

Aggregating Distortions in Networks with Multi-Product Firms

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Abstract

We investigate the role of multiproduct firms in shaping resource misallocation and its impact on aggregate total factor productivity (TFP) growth. Using administrative data on product transactions between all formal Chilean firms, we provide evidence that demand shocks to one product affect the production of other products within the same firm, suggesting that firms engage in joint production. We develop a framework to measure resource misallocation in production networks with joint production, deriving non-parametric sufficient statistics to quantify these effects. Applying the framework to Chile, we find that reallocation effects explain 86% of the observed aggregate TFP growth for the 2016-2022 period. Ignoring joint production leads to overestimation of resource misallocation.

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1 Introduction

Resource misallocation across heterogeneous producers has been recognized as a driver of aggregate total factor productivity (TFP) differences across countries and over time. To quantify the extent of misallocation, the recent literature has made extensive use of granular firm-level data.

However, despite its emphasis on micro data, this literature typically ignores the fact that most firms sell multiple products. For example, 75% of formal firms in Chile report selling multiple products, and these firms collectively account for 99% of all firm-to-firm transactions in Chilean tax data. The ubiquity of multi-product firms introduces new challenges to understanding resource allocation. Specifically, researchers must consider how the allocation of resources across products within firms affects allocative efficiency and aggregate TFP. Measuring resource allocation within firms often requires determining how to assign inputs to specific outputs.

The literature on multi-product firms often assumes product line independence (Klette and Kortum (2004); Bernard et al. (2011); De Loecker et al. (2016); Hottman et al. (2016); Mayer et al. (2021)). If firms are collections of independent products, the challenge is reduced to a measurement problem, and existing theories for single-product firms can be applied by treating different products as-if they are separate firms. However, firms often simultaneously produce multiple outputs using shared inputs, making it impossible to assign inputs to specific outputs. Consider an oil refinery that produces diesel and gasoline concurrently: the inputs—crude oil, labor, and capital—are used to produce both outputs and cannot be accounted for separately.

How do multi-product firms with non-separable production technologies affect the measurement of the extent of resource misallocation? We model firms' technology via joint non-separable production functions that map bundles of inputs into bundles of multiple outputs. This approach eliminates the need to define individual product-level production functions. The joint production function describes the firm's flexibility in adjusting its product mix, which then determines the importance of resource allocation within the firm.

We generalize previous work to accommodate multi-product firms with joint production technologies. We provide nonparametric sufficient statistics to measure changes in allocative efficiency using ex-post data. Our framework is general enough to accommodate firm-to-firm linkages. We validate and implement our framework using a granular firm-to-firm transactions database for Chile. We show that the extent of resource misallo-

cation is overstated if we abstract from joint production, as is standard in the literature. While our primary focus is ex-post analysis, we also develop a complementary parametric framework to understand ex-ante counterfactuals, like the gains from eliminating all distortions taking into account joint production technologies.

We first describe the theoretical contributions of the paper, before turning to our empirical validation and application. In our model, products within firms can be under- or overproduced because they face different wedges (e.g., markups). Loosely speaking, products with relatively high wedges are underproduced. However, since firms interact with one another, the relevant wedges that affect resource allocation and, hence, aggregate TFP are not just the firm's own wedges, but the entire chain of cumulative wedges leading from final demand to the production of the product.

The impact of changes in these cumulative wedges on resource misallocation depends on how easily firms can adjust their product mix. Consider again the oil refinery example. If the oil refinery raises the markup on gasoline, thereby lowering its demand, it cannot redirect production resources toward diesel, because the production technology yields gasoline and diesel in nearly fixed proportions from crude oil. This technological constraint limits the firm's ability to reallocate resources in response to demand changes and hence limits the extent of misallocation within the firm. Therefore, joint production technology can attenuate the extent of resource misallocation and its contribution to aggregate TFP.

To quantify the extent of resource misallocation in the presence of joint production, we develop a nonparametric sufficient statistics approach. Instead of estimating the firm's production technology parameters directly, we rely on observed changes in product-level prices within the firm. Intuitively, if prices for certain products rise within the firm, net of the markup, then this captures the firm's inability to easily substitute production across products. The covariance of cumulative wedges at the product level with relative price changes captures the attenuation of resource misallocation due to joint production technology. Intuitively, if prices rise for products with high (cumulative) wedges, then the scope for reallocation is limited. Theoretically, these price movements, net of markups, trace out the production possibility frontier, whose slope captures each firm's technological constraints when adjusting their product mix. This provides a way to quantify misallocation without imposing parametric assumptions on firms' production technologies.

The growth-accounting formula we develop generalizes previous approaches: it col-

lapses to the Baqaee and Farhi (2020) growth accounting result under the single-product firm assumption and to the Hulten (1978) benchmark under perfect competition.

To implement our framework, we use administrative firm-to-firm transactions data from the Chilean Internal Revenue Service. The dataset contains product-level prices, firm-product input-output linkages, and balance sheet variables, enough to construct our sufficient statistics.

We first validate joint production technology in the data. We examine whether the standard assumption in the literature that firms operate as independent single-product lines, holds. We find that demand shocks to one product significantly affect the production of other products within the same firm, indicating that firms employ joint production technology. In fact, we find that a negative demand shock to a firm’s main product reduces production of alternative products — as predict by our framework.

Having established the presence of joint production, we implement our sufficient statistics to conduct a growth accounting exercise. We find that changes in allocative efficiency explain 86% of TFP growth in Chile from 2016 to 2022. Ignoring joint production provides a misleading assessment and substantially overestimates the importance of resource reallocation.

We attribute this finding to multi-product firms facing constraints when adjusting their product mix. These constraints limit the scope for product-level resource reallocation within firms, which affects allocative efficiency and aggregate TFP growth.

As mentioned above, our primary focus in the paper is nonparametric ex-post analysis using observed data. However, we also develop a complementary ex-ante approach that enables counterfactual analysis by imposing a parametric structure for production technologies and requires knowledge of the curvature in the firm’s production possibility frontier. We provide an analytical characterization of the extent of misallocation, measured as the distance to the Pareto-efficient frontier, for multi-product firms with joint production. Consistent with our earlier finding that joint production constrains reallocation, we find that assuming separable production technologies overestimates the extent of resource misallocation caused by a given set of wedges.

Related Literature

Our paper contributes to and connects different strands of the literature. We incorporate multiproduct firms and joint production to extend the literature on misallocation. The work by Restuccia and Rogerson (2008); Hsieh and Klenow (2009) shows the potential

importance of resource misallocation in accounting for TFP differences. We contribute to this literature by developing a framework that allows for multi-product firms (Bernard et al. (2010, 2011); Mayer et al. (2014); De Loecker et al. (2016); Hottman et al. (2016); Mayer et al. (2021); Wang and Yang (2023)) and joint production (Powell and Gruen (1968); Diewert (1971); Lau (1972); Hall (1973, 1988); Boehm and Oberfield (2023); Carrillo et al. (2023); Ding (2023)) and quantifying the impact of resource misallocation.

Recent work has emphasized the importance of taking into account input-output linkages when the extent of these losses is quantified (Baqae and Farhi (2020); Bigio and La'O (2020)). Our theory provides a flexible framework that allows for arbitrary production structures, including input-output networks while incorporating joint production, and it enables the quantification of these effects.¹

Our theory provides a tool for growth accounting (Solow (1957); Hulten (1978); Basu and Fernald (2002); Petrin and Levinsohn (2012); Baqae and Farhi (2020); Baqae et al. (2023)) that decomposes aggregate TFP growth into technology and allocative efficiency under joint production in networks and generalizes existing methods to consider multi-product firms.

Our empirical application uses Chile's comprehensive product-level transaction database to quantify misallocation. This approach contrasts with the prior literature on production networks and misallocation, which typically uses industry-level input-output tables instead of firm-to-firm data. For example, Baqae and Farhi (2020) impute US Compustat data using an industry-level input-output table. Finally, Burstein et al. (2024) uses the same dataset as ours but complements our work by analyzing misallocations that arise from different buyers receiving varying prices for the same product.²

Lastly our empirical strategy is related to the literature on shock transmission between firms. While much of this literature focuses on how supply shocks propagate downstream across firms (Boehm et al. (2019); Carvalho et al. (2020); Fujiy et al. (2022); Bai et al. (2024)) using the granular a firm-to-firm transaction dataset, our empirical strategy instead analyzes how external demand shocks transmit within firms across their products. We compare our results with existing studies that examine within-firm spillovers of demand shocks (Giroud and Mueller (2019); Almunia et al. (2021); Ding (2023)).³

¹Our work is also related to the work of Liu (2019) and Dávila and Schaab (2023), which analyzed the effects of misallocation in input-output linkages on welfare.

²Kikkawa (2022) examines firm pair-specific markups based on a theoretical model using Belgian firm-to-firm transaction data.

³This work is related to the literature on production function estimation for multiproduct firms. Notably, estimation methods for joint production have been developed recently Dhyne et al. (2017, 2022);

The rest of the paper is organized as follows. Section 2 presents the data and motivating facts, highlighting the prevalence of multiproduct firms. It also presents empirical evidence that suggests that firms engage in joint production. Section 3 outlines the theoretical framework, deriving the nonparametric sufficient statistics for measuring allocative efficiency explained by multiproduct firms. Section 4 details the data and the construction of sufficient statistics. Section 5 applies the framework to decompose aggregate TFP growth in Chile for the 2016–2022 period. Section 6 presents ex-ante structural results. Finally, Section 7 concludes.

2 Reduced Form Evidence

This Section presents empirical facts that motivate our theory of multiproduct firms engaging in joint production.

The primary aim of our empirical analysis is to test whether firms can be treated as collections of independent single-product lines. If this were the case, existing aggregation theorems could be applied by considering each product as a separate firm. Our findings, however, reject this separability assumption, revealing interdependencies among products within firms. This result motivates the need for a new aggregation theorem that we develop in subsequent sections.

We use data from the Chilean Internal Revenue Service (SII), covering all formal firms in Chile.⁴ The cross-sectional analysis uses 2018, a year unaffected by any major shock. We then employ monthly data from January 2019 to December 2021 to test for joint production. We exploit the unexpected nature of early COVID-19 lockdowns as a source of exogenous variation in product-specific demands.

The SII provides detailed information on firm-to-firm transactions through electronic tax documents. This dataset, which captures every product, quantity, and price traded

Valmari (2023); Cairncross and Morrow (2023). We estimate the universe of products traded in Chile by formal firms from 2016 to 2022.

⁴This study was developed within the scope of the research agenda conducted by the Central Bank of Chile (CBC) in economic and financial affairs of its competence. The CBC has access to anonymized information from various public and private entities by virtue of collaboration agreements signed with these institutions. To secure the privacy of workers and firms, the CBC mandates that the development, extraction, and publication of the results should not allow the identification, directly or indirectly, of natural or legal persons. Officials of the CBC processed the disaggregated data. All the analysis was implemented by the authors and did not involve nor compromise the Chilean IRS. The information contained in the databases of the Chilean IRS is of a tax nature originating in self-declarations of taxpayers presented to the Service; therefore, the veracity of the data is not the responsibility of the Service.

between formal Chilean firms, contains data on over 15 million unique firm-specific product descriptions.⁵ For 2018, we sum the real-time quantity traded and the value for every buyer firm–seller firm–product triplet transaction. We divide the value traded over its quantity to obtain average yearly prices for every triplet. We use the 2018 data to describe the main features of the firm-to-firm trade patterns.

Fact 1: Multiproduct Firms Dominate Domestic Intermediate Inputs Trade

Of all firms, 75% produce multiple products, and these firms account for 98.94% of intermediate input transaction value. Table 1 illustrates the distribution of products per firm, weighted by firm-to-firm transaction values.

Table 1: Distribution of Product Numbers

| Percentile | Number of products | Number of products |
|------------|--------------------|---------------------------------|
| | (Unweighted) | (Weighted by transaction value) |
| 1% | 1 | 1 |
| 5% | 1 | 2 |
| 10% | 1 | 4 |
| 25% | 2 | 36 |
| 50% | 7 | 475 |
| 75% | 26 | 2,459 |
| 90% | 119 | 32,195 |
| 95% | 290 | 37,422 |
| 99% | 1,253 | 62,372 |

Notes: The Table presents the distribution of product numbers for 2018. The left column shows the number of products without weighting, while the right column displays the number of products weighted by the intermediate product transaction volumes of the firms.

⁵The specific invoice variable is called “detail”, which is inherently firm-specific and can differ between firms even for the same product. For example, one supermarket might declare selling “Sprite can 330cc” while another declares selling “Sprite 330”. This variation across sellers does not affect our analysis in this Section as we do not compare identical products across firms.

Fact 2: Demand Shocks to One Product Affect the Production of Other Products within the Same Firm

Given the prevalence of multiproduct firms established in Fact 1, we examine whether we can treat multiproduct firms as separate product lines (Bernard et al. (2010, 2011); Hottman et al. (2016); Mayer et al. (2021)). Under this assumption, each product can be relabeled as a fictitious firm, which can be analyzed using standard single-product firm frameworks. We examine this assumption's validity and determine whether multiproduct firms differ from collections of single-product firms.

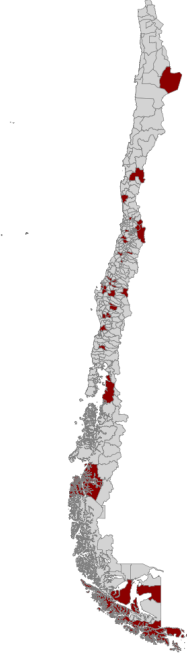
To investigate this, we exploit heterogeneous exposure to local buyer shocks for each firm's products and investigate whether firms engage in joint production in the spirit of Ding (2023) but in a production network context. Specifically, We examine how demand shocks to specific products affect the production of other products within the same firm. We use monthly data from 2019 to 2021. We treat different establishments of the same firm as different firms, allowing us to treat regional differences in COVID-19 lockdowns as exogenous shocks. We sum the monthly quantity traded and value for every buyer firm-seller firm-product triplet transaction. We divide the value traded over its quantity to obtain average monthly prices for each triplet.

Our empirical strategy consists of two main components: First, we employ a validation step to confirm that lockdowns indeed caused a decrease in intermediate input purchases, which can be interpreted as demand shocks from the perspective of supplier firms with buyers in lockdown areas. Second, we conduct a main event study that examines how these shocks affect the production of other products within the same firm.

COVID-19 Lockdowns in Chile

Starting in March 2020, the Chilean government declared county-specific lockdowns due to COVID-19. We focus on March 2020, when the first lockdowns were imposed, to ensure the shock was unexpected. Figure 1 shows regions that experienced lockdowns in March 2020, illustrating spatial lockdown heterogeneity.

Figure 1: Distribution of Early Covid-19 lockdown in Chile



Notes: Lockdown counties as of March 2020 are red; all others are gray.

Lockdown Effects on Intermediate Input Purchases

Before proceeding to our main analysis, we establish that COVID-19 lockdowns resulted in reduced intermediate input purchases by buyers in lockdown areas, which can be interpreted as a negative demand shock to suppliers in non-lockdown areas.

We posit that intermediate input transactions declined between suppliers in unaffected (gray) counties and buyers in counties that experienced early COVID-19 lockdowns (red). To test this hypothesis, we estimate the following reduced-form specification at the buyer level:

$$\log M_{it} = \beta \text{Lockdown}_{it} + FE_t + FE_i + \varepsilon_{it}, \quad (1)$$

where M_{it} denotes total intermediate input purchases of a firm i at time t and Lockdown_{it} is a dummy variable equal to one if firm i 's location was under lockdown at time t , and is zero otherwise. To address potential bias arising from buyers in lockdown areas who purchased from suppliers in lockdown areas, we restricted the sample by including only buyers with suppliers in non-lockdown areas.

Table 2: Lockdown and intermediate input purchases

| | (1) | (2) | (3) |
|-------------------------|-----------------------|------------------------|-----------------------|
| Lockdown Dummy | -0.222*** (0.0524) | -0.230*** (0.00521) | -0.191*** (0.0589) |
| Firm FE | Y | Y | Y |
| Time FE | N | N | Y |
| Sector \times Time FE | N | Y | N |
| Restricted sample | N | N | Y |
| Observations | 4,345,534 | 4,345,534 | 378,646 |

Notes: The Table reports the results of estimating equation (1) by ordinary least squares (OLS), clustered at the firm-municipality level. The sample periods are January 2019 to March 2020. Columns (1) and (2) report results for the full sample. Column (3) presents the results restricted to firms with no suppliers in the lockdown area. Three stars indicate statistical significance at the 1% level.

The results confirm our hypothesis: The coefficient of interest, β , is negative, indicating that purchases of intermediate inputs from lockdown counties decreased by about 20% on average. This result confirms that we can interpret the decrease in purchases as a negative demand shock to intermediate inputs sold by firms in non-lockdown regions to buyers in lockdown regions.

Empirical Evidence for Joint Production

Having validated our use of lockdowns as a source of demand shocks, we now focus on the central question of our analysis: do firms engage in joint production?

To investigate this, we examine how a demand shock to one product affects the production of other products within the same firm. We focus on shocks to a firm's main product, defined as the product with the highest sales from January 2019 to December 2021. We define a firm as experiencing a demand shock to its main product if at least one buyer of its main product is located in an area that implemented a lockdown in March 2020 due to the initial COVID-19 outbreak. This definition focuses on the early lockdowns to ensure the exogenous nature of the shock, as these initial closures were largely unexpected.

