Aggregating Distortions in Networks with Multi-Product Firms

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Abstract

We investigate the role of multiproduct firms in shaping resource misallocation and its impact on aggregate total factor productivity (TFP) growth. Using administrative data on product transactions between all formal Chilean firms, we provide evidence that demand shocks to one product affect the production of other products within the same firm, suggesting that firms engage in joint production. We develop a framework to measure resource misallocation in production networks with joint production, deriving sufficient statistics to quantify these effects. Applying the framework to Chile, we find that changes in allocative efficiency explain 86% of the observed aggregate TFP growth for the 2016-2022 period. Ignoring joint production leads to overestimation of changes in allocative efficiency.

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1 Introduction

Resource misallocation across heterogeneous producers has been recognized as a driver of aggregate total factor productivity (TFP) differences across countries and over time. To quantify the extent of misallocation, recent literature has made extensive use of granular firm-level data.

However, despite its emphasis on micro data, this literature typically ignores the fact that most firms sell multiple products. For example, 75% of formal firms in Chile report selling multiple products, and these firms collectively account for 99% of all firm-to-firm transactions in Chilean tax data. The ubiquity of multi-product firms introduces new challenges to understanding resource allocation. Specifically, researchers must consider how the allocation of resources across products within firms affects allocative efficiency and aggregate TFP. Measuring resource allocation within firms often requires determining how to assign inputs to specific outputs.

The literature on multi-product firms often assumes product line independence (Klette and Kortum (2004); Bernard et al. (2011); De Loecker et al. (2016); Hottman et al. (2016); Mayer et al. (2021)). If firms are collections of independent products, the challenge is reduced to a measurement problem, and existing theories for single-product firms can be applied by treating different products as if they are separate firms. However, firms often simultaneously produce multiple outputs using shared inputs, making it impossible to assign inputs to specific outputs. Consider an oil refinery that produces diesel and gasoline concurrently: the inputs—crude oil, labor, and capital—are used to produce both outputs and cannot be accounted for separately.

How do multi-product firms with non-separable production technologies affect the measurement of the extent of resource misallocation? We model firms' technology via joint non-separable production functions that map bundles of inputs into bundles of multiple outputs. This approach eliminates the need to define individual product-level production functions. The joint production function describes the firm's flexibility in adjusting its product mix, which then determines the importance of resource allocation within the firm.

We generalize previous work to accommodate multi-product firms with joint production technologies. We provide sufficient statistics to measure changes in allocative efficiency using ex-post data. Our framework is general enough to accommodate firm-to-firm linkages. We validate and implement our framework using a granular firm-to-firm transactions database for Chile. We show that the extent of resource misallocation is overstated if we abstract from joint production, as is standard practice in the literature. While our primary focus is ex-post analysis, we also develop a complementary parametric framework to understand ex-ante counterfactuals, like the gains from eliminating all distortions, taking into account joint production technologies.

We first describe the theoretical contributions of the paper, then turning to our empirical validation and application. In our model, products within firms can be under- or overproduced because they face different wedges (e.g., markups). Loosely speaking, products with relatively high wedges are underproduced. However, since firms interact with one another, the relevant wedges that affect resource allocation and, hence, aggregate TFP are not just the firm's own wedges, but the entire chain of cumulative wedges leading from final demand to the production of the product.

The impact of changes in these cumulative wedges on resource misallocation depends on how easily firms can adjust their product mix. Consider again the oil refinery example. If the oil refinery raises the markup on gasoline, thereby lowering its demand, it cannot redirect production resources toward diesel because the production technology yields gasoline and diesel in nearly fixed proportions from crude oil. This technological constraint limits the firm's ability to reallocate resources in response to demand changes and hence limits the extent of misallocation within the firm. Therefore, joint production technology can attenuate the extent of resource misallocation and its contribution to aggregate TFP.

To quantify the extent of resource misallocation in the presence of joint production, we develop a nonparametric sufficient statistics approach. Our approach relies on observed changes in product-level prices within the firm. Theoretically, these price movements, net of markups, trace out the production possibility frontier, whose slope captures each firms' technological constraints when adjusting their product mix. When firms have flexibility to adjust their product mix, changes in prices net of markups will be small as firms can easily substitute production between products.

The covariance of relative price changes with cumulative wedges at the product level captures the attenuation of resource misallocation due to joint production technology. Intuitively, if prices rise for products with high (cumulative) wedges, then the scope for reallocation is limited. Rather than directly estimating the production possibility frontier, our approach infers its shape from observed price changes, providing a way to quantify misallocation without imposing parametric assumptions about firms' production technologies.

The growth-accounting formula we develop generalizes previous approaches: it collapses to the Baqaee and Farhi (2020) growth accounting result under the single-product firm assumption and to the Hulten (1978) benchmark under perfect competition.

To implement our framework, we use administrative firm-to-firm transactions data from the Chilean Internal Revenue Service. The dataset contains product-level prices, firm-product input-output linkages, and balance sheet variables — enough to construct our sufficient statistics.

We first validate joint production technology in the data. We examine whether the standard assumption in the literature — that firms operate as independent single-product lines — holds. We find that demand shocks to one product significantly affect the production of other products within the same firm, indicating that firms employ joint production technology. In fact, we find that a negative demand shock to a firm's main product reduces the production of alternative products, as predicted by our framework.

Having established the presence of joint production, we implement our sufficient statistics to conduct a growth accounting exercise. We find that changes in allocative efficiency explain 86% of TFP growth in Chile from 2016 to 2022. Ignoring joint production provides a misleading assessment and substantially overestimates the importance of resource reallocation.

We attribute this finding to multi-product firms face that constraints when they adjust their product mix. These constraints limit the scope for product-level resource reallocation within firms, and this affects allocative efficiency and aggregate TFP growth.

As mentioned above, our primary focus is nonparametric ex-post analysis that employs observed data. However, we also develop a complementary ex-ante approach that enables counterfactual analysis by imposing a parametric structure for production technologies and requires knowledge of the curvature in the firm's production possibility frontier. We provide an analytical characterization of the extent of misallocation, measured as the distance to the Pareto-efficient frontier, for multi-product firms with joint production. Consistent with our earlier finding that joint production constrains reallocation, we find that assuming separable production technologies overestimates the extent of resource misallocation caused by a given set of wedges.

Related Literature

Our paper contributes to and connects different strands of the literature. We incorporate multiproduct firms and joint production to extend the literature on misallocation. The work by Restuccia and Rogerson (2008) and Hsieh and Klenow (2009) show the potential importance of resource misallocation in accounting for TFP differences. We contribute to this literature by developing a framework that allows for multi-product firms (Bernard et al. (2010, 2011); Mayer et al. (2014); De Loecker et al. (2016); Hottman et al. (2016); Mayer et al. (2021); Wang and Yang (2023)) and joint production (Powell and Gruen (1968); Diewert (1971); Lau (1972); Hall (1973, 1988); Boehm and Oberfield (2023); Carrillo et al. (2023); Ding (2023)) and quantifying the impact of resource misallocation.

Recent work has emphasized the need to take into account input-output linkages when quantifying the extent of these losses (Baqaee and Farhi (2020); Bigio and La'O (2020)). Our theory provides a flexible framework that allows for arbitrary production structures, including input-output networks while incorporating joint production, and it enables the quantification of these effects. [1]

Our theory provides a tool for growth accounting (Solow (1957); Hulten (1978); Basu and Fernald (2002); Petrin and Levinsohn (2012); Baqaee and Farhi (2020); Baqaee et al. (2023)) that decomposes aggregate TFP growth into technology and allocative efficiency under joint production in networks and generalizes existing methods to consider multi-product firms.

Our empirical application uses Chile's comprehensive product-level transaction database to quantify misallocation. This approach contrasts with the prior literature on production networks and misallocation, which typically uses industry-level input-output table. For example, Baqaee and Farhi (2020) impute US Compustat data using an industry-level input-output table. Finally, Burstein et al. (2024) uses the same dataset as ours but complements our work by analyzing misallocations that arise from different buyers receiving varying prices for the same product. [2]

Lastly our empirical strategy is related to the literature on shock transmission between firms. While much of this literature focuses on how supply shocks propagate downstream across firms (Boehm et al. (2019); Carvalho et al. (2020); Fujiy et al. (2022); Bai et al. (2024)) using the granular a firm-to-firm transaction dataset, our empirical strategy instead analyzes how external demand shocks transmit within firms across their products. We compare our results with existing studies that examine within-firm spillovers of demand shocks (Giroud and Mueller (2019); Almunia et al. (2021); Ding (2023)). [3]

[1] Our work is also related to the work of Liu (2019) and Dávila and Schaab (2023), which analyzed the effects of misallocation in input-output linkages on welfare.

[2] Using a theoretical model grounded in Belgian firm-to-firm transaction data, Kikkawa (2022) examines firm pair-specific markups.

[3] This work is related to the literature on production function estimation for multiproduct firms. No-

The rest of the paper is organized as follows. Section 2 outlines the theoretical framework, deriving the nonparametric sufficient statistics for measuring allocative efficiency explained by multiproduct firms. Section 3 presents the empirical evidence on joint production. Section 4 details the data and the construction of sufficient statistics. Section 5 applies the framework to decompose aggregate TFP growth in Chile for the 2016–2022 period. Section 6 presents ex-ante structural results. Finally, Section 7 concludes.

2 A Theory to Aggregate Distortions in Networks with Multiproduct Firms

We develop a theoretical framework to analyze resource misallocation in production networks with multiproduct firms that use joint production technologies. Given that firms use shared inputs to produce multiple outputs simultaneously, even with suitable data, it is impossible to assign inputs to products separately. We generalize previous frameworks to accommodate multi-product firms with joint production technologies. We provide nonparametric sufficient statistics to measure changes in allocative efficiency using ex-post data. Our framework is general enough to accommodate firm-to-firm linkages.

We use markup and wedge interchangeably to refer to any distortion that creates a gap between price and marginal cost. This can include any distortions such as taxes, subsidies, or financial frictions.

The section proceeds as follows. First, we formalize the concept of joint production technology. To build intuition, we present parametric examples illustrating how joint production affects resource allocation and aggregate TFP. Then, using the parametric model data generation process, we discuss the nonparametric sufficient statistics needed to measure allocative efficiency. Next, we present our general model, and main proposition, which unpacks allocative efficiency into a single-product term and a multiple-product term.

tably, estimation methods for joint production recently have been developed by Dhyne et al. (2017, 2022); Valmari (2023); Cairncross and Morrow (2023). We estimate the universe of products traded by formal firms in Chile by formal firms from 2016 to 2022.

2.1 Joint Production

We begin by formalizing the concept of joint production, where firms simultaneously use shared inputs to produce different products.

To formalize this concept, we follow Hall (1973)'s approach to joint production technology. Let $J(\boldsymbol{q}, \boldsymbol{x})$ be a joint production function, where $\boldsymbol{q}$ is a vector of outputs and $\boldsymbol{x}$ is a vector of inputs. The joint cost function is derived from the firm's cost minimization problem, as follows:

$$C(\boldsymbol{q}, \boldsymbol{p}) \equiv \min_{x \in V(q)} \boldsymbol{p}'\boldsymbol{x},$$

where $V(\boldsymbol{q})$ is the input requirement set, $V(\boldsymbol{q}) = \{\boldsymbol{x} | J(\boldsymbol{q}, \boldsymbol{x}) \geq 0\}$ and $\boldsymbol{p}$ is a vector of input prices. We introduce two assumptions about the shape of a joint production function, which will be used throughout this paper.

Assumption 1 (Constant Return to Scale (CRS))**.** $J(\boldsymbol{q}, \boldsymbol{x}) = 0$ implies $J(\lambda\boldsymbol{q}, \lambda\boldsymbol{x}) = 0$ for any scalar λ. [4]

Unlike a single-output production function, the output is a vector. Note that we do not assume CRS for each single-output production function.

Assumption 2 (Separability between Input and Output Functions)**.** The joint production function can be written as $J(\boldsymbol{q}, \boldsymbol{x}) = -F^Q(\boldsymbol{q}) + F^X(\boldsymbol{x})$, and the joint cost function as $C(\boldsymbol{q}, \boldsymbol{x}) = H(\boldsymbol{q})\varphi(\boldsymbol{p})$.

Note that this differs from assuming separable production functions, where the output, q, is a single product, not a vector; it degenerates to $F^Q(q) = q$. Example 1 illustrates a joint production function satisfying assumptions 1 and 2:

Example 1 (Constant Elasticity of Transformation Output Bundle and Constant Elasticity of Substitution Input Bundle (CET-CES))**.**

$$\underbrace{\left(\sum_g q_g^{\frac{\sigma+1}{\sigma}}\right)^{\frac{\sigma}{\sigma+1}}}_{\text{Output bundle}} = A\underbrace{\left(L^{\frac{\theta-1}{\theta}} + K^{\frac{\theta-1}{\theta}}\right)^{\frac{\theta}{\theta-1}}}_{\text{Input Bundle}},$$

[4] While we assume CRS, this is not theoretically restrictive. Variable returns to scale can be accommodated through constant returns and producer-specific fixed factors. However, our empirical application to Chile adopts constant returns with respect to observable inputs — labor, capital, and intermediates — as we cannot measure producer-specific factors in our data.

The associated cost function is

$$C(\boldsymbol{q}, w, r) = \frac{1}{A}\left(\sum_{g} q_g^{\frac{\sigma+1}{\sigma}}\right)^{\frac{\sigma}{\sigma+1}} \left(w^{1-\theta} + r^{1-\theta}\right)^{\frac{1}{1-\theta}},$$

where L and K are the two inputs, w and r are their prices, and $\boldsymbol{q}$ is a vector of outputs.

The input bundle takes a standard CES function with elasticity of substitution θ, and the output is a vector of products rather than a scalar. The parameter σ is called the constant elasticity of transformation; it gives a constant value to the production possibility frontier's curvature of the products within a firm. This example is illustrative as our theoretical framework requires no parametric assumption.

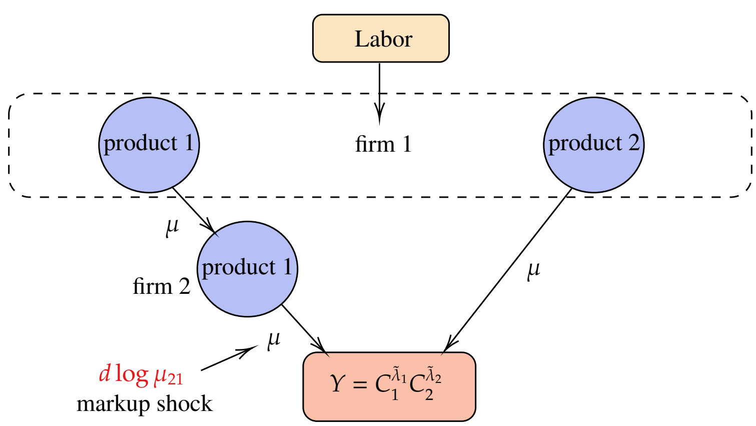
2.2 Parametric Examples of Misallocation with Joint Production

Before presenting the general framework, We provide simplified examples to help gain an intuition about how joint production affects resource allocation and aggregate TFP. Proofs are provided in Appendix E.

2.2.1 An Example without Joint Production

We begin with an example of a production network with multiproduct firms but without joint production. Consider an economy with two firms, as illustrated in Figure 1.

Figure 1: A simplified economy with production networks and multiproduct firms



Firm 1 uses labor to produce two differentiated products using labor (L) as unique input, $q_{11} = L_{11}$, $q_{12} = L_{12}$, where $L = L_{11} + L_{12}$. Product 1 is sold to firm 2, while product

2 is sold directly to households. For simplicity, we assume that both products have the same markup, μ. Firm 2 uses product 1 from firm 1 as a production input and produces a different product using a linear technology ($q_{21} = q_{11}$) that sells to households with markup μ. Final consumption goods are aggregated using a Cobb-Douglas function $Y = c_1^{\tilde{\lambda}_1} c_2^{\tilde{\lambda}_2}$, where $c_1 = q_{21}$, $c_2 = q_{12}$. In this simple economy, Y is the real GDP, and aggregated TFP can be defined as $TFP = Y/L$.

In a production network environment, distorted resource allocation arises from both firms' own markups and downstream firms' markups. In this example, product 1 is sold to households with double marginalization; firm 1 charges a markup to firm 2, and firm 2 charges a markup to the household. As a result, product 1 from firm 1 suffers from a higher distortion than product 2 from firm 1, both relative to a perfect competition setup. To capture this concept in our simplified economy, we introduce the notion of cumulative wedges.

Definition 1 (Cumulative Wedge in a Simplified Economy)**.** In this simplified two-firm economy, we define the cumulative wedge for each product g produced by firm 1, denoted by Γ_{1g}, as the product of markups along the production path from the initial producer to the final consumer. Specifically:

$$\Gamma_{11} = \mu^2 \quad \text{and} \quad \Gamma_{12} = \mu,$$

where μ represents the markup applied by each firm. Here, Γ_{11} captures the cumulative wedge for product 1, which is sold to firm 2 before reaching households, thus incorporating both firm 1's and firm 2's markups (double marginalization). In contrast, Γ_{12} represents the wedge for product 2, which is sold directly from firm 1 to the household with only a single markup.

Now consider a shock that changes firm 2's markup on product 1 ($d \log \mu_{21}$). The first-order response of aggregate TFP to this markup shock can be expressed as:

$$\Delta \log TFP = \tilde{\lambda}_1 \left(\frac{\bar{\Gamma}_1}{\Gamma_{11}} - 1 \right) d \log \mu_{21},$$

where $\bar{\Gamma}_1$ is the weighted harmonic mean of cumulative wedges, defined as:

$$\bar{\Gamma}_1 = \left(\tilde{\lambda}_1 \Gamma_{11}^{-1} + \tilde{\lambda}_2 \Gamma_{12}^{-1} \right)^{-1}.$$

Since $\Gamma_{11} = \mu^2 > \Gamma_{12} = \mu$, we know that $\Gamma_{11} > \bar{\Gamma}_1$, making the term $(\bar{\Gamma}_1/\Gamma_{11} - 1)$ negative. Consequently, an increase in markup ($d\log\mu_{21} > 0$) for product 1, which already faces higher cumulative distortions, reduces aggregate TFP. This occurs because the markup increase further distorts the allocation of resources away from the more distorted product, exacerbating existing misallocation.

Conversely, a decrease in markup ($d\log\mu_{21} < 0$) for product 1 increases aggregate TFP. The reduction in markup allows for increased production of product 1, which was previously underproduced due to higher cumulative distortions. As the relative price of product 1 decreases, households shift consumption away from product 2 toward product 1 (through firm 2's product), leading to improved allocative efficiency and higher aggregate TFP.

2.2.2 An Example with Joint Production

Next, we introduce joint production into our simplified economy. Instead of separable production functions for each product, firm 1 uses a joint production technology to produce both products simultaneously using a constant elasticity of transformation (CET) function:

$$\left(q_{11}^{\frac{\sigma+1}{\sigma}} + q_{12}^{\frac{\sigma+1}{\sigma}}\right)^{\frac{\sigma}{\sigma+1}} = L,$$

where σ represents the elasticity of transformation between the two products.

Consider the same markup shock to product 1 from firm 2 as in the previous case ($d\log\mu_{21}$). By taking a first-order approximation of the change in TFP, we obtain the following response:

$$\Delta\log TFP = \left(1 - \frac{1}{\sigma+1}\right)\tilde{\lambda}_1\left(\frac{\bar{\Gamma}_1}{\Gamma_{11}} - 1\right)d\log\mu_{21}, \tag{1}$$

where $\bar{\Gamma}_1$ is defined as before. This expression reveals how joint production affects the transmission of markup shocks to aggregate TFP.

The magnitude of the TFP response depends critically on the elasticity of transformation σ, which governs how easily firm 1 can adjust its product mix. Since $\Gamma_{11} > \bar{\Gamma}_1$, an increase in the markup ($d\log\mu_{21} > 0$) reduces TFP by distorting the allocation of resources away from the more distorted product 1, while a reduction in the markup ($d\log\mu_{21} < 0$) increases TFP by shifting production toward the more distorted product 1. However, joint production attenuates these TFP responses through the term $(1 - \frac{1}{\sigma+1})$. This attenuation factor reduces the magnitude of the TFP response regardless of the direction of the markup shock, reflecting the technological constraints firms face when adjusting their product mix.

The role of these technological constraints becomes particularly clear when we examine two extreme cases:

$$\Delta \log TFP = \begin{cases} \tilde{\lambda}_1 \left(\frac{\bar{\Gamma}_1}{\Gamma_{11}} - 1 \right) d \log \mu_{21} & \text{if } \sigma \to \infty \\ 0 & \text{if } \sigma \to 0. \end{cases}$$

As σ approaches infinity, the case converges to our previous example without joint production. The firm can freely adjust its product mix in response to the markup shock, allowing for the maximum possible reallocation of resources. The TFP response in this case represents the upper bound of the potential efficiency impact.

Conversely, as σ approaches zero, the production technology becomes Leontief in outputs, meaning the products must be produced in fixed proportions. Consider, for example, an oil refinery that produces both gasoline (product 1) and diesel (product 2). Due to the chemical properties of crude oil and technological constraints of the refining process, the refinery cannot easily adjust the ratio of gasoline to diesel production in response to price changes. In this limit case, even if relative prices change due to markup shocks, the firm cannot adjust its product mix, eliminating any potential gains or losses from resource reallocation.

