Review on classification of symmetry protected topological(SPT) phase and lattice model construction for $G \times U(1)$ symmetry group

Lau Yat Fan 1155109378
The Chinese university of Hong Kong
Supervisor: Prof.Gu ZhengCheng

Abstract

Phases of matter is an important concept in modern condensed matter physics. In this thesis, we mainly focus on the short range entangled phases with symmetry. First, the classification of the bosonic SPT phases with a symmetry group via group cohomology is reviewed. Then we reviewed the construction of a physical lattice model for those SPT phases in 1D with discrete symmetry group G, that is to construct the fixed point wavefunction and also the Hamiltonian of the system. However, the construction for a discrete group tensor product with a Lie group is not established yet. Thus, we try to construct the fixed point wavefunction and Hamiltonian for $G \times U(1)$ symmetry by introducing the decoration picture with $U(1)$ charge. It turns out this is a beautiful physical interpretation for the Kunneth formula in group cohomology.

1 Introduction

In 1989, people discovered that some quantum phases possess new kind of orders which cannot be explained by the Landau's symmetry breaking theory. This new order is called topological order and it is related to the long-range entanglement which is defined by local unitary(LU) transformations. In this thesis, we mainly focus on gapped systems as gapeless systems are more complicated and not well explored yet.

For gapped phases **without any symmetry**, their quantum phases can be *short range entangled*(SRE) and *long range entangled*(LRE). All SRE states belong to the same phase as they can be transformed into each other through LU transformations. They are trivial as they can all be transformed into the direct product state by the LU transformation. On the other hand, LRE states cannot be transformed into each other by LU transformations and thus different states correspond to different quantum phases. They are call topologically ordered states(topological order). For example, fractional quantum Hall states, chiral and Z_2 spin liquids etc. belongs to LRE states. The mathematical foundation for topologically ordered states is tensor category and simple current

algebra.

In this thesis, we are more interested in gapped phases **with symmetry**. With symmetry the physics is richer. Now even the SRE states belong to different phases. Indeed the Landau symmetry breaking states belong to this class of phases. More interestingly, even SRE states that do not break any symmetry can belong to different phases. Examples are the 1D Haldane phase and topological insulators as they do not break symmetry and we call these phases symmetry protected topological(SPT) phases, which is our main focus in this thesis. The mathematical foundation in this case is the group cohomology. For gapped system with symmetry, the LRE states will be even more complicated and those phases are called symmetry enriched topological(SET) phases, which will not be our focus.

2 Local unitary transformation

Local unitary transformation is an important concept that can help us to define quantum phases. Before that we first define local unitary evolution as the following unitary operator that acts on the degrees of freedom in a quantum system:

$$\mathcal{T}[e^{-i\int_0^1 dg\tilde{H}(g)}] \tag{1}$$

where $\mathcal{T}$ is the path-ordering operator and $\tilde{H}(g) = \sum_i O_i(g)$ is a sum of local Hermitian operators. **Two gapped states can be classified as the same phase if and only if they are related by a LU evolution.** More concretely, we can consider the following quantum circuit with finite length.
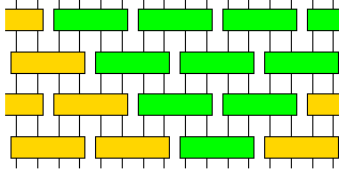

Figure 1: Fig.1 Quantum circuit for LU transformation[1]

We can define the piecewise local unitary operator as

$$U_{pwl} = \prod_i U^i \tag{2}$$

where $\{U^i\}$ is the set of unitary operators that act on non-overlapping region. The size of each region is finite. For a quantum circuit with depth M we just need to multiply U_{pwl} for all layers. That is

$$U_{circ}^M = U_{pwl}^{(1)} U_{pwl}^{(2)} \cdots U_{pwl}^{(M)} \tag{3}$$

This is called the unitary local(LU) transformation. Theory from quantum information allows a finite time evolution with local Hamiltonian be simulated with constant depth quantum circuit and vice versa:

$$\mathcal{T}[e^{-i\int_0^1 dg\tilde{H}(g)}] = U_{circ}^M \tag{4}$$

That's why two gapped quantum states belong to the same phase if they can be related via the LU transformation.

3 Group cohomology and relation to classification of SPT phase

In this section, the goal is to introduce the basic concept of group cohomology and its relation to classification of SPT phases.