To quantify this effect, we study the impact of demand shocks to a firm's main product on the production of its other products using an event-study specification for all products $g \neq m$:

$$\log X_{igt} = \sum_{\substack{j=-11 \\ j \neq -1}}^{10} \beta_j D_{i,t-j} + FE_{ig} + FE_t + \varepsilon_{igt}, \quad (2)$$

where X_{igt} represents either the quantity or price of product g for firm i at time t . $D_{i,t-j}$ is a treatment indicator equal to one if firm i was treated j months ago. FE_{ig} and FE_t are firm-product and time fixed effects, respectively. The coefficients of interest are β_t , which capture the effect of the main product's demand shock on other products' quantities or prices at different time points relative to the shock.

To obtain unbiased estimates of β_j , the treatment indicator $D_{i,t-j}$ must be conditionally orthogonal to the error term ε_{igt} . A key concern is that supply-side shocks could be correlated with the lockdown if suppliers and main product buyers are located in the same area, potentially confounding our results. To address this issue and isolate the impact of demand shocks from the main product while ruling out direct supply shocks, we impose the following restrictions:

1. **Firm Location:** The firm itself is not located in an area under lockdown.
2. **Supplier Location:** The firm's direct suppliers are not subject to lockdown shocks.
3. **Buyer Location for Product g :** None of the buyers of product g are located in lockdown areas.⁶

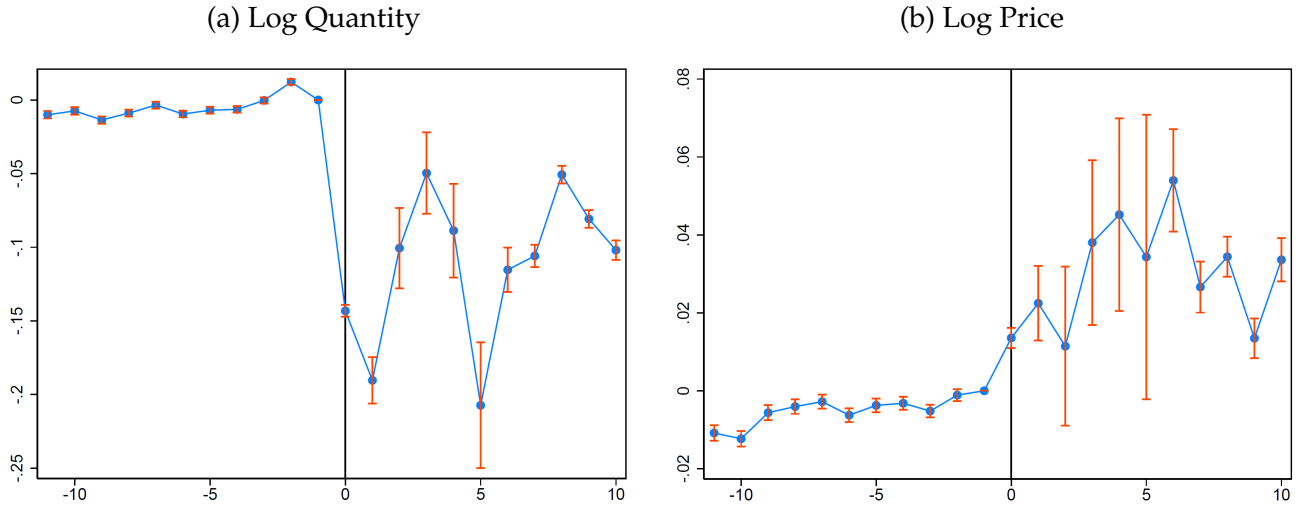
Restrictions 1 and 2 help eliminate direct supply-side effects, ensuring that any observed changes in production are not due to supply disruptions that affect the firm or its suppliers. Restriction 3 ensures that product g is not subject to a direct demand shock, allowing us to attribute any changes in its production to the demand shock that affects the main product m .

Our treatment group consists of firms that meet these conditions and experience a demand shock to their main product in March 2020. The control group includes firms that satisfy the conditions but whose main products did not experience a demand shock

⁶Within the same firm, the set of buyers often differs across products. Therefore, even if the main product's buyers are affected by the lockdown, it does not necessarily imply that buyers of other products are similarly affected. This distinction is further detailed in Figure A1 of Appendix A.1.

until t (i.e., the buyers of the main product have never been in a lockdown area from March 2020 until t). Figure 2 presents the results of regression (2) ⁷.

Figure 2: The effects of demand shocks to the main product on the production of other products within the firm



Notes: Standard errors are clustered at the firm-county level, and the error bands represent 95% confidence intervals. The X-axis represents the time to treat, with 0 denoting March 2020, when the main product experienced the demand shock. The other values indicate the number of months before or after this event.

First, prior to the shock, the differences in quantities and prices between the treatment and control groups remained stable and close to zero. This observation supports the parallel trends assumption and aligns with the interpretation that the initial closures were unanticipated. Second, we detect a significant quantity effect: at the time of the shock ($t = 0$), there is a 14% decrease in the quantity of non-main products when the main product experiences a demand shock. Notably, this effect persists throughout the post-shock period, indicating that the impact is not transitory. Third, we observe an increase in the price of non-main products on impact. This price increase remains elevated above the pre-shock level even ten months after the initial shock. In addition to the quantity response, this price response is consistent with the assumption of joint production, which will be introduced in the next section.

⁷A comparison of observable characteristics between the treatment and control groups is provided in Table A1 of Appendix A.1

Discussion on Other Within-Firm Spillover Mechanisms

Our empirical results are inconsistent with separable production functions. This outcome suggests that demand shocks to one product affect the production of other products within the same firm. Next, we compare our findings to those of previous studies.

First, Almunia et al. (2021) propose a model of diminishing returns to scale or firm-specific factors at the firm level to explain how a decline in domestic demand in Spain affects exports. Their model predicts that when there is a negative demand shock for a product in one market, firm-specific factors are reallocated to another product in another market, positively affecting the production of the same product in other markets. This prediction contrasts with our findings, which show negative spillovers across products within the same firm.

Second, Ding (2023) focuses on industries that share knowledge-intensive inputs to examine joint production effects in the US using Census data. This paper, like ours, predicts that when a product faces a negative demand shock, it negatively affects other products. The study interprets the model prediction as knowledge spillovers across industries sharing intangible inputs; however, knowledge spillovers are unlikely to explain our results. The differences in time horizon (five years vs. monthly data) and research and development (R&D) intensity (Chile's R&D spending is less than one-tenth that of the US as a percentage of GDP) limit its applicability to our context.

Giroud and Mueller (2019) use US multiregion firm data to model demand-driven regional spillovers based on financial constraints. In this model, firms facing credit constraints optimize resource allocation across regions. A negative demand shock in one region reduces employment in other regions due to shared financial constraints within the firm. This mechanism predicts a negative response: a negative shock to one product would lead to reduced production of other products through financial constraints. In addition, although Giroud and Mueller (2019) do not report price reactions, Kim (2020) finds that firms in financial distress reduce product prices by selling off inventory, which is contrary to the results of this paper and suggests a negative price response. In Appendix A.1, we conduct the same event study using a subset of firms unlikely to be under financial constraints. The results do not change significantly, suggesting they are unlikely to be driven solely by financial constraints.

Our results suggest that multiproduct firms are not collections of independent product lines. These empirical results motivate the theoretical framework we develop in the following section.

3 A Theory to Aggregate Distortions in Networks with Multiproduct Firms

Having established empirically that firms engage in joint production, we now develop a theoretical framework to analyze resource misallocation in production networks with multiproduct firms. The key challenge is that firms use shared inputs to produce multiple outputs simultaneously, making it impossible to separately account for inputs used for each product. Our framework addresses this challenge by developing a general network model that accommodates joint production.

In our theoretical framework, we use the terms “markup” and “wedge” interchangeably to refer to any distortion that creates a gap between price and marginal cost. This can include various distortions such as taxes, subsidies, or financial frictions.

The section proceeds as follows. First, we formalize the concept of joint production and introduce key assumptions. To build intuition, we then present simplified examples that illustrate how joint production affects resource allocation and aggregate TFP. Next, we develop our general network model with multiproduct firms and define the relevant input-output concepts. Finally, we introduce cumulative wedges - our key theoretical object for measuring misallocation - and present our main aggregation theorem.

3.1 Joint Production

We begin by formalizing the concept of joint production, where firms simultaneously use shared inputs to produce different products.

To formalize this concept, we follow Hall (1973)’s approach to joint production technology. Let $J(q, x)$ be a joint production function, where q is a vector of outputs and x is a vector of inputs. The joint cost function is derived from the firm’s cost minimization problem, as follows:

$$C(q, p) \equiv \min_{x \in V(q)} p'x,$$

where $V(q)$ is the input requirement set, $V(q) = \{x | J(q, x) \geq 0\}$ and p is a vector of input prices. We introduce two assumptions about the shape of a joint production function, which will be used throughout this paper.

Assumption 1. Constant return to scale (CRS): $J(q, x) = 0$ implies $J(\lambda q, \lambda x) = 0$ for any

scalar λ .

Unlike a single-output production function, the output is a vector. Note that we do not assume CRS for each single-output production function.

Assumption 2. Separability between Input and Output Functions: The joint production function can be written as $J(\mathbf{q}, \mathbf{x}) = -F^Q(\mathbf{q}) + F^X(\mathbf{x})$, and the joint cost function as $C(\mathbf{q}, \mathbf{x}) = H(\mathbf{q})\varphi(\mathbf{p})$.

Note that this differs from assuming separable production functions, where the output, q , is a single product, not a vector; it degenerates to $F^Q(q) = q$. Example 1 illustrates a joint production function satisfying assumptions 1 and 2:

Example 1. Constant Elasticity of Transformation Output Bundle and Constant Elasticity of Substitution Input Bundle (CET-CES):

$$\underbrace{\left(\sum_g q_g^{\frac{\sigma+1}{\sigma}} \right)^{\frac{\sigma}{\sigma+1}}}_{\text{Output bundle}} = A \underbrace{\left(L^{\frac{\theta-1}{\theta}} + K^{\frac{\theta-1}{\theta}} \right)^{\frac{\theta}{\theta-1}}}_{\text{Input Bundle}},$$

The associated cost function is

$$C(\mathbf{q}, w, r) = \frac{1}{A} \left(\sum_g q_g^{\frac{\sigma+1}{\sigma}} \right)^{\frac{\sigma}{\sigma+1}} \left(w^{1-\theta} + r^{1-\theta} \right)^{\frac{1}{1-\theta}},$$

where L and K are the two inputs, w and r are their prices, and \mathbf{q} is a vector of outputs.

The input bundle takes a standard CES function with elasticity of substitution θ , and the output is a vector of products rather than a scalar. The parameter σ is called the constant elasticity of transformation; it gives a constant value to the production possibility frontier's curvature of the products within a firm. This example is illustrative as our theoretical framework requires no parametric assumption.

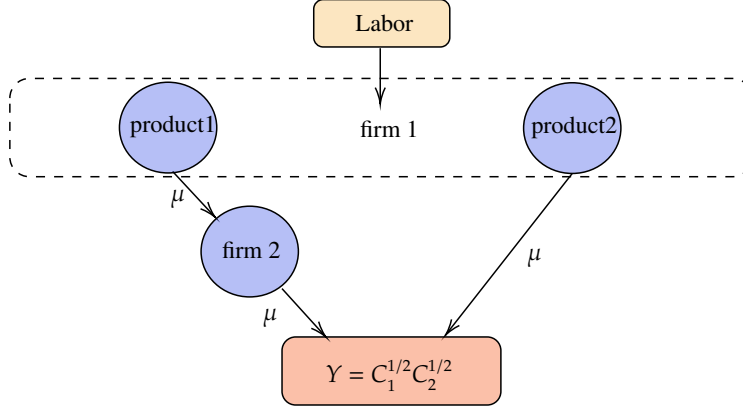
3.2 Parametric Examples of Misallocation with Joint Production

Before presenting the general framework, We provide simplified examples to help gain an intuition about how joint production affects resource allocation and aggregate TFP. Proofs are provided in Appendix D.

3.2.1 Without joint production

We begin with an example of a production network with multiproduct firms but without joint production. Consider an economy with two firms, as illustrated in Figure 3.

Figure 3: A simplified economy with production networks and multiproduct firms



Firm 1 uses labor to produce two differentiated products using labor (L) as unique input, $q_{11} = L_1$, $q_{12} = L_2$, where $L = L_1 + L_2$. Product 1 is sold to firm 2, while product 2 is sold directly to households. For simplicity, we assume that both products have the same markup, μ . Firm 2 uses product 1 from firm 1 as a production input and produces a different product using a linear technology ($q_2 = q_{11}$) that sells to households with markup μ . Final consumption goods are aggregated using a Cobb-Douglas function $Y = c_1^{1/2} c_2^{1/2}$, where $c_1 = q_2$, $c_2 = q_{12}$. In this simple economy, Y is the real GDP, and aggregated TFP can be defined as $TFP = Y/L$.

In a production network environment, distorted resource allocation arises from firms' own markups and downstream firm's markup. In this example, product 1 is sold to households with double marginalization; firm 1 charges a markup to firm 2, and firm 2 charges a markup to the household. As a result, product 1 from firm 1 suffered from a higher distortion than product 2 from firm 1, both relative to a perfect competition setup.

Assume that product 1 from firm 2 receives a shock that reduces its markup ($\Delta \log \mu_{21} = -\epsilon$). This shock increases demand for product 1, which is too small to begin with. By taking a first-order approximation of the change in aggregate TFP, we obtain the following response:

$$\Delta \log TFP = \frac{\mu - 1}{2(1 + \mu)} \epsilon > 0.$$

As product 1's markup decreased, the production of product 1 increased, increasing a portion of the economy that was too small to begin with because of a higher initial markup. Also, product 1's relative price decreased. As a result, the household shifts away from product 2 and increases its consumption of product 1 (indirectly through firm 2's product), which boosts aggregate TFP.

3.2.2 Joint Production

Next, we introduce joint production into our simplified economy. Instead of separate production functions for each product, firm 1 uses a joint production technology to produce both products simultaneously:

$$\left(q_{11}^{\frac{\sigma-1}{\sigma}} + q_{12}^{\frac{\sigma-1}{\sigma}} \right)^{\frac{\sigma}{\sigma-1}} = L_1.$$

Assume that product 1 from firm 2 receives the same shock as before, and that reduces its markup ($\Delta \log \mu_{21} = -\epsilon$). By taking a first-order approximation of the change in TFP, we obtain the following response:

$$d \log TFP = \left(\frac{\sigma}{\sigma + 1} \right) \left(\frac{\mu - 1}{2(1 + \mu)} \right) \epsilon > 0.$$

A negative markup shock induces a relative price decline, triggering a substitution effect that shifts household expenditures towards product 1 (indirectly through firm 2 product). Parameter σ governs the elasticity of transformation between the two products of firm 1—that is, how easily firm 1 can adjust its product mix for its two products. The elasticity of transformation will determine the changes in aggregate TFP, which we illustrate in two extreme cases.

As σ approaches infinity, firm 1 can freely choose the ratio of products in response to a markup decrease to one of its products. The change in aggregate TFP converges to the maximum possible improvement as firm 1 can freely adjust its production inputs for each of its products.

Conversely, as σ approaches zero, the production technology becomes Leontief in outputs, with fixed proportions. To illustrate this, consider a petroleum refining firm that produces gasoline (product 1) and diesel (product 2). Diesel production is constrained by the refining process's specific technological and chemical properties of crude oil; the firm cannot easily switch the oil used to produce gasoline to produce diesel.

This rigidity in production means that refineries remain bound to produce gasoline and diesel fuel in fixed proportions despite relative prices or demand changes. Consequently, the potential for production factors reallocation disappears, eliminating aggregate TFP growth changes in response to markup changes.

While an infinite or zero σ are extremes, these examples illustrate that ignoring the rigidities associated with joint production may lead to overestimating aggregate TFP changes in response to shocks; this is consistent with most literature, which implicitly assumes σ approaches infinity.

The degree to which joint production constrains aggregate TFP growth is an empirical question. It depends on how easily firms can adjust inputs to produce different outputs, which may vary between firms and within firms for different products.

In the next section, we develop a framework to measure these effects in a general production network setting without requiring parametric assumptions about the joint production technology.

3.3 General Network Setup

Regarding input-output notation and definitions, we follow Baqaee and Farhi (2020) to present our generalization and add product-level (instead of firm-level) objects. Without joint production, every product can be considered a fictitious firm so that the Baqaee and Farhi (2020) setup applies.

Multiproduct Firms

Firm $i \in \mathcal{N}$ produces product $g \in \mathcal{G}$ and uses products $g' \in \mathcal{G}$ from other firms $j \in \mathcal{N}$ and factors (Labor, L and Capital, K) as production inputs.⁸ We assume the following production set with CRS and separability between input and output functions:

$$F_i^Q \left(\underbrace{\{q_{ig}\}_{g \in \mathcal{G}}}_{\text{outputs}} \right) = A_i F_i^X \left(\underbrace{\{x_{i,jg'}\}_{j \in \mathcal{N}, g' \in \mathcal{G}}}_{\text{Intermediate product } g' \text{ from } j}, L_i, K_i \right), \quad (3)$$

Firms charge a product-specific markup, μ_{ig} , over its product-specific marginal cost; thus, the price is defined as $p_{ig} = mc_{ig}\mu_{ig}$.

⁸We treat factors exhibiting zero return to scale production functions; they generate production inputs without using inputs from other firms.