Joint production thus attenuates the TFP response to markup shocks. The degree of attenuation, captured by the term $-\frac{1}{\sigma+1}$, reduces the magnitude of TFP response regardless of whether the markup shock is positive or negative. This suggests that previous studies, which implicitly assume infinite substitutability across products ($\sigma \to \infty$), may overestimate the impact of misallocation — both positive and negative — on aggregate TFP.

2.2.3 Towards a Theory for Measurement

While our previous results help us understand how joint production affects TFP responses, the structural results depend on the elasticity of transformation σ, which is difficult to estimate. We seek to express these results in terms of prices, which are easier to obtain from data.

With joint production, changes in relative prices are associated with changes in production ratios:

$$d\log(p_{11}/p_{12}) = \frac{1}{\sigma} d\log(q_{11}/q_{12}). \tag{2}$$

These price movements effectively trace out the production possibility frontier, whose slope captures each firm's technological constraints when adjusting their product mix.

In our simple example with joint production, let λ_{ij} denote the GDP share of product j of firm i. The downstream markup change implies that the GDP share of product 1 of firm 1 changes by $d\log\lambda_{11} = -d\log\mu_{21}$. Due to the Cobb-Douglas specification of final demand, the GDP share of product 2 does not change ($d\log\lambda_{12} = 0$). Combining these observations with the relationship between relative prices and quantities gives us[5]:

$$d\log(p_{11}/p_{12}) = -\frac{1}{\sigma+1} d\log\mu_{21}. \tag{3}$$

Using this relationship, we can rewrite the TFP response in equation (1):

$$\Delta\log TFP = \left(1 - \frac{1}{\sigma+1}\right)\tilde{\lambda}_1\left(\frac{\bar{\Gamma}_1}{\Gamma_{11}} - 1\right) d\log\mu_{21}. \tag{4}$$

This can be further decomposed into two terms:

$$\Delta\log TFP = \underbrace{\tilde{\lambda}_1\left(\frac{\bar{\Gamma}_1}{\Gamma_{11}} - 1\right) d\log\mu_{21}}_{\text{Single-Product Term}} + \underbrace{d\log(p_{11}/p_{12})\left(\frac{\bar{\Gamma}_1}{\Gamma_{11}} - 1\right)}_{\text{Multi-Product Term}}.$$

These terms can be expressed in terms of observable variables. The single-product term becomes:

$$\tilde{\lambda}_1\left(\frac{\bar{\Gamma}_1}{\Gamma_{11}} - 1\right) d\log\mu_{21} = -d\log\Lambda - \tilde{\lambda}_1 d\log\mu_{21}.$$

where Λ is the labor share. The multi-product term can be written as a covariance between price changes and cumulative wedges:

$$d\log(p_{11}/p_{12})\left(\frac{\bar{\Gamma}_1}{\Gamma_{11}} - 1\right) = \mathrm{Cov}_{s_1}\left(d\log p_{(1,\cdot)}, \frac{\bar{\Gamma}_1}{\Gamma_{(1,\cdot)}}\right).$$

The single-product term captures the resource misallocation effects that would exist

[5]Taking logs and differentiating gives $d\log\lambda_{21} = d\log\mu_{21} + d\log\lambda_{11}$. Since λ_{21} is constant under Cobb-Douglas demand, we have $d\log\lambda_{11} = -d\log\mu_{21}$. Then, from equation (2) and $d\log\lambda = d\log q + d\log p$, we have $d\log(q_{11}/q_{12}) = \frac{\sigma}{\sigma+1} d\log(\lambda_{11}/\lambda_{12})$.

even in an economy without joint production. When the initial equilibrium is inefficient, products with high markups are underproduced. A decline in factor shares indicates resources shifting toward these high-markup activities, but we must adjust for mechanical changes in factor shares caused directly by markup changes.

The multi-product term captures how joint production affects firms' ability to reallocate across products. Instead of estimating the firm's production technology parameters directly, we rely on observed changes in product-level prices within the firm. Intuitively, if prices for certain products rise within the firm then this captures the firm's inability to easily substitute production across products. The covariance between these price changes and cumulative wedges reveals how joint production constraints affect misallocation — if prices rise for products with high cumulative wedges, then the scope for reallocation is limited. This leads to our main result:

Proposition 1 (Sufficient Statistics in a Simplified Economy)**.** *In this simple economy, TFP response to the markup shock to the downstream firm can be expressed as:*

$$\Delta TFP = \underbrace{\mathrm{Cov}_{s_1}\left(d\log p_{(1,\cdot)}, \frac{\bar{\Gamma}_1}{\Gamma_{(1,\cdot)}}\right)}_{\textit{Multi-Product Term}} \underbrace{- d\log\Lambda - \tilde{\lambda}_1 d\log\mu_{21}}_{\textit{Single-Product Term}},$$

where $s_1 = (\tilde{\lambda}_1, \tilde{\lambda}_2)$ and $\bar{\Gamma}_1 = (\tilde{\lambda}_1\Gamma_{11}^{-1} + \tilde{\lambda}_2\Gamma_{12}^{-1})^{-1}$ is the weighted harmonic mean of cumulative wedges.

This result provides a nonparametric approach to quantifying misallocation in the presence of joint production. Rather than directly estimating technological parameters, we can infer the constraints on resource reallocation from observed price movements within firms.

2.2.4 Another Example: Response to Taste Shocks

To further illustrate how joint production affects resource allocation, we consider a case where the economy with CET joint production technology experiences taste shocks rather than markup shocks. Specifically, we examine how changes in household preferences affect aggregate TFP. Under the Cobb-Douglas utility function, when the preference weight for product 1 changes by $d\tilde{\lambda}_1$, the weight for the product 2 adjusts by $d\tilde{\lambda}_2 = -d\tilde{\lambda}_1$.

The first-order response of aggregate TFP to a taste shock can be expressed as:

$$\Delta \log TFP = \left(1 - \frac{1}{\sigma + 1}\right) \frac{\tilde{\lambda}_1}{\tilde{\lambda}_2} \left(\frac{\bar{\Gamma}_1}{\Gamma_{11}} - 1\right) d \log \tilde{\lambda}_1. \tag{5}$$

As before, the degree of attenuation depends critically on σ. When σ approaches infinity, firms can freely adjust their product mix to match changes in consumer preferences. When σ approaches zero, firms must maintain fixed production proportions regardless of taste shifts, eliminating any potential efficiency gains from demand-driven reallocation.

To express this in terms of observable variables, we first note that with joint production, changes in relative prices are associated with changes in production ratios from equation (2):

$$d \log(p_{11}/p_{12}) = \frac{1}{\sigma} d \log(q_{11}/q_{12}).$$

Using this equation, we can derive the relationship between relative prices and taste shocks[6]:

$$d \log(p_{11}/p_{12}) = \frac{1}{\sigma + 1} \frac{1}{\tilde{\lambda}_2} d \log \tilde{\lambda}_1.$$

This allows us to decompose the TFP response:

$$\Delta \log TFP = \underbrace{\frac{\tilde{\lambda}_1}{\tilde{\lambda}_2} \left(\frac{\bar{\Gamma}_1}{\Gamma_{11}} - 1\right) d \log \tilde{\lambda}_1}_{\text{Single-Product Term}} + \underbrace{\tilde{\lambda}_1 \left(\frac{\bar{\Gamma}_1}{\Gamma_{11}} - 1\right) d \log p_{11}/p_{12}}_{\text{Mutil-Product Term}}.$$

Following similar calculations as in the markup shock example, we can express the single-product term using the labor share and the multi-product term as a covariance between prices and cumulative wedges. We formalize this result in the following proposition.

Proposition 2 (Sufficient Statistics with Taste Shocks)**.** *In this simple economy, the TFP re-*

[6]The relationship between taste shocks and relative prices can be derived as follows. Under Cobb-Douglas preferences, changes in expenditure shares directly reflect taste shocks: $d \log \lambda_{11} = \frac{d\tilde{\lambda}_1}{\tilde{\lambda}_1}$ and $d \log \lambda_{12} = -\frac{d\tilde{\lambda}_1}{\tilde{\lambda}_2}$. The relative price change is related to quantity changes through the elasticity of transformation: $d \log(p_{11}/p_{12}) = \frac{1}{\sigma} d \log(q_{11}/q_{12})$. Combining these with the relationship $d \log \lambda = d \log p + d \log q$ yields $d \log(p_{11}/p_{12}) = \frac{1}{\sigma+1} \frac{1}{\tilde{\lambda}_2} d \log \tilde{\lambda}_1$.

sponse to taste shocks can be expressed as:

$$\Delta TFP = \underbrace{\mathrm{Cov}_{s_1}\left(d\log p_{(1,\cdot)}, \frac{\bar{\Gamma}_1}{\Gamma_{(1,\cdot)}}\right)}_{\textit{Multi-Product Term}} - \underbrace{d\log\Lambda_L}_{\textit{Single-Product Term}},$$

where $s_1 = (\tilde{\lambda}_1, \tilde{\lambda}_2)$ *and* $\bar{\Gamma}_1 = (\tilde{\lambda}_1\Gamma_{11}^{-1} + \tilde{\lambda}_2\Gamma_{12}^{-1})^{-1}$ *is the weighted harmonic mean of cumulative wedges.*

These examples of markup shocks and taste shocks demonstrate a notable feature of our sufficient statistics approach: despite the different nature of the underlying shocks, their impact on allocative efficiency can be measured using the same statistics.

Moreover, when multiple shocks occur simultaneously, prices and factor shares reflect the combined impact of these shocks. This is particularly useful because real-world data can be considered to be generated as a consequence of compound shocks, meaning we can infer changes in allocative efficiency from observed data.

While our examples have focused on a simplified economy, many of these insights carry over to more general settings. We now turn to introducing a more general economy.

2.3 General Production Network Setup

We present our general framework to measure misallocation without imposing parametric assumptions on firms' technologies. We allow for arbitrary firm-to-firm linkages and arbitrary joint production technologies and, hence, heterogeneous transformation elasticities across products within firms.

Multiproduct Firms

Firm $i \in \mathcal{N}$ produces product $g \in \mathcal{G}$ and uses products $g' \in \mathcal{G}$ from other firms $j \in \mathcal{N}$ and factors (Labor, L and Capital, K) as production inputs. [7] We assume the following production set with CRS and separability between input and output functions:

[7] We treat factors exhibiting zero return to scale production functions; they generate production inputs without using inputs from other firms.

$$F_i^Q\left(\underbrace{\left\{q_{ig}\right\}_{g\in G}}_{\text{outputs}}\right) = A_i F_i^X\left(\underbrace{\left\{x_{i,jg'}\right\}_{j\in N, g'\in G}}_{\text{Intermediate product } g' \text{ from } j}, L_i, K_i\right), \tag{6}$$

where A_i represents the productivity at the firm level.[8] While the model specifies Hicks-neutral productivity, this formulation can accommodate input-biased productivity.[9] However, incorporating product-specific productivity differences requires that such productivity be uncorrelated with the cumulative wedge introduced in Section 2.5 for Proposition 3 to apply.

Firms charge a product-specific markup, μ_{ig}, over its product-specific marginal cost; thus, the price is defined as $p_{ig} = mc_{ig}\mu_{ig}$.

Final Demand

Real GDP is the maximizer of a constant-returns homothetic aggregator of final uses of products: $Y = \max_{(c_{i1},\ldots,c_{NG})} U(c_{i1}, \ldots, c_{NG})$ subject to the budget constraint

$$\sum_{i\in\mathcal{N}}\sum_{g\in\mathcal{G}} p_{ig}c_{ig} = \sum_{f\in\{L,K\}} w_f L_f + \sum_{i\in\mathcal{N}}\sum_{g\in\mathcal{G}}\left(1 - 1/\mu_{ig}\right)p_{ig}q_{ig},$$

where w_f is the price of factor f.

Each product can be consumed by final consumers (c_{ig}) or used as an input in production by other firms ($x_{ji,g}$). The following resource constraint applies:

$$q_{ig} = c_{ig} + \sum_{j\in\mathcal{N}} x_{jig}, \quad \sum_{i\in\mathcal{N}} L_i = L, \quad \sum_{i\in\mathcal{N}} K_i = K.$$

Figure 2 presents a stylized representation, showing the flow of products.

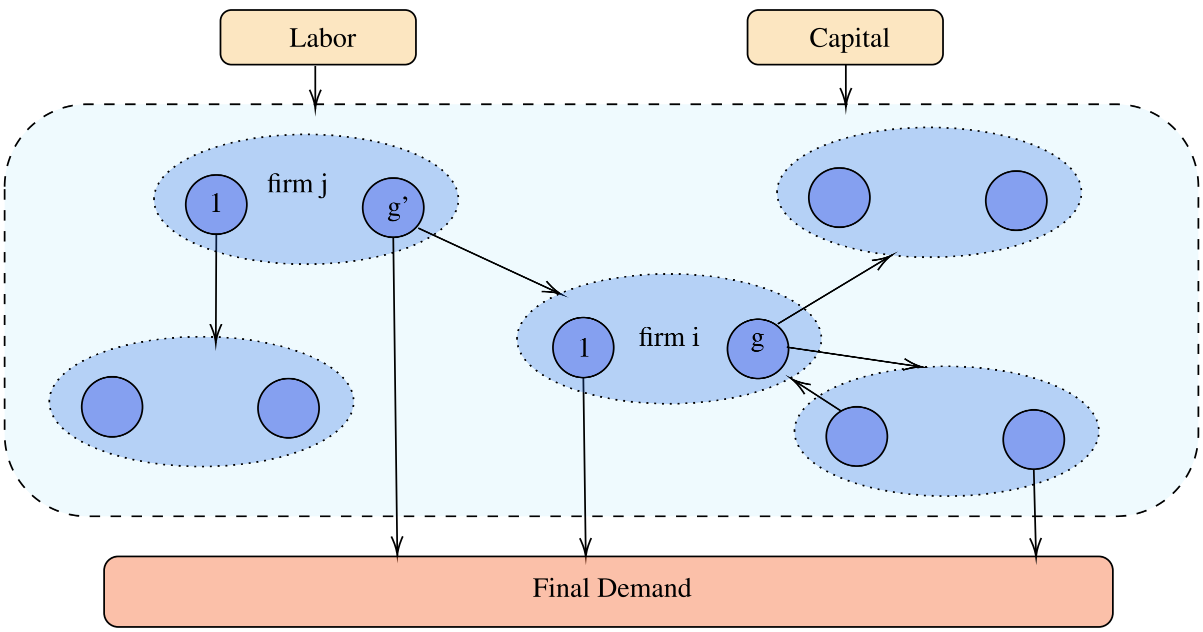
[8] While this formulation assumes common input intensities across different production activities within a firm, this is not a theoretical restriction but rather a measurement constraint. If data exists for separable production activities, these can be treated as if they were (joint) production activities of different firms. Since such data is not available in our application, we assume common input bundles for each firm.

[9] Input-specific productivity can be captured by introducing a fictitious producer who purchases input j and sells to producer i using a linear technology, with Hicks-neutral shocks applied to this fictitious producer.

General Equilibrium

Given a vector of firm-level productivity, $\boldsymbol{A}$, and vector of product-level markups, $\boldsymbol{\mu}$, for all $i \in \mathcal{N}$ and $g \in \mathcal{G}$, the general equilibrium is a set of prices (p_{ig}) intermediate input choices ($x_{ijg'}$), factor input choices (L_i, K_i), output, (q_{ig}), and consumption choices (c_{ig}). As such, (i) the price of each product is equal to its markup multiplied by its marginal cost; (ii) households maximize utility under budget constraints, given prices; and (iii) markets are clear for all products and factors.

Figure 2: Graphical illustration of networks with multiproduct firms



Notes: The dashed line represents firms' universe $\mathcal{N}$, the dotted circled line represents each firm's boundary, and the circled line represents each product within a firm. The two top nodes represent factors, and the bottom node represents households. Arrows represent the direction of input flows.

2.4 Input–Output Definitions

To state our decomposition results, we introduce notation for input-output relationships at the product level.

Product-Level Input–Output Matrix

The product-level input–output matrix $\tilde{\Omega}$ is a $(\mathcal{N}\mathcal{G} + \mathcal{F})$ square matrix. Here, $\mathcal{N}$ is the number of firms, $\mathcal{G}$is the number of products, and $\mathcal{F}$ is the number of factors. $\tilde{\Omega}$ has at its

ig, jg'^{th} element the expenditure share of product g' from firm j and factor $f \in \mathcal{F}$ used by firm i in production over firm i total costs (of producing all its products). The separability assumption indicates that the same expenditure share applies for all products, g, that firm i produces; thus, $\tilde{\Omega}_{ig,jg'}$ and $\tilde{\Omega}_{ig,f}$ are as follows.

$$\tilde{\Omega}_{ig,jg'} = \frac{p_{jg'}x_{i,jg'}}{\sum_{j,p} p_{jg'}x_{i,jg'} + \sum_f w_f L_{if}}, \quad \tilde{\Omega}_{ig,f} = \frac{w_f L_{if}}{\sum_{j,p} p_{jg'}x_{i,jg'} + \sum_f w_f L_{if}}.$$

The product cost-based Leontief inverse $\tilde{\Psi}$ captures each firm-product pair's direct and indirect cost exposures through production networks. We use each $\tilde{\Psi}$ element to measure the weighted sum of all paths between two nonzero firm-product pairs.

$$\tilde{\Psi} \equiv (I - \tilde{\Omega})^{-1} = i + \tilde{\Omega} + \tilde{\Omega}^2 + \dots$$

We define the final consumption share vector, b, as follows:

$$b_{ig} = \begin{cases} \frac{p_{ig}c_{ig}}{GDP} & \text{if } i \in \mathcal{N}, g \in \mathcal{G} \\ 0 & \text{otherwise} \end{cases}$$

where $GDP = \sum_{i \in \mathcal{N}} \sum_{g \in \mathcal{G}} p_{ig}c_{ig}$. We set GDP to be the numeraire and define the product-level cost-based Domar weight, $\tilde{\lambda}_{ig}$. [10] This measures the importance of product g from firm i in final demand in two dimensions: directly when sold to final consumers, and indirectly through the production network when product g is sold to other firms and eventually reaches final consumers via downstream production networks.

$$\tilde{\boldsymbol{\lambda}}' \equiv \boldsymbol{b}'\tilde{\Psi} = \boldsymbol{b}' + \boldsymbol{b}'\tilde{\Omega} + \boldsymbol{b}'\tilde{\Omega}^2 + \dots$$

Factor shares are defined as

$$\Lambda_L = \frac{wL}{GDP}, \quad \Lambda_K = \frac{rK}{GDP}.$$

Firm-Level Aggregation

Summing over products by firms allows us to recover the firm-level cost-based Domar weight $\tilde{\lambda}_i$, which we use to compute the within-firm product-level Domar weight share

[10]We denote $\tilde{\Lambda}_f$ with $f \in \{L, K\}$.

s_{ig}:

$$\tilde{\lambda}_i = \sum_{g \in \mathcal{G}} \tilde{\lambda}_{ig}, \quad s_{ig} = \frac{\tilde{\lambda}_{ig}}{\tilde{\lambda}_i}.$$

Finally, we define firm-level aggregate markup as follows:

$$\mu_i = \frac{\text{sales of } i}{\text{total cost of i}},$$

2.5 Cumulative Wedges

Building on our insights from the simple example, we now generalize the concept of cumulative wedges to arbitrary production networks with multiproduct firms. The example shows that products can face different cumulative distortions depending on their downstream supply chain. We now formalize this notion for arbitrary production networks.

Definition 2 (Cumulative Wedge)**.** For product g of firm i, the cumulative wedge is defined as:

$$\Gamma_{ig} \equiv \underbrace{\tilde{\lambda}_{ig}/\lambda_{ig}}_{\text{downstream wedges}} \times \underbrace{\mu_{ig}}_{\text{own wedge}},$$

where λ_{ig} denotes sales share of firm i's product g over GDP.

The cumulative wedge summarizes the cumulative distortion in the downstream supply chain of product g sold by firm i. In efficient economies with no markups, the product cost-based Domar weight equals observed sales shares, generating a cumulative wedge equal to one for all products and firms. Conversely, in an inefficient economy, a portion of the indirect demand transmitted from downstream firm-product pairs to upstream firm-product firms is absorbed as profit by downstream firms. This effect accumulates in each supply chain transaction upstream until indirect demand reaches product g sold by firm i; thus, the sales share of a product is smaller relative to an efficient economic outcome. Therefore, the larger the ratio, the greater the cumulative wedges in the downstream supply chain.

For our aggregation result, we compare distortions across products within the same firm. We do this by defining the weighted harmonic mean of cumulative wedges for firm i:

$$\bar{\Gamma}_i = \mathbb{E}_{s_i}[\Gamma_{(i,g)}^{-1}]^{-1}$$

where s_i represents the vector of within-firm cost-based Domar weight shares.

Cumulative Wedges in a Simplified Example

To illustrate how cumulative wedges capture distortions, we revisit our simplified economy composed of two firms and a representative household. We normalize GDP to 1; hence, in this setup, sales (shares) to final consumption are $\tilde{\lambda}_1$ and $\tilde{\lambda}_2$ for products 1 and 2 respectively; however, firm 1's sales of product 1 are reduced by the markup charged by firm 2, which is $\tilde{\lambda}_1/\mu$.