3.1 G-module and algebraic definition of group cohomology

For a group G, we define M to be the G-module which is an Abelian group such that

$$g \cdot (ab) = (g \cdot a)(g \cdot b), \qquad g \in G, \quad a, b \in M$$

In our case we always choose M to be the $U(1)$ group and ab is just the usual multiplication of $U(1)$ elements. In this case, the group action is trivial: $g \cdot a = a$.

If the group G contains time-reversal operation, a non-trivial G-module can be defined. We denote it as $M = U_T(1)$. Now the group action is also non-trivial and $g \cdot a = a^{s(g)}$, where $s(g) = 1$ if g contains no time-reversal transformation T and $s(g) = -1$ if g does contain one time-reversal transformation.

Definition of group cohomology

Let $\omega_n(g_1, ..., g_n)$ be a function of n group elements whose value is in the G-module M. i.e. $\omega_n : G^n \to M$. Let $\mathcal{C}^n(G, M) = \{\omega_n\}$ be the space of all such functions. Note that ω_n is an Abelian group with the usual multiplication rule of functions. We can define a map d_n such that $d_n : \mathcal{C}^n[G, U_T(1)] \to \mathcal{C}^{n+1}[G, U_T(1)]$ and we have

$$(d_n\omega_n)(g_1, ..., g_{n+1}) = [g_1 \cdot \omega_n(g_2, ..., g_{n+1})]\omega_n^{(-1)^{n+1}}(g_1, ..., g_n) \times$$
$$\prod_{i=1}^{n} \omega_n^{(-1)^i}(g_1, ..., g_{i-1}, g_i g_{i+1}, g_{i+2}, ..., g_{n+1}) \tag{5}$$

Next we define

$$\mathcal{B}^n(G, M) = \{\omega_n | \omega_n = d_{n-1}\omega_{n-1} | \omega_{n-1} \in \mathcal{C}^{n-1}(G, M)\} \tag{6}$$

and

$$\mathcal{Z}^n(G, M) = \{\omega_n | d_n\omega_n = 1, \omega \in \mathcal{C}^n(G, M) \tag{7}$$

Then the n-cohomology group of G is defined as

$$\mathcal{H}^n(G, M) = \mathcal{Z}^n(G, M)/\mathcal{B}^n(G, M) \tag{8}$$

The above definition is abstract so let us look at some examples. First consider $n = 0$, we choose $M = U_T(1)$ and $g \cdot a = a^{s(g)}, \quad a \in U_T(1)$. From Eq(1), we have

$$(d_0\omega_0)(g_1) = \omega_0^{s(g_1)}/\omega_0 \tag{9}$$

So $\mathcal{Z}^0[G, U_T(1)] = \{\omega_0 | \omega_0^{s(g_1)} = \omega_0\} \equiv U_T^G(1)$. If G contains time reversal, then $\omega_0^{-1} = \omega_0$ which means $U_T^G(1) = \{1, -1\}$. If it does not contain time reversal, we can choose any value of ω_0 and $U_T^G(1) = U(1)$ Since $\mathcal{B}^0[G, U_T(1)] = \{1\}$ is trivial, thus we have $\mathcal{H}^0[G, U_T(1)] = U_T^G(1)$

For $n = 1$, a similar computation gives

$$\mathcal{Z}^1[G, U_T(1)] = \{\omega_1 | \omega_1(g_1)\omega_1^{s(g_1)}(g_2) = \omega_1(g_1 g_2)\} \tag{10}$$

and

$$\mathcal{B}^1[G, U_T(1)] = \{\omega_1 | \omega_1(g_1) = \omega_0^{s(g_1)}/\omega_0\} \tag{11}$$

The cohomology group $\mathcal{H}^1[G, U_T(1)] = \mathcal{Z}^1[G, U_T(1)]/\mathcal{B}^1[G, U_T(1)]$ corresponds to the **the set of all inequivalent 1D representation of** G

For $n = 2$,

$$\mathcal{Z}^2[G, U_T(1)] = \{\omega_2 | \omega_2(g_1, g_2 g_3)\omega_2^{s(g_1)}(g_2, g_3) = \omega_2(g_1 g_2, g_3)\omega_2(g_1, g_2)\} \tag{12}$$

and

$$\mathcal{B}^2[G, U_T(1)] = \{\omega_2(g_1, g_2) = \omega_1^{s(g_1)}(g_2)\omega_1(g_1)/\omega_1(g_1 g_2)\} \tag{13}$$