Final Demand

A representative household with a homothetic utility function of $U(c_{ig}, \dots, c_{NG})$ receives income from factor payments and profits from firms they own, following a budget constraint:

$$\sum_{i \in \mathcal{N}} \sum_{g \in \mathcal{G}} p_{ig} c_{ig} = \sum_{f \in \{L, K\}} w_f L_f + \sum_{i \in \mathcal{N}} \sum_{g \in \mathcal{G}} (1 - 1/\mu_{ig}) p_{ig} q_{ig}.$$

Each product can be consumed by final consumers (c_{ig}) or used as an input in production by other firms ($x_{ji,g}$). The following resource constraint applies:

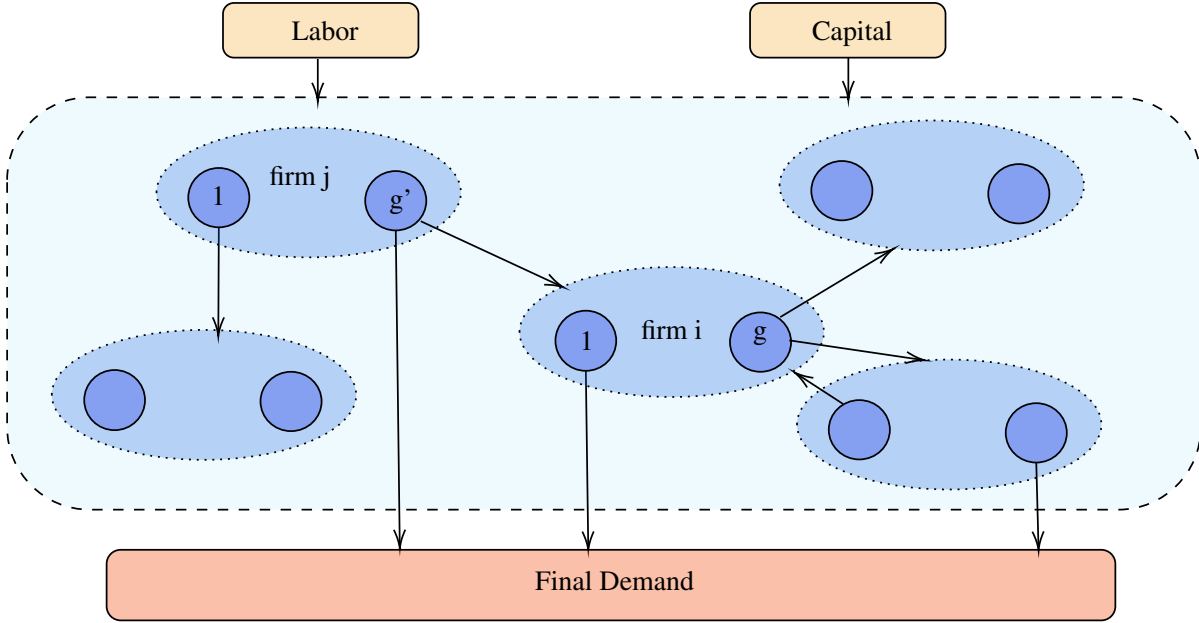
$$q_{ig} = c_{ig} + \sum_{j \in \mathcal{N}} x_{ji,g}, \quad \sum_{i \in \mathcal{N}} L_i = L, \quad \sum_{i \in \mathcal{N}} K_i = K.$$

Figure 4 presents a stylized representation, showing the flow of products.

General Equilibrium

Given a vector of firm-level productivity, A , and vector of product-level markups, μ , for all $i \in \mathcal{N}$ and $g \in \mathcal{G}$, the general equilibrium is a set of prices (p_{ig}) intermediate input choices (x_{ijg}), factor input choices (L_i, K_i), output, (q_{ig}), and consumption choices (c_{ig}). As such, (i) the price of each product is equal to its markup multiplied by its marginal cost; (ii) households maximize utility under budget constraints, given prices; and (iii) markets are clear for all products and factors.

Figure 4: Graphical illustration of networks with multiproduct firms



Notes: The dashed line represents firms' universe \mathcal{N} , the dotted circled line represents each firm's boundary, and the circled line represents each product within a firm. The two top nodes represent factors, and the bottom node represents households. Arrows represent the direction of input flows.

3.4 Input-Output Definitions

To state our decomposition results, we introduce notation for input-output relationships at the product level.

Product-Level Input-Output Matrix

The product-level input-output matrix $\tilde{\Omega}$ is a $(\mathcal{N}\mathcal{G} + \mathcal{F})$ square matrix. Here, \mathcal{N} is the number of firms, \mathcal{G} is the number of products, and \mathcal{F} is the number of factors. $\tilde{\Omega}$ has at its ig, jg'^{th} element the expenditure share of product g' from firm j and factor $f \in \mathcal{F}$ used by firm i in production over firm i total costs (of producing all its products). The separability assumption indicates that the same expenditure share applies for all products, g , that firm i produces; thus, $\tilde{\Omega}_{ig, jg'}$ and $\tilde{\Omega}_{ig, f}$ are as follows.

$$\tilde{\Omega}_{ig, jg'} = \frac{p_{jg'} x_{i, jg'}}{\sum_{j, p} p_{jg'} x_{i, jg'} + \sum_f w_f L_{if}}, \quad \tilde{\Omega}_{ig, f} = \frac{w_f L_{if}}{\sum_{j, p} p_{jg'} x_{i, jg'} + \sum_f w_f L_{if}}.$$

The product cost-based Leontief inverse $\tilde{\Psi}$ captures each firm-product pair's direct and indirect cost exposures through production networks. We use each $\tilde{\Psi}$ element to measure the weighted sum of all paths between two nonzero firm-product pairs.

$$\tilde{\Psi} \equiv (I - \tilde{\Omega})^{-1} = I + \tilde{\Omega} + \tilde{\Omega}^2 + \dots$$

We define the final consumption share vector, b , as follows:

$$b_{ig} = \begin{cases} \frac{p_{ig}c_{ig}}{GDP} & \text{if } i \in \mathcal{N}, g \in \mathcal{G} \\ 0 & \text{otherwise} \end{cases}$$

We set GDP to be the numeraire and define the product-level cost-based Domar weight, $\tilde{\lambda}_{ig}$.⁹ This measures the importance of product g from firm i in final demand in two dimensions: directly when sold to final consumers, and indirectly through the production network when product g is sold to other firms and eventually reaches final consumers via downstream production networks.

$$\tilde{\lambda}' \equiv b' \tilde{\Psi} = b' + b' \tilde{\Omega} + b' \tilde{\Omega}^2 + \dots$$

Firm-Level Aggregation

Summing over products by firms allows us to recover the firm-level cost-based Domar weight $\tilde{\lambda}_i$, which we use to compute the within-firm product-level Domar weight share s_{ig} :

$$\tilde{\lambda}_i = \sum_{g \in \mathcal{G}} \tilde{\lambda}_{ig}, \quad s_{ig} = \frac{\tilde{\lambda}_{ig}}{\tilde{\lambda}_i}.$$

Finally, we define firm-level aggregate markup as follows:

$$\mu_i = \frac{\text{sales of } i}{\text{total cost of } i'}$$

⁹We denote $\tilde{\Lambda}_f$ with $f \in \{L, K\}$.

National Accounts

GDP is defined as the sum of all product values consumed by final consumers: $GDP = \sum_{i \in N} \sum_{g \in \mathcal{G}} p_{ig} c_{ig}$. Real GDP (Y) changes can be computed as follows:

$$d \log Y = d \log GDP - \sum_{i \in N} \sum_{g \in \mathcal{G}} \frac{p_{ig} c_{ig}}{GDP} d \log p_{ig}.$$

Factor shares are defined as

$$\Lambda_L = \frac{wL}{GDP}, \quad \Lambda_K = \frac{rK}{GDP}.$$

3.5 Cumulative Wedges

We now define the cumulative wedges, Γ_{ig} , which is the critical input to our proposed sufficient statistic strategy. The cumulative wedge is defined as the ratio of the product cost-based Domar weight, $\tilde{\lambda}_{ig}$, to the product sales share (to GDP), adjusted by the product-level wedge.

Definition 1. Cumulative Wedge of firm i 's product g :

$$\Gamma_{ig} \equiv \underbrace{\frac{\tilde{\lambda}_{ig}}{\lambda_{ig}}}_{\text{downstream wedges}} \times \underbrace{\mu_{ig}}_{\text{own wedge}}.$$

where λ_{ig} denote sales share of firm i 's product g over GDP.

The cumulative wedge summarizes the cumulative distortion in the downstream supply chain of product g sold by firm i . In efficient economies, with no markups, the product cost-based Domar weight equals observed sales shares, generating a cumulative wedge equal to one for all products and firms. Conversely, in an inefficient economy, a portion of the indirect demand transmitted from downstream firm-product pairs to upstream firm-product firms is absorbed as profit by downstream firms. This effect accumulates in each supply chain transaction upstream until indirect demand reaches product g sold by firm i ; thus, the sales share of a product is smaller relative to an efficient economic outcome. Therefore, the larger the ratio, the greater the cumulative wedges in the downstream supply chain.

Next, we define the relative product cumulative wedges, which ranks product distortions within a given firm. It is measured as the relative downstream wedge of product g with respect to the average distortion of all products within firm i , Γ_i .

Definition 2. Relative Cumulative Wedges

$$\gamma_{ig} \equiv \frac{\Gamma_{ig}}{\Gamma_i},$$

where the average wedge of firm i is defined as: $\Gamma_i \equiv \sum_g \tilde{\lambda}_{ig} / \sum_g (\text{salesshare}_{ig} / \mu_{ig})$.

A Simple Example of Cumulative Wedge

To illustrate how cumulative wedges capture distortions, we revisit our simplified economy composed of two firms and a representative household. We normalize GDP to 1; hence, in this setup, sales (shares) to final consumption are $(1/2)$ for products 1 and 2; however, firm 1's sales of product 1 are reduced by the markup charged by firm 2, which is $(1/2)/\mu$.

The product cost-based Domar weights are $1/2$ for products 1 and 2. In matrix notation, the value-added share vector (b) and the product cost-based input-output matrix ($\tilde{\Omega}$) are:

$$b = \begin{bmatrix} 1/2 \\ 0 \\ 1/2 \\ 0 \end{bmatrix}, \quad \tilde{\Omega} = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

where the matrix and vector components are arranged in the following order: product 1 and 2 of firm 1, firm 2, and labor. Therefore, the product cost-based Domar weights can be computed as:

$$\begin{aligned} \tilde{\lambda}' &= b' + b'\tilde{\Omega} + b'\tilde{\Omega}^2 + \dots, \\ &= \underbrace{\begin{bmatrix} 1/2 & 0 & 1/2 & 0 \end{bmatrix}}_{\text{Final demand}} + \underbrace{\begin{bmatrix} 0 & 1/2 & 0 & 0 \end{bmatrix}}_{\text{Indirect demand}}. \\ &= \begin{bmatrix} 1/2 & 1/2 & 1/2 & 0 \end{bmatrix}. \end{aligned}$$

These weights represent the counterfactual sales shares if markups were removed while keeping expenditure shares constant. Following the definition, the cumulative wedge is $\Gamma_{ig} = \frac{\bar{\lambda}_{ig}}{\text{sales share}_{ig}} \times \mu_{ig}$ for firm 1:

$$\Gamma_{11} = \frac{1/2}{(1/2)/\mu} \mu = \mu^2, \quad \Gamma_{12} = \frac{1/2}{(1/2)} \mu = \mu.$$

The following Table 3 summarizes the results.

Table 3: Sales share, cost-based Domar weight, and cumulative wedge in this example

| | product 1 of firm 1 | product 2 of firm 1 |
|---|---------------------|---------------------|
| (1) Sales share | $(1/2)/\mu$ | $1/2$ |
| (2) Cost-based Domar weights | $1/2$ | $1/2$ |
| (3) Cumulative Wedge: (2)/(1) \times own markup | μ^2 | μ |

The markup of product 2 from firm 1 and the product from firm 2 equal μ . Comparatively, product 1 from firm 1 has a larger cumulative wedge of μ^2 than that of product 2, reflecting the product's markup and the downstream distortions the product faces. In this case, product 1 from firm 1 generates a distortion by charging a markup; it is subject to an additional distortion downstream production networks because firm 2 uses a marked-up input on its production. The sum of both distortions is the main driver when the multi-product channel of allocative efficiency, discussed in the next section, is characterized.

3.6 Aggregation Theorem with Multiproduct Firms within Production Networks

To allow for joint production and derive sufficient statistics, this section generalizes the concept of an allocation matrix introduced by Baqaee and Farhi (2020).

Let \mathcal{X} be an $(N + \mathcal{F}) \times (\mathcal{NG} + \mathcal{F})$ admissible input allocation matrix; the columns are buyer firms, and the rows are seller-product pairs. Each element, $\mathcal{X}_{ijg} = \frac{x_{ijg}}{q_{jg}}$, is the share of the output of product g produced by firm j that firm i uses as a production input.

A productivity shock ($d \log A$) and a markup shock ($d \log \mu$) effect in real GDP, Y , can be decomposed into a pure change in productivity ($d \log A$) for a given fixed allocation matrix \mathcal{X} and changes in the distribution of \mathcal{X} ($d\mathcal{X}$), holding productivity constant. In

vector notation:

$$d \log \mathcal{Y} = \underbrace{\frac{\partial \log \mathcal{Y}}{\partial \log A} d \log A}_{\Delta \text{ Technology}} + \underbrace{\frac{\partial \log \mathcal{Y}}{\partial \log \mathcal{X}} d \log \mathcal{X}}_{\Delta \text{ Allocative Efficiency}}. \quad (4)$$

We now present a decomposition of changes in aggregate TFP that considers multi-product firms and arbitrary production networks with product-level distortions.

Proposition 1. *Growth Accounting in Production Networks with Multiproduct Firms: To a first order, aggregate TFP can be decomposed into technology and allocative efficiency terms as follows:*

$$d \log TFP = \underbrace{\sum_{i \in N} \tilde{\lambda}_i d \log A_i}_{\Delta \text{ Technology}} + \underbrace{\sum_{i \in N} \tilde{\lambda}_i \text{Cov}_{s_i} \left(d \log \mathbf{p}_i, \frac{1}{\gamma_i} \right)}_{\text{Multiproduct Term}} - \underbrace{\sum_{i \in N} \tilde{\lambda}_i d \log \mu_i - \sum_{f \in \mathcal{F}} \tilde{\Lambda}_f d \log \Lambda_f}_{\text{Single Product Term}} \quad (5)$$

$\Delta \text{ Allocative Efficiency}$

where $\gamma_i = (\gamma_{i1}, \dots, \gamma_{iG})$ and $d \log \mathbf{p}_i = (d \log p_{i1}, \dots, d \log p_{iG})$.

Appendix D presents the proof. The change in aggregate TFP can be decomposed into technology and allocative efficiency terms. The technology term represents a weighted average of changes in firm-level Hicks-neutral productivity using cost-based Domar weights. The allocative efficiency term is further decomposed into a multiproduct firm term, a change in aggregate factor shares, and firm-level average markup changes.

The multiproduct term captures the allocative efficiency implications of firm-level product mix adjustments. The contribution to each product's allocative efficiency depends on the product's price relative to the firm's average price change and the product's relative cumulative wedge in the product's product mix. In economies with joint production, the opportunity cost of changing the product mix is generally not constant. This means that it may vary from firm to firm and product pair to product pair, affecting the product level unit cost, depending on the curvature of the production possibility frontier.

For example, an oil refinery producing gasoline and diesel may face different constraints when adjusting its production mix compared to a dairy farmer producing milk and meat. The observed relative price changes per product contain all the information on the production possibility frontier for this firm. The implication for allocative efficiency of the price change associated with a change in the product mix can then be computed us-

ing the covariance of the relative price changes and the cumulative wedges of the product whose price has changed.

Suppose the price change for a given product g in firm i is higher than the average price change for firm i , and product g is more distorted than the average product within firm i . In this case, the covariance will be positive. Intuitively, the covariance of cumulative wedges at the product level with relative price changes captures the attenuation of resource misallocation due to joint production technology. Intuitively, if prices rise for products with high (cumulative) wedges, then the scope for reallocation is limited. Theoretically, these price movements, net of markups, trace out the production possibility frontier, whose slope captures each firms' technological constraints when adjusting their product mix. To calculate the aggregate effect across the economy, we can sum these firm-level covariances using Domar weights, which indicate its macroeconomic importance. This aggregation allows us to quantify the overall impact of product mix changes on allocative efficiency in the economy.

Regarding the single product term, which consists of factor shares and firm-level markup, if the initial equilibrium is inefficient, the products charging markups are under-produced relative to an efficient economy. Improving the allocation involves reallocating resources to a more distorted part of the economy, such as firms' product pairs that charge relatively high markups. A decrease in factor shares implies reallocating resources to the portion of the economy that has relatively high markups; however, if the change in factor share is due to a change in markup, this is a mechanical change and does not imply reallocation. Therefore, the contribution of the change must be purged, which the firm-level markup term captures. The factor shares and firm-level markup terms are proposed by Baqaee and Farhi (2020). Both terms are valid under a joint production approach and, together with the multiproduct term this work introduces, constitute allocative efficiency.

Relation to Existing Aggregation Theorems

Proposition 1 nests existing aggregation theorems for production networks as a special case.

Corollary 1. *Baqaee and Farhi (2020): If no firms engage in joint production and impose the same markup on all their products (the single-product firm assumption), then to a first order, aggregate*

TFP growth can be decomposed into technology and allocative efficiency terms as follows.

$$d \log TFP = \underbrace{\sum_{i \in N} \tilde{\lambda}_i d \log A_i}_{\text{Technology}} - \underbrace{\sum_{i \in N} \tilde{\lambda}_i d \log \mu_i + \sum_{f \in \mathcal{F}} \tilde{\Lambda}_f d \log \Lambda_f}_{\text{Allocative Efficiency}}.$$

The proof follows from the fact that the covariance term from proposition 1 is zero because the changes in marginal cost and markup for all products within a firm are equal.