The product cost-based Domar weights are $\tilde{\lambda}_1$ for both products 1 and 2. In matrix notation, the value-added share vector (b) and the product cost-based input-output matrix ($\tilde{\Omega}$) are:

$$b = \begin{bmatrix} \tilde{\lambda}_1 \\ 0 \\ \tilde{\lambda}_2 \\ 0 \end{bmatrix}, \quad \tilde{\Omega} = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

where the matrix and vector components are arranged in the following order: product 1 and 2 of firm 1, firm 2, and labor. Therefore, the product cost-based Domar weights can be computed as:

$$\begin{aligned} \tilde{\lambda}' &= b' + b'\tilde{\Omega} + b'\tilde{\Omega}^2 + \dots, \\ &= \underbrace{[\tilde{\lambda}_1, 0, \tilde{\lambda}_2, 0]}_{\text{Final demand}} + \underbrace{[0, \tilde{\lambda}_1, 0, 0]}_{\text{Indirect demand}}. \\ &= [\tilde{\lambda}_1, \tilde{\lambda}_1, \tilde{\lambda}_2, 0]. \end{aligned}$$

These weights represent the counterfactual sales shares if markups were removed while keeping expenditure shares constant. Following the definition, the cumulative wedge for firm 1's products is:

$$\Gamma_{11} = \frac{\tilde{\lambda}_1}{(\tilde{\lambda}_1/\mu)}\mu = \mu^2, \qquad \Gamma_{12} = \frac{\tilde{\lambda}_1}{\tilde{\lambda}_1}\mu = \mu.$$

The markup of product 2 from firm 1 and the product from firm 2 equal μ. Comparatively, product 1 from firm 1 has a larger cumulative wedge of μ^2 than that of product 2, reflecting both the product's own markup and the downstream distortions the product faces. In this case, product 1 from firm 1 generates a distortion by charging a markup and is subject to an additional distortion through downstream production networks because firm 2 uses the marked-up input in its production.

The next section shows how these wedges enter our main aggregation result for arbitrary production networks with multiproduct firms.

2.6 Aggregation Theorem with Multiproduct Firms within Production Networks

To allow for joint production and derive sufficient statistics, this section generalizes the concept of an allocation matrix introduced by Baqaee and Farhi (2020).

Let $\mathcal{X}$ be an $(\mathcal{N}+\mathcal{F})\times(\mathcal{N}\mathcal{G}+\mathcal{F})$ admissible input allocation matrix; the columns are buyer firms, and the rows are seller-product pairs. Each element, $\mathcal{X}_{ijg} = \frac{x_{ijg}}{q_{jg}}$, is the share of the output of product g produced by firm j that firm i uses as a production input.

A productivity shock ($\mathrm{d}\log A$) and a markup shock ($\mathrm{d}\log\mu$) effect in real GDP, Y, can be decomposed into changes in the distribution of $\mathcal{X}$ ($\mathrm{d}\mathcal{X}$), holding productivity constant, and a pure change in productivity ($\mathrm{d}\log A$) for a given fixed allocation matrix $\mathcal{X}$. In vector notation:

$$\mathrm{d}\log\mathcal{Y} = \underbrace{\frac{\partial\log\mathcal{Y}}{\partial\mathcal{X}}\mathrm{d}\log\mathcal{X}}_{\Delta\text{ Allocative Efficiency}} + \underbrace{\frac{\partial\log\mathcal{Y}}{\partial\log A}\mathrm{d}\log A}_{\Delta\text{ Technology}}. \tag{7}$$

We now present a decomposition of changes in aggregate TFP that considers multiproduct firms and arbitrary production networks with product-level distortions.

Proposition 3 (Growth Accounting in Networks with Multiproduct Firms)**.** *To the first order, aggregate TFP can be decomposed into technology and allocative efficiency terms as follows:*

$$d\log TFP = \underbrace{\underbrace{\sum_i \tilde{\lambda}_i \mathrm{Cov}_{s_i}\left(d\log p_{(i,\cdot)}, \frac{\bar{\Gamma}_i}{\Gamma_{(i,\cdot)}}\right)}_{\textit{Multi-Product Term}} \underbrace{- \sum_f \tilde{\Lambda}_f d\log\Lambda_f - \sum_i \tilde{\lambda}_i d\log\mu_i}_{\textit{Single-Product Term}}}_{\Delta\textit{ Allocative Efficiency}} + \underbrace{\sum_i \tilde{\lambda}_i d\log A_i}_{\Delta\textit{Technology}},$$

where $d\log p_{(i,\cdot)} = (d\log p_{i1},...,d\log p_{iG})$ denotes the vector of price changes, $\Gamma_{(i,\cdot)} = (\Gamma_{i1},...,\Gamma_{iG})$ represents the vector of cumulative wedges for firm i's products, and $\bar{\Gamma}_i = \mathbb{E}_{s_i}[\Gamma^{-1}_{(i,g)}]^{-1}$ is the weighted harmonic mean of cumulative wedges.

Appendix E presents the proof. The change in aggregate TFP can be decomposed into technology and allocative efficiency terms. The technology term represents a weighted average of changes in firm-level Hicks-neutral productivity using cost-based Domar weights. The allocative efficiency term is further decomposed into a multiproduct firm term, a change in aggregate factor shares, and firm-level average markup changes.

The multiproduct term captures the allocative efficiency implications of firm-level product mix adjustments. When a firm adjusts its product mix, the relative prices of its products change to reflect the reallocation costs imposed by technological constraints. These price changes interact with existing distortions: if prices rise more for products with higher cumulative wedges (captured by a positive covariance), technological constraints limit reallocation precisely where it would be most beneficial for efficiency. In this general setting, these opportunity costs vary across firms and product pairs, reflecting differences in the curvature of their production possibility frontiers. For example, an oil refinery producing gasoline and diesel may face different constraints when adjusting its production mix than a dairy farmer producing milk and meat.

To calculate the aggregate effect across the economy, we sum these firm-level covariances using Domar weights, which indicate its macroeconomic importance. This aggregation allows us to quantify the overall impact of product mix changes on allocative efficiency in the economy.

Regarding the single product term, which consists of factor shares and firm-level markup, if the initial equilibrium is inefficient, the products charging markups are under-produced relative to an efficient economy. Improving the allocation involves reallocating resources to a more distorted part of the economy, such as firms' product pairs that charge relatively high markups. A decrease in factor shares implies reallocating resources to the portion of the economy that has relatively high markups; however, if the change in factor share is due to a change in markup, this is a mechanical change and does not imply reallocation. Therefore, the contribution of the change must be purged, which the firm-level markup term captures. The factor shares and firm-level markup terms are proposed by Baqaee and Farhi (2020). Both terms are valid under a joint production approach and, together with the multiproduct term this work introduces, constitute allocative efficiency.

Relation to Existing Aggregation Theorems

Proposition 1 nests existing aggregation theorems for production networks as a special case.

Corollary 1 (Baqaee and Farhi (2020))**.** *If no firms engage in joint production and impose the same markup on all their products (the single-product firm assumption), then to a first order, ag-*

gregate TFP growth can be decomposed into technology and allocative efficiency terms as follows.

$$d\log TFP = \underbrace{\sum_{i\in\mathcal{N}} \tilde{\lambda}_i d\log A_i}_{\textit{Technology}} \underbrace{- \sum_{i\in\mathcal{N}} \tilde{\lambda}_i d\log \mu_i - \sum_{f\in\mathcal{F}} \tilde{\Lambda}_f d\log \Lambda_f}_{\textit{Allocative Efficeincy}} .$$

The proof follows from the fact that the covariance term from proposition 3 is zero because the changes in marginal cost and markup for all products within a firm are equal.

Our approach quantifies misallocation through the multiproduct channel by measuring deviations from the single-product, single-markup assumption when product-level data are available; if this assumption holds, the multiproduct term becomes zero. The assumption of uniform marginal costs and markups is unlikely to hold in practice; however, its quantitative relevance remains an empirical question. Our decomposition quantifies the extent to which this assumption is violated and isolates the impact of existing misallocation literature.

Finally, without markups, when prices equal marginal costs, allocative efficiency converges to zero. In this case, all aggregate TFP changes are attributed to technology, aligning with Hulten (1978).

Corollary 2 (Hulten (1978))**.** *Growth Accounting in an Efficient Economy:*

$$d\log TFP = \underbrace{\sum_{i\in\mathcal{N}} \tilde{\lambda}_i d\log A_i}_{\textit{Technology}} .$$

The proof follows from the fact that the markup is always 1, the markup change term is 0, and the sum of factor shares is always 1. Therefore, the sum of factor changes is always 0, and the covariance of the multiproduct term is 0 because $\Gamma_{(i,\cdot)} = (\Gamma_{i1}, ..., \Gamma_{iG})$ are all 1 in an efficient economy.

Proposition 3's formula converges to Hulten's theorem when the economy is efficient. Measured aggregate TFP growth equals the Domar weighted sum of firm-level productivity changes.

3 Reduced Form Evidence on Joint Production

In Section 2, we developed a theoretical framework that accommodates multiproduct firms with joint production technology. When products within firms are separable, our framework collapses to existing aggregation theorems that treat each product as a separate firm. The literature on multiproduct firms often assumes such product line independence (Bernard et al. (2010, 2011); Hottman et al. (2016); Mayer et al. (2021)).

To validate our theoretical framework and test the separability assumption, we provide a partial equilibrium setting that delivers sharp predictions about how firms adjust their product mix in response to demand shocks. We then test these predictions using detailed Chilean firm-to-firm transaction data. We exploit heterogeneous exposure to local buyer shocks for each firm's products and investigate whether firms engage in joint production in the spirit of Ding (2023). Specifically, We examine how demand shocks to specific products affect the production of other products within the same firm. We find that demand shocks to one product significantly reduce the production of other products within the same firm, indicating that firms employ joint production technology.

3.1 Partial Equilibrium Setting for Reduced-Form Regression

Consider a firm operating with the joint production technology characterized by the CET function introduced in Section 2:

$$C(q_1, \ldots, q_N) = \frac{P_M}{A}\left(\sum_{i=1}^{N}\left(\frac{q_i}{a_i}\right)^{\frac{\sigma+1}{\sigma}}\right)^{\frac{\sigma}{\sigma+1}} \quad \sigma > 0, \tag{8}$$

where q_g represents the output of product g, and P_M denotes a composite input price index combining inputs, potentially any combination of intermediate inputs, labor, and capital. The parameter σ represents the elasticity of transformation between outputs in production.

For each product g, we assume a standard isoelastic demand with no cross-price effects:

$$q_g \;=\; D_g\, p_g^{-\theta_g}, \quad \theta_g > 1, \tag{9}$$

where $D_g > 0$ is the demand shifter for product g, θ_g is the own-price elasticity, and p_g is the price of product g. We allow permit a reduced-form wedge μ_g such that $p_g = \mu_g \frac{\partial C}{\partial q_g}$. We assume changes in D_g are uncorrelated with changes in μ_g, so that exogenous variation in D_g shifts the firm's output choice for good g without directly affecting the markup wedge.

Proposition 4 (Within-Firm Demand Shock Spillover via Joint Production)**.** *Consider a negative demand shock to product k ($d \log D_k < 0$). For products $g \neq k$, this leads to:*

(i) Quantity response: $d \log q_g < 0$
(ii) Price response: $d \log p_g > 0$

The economic mechanism operates through the joint production technology introduced in Section 2. When products share production technology through the CET function, a negative demand shock to one product affects the marginal costs of other products through technological complementarities, leading to lower quantities and higher prices of other products.

In the extreme, if $\sigma = 0$ in (8), product line is independent. Then a shock to D_k *does not* alter mc_g for $g \neq k$, so there is *no cross-product spillover*. Formally:

Lemma 1 (Product Separability)**.** *When products are perfectly separable in production* ($\sigma = 0$), *a demand shock to product k has no effect on other products $g \neq k$:*

$$d \log q_g = 0, \quad d \log p_g = 0, \quad \forall g \neq k.$$

This lemma corresponds to the standard assumption in the multiproduct firm literature, where product lines operate independently.

Complete proofs of Proposition 4 and Lemma 1 are provided in Appendix D. Appendix B presents examples of market structures that generate our assumed relationships between wedges and demand shifters.

3.2 Data and Empirical Strategy

We use data from the Chilean Internal Revenue Service (SII), covering all formal firms in Chile.[11] We then employ monthly data from January 2019 to December 2021 to test for joint production. The SII provides detailed information on firm-to-firm transactions through electronic tax documents. This dataset, which captures every product, quantity, and price traded between formal Chilean firms, contains data on over 15 million unique

[11] This study was developed within the scope of the research agenda conducted by the Central Bank of Chile (CBC) in economic and financial affairs of its competence. The CBC has access to anonymized information from various public and private entities by virtue of collaboration agreements signed with these institutions. To secure the privacy of workers and firms, the CBC mandates that the development, extraction, and publication of the results should not allow the identification, directly or indirectly, of natural or legal persons. Officials of the CBC processed the disaggregated data. All the analysis was implemented by the authors and did not involve nor compromise the Chilean IRS. The information contained in the databases of the Chilean IRS is of a tax nature originating in self-declarations of taxpayers presented to the Service; therefore, the veracity of the data is not the responsibility of the Service.

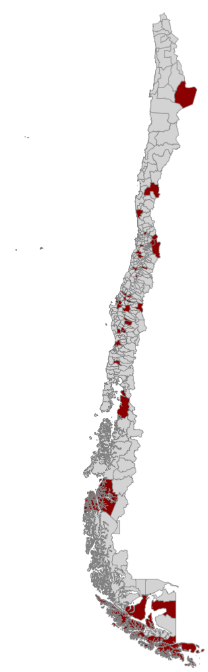
firm-specific product descriptions.[12] We divide the value traded over its quantity to obtain average yearly prices for every triplet.[13]

To test the model's predictions about spillovers across products, we exploit geographic variation in COVID-19 lockdowns across Chilean counties in March 2020 as exogenous demand shocks. These lockdowns represent negative shocks to the demand shifters D_g in our theoretical framework, allowing us to examine how firms adjust their production of other products in response.

COVID-19 Lockdowns in Chile

The Chilean government implemented county-specific lockdowns beginning in March 2020. We focus on this initial period to ensure the shock was unexpected. Figure 3 illustrates the spatial heterogeneity of these lockdowns.

Figure 3: Distribution of early Covid-19 lockdown in Chile



Notes: Lockdown counties as of March 2020 are red; all others are gray.

[12] The specific invoice variable is called "detail", which is inherently firm-specific and can differ between firms even for the same product. For example, one supermarket might declare selling "Sprite can 330cc" while another declares selling "Sprite 330". This variation across sellers does not affect our analysis in this section as we do not compare identical products across firms.

[13] In Appendix A.1, the distribution of the number of products is provided

We exploit this geographic variation as a source of demand shocks to intermediate input transactions, treating lockdowns as negative demand shocks (reductions in D_g) from buyers in lockdown areas to their suppliers in non-lockdown areas. Appendix A.1 validates this approach, demonstrating that firms in lockdown areas reduced their intermediate input purchases by approximately 20%.

Emprical Evidence for Joint Production

To investigate how demand shocks to one product affect the production of other products within firms, we focus on shocks to firms' main products (defined by highest sales from January 2019 to December 2021). We classify a firm as experiencing a main product demand shock if at least one buyer of its main product is located in an area that implemented a March 2020 lockdown.

To quantify this effect, we study the impact of demand shocks to a firm's main product on the production of its other products using an event-study specification for all products $g \neq m$:

$$\log X_{igt} = \sum_{\substack{j=-11 \\ j \neq -1}}^{10} \beta_j D_{i,t-j} + FE_{ig} + FE_t + \varepsilon_{igt}, \tag{10}$$

where X_{igt} represents either the quantity or price of product g for firm i at time t. $D_{i,t-j}$ is a treatment indicator equal to one if firm i was treated j months ago. FE_{ig} and FE_t are firm-product and time fixed effects, respectively. The coefficients of interest are β_j, which capture the effect of the main product's demand shock on other products' quantities or prices at different time points relative to the shock.

To obtain unbiased estimates of β_j, the treatment indicator $D_{i,t-j}$ must be conditionally orthogonal to the error term ε_{igt}. A concern is that supply-side shocks could be correlated with the lockdown if suppliers and main product buyers are located in the same area, potentially confounding our results. To address this issue and isolate the impact of demand shocks from the main product while ruling out direct supply shocks, we impose the following restrictions:

1. **Firm Location**: The firm itself is not located in an area under lockdown.

2. **Supplier Location**: The firm's direct suppliers are not subject to lockdown shocks.

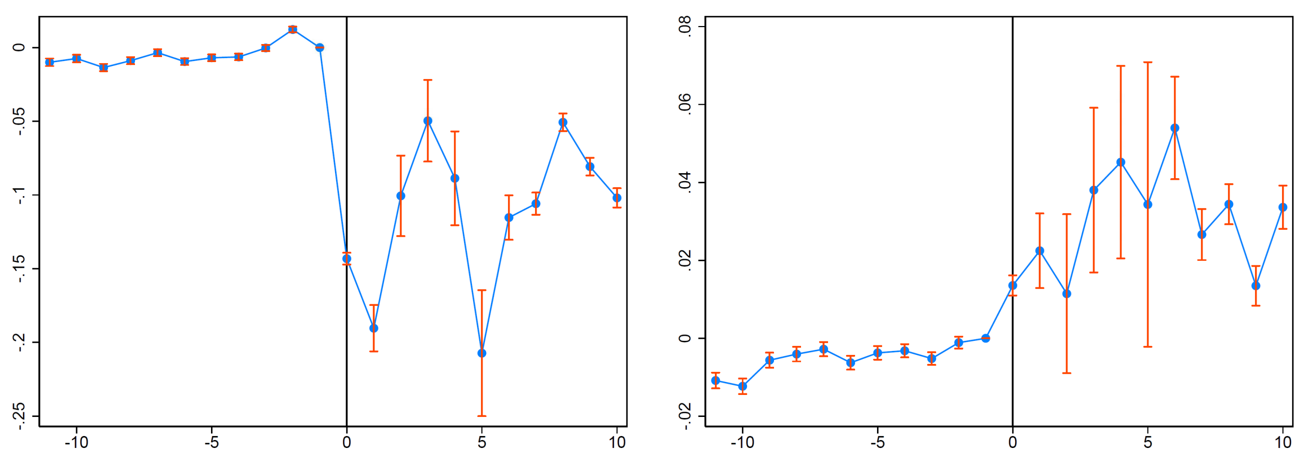
3. **Buyer Location for Product** g: None of the buyers of product g are located in lockdown areas.[14]

Restrictions 1 and 2 help eliminate direct supply-side effects, ensuring that any observed changes in production are not due to supply disruptions that affect the firm or its suppliers. Restriction 3 ensures that product g is not subject to a direct demand shock, allowing us to attribute any changes in its production to the demand shock that affects the main product m. It also ensures that buyers of the main product and product g are different, eliminating the impact of shocks to the main product on product g through demand complementarities and justifying equation (9).

Our treatment group comprises firms meeting these conditions with main products experiencing March 2020 demand shocks. The control group includes firms satisfying the conditions but whose main products remained unaffected by lockdowns. Figure 4 presents the regression results.[15].

Figure 4: The effects of demand shocks to the main product on the production of other products within the firm

(a) Log quantity (b) Log price



Notes: Standard errors are clustered at the firm-county level, and the error bands represent 95% confidence intervals. The X-axis represents the time to treat, with 0 denoting March 2020, when the main product experienced the demand shock. The other values indicate the number of months before or after this event.

[14] Within the same firm, each product typically has its own set of buyers. As a result, when buyers of one product are affected by the lockdown, buyers of other products may remain unaffected. This distinction is further detailed in Figure A1 of Appendix A.1.

[15] A comparison of observable characteristics between the treatment and control groups is provided in Table A2 of Appendix A.1.

The results strongly support our joint production model with finite elasticity of transformation. First, the pre-shock stability of quantities and prices between treatment and control groups supports the parallel trends assumption and the unanticipated nature of initial closures. Second, consistent with Proposition 4(i), we find a significant ten percent decrease in non-main product quantities following main product demand shocks, with persistent effects. Third, aligning with Proposition 4(ii), we observe sustained price increases for other products. These combined quantity and price spillovers match our theoretical predictions under finite σ and reject the product line separability hypothesis.

Discussion on Other Within-Firm Spillover Mechanisms

Our findings of negative quantity spillovers and positive price spillovers across products within firms contrast with several alternative mechanisms in the literature:

First, Almunia et al. (2021) propose a model of diminishing returns to scale or firm-specific factors at the firm level to explain how a decline in domestic demand in Spain affects exports. Their model predicts that when there is a negative demand shock for a product in one market, firm-specific factors are reallocated to another product in another market, positively affecting the production of the same product in other markets. This prediction contrasts with our findings, which show negative spillovers across products within the same firm.

Second, Ding (2023) focuses on industries that share knowledge-intensive inputs to examine joint production effects in the US using Census data. This paper, like ours, predicts that when a product faces a negative demand shock, it negatively affects other products. The study interprets the model prediction as knowledge spillovers across industries sharing intangible inputs; however, knowledge spillovers are unlikely to explain our results. The differences in time horizon (five years vs. monthly data) and research and development (R&D) intensity (Chile's R&D spending is less than one-tenth that of the US as a percentage of GDP) limit its applicability to our context.