The cohomology group $\mathcal{H}^2[G, U_T(1)] = \mathcal{Z}^2[G, U_T(1)]/\mathcal{B}^2[G, U_T(1)]$ corresponds to **the set of all inequivalent projective representations of** G

Importantly, it turns out that the group cohomology

$$\mathcal{H}^{1+d}[G, U_T(1)] = \mathcal{Z}^{1+d}[G, U_T(1)]/\mathcal{B}^{1+d}[G, U_T(1)]$$

can classify SPT phases in d**-spatial dimensions with an onsite unitary symmetry group** G. The identity element in $\mathcal{H}^{1+d}[G, U_T(1)]$ corresponds to the trivial SPT phase.

For example, a 1D integer spin chain with $SO(3)$ spin rotation symmetry(but no translation symmetry) has two kinds of SPT phases: one is the trival phase, another one is the well-known Haldane phase. That means $\mathcal{H}^2[SO(3), U(1)] = Z_2$

3.2   Computation of group cohomology

To compute the cohomology of a symmetry group G, of course we can start from the definition of group cohomology, but it will be much easier if we apply the Kunneth formula:

$$\mathcal{H}^d(G \times G', M \otimes_R M')$$
$$= \left[\prod_{p=0}^{d} \mathcal{H}^p(G, M) \otimes_R \mathcal{H}^{d-p}(G', M') \right] \times$$
$$\left[\prod_{p=0}^{d+1} \mathrm{Tor}_1^R(\mathcal{H}^p(G, M), \mathcal{H}^{d-p+1}(G', M')) \right] \tag{14}$$

4

where M and M' are two R-free modules and Tor is the torsion product with the following properties: (We will not discuss the definition of torsion product here but just treat it as a mathematical device)

$$
\begin{aligned}
&\mathrm{Tor}_1^R(A, B) \simeq \mathrm{Tor}_1^R(B, A), \\
&\mathrm{Tor}_1^{\mathbb{Z}}(\mathbb{Z}, M) = \mathrm{Tor}_1^{\mathbb{Z}}(M, \mathbb{Z}) = 0, \\
&\mathrm{Tor}_1^{\mathbb{Z}}(\mathbb{Z}_n, M) = \{m \in M | nm = 0\}, \\
&\mathrm{Tor}_1^{\mathbb{Z}}(\mathbb{Z}_m, \mathbb{Z}_n) = \mathbb{Z}_{(m,n)}, \\
&\mathrm{Tor}_1^R(A \times B, M) = \mathrm{Tor}_1^R(A, M) \times \mathrm{Tor}_1^R(B, M), \\
&\mathrm{Tor}_1^R(M, A \times B) = \mathrm{Tor}_1^R(M, A) \times \mathrm{Tor}_1^R(M, B),
\end{aligned}
\tag{15}
$$

Let us try to apply the Kunneth formula to compute $\mathcal{H}^d[Z_m \times Z_n, Z]$:

$$
\mathcal{H}^d(\mathbb{Z}_m \times Z_n; \mathbb{Z}) \cong \left(\prod_{i=0}^{d} \mathcal{H}^i(Z_m; \mathbb{Z}) \otimes_{\mathbb{Z}} \mathcal{H}^{d-i}(Z_n; \mathbb{Z}) \right) \times \left(\prod_{p=0}^{n+1} \mathrm{Tor}_1^{\mathbb{Z}}(\mathcal{H}^p(Z_m; \mathbb{Z}), \mathcal{H}^{d+1-p}(Z_n; \mathbb{Z})) \right)
\tag{16}
$$

For $d = 0$ to $d = 2$, the above can be simplified as

$$
\begin{aligned}
&\mathcal{H}^0(\mathbb{Z}_m \times Z_n; \mathbb{Z}) \\
&\cong (\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}) \times \mathrm{Tor}_1^{\mathbb{Z}}(\mathbb{Z}_1, \mathbb{Z}) \times \mathrm{Tor}_1^{\mathbb{Z}}(\mathbb{Z}, \mathbb{Z}_1) \\
&= \mathbb{Z}.
\end{aligned}
\tag{17}
$$