Our approach quantifies misallocation through the multiproduct channel by measuring deviations from the single-product, single-markup assumption when product-level data are available; if this assumption holds, the multiproduct term becomes zero. The assumption of uniform marginal costs and markups is unlikely to hold in practice; however, its quantitative relevance remains an empirical question. Our decomposition quantifies the extent to which this assumption is violated and isolates the impact of existing misallocation literature.

Finally, without markups, when prices equal marginal costs, allocative efficiency converges to zero. In this case, all aggregate TFP changes are attributed to technology, aligning with Hulten (1978).

Corollary 2. *Hulten (1978): Growth Accounting in an Efficient Economy:*

$$d \log TFP = \underbrace{\sum_{i \in N} \tilde{\lambda}_i d \log A_i}_{\text{Technology}}.$$

The proof follows from the fact that the markup is always 1, the markup change term is 0, and the sum of factor shares is always 1. Therefore, the sum of factor changes is always 0, and the covariance of the multiproduct term is 0 because $\gamma_i = (\gamma_{i1}, \dots, \gamma_{iG})$ are all 1 in an efficient economy.

Proposition 1's formula converges to Hulten's theorem when the economy is efficient. Measured aggregate TFP growth equals the Domar weighted sum of firm-level productivity changes.

Applying Our Formula to the Simple Example

We revisit our simple example to illustrate how we can apply our decomposition formula without knowing the specific joint production structure. Appendix D provides the proofs

for this example. Consider the economy where firm 1 employs workers to produce two differentiated products using a joint production technology:

$$\left(q_{11}^{\frac{\sigma+1}{\sigma}} + q_{12}^{\frac{\sigma+1}{\sigma}}\right)^{\frac{\sigma}{\sigma+1}} = L_1.$$

Applying our decomposition, we find the following steps. First, we compute the multiproduct term as follows:

$$Cov_{s_i}\left(d\log p_i, \frac{1}{\gamma_i}\right) = -\left(\frac{1}{\sigma+1}\right)\left(\frac{\mu-1}{2(1+\mu)}\right)\epsilon > 0.$$

The single product term is:

$$\underbrace{-\sum_{i \in N} \tilde{\lambda}_i d\log \mu_i}_{\text{Firm-level Markup}} - \underbrace{d\log \Lambda_f}_{\text{Aggregate Labor Shares}} = \left(\frac{\mu-1}{2(1+\mu)}\right)\epsilon.$$

Therefore,

$$\Delta \log TFP = \frac{\sigma}{\sigma+1} \left(\frac{\mu-1}{2(1+\mu)}\right)\epsilon > 0.$$

We recover the results from subsection 3.2.2 using Proposition 1. Importantly, we can see that the multiproduct term partially offsets the single-product term. The single-product term suggests a positive reallocation effect due to the markup reduction. However, the multiproduct term, which is negative, dampens this effect. This dampening occurs because joint production constrains the firm's ability to freely adjust its product mix in response to the markup shock. The net effect on TFP, although still positive, is smaller than what would be predicted if only the single product term were considered.

This example illustrates that ignoring the rigidities associated with joint production can lead to overestimating the TFP contribution of misallocation. Our sufficient statistics automatically detect whether each firm engages in joint production without the need to identify a parametric elasticity of transformation. Some firms may operate independent product lines, while others face strict constraints when they adjust their product mix. The relative price variations of products sold to different buyers within the firm contain all relevant information about the cost of exchanging two goods.

4 Data and Construction of Sufficient Statistics

Our analysis relies on a dataset that covers the universe of formal firms operating in Chile from 2016 to 2022; the data are from the Chilean Internal Revenue Service (Servicio de Impuestos Internos, SII). The Chilean tax system requires all formal firms to declare their invoices for transactions with other firms, which provides detailed information on every product, quantity, and price traded. Additionally, we use tax accounting declarations, which provide monthly data on each firm's revenue and input expenditures, including capital and labor costs. A key advantage of the SII data is its use of unique identifiers for firms and workers, which allows individual and firm data to be merged across datasets. We utilize four distinct sources from SII.

The first is the value-added tax form, which includes gross monthly firm sales, materials expenditures, and investment.

Second, the SII provides information from a matched employer–employee census of Chilean firms from 2005 to 2022. Specifically, firms must report all payments to individual workers, including the sum of taxable wages, overtime, bonuses, and any other labor earnings for each fiscal year. All legal firms must report to the SII; thus, the data cover the total labor force with a formal wage contract, representing roughly 65% of employment in Chile. For any given month, it is possible to identify an individual worker's employment status, their average monthly labor income that year, a monthly measure of total employment, and the distribution of average monthly earnings within the firm.

Third, income tax form data includes yearly information on all sources of a firm's income and expenses. This form allows for computing every individual's actual tax payments for each year. Details on sales and employment are available on this form; however, we use only data on capital stock for each firm and year. This approach allows us to build perpetual inventories using data from the monthly F22 form. We obtain the user cost of capital by multiplying nominal capital stock by the real rental rate of capital, which is built using publicly available data. We use the 10-year government bond interest rate minus expected inflation plus the external financing premium. Finally, we use the capital depreciation rate from the LA-Klems database.

Fourth are electronic tax documents from 2016 onward. These documents provide information on each product (price and quantity) traded domestically or internationally with at least one Chilean firm. We only use domestic transactions and observe the firm-to-firm transactions and a firm's sales (including firm-to-firm and firm-to-consumer sales). We compute firm-specific product shares for firm-to-firm transactions and assume that

their distributions are equivalent to firm-to-consumer transactions to recover the complete distribution of firm sales by product. Each firm-to-firm transaction includes a “detail” column that records the name of each product.

Building on the data cleaning process described in section 2, we process the data to construct product code-level output and input-price indices for each firm using standard Tornqvist indices. We aggregate products into a 290 product-code identifier to facilitate comparison between firms, allowing us to estimate product production functions that use the same product across firms.

4.1 Data Cleaning and Implementation Strategy

We begin the data processing by applying filters to the raw data to obtain the final database for empirical analysis. We define a firm as a taxpayer with a tax ID, positive sales, positive materials, positive wage bill, and capital for any given year. We exclude firms that hire less than two employees a year or have capital valued below US\$20 in a year. All variables are winsorized at the 1% and 99% levels to mitigate measurement error.

We selected 2016 as the base year for price indices because it was the first year we observed prices for firm-to-firm transactions. This method is widely recognized for estimating aggregate production functions at the firm or plant level when price data is accessible. We use crosswalks developed at the Central Bank of Chile (Acevedo et al. (2023)) to address the challenge of product aggregation (from around 15 million products to 290 product codes). We create aggregated product-level quantity produced and material usage indices, matching product descriptions and characteristics to ensure consistency across firms and over time.

4.2 Construction of Sufficient Statistics

We measure five distinct objects to implement the growth accounting framework that includes the multiproduct channel: (1) product-level cost-based Domar weights $\tilde{\lambda}$, (2) product-firm level price indices, (3) product-level markups μ , (4) cumulative wedges, and (5) aggregate objects. We discuss each of these in the following subsection.

4.2.1 Product-Level Cost-Based Domar Weights

The product cost-based Domar weights can be calculated using the following equation:

$$\tilde{\lambda}' \equiv b' \tilde{\Psi} = b' + b' \tilde{\Omega} + b' \tilde{\Omega}^2 + \dots$$

To compute these weights, we must measure value-added shares (b) and the input-output matrix ($\tilde{\Omega}$). We measure these objects directly from the data.

Final expenditure shares (b) are represented by a vector of dimension $(N\mathcal{G} + \mathcal{F}) \times 1$. Here, N is the number of firms, \mathcal{G} is the number of products, and \mathcal{F} is the number of factors. The first $N\mathcal{G}$ entries are calculated as the residual between a firm product's total sales and its intermediate sales to other firms (measured from the firm-to-firm data). This approach provides a theory-consistent measure of final expenditures. The final \mathcal{F} entries are set to zero because households do not directly purchase factors. Using firm-to-firm records and factor expenditures, we construct the input-output matrix $\tilde{\Omega}$ at the product-firm level.

Specifically, we compute an annual cost-based input-output matrix by product. We calculate the denominator of each element (indexed by ig, jg') by summing a firm's purchases from all its suppliers, its wage bill, and its capital multiplied by the relevant user cost rental rate of capital. The last two elements of the matrix have wage bills and capital expenditures as their numerators.

The resulting $\tilde{\Omega}$ is a $(N\mathcal{G} + 2) \times (N\mathcal{G} + 2)$ matrix that can be expressed as follows:

$$\tilde{\Omega} = \left[\begin{array}{ccc|cc} \tilde{\Omega}_{11,11} & \cdots & \tilde{\Omega}_{11,N\mathcal{G}} & \tilde{\Omega}_{11,N\mathcal{G}+1} & \tilde{\Omega}_{11,N\mathcal{G}+2} \\ & & \ddots & & \\ & & & \tilde{\Omega}_{N\mathcal{G},N\mathcal{G}+1} & \tilde{\Omega}_{N\mathcal{G},N\mathcal{G}+2} \\ \hline 0 & \cdots & 0 & 0 & 0 \\ 0 & \cdots & 0 & 0 & 0 \end{array} \right].$$

Based on the separability assumption, the same expenditure share applies to all products g that firm i produces. The expressions for $\tilde{\Omega}_{ig,jg'}$ and $\tilde{\Omega}_{ig,f}$ are as follows:

$$\tilde{\Omega}_{ig,jg'} = \frac{p_{jg'} x_{i,jg'}}{\sum_{j,p} p_{jg'} x_{i,jg'} + \sum_f w_f L_{if}}, \quad \tilde{\Omega}_{ig,f} = \frac{w_f L_{if}}{\sum_{j,p} p_{jg'} x_{i,jg'} + \sum_f w_f L_{if}}.$$

Factors do not require inputs; thus, the last row of the matrix is zero.

After calculating the product-level cost-based Domar weights, we sum them for the same firms to compute the firm-level cost-based Domar weights and their shares. These

will be inputs for Proposition 1.

$$\tilde{\lambda}_i = \sum_{g \in \mathcal{G}} \tilde{\lambda}_{ig}, \quad s_{ig} = \frac{\tilde{\lambda}_{ig}}{\tilde{\lambda}_i}.$$

4.2.2 Product-Firm Level Price Indices

We observe prices for each transaction and aggregate them into 290 product categories. We construct two types of price indices: output and input price indices. We compute firm-product-specific annual price indices for the output price index, which is an input to sufficient statistics that deflates product output for production function estimation. The original data are at the “detail” product level, which we aggregate to a Tornqvist index for each 290 product category the firm owns. Specifically, we construct the following price index:

$$\Delta \log P_{igt} = \sum_{d \in g} \frac{s_{idt} + s_{idt-1}}{2} \Delta \log P_{idt},$$

where d is the detailed category belonging to the upper product category (290 product codes). $\Delta \log P_{idt}$ is the price change, and s_{idt} is the share at time t in the continuing products in category g . We construct our price index with 2016, the starting year of the data, as the base year. We also construct an input price index to deflate material costs for production function estimation. We define one aggregate index per firm because aggregate materials are used as inputs in production function estimation. The construction method is the same as for the output price index.

4.2.3 Cumulative Wedges

To construct the cumulative wedge measure, we need product cost-based Domar weights, product sales shares, and product markups :

$$\Gamma_{ig} \equiv \underbrace{\frac{\tilde{\lambda}_{ig}}{salesshare_{ig}}}_{\text{Downstream wedge}} \times \underbrace{\mu_{ig}}_{\text{own markup}}.$$

As discussed in Section 3, the ratio of cost-based Domar weights to sales share represents the cumulative wedge accumulated downstream of a product. These two components can be computed from our data. However, the markup requires estimation. We

address it by two strategies.

First, we estimate product-level markups using the production function approach based on Dhyne et al. (2022). This method extends the Akerberg et al. (2015) production function estimation technique to a joint production setting. The approach relies on cost minimization principles to identify unobserved marginal costs for each firm’s product. We employ a Cobb–Douglas production function with three factors: capital, labor, and materials.¹⁰

The second method is the accounting approach. Specifically, we assume that the markup within a firm is the same and uses firm-level sales divided by firm-level costs (the accounting approach). This method recovers the markup without using any parametric assumptions. Note that while we fix the initial level of markups in both approaches, we allow for within-firm markup changes as captured by prices.

The cumulative wedge, Γ , is essential in constructing the multiproduct term, and it is vital to understand whether this variation arises from downstream wedges or a product’s markup. For this analysis, we use markup estimated by the production function technique.

¹⁰Our results show a product-level markup median of 1.22. Please refer to Appendix C for a detailed explanation of the methodology.

Table 4: Variance decomposition of $\log \Gamma$

| Year | Downstream wedge | Own markup | Covariance |
|------|------------------|------------|------------|
| 2016 | 103.3% | 0.6% | -3.9% |
| 2017 | 102.3% | 0.7% | -3.0% |
| 2018 | 102.5% | 0.6% | -3.1% |
| 2019 | 102.8% | 0.6% | -3.5% |
| 2020 | 103.2% | 0.7% | -3.8% |
| 2021 | 103.7% | 0.6% | -4.3% |
| 2022 | 104.9% | 0.7% | -5.6% |

Notes: We compute the variance decomposition of the logarithm of $\Gamma_{ig} \equiv \frac{\tilde{\lambda}_{ig}}{salesshare_{ig}} \times \mu_{ig}$ for each year. $Var(\log \Gamma_{ig}) = Var(\log(\tilde{\lambda}_{ig}/salesshare_{ig})) + Var(\log \mu_{ig}) + 2Cov(\log(\tilde{\lambda}_{ig}/salesshare_{ig}), \log \mu_{ig})$. The first term on the right-hand side is the variance of downstream wedges. The second term is the variance of their own markup, and the last is the covariance of both. We report the percentage of each term on the right-hand side that explains the total variance.

Table 4 presents the variance decomposition of Γ by year. The results show that most variation in Γ stems from downstream distortions, with a minimal contribution from the product’s markup. This finding is unsurprising, given that downstream distortions represent cumulative wedges throughout the downstream supply chain of the entire economy. In contrast, μ represents a product’s own markup. This result implies that the downstream distortions faced by each pair of firms and products are highly heterogeneous when considering product- and firm-level production networks.

When we apply Proposition 1 to the data, the cumulative wedges (the multiproduct term using Γ as input) will be less sensitive to markup estimates. In our application, we show the two methods produce almost identical aggregate implications.

Ranking of Downstream Wedges

The analysis reveals that cumulative wedges primarily represent accumulated downstream wedges rather than product-level markups. We ranked products by their cumulative wedges to better understand which products face increased downstream wedge. Below, we describe and discuss the major product categories. Appendix B presents the

complete list of the 30 top and bottom items.

The product categories with the greatest (downstream) wedge mainly comprise business services. For example, insurance brokerage services top the list, followed by employment services (recruitment and supply), electricity distribution to businesses, and postal and courier services. These products are usually upstream inputs that other firms use in production, suggesting insufficient size as wedges accumulate through the supply chain before they reach final demand. An important exception is tobacco, a product close to final demand but ranked high (15th) because of the large wedge accumulated (59.7% tax rate vs 19% VAT tax for other products).

Conversely, the least distorted products include cakes, beer, pet food, personal services such as hospitals, and minerals (copper, silver, and molybdenum), that are Chile's primary export industry. These products are common downstream products close to Chile's final demand. As a result, the number of supply chains that reach the final consumer is relatively small, and inefficiencies are relatively less likely to accumulate.

4.2.4 Aggregate Objects

In addition to product cost-based Domar weights and cumulative wedges, we must measure aggregate objects to implement the sufficient statistics presented in Proposition 1. In particular, Y , L , K , Λ_L , and Λ_K denote aggregate value-added, employment, capital, and labor and capital shares, respectively. We measure Y , L , and K as the sum of value added, employment, and capital, respectively, for all firms in the economy. Factor shares of GDP, Λ_L and Λ_K , are measured as total compensation and capital with user cost of capital divided by GDP. Real GDP is calculated by deflating GDP with the official GDP deflator.

5 Application: Decomposing Aggregate TFP Growth

This section applies Proposition 1 to analyze aggregate TFP growth for the Chilean economy. Our analysis covers 2016 to 2022, during which Chile's aggregate TFP stagnated and decreased at the margin. This productivity trend aligns with the pattern of productivity stagnation observed in Chile using different computation methods.¹¹

We begin by presenting results using the standard assumption in the literature of

¹¹CNEP (2023)

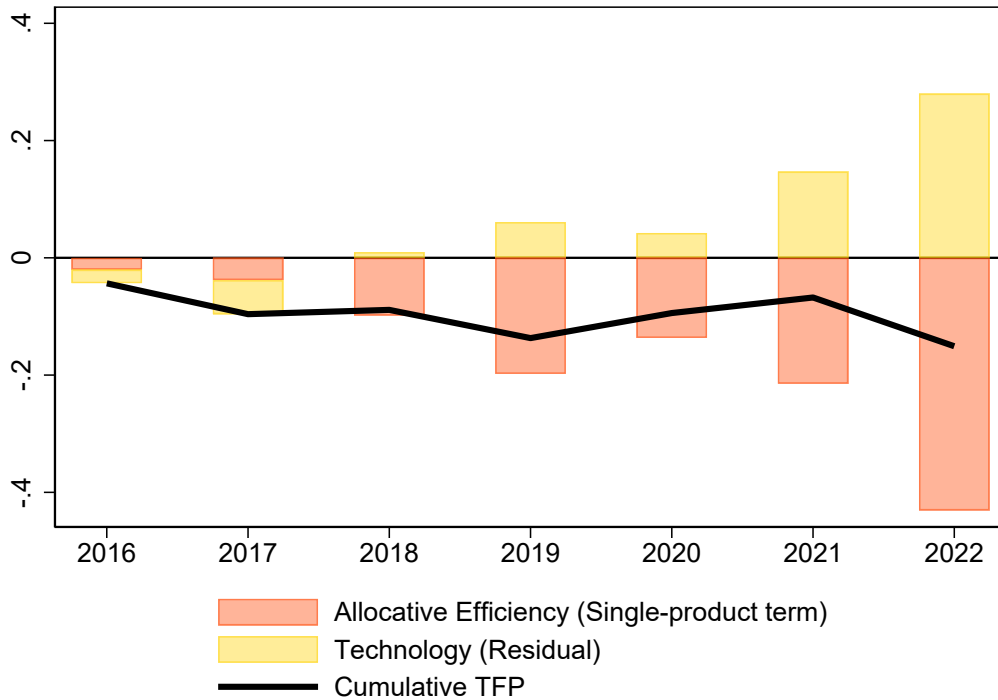
single-product firms. If firms produce a single product, then Corollary 1 applies:

$$d \log TFP = \underbrace{\sum_{i \in N} \tilde{\lambda}_i d \log A_i}_{\text{Technology}} - \underbrace{\sum_{i \in N} \tilde{\lambda}_i d \log \mu_i + \sum_{f \in \mathcal{F}} \tilde{\Lambda}_f d \log \Lambda_f}_{\text{Allocative Efficiency}}.$$

This approach implements growth accounting but overlooks multiproduct firms engaged in joint production. Figure 5 illustrates the decomposition of cumulative changes in aggregate TFP from 2016 to 2022 under this assumption.