Third, Giroud and Mueller (2019) model demand-driven regional spillovers through financial constraints, predicting negative quantity responses across regions. While this aligns with our quantity findings, related work by Kim (2020) suggests financially constrained firms reduce prices, contrary to our observed price increases. Our results persist in a subset of financially unconstrained firms (see Appendix A.1), suggesting financial constraints are not the primary driver.

4 Construction of Sufficient Statistics

Having established the presence of joint production, we implement our sufficient statistics framework developed in Section 2 using a dataset from the Chilean Internal Revenue Service (Servicio de Impuestos Internos, SII). As discussed in Section 3, the dataset primarily relies on electronic tax invoices, which provide detailed records of all firm-to-firm transactions, including product descriptions, quantities, and prices. These invoices allow us to observe the complete structure of firm-to-firm relationships and compute firm-specific product shares.

To construct a full input output matrix, and cumulative wedge, we additionally use tax accounting declarations, which provide monthly data on each firm's revenue and input expenditures, including capital and labor costs. A key advantage of the SII data is its use of unique identifiers for firms and workers, which allows individual and firm data to be merged across datasets. We utilize four distinct sources from SII.

The first is the value-added tax form, which includes gross monthly firm sales, materials expenditures, and investment.

Second, the SII provides information from a matched employer–employee census of Chilean firms from 2005 to 2022. Specifically, firms must report all payments to individual workers, including the sum of taxable wages, overtime, bonuses, and any other labor earnings for each fiscal year. All legal firms must report to the SII; thus, the data cover the total labor force with a formal wage contract, representing roughly 65% of employment in Chile. For any given month, it is possible to identify an individual worker's employment status, their average monthly labor income that year, a monthly measure of total employment, and the distribution of average monthly earnings within the firm.

Third, income tax form data includes yearly information on all sources of a firm's income and expenses. This form allows for computing every individual's actual tax payments for each year. Details on sales and employment are available on this form; however, we use only data on capital stock for each firm and year. This approach allows us to build perpetual inventories using data from the monthly F22 form. We obtain the user cost of capital by multiplying nominal capital stock by the real rental rate of capital, which is built using publicly available data. We use the 10-year government bond interest rate minus expected inflation plus the external financing premium. Finally, we use the capital depreciation rate from the LA-Klems database.

Fourth are electronic tax documents from 2016 onward. These documents provide information on each product (price and quantity) traded domestically or internationally with at least one Chilean firm. We only use domestic transactions and observe the firm-to-firm transactions and a firm's sales (including firm-to-firm and firm-to-consumer sales). We compute firm-specific product shares for firm-to-firm transactions and assume that their distributions are equivalent to firm-to-consumer transactions to recover the complete distribution of firm sales by product. Each firm-to-firm transaction includes a "detail" column that records the name of each product.

Building on the data cleaning process described in Section 3, we process the data to construct product code-level output and input-price indices for each firm using standard Tornqvist indices. We aggregate products into a 290 product-code identifier to facilitate comparison between firms, allowing us to estimate product production functions that use the same product across firms.

4.1 Data Cleaning and Implementation Strategy

We begin the data processing by applying filters to the raw data to obtain the final database for empirical analysis. We define a firm as a taxpayer with a tax ID, positive sales, positive materials, positive wage bill, and capital for any given year. We exclude firms that hire less than two employees a year or have capital valued below US$20 in a year. All variables are winsorized at the 1% and 99% levels to mitigate measurement error.

We selected 2016 as the base year for price indices because it was the first year we observed prices for firm-to-firm transactions. This method is widely recognized for estimating aggregate production functions at the firm or plant level when price data is accessible. We use crosswalks developed at the Central Bank of Chile (Acevedo et al. (2023)) to address the challenge of product aggregation (from around 15 million products to 290 product codes). We create aggregated product-level quantity produced and material usage indices, matching product descriptions and characteristics to ensure consistency across firms and over time.

4.2 Construction of Sufficient Statistics

We measure five distinct objects to implement the growth accounting framework that includes the multiproduct channel: (1) product-level cost-based Domar weights $\tilde{\lambda}$, (2) product-firm level price indices, (3) product-level markups μ, (4) cumulative wedges, and (5) aggregate objects. We discuss each of these in the following subsection.

4.2.1 Product-Level Cost-Based Domar Weights

The product cost-based Domar weights can be calculated using the following equation:

$$\tilde{\lambda}' \equiv \boldsymbol{b}'\tilde{\Psi} = \boldsymbol{b}' + \boldsymbol{b}'\tilde{\Omega} + \boldsymbol{b}'\tilde{\Omega}^2 + \dots$$

To compute these weights, we must measure value-added shares ($\boldsymbol{b}$) and the input–output matrix ($\tilde{\Omega}$). We measure these objects directly from the data.

Final expenditure shares ($\boldsymbol{b}$) are represented by a vector of dimension $(\mathcal{NG} + \mathcal{F}) \times 1$. Here, $\mathcal{N}$ is the number of firms, $\mathcal{G}$ is the number of products, and $\mathcal{F}$ is the number of factors. The first $\mathcal{NG}$ entries are calculated as the residual between a firm product's total sales and its intermediate sales to other firms (measured from the firm-to-firm data). This approach provides a theory-consistent measure of final expenditures. The final $\mathcal{F}$ entries are set to zero because households do not directly purchase factors. Using firm-to-firm records and factor expenditures, we construct the input–output matrix $\tilde{\Omega}$ at the product-firm level.

Specifically, we compute an annual cost-based input–output matrix by product. We calculate the denominator of each element (indexed by ig, jg') by summing a firm's purchases from all its suppliers, its wage bill, and its capital multiplied by the relevant user cost rental rate of capital. The last two elements of the matrix have wage bills and capital expenditures as their numerators.

The resulting $\tilde{\Omega}$ is a $(\mathcal{NG} + 2) \times (\mathcal{NG} + 2)$ matrix that can be expressed as follows:

$$\tilde{\Omega} = \left[\begin{array}{ccc|cc} \tilde{\Omega}_{11,11} & \cdots & \tilde{\Omega}_{11,\mathcal{NG}} & \tilde{\Omega}_{11,\mathcal{NG}+1} & \tilde{\Omega}_{11,\mathcal{NG}+2} \\ & \ddots & & & \\ \tilde{\Omega}_{\mathcal{NG},11} & & \tilde{\Omega}_{\mathcal{NG},\mathcal{NG}} & \tilde{\Omega}_{\mathcal{NG},\mathcal{NG}+1} & \tilde{\Omega}_{\mathcal{NG},\mathcal{NG}+2} \\ \hline 0 & \cdots & 0 & 0 & 0 \\ 0 & \cdots & 0 & 0 & 0 \end{array}\right].$$

Based on the separability assumption, the same expenditure share applies to all products g that firm i produces. The expressions for $\tilde{\Omega}_{ig,jg'}$ and $\tilde{\Omega}_{ig,f}$ are as follows:

$$\tilde{\Omega}_{ig,jg'} = \frac{p_{jg'}x_{i,jg'}}{\sum_{j,p} p_{jg'}x_{i,jg'} + \sum_f w_f L_{if}}, \quad \tilde{\Omega}_{ig,f} = \frac{w_f L_{if}}{\sum_{j,p} p_{jg'}x_{i,jg'} + \sum_f w_f L_{if}}.$$

Factors do not require inputs; thus, the last row of the matrix is zero.

After calculating the product-level cost-based Domar weights, we sum them for the same firms to compute the firm-level cost-based Domar weights and their shares. These will be inputs for Proposition 3.

$$\tilde{\lambda}_i = \sum_{g \in \mathcal{G}} \tilde{\lambda}_{ig}, \quad s_{ig} = \frac{\tilde{\lambda}_{ig}}{\tilde{\lambda}_i}.$$

4.2.2 Product-Firm Level Price Indices

We observe prices for each transaction and aggregate them into 290 product categories. We construct two types of price indices: output and input price indices. We compute firm-product-specific annual price indices for the output price index, which is an input to sufficient statistics that deflates product output for production function estimation. The original data are at the "detail" product level, which we aggregate to a Tornqvist index for each 290 product category the firm owns. Specifically, we construct the following price index:

$$\Delta \log P_{igt} = \sum_{d \in g} \frac{s_{idt} + s_{idt-1}}{2} \Delta \log P_{idt},$$

where d is the detailed category belonging to the upper product category (290 product codes). $\Delta \log P_{idt}$ is the price change, and s_{idt} is the share at time t in the continuing products in category g. We construct our price index with 2016, the starting year of the data, as the base year. We also construct an input price index to deflate material costs for production function estimation. We define one aggregate index per firm because aggregate materials are used as inputs in production function estimation. The construction method is the same as for the output price index.

4.2.3 Cumulative Wedges

To construct the cumulative wedge measure, we need product cost-based Domar weights, product sales shares, and product markups :

$$\Gamma_{ig} \equiv \underbrace{\frac{\tilde{\lambda}_{ig}}{salesshare_{ig}}}_{\text{Downstream wedge}} \times \underbrace{\mu_{ig}}_{\text{own markup}}.$$

As discussed in Section 2, the ratio of cost-based Domar weights to sales share represents the cumulative wedge accumulated downstream of a product. The downstream wedge is calculated using cost-based Domar weight and its sales share. The remaining markup for the own markup needs to be estimated.

As our baseline specification, we employ the accounting approach, wherein markups are assumed to be homogeneous within firms and are computed as the ratio of firm-level sales to firm-level costs. In this setting, variation in cumulative wedges across products within firms emerges solely from heterogeneity in downstream wedges. This setting is consistent with single-product firm models (e.g., Baqaee and Farhi (2020)), thereby facilitating direct comparison of our findings with the existing literature under equivalent conditions in the next Section.

In the Appendix D, we present alternative results using markup estimates based on Dhyne et al. (2022)'s methodology. We find that this approach yields nearly identical aggregate results, which can be attributed to the fact that most of the variation in cumulative wedges stems from downstream wedges. [16]

Ranking of Downstream Wedges

Which products have greater downstream wedges? We ranked products by their cumulative wedges to better understand which products face increased downstream wedge. Below, we describe and discuss the major product categories. Appendix C presents the complete list of the 30 top and bottom items.

The product categories with the greatest (downstream) wedge mainly comprise business services. For example, insurance brokerage services top the list, followed by employment services (recruitment and supply), electricity distribution to businesses, and postal and courier services. These products are usually upstream inputs that other firms use in production, suggesting insufficient size as wedges accumulate through the supply chain before they reach final demand.

Conversely, the least distorted products include cakes, beer, pet food, personal services such as hospitals, and minerals (copper, silver, and molybdenum), that are Chile's primary export industry. These products are common downstream products close to Chile's final demand. As a result, the number of supply chains that reach the final consumer is relatively small, and inefficiencies are relatively less likely to accumulate.

[16] We estimate product-level markups using the production function approach developed by Dhyne et al. (2022), which extends the Ackerberg et al. (2015) production function estimation technique to a joint production setting.

4.2.4 Aggregate Objects

In addition to product cost-based Domar weights and cumulative wedges, we must measure aggregate objects to implement the sufficient statistics presented in Proposition 3. In particular, Y, L, K, Λ_L, and Λ_K denote aggregate value-added, employment, capital, and labor and capital shares, respectively. We measure Y, L, and K as the sum of value added, employment, and capital, respectively, for all firms in the economy. Factor shares of GDP, Λ_L and Λ_K, are measured as total compensation and capital with user cost of capital divided by GDP. Real GDP is calculated by deflating GDP with the official GDP deflator.

5 Application: Decomposing Aggregate TFP Growth

This section applies Proposition 3 to analyze aggregate TFP growth for the Chilean economy. Our analysis covers 2016 to 2022, during which Chile's aggregate TFP stagnated and decreased at the margin. This productivity trend aligns with the pattern of productivity stagnation observed in Chile using different computation methods. [17]

We begin by presenting results using the standard assumption in the literature of single-product firms. If firms produce a single product, then Corollary 1 applies:

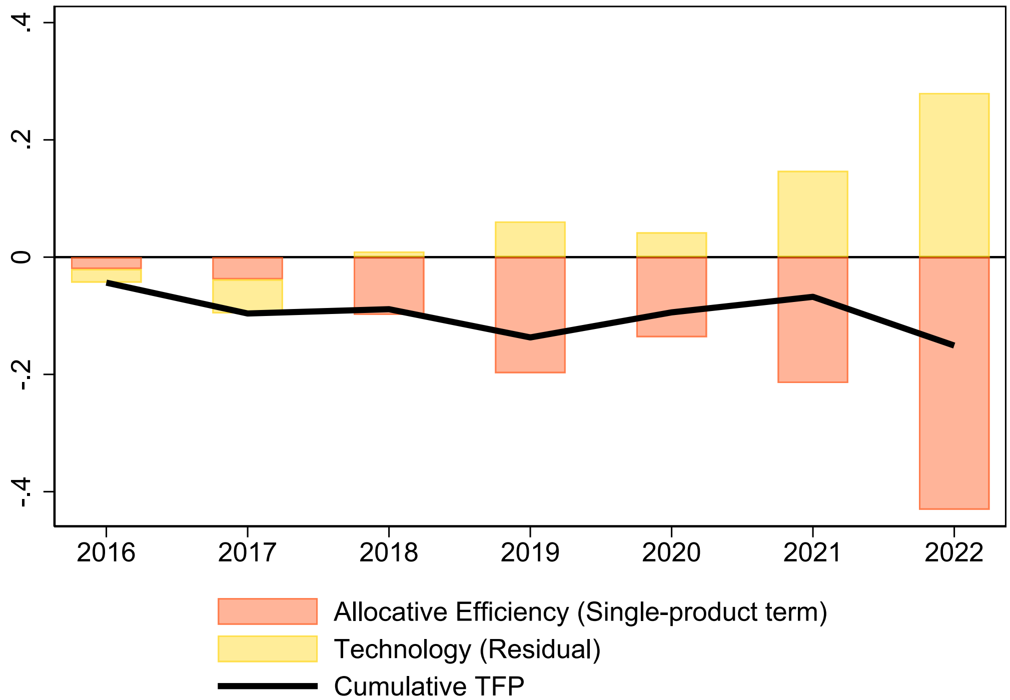
$$d \log TFP = \underbrace{-\sum_{i \in \mathcal{N}} \tilde{\lambda}_i d \log \mu_i - \sum_{f \in \mathcal{F}} \tilde{\Lambda}_f d \log \Lambda_f}_{\Delta \text{Allocative Efficeincy}} + \underbrace{\sum_{i \in \mathcal{N}} \tilde{\lambda}_i d \log A_i}_{\Delta \text{Technology (Residual)}} .$$

This approach implements growth accounting but overlooks multiproduct firms engaged in joint production. Figure 5 illustrates the decomposition of cumulative changes in aggregate TFP from 2016 to 2022 under this assumption.

Figure 5 shows that the allocative efficiency term (in red) declined over this period. This outcome suggests that high-markup firms contracted further, resulting in a negative reallocation effect; however, the contribution of allocative efficiency exceeds that of the technology (residual) component, particularly during the COVID-19 pandemic and the subsequent high inflation period. To rationalize this disparity, the technology term, measured as a residual, must have increased by about 20%.

[17] CNEP (2023)

Figure 5: Cumulative TFP growth decomposition: ignoring multiproduct term



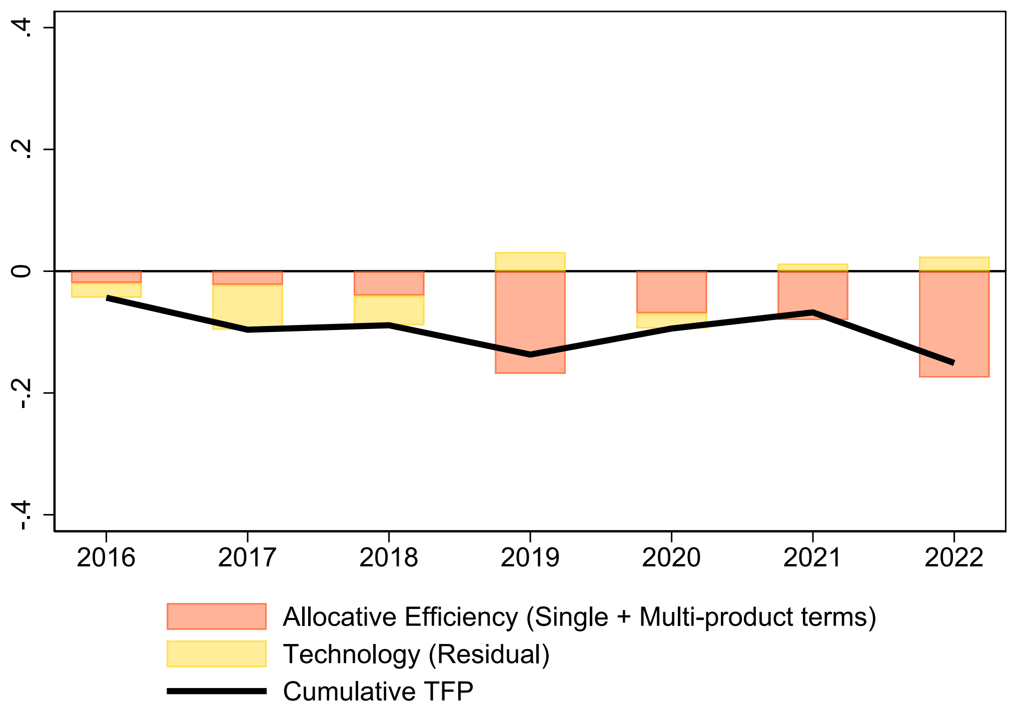
Notes: This Figure shows the cumulative change calculated by applying Corollary 1 repeatedly each year. Technology (residual) is calculated by subtracting allocative efficiency from TFP growth.

Next, we incorporate the multiproduct term using Proposition 3:

$$d\log TFP = \underbrace{\underbrace{\sum_{i\in\mathcal{N}} \tilde{\lambda}_i \mathrm{Cov}_{s_i}\left(d\log p_{(i,\cdot)}, \frac{\bar{\Gamma}_i}{\Gamma_{(i,\cdot)}}\right)}_{\text{Multiproduct term}} - \underbrace{\sum_{i\in\mathcal{N}} \tilde{\lambda}_i d\log \mu_i}_{\text{Firm-level Markup}} - \underbrace{\sum_{f\in\mathcal{F}} \tilde{\Lambda}_f d\log \Lambda_f}_{\text{Aggrregate Factor Shares}}}_{\Delta \text{ Allocative Efficiency}} + \underbrace{\sum_{i\in\mathcal{N}} \tilde{\lambda}_i d\log A_i}_{\Delta \text{ Technology (Residual)}} .$$

Figure 6 presents the results incorporating the multiproduct term, which reduces the magnitude of the technology (residual) observed in Figure 5. In other words, the multiproduct and single-product misallocation terms account for a larger portion of aggregate TFP movements during the COVID-19 pandemic and the resultant high inflation period.

Figure 6: Cumulative TFP growth decomposition with multiproduct term

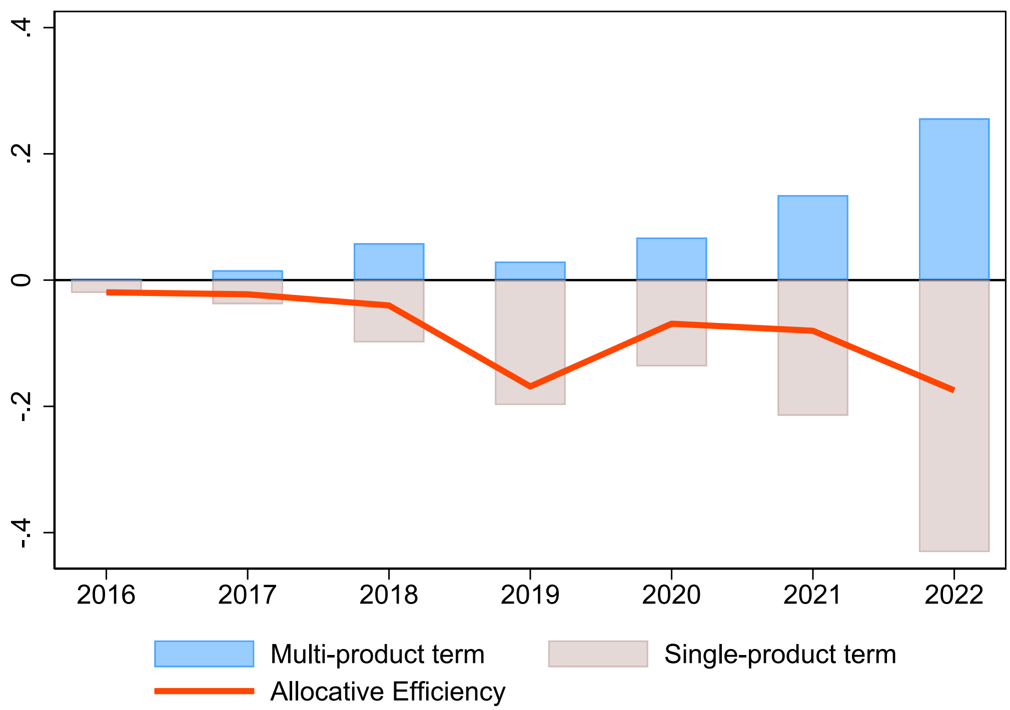


Notes: This Figure shows the cumulative log change calculated by repeatedly applying the equation from Proposition 1 each year. Technology (residual) is calculated by subtracting allocative efficiency from aggregate TFP growth.

