{1}{k!} \frac{d^k f}{dx^k}(a)\,(b-a)^k} \right] = 0,
\end{aligned}$$

and

$$\begin{aligned}
g(b) &= f(b) - \sum_{k=0}^{n-1} \frac{1}{k!} \frac{d^k f}{dx^k}(b)\,(b-b)^k - R\,(b-b)^n \\
&= \cancel{f(b)} - \cancel{f(b)} - \sum_{k=1}^{n-1} \frac{1}{k!} \frac{d^k f}{dx^k}(b)\,\cancelto{0}{(b-b)^k} - R\,\cancelto{0}{(b-b)^n} = 0,
\end{aligned}$$

one finds

$$g(a) = g(b) = 0.$$

(2)

$$
\begin{aligned}
\frac{dg}{dx}(x) &= \frac{d}{dx}\left[f(b) - \sum_{k=0}^{n-1} \frac{1}{k!}\frac{d^k f}{dx^k}(x)\,(b-x)^k - R\,(b-x)^n\right] \\
&= -\sum_{k=0}^{n-1} \frac{1}{k!}\frac{d}{dx}\left[\frac{d^k f}{dx^k}(x)\,(b-x)^k\right] - R\,\frac{d}{dx}\left[(b-x)^n\right] \\
&= -\cancelto{1}{\frac{1}{0!}}\frac{d}{dx}\left[\cancelto{f(x)}{\frac{d^0 f}{dx^0}(x)}\,\cancelto{1}{(b-x)^0}\right] - \sum_{k=1}^{n-1}\frac{1}{k!}\frac{d}{dx}\left[\frac{d^k f}{dx^k}(x)\,(b-x)^k\right] - R\,\frac{d}{dx}\left[(b-x)^n\right] \\
&= -\frac{df}{dx}(x) - \sum_{k=1}^{n-1}\frac{1}{k!}\left[\frac{d^{k+1} f}{dx^{k+1}}(x)\,(b-x)^k + \frac{d^k f}{dx^k}(x)\,k(b-x)^{k-1}\times(-1)\right] - R\,n(b-x)^{n-1}\times(-1) \\
&= -\frac{df}{dx}(x) - \sum_{k=1}^{n-1}\frac{1}{k!}\left[\frac{d^{k+1} f}{dx^{k+1}}(x)\,(b-x)^k\right] + \sum_{k=1}^{n-1}\frac{1}{k!}\left[\frac{d^k f}{dx^k}(x)\,k(b-x)^{k-1}\right] + R\,n(b-x)^{n-1} \\
&= -\frac{df}{dx}(x) \underbrace{- \sum_{k=2}^{n}\frac{1}{(k-1)!}\left[\frac{d^k f}{dx^k}(x)\,(b-x)^{k-1}\right]}_{\text{(i)}} \underbrace{+ \sum_{k=1}^{n-1}\frac{1}{(k-1)!}\left[\frac{d^k f}{dx^k}(x)\,(b-x)^{k-1}\right]}_{\text{(ii)}} + R\,n(b-x)^{n-1} \\
&= -\frac{df}{dx}(x) \underbrace{\cancel{- \sum_{k=2}^{n-1}\frac{1}{(k-1)!}\left[\frac{d^k f}{dx^k}(x)\,(b-x)^{k-1}\right]} - \frac{1}{(n-1)!}\left[\frac{d^n f}{dx^n}(x)\,(b-x)^{n-1}\right]}_{\text{(i)}} \\
&\qquad \underbrace{+ \cancelto{1}{\frac{1}{0!}}\left[\cancelto{\frac{df}{dx}(x)}{\frac{d^1 f}{dx^1}(x)}\quad \cancelto{1}{(b-x)^0}\right] + \cancel{\sum_{k=2}^{n-1}\frac{1}{(k-1)!}\left[\frac{d^k f}{dx^k}(x)\,(b-x)^{k-1}\right]}}_{\text{(ii)}} + R\,n(b-x)^{n-1} \\
&= -\cancel{\frac{df}{dx}(x)} + \cancel{\frac{df}{dx}(x)} - \frac{1}{(n-1)!}\left[\frac{d^n f}{dx^n}(x)\,(b-x)^{n-1}\right] + R\,n(b-x)^{n-1} \\
&= -\left[\frac{1}{(n-1)!}\frac{d^n f}{dx^n}(x) - nR\right](b-x)^{n-1}
\end{aligned}
$$

(3) Function $g(x)$ is continuous on the closed interval $[a, b]$ and differentiable on the open interval (a, b). Furthermore, one finds $g(a) = g(b)$. Therefore, the Rolle's theorem can be applied and one finds that there exists $c \in (a, b)$ such that

$$\frac{dg}{dx}(c) \quad \boxed{=} \quad \boxed{0}.$$

Hence, one has

$$R = \boxed{\tfrac{1}{n!}\tfrac{d^n f}{dx^n}(c)}.$$

Inserting this to

$$R = \frac{1}{(b-a)^n}\left[f(b) - \sum_{k=0}^{n-1}\frac{1}{k!}\frac{d^k f}{dx^k}(a)\,(b-a)^k\right],$$

then finally, one finds

$$f(b) = \sum_{k=0}^{n-1}\frac{1}{k!}\frac{d^k f}{dx^k}(a)\ (b-a)^k + \frac{1}{n!}\frac{d^n f}{dx^n}(c)\ (b-a)^n.$$