Figure 5 shows that the allocative efficiency term (in red) declined over this period. This outcome suggests that high-markup firms contracted further, resulting in a negative reallocation effect; however, the contribution of allocative efficiency exceeds that of the technology (residual) component, particularly during the COVID-19 pandemic and the subsequent high inflation period. To rationalize this disparity, the technology term, measured as a residual, must have increased by about 20%.

Figure 5: Cumulative TFP growth decomposition: Ignoring multiproduct term



Notes: This Figure shows the cumulative change calculated by applying Corollary 1 repeatedly each year. Technology (residual) is calculated by subtracting allocative efficiency from TFP growth.

Next, we incorporate the multiproduct term using Proposition 1:

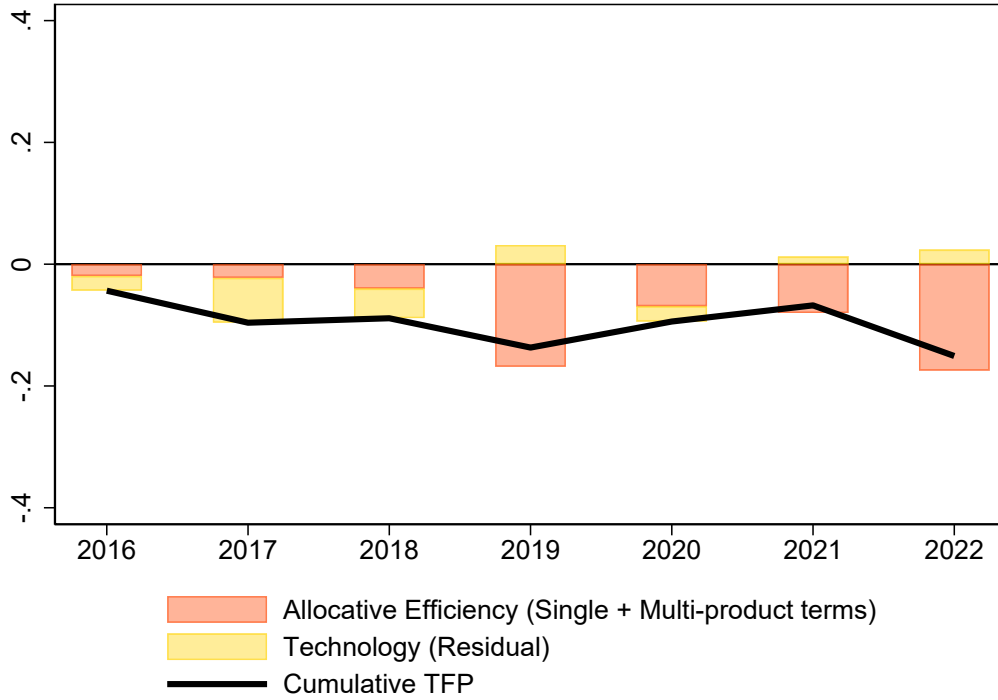
$$\begin{aligned}
d \log TFP = & \underbrace{\sum_{i \in \mathcal{N}} \tilde{\lambda}_i d \log A_i}_{\Delta \text{ Technology}} + \underbrace{\sum_{i \in \mathcal{N}} \tilde{\lambda}_i Cov_{s_i} \left(d \log p_i, \frac{1}{\gamma_i} \right)}_{\text{Multiproduct term}} - \underbrace{\sum_{i \in \mathcal{N}} \tilde{\lambda}_i d \log \mu_i}_{\text{Firm-level Markup}} - \underbrace{\sum_{f \in \mathcal{F}} \tilde{\Lambda}_f d \log \Lambda_f}_{\text{Aggregate Factor Shares}} . \\
& \underbrace{\hspace{10em}}_{\Delta \text{ Allocative Efficiency}}
\end{aligned} \tag{6}$$

Figure 6 presents the results incorporating the multiproduct term, which reduces the magnitude of the technology (residual) observed in Figure 5. In other words, the multiproduct and single-product misallocation terms account for a larger portion of aggregate TFP movements during the COVID-19 pandemic and the resultant high inflation period.

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¹²Appendix A.2 shows that the accounting markup approach has almost the same aggregation implications.

Figure 6: Cumulative TFP growth decomposition with multiproduct term

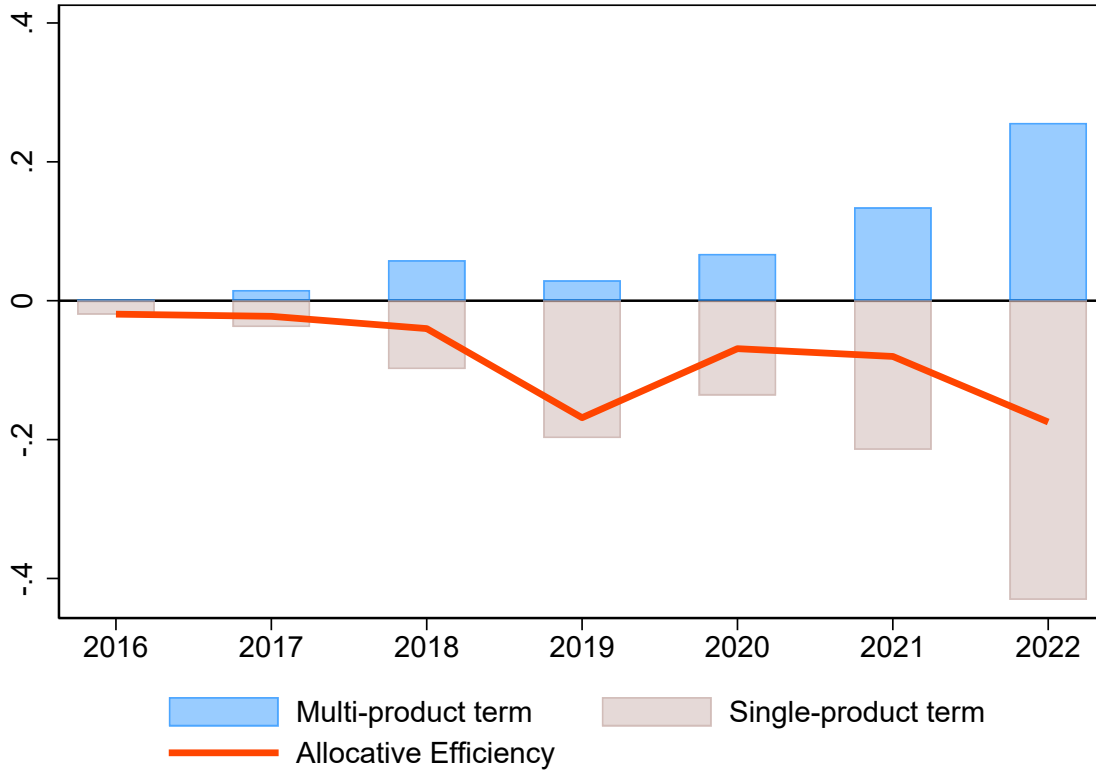


Notes: This Figure shows the cumulative log change calculated by repeatedly applying the equation from Proposition 1 each year. Technology (residual) is calculated by subtracting allocative efficiency from aggregate TFP growth.

Reallocation effects that consider joint production explain 86% of the observed aggregate TFP growth. Conversely, as shown in Figure 5, ignoring joint production leads to overestimating resource misallocation. This result suggests that considering joint production considerably decreases the reallocation implied under the traditional assumption that firms produce only single products.

This finding is consistent with the joint production mechanism described in Section 3. When firms engage in joint production, they create multiple products using common inputs. When a given product receives a shock, if firms face technological constraints to adjust their product mix (non-infinite elasticity of transformation), firms will struggle to reallocate productive resources from one product to another. The reallocation through substitution among products within multi-product firms is dampened, and reallocation is not materialized to the extent suggested under the single-product firm assumption.

Figure 7: Cumulative TFP growth Decomposition with multiproduct term



Notes: This Figure decomposes the cumulative change in allocative efficiency in Figure 6 into single-product and multi-product terms.

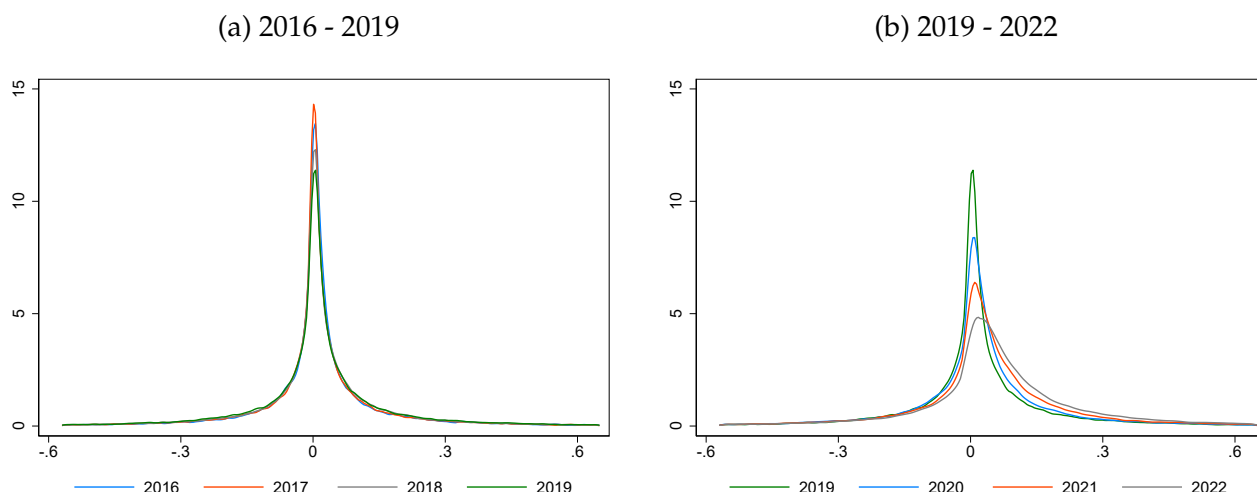
Furthermore, Figure 7 breaks down the allocative efficiency in Figure 6 into multi-product and single product terms. It shows that the offsetting of reallocation due to joint production will become particularly strong after 2020. During this period, the economy was disrupted by COVID-19 and subsequent high inflation. We interpret the latter as firms facing changes in product-specific demands, which changed their total demand composition. In response, firms were willing to readjust their product mix by reallocating productive resources. However, due to the non-infinite elasticity of transformation, firms were constrained to change their product mix.

Finally, the granularity of the data allows us to track the distributional changes of joint production (the multiproduct term) that limit the extent of resource reallocation. Since the covariance degenerates to zero under the single-product firm assumption, the dispersion of covariance implies that joint-production forces are active. These distributions vary from period to period. Figure 8a plots the distribution for pre-COVID-19 (2016–2019),

which is symmetric around 0, with slight differences from year to year.

Figure 8b presents the distribution after the onset of COVID-19, showing a shift to the right from year to year, resulting in a right-skewed distribution. This result suggests that the increase in the contribution from joint-production forces (the multiproduct term) was not driven by a few specific firms.

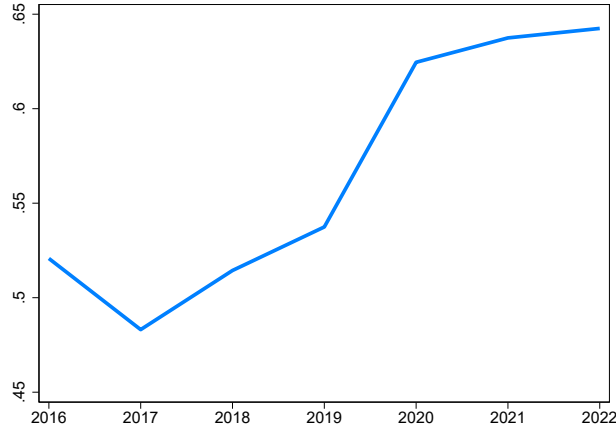
Figure 8: $Cov_{s_i} \left(d \log p_i, \frac{1}{\gamma_i} \right)$ distributions by year



Notes: These Figures plot the distribution of firm level $Cov_{s_i} \left(d \log p_i, \frac{1}{\gamma_i} \right)$ for each year.

Finally, Figure 9 plots the median variance of product-specific production changes across firms from 2016 to 2022. This figure provides suggestive evidence that aligns with the changing distribution of multi-product firms shown in Figure 8b and corresponds to the period of significant contribution from the multi-product term in our decomposition. The increasing variance, particularly the sharp rise from 2019 to 2020 and its sustained high level thereafter, indicates that firms have been under greater pressure to adjust their product mix. This trend coincides with the timeframe when we observe the most substantial impact of the multi-product term on allocative efficiency. The temporal consistency between the increased variance in product-specific production changes and the heightened contribution of the multi-product term reinforces our model's emphasis on the importance of multi-product firms engaged in joint production, especially during major economic shocks like the COVID-19 pandemic.

Figure 9: Large Product Mix Adjustments and Efficiency: Suggestive Evidence



Notes: This figure depicts the evolution of the median variance of product quantity changes, denoted as $Var_{\lambda_i}(d \log q_{ig})$, from 2016 to 2022.

6 Extension: Ex-Ante Structural Results

This section develops a structural framework to predict how economies with multiproduct firms respond to shocks. While our previous analysis relied on observed price and factor share changes, we now model these endogenous responses explicitly. This theoretical extension allows us to move beyond ex-post measurement to ex-ante prediction of counterfactual scenarios. The framework complements our earlier results.

We show how to apply this framework to study the distance to the Pareto-efficient frontier when firms use joint production technology. This method compares output in an efficient equilibrium (with all markup wedges removed) to that in a distorted decentralized economy. Our analysis demonstrates how the theoretical results of previous studies, such as those of Hsieh and Klenow (2009) and Baqaee and Farhi (2020), change when firms engage in joint production. Since an economy without markups is unobservable, a model is necessary to analyze this counterfactual case.

6.1 The Nested CET-CES Model

We propose the nested CET-CES model, which provides a tractable framework for our analysis. We use the same setup with subsection 3.3 but impose the CET-CES functional

form to the joint production function to the equation 3. We then derive a linear system for price and sales responses, allowing us to characterize the economy's response to shocks. The production technology is given by:

$$\underbrace{\left(\sum_g \delta_{ig} [q_{ig}]^{\frac{\sigma_i-1}{\sigma_i}} \right)^{\frac{\sigma_i}{\sigma_i-1}}}_{\text{Output bundle}} = A_i \underbrace{\left(\sum_g \omega_{i,jg'} q_{i,jg'}^{\frac{\theta_i-1}{\theta_i}} + \omega_{i,L} L_i^{\frac{\theta_i-1}{\theta_i}} + \omega_{i,K} K_i^{\frac{\theta_i-1}{\theta_i}} \right)^{\frac{\theta_i}{\theta_i-1}}}_{\text{Input Bundle}}. \quad (7)$$

Here, σ_i represents the elasticity of transformation between different outputs, A_i denotes the productivity of firm i , and δ_{ig} are the output share parameters. The input bundle comprises intermediate inputs $q_{i,jg'}$, labor L_i , and capital K_i , aggregated using a CES function with an elasticity of substitution θ_i . Note that this class of models is highly general, nesting the nested CES system widely used in macroeconomics and international economics as a special case. For single-output firms, the production function degenerates to:

$$q_i = A_i \underbrace{\left(\sum_{j \in N, g' \in \mathcal{G}} \omega_{i,jg'} q_{i,jg'}^{\frac{\theta_i-1}{\theta_i}} + \omega_{i,L} L_i^{\frac{\theta_i-1}{\theta_i}} + \omega_{i,K} K_i^{\frac{\theta_i-1}{\theta_i}} \right)^{\frac{\theta_i}{\theta_i-1}}}_{\text{Input bundle}}. \quad (8)$$

Furthermore, we specify the household utility function as a CES aggregator over final consumption goods. Formally, the representative household's utility function is given by:

$$U(c_{11}, \dots, c_{ig}, \dots, c_{NG}) = \left(\sum_{i \in N, g \in \mathcal{G}} \psi_{ig} c_{ig}^{\frac{\theta_0-1}{\theta_0}} \right)^{\frac{\theta_0}{\theta_0-1}} \quad (9)$$

where c_{ig} represents the consumption of product g from firm i , ω_{ig} represents the taste parameter for each product, and θ_0 is the elasticity of substitution between products.