Reallocation effects that consider joint production explain 86% of the observed aggregate TFP growth. Conversely, as shown in Figure 5, ignoring joint production leads to overestimating resource misallocation. This result suggests that considering joint production considerably decreases the reallocation implied under the traditional assumption that firms produce only single products.

This finding is consistent with the joint production mechanism described in Section 3. When firms engage in joint production, they create multiple products using common inputs. When a given product receives a shock, if firms face technological constraints to adjust their product mix (non-infinite elasticity of transformation), firms will struggle to reallocate productive resources from one product to another. The reallocation through substitution among products within multi-product firms is attenuated, and reallocation is not materialized to the extent suggested under the single-product firm assumption.

Figure 7: Cumulative TFP growth decomposition with multiproduct term



Notes: This Figure decomposes the cumulative change in allocative efficiency in Figure 6 into single-product and multi-product terms.

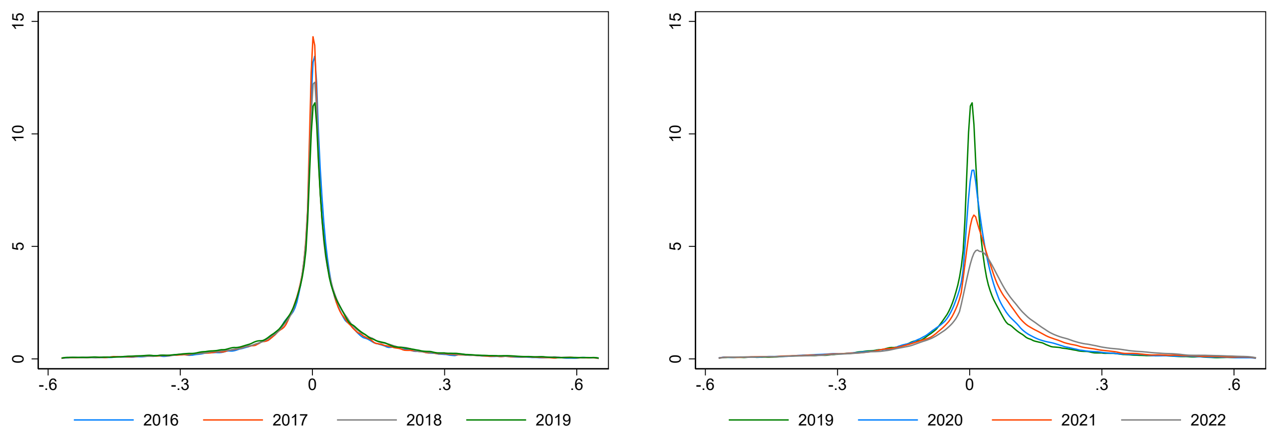
Furthermore, Figure 7 breaks down the allocative efficiency in Figure 6 into multi-product and single product terms. It shows that the offsetting of reallocation due to joint production will become particularly strong after 2020. During this period, the economy was disrupted by COVID-19 and subsequent high inflation. We interpret the latter as firms facing changes in product-specific demands, which changed their total demand composition. In response, firms were willing to readjust their product mix by reallocating productive resources. However, due to the non-infinite elasticity of transformation, firms were constrained to change their product mix.

Finally, the granularity of the data allows us to track the distributional changes of joint production (the multiproduct term) that limit the extent of resource reallocation. Since the covariance degenerates to zero under the single-product firm assumption, the dispersion of covariance implies that joint-production forces are active. These distributions vary from period to period. Figure 8a plots the distribution for pre-COVID-19 (2016–2019), which is symmetric around 0, with slight differences from year to year.

Figure 8b presents the distribution after the onset of COVID-19, showing a shift to the right from year to year, resulting in a right-skewed distribution. This result suggests that the increase in the contribution from joint-production forces (the multiproduct term) was not driven by a few specific firms.

Figure 8: $\text{Cov}_{s_i}\left(d\log p_{(i,\cdot)}, \frac{\bar{\Gamma}_i}{\Gamma_{(i,\cdot)}}\right)$ distributions by year

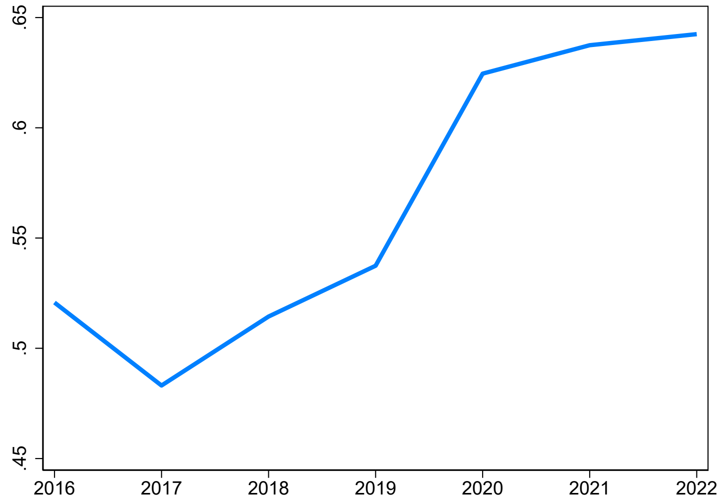
(a) 2016 - 2019 (b) 2019 - 2022



Notes: These Figures plot the distribution of firm level $\text{Cov}_{s_i}\left(d\log p_{(i,\cdot)}, \frac{\bar{\Gamma}_i}{\Gamma_{(i,\cdot)}}\right)$ for each year.

Finally, Figure 9 plots the median variance of product-specific production changes across firms from 2016 to 2022. This figure provides suggestive evidence that aligns with the changing distribution of multi-product firms shown in Figure 8b and corresponds to the period of significant contribution from the multi-product term in our decomposition. The increasing variance, particularly the sharp rise from 2019 to 2020 and its sustained high level thereafter, indicates that firms have been under greater pressure to adjust their product mix. This trend coincides with the timeframe when we observe the most substantial impact of the multi-product term on allocative efficiency. The temporal consistency between the increased variance in product-specific production changes and the heightened contribution of the multi-product term reinforces our model's emphasis on the importance of multi-product firms engaged in joint production, especially during major economic shocks like the COVID-19 pandemic.

Figure 9: Large product mix adjustments: suggestive evidence



Notes: This figure depicts the evolution of the median variance of product quantity changes, denoted as $Var_{\lambda_i}\left(d\log q_{ig}\right)$, from 2016 to 2022.

6 Extension: Ex-Ante Structural Results

This section develops a structural framework to predict how economies with multiproduct firms respond to shocks. While our previous analysis relied on observed price and factor share changes, we now model these endogenous responses explicitly. This theoretical extension allows us to move beyond ex-post measurement to ex-ante prediction of counterfactual scenarios. The framework complements our earlier results.

We show how to apply this framework to study the distance to the Pareto-efficient frontier when firms use joint production technology. This method compares output in an efficient equilibrium (with all markup wedges removed) to that in a distorted decentralized economy. Our analysis demonstrates how the theoretical results of previous studies, such as those of Hsieh and Klenow (2009) and Baqaee and Farhi (2020), change when firms engage in joint production. Since an economy without markups is unobservable, a model is necessary to analyze this counterfactual case.

6.1 The Nested CET-CES Model

We propose the nested CET-CES model, which provides a tractable framework for our analysis. We use the same setup with subsection 2.3 but impose the CET-CES functional form to the joint production function to the equation 6. We then derive a linear system for price and sales responses, allowing us to characterize the economy's response to shocks. The production technology is given by:

$$\underbrace{\left(\sum_{g\in\mathcal{G}} \delta_{ig}\left[q_{ig}\right]^{\frac{\sigma_i+1}{\sigma_i}}\right)^{\frac{\sigma_i}{\sigma_i+1}}}_{\text{Output bundle}} = \underbrace{A_i\left(\sum_{j\in\mathcal{N},g'\in\mathcal{G}} q_{i,jg'}^{\frac{\theta_i-1}{\theta_i}} + \omega_{i,L}L_i^{\frac{\theta_i-1}{\theta_i}} + \omega_{i,K}K_i^{\frac{\theta_i-1}{\theta_i}}\right)^{\frac{\theta_i}{\theta_i-1}}}_{\text{Input Bundle}}. \tag{11}$$

Here, σ_i represents the elasticity of transformation between different outputs, A_i denotes the productivity of firm i, and δ_{ig} are the output share parameters. The input bundle comprises intermediate inputs $q_{i,jg'}$, labor L_i, and capital K_i, aggregated using a CES function with an elasticity of substitution θ_i. Note that this class of models is highly general, nesting the nested CES system widely used in macroeconomics and international economics as a special case. For single-output firms, the production function degenerates to:

$$q_i = \underbrace{A_i\left(\sum_{j\in\mathcal{N},g'\in\mathcal{G}} \omega_{i,jg'}q_{i,jg'}^{\frac{\theta_i-1}{\theta_i}} + \omega_{i,L}L_i^{\frac{\theta_i-1}{\theta_i}} + \omega_{i,K}K_i^{\frac{\theta_i-1}{\theta_i}}\right)^{\frac{\theta_i}{\theta_i-1}}}_{\text{Input bundle}}. \tag{12}$$

Furthermore, we specify the household utility function as a CES aggregator over final consumption goods. Formally, the representative household's utility function is given by:

$$U\left(c_{11},...,c_{ig},...,c_{NG}\right) = \left(\sum_{i\in\mathcal{N},g\in\mathcal{G}} \psi_{ig}c_{ig}^{\frac{\theta_0-1}{\theta_0}}\right)^{\frac{\theta_0}{\theta_0-1}} \tag{13}$$

where c_{ig} represents the consumption of product g from firm i, ω_{ig} represents the taste parameter for each product, and θ_0 is the elasticity of substitution between products.

For analytical simplicity, we assume a uniform substitution elasticity within the firm's CES structure, though extending the model to incorporate further nesting would be straightforward.

6.2 Linear System for Price and Sales Responses

Using this model, we construct a system for solving the first-order response to primitive shocks $(\boldsymbol{A}, \boldsymbol{\mu})$ to the endogenous variables. This system of linear equations, derived from the model's first-order conditions, enables us to generate ex-ante predictions of how the economy will respond to counterfactual shocks. We begin with the multi-product firm's forward equation under the CET output function:

Proposition 5 (Multi-Product Firm's Forward Equation under CET Output Function)**.**

$$d\log p_{ig} = \underbrace{-\sum_{j\in\mathcal{N},g'\in\mathcal{G}} \tilde{\Psi}_{ig,jg'}\left(d\log\mu_j - d\log A_j\right) + \sum_{f\in\mathcal{F}} \tilde{\Psi}_{ig,f} d\log\Lambda_f}_{\textit{Indirect cost exposure}}$$
$$+ \underbrace{\sum_{j\in\mathcal{N},g'\in\mathcal{G}} c_j^R \tilde{\Psi}_{ig,jg'} d\log\Theta_{jg'}}_{\textit{Indirect exposure to the Product Mix adjustment}},$$

where

$$d\log\Theta_{jg'} = \left(\frac{\sigma_j}{\sigma_j+1}\right) d\log\mu_{jg'}/\mu_{jr} + \frac{1}{\left(1+\sigma_j\right)}\left[d\log\lambda_{jg'}/\lambda_{jr}\right],$$

and $c_j^R = \frac{mc_{jr}q_{jr}}{\sum mc_{jg}q_{jg}}$ *is a reference product cost share. Here, r denotes a reference good.*

This equation describes how changes in unit prices within a firm due to markups, productivity shocks, and price changes associated with endogenous product mix adjustments are transmitted through production networks to other firms' products. The first term illustrates the effect of exposure to common cost shocks on prices, a force present in standard production network models. The second term, unique to the joint production model, indicates the exposure of firms with nonlinear production possibility frontiers to modify their product mix due to reallocation, which affects endogenous unit costs. The magnitude of this effect depends on the transformation elasticity, with lower elasticities leading to larger cost effects. As σ approaches infinity, this endogenous effect disappears, as firms can freely adjust their product mix.

Next, we consider backward propagation:

Proposition 6 (Backward Propagation)**.**

$$\lambda_{ig} d\log\lambda_{ig} = -\sum_{j\in\mathcal{N},g'\in\mathcal{G}} \lambda_{jg'}\left(\Psi_{jg',ig} - \mathbf{1}\left(ig = jg'\right)\right) d\log\mu_{jg'}$$

$$+ \sum_{k \in \mathcal{N}, g'' \in \mathcal{G}} \frac{\lambda_{kg''}}{\mu_{kg''}} \left(1 - \theta^k\right) \mathrm{Cov}_{\tilde{\Omega}_{(kg'',:)}} \left(d \log p, \Psi_{(:,ig)}\right).$$

This equation, based Baqaee and Farhi (2020) methodology, describes the sales and factor share response. Notably, it does not contain σ, indicating that joint production only affects the changes in sales share via substitution effects. The equation shows how shocks propagate through the production network, affecting markups and quantities of each product via changes in upstream suppliers' price indices and productivity.

By combining the forward equation from Proposition 1 and the backward equation from Proposition 2, we obtain a complete system of linear equations that characterizes the economy's response to shocks. This system consists of $2 \times (1 + \mathcal{N} \times \mathcal{G} + \mathcal{F})$ equations and $2 \times (1 + \mathcal{N} \times \mathcal{G} + \mathcal{F})$ unknowns, where $\mathcal{N}$ is the number of firms, $\mathcal{G}$ is the number of products, and $\mathcal{F}$ is the number of factors. This system of equations fully characterizes the first-order response of all endogenous variables to any combination of productivity or markup shocks. By solving this linear system using standard matrix algebra, we can conduct counterfactual analyses and evaluate the impact of various shocks on the economy's equilibrium outcomes.

6.3 Distance to the Pareto-Efficient Frontier

Using our model, we can characterize the distance to the Pareto-efficient frontier when introducing distortions, allowing us to predict efficiency losses from counterfactual changes in markups or other distortions:

Proposition 7 (Distance to the Pareto-Efficient Frontier)**.** *Under joint production, starting at an efficient equilibrium, up to second order, and in response to the introduction of distortions, changes in the TFP are given by Domar-weighted Harberger triangles:*

$$\mathcal{L} = \frac{1}{2} \sum_{ig} \lambda_{ig} d \log q_{ig} d \log \mu_{ig}, \tag{14}$$

where λ_{ig} is the Domar weight of product g from firm i, q_{ig} is the quantity, and μ_{ig} is the markup.

This result demonstrates that TFP changes resulting from the introduction of distortions are solely determined by three statistics: Domar weights of each product; the magnitude of the wedges; and the change in the quantity of the product. The quantity change can be derived from sales and price changes given by our linear system, using the relationship $d\log q = d\log\lambda - d\log p$. Since this system includes transformation elasticity σ, it is generally affected by the elasticity value. To illustrate how joint production affects the Distance to the Frontier, we provide analytical solutions for two examples.

6.3.1 Horizontal Economy with Joint Production

We consider a horizontal economy similar to Hsieh and Klenow (2009) but with use joint production technologies. This allows us to investigate within-firm markup heterogeneity in the presence of production transformation constraints. In an economy with a representative consumer (CES utility with elasticity θ), $\mathcal{N}$ firms each use a shared input L to produce $\mathcal{G}$ products using CET technology (elasticity σ). Markups μ_{ig} are heterogeneous across products and firms.

Proposition 8 (The Distance to the Frontier in the Horizontal Economy)**.**

$$\mathcal{L} = -\frac{1}{2}\theta\left(\mathrm{Var}_{\lambda}(d\log\mu_{ig}) - \frac{1}{\sigma+1}\mathbb{E}_{\bar{\lambda}}\left\{\mathrm{Var}_{s_i}(d\log\mu_{ig})\right\}\right),$$

where $\mathrm{Var}_{\lambda}(d\log\mu_{ig})$ *and* $\mathrm{Var}_{\lambda i}(d\log\mu_{ig})$ *represent the Domar weighted variance of markup change and the average of within-firm variances of markup changes, respectively. Vectors* $\boldsymbol{\lambda} = (\lambda_{11}, \lambda_{12}, \ldots, \lambda_{NG})$, $\bar{\boldsymbol{\lambda}} = (\lambda_1, \lambda_2, \ldots, \lambda_N)$ *are the vectors of firm-level revenue and cost Domar weights with* $\lambda_i = \sum_g \lambda_{ig}$, *and* $\boldsymbol{s}_i = (\lambda_{i1}/\lambda_i, \lambda_{i2}/\lambda_i, \ldots, \lambda_{iG}/\lambda_i)$.

Proposition 8 characterizes the distance to the frontier in a horizontal economy with joint production and heterogeneous markups. The distance to the frontier comprises two markup variances. The first term means the product-level markup's Domer-weighted variance gives the distance to the frontier. On the other hand, the second term, which is related to joint production, means that the finite elasticity of transformation σ attenuates the effect of the variance of the markup within the firm. As σ increases and approaches infinity, the force of attenuation associated with joint production approaches zero. Conversely, as σ approaches zero (which implies Leontief production technology), the importance of within-firm markup dispersion decreases. We give this result in the following formal corollary.

Corollary 3 (Limit Cases)**.** *The distance to the frontier simplifies in extreme cases of the elasticity of transformation:*

1. *As $\sigma \to \infty$ (perfect substitution between products):*

$$\mathcal{L} = -\frac{1}{2}\theta \operatorname{Var}_{\lambda}(d \log \mu_{ig}).$$

2. *As $\sigma \to 0$ (Leontief production technology):*

$$\mathcal{L} = -\frac{1}{2}\theta \operatorname{Var}_{\bar{\lambda}}(\mathbb{E}_{s_i}(d \log \mu_i)).$$

In the case of perfect substitutes, misallocation depends on the variance of markups across all products. This term can be obtained by considering the product as an independent firm and applying the results of Baqaee and Farhi (2020). Conversely, in the Leontief case, only the variance of markups between firms is relevant. This is a consequence of the law of total variance.

These results imply that, in the absence of within-firm markup dispersion, the term related to joint production disappears regardless of the value of σ. However, this reasoning is specific to horizontal economies. This relationship easily breaks down in more complex economic structures that include firm-to-firm networks, and σ remains essential even when markups within firms are homogeneous. To illustrate this, we consider a simplified network example examined in Section 1.

6.3.2 Simplified Network Economy

Proposition 9 (The Distance to the Frontier in a Simplified Network Economy)**.**

$$\mathcal{L} = -\frac{1}{2}\tilde{\lambda}_1\tilde{\lambda}_2\left(\frac{\sigma}{\sigma+1}\right)(d \log \mu)^2 \tag{15}$$

Despite no within-firm markup dispersion, σ appears in the loss function, demonstrating that network structure and transformation elasticity jointly determine the distance to the frontier.

Welfare losses decrease as σ increases. We obtain an upper bound on social loss as σ approaches infinity. When σ approaches zero (Leontief case), no misallocation occurs regardless of markup size.

These findings emphasize the importance of considering network structures and transformation elasticities when evaluating misallocation and efficiency in economies with joint production, even without within-firm markup dispersion.

6.4 Application to Chile

7 Conclusion

This paper develops a theoretical framework to aggregate distortions in production networks with multiproduct firms. We assess their impact on aggregate TFP growth and derive a nonparametric sufficient statistic to describe allocative efficiency with multiproduct firms engaging in joint production.

We apply the framework to a comprehensive Chilean firm-to-firm transaction database. Reallocation effects considering joint production explain 86% of the observed aggregate TFP growth. Conversely, ignoring joint production leads to overestimating resource misallocation.

We demonstrate the importance of considering joint production in understanding aggregate TFP dynamics, especially during economic disruptions. The constraints multiproduct firms face in adjusting their product portfolios reduce reallocation within the network that single-product models would predict.

Our analysis reveals that joint production, a previously understudied source of TFP growth, can be of first-order importance. Our results demonstrate how aggregating granular microdata, through the lens of theory, can reduce the measure of our ignorance as captured by aggregate TFP and provide new insights into the drivers of economic growth.

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Appendix for "Aggregating Distortions in Networks with Multi-Product Firms"

A Additional Figures and Tables

A.1 Additional Empirical Results for Reduced-Form Evidence

In this appendix section, we present additional Figures and Table for the event study analysis shown in Section 3.

Validation of COVID-19 Lockdowns as Demand Shocks

This appendix validates our use of Chilean COVID-19 lockdowns in March 2020 as demand shocks to intermediate input transactions. We demonstrate empirically, using firm-level transaction data, that these lockdowns led to substantial reductions in intermediate input purchases.

We posit that intermediate input transactions declined between suppliers in unaffected (gray) counties and buyers in counties that experienced early COVID-19 lockdowns (red). To test this hypothesis, we estimate the following reduced-form specification at the buyer level:

$$\log M_{it} = \beta\, Lockdown_{it} + FE_t + FE_i + \varepsilon_{it}, \tag{16}$$

where M_{it} denotes total intermediate input purchases of a firm i at time t and $Lockdown_{it}$ is a dummy variable equal to one if firm i's location was under lockdown at time t, and is zero otherwise. To address potential bias arising from buyers in lockdown areas who purchased from suppliers in lockdown areas, we restricted the sample by including only buyers with suppliers in non-lockdown areas.