$$
\begin{aligned}
&\mathcal{H}^1(\mathbb{Z}_m \times Z_n; \mathbb{Z}) \\
&\cong (\mathbb{Z}_1 \otimes_{\mathbb{Z}} \mathbb{Z}) \times (\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}_1) \times \mathrm{Tor}_1^{\mathbb{Z}}(\mathbb{Z}, \mathbb{Z}_n) \\
&\qquad \times \mathrm{Tor}_1^{\mathbb{Z}}(\mathbb{Z}_1, \mathbb{Z}_1) \times \mathrm{Tor}_1^{\mathbb{Z}}(\mathbb{Z}_m, \mathbb{Z}) \\
&= \mathbb{Z}_1.
\end{aligned}
\tag{18}
$$

$$
\begin{aligned}
&\mathcal{H}^2(\mathbb{Z}_m \times Z_n; \mathbb{Z}) \\
&\cong (\mathbb{Z}_m \otimes_{\mathbb{Z}} \mathbb{Z}) \times (\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}_n) \times \mathrm{Tor}_1^{\mathbb{Z}}(\mathbb{Z}, \mathbb{Z}_1) \times \mathrm{Tor}_1^{\mathbb{Z}}(\mathbb{Z}_1, \mathbb{Z}_n) \\
&\qquad \times \mathrm{Tor}_1^{\mathbb{Z}}(\mathbb{Z}_m, \mathbb{Z}_1) \times \mathrm{Tor}_1^{\mathbb{Z}}(\mathbb{Z}_1, \mathbb{Z}) \\
&= \mathbb{Z}_m \times \mathbb{Z}_n.
\end{aligned}
\tag{19}
$$

Using the relation $\mathcal{H}^d[Z_m \times Z_n; U(1)] = \mathcal{H}^{d+1}(Z_m \times Z_n; Z)$ we obtain

$$
\mathcal{H}^d[Z_m \times Z_n, U(1)] = \begin{cases} U(1) & d = 0, \\ \mathbb{Z}_{(m,n)}^{d/2} & d = \text{ even} \\ \mathbb{Z}_m \times \mathbb{Z}_n \times \mathbb{Z}_{(m,n)}^{(d-1)/2} & d = \text{ odd.} \end{cases}
\tag{20}
$$

4 Construction of SPT phases

4.1 Definition of group cocycles and related concepts

Group cocycle is an important concept in group cohomology. In the following we will give the key definitions first and latter we will see how they are related to the construction of SPT phases. First we define a d-cochain of a group G as a complex function $\nu_d(g_0, g_1, ..., g_d)$ of $1 + d$ variables in G that satisfy

$$
|\nu_d(g_0, g_1, ..., g_d)| = 1,
\tag{21}
$$

$$
\nu_d^{s(g)}(g_0, g_1, ..., g_d) = \nu_d(gg_0, gg_1, ..., gg_d), \qquad g \in G
\tag{22}
$$

where $s(g) = 1$ if g contains no anti-unitary time reversal transformation T and $s(g) = -1$ if there is one anti-unitary time reversal transformation.

Next, the d-cocycles can be defined as special d-cochain that satisfy

$$\prod_{i=0}^{d+1} \nu_d^{(-1)^i}(g_0, ..., g_{i-1}, g_{i+1}, ..., g_{d+1}) = 1 \tag{23}$$

Now let us look at some simple examples. For $d = 1$, the 1-cocycles satisfy

$$\nu_1(g_1, g_2)\nu_1(g_0, g_1)/\nu_1(g_0, g_2) = 1 \tag{24}$$

For $d = 2$, we have

$$\frac{\nu_2(g_1, g_2, g_3)\nu_2(g_0, g_1, g_3)}{\nu_2(g_0, g_2, g_3)\nu_2(g_0, g_1, g_2)} = 1 \tag{25}$$

and for $d = 3$ we have

$$\frac{\nu_3(g_1, g_2, g_3, g_4)\nu_3(g_0, g_1, g_3, g_4)\nu_3(g_0, g_1, g_2, g_3)}{\nu_3(g_0, g_2, g_3, g_4)\nu_3(g_0, g_1, g_2, g_4)} = 1 \tag{26}$$

Another important concept is the d-boundaries λ_d which is also special d-cocycles that can be constructed from the $(d-1)$-cochains μ_{d-1}:

$$\lambda_d(g_0, ..., g_d) = \prod_{i=0}^{d} \mu_{d-1}^{(-1)^i}(g_0, ..., g_{i-1}, g_{i+1}, ..., g_d) \tag{27}$$