For analytical simplicity, we assume a uniform substitution elasticity within the firm's CES structure, though extending the model to incorporate further nesting would be straightforward.

6.2 Linear System for Price and Sales Responses

Using this model, we construct a system for solving the first-order response to primitive shocks (A, μ) to the endogenous variables. This system of linear equations, derived from

the model's first-order conditions, enables us to generate ex-ante predictions of how the economy will respond to counterfactual shocks. We begin with the multi-product firm's forward equation under the CET output function:

Proposition 2. *Multi-Product firm's forward equation under CET output function:*

$$\begin{aligned}
d \log p_{ig} = & - \underbrace{\sum_{j \in \mathcal{N}, g' \in \mathcal{G}} \tilde{\Psi}_{ig,jg'} (d \log \mu_j - d \log A_j) + \sum_{f \in \mathcal{F}} \tilde{\Psi}_{ig,f} d \log \Lambda_f}_{\text{Indirect cost exposure}} \\
& + \underbrace{\sum_{j \in \mathcal{N}, g' \in \mathcal{G}} c_j^R \tilde{\Psi}_{ig,jg'} d \log \Theta_{jg'}}_{\text{Indirect exposure to the Product Mix adjustment}},
\end{aligned}$$

where

$$d \log \Theta_{jg'} = \left(\frac{\sigma_j}{\sigma_j + 1} \right) d \log \mu_{jg'} / \mu_{jr} + \frac{1}{(1 + \sigma_j)} \left[d \log \lambda_{jg'} / \lambda_{jr} \right],$$

and $c_j^R = \frac{mc_{jr}q_{jr}}{\sum mc_{jg}q_{jg}}$ is a reference product cost share. Here, r denotes a reference good.

This equation describes how changes in unit prices within a firm due to markups, productivity shocks, and price changes associated with endogenous product mix adjustments are transmitted through production networks to other firms' products. The first term illustrates the effect of exposure to common cost shocks on prices, a force present in standard production network models. The second term, unique to the joint production model, indicates the exposure of firms with nonlinear production possibility frontiers to modify their product mix due to reallocation, which affects endogenous unit costs. The magnitude of this effect depends on the transformation elasticity, with lower elasticities leading to larger cost effects. As σ approaches infinity, this endogenous effect disappears, as firms can freely adjust their product mix.

Next, we consider backward propagation:

Proposition 3. *Backward propagation:*

$$\begin{aligned}
\lambda_{ig} d \log \lambda_{ig} = & - \sum_{j \in \mathcal{N}, g' \in \mathcal{G}} \lambda_{jg'} (\Psi_{jg',ig} - \mathbf{1}(ig = jg')) d \log \mu_{jg'} \\
& + \sum_{k \in \mathcal{N}, g'' \in \mathcal{G}} \frac{\lambda_{kg''}}{\mu_{kg''}} (1 - \theta^k) \text{Cov}_{\tilde{\mathcal{Q}}_{(kg'', :)}} (d \log p, \Psi_{(:,ig)}).
\end{aligned}$$

This equation, based Baqaee and Farhi (2020) methodology, describes the sales and factor share response. Notably, it does not contain σ , indicating that joint production only affects the changes in sales share via substitution effects. The equation shows how shocks propagate through the production network, affecting markups and quantities of each product via changes in upstream suppliers' price indices and productivity.

By combining the forward equation from Proposition 1 and the backward equation from Proposition 2, we obtain a complete system of linear equations that characterizes the economy's response to shocks. This system consists of $2 \times (1 + \mathcal{N} \times \mathcal{G} + \mathcal{F})$ equations and $2 \times (1 + \mathcal{N} \times \mathcal{G} + \mathcal{F})$ unknowns, where \mathcal{N} is the number of firms, \mathcal{G} is the number of products, and \mathcal{F} is the number of factors. This system of equations fully characterizes the first-order response of all endogenous variables to any combination of productivity or markup shocks. By solving this linear system using standard matrix algebra, we can conduct counterfactual analyses and evaluate the impact of various shocks on the economy's equilibrium outcomes.

6.3 Distance to the Pareto-Efficient Frontier

Using our model, we can characterize the distance to the Pareto-efficient frontier when introducing distortions, allowing us to predict efficiency losses from counterfactual changes in markups or other distortions:

Proposition 4. *Under joint production, starting at an efficient equilibrium, up to second order, and in response to the introduction of distortions, changes in the TFP are given by Domar-weighted Harberger triangles:*

$$\mathcal{L} = \frac{1}{2} \sum_{ig} \lambda_{ig} d \log q_{ig} d \log \mu_{ig}, \quad (10)$$

where λ_{ig} is the Domar weight of product g from firm i , q_{ig} is the quantity, and μ_{ig} is the markup.

This result demonstrates that TFP changes resulting from the introduction of distortions are solely determined by three statistics: Domar weights of each product; the magnitude of the wedges; and the change in the quantity of the product. The quantity change can be derived from sales and price changes given by our linear system, using the relationship $d \log q = d \log \lambda - d \log p$. Since this system includes transformation elasticity σ , it is generally affected by the elasticity value. To illustrate how joint production affects the Distance to the Frontier, we provide analytical solutions for two examples.

6.3.1 Horizontal Economy with Joint Production

We consider a horizontal economy similar to Hsieh and Klenow (2009) but with use joint production technologies. This allows us to investigate within-firm markup heterogeneity in the presence of production transformation constraints. In an economy with a representative consumer (CES utility with elasticity θ), N firms each use a shared input \mathcal{L} to produce \mathcal{G} products using CET technology (elasticity σ). Markups μ_{ig} are heterogeneous across products and firms.

Proposition 5. *The distance to the frontier in the horizontal economy is given by:*

$$\mathcal{L} = -\frac{1}{2}\theta \left(\text{Var}_{\lambda}(d \log \mu_{ig}) - \frac{1}{\sigma + 1} \mathbb{E}_{\bar{\lambda}} \left\{ \text{Var}_{s_i}(d \log \mu_{ig}) \right\} \right),$$

where $\text{Var}_{\lambda}(d \log \mu_{ig})$ and $\text{Var}_{\lambda_i}(d \log \mu_{ig})$ represent the Domar weighted variance of markup change and the average of within-firm variances of markup changes, respectively. Vectors $\lambda = (\lambda_{11}, \lambda_{12}, \dots, \lambda_{NG})$, $\bar{\lambda} = (\lambda_1, \lambda_2, \dots, \lambda_N)$ are the vectors of firm-level revenue and cost Domar weights with $\lambda_i = \sum_g \lambda_{ig}$, and $s_i = (\lambda_{i1}/\lambda_i, \lambda_{i2}/\lambda_i, \dots, \lambda_{iG}/\lambda_i)$.

Proposition 5 characterizes the distance to the frontier in a horizontal economy with joint production and heterogeneous markups. The distance to the frontier comprises two markup variances. The first term means the product-level markup's Domer-weighted variance gives the distance to the frontier. On the other hand, the second term, which is related to joint production, means that the finite elasticity of transformation σ attenuates the effect of the variance of the markup within the firm. As σ increases and approaches infinity, the force of attenuation associated with joint production approaches zero. Conversely, as σ approaches zero (which implies Leontief production technology), the importance of within-firm markup dispersion decreases. We give this result in the following formal corollary.

Corollary 3 (Limit Cases). *The distance to the frontier simplifies in extreme cases of the elasticity of transformation:*

1. As $\sigma \rightarrow \infty$ (perfect substitution between products):

$$\mathcal{L} = -\frac{1}{2}\theta \text{Var}_{\lambda}(d \log \mu_{ig}).$$

2. As $\sigma \rightarrow 0$ (Leontief production technology):

$$\mathcal{L} = -\frac{1}{2}\theta \text{Var}_\lambda(\mathbb{E}_{s_i}(d \log \mu_i)).$$

In the case of perfect substitutes, misallocation depends on the variance of markups across all products. This term can be obtained by considering the product as an independent firm and applying the results of Baqaee and Farhi (2020). Conversely, in the Leontief case, only the variance of markups between firms is relevant. This is a consequence of the law of total variance.

These results imply that, in the absence of within-firm markup dispersion, the term related to joint production disappears regardless of the value of σ . However, this reasoning is specific to horizontal economies. This relationship easily breaks down in more complex economic structures that include firm-to-firm networks, and σ remains essential even when markups within firms are homogeneous. To illustrate this, we consider a simplified network example examined in Section 3.

6.3.2 Simplified Network Economy

Example 2. TFP Losses in a Simplified Network Economy: The second-order social loss is given by:

$$\mathcal{L} = -\frac{1}{4}\left(\frac{\sigma}{\sigma+1}\right)(d \log \mu)^2. \quad (11)$$

Despite no within-firm markup dispersion, σ appears in the loss function, demonstrating that network structure and transformation elasticity jointly determine the distance to the frontier.

Welfare losses decrease as σ increases. We obtain an upper bound on social loss as σ approaches infinity. When σ approaches zero (Leontief case), no misallocation occurs regardless of markup size.

These findings emphasize the importance of considering network structures and transformation elasticities when evaluating misallocation and efficiency in economies with joint production, even without within-firm markup dispersion.

7 Conclusion

This paper develops a theoretical framework to aggregate distortions in production networks with multiproduct firms. We assess their impact on aggregate TFP growth and derive a nonparametric sufficient statistic to describe allocative efficiency with multiproduct firms engaging in joint production.

We apply the framework to a comprehensive Chilean firm-to-firm transaction database. Reallocation effects considering joint production explain 86% of the observed aggregate TFP growth. Conversely, ignoring joint production leads to overestimating resource misallocation.

We demonstrate the importance of considering joint production in understanding aggregate TFP dynamics, especially during economic disruptions. The constraints multiproduct firms face in adjusting their product portfolios reduce reallocation within the network that single-product models would predict.

Our analysis reveals that joint production, a previously understudied source of TFP growth, can be of first-order importance. Our results demonstrate how aggregating granular microdata, through the lens of theory, can reduce the measure of our ignorance as captured by aggregate TFP and provide new insights into the drivers of economic growth.

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Appendix for “Aggregating Distortions in Networks with Multi-Product Firms”

A Additional Figures and Tables

A.1 Additional Empirical Results for Reduced-Form Evidence

In this appendix section, we present additional Figures and Table for the event study analysis shown in Section 2.

Characteristics of Treatment Firms

Table A1 displays the characteristics of treated and control firms.

Table A1: Characteristics of Treatment Firms

| | Treatment Firms | Control Firms |
|---|-----------------|---------------|
| Number of firms | 26,411 | 96,321 |
| Number of workers | 6 | 4 |
| Number of products sold | 16 | 10 |
| Number of producers | 107 | 119 |
| Number of buyers | 59 | 26 |
| Annual revenue (million pesos) | 186 | 101 |
| Annual total intermediate purchases (million pesos) | 107 | 59 |
| Share of firms in manufacturing | 0.21 | 0.24 |
| Share of firms in Retail and wholesale | 0.44 | 0.39 |
| Share of firms in Services | 0.22 | 0.21 |

Notes: This Table presents the characteristics of treated firms (those whose major product buyers experienced lockdowns in March 2020) and control firms, showing values from February 2020, the month before the shock. The rows for the number of workers, products sold, providers, buyers, revenue, and total intermediate purchases display the median of each statistic. The industry shares indicate the proportion of firms within each group belonging to specific industries.

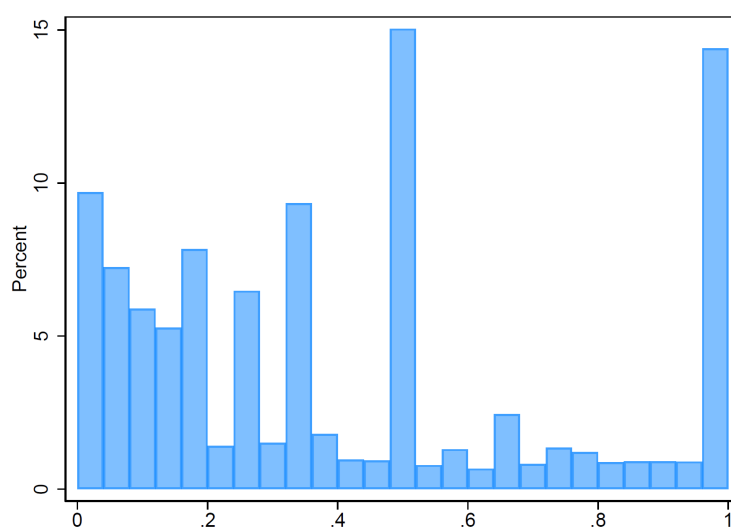
Firms Sell Different Products to Distinct Sets of Buyers

We construct the following measure to characterize the heterogeneity from the intermediate inputs buyer perspective across products and within firms:

$$S_i = \frac{\text{number of buyers of the main product of firm } i}{\text{number of buyers of firm } i},$$

where the main product is that with the largest sales within firm i in 2018. Figure A1 presents the distribution of this measure across firms.

Figure A1: Buyer heterogeneity across product within firm



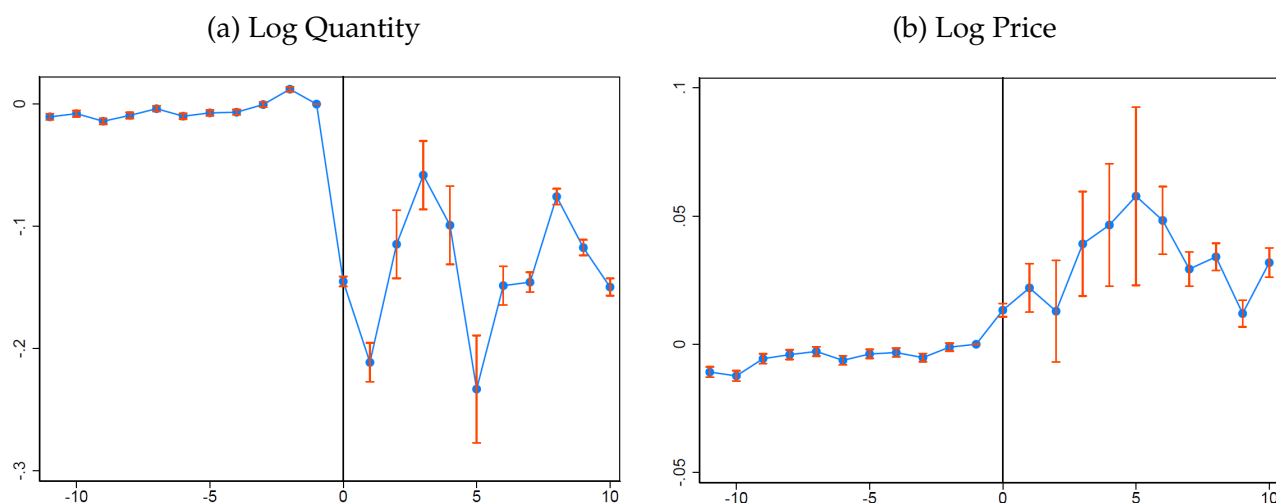
Notes: Histogram of the number of buyers buying the main product of firm i / number of buyers in firm i for a multiproduct firm. The main product is the product with the highest sales within that firm. Data are from 2018.

If the buyers of the seller's main product and its other products were the same, S_i would be one. Some mass exists at $S_i = 1$ but for more than 50% of multiproduct firms; however, buyers of the main product constitute less than 50% of the total buyer-firms base. The fact that each product has a distinct set of buyers ensures that we can construct a sample where the main product experiences a demand shock while the other products do not.

Event study excluding small firms

To address potential bias from financially constrained firms in our event study, we conduct a robustness check focusing on larger firms, which we assume to be relatively less financially constrained. We calculate total firm sales, including both network and final consumer sales, for each year of the study period (2019-2021). We then isolate firms above the 90th percentile of this distribution and replicate our event study analysis from Section 2 using only this subset of larger firms. The results of this robustness check are presented in Figure A2.

Figure A2: The effects of demand shocks to the main product on the production of other products within the firm: Robustness check



Notes: Standard errors are clustered at the firm-county level, and the error bands represent 95% confidence intervals. The X-axis represents the time to treat, with 0 denoting March 2020 when the main product experienced the demand shock, and other values indicating the number of months before or after this event.

The event study results excluding small firms are similar to the event study including all firms, suggesting that financial constraints are not driving the results.

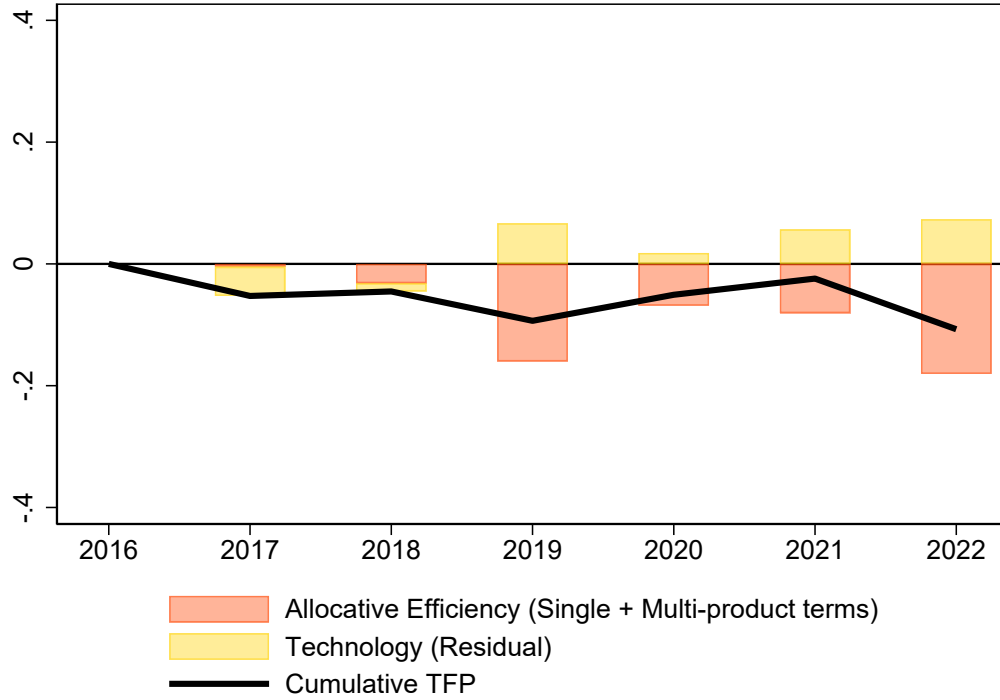
A.2 Additional Empirical Results for Growth accounting

Growth accounting results assuming that markups within firms are equal in initial equilibrium

We show the results when the equilibrium is approximated around $\mu_{ig} = \mu_i$. Note that the level of μ_i does not affect the results since Γ is normalized for covariance.