Table A1: Lockdown and intermediate input purchases

	(1)	(2)	(3)
Lockdown Dummy	-0.222***	-0.230***	-0.191***
	(0.0524)	(0.00521)	(0.0589)
Firm FE	Y	Y	Y
Time FE	N	N	Y
Sector × Time FE	N	Y	N
Restricted sample	N	N	Y
Observations	4,345,534	4,345,534	378,646

Notes: The Table reports the results of estimating equation (16) by ordinary least squares (OLS), clustered at the firm-municipality level. The sample periods are January 2019 to March 2020. Columns (1) and (2) report results for the full sample. Column (3) presents the results restricted to firms with no suppliers in the lockdown area. Three stars indicate statistical significance at the 1% level.

The results confirm our hypothesis: The coefficient of interest, β, is negative, indicating that purchases of intermediate inputs from lockdown counties decreased by about 20% on average. This result confirms that we can interpret the decrease in purchases as a negative demand shock to intermediate inputs sold by firms in non-lockdown regions to buyers in lockdown regions.

Characteristics of Treatment Firms

Table A2 displays the characteristics of treated and control firms.

Table A2: Characteristics of treatment firms

	Treatment Firms	Control Firms
Number of firms	26,411	96,321
Number of workers	6	4
Number of products sold	16	10
Number of producers	107	119
Number of buyers	59	26
Annual revenue (million pesos)	186	101
Annual total intermediate purchases (million pesos)	107	59
Share of firms in manufacturing	0.21	0.24
Share of firms in Retail and wholesale	0.44	0.39
Share of firms in Services	0.22	0.21

Notes: This Table presents the characteristics of treated firms (those whose major product buyers experienced lockdowns in March 2020) and control firms, showing values from February 2020, the month before the shock. The rows for the number of workers, products sold, providers, buyers, revenue, and total intermediate purchases display the median of each statistic. The industry shares indicate the proportion of firms within each group that belong to specific industries.

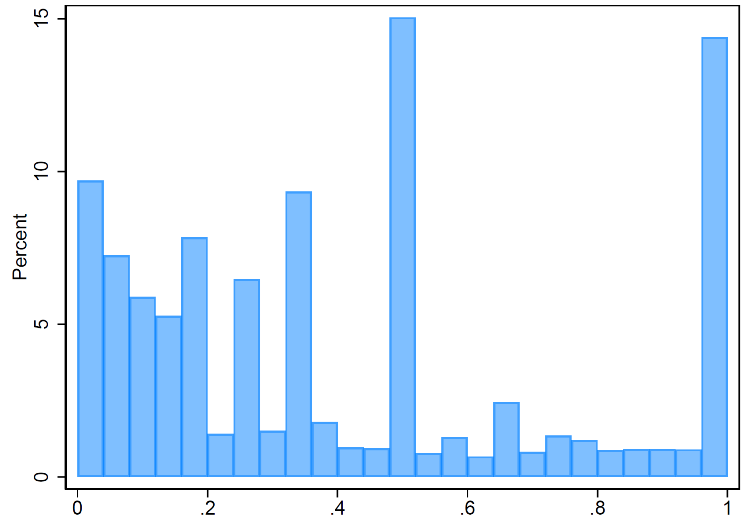
Firms Sell Different Products to Distinct Sets of Buyers

We construct the following measure to characterize the heterogeneity from the intermediate inputs buyer perspective across products and within firms:

$$S_i = \frac{\text{number of buyers of the main product of firm } i}{\text{number of buyers of firm } i},$$

where the main product is the one that has the largest sales within firm i in 2018. Figure A1 presents the distribution of this measure across firms.

Figure A1: Buyer heterogeneity across product within firm



Notes: Histogram of the number of buyers buying the main product of firm i / number of buyers in firm i for a multiproduct firm. The main product is the product with the highest sales within that firm. Data are from 2018.

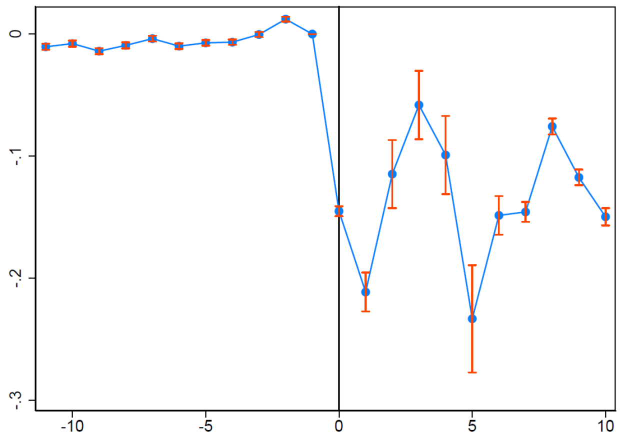
If the buyers of the seller's main product and its other products were the same, S_i would be one. Some mass exists at $S_i = 1$ but for more than 50% of multiproduct firms; however, buyers of the main product constitute less than 50% of the total buyer-firms base. The fact that each product has a distinct set of buyers ensures that we can construct a sample where the main product experiences a demand shock while the other products do not.

Event Study Excluding Small Firms

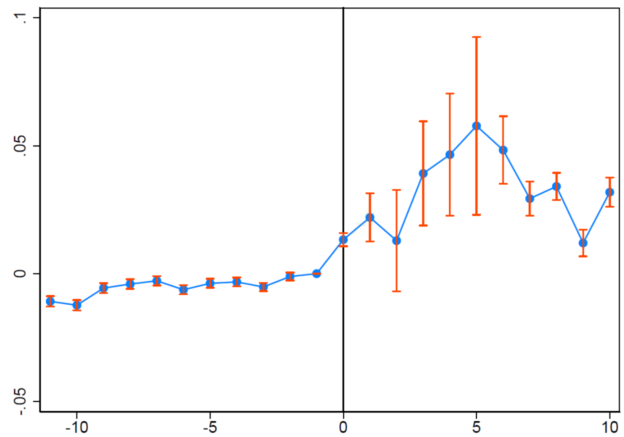
To address potential bias from financially constrained firms in our event study, we conduct a robustness check that focuses on larger firms, which we assume to be relatively less financially constrained. We calculate total firm sales, including both network and final consumer sales, for each year of the study period (2019-2021). We then isolate firms above the 90th percentile of this distribution and replicate our event study analysis from Section 3 using only this subset of larger firms. Figure A2 presents the results of this robustness check.

Figure A2: The effects of demand shocks of the main product on the production of other products within the firm: Robustness check

(a) Log quantity



(b) Log price



Notes: Standard errors are clustered at the firm-county level, and the error bands represent 95% confidence intervals. The X-axis represents the time to treat, with 0 denoting March 2020 when the main product experienced the demand shock. Other values indicate the number of months before or after this event.

The event study results that exclude small firms are similar to the event study that includes all firms, suggesting that financial constraints are not driving the results.

Table A3: Aggregate firm-level statistics

Year	Count	Sales	Wagebill	Employment
2016	110,451	262,506	40,260	4,242,555
2017	114,480	277,960	43,691	4,349,248
2018	115,916	330,486	44,688	4,349,454
2019	116,706	336,386	47,299	4,425,780
2020	102,306	310,317	44,053	3,935,883
2021	105,651	376,220	51,642	4,166,838
2022	105,032	454,818	59,148	4,266,972

Notes: Count stands by the number of firms when sales and wage bills are yearly aggregates expressed in millions of pesos. Employment represents the headcount of yearly workers included in the sample.

Distribution of the Number of Products

We use the 2018 data to describe the main features of the firm-to-firm trade patterns.

Of all firms, 75% produce multiple products, and these firms account for 98.94% of intermediate input transaction value. Table A4 illustrates the distribution of products per firm, weighted by firm-to-firm transaction values.

Table A4: Distribution of product numbers

Percentile	**Number of products (Unweighted)**	**Number of products (Weighted by transaction value)**
1%	1	1
5%	1	2
10%	1	4
25%	2	36
50%	7	475
75%	26	2,459
90%	119	32,195
95%	290	37,422
99%	1,253	62,372

Notes: The Table presents the distribution of product numbers for 2018. The left column shows the number of products without weighting, while the right column displays the number of products weighted by the intermediate product transaction volumes of the firms.

B A CET Cost Function and Lerner Markups

We solve the firm's profit maximization problem in the setting at Section 3 and show that the optimal markup in joint production follows the standard Lerner formula.

Profit Function. The firm's profit is

$$\Pi = \sum_{i=1}^{N} \left[p_i \, q_i \right] - C(q_1, \ldots, q_N).$$

We want to solve for each p_i that maximizes Π, showing that the ratio $p_i/\big(\partial C/\partial q_i\big)$ is given by a standard Lerner markup formula.

We consider how p_i affects profit. Because q_i depends only on p_i, we have

$$\frac{\partial \Pi}{\partial p_i} = \frac{\partial}{\partial p_i}\big[p_i\, q_i(p_i)\big] \;-\; \frac{\partial}{\partial p_i} C(\ldots).$$

The cost $C(\ldots)$ depends on p_i only through q_i, so

$$\frac{\partial}{\partial p_i} C(\ldots) = \frac{\partial C}{\partial q_i}\,\frac{dq_i}{dp_i} \quad \text{(the partials w.r.t. } q_j \text{ for } j \neq i \text{ vanish, since } p_j \text{ does not enter } q_i\text{).}$$

Hence,

$$\frac{\partial}{\partial p_i}\big[p_i\, q_i\big] = q_i + p_i\,\frac{dq_i}{dp_i}.$$

But from $q_i = D_i\, p_i^{-\theta_i}$,

$$\frac{dq_i}{dp_i} = -\theta_i\, D_i\, p_i^{-\theta_i - 1} = -\theta_i\,\frac{q_i}{p_i}.$$

Thus

$$\frac{\partial}{\partial p_i}\big[p_i\, q_i\big] = p_i\Big[-\theta_i\,\frac{q_i}{p_i}\Big] + q_i = (1-\theta_i)\, q_i.$$

Meanwhile,

$$\frac{\partial}{\partial p_i} C(q_1, \ldots, q_N) = mc_i\,\frac{dq_i}{dp_i} = mc_i\Big[-\theta_i\,\frac{q_i}{p_i}\Big].$$

Therefore

$$\frac{\partial \Pi}{\partial p_i} = (1-\theta_i)\, q_i \;-\; mc_i\Big[-\theta_i\,\tfrac{q_i}{p_i}\Big] = (1-\theta_i)\, q_i + \theta_i\,\tfrac{q_i}{p_i}\, mc_i.$$

Setting $\frac{\partial \Pi}{\partial p_i} = 0$ gives

$$(1-\theta_i)\, q_i + \theta_i \frac{q_i}{p_i}\, mc_i = 0.$$

Divide through by $q_i > 0$:

$$(1-\theta_i) + \theta_i\,\frac{mc_i}{p_i} = 0 \quad\Longrightarrow\quad \theta_i\,\frac{mc_i}{p_i} = \theta_i - 1 \quad\Longrightarrow\quad \frac{p_i}{mc_i} = \frac{\theta_i}{\theta_i - 1}.$$

So the optimal price for good i is

$$p_i = \frac{\theta_i}{\theta_i - 1}\, mc_i,$$

matching the usual Lerner markup ratio $\frac{\theta_i}{\theta_i - 1}$, even though the cost function couples all outputs $(q_1, \ldots, q_N)$.

C Downstream Wedge

Table A5: The 30 most distorted products

Ranking	Description
1	Insurance brokerage services
2	Other services
3	Passenger air transport services
4	Wholesale trade intermediary services
5	Electricity distribution and other related services
6	Investigation and security services
7	Airport services
8	Radio and open TV broadcast services
9	Wastewater treatment services
10	Online content services
11	Cleaning services
12	Liquefied Natural Gas
13	Employment services (placement and supply)
14	Postal and courier services
15	Tobacco
16	Paper and cardboard containers, paper or cardboard for recycling
17	Other IT services
18	News agency services
19	Margarine and similar preparations, other residues and waste from fats
20	General insurance
21	Other rubber products
22	Other auxiliary and complementary services for education services
23	Other goods or services not classified elsewhere
24	Long-distance passenger transport services
25	Gas distribution services and other related services
26	Some other product
27	Maritime passenger transport services
28	Research and development services
29	Repair and installation of machinery and equipment, except for the textile industry
30	Database software licensing services

Notes: For 2018, products are ranked using the downstream wedge medians for the product categories, and products with the top 30 downstream markup sizes are reported.

Table A6: The 30 least distorted product

Ranking	Description
1	Molybdenum minerals and their concentrates
2	Other non-metallic minerals
3	Gaseous natural gas
4	Crude oil
5	Mining works
6	Unrefined copper, ashes, residues and wastes of copper
7	Silver
8	Public administration and defense services; compulsory social security plans
9	Pet food
10	Bird food
11	Fish meal, crustacean, mollusk and other aquatic invertebrate meal
12	Ammonium nitrate
13	Lease services with or without purchase option
14	Bread
15	Veterinary services
16	Poultry meat and edible offal
17	Integrated telecommunications services (packs)
18	Fuel oil
19	Beers
20	Life insurance
21	Cakes, cakes and cookies
22	Hake
23	Consultancy and post services
24	Copper minerals and their concentrates
25	Public hospital services
26	Social and association services
27	Petroleum gas and other gaseous hydrocarbons, except natural gas
28	Fish oil
29	Mining exploration and evaluation services
30	Housing services

Notes: For 2018, products are ranked using the downstream wedge medians for the product categories, and products with the top 30 downstream markup sizes are reported.

D Additional Empirical Results for Growth accounting

In this section, we analyze the results obtained using the product-level markup estimation methodology developed by Dhyne et al. (2022). We first delineate the markup estimation procedure and subsequently perform growth accounting analysis employing Proposition 3, consistent with our main specification. Our findings indicate that the results remain largely invariant between our baseline specification and the case accounting for markup heterogeneity, which can be attributed to the fact that the majority of variation in cumulative wedges stems from downstream markups rather than firms' own markup decisions.

D.1 Detailed Methodology for Product-Level Markup Estimation

We estimate product-level markups following a production approach based on Dhyne et al. (2022). In a joint production setup, firms use common inputs to produce a product portfolio, meaning that some inputs can simultaneously be used to produce multiple products. Dhyne et al. (2022) proposed a production function estimation method that is like that of Ackerberg et al. (2015) yet is based on the production set of Diewert (1973).

A firm has a production possibilities set, V, that consists of a set of feasible inputs $\boldsymbol{x} = (x_1, ..., x_M)$ and outputs of the product,$\boldsymbol{q} = (q_1, ..., q_G)$. For any (q_g, x) the transformation function is defined as

$$q_g^* = f_g\left(q_g, x\right) \equiv \max\left\{q_g | \left(q_g, q_{-g}, x\right) \in V\right\}$$

To identify the unobserved marginal cost for each firm's product, we rely on (variable) cost minimization. Firms have $N-1$ freely variable inputs and one fixed input, capital (K), so the problem that a firm faces to minimize its variables cost to produce its output vector q_i^* given the input prices vector $\boldsymbol{p_x} = (p_{x1}, ..., p_{xM})$ and unobserved productivity for products is $\boldsymbol{\omega} = (\omega_1, ..., \omega_G)$.

Defining the Lagrangian multiplier of the cost minimization problem, mc_g, as the marginal cost of product g, the first order condition for every optimal input demand yield is:

$$p_m = mc_g \frac{\partial f(q_{-g}^*, x, K, \boldsymbol{\omega})}{\partial x_m} \;\; \forall m = 1, .., M,$$

It is possible to solve for the product g marginal cost as the expenditure on production input m divided by its output elasticity (β_n^g) times product g production:

$$mc_g = \frac{p_m}{\frac{\partial f(q^*_{-g}, x, K, \omega)}{\partial x_m}} = \frac{p_m x^*_m}{\beta^g_m q^*_g},$$

Multiplying the marginal cost expression by $\frac{1}{p_g}$, where p_g is the product g price, the product g markup is given by:

$$\mu_g = \beta^g_m \frac{p_g q^*_g}{p_m x^*_m},$$

We use control functions for the unobserved productivity terms (i.e., Ackerberg et al. (2015)) to account for unobserved productivity with the difference of the need for instruments for q_{-g}; following Dhyne et al. (2022) we use lagged values of q_{-g}. We assume that firms use a Cobb-Douglas production function with three factors: (Capital K, Labor L, and Materials M). A multi-product firm will produce physical units of product g using the following production function:

$$\log q_{gt} = \beta^g_0 + \beta^g_k \log k_t + \beta^g_l \log l_t + \beta^g_m \log m^j_t + \gamma^g_{-g} \ \log q_{-gt} + \omega_{gt}$$

We pool together products at one digit (12 aggregate product categories) and perform production function estimations separately for each category following ACF using a GMM estimator.

Product-level markup distribution is concentrated around 1, with a 1.22 median.

The cumulative wedge, Γ, is essential in constructing the multiproduct term, and it is vital to understand whether this variation arises from downstream wedges or a product's markup. For this analysis, we use markup estimated by the production function technique.

Table A7: Variance decomposition of $\log \Gamma$

Year	Downstream wedge	Own markup	Covariance
2016	103.3%	0.6%	-3.9%
2017	102.3%	0.7%	-3.0%
2018	102.5%	0.6%	-3.1%
2019	102.8%	0.6%	-3.5%
2020	103.2%	0.7%	-3.8%
2021	103.7%	0.6%	-4.3%
2022	104.9%	0.7%	-5.6%

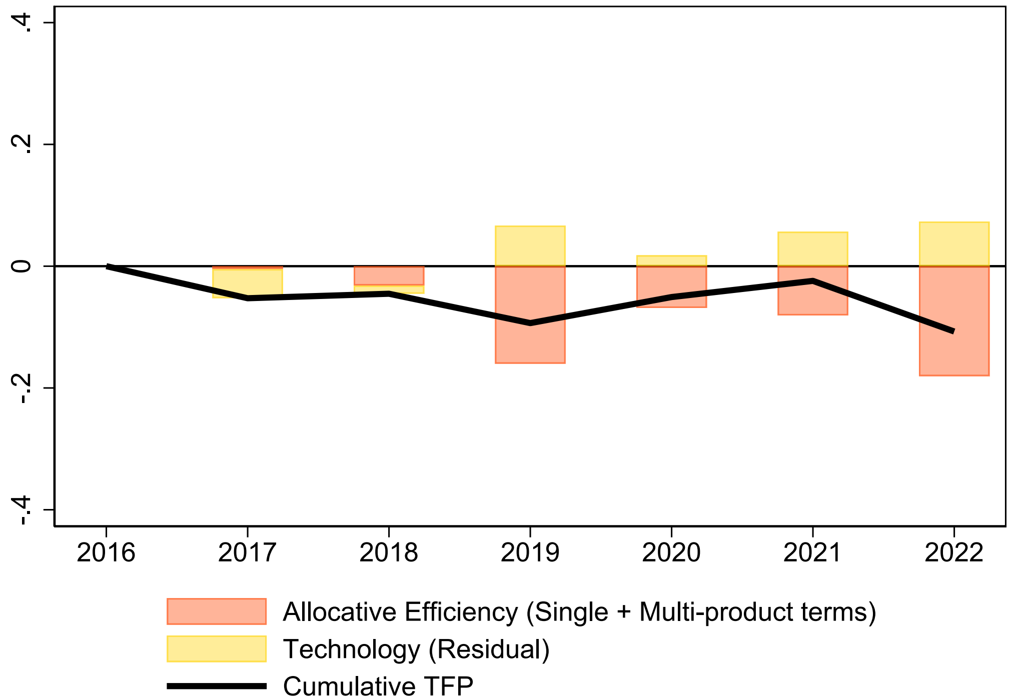
Notes: We compute the variance decomposition of the logarithm of $\Gamma_{ig} \equiv \frac{\tilde{\lambda}_{ig}}{salesshare_{ig}} \times \mu_{ig}$ for each year. $Var\left(\log \Gamma_{ig}\right) = Var\left(\log\left(\tilde{\lambda}_{ig}/salesshare_{ig}\right)\right) + Var\left(\log \mu_{ig}\right) + 2\text{Cov}\left(\log\left(\tilde{\lambda}_{ig}/salesshare_{ig}\right), \log \mu_{ig}\right)$. The first term on the right-hand side is the variance of downstream wedges. The second term is the variance of their own markup, and the last is the covariance of both. We report the percentage of each term on the right-hand side that explains the total variance.

Table A7 presents the variance decomposition of Γ by year. The results show that most variation in Γ stems from downstream distortions, with a minimal contribution from the product's markup. This finding is unsurprising, given that downstream distortions represent cumulative wedges throughout the downstream supply chain of the entire economy. In contrast, μ represents a product's own markup. This result implies that the downstream distortions faced by each pair of firms and products are highly heterogeneous when considering product- and firm-level production networks.