Also, we can look at the simple cases. For $d = 1$, the 1-coboundaries are given by

$$\lambda_1(g_0.g_1) = \frac{\mu_0(g_1)}{\mu_0(g_0)} \tag{28}$$

For $d = 2$ and $d = 3$ the 2- and 3-coboundaries are given by

$$\lambda_2(g_0, g_1, g_2) = \frac{\mu_1(g_1, g_2)\mu_1(g_0, g_1)}{\mu_1(g_0, g_2)} \tag{29}$$

and

$$\lambda_3(g_0, g_1, g_2, g_3) = \frac{\mu_2(g_1, g_2, g_3)\mu_2(g_0, g_1, g_3)}{\mu_2(g_0, g_2, g_3)\mu_2(g_0, g_1, g_2)} \tag{30}$$

respectively. It turns out that two d-cocycles ν_d and ν_d' are equivalent iff they only differ by a coboundary $\lambda_d : \nu_d = \nu_d'\lambda_d$. Here's the important consequence: **The equivalent classes of cocycles can be classified by elements in the d-cohomology group $\mathcal{H}^d[G, U_T(1)]$.** We will see that the construction of the wavefunction for SPT phases is closely related to the group cocycles, and thus related to the cohomology group.

4.2 Construction of wavefunction and Hamiltonian for discrete symmetry groups G

Normally, we are given a Hamiltonian of a system and we want to study its properties. However, now we are going to do the opposite: we have the classification results and we wish to construct a lattice model(say constructing an

exactly solvable Hamiltonian) for the corresponding SPT phases. For simplicity, we only consider the 1D case in this thesis.

To begin with, we consider a 1D lattice that has bosonic degrees of freedom labeled by a group element $g_i \in G$. The total wavefunction can be written as the superposition of all possible configuration of states(i.e. choosing different group elements at each site)

$$|\Psi\rangle = \sum_{\text{all conf.}} \Psi \left(\begin{array}{cccccc} g_0 & g_1 & g_2 & g_3 & g_4 & g_5 \\ \bullet\!\rightarrow\!\bullet\!\rightarrow\!\bullet\!\rightarrow\!\bullet\!\rightarrow\!\bullet\!\rightarrow\!\bullet \end{array} \right) \left| \begin{array}{cccccc} g_0 & g_1 & g_2 & g_3 & g_4 & g_5 \\ \bullet\!\rightarrow\!\bullet\!\rightarrow\!\bullet\!\rightarrow\!\bullet\!\rightarrow\!\bullet\!\rightarrow\!\bullet \end{array} \right\rangle \tag{31}$$

Indeed, the wavefunction here is called the *fixed point wavefunction*. That means under renormalization the wavefunction is in some sense invariant. Firstly, the transformation between two different 1D graphs can be given by

$$\Psi \left(\begin{array}{ccc} g_0 & g_1 & g_2 \\ \bullet\!\!\rightarrow\!\!\bullet\!\!\rightarrow\!\!\bullet \end{array} \right) = \frac{1}{|G|^{1/2}} \nu_2(g_0, g_1, g_2) \Psi \left(\begin{array}{cc} g_0 & g_2 \\ \bullet\!\!\rightarrow\!\!\bullet \end{array} \right) \tag{32}$$

This is the so-called (2-1) Pachner move which is a symmetric LU transformation between two states. Note that $|G|$ is the order of the group. The factor $1/|G|^{1/2}$ is just a normalization factor. The $\nu_2(g_0, g_1, g_2)$ is a $U(1)$-valued function. Since we are constructing a fixed point wavefunction, it should be invariant under renormalization. For example, we can use the above Pachner move to obtain

$$\Psi \left(\begin{array}{cccc} g_0 & g_1 & g_2 & g_3 \\ \bullet\!\!\rightarrow\!\!\bullet\!\!\rightarrow\!\!\bullet\!\!\rightarrow\!\!\bullet \end{array} \right)$$
$$= \frac{1}{|G|^{1/2}} \nu_2(g_1, g_2, g_3) \Psi \left(\begin{array}{ccc} g_0 & g_1 & g_3 \\ \bullet\!\!\rightarrow\!\!\bullet\!\!\rightarrow\!\!\bullet \end{array} \right)$$
$$= \frac{1}{|G|} \nu_2(g_1, g_2, g_3) \nu_2(g_0, g_1, g_3) \Psi \left(\begin{array}{cc} g_0 & g_3 \\ \bullet\!\!\rightarrow\!\!\bullet \end{array} \right) \tag{33}$$