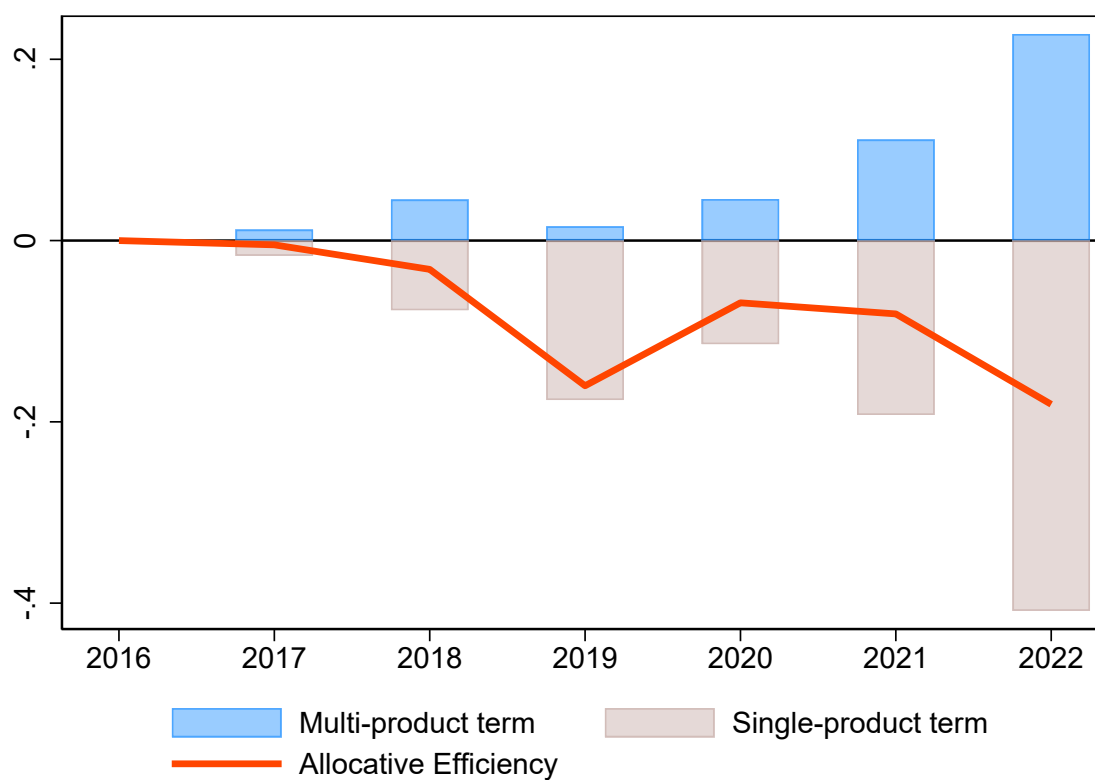
$$\Gamma_{ig} \equiv \underbrace{\frac{\tilde{\lambda}_{ig}}{salesshare_{ig}}}_{\text{Downstream distortion}} \times \underbrace{\mu_i}_{\text{own markup}},$$

Figure A3: Cumulative TFP Decomposition with Multi-Product Term



Notes: This Figure shows the cumulative log change calculated by repeatedly applying the equation from Proposition 1 each year. Technology (residual) is calculated by subtracting allocative efficiency from TFP.

Figure A4: Cumulative TFP growth Decomposition with multiproduct term



Notes: This Figure decomposes the cumulative change in allocative efficiency in Figure A3 into single-product and multi-product terms.

Table A2: Aggregate firm-level statistics

| Year | Count | Sales | Wagebill | Employment |
|------|---------|---------|----------|------------|
| 2016 | 110,451 | 262,506 | 40,260 | 4,242,555 |
| 2017 | 114,480 | 277,960 | 43,691 | 4,349,248 |
| 2018 | 115,916 | 330,486 | 44,688 | 4,349,454 |
| 2019 | 116,706 | 336,386 | 47,299 | 4,425,780 |
| 2020 | 102,306 | 310,317 | 44,053 | 3,935,883 |
| 2021 | 105,651 | 376,220 | 51,642 | 4,166,838 |
| 2022 | 105,032 | 454,818 | 59,148 | 4,266,972 |

Notes: Count stands by the number of firms, while sales and wage bill are yearly aggregates expressed in million pesos. Employment represents the headcount of yearly workers included in the sample.

B Downstream wedge

Table A3: The 30 most distorted products

| Ranking | Description |
|---------|---|
| 1 | Insurance brokerage services |
| 2 | Other services |
| 3 | Passenger air transport services |
| 4 | Wholesale trade intermediary services |
| 5 | Electricity distribution and other related services |
| 6 | Investigation and security services |
| 7 | Airport services |
| 8 | Radio and open TV broadcast services |
| 9 | Wastewater treatment services |
| 10 | Online content services |
| 11 | Cleaning services |
| 12 | Liquefied Natural Gas |
| 13 | Employment services (placement and supply) |
| 14 | Postal and courier services |
| 15 | Tobacco |
| 16 | Paper and cardboard containers, paper or cardboard for recycling |
| 17 | Other IT services |
| 18 | News agency services |
| 19 | Margarine and similar preparations, other residues and waste from fats |
| 20 | General insurance |
| 21 | Other rubber products |
| 22 | Other auxiliary and complementary services for education services |
| 23 | Other goods or services not classified elsewhere |
| 24 | Long-distance passenger transport services |
| 25 | Gas distribution services and other related services |
| 26 | Some other product |
| 27 | Maritime passenger transport services |
| 28 | Research and development services |
| 29 | Repair and installation of machinery and equipment, except for the textile industry |
| 30 | Database software licensing services |

Notes: For 2018, products are ranked using the downstream wedge medians for the product categories, and products with the top 30 downstream markup sizes are reported.

Table A4: The 30 Least distorted product

| Ranking | Description |
|---------|--|
| 1 | Molybdenum minerals and their concentrates |
| 2 | Other non-metallic minerals |
| 3 | Gaseous natural gas |
| 4 | Crude oil |
| 5 | Mining works |
| 6 | Unrefined copper, ashes, residues and wastes of copper |
| 7 | Silver |
| 8 | Public administration and defense services; compulsory social security plans |
| 9 | Pet food |
| 10 | Bird food |
| 11 | Fish meal, crustacean, mollusk and other aquatic invertebrate meal |
| 12 | Ammonium nitrate |
| 13 | Lease services with or without purchase option |
| 14 | Bread |
| 15 | Veterinary services |
| 16 | Poultry meat and edible offal |
| 17 | Integrated telecommunications services (packs) |
| 18 | Fuel oil |
| 19 | Beers |
| 20 | Life insurance |
| 21 | Cakes, cakes and cookies |
| 22 | Hake |
| 23 | Consultancy and post services |
| 24 | Copper minerals and their concentrates |
| 25 | Public hospital services |
| 26 | Social and association services |
| 27 | Petroleum gas and other gaseous hydrocarbons, except natural gas |
| 28 | Fish oil |
| 29 | Mining exploration and evaluation services |
| 30 | Housing services |

Notes: For 2018, products are ranked using the downstream wedge medians for the product categories, and products with the top 30 downstream markup sizes are reported.

C Detailed Methodology for Product-Level Markup Estimation

We estimate product-level markups following the production approach based on Dhyne et al. (2022). In a joint production setup, firms use common inputs to produce a product portfolio, meaning that some inputs may simultaneously be used to produce multiple products. They proposed an Akerberg et al. (2015) like production function estimation method based on Diewert (1973)'s production set. The following is an overview of Dhyne et al. (2022)'s methodology.

A firm has production possibilities set, V , that consists of a set of feasible inputs $x = (x_1, \dots, x_M)$ and outputs of the product, $q = (q_1, \dots, q_G)$. For any (q_g, x) the transformation function is defined as

$$q_g^* = f_g(q_g, x) \equiv \max \{q_g | (q_g, q_{-g}, x) \in V\}$$

To identify the unobserved marginal cost for each firm's product, we rely on (variable) cost minimization. Firms have $N - 1$ freely variable inputs and one fixed input, capital (K), so the problem that a firm faces to minimize its variables cost to produce its output vector q_i^* given the input prices vector $p_x = (p_{x1}, \dots, p_{xM})$ and unobserved productivity for products, $\omega = (\omega_1, \dots, \omega_G)$.

Defining the Lagrangian multiplier of the cost minimization problem, mc_g , as the marginal cost of product g , the first order condition for every optimal input demand yield:

$$p_m = mc_g \frac{\partial f(q_{-g}^*, x, K, \omega)}{\partial x_m} \quad \forall m = 1, \dots, M,$$

It is possible to solve for product g marginal cost as the expenditure on production input m divided by its output elasticity (β_n^g) times product g production:

$$mc_g = \frac{p_m}{\frac{\partial f(q_{-g}^*, x, K, \omega)}{\partial x_m}} = \frac{p_m x_m^*}{\beta_m^g q_g^*},$$

Multiplying the marginal cost expression by $\frac{1}{p_g}$, where p_g is product g price, product g markup is given by:

$$\mu_g = \beta_m^g \frac{p_g q_g^*}{p_m x_m^*},$$

We use control functions for the unobserved productivity terms (i.e., Akerberg et al. (2015)) to account for unobserved productivity with the difference of the need for instruments for q_{-g} ; following Dhyne et al. (2022) we use lagged values of q_{-g} . We assume that firms use a Cobb-Douglas production function with three factors: (Capital K , Labor L , and Materials M). A multi-product firm will produce physical units of product g using the following production function:

$$\log q_{gt} = \beta_0^g + \beta_k^g \log k_t + \beta_l^g \log l_t + \beta_m^g \log m_t^j + \gamma_{-g}^g \log q_{-gt} + \omega_{gt}$$

We pool together products at one digit (12 aggregate product categories) and perform production function estimations separately for each category following ACF using a GMM estimator.

Product-level markup distribution concentrated around 1, with a 1.22 median.

D Proofs

Proof of Example 1 in subsection 3.6:

Proof. First, we compute relative network distortions:

$$\gamma_{ig} \equiv \frac{\Gamma_{ig}}{\Gamma_i},$$

where the average distortion of firm i is defined as: $\Gamma_i \equiv \sum_g \tilde{\lambda}_{ig} / \sum_g (\text{salesshare}_{ig} / \mu_{ig})$. So Γ_1 is

$$\Gamma_1 \equiv \frac{1}{\frac{1}{\mu} + \frac{1}{\mu^2}} 2 = \frac{2}{\frac{1}{\mu} \left(1 + \frac{1}{\mu}\right)}.$$

and

$$\gamma_{11} = \frac{\frac{\mu^2}{2}}{\frac{\frac{1}{\mu}(1+\frac{1}{\mu})}{2}} = \frac{(\mu+1)}{2},$$

$$\gamma_{12} = \frac{\frac{\mu}{2}}{\frac{\frac{1}{\mu}(1+\frac{1}{\mu})}{2}} = \frac{\mu \frac{1}{\mu} (1 + \frac{1}{\mu})}{2} = \frac{(1 + \frac{1}{\mu})}{2},$$

Thus,

$$\frac{1}{\gamma_{11}} = \frac{2}{\frac{1+\mu}{\mu}} = 2 \frac{1}{(\mu+1)},$$

$$\frac{1}{\gamma_{12}} = \frac{2}{\frac{1+\mu}{\mu}} = 2\frac{\mu}{1+\mu}.$$

Substitute them into covariance:

$$\begin{aligned} Cov_{s_i} \left(d \log p_i, \frac{1}{\gamma_i} \right) &= Cov_{[\frac{1}{2}, \frac{1}{2}]} \left(\begin{bmatrix} -\epsilon \\ 0 \end{bmatrix}, \begin{bmatrix} 2\frac{1}{(\mu+1)} \\ 2\frac{\mu}{(\mu+1)} \end{bmatrix} \right), \\ &= \left(\frac{\mu-1}{2(1+\mu)} \right) \epsilon. \end{aligned}$$

Next, we compute the endogenous response of Labor share $d \log \Lambda$ to markup shock. By factor share identity, we know

$$\Lambda_L = 1 - \frac{1}{2} \left(1 - \frac{1}{\mu_{12}} \right) - \frac{1}{2} \left(1 - \frac{1}{\mu_{21}} \right) - \frac{1}{2} \frac{1}{\mu_{21}} \left(1 - \frac{1}{\mu_{11}} \right),$$

Evaluating markup other than μ_{21} with μ and take the logarithm, we have

$$\log \Lambda = \log \frac{1}{\mu} + \log \frac{1}{2} \left(1 + \frac{1}{\mu_{21}} \right).$$

Taking the derivative with respect to $\log \mu_{21}$,

$$\begin{aligned} \frac{d \log \Lambda}{d \log \mu_{21}} &= \frac{d \log \left(\frac{1}{\mu} \frac{1}{2} \left(1 + \frac{1}{\mu_{21}} \right) \right)}{d \mu_{21}} \frac{d \mu_{21}}{d \log \mu_{21}}, \\ &= \frac{-\frac{1}{\mu_{21}}}{\frac{1}{2} \left(1 + \frac{1}{\mu_{21}} \right)}, \\ &= \frac{-\frac{1}{\mu_{21}}}{\frac{1+\mu}{\mu}}. \end{aligned}$$

Evaluate at $\mu_{11} = \mu$ gives

$$\frac{d \log \Lambda}{d \log \mu_{21}} = -\frac{1}{\mu+1}.$$

And

$$\sum_i \tilde{\lambda}_i d \log \mu_i = \frac{1}{2}(-\epsilon).$$

Therefore,

$$-d \log \Lambda - \sum_i \tilde{\lambda}_i d \log \mu_i = \left(\frac{\mu - 1}{2(1 + \mu)} \right) \epsilon.$$

□

Proof of Example 2 in subsection 3.6:

Proof. Pick product 2 to be a reference product for firm 1. Then, we know

$$d \log p_{11}/p_{12} = d \log \mu_{11}/\mu_{12} + \frac{1}{\sigma} d \log y_{11}/y_{12}.$$

Using $d \log \lambda = d \log p + d \log y$

$$d \log p_{11}/p_{12} = \left(\frac{\sigma}{\sigma + 1} \right) d \log \mu_{11}/\mu_{12} + \frac{1}{\sigma + 1} d \log \lambda_{11}/\lambda_{12},$$

where $d \log \mu_{11} = 0$, $d \log \mu_{12} = 0$, and $d \log \mu_{21} = 0$. By Cobb Douglas assumption, we know $d \log \lambda_{11}/\lambda_{12} = -\epsilon$. Thus we have

$$d \log p_{11}/p_{12} = -\left(\frac{1}{\sigma + 1} \right) \epsilon.$$

$$\begin{aligned} d \log TFP &= Cov_{[\frac{1}{2}, \frac{1}{2}]} \left(\begin{bmatrix} d \log p_{11} \\ d \log p_{12} \end{bmatrix}, \begin{bmatrix} \frac{1}{\gamma_1} \\ \frac{1}{\gamma_2} \end{bmatrix} \right) - d \log \Lambda - \sum_i \tilde{\lambda}_i d \log \mu_i, \\ &= Cov_{[\frac{1}{2}, \frac{1}{2}]} \left(\begin{bmatrix} d \log p_{11}/p_{12} \\ d \log p_{12}/p_{12} \end{bmatrix}, \begin{bmatrix} 2 \frac{1}{1+\mu} \\ 2 \frac{\mu}{(\mu+1)} \end{bmatrix} \right) + \left(\frac{\mu - 1}{2(1 + \mu)} \right) \epsilon, \\ &= Cov_{[\frac{1}{2}, \frac{1}{2}]} \left(\begin{bmatrix} -\left(\frac{1}{\sigma+1} \right) \epsilon \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \frac{1}{1+\mu} \\ 2 \frac{\mu}{(\mu+1)} \end{bmatrix} \right) + \left(\frac{\mu - 1}{2(1 + \mu)} \right) \epsilon, \\ &= \left(\frac{\sigma}{\sigma + 1} \right) \left(\frac{\mu - 1}{2(1 + \mu)} \right) \epsilon. \end{aligned}$$

□

Proof of Proposition 1

Lemma 1. Price equation with multi-product firms for some reference product r of firm i :

$$\begin{aligned} \frac{y_{ir}mc_{ir}}{C(y_i, p)} d \log p_{ir} = & -d \log A_i / \mu_i + \underbrace{\sum_{j,k} \frac{p_{jg'} x_{i,jg'}}{C(y_i, p)} d \log p_{jg'}}_{\text{intermediate and factorprice}} + \sum_f \frac{w_f l_{if}}{C(y_i, p)} d \log w_f \\ & + \underbrace{\sum_{g \neq r} \left(-\frac{y_{ig} mc_{ig}}{C(y_i, p)} \right) d \log p_{ig}}_{\text{other product from the same firm}} \end{aligned}$$

Proof. By CRS, we know

$$C_i(q_i, p_i) = \sum_g q_{ig} mc_{ig}.$$

Total derivative:

$$\begin{aligned} RHS &= \sum_g \frac{\partial \log(\sum q_{ig} mc_{ig})}{\partial \log y_i} y_i d \log y_i + \sum_i \frac{\partial \log(\sum q_{ig} mc_{ig})}{\partial \log mc_{ig}} mc_{ig} d \log mc_{ig}, \\ &= \sum_g \frac{q_{ig} mc_{ig}}{C_i(q_i, p_i)} d \log q_{ig} + \sum_g \frac{q_{ig} mc_{ig}}{C_i(q_i, p_i)} d \log mc_{ig}. \end{aligned}$$

and

$$LHS = -d \log A_i + \sum_{j,g'} \frac{p_{jg'} x_{i,jg'}}{C_i(q_i, p_i)} d \log p_{jg'} + \sum_f \frac{w_f l_{if}}{C_i(q_i, p_i)} d \log w_f + \sum_g \frac{q_{ig} mc_{ig}}{C_i(q_i, p_i)} d \log q_{ig}.$$

Hence,

$$\sum_g \frac{y_{ig} mc_{ig}}{C_i(q_i, p_i)} d \log mc_{ig} = -d \log A_i + \sum_{j,g'} \frac{p_{jg'} x_{i,jg'}}{C_i(q_i, p_i)} d \log p_{jg'} + \sum_f \frac{w_f l_{if}}{C_i(q_i, p_i)} d \log w_f. \quad (12)$$

□

Cost minimization with joint production function:

Pick some reference product r of firm i . Following Hall (1973), as a consequence of cost minimization, the following condition holds:

$$\frac{mc_{ig}}{mc_{ir}} = \frac{\partial F_i^Q(q) / \partial q_{ig}}{\partial F_i^Q(q) / \partial q_{ir}},$$

By taking the log difference and adjusting it with markup, we get the following formula:

$$d \log(p_{ir}/p_{ig}) = d \log(\mu_{ir}/\mu_{ig}) + d \log\left(\frac{\partial F_i^Q(q) / \partial q_{ig}}{\partial F_i^Q(q) / \partial q_{ir}}\right). \quad (13)$$

This pins down the equilibrium prices with equation 11. For later proof, we define the RHS of the equation 13 as Θ_{ig} .

$$d \log p_{ig}/p_{ir} = \Theta_{ig}. \quad (14)$$

Then, we proceed to the main proof.