Next, we apply Proposition 3 to the cumulative wedges. We found that

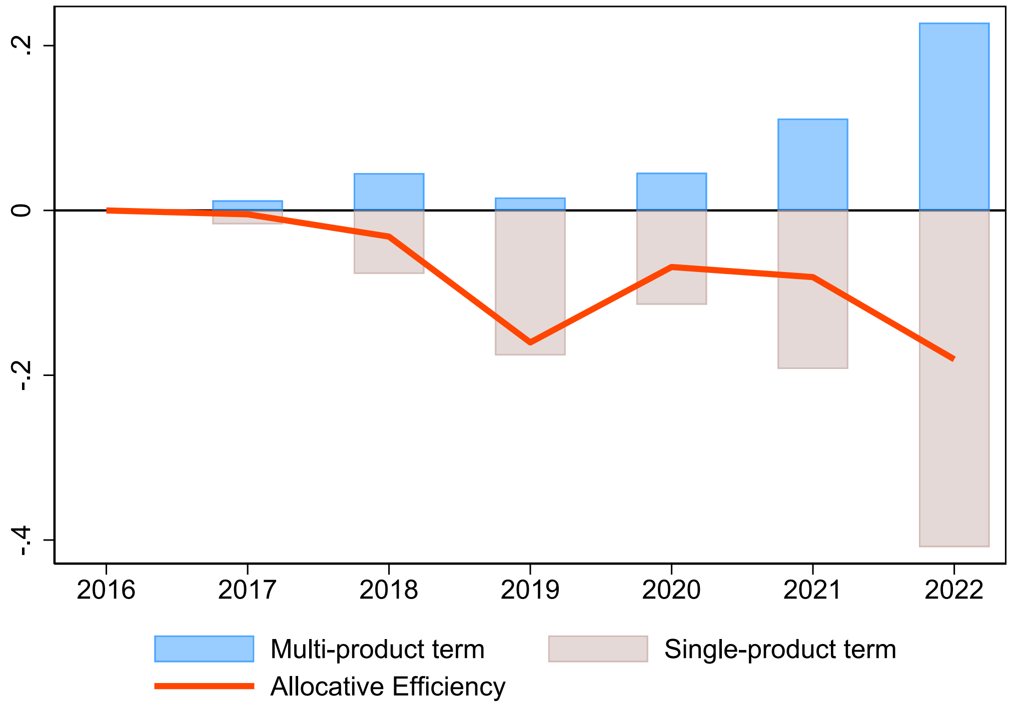
(the multiproduct term using Γ as input) will be less sensitive to markup estimates. In our application, we show the two methods produce almost identical aggregate implications.

Figure A3: Cumulative TFP decomposition with multi-product term



Notes: This Figure shows the cumulative log change calculated by repeatedly applying the equation from Proposition 1 each year. Technology (residual) is calculated by subtracting allocative efficiency from TFP.

Figure A4: Cumulative TFP growth decomposition with multiproduct term



Notes: This Figure decomposes the cumulative change in allocative efficiency in Figure A3 into single-product and multi-product terms.

E Proofs

Proof of Example 1 in Subsection 2.2. This proof uses Proposition 3. First, the multi-product term is zero since that example has no price variation. To compute the endogenous response of the labor share Λ_L to a change in the markup μ_{21}, we begin by expressing Λ_L solely in terms of the cumulative wedges Γ. Recall that the cumulative wedges are defined as:

$$\Gamma_{11} = \mu_{11}\mu_{21}, \quad \Gamma_{12} = \mu_{12},$$

where μ_{ig} denotes the markup of firm i on product g.

The labor share Λ_L can be written as:

$$\Lambda_L = 1 - \tilde{\lambda}_2\left(1 - \frac{1}{\mu_{12}}\right) - \tilde{\lambda}_1\left(1 - \frac{1}{\mu_{21}}\right) - \tilde{\lambda}_1\frac{1}{\mu_{21}}\left(1 - \frac{1}{\mu_{11}}\right).$$

Simplifying, we have:

$$\Lambda_L = \tilde{\lambda}_2 \left(\frac{1}{\mu_{12}}\right) + \tilde{\lambda}_1 \left(\frac{1}{\mu_{21}\mu_{11}}\right).$$

Using the definitions of the cumulative wedges, this becomes:

$$\Lambda_L = \tilde{\lambda}_2 \left(\frac{1}{\Gamma_{12}}\right) + \tilde{\lambda}_1 \left(\frac{1}{\Gamma_{11}}\right).$$

Since Γ_{12} does not depend on μ_{21}, the dependence of Λ_L on μ_{21} is solely through Γ_{11}. Differentiating Λ_L with respect to μ_{21}, we obtain:

$$\frac{d\Lambda_L}{d\mu_{21}} = -\tilde{\lambda}_1 \frac{1}{\Gamma_{11}^2} \mu_{11}.$$

Therefore, the derivative of $\log \Lambda_L$ with respect to $\log \mu_{21}$ is:

$$\frac{d \log \Lambda_L}{d \log \mu_{21}} = \frac{1}{\Lambda_L} \frac{d\Lambda_L}{d\mu_{21}} \mu_{21} = -\frac{\tilde{\lambda}_1 \mu_{11} \mu_{21}}{\Lambda_L \Gamma_{11}^2}.$$

Since $\Gamma_{11} = \mu_{11}\mu_{21}$, we have $\Gamma_{11}^2 = (\mu_{11}\mu_{21})^2$, so the expression simplifies to:

$$\frac{d \log \Lambda_L}{d \log \mu_{21}} = -\frac{\tilde{\lambda}_1}{\Lambda_L \Gamma_{11}}.$$

Substituting the expression for Λ_L in terms of Γ:

$$\Lambda_L = \tilde{\lambda}_2 \left(\frac{1}{\Gamma_{12}}\right) + \tilde{\lambda}_1 \left(\frac{1}{\Gamma_{11}}\right),$$

we find that:

$$\Lambda_L \Gamma_{11} = \tilde{\lambda}_2 \left(\frac{\Gamma_{11}}{\Gamma_{12}}\right) + \tilde{\lambda}_1.$$

Therefore, the derivative simplifies to:

$$\begin{aligned}
\frac{d \log \Lambda_L}{d \log \mu_{21}} &= -\frac{\tilde{\lambda}_1}{\tilde{\lambda}_2 \left(\frac{\Gamma_{11}}{\Gamma_{12}}\right) + \tilde{\lambda}_1}, \\
&= -\frac{\tilde{\lambda}_1 \Gamma_{11}^{-1}}{\tilde{\lambda}_2 \Gamma_{12}^{-1} + \tilde{\lambda}_1 \Gamma_{11}^{-1}},
\end{aligned}$$

$$= -\tilde{\lambda}_1 \frac{\bar{\Gamma}_1}{\Gamma_{11}}.$$

And

$$\sum_i \tilde{\lambda}_i d \log \mu_i = \lambda_1 d \log \mu_{21}.$$

Therefore,

$$d \log TFP = \tilde{\lambda}_{11} \left(\frac{\bar{\Gamma}_1}{\Gamma_{11}} - 1 \right) d \log \mu_{21}.$$

$\square$

Proof of an Example 2 in Subsection 2.2. Let us pick product 2 to be a reference product for firm 1. Then, we obtain

$$d \log p_{11}/p_{12} = d \log \mu_{11}/\mu_{12} + \frac{1}{\sigma} d \log y_{11}/y_{12}.$$

Using the relation $d \log \lambda = d \log p + d \log y$, we derive

$$d \log p_{11}/p_{12} = \left(\frac{\sigma}{\sigma + 1} \right) d \log \mu_{11}/\mu_{12} + \frac{1}{\sigma + 1} d \log \lambda_{11}/\lambda_{12},$$

where $d \log \mu_{11} = 0$ and $d \log \mu_{12} = 0$. From the Cobb-Douglas assumption, we know that $d \log \lambda_{11}/\lambda_{12} = d \log \mu_{21}$. Therefore, we have

$$d \log p_{11}/p_{12} = \left(\frac{1}{\sigma + 1} \right) d \log \mu_{21}.$$

Now, we can write

$$\begin{aligned}
\mathrm{Cov}_{s_i}\left(d \log p_{(i,\cdot)}, \frac{\bar{\Gamma}_i}{\Gamma_{(i,\cdot)}} \right) &= \mathrm{Cov}_{[\tilde{\lambda}_1, \tilde{\lambda}_2]}\left(\begin{bmatrix} d \log p_{11} \\ d \log p_{12} \end{bmatrix}, \begin{bmatrix} \frac{\Gamma_1}{\Gamma_{11}} \\ \frac{\Gamma_1}{\Gamma_{12}} \end{bmatrix} \right) \\
&= \mathrm{Cov}_{[\tilde{\lambda}_1, \tilde{\lambda}_2]}\left(\begin{bmatrix} d \log p_{11}/p_{12} \\ d \log p_{12}/p_{12} \end{bmatrix}, \begin{bmatrix} \frac{\Gamma_1}{\Gamma_{11}} \\ \frac{\Gamma_1}{\Gamma_{12}} \end{bmatrix} \right) \\
&= \mathrm{Cov}_{[\tilde{\lambda}_1, \tilde{\lambda}_2]}\left(\begin{bmatrix} \left(\frac{1}{\sigma+1} \right) d \log \mu_{21} \\ 0 \end{bmatrix}, \begin{bmatrix} \frac{\Gamma_1}{\Gamma_{11}} \\ \frac{\Gamma_1}{\Gamma_{12}} \end{bmatrix} \right).
\end{aligned}$$

Observe that

$$\mathbb{E}_{[\tilde{\lambda}_1,\tilde{\lambda}_2]}\begin{pmatrix} \left(\frac{1}{\sigma+1}\right) d\log\mu_{21}\frac{\Gamma_1}{\Gamma_{11}} \\ 0 \end{pmatrix} = \left(\frac{1}{\sigma+1}\right) d\log\mu_{21}\frac{\Gamma_1}{\Gamma_{11}}\tilde{\lambda}_{11},$$

and

$$\begin{aligned}\mathbb{E}_{[\tilde{\lambda}_1,\tilde{\lambda}_2]}\begin{pmatrix} d\log p_{11}/p_{12} \\ d\log p_{12}/p_{12} \end{pmatrix}\mathbb{E}_{[\tilde{\lambda}_1,\tilde{\lambda}_2]}\begin{pmatrix} \frac{\Gamma_1}{\Gamma_{11}} \\ \frac{\Gamma_1}{\Gamma_{12}} \end{pmatrix} &= \left(\frac{1}{\sigma+1}\right) d\log\mu_{21}\tilde{\lambda}_1\left(\tilde{\lambda}_1\frac{\Gamma_1}{\Gamma_{11}}+\tilde{\lambda}_2\frac{\Gamma_1}{\Gamma_{12}}\right) \\ &= \left(\frac{1}{\sigma+1}\right) d\log\mu_{21}\tilde{\lambda}_1.\end{aligned}$$

Therefore, the multi-product term is given by

$$\mathrm{Cov}_{s_i}\left(d\log p_{(i,\cdot)},\frac{\bar{\Gamma}_i}{\Gamma_{(i,\cdot)}}\right) = -\left(\frac{1}{\sigma+1}\right)\tilde{\lambda}_{11}\left(\frac{\bar{\Gamma}_1}{\Gamma_{11}}-1\right) d\log\mu_{21}.$$

Using the single product term's result,

$$\underbrace{-\sum_{i\in\mathcal{N}}\tilde{\lambda}_i d\log\mu_i}_{\text{Firm-level Markup}} - \underbrace{d\log\Lambda_f}_{\text{Aggregate Labor Shares}} = \tilde{\lambda}_{11}\left(\frac{\bar{\Gamma}_1}{\Gamma_{11}}-1\right) d\log\mu_{21}.$$

Finally, we obtain

$$\Delta TFP = \left(1-\frac{1}{\sigma+1}\right)\tilde{\lambda}_{11}\left(\frac{\bar{\Gamma}_1}{\Gamma_{11}}-1\right) d\log\mu_{21}.$$

□

Proof of an Example with Taste Shocks in Subsection 2.2.4: The proof follows similar steps to the markup shock case. Let us first calculate how the labor share responds to taste shocks. Starting from the labor share expression:

$$\Lambda_L = \tilde{\lambda}_2\left(\frac{1}{\Gamma_{12}}\right) + \tilde{\lambda}_1\left(\frac{1}{\Gamma_{11}}\right) = \frac{1}{\bar{\Gamma}_1},$$

differentiating with respect to $\tilde{\lambda}_1$ and using the fact that $d\tilde{\lambda}_2 = -d\tilde{\lambda}_1$, we obtain:

$$d\log\Lambda_L = d\log\tilde{\lambda}_1\left(1 - \frac{1}{\tilde{\lambda}_2} + \frac{\tilde{\lambda}_1}{\tilde{\lambda}_2}\frac{\bar{\Gamma}_1}{\Gamma_{11}}\right) = d\log\tilde{\lambda}_1\frac{\tilde{\lambda}_1}{\tilde{\lambda}_2}\left(\frac{\bar{\Gamma}_1}{\Gamma_{11}} - 1\right)$$

Next, to express the multi-product term as a covariance, we write:

$$\begin{aligned}
\mathrm{Cov}_{s_1}\left(d\log p_{(1,\cdot)}, \frac{\bar{\Gamma}_1}{\Gamma_{(1,\cdot)}}\right) &= \mathrm{Cov}_{[\tilde{\lambda}_1,\tilde{\lambda}_2]}\left(\begin{bmatrix} d\log p_{11}/p_{12} \\ d\log p_{12}/p_{12} \end{bmatrix}, \begin{bmatrix} \frac{\bar{\Gamma}_1}{\Gamma_{11}} \\ \frac{\bar{\Gamma}_1}{\Gamma_{12}} \end{bmatrix}\right) \\
&= \mathrm{Cov}_{[\tilde{\lambda}_1,\tilde{\lambda}_2]}\left(\begin{bmatrix} \frac{1}{\sigma+1}\frac{1}{\tilde{\lambda}_2}d\log\tilde{\lambda}_1 \\ 0 \end{bmatrix}, \begin{bmatrix} \frac{\bar{\Gamma}_1}{\Gamma_{11}} \\ \frac{\bar{\Gamma}_1}{\Gamma_{12}} \end{bmatrix}\right) \\
&= -\frac{1}{\sigma+1}\tilde{\lambda}_1\left(\frac{\bar{\Gamma}_1}{\Gamma_{11}} - 1\right)d\log p_{11}/p_{12}
\end{aligned}$$

Combining these expressions with equation (5) yields the result. □

Proof of Proposition 3

Lemma 2. *The price equation with multi-product firms for some reference product r of firm i:*

$$\frac{y_{ir}mc_{ir}}{C(y_i,p)}d\log p_{ir} = -d\log A_i/\mu_i + \underbrace{\sum_{j,k}\frac{p_{jg}x_{i,jg'}}{C(y_i,p)}d\log p_{jg'} + \sum_f \frac{w_f l_{if}}{C(y_i,p)}d\log w_f}_{\text{intermediate and factorprice}}$$

$$+ \underbrace{\sum_{g\neq r}\left(-\frac{y_{ig}mc_{ig}}{C(y_i,p)}\right)d\log p_{ig}}_{\text{other product from the same firm}},$$

Proof. By definition, we know

$$C_i(\boldsymbol{q}_i, \boldsymbol{p}_i) = \sum_g q_{ig}mc_{ig}.$$

Total derivative:

$$RHS = \sum_g \frac{\partial \log\left(\sum q_{ig} mc_{ig}\right)}{\partial \log y_i} y_i d\log y_i + \sum_i \frac{\partial \log\left(\sum q_{ig} mc_{ig}\right)}{\partial \log mc_{ig}} mc_{ig} d\log mc_{ig},$$
$$= \sum_g \frac{q_{ig} mc_{ig}}{C_i(\boldsymbol{q}_i, \boldsymbol{p}_i)} d\log q_{ig} + \sum_g \frac{q_{ig} mc_{ig}}{C_i(\boldsymbol{q}_i, \boldsymbol{p}_i)} d\log mc_{ig}.$$

and

$$LHS = -d\log A_i + \sum_{j,g'} \frac{p_{jg'} x_{i,jg'}}{C_i(\boldsymbol{q}_i, \boldsymbol{p}_i)} d\log p_{jg'} + \sum_f \frac{w_f l_{i,f}}{C_i(\boldsymbol{q}_i, \boldsymbol{p}_i)} d\log w_f + \sum_g \frac{q_{ig} mc_{ig}}{C_i(\boldsymbol{q}_i, \boldsymbol{p}_i)} d\log q_{ig}.$$

Hence,

$$\sum_g \frac{y_{ig} mc_{ig}}{C_i(\boldsymbol{q}_i, \boldsymbol{p}_i)} d\log mc_{ig} = -d\log A_i + \sum_{j,g'} \frac{p_{jg'} x_{i,jg'}}{C_i(\boldsymbol{q}_i, \boldsymbol{p}_i)} d\log p_{jg'} + \sum_f \frac{w_f L_{if}}{C_i(\boldsymbol{q}_i, \boldsymbol{p}_i)} d\log w_f. \tag{17}$$

□

Pick some reference product r of firm i. Following Hall (1973), as a concequence of cost minimization, the following condition holds:

$$\frac{mc_{ig}}{mc_{ir}} = \frac{\partial F_i^Q(\boldsymbol{q}) / \partial q_{ig}}{\partial F_i^Q(\boldsymbol{q}) / \partial q_{ir}},$$

By taking the log difference and adjusting it with the markup, we get the following formula:

$$d\log\left(p_{ig}/p_{ir}\right) = d\log\left(\mu_{ig}/\mu_{ir}\right) + d\log\left(\frac{\partial F^Q(\boldsymbol{q}) / \partial q_{ig}}{\partial F^Q(\boldsymbol{q}) / \partial q_{ir}}\right). \tag{18}$$

This pins down the equilibrium prices with equation 11. For later proof, we define the RHS of the equation 18 as Θ_{ig}.

$$d\log p_{ig}/p_{ir} = \Theta_{ig}. \tag{19}$$

Then, we proceed to the main proof.

Proof. From Lemma 2 We know for one reference product r:

$$\frac{q_{ir}mc_{ir}}{C_i(\boldsymbol{q}_i,\boldsymbol{p}_i)}d\log mc_{ir} = -\,d\log A_i + \sum_{j,p}\frac{p_{jp}x_{jp}}{C_i(\boldsymbol{q}_i,\boldsymbol{p}_i)}d\log p_{jp} + \sum_f \frac{w_f L_{if}}{C_i(\boldsymbol{q}_i,\boldsymbol{p}_i)}d\log w_f + \sum_{g\neq r}\left(-\frac{q_{ig}mc_{ig}}{C_i(\boldsymbol{q}_i,\boldsymbol{p}_i)}\right)d\log mc_{ig},$$

$$\begin{aligned}\frac{q_{ir}mc_{ir}}{C_i(\boldsymbol{q}_i,\boldsymbol{p}_i)}d\log p_{ir} &= -\,d\log A_i + d\log\mu_i + \sum_{j,p}\frac{p_{jp}x_{i,jp}}{C_i(\boldsymbol{q}_i,\boldsymbol{p}_i)}d\log p_{jp} + \sum_f \frac{w_f L_{if}}{C_i(\boldsymbol{q}_i,\boldsymbol{p}_i)}d\log w_f \\ &\quad + \sum_{g\neq r}\left(-\frac{q_{ig}mc_{ig}}{C_i(\boldsymbol{q}_i,\boldsymbol{p}_i)}\right)d\log p_{ig}.\end{aligned}$$

From equation 19, we have

$$d\log p_{ig}/p_{ir} = \Theta_{ig}.$$

Combining 17 with 19 yields

$$\begin{aligned}\frac{q_{ir}mc_{ir}}{C_i(\boldsymbol{q}_i,\boldsymbol{p}_i)}d\log p_{ir} &= -d\log A_i + \sum_g \frac{q_{ig}mc_{ig}}{C_i(\boldsymbol{q}_i,\boldsymbol{p}_i)}d\log\mu_{ig} + \sum_{j,p}\frac{p_{jp}x_{jp}}{C_i(\boldsymbol{q}_i,\boldsymbol{p}_i)}d\log p_{jp} + \sum_f \frac{w_f l_{if}}{C_i(\boldsymbol{q}_i,\boldsymbol{p}_i)}d\log w_f \\ &\quad + \sum\left(-\frac{q_{ig}mc_{ig}}{C_i(\boldsymbol{q}_i,\boldsymbol{p}_i)}\right)d\log p_{ig}, \\ \frac{q_{ir}mc_{ir}}{C_i(\boldsymbol{q}_i,\boldsymbol{p}_i)}d\log p_{ir} &= -d\log A_i + d\log\mu_i + \sum_{j,p}\tilde{\Omega}_{ig,jp}d\log p_{jp} + \sum_f \tilde{\Omega}_{ig,f}d\log w_f + \sum_{g\neq r}\left(-\frac{q_{ig}mc_{ig}}{C_i(\boldsymbol{q}_i,\boldsymbol{p}_i)}\right)\left[d\log p_{ir} + \Theta_{ig}\right], \\ d\log p_{ir} &= -d\log A_i + d\log\mu_i + \sum_{j,p}\tilde{\Omega}_{ig,jp}d\log p_{jp} + \sum_f \tilde{\Omega}_{ig,f}d\log w_f + \sum_{g\neq r}\left(-\frac{q_{ig}mc_{ig}}{C_i(\boldsymbol{q}_i,\boldsymbol{p}_i)}\right)\Theta_{ig}.\end{aligned}$$

Because $\Theta_{ig} = 0$ if g is n reference product, the price equations could be written as

$$d\log p_{ig} = -d\log A_i + d\log\mu_i + \sum_{j,g'}\tilde{\Omega}_{ig,jg'}d\log p_{jg'} + \sum_f \tilde{\Omega}_{ig,f}d\log w_f + \left\{\mathbb{I}_i(g) - \sum_{g\neq r}\left(\frac{q_{ig}mc_{ig}}{C_i(\boldsymbol{q}_i,\boldsymbol{p}_i)}\right)\right\}\Theta_{ig}.$$