Similarly, we also have

$$\Psi \left(\begin{array}{cccc} g_0 & g_1 & g_2 & g_3 \\ \bullet\!\!\rightarrow\!\!\bullet\!\!\rightarrow\!\!\bullet\!\!\rightarrow\!\!\bullet \end{array} \right)$$
$$= \frac{1}{|G|^{1/2}} \nu_2(g_0, g_1, g_2) \Psi \left(\begin{array}{ccc} g_0 & g_2 & g_3 \\ \bullet\!\!\rightarrow\!\!\bullet\!\!\rightarrow\!\!\bullet \end{array} \right)$$
$$= \frac{1}{|G|} \nu_2(g_0, g_1, g_2) \nu_2(g_0, g_2, g_3) \Psi \left(\begin{array}{cc} g_0 & g_3 \\ \bullet\!\!\rightarrow\!\!\bullet \end{array} \right) \tag{34}$$

For fixed point wavefunction of course the above two expressions should be equal, which leads to

$$\nu_2(g_0, g_1, g_3) \nu_2(g_1, g_2, g_3) = \nu_2(g_0, g_2, g_3) \nu_2(g_0, g_1, g_2) \tag{35}$$

Moving terms around we have

$$d\nu_2(g_0, g_1, g_2, g_3) = \frac{\nu_2(g_1, g_2, g_3) \nu_2(g_0, g_1, g_3)}{\nu_2(g_0, g_2, g_3) \nu_2(g_0, g_1, g_2)} = 1 \tag{36}$$

This is exactly the same as the cocycle equation in Eq[18]. This means ν_2 here is a $U(1)$-valued 2-cocycle. Now we realize the fixed point wavefunction of SPT

phases is related to the cocycle we introduced in previous section.

In addition, we can use a symmetric LU transformation to redefine the basis state $|\{g_l\}\rangle$ as $|\{g_l\}\rangle' = \prod_{<ij>} \mu_1(g_i, g_j) |\{g_l\}\rangle$. In the new basis, the phase factor now would become

$$\nu_2'(g_0, g_1, g_2) \equiv \nu_2(g_0, g_1, g_2) \frac{\mu_1(g_1, g_2)\mu_1(g_0, g_1)}{\mu_1(g_0, g_2)} \tag{37}$$

Recall that state related by LU transformation belong to the same phase and the above ν and ν' just differ by a coboundary(i.e. ν and ν' belong to the same classes). Thus, we can conclude that **the elements ν_2 in the same group cohomology class in $\mathcal{H}^2[G, U_T(1)]$ belong to the same 1D bosonic SPT phase**.

After constructing the fixed point wavefunction, we now turn to the Hamiltonian. In 1D, the Hamiltonian can be written as

$$H = -\sum_i H_i \tag{38}$$

where H_i is the local Hamiltonian that act on site i and its neighbouring. By using the Pachner move, it's straightforward to obtain

$$
\begin{aligned}
&\Psi\left(\overset{g_1 \quad\quad g_i \quad\quad g_2}{\bullet\!\!\longrightarrow\!\!\bullet\!\!\longrightarrow\!\!\bullet} \right) \\
&= \frac{1}{|G|^{1/2}} \nu_2(g_1, g_i, g_2) \Psi\left(\overset{g_1 \quad\quad g_2}{\bullet\!\!\longrightarrow\!\!\bullet} \right) \\
&= \frac{1}{|G|} \sum_{g_i'} \frac{\nu_2(g_1, g_i, g_2)}{\nu_2(g_1, g_i', g_2)} \Psi\left(\overset{g_1 \quad\quad g_i' \quad\quad g_2}{\bullet\!\!\longrightarrow\!\!\bullet\!\!\longrightarrow\!\!\bullet} \right)
\end{aligned}
\tag{39}
$$

and it implies

$$
\begin{aligned}
\langle g_i', g_1 g_2 | H_i | g_i, g_1 g_2 \rangle &= \frac{1}{|G|} \frac{\nu_2(g_1, g_i, g_2)}{\nu_2(g_1, g_i', g_2)} \\
&= \frac{1}{|G|} \frac{\nu_2(g_1, g_i, g_i')}{\nu_2(g_i, g_i', g_2)}
\end{aligned}
\tag{40}
$$

Thus, the Hamiltonian can be given by

$$H_i |g_i, g_1 g_2\rangle = \frac{1}{|G|} \sum_{g_i'} \frac{\nu_2(g_1, g_i, g_i')}{\nu_2(g_i, g_i', g_2)} |g_i', g_1 g_2\rangle \tag{41}$$

4.3 Construction for symmetry group $G \times U(1)$

The above construction only works for discrete symmetry group G. The question is what if the discrete group is in tensor product with a Lie group(say U(1) group in our case). Although the classification is quite complete, it still lacks a lattice model similar to what we have above. The main difficulty is $U(1)$ is continuous with infinitely many group elements so we cannot just use the group elements to label the state. However, by introducing the *decoration picture*, the problem may be solved.