Proof. From Lemma 1 We know for one reference product r :

$$\begin{aligned} \frac{q_{ir}mc_{ir}}{C_i(q_i, p_i)} d \log mc_{ir} &= -d \log A_i + \sum_{j,p} \frac{p_{jp}x_{jp}}{C_i(q_i, p_i)} d \log p_{jp} + \sum_f \frac{w_f L_{if}}{C_i(q_i, p_i)} d \log w_f + \sum_{g \neq r} \left(-\frac{q_{ig}mc_{ig}}{C_i(q_i, p_i)} \right) d \log mc_{ig}, \\ \frac{q_{ir}mc_{ir}}{C_i(q_i, p_i)} d \log p_{ir} &= -d \log A_i + d \log \mu_i + \sum_{j,p} \frac{p_{jp}x_{i,jp}}{C_i(q_i, p_i)} d \log p_{jp} + \sum_f \frac{w_f L_{if}}{C_i(q_i, p_i)} d \log w_f \\ &\quad + \sum_{g \neq r} \left(-\frac{q_{ig}mc_{ig}}{C_i(q_i, p_i)} \right) d \log p_{ig}. \end{aligned} \quad (15)$$

From equation 14, we have

$$d \log p_{ig}/p_{ir} = \Theta_{ig}.$$

Combining 12 with 14 yields

$$\begin{aligned} \frac{q_{ir}mc_{ir}}{C_i(q_i, p_i)} d \log p_{ir} &= -d \log A_i + \sum_g \frac{q_{ig}mc_{ig}}{C_i(q_i, p_i)} d \log \mu_{ig} + \sum_{j,p} \frac{p_{jp}x_{jp}}{C_i(q_i, p_i)} d \log p_{jp} + \sum_f \frac{w_f L_{if}}{C_i(q_i, p_i)} d \log w_f \\ &\quad + \sum \left(-\frac{q_{ig}mc_{ig}}{C_i(q_i, p_i)} \right) d \log p_{ig}, \\ \frac{q_{ir}mc_{ir}}{C_i(q_i, p_i)} d \log p_{ir} &= -d \log A_i + d \log \mu_i + \sum_{j,p} \tilde{\Omega}_{ig,jp} d \log p_{jp} + \sum_f \tilde{\Omega}_{ig,f} d \log w_f + \sum_{g \neq r} \left(-\frac{q_{ig}mc_{ig}}{C_i(q_i, p_i)} \right) [d \log p_{ir} + \Theta_{ig}], \end{aligned}$$

$$d \log p_{ir} = -d \log A_i + d \log \mu_i + \sum_{j,p} \tilde{\Omega}_{ig,jp} d \log p_{jp} + \sum_f \tilde{\Omega}_{ig,f} d \log w_f + \sum_{g \neq r} \left(-\frac{q_{ig} m c_{ig}}{C_i(q_i, p_i)} \right) \Theta_{ig}.$$

Since $\Theta_{ig} = 0$ if g is an reference products, the price equations could be written by

$$d \log p_{ig} = -d \log A_i + d \log \mu_i + \sum_{j,g'} \tilde{\Omega}_{ig,jg'} d \log p_{jg'} + \sum_f \tilde{\Omega}_{ig,f} d \log w_f + \left\{ \mathbb{I}_i(g) - \sum_{g \neq r} \left(\frac{q_{ig} m c_{ig}}{C_i(q_i, p_i)} \right) \right\} \Theta_{ig}.$$

In vector notation

$$d \log \mathbf{p} = (I - \tilde{\Omega}^{\mathcal{NG} \times \mathcal{NG}})^{-1} \left\{ -d \log \mathbf{A}^{\mathcal{NG} \times 1} + d \log \boldsymbol{\mu}^{\mathcal{NG} \times 1} + \tilde{\Omega}_f^{\mathcal{NG} \times \mathcal{F}} d \log \mathbf{w} + (\mathbf{1} - \mathbf{C}) \circ \boldsymbol{\Theta}^{\mathcal{NG} \times 1} \right\},$$

where \circ represents Hadamard product and \mathbf{C} is a vector of $\mathcal{NG} \times 1$, with the following C_i common elements for firm $i \in \mathcal{N}$,

$$C_i = \sum_{g \neq r} \left(\frac{q_{ig} m c_{ig}}{C(q_i, p_i)} \right).$$

We know

$$d \log Y = -\mathbf{b}' d \log \mathbf{p}.$$

$$\begin{aligned} d \log Y &= -\mathbf{b}' \tilde{\Psi}^{\mathcal{NG} \times \mathcal{NG}} \left\{ -d \log \mathbf{A} + d \log \boldsymbol{\mu} + \tilde{\Omega}_f d \log \mathbf{w} + (\mathbf{1} - \mathbf{C}) \circ \boldsymbol{\Theta}^{\mathcal{NG} \times 1} \right\}, \\ &= -\tilde{\lambda}' \left\{ -d \log \mathbf{A} + d \log \boldsymbol{\mu} + \tilde{\Omega}_f d \log \mathbf{w} + (\mathbf{1} - \mathbf{C}) \circ \boldsymbol{\Theta}^{\mathcal{NG} \times 1} \right\}. \end{aligned}$$

subtracting $\sum_f \tilde{\Lambda}_f d \log L_f$ from both sides yields

$$\begin{aligned} d \log TFP &= \sum_{i \in \mathcal{N}} \tilde{\lambda}_i d \log A_i - \sum_{i \in \mathcal{N}} \tilde{\lambda}_i d \log \mu_i - \sum_{f \in \mathcal{F}} \tilde{\Lambda}_f d \log \Lambda_f, \\ &\quad - \sum_i \left(\sum_{g \in \mathcal{G}} \tilde{\lambda}_{ig} d \log p_{ig}/p_{ir} - \sum_{g \neq r} \frac{q_{ig} m c_{ig}}{C_i(q_i, p_i)} \tilde{\lambda}_i d \log p_{ig}/p_{ir} \right). \end{aligned}$$

$$\begin{aligned}
\left(\sum_g \tilde{\lambda}_{ig} d \log p_{ig}/p_{ir} - \sum_{g \neq r} \frac{q_{ig} m c_{ig}}{C_i(y, p)} \tilde{\lambda}_i d \log p_{ig}/p_{ir} \right) &= \tilde{\lambda}_i \left(\sum_{g \in \mathcal{G}} s_{ig} d \log p_{ig}/p_{ir} - \sum_{g \neq r} c_{ig} d \log p_{ig}/p_{ir} \right), \\
&= \tilde{\lambda}_i \left(\sum_{g \in \mathcal{G}} s_{ig} d \log p_{ig}/p_{ir} - \sum_{g \in \mathcal{G}} c_{ig} d \log p_{ig}/p_{ir} \right), \\
&= \tilde{\lambda}_i \left(\sum_{g \in \mathcal{G}} (s_{ig} - c_{ig}) d \log p_{ig} \right), \\
&= \tilde{\lambda}_i \left(\sum_{g \in \mathcal{G}} \left(s_{ig} - \frac{\frac{q_{ig} m c_{ig}}{C(y, p)}}{s_{ig}} s_{ig} \right) d \log p_{ig} \right), \\
&= \tilde{\lambda}_i \left(\sum_{g \in \mathcal{G}} \left(s_{ig} - \frac{1}{\gamma_{ig}} s_{ig} \right) d \log p_{ig} \right), \\
&= \tilde{\lambda}_i \left(E_{s_i} [d \log \mathbf{p}_i] E_{s_i} \left[\frac{1}{\gamma_i} \right] - E_{s_i} \left[d \log \mathbf{p}_{ir} \frac{1}{\gamma_i} \right] \right), \\
&= -\tilde{\lambda}_i \text{Cov}_{s_i} \left(d \log \mathbf{p}_{ir}, \frac{1}{\gamma_i} \right).
\end{aligned}$$

where $c_{ig} = \frac{q_{ig} m c_{ig}}{C(\mathbf{q}_i, \mathbf{p}_i)}$.

□

Proof of Proposition 4

Proof. From resource constraint,

$$q_{ig} = y_{ig} + \sum_{j \in N} x_{jig}.$$

$$d \log y_{ig} = \frac{q_{ig}}{y_{ig}} d \log q_{ig} - \sum_j \frac{x_{jig}}{y_{ig}} d \log x_{jig}.$$

From cost minimization of joint production, we know

$$\sum_g q_{ig} m c_{ig} d \log q_{ig} = \sum_{jp} x_{i,jp} p_{jp} d \log x_{i,jp} + \sum_f w_f x_{i,f} d \log L_{if}.$$

$$\sum_{jp} \frac{x_{i,jp} p_{jp}}{GDP} d \log x_{i,jp} = \sum_g \frac{1}{\mu_{ig}} \frac{q_{ig} p_{ig}}{GDP} d \log q_{ig} - \sum_f \frac{w_f x_{i,f}}{GDP} d \log L_{if}.$$

By definition,

$$\begin{aligned} d \log Y &= \sum_{ig} \frac{p_{ig} y_{ig}}{GDP} d \log y_{ig} - \sum_f \frac{w_f x_{i,f}}{q_{ir} m c_{ig}} d \log L_f, \\ &= \sum_{ig} \frac{p_{ig} y_{ig}}{GDP} \left(\frac{q_{ir}}{y_{ig}} d \log q_{ig} - \sum_j \frac{x_{jig}}{y_{ig}} d \log x_{jig} \right) - \sum_f \frac{w_f L_f}{GDP} d \log L_f, \\ &= \sum_{ig} \frac{p_{ig} y_{ig}}{GDP} \frac{q_{ir}}{y_{ig}} d \log q_{ig} - \sum_{ig} \sum_j \frac{p_{ig} y_{ig}}{GDP} \frac{x_{jig}}{y_{ig}} d \log x_{jig} - \sum_f \frac{w_f L_f}{GDP} d \log L_f, \\ &= \sum_{ig} \frac{p_{ig} q_{ig}}{GDP} d \log q_{ig} - \sum_{ig} \frac{1}{\mu_{ig}} \frac{p_{ig} q_{ig}}{GDP} d \log q_{ig} + \sum_{if} \frac{w_f x_{i,f}}{GDP} d \log L_{if} - \sum_f \frac{w_f L_f}{GDP} d \log L_f, \\ &= \sum_{ig} \lambda_{ig} (1 - \mu_{ig}^{-1}) d \log q_{ig}. \end{aligned}$$

Therefore,

$$\begin{aligned} \frac{\partial \log Y}{\partial \log \mu_{ig'}} &= \sum_{ig} \lambda_{ig} (1 - \mu_{ig}^{-1}) \frac{d \log q_{ig}}{d \log \mu_{jg'}}, \\ \frac{\partial \log Y}{\partial \log \mu_{ig} \partial \log \mu_{jg'}} &= \sum_{ig, jg'} \lambda_{ig} d \log \mu_{ig} \frac{d \log q_{ig}}{d \log \mu_{jg'}}. \\ \frac{\partial \log Y}{\partial \log \mu_{ig} \partial \log \mu_{jg'}} &= \sum_{ig} \sum_{jg'} \lambda_{ig} d \log \mu_{ig} d \log \mu_{jg'} \frac{d \log q_{ig}}{d \log \mu_{jg'}}, \\ &= \sum_{ig} \lambda_{ig} d \log q_{ig} d \log \mu_{ig}. \end{aligned}$$

□

Proof of Proposition 5

Proof. Define the firm-level Domar weight:

$$\lambda_i = \sum_g \lambda_{ig}.$$

Let $d \log \mu_{ig}$ denote the exogenous change in the markup for product g of firm i . The aggregate markup change for firm i is:

$$d \log \mu_i = \frac{1}{\lambda_i} \sum_g \lambda_{ig} d \log \mu_{ig}.$$

The weighted average markup change is:

$$d \bar{\log} \mu = \sum_i \lambda_i d \log \mu_i = \sum_i \sum_g \lambda_{ig} d \log \mu_{ig}.$$

Let $d \log p_{ig}$ be the change in the price of product g by firm i , and $d \log w$ be the change in the wage rate.

The aggregate price index change is:

$$d \log p = \sum_i \sum_g \lambda_{ig} d \log p_{ig}.$$

The price change for product g of firm i is:

$$d \log p_{ig} = d \log \mu_i + d \log w + \frac{\kappa}{\lambda_i} (d \log \mu_{ig} - d \log \mu_i),$$

where

$$\kappa = \frac{\sigma}{\sigma + 1}.$$

Using the zero-profit condition and labor market clearing, we have:

$$d \log w = -d \bar{\log} \mu.$$

Therefore, the aggregate price index change simplifies to:

$$d \log p = \sum_i \lambda_i d \log \mu_i - d \bar{\log} \mu = 0,$$

so $d \log p = 0$.

The change in the Domar weight of product g is:

$$d \log \lambda_{ig} = (1 - \theta) d \log p_{ig}.$$

The welfare change is:

$$\mathcal{L} = \frac{1}{2} \sum_i \sum_g \lambda_{ig} d \log q_{ig} d \log \mu_{ig} = -\frac{1}{2} \theta \sum_i \sum_g \lambda_{ig} d \log p_{ig} d \log \mu_{ig}.$$

Our goal is to express \mathcal{L} in terms of $d \log \mu_{ig}$. Using $d \log w = -d \bar{\log} \mu$, we have:

$$d \log p_{ig} = (d \log \mu_i - d \bar{\log} \mu) + \frac{\kappa}{\lambda_i} (d \log \mu_{ig} - d \log \mu_i).$$

Let

$$d \tilde{\log} \mu_i = d \log \mu_i - d \bar{\log} \mu,$$

and

$$d \tilde{\log} \mu_{ig} = d \log \mu_{ig} - d \log \mu_i,$$

then:

$$d \log p_{ig} = d \tilde{\log} \mu_i + \frac{\kappa}{\lambda_i} d \tilde{\log} \mu_{ig}.$$

Substituting into the welfare change:

$$\mathcal{L} = -\frac{1}{2} \theta \sum_i \sum_g \lambda_{ig} \left(d \tilde{\log} \mu_i + \frac{\kappa}{\lambda_i} d \tilde{\log} \mu_{ig} \right) d \log \mu_{ig}.$$

This simplifies to:

$$\mathcal{L} = -\frac{1}{2} \theta \left(\sum_i \lambda_i d \tilde{\log} \mu_i d \log \mu_i + \kappa \sum_i \frac{1}{\lambda_i} \sum_g \lambda_{ig} d \tilde{\log} \mu_{ig}^2 \right).$$

By definitions of total variance and conditional variance, the welfare change becomes:

$$\mathcal{L} = -\frac{1}{2} \theta \left(\text{Var}_\lambda(d \log \mu_i) + \frac{\sigma}{\sigma + 1} \sum_i \lambda_i \text{Var}_{s_i}(d \log \mu_{ig}) \right).$$

By the law of total variance, we know

$$\text{Var}_\lambda(d \log \mu_{ig}) = \text{Var}_\lambda(d \log \mu_i) + \sum_i \lambda_i \text{Var}_{s_i}(d \log \mu_{ig}).$$

Using this, we obtain

$$\mathcal{L} = -\frac{1}{2}\theta \left(\text{Var}_\lambda(d \log \mu_{ig}) - \frac{1}{\sigma + 1} \sum_i \lambda_i \text{Var}_{s_i}(d \log \mu_{ig}) \right).$$

□