In vector notation

$$d\log\boldsymbol{p} = (I - \tilde{\Omega}^{\mathcal{NG}\times\mathcal{NG}})^{-1}\left\{-d\log\boldsymbol{A}^{\mathcal{NG}\times 1} + d\log\boldsymbol{\mu}^{\mathcal{NG}\times 1} + \tilde{\Omega}_f^{\mathcal{NG}\times\mathcal{F}}d\log\boldsymbol{w} + (\mathbf{1}-\mathbf{C})\circ\boldsymbol{\Theta}^{\mathcal{NG}\times 1}\right\},$$

where $\circ$ represents the Hadamard product and and C is a vector of $\mathcal{NG}\times 1$, with the following C_i common elements for firm $i \in \mathcal{N}$,

$$C_i = \sum_{g \neq r} \left(\frac{q_{ig} mc_{ig}}{C\left(\boldsymbol{q}_i, \boldsymbol{p}_i\right)} \right).$$

We know

$$d \log Y = -\boldsymbol{b}' d \log \boldsymbol{p}.$$

$$\begin{aligned} d \log Y &= -\boldsymbol{b}' \tilde{\Psi}^{\mathcal{NG} \times \mathcal{NG}} \left\{ -d \log \boldsymbol{A} + d \log \boldsymbol{\mu} + \tilde{\Omega}_f d \log \boldsymbol{w} + (\mathbf{1} - \mathbf{C}) \circ \boldsymbol{\Theta}^{\mathcal{NG} \times 1} \right\}, \\ &= -\tilde{\boldsymbol{\lambda}}' \left\{ -d \log \boldsymbol{A} + d \log \boldsymbol{\mu} + \tilde{\Omega}_f d \log \boldsymbol{w} + (\mathbf{1} - \mathbf{C}) \circ \boldsymbol{\Theta}^{\mathcal{NG} \times 1} \right\}. \end{aligned}$$

subtracting $\sum_f \tilde{\Lambda}_f d \log L_f$ from both sides yields

$$\begin{aligned} d \log TFP = &\sum_{i \in \mathcal{N}} \tilde{\lambda}_i d \log A_i - \sum_{i \in \mathcal{N}} \tilde{\lambda}_i d \log \mu_i - \sum_{f \in \mathcal{F}} \tilde{\Lambda}_f d \log \Lambda_f, \\ &- \sum_i \left(\sum_{g \in \mathcal{G}} \tilde{\lambda}_{ig} d \log p_{ig} / p_{ir} - \sum_{g \neq r} \frac{q_{ig} mc_{ig}}{C_i\left(\boldsymbol{q}_i, \boldsymbol{p}_i\right)} \tilde{\lambda}_i d \log p_{ig} / p_{ir} \right). \end{aligned}$$

$$\begin{aligned} \left(\sum_g \tilde{\lambda}_{ig} d \log p_{ig} / p_{ir} - \sum_{g \neq r} \frac{q_{ig} mc_{ig}}{C_i\left(y, p\right)} \tilde{\lambda}_i d \log p_{ig} / p_{ir} \right) &= \tilde{\lambda}_i \left(\sum_{g \in \mathcal{G}} s_{ig} d \log p_{ig} / p_{ir} - \sum_{g \neq r} c_{ig} d \log p_{ig} / p_{ir} \right), \\ &= \tilde{\lambda}_i \left(\sum_{g \in \mathcal{G}} s_{ig} d \log p_{ig} / p_{ir} - \sum_{g \in \mathcal{G}} c_{ig} d \log p_{ig} / p_{ir} \right), \\ &= \tilde{\lambda}_i \left(\sum_{g \in \mathcal{G}} \left(s_{ig} - c_{ig} \right) d \log p_{ig} \right), \\ &= \tilde{\lambda}_i \left(\sum_{g \in \mathcal{G}} \left(s_{ig} - \frac{\frac{q_{ig} mc_{ig}}{C(y,p)}}{s_{ig}} s_{ig} \right) d \log p_{ig} \right), \\ &= \tilde{\lambda}_i \left(\sum_{g \in \mathcal{G}} \left(s_{ig} - \frac{\bar{\Gamma}_i}{\Gamma_{ig}} s_{ig} \right) d \log p_{ig} \right), \\ &= \tilde{\lambda}_i \left(\mathbb{E}_{\boldsymbol{s}_i} \left[d \log p_{(i,\cdot)} \right] \mathbb{E}_{\boldsymbol{s}_i} \left[\frac{\bar{\Gamma}_i}{\Gamma_{(i,\cdot)}} \right] - \mathbb{E}_{\boldsymbol{s}_i} \left[d \log p_{(i,\cdot)}, \frac{\bar{\Gamma}_i}{\Gamma_{(i,\cdot)}} \right] \right), \end{aligned}$$

$$= -\tilde{\lambda}_i \mathrm{Cov}_{s_i}\left(d\log p_{(i,\cdot)}, \frac{\bar{\Gamma}_i}{\Gamma_{(i,\cdot)}}\right).$$

where $c_{ig} = \frac{q_{ig} mc_{ig}}{C(\boldsymbol{q}_i, \boldsymbol{p}_i)}$.

□

Proof of Proposition 4. Let $i = 1, \ldots, N$ index N products. Each product i faces the isoelastic demand

$$q_i \;=\; D_i\, p_i^{-\theta_i}, \quad \theta_i > 1,$$

and its price satisfies $p_i = \mu_i \frac{\partial C}{\partial q_i}$, where $\mu_i > 0$ is a wedge (e.g. a markup). The cost function

$$C(q_1, \ldots, q_N) \;=\; \frac{w}{A}\Big(\sum_{j=1}^{N} q_j^{\frac{\sigma+1}{\sigma}}\Big)^{\frac{\sigma}{\sigma+1}}, \quad \sigma > 0,$$

implies that

$$d\log p_i \;=\; d\log\Big[\mu_i \frac{\partial C}{\partial q_i}\Big] \;=\; d\log\Big[\frac{\partial C}{\partial q_i}\Big] \quad (\mu_i \text{ fixed}).$$

To derive this marginal-cost term explicitly, define

$$S_j \;:=\; \frac{q_j^{(\sigma+1)/\sigma}}{\sum_{\ell=1}^{N} q_\ell^{(\sigma+1)/\sigma}}, \quad \sum_{j=1}^{N} S_j \;=\; 1,$$

and let

$$\overline{d\log q} \;:=\; \sum_{j=1}^{N} S_j\, d\log q_j.$$

A standard differentiation of the CET cost function shows that

$$d\log p_i \;=\; \frac{1}{\sigma}\Big[d\log q_i \;-\; \overline{d\log q}\Big]. \tag{20}$$

Since the shock is exclusively to product k, we have $d\log D_k < 0$ and $d\log D_i = 0$ for $i \neq k$. From the isoelastic demand,

$$d\log q_i \;+\; \theta_i\, d\log p_i = d\log D_i,$$

so for $i \neq k$,

$$d\log q_i + \theta_i\, d\log p_i \;=\; 0, \quad \text{and for } i = k, \quad d\log q_k + \theta_k\, d\log p_k \;=\; d\log D_k < 0.$$

Combining $d \log q_i + \theta_i \, d \log p_i = 0$ with (20) gives

$$\theta_i \Big[\tfrac{1}{\sigma} \Big(d \log q_i - \overline{d \log q} \Big) \Big] \;=\; -\, d \log q_i \quad (i \neq k),$$

which simplifies to

$$(\theta_i + \sigma) \, d \log q_i \;=\; \theta_i \, \overline{d \log q} \quad (i \neq k).$$

An analogous expression arises for $i = k$, except that the right-hand side involves $d \log D_k$:

$$(\theta_k + \sigma) \, d \log q_k \;=\; \sigma \, d \log D_k \;+\; \theta_k \, \overline{d \log q}.$$

Solving these equations jointly forces $\overline{d \log q}$ to have the same sign as $d \log D_k$. In particular, since $d \log D_k < 0$ by assumption, one can verify that the unique solution consistent with marginal-cost equality implies

$$\overline{d \log q} \;<\; 0.$$

Then for each $i \neq k$, the equation

$$(\theta_i + \sigma) \, d \log q_i \;=\; \theta_i \, \overline{d \log q}$$

implies $d \log q_i < 0$ (because both $\theta_i + \sigma > 0$ and $\theta_i > 0$). This confirms part (i) of the proposition.

Finally, we substitute $d \log q_i < 0$ and $\overline{d \log q} < 0$ into (20) to find

$$d \log p_i \;=\; \frac{1}{\sigma} \Big[d \log q_i - \overline{d \log q} \Big].$$

Since $| d \log q_i | < | \overline{d \log q} |$ for $i \neq k$ but $d \log q_i$ and $\overline{d \log q}$ are both negative, the difference in brackets is strictly positive, and hence $d \log p_i > 0$. This establishes part (ii).

Thus for each product $g \neq k$, we have $d \log q_g < 0$ and $d \log p_g > 0$ when $d \log D_k < 0$. □

Proof of Proposition 7. From the resource constraint,

$$q_{ig} = y_{ig} + \sum_{j \in N} x_{jig}.$$

$$d\log y_{ig} = \frac{q_{ig}}{y_{ig}} d\log q_{ig} - \sum_j \frac{x_{jig}}{y_{ig}} d\log x_{jig}.$$

From the cost minimization of joint production we know

$$\sum_g q_{ig} mc_{ig} d\log q_{ig} = \sum_{jp} x_{i,jp} p_{jp} d\log x_{i,jp} + \sum_f w_f x_{i,f} d\log L_{if}.$$

$$\sum_{jp} \frac{x_{i,jp} p_{jp}}{GDP} d\log x_{i,jp} = \sum_g \frac{1}{\mu_{ig}} \frac{q_{ig} p_{ig}}{GDP} d\log q_{ig} - \sum_f \frac{w_f x_{i,f}}{GDP} d\log L_{if}.$$

By cost minimization assumption,

$$\begin{aligned}
d\log Y &= \sum_{ig} \frac{p_{ig} y_{ig}}{GDP} d\log y_{ig} - \sum_f \frac{w_f x_{i,f}}{q_{ir} mc_{ig}} d\log L_f, \\
&= \sum_{ig} \frac{p_{ig} y_{ig}}{GDP} \left(\frac{q_{ir}}{y_{ig}} d\log q_{ig} - \sum_j \frac{x_{jig}}{y_{ig}} d\log x_{jig} \right) - \sum_f \frac{w_f L_f}{GDP} d\log L_f, \\
&= \sum_{ig} \frac{p_{ig} y_{ig}}{GDP} \frac{q_{ir}}{y_{ig}} d\log q_{ig} - \sum_{ig} \sum_j \frac{p_{ig} y_{ig}}{GDP} \frac{x_{jig}}{y_{ig}} d\log x_{jig} - \sum_f \frac{w_f L_f}{GDP} d\log L_f, \\
&= \sum_{ig} \frac{p_{ig} q_{ig}}{GDP} d\log q_{ig} - \sum_{ig} \frac{1}{\mu_{ig}} \frac{p_{ig} q_{ig}}{GDP} d\log q_{ig} + \sum_{if} \frac{w_f x_{i,f}}{GDP} d\log L_{if} - \sum_f \frac{w_f L_f}{GDP} d\log L_f, \\
&= \sum_{ig} \lambda_{ig} \left(1 - \mu_{ig}^{-1}\right) d\log q_{ig}.
\end{aligned}$$

Therefore,

$$\begin{aligned}
\frac{\partial \log Y}{\partial \log \mu_{ig'}} &= \sum_{ig} \lambda_{ig} \left(1 - \mu_{ig}^{-1}\right) \frac{d\log q_{ig}}{d\log \mu_{jg'}}, \\
\frac{\partial \log Y}{\partial \log \mu_{ig} \partial \log \mu_{jg'}} &= \sum_{ig,jg'} \lambda_{ig} d\log \mu_{ig} \frac{d\log q_{ig}}{d\log \mu_{jg'}}.
\end{aligned}$$

$$\begin{aligned}
\frac{\partial \log Y}{\partial \log \mu_{ig} \partial \log \mu_{jg'}} &= \sum_{ig} \sum_{jg'} \lambda_{ig} d\log \mu_{ig} d\log \mu_{jg'} \frac{d\log q_{ig}}{d\log \mu_{jg'}}, \\
&= \sum_{ig} \lambda_{ig} d\log q_{ig} d\log \mu_{ig}.
\end{aligned}$$

□

Proof of Proposition 8. Define the firm-level Domar weight:

$$\lambda_i = \sum_g \lambda_{ig}.$$

Let $d \log \mu_{ig}$ denote the exogenous change in the markup for product g of firm i. The aggregate markup change for firm i is:

$$d \log \mu_i = \frac{1}{\lambda_i} \sum_g \lambda_{ig} \, d \log \mu_{ig}.$$

The weighted average markup change is:

$$d \bar{\log} \mu = \sum_i \lambda_i \, d \log \mu_i = \sum_i \sum_g \lambda_{ig} \, d \log \mu_{ig}.$$

Let $d \log p_{ig}$ be the change in the price of product g by firm i, and let $d \log w$ be the change in the wage rate.

The aggregate price index change is:

$$d \log p = \sum_i \sum_g \lambda_{ig} \, d \log p_{ig}.$$

The price change for product g of firm i is:

$$d \log p_{ig} = d \log \mu_i + d \log w + \frac{\kappa}{\lambda_i} \left(d \log \mu_{ig} - d \log \mu_i \right),$$

where

$$\kappa = \frac{\sigma}{\sigma + 1}.$$

Using the zero-profit condition and labor market clearing, we have:

$$d \log w = -d \bar{\log} \mu.$$

Therefore, the aggregate price index change simplifies to:

$$d \log p = \sum_i \lambda_i \, d \log \mu_i - d \bar{\log} \mu = 0,$$

so $d\log p = 0$.

The change in the Domar weight of product g is:

$$d\log\lambda_{ig} = (1-\theta)\, d\log p_{ig}.$$

The welfare change is:

$$\mathcal{L} = \frac{1}{2}\sum_i \sum_g \lambda_{ig}\, d\log q_{ig}\, d\log\mu_{ig} = -\frac{1}{2}\theta \sum_i \sum_g \lambda_{ig}\, d\log p_{ig}\, d\log\mu_{ig}.$$

Our goal is to express $\mathcal{L}$ in terms of $d\log\mu_{ig}$. Using $d\log w = -d\,\bar{\log}\,\mu$, we have:

$$d\log p_{ig} = (d\log\mu_i - d\,\bar{\log}\,\mu) + \frac{\kappa}{\lambda_i}\left(d\log\mu_{ig} - d\log\mu_i\right).$$

Let

$$d\,\tilde{\log}\,\mu_i = d\log\mu_i - d\,\bar{\log}\,\mu,$$

and

$$d\,\tilde{\log}\,\mu_{ig} = d\log\mu_{ig} - d\log\mu_i,$$

then:

$$d\log p_{ig} = d\,\tilde{\log}\,\mu_i + \frac{\kappa}{\lambda_i} d\,\tilde{\log}\,\mu_{ig}.$$

Substituting into the welfare change:

$$\mathcal{L} = -\frac{1}{2}\theta \sum_i \sum_g \lambda_{ig}\left(d\,\tilde{\log}\,\mu_i + \frac{\kappa}{\lambda_i} d\,\tilde{\log}\,\mu_{ig}\right) d\log\mu_{ig}.$$

This simplifies to:

$$\mathcal{L} = -\frac{1}{2}\theta\left(\sum_i \lambda_i d\,\tilde{\log}\,\mu_i d\log\mu_i + \kappa \sum_i \frac{1}{\lambda_i}\sum_g \lambda_{ig} d\,\tilde{\log}\,\mu_{ig}^2\right).$$

Given the definitions of total variance and conditional variance, the welfare change becomes:

$$\mathcal{L} = -\frac{1}{2}\theta\left(\mathrm{Var}_{\bar{\lambda}}(d\log\mu_i) + \frac{\sigma}{\sigma+1}\sum_i \lambda_i\, \mathrm{Var}_{s_i}(d\log\mu_{ig})\right).$$

From the law of total variance we know

$$\mathrm{Var}_\lambda(d\log\mu_{ig}) = \mathrm{Var}_\lambda(d\log\mu_i) + \sum_i \lambda_i \,\mathrm{Var}_{s_i}(d\log\mu_{ig}).$$

Using this, we obtain

$$\mathcal{L} = -\frac{1}{2}\theta\left(\mathrm{Var}_\lambda(d\log\mu_{ig}) - \frac{1}{\sigma+1}\sum_i \lambda_i \,\mathrm{Var}_{s_i}(d\log\mu_{ig})\right).$$

□

Proof of Proposition 9. By factor share identity and given $\tilde{\lambda}_1 + \tilde{\lambda}_2 = 1$, we have:

$$d\log w = -\left(1+\tilde{\lambda}_1\right) d\log\mu.$$

Under the Cobb-Douglas specification:

$$\begin{aligned} d\log\lambda_{11} &= -d\log\mu \\ d\log\lambda_{21} &= 0 \\ d\log\lambda_{12} &= 0. \end{aligned}$$

The relative price adjustment equation and the wage equation give us:

$$\begin{aligned} d\log p_{11} - d\log p_{12} &= -\frac{1}{\sigma+1} d\log\mu \\ \tilde{\lambda}_1 d\log p_{11} + \tilde{\lambda}_2 d\log p_{12} &= -\tilde{\lambda}_1 d\log\mu. \end{aligned}$$

From the second equation:

$$\begin{aligned} \tilde{\lambda}_1\left(d\log p_{12} - \frac{1}{\sigma+1} d\log\mu\right) + \tilde{\lambda}_2 d\log p_{12} &= -\tilde{\lambda}_1 d\log\mu \\ d\log p_{12} - \tilde{\lambda}_1 \frac{1}{\sigma+1} d\log\mu &= -\tilde{\lambda}_1 d\log\mu \\ d\log p_{12} &= -\frac{\sigma}{\sigma+1}\tilde{\lambda}_1 d\log\mu. \end{aligned}$$

Using the relative price adjustment equation:

$$d\log p_{11} = d\log p_{12} - \frac{1}{\sigma+1} d\log\mu$$

$$= -\frac{\sigma}{\sigma+1}\tilde{\lambda}_1 d\log\mu - \frac{1}{\sigma+1}d\log\mu$$
$$= -\left(\frac{1+\sigma\tilde{\lambda}_1}{\sigma+1}\right)d\log\mu.$$

For $d\log p_{21}$:

$$\begin{aligned} d\log p_{21} &= d\log p_{11} + d\log\mu \\ &= -\left(\frac{1+\sigma\tilde{\lambda}_1}{\sigma+1}\right)d\log\mu + d\log\mu. \end{aligned}$$

The quantity adjustments follow from the factor share changes:

$$\begin{aligned} d\log q_{11} &= d\log\lambda_{11} - d\log p_{11} \\ &= -d\log\mu + \left(\frac{1+\sigma\tilde{\lambda}_1}{\sigma+1}\right)d\log\mu. \end{aligned}$$

$$\begin{aligned} d\log q_{12} &= d\log\lambda_{12} - d\log p_{12} \\ &= 0 + \frac{\sigma}{\sigma+1}\tilde{\lambda}_1 d\log\mu \\ &= \frac{\sigma}{\sigma+1}\tilde{\lambda}_1 d\log\mu. \end{aligned}$$

$$\begin{aligned} d\log q_{21} &= d\log\lambda_{21} - d\log p_{21} \\ &= 0 - \left(-\left(\frac{1+\sigma\tilde{\lambda}_1}{\sigma+1}\right)d\log\mu + d\log\mu\right) \\ &= \left(\frac{1+\sigma\tilde{\lambda}_1}{\sigma+1}\right)d\log\mu - d\log\mu. \end{aligned}$$

The social loss is given by:

$$\begin{aligned} \mathcal{L} &= \frac{1}{2}\left(\tilde{\lambda}_1 d\log q_{11} d\log\mu + \tilde{\lambda}_2 d\log q_{12} d\log\mu + \tilde{\lambda}_1 d\log q_{21} d\log\mu\right) \\ &= \frac{1}{2}\Bigg(\tilde{\lambda}_1\left(-1 + \frac{1+\sigma\tilde{\lambda}_1}{\sigma+1}\right) + \tilde{\lambda}_2\frac{\sigma}{\sigma+1}\tilde{\lambda}_1 \\ &\quad + \tilde{\lambda}_1\left(\frac{1+\sigma\tilde{\lambda}_1}{\sigma+1} - 1\right)\Bigg)(d\log\mu)^2 \end{aligned}$$

$$= \frac{1}{2}\left(2\tilde{\lambda}_1\left(-\frac{\sigma}{\sigma+1} + \left(\frac{\sigma}{\sigma+1}\right)\tilde{\lambda}_1\right) + \frac{\sigma}{\sigma+1}\tilde{\lambda}_1\tilde{\lambda}_2\right)(d\log\mu)^2$$
$$= -\frac{1}{2}\tilde{\lambda}_1\tilde{\lambda}_2\left(\frac{\sigma}{\sigma+1}\right)(d\log\mu)^2.$$

$\square$