4.4 Decoration of U(1) charge

Because of the additional $U(1)$ symmetry, we can introduce a bosonic $U(1)$ charge between each two neighbouring lattice points.

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Recall that the Pachner move in the previous section just results in a complex phase between the conversion of different fixed point wavefunction. For this case we need to introduce creation and annihilation operators for the $U(1)$ charge. Thus, the transformation between different fixed point wavefunction will be given by

$$\Psi\left(\underset{g_0 \qquad g_2}{\longmapsto}\right) \quad = \quad F(g_0, g_1, g_2) \quad \Psi\left(\underset{g_0 \qquad g_1 \qquad g_2}{\longmapsto}\right) \tag{42}$$

Here the F operator can be explicitly expressed as

$$F(g_0, g_1, g_2) = |G|^{\frac{1}{2}} \nu_2(g_0, g_1, g_2)(a_{02}^{n_{02}})^\dagger (a_{01}^{n_{01}})(a_{12}^{n_{12}}) \tag{43}$$

Note that the normalization factor and the complex phase ν_2 are just the same as the previous case. What's more is the bosonic creation and annihilation operators. Basically it means we first annihilate the $U(1)$ charges between site 1 and 2, then site 0 and 1(which corresponds to the RHS in Eq[35]) while the creation operator just creates a $U(1)$ charge that exists on LHS of Eq[35]. The upper indices on the bosonic operators means the total number of charges created or annihilated on each site. Of course due to the conservation of total $U(1)$ charge we must have

$$n_{02} = n_{01} + n_{12} \tag{44}$$

Similar to the procedure in previous section, we can derive the Hamiltonian as

$$\Psi(\underset{g_1 \quad g_i \quad g_2}{\longmapsto}) = \frac{1}{|G|^{1/2}} \nu_2^{-1}(g_1, g_i, g_2)(a_{1i}^{n_{1i}})^\dagger (a_{i2}^{n_{i2}})^\dagger (a_{12}^{n_{12}}) \ \Psi(\underset{g_1 \quad g_2}{\longmapsto})$$

$$= \frac{1}{|G|^{1/2}} \nu_2^{-1}(g_1, g_i, g_2)(a_{1i}^{n_{1i}})^\dagger (a_{i2}^{n_{i2}})^\dagger (a_{12}^{n_{12}})|G|^{\frac{1}{2}} \nu_2(g_0, g_i', g_2)(a_{02}^{n_{02}})^\dagger (a_{01}^{n_{01}})(a_{12}^{n_{12}}) \ \Psi(\underset{g_1 \quad g_i' \quad g_2}{\longmapsto})$$

$$= \frac{1}{|G|} \sum_{g_i'} \frac{\nu_2(g_1, g_i', g_2)}{\nu_2(g_1, g_i, g_2)} (a_{1i}^{n_{1i}})^\dagger (a_{i2}^{n_{i2}})^\dagger (a_{12}^{n_{12}})(a_{12}^{n_{12}})^\dagger (a_{i2}^{n_{i2}})(a_{1i}^{n_{1i}}) \ \Psi(\underset{g_1 \quad g_i' \quad g_2}{\longmapsto})$$

$$= \frac{1}{|G|} \sum_{g_i'} \frac{\nu_2(g_1, g_i', g_2)}{\nu_2(g_1, g_i, g_2)} (a_{1i}^{n_{1i}})^\dagger (a_{i2}^{n_{i2}})^\dagger (a_{i2}^{n_{i2}})(a_{1i}^{n_{1i}}) \ \Psi(\underset{g_1 \quad g_i' \quad g_2}{\longmapsto})$$

$$\tag{45}$$

In the last equality, the middle two bosonic operators cancel out as they create and annihilate $U(1)$ charges at the intermediate steps. Note that in the above

expression, the total $U(1)$ charge is always conserved. Thus, the Hamiltonian can be defined as

$$H_i = \frac{1}{|G|} \sum_{g_i'} \frac{\nu_2(g_1, g_i', g_2)}{\nu_2(g_1, g_i, g_2)} (a_{1i}^{n_{1i}})^\dagger (a_{i2}^{n_{i2}})^\dagger (a_{i2}^{n_{i2}})(a_{1i}^{n_{1i}}) \tag{46}$$

where the total Hamiltonian is just the sum over all H_i

$$H = -\sum_i H_i = -\sum_{i,g_i'} \frac{1}{|G|} \frac{\nu_2(g_1, g_i', g_2)}{\nu_2(g_1, g_i, g_2)} (a_{1i}^{n_{1i}})^\dagger (a_{i2}^{n_{i2}})^\dagger (a_{i2}^{n_{i2}})(a_{1i}^{n_{1i}}) \tag{47}$$

Note that the a operators we consider here work only for tensor product of $U(1)$, for semi-product there will be some subtlety as there will be non-trivial action. Now we can look at some simple examples. We can take G to be different discrete groups, say Z_2 or Z_2^T and more generally we can also consider semi-product($\rtimes$) instead of tensor product($\times$). Note that the four symmetry groups above can be defined by

$$\begin{aligned} U(1) \times Z_2 &: \quad tU_\theta = U_\theta t; \\ U(1) \times Z_2^T &: \quad TU_\theta = U_\theta T; \\ U(1) \rtimes Z_2 &: \quad tU_\theta = U_{-\theta} t; \\ U(1) \rtimes Z_2^T &: \quad TU_\theta = U_{-\theta} T. \end{aligned} \tag{48}$$

where t is the generator of Z_2 which corresponds to unitary representation. T is a generator of Z_2^T which corresponds to the anti-unitary representation. In the case of semi-product, we observe the action will change the θ of the $U(1)$ phase to $-\theta$

The table below shows the classification results for these four possibilities:

Symmetry group	d=1
$U(1) \rtimes Z_2^T$	Z_2
$U(1) \times Z_2^T$	$Z_2 \times Z_2$
$U(1) \rtimes Z_2$	Z_2
$U(1) \times Z_2$	Z_1

We can see that for $U(1) \times Z_2$ symmetry, there is no non-trivial result in 1D as the Z_2 itself has trivial result in 1D. Indeed only the middle two cases have non-trivial SPT phases due to decoration scheme. For the $U(1) \times Z_2^T$, since Z_2^T has anti-unitary representation, the T action will change i to $-i$ in $e^{in\theta}$ (n is just an integer). Another Z_2 in $Z_2 \times Z_2$ is due to the symmetry group Z_2^T itself(but not decoration). Similarly, another non-trivial case is the $U(1) \rtimes Z_2$, in this case θ will change to $-\theta$ in $e^{in\theta}$, thus it also has a non-trivial result in 1D due to decoration scheme. For the first case, although the result is Z_2 but that's due to the Z_2^T group itself. In this case both θ and i change signs so there is no non-trivial action.

4.5 Relation to the Kunneth formula

In section 3 we have introduced the Kunneth formula for calculating group cohomology. Although it is just a mathematical formula, the beautiful side

of this formula is that it has a physical meaning that is consistent with the decoration picture. To see this, first we rewrite the kunneth formula in a more convenient form:

$$
\begin{aligned}
&\mathcal{H}^d(G_s \times G_g, \mathbb{R}/\mathbb{Z}) \\
&= \mathcal{H}^{d+1}(G_s \times G_g, \mathbb{Z}) \\
&= \left[\oplus_{p=0}^{d+1} \mathcal{H}^p(G_s, \mathbb{Z}) \otimes_{\mathbb{Z}} \mathcal{H}^{d+1-p}(G, \mathbb{Z}) \right] \oplus \\
&\quad \left[\oplus_{p=0}^{d+2} \mathrm{Tor}_1^{\mathbb{Z}}[\mathcal{H}^p(G_s, \mathbb{Z}), \mathcal{H}^{d-p+2}(G, \mathbb{Z})] \right] \\
&= \mathcal{H}^d(G_s, \mathbb{R}/\mathbb{Z}) \oplus \mathcal{H}^d(G_g, \mathbb{R}/\mathbb{Z}) \oplus \\
&\quad \left[\oplus_{p=1}^{d-1} \mathcal{H}^{d-p}(G_s, \mathbb{Z}) \otimes_{\mathbb{Z}} \mathcal{H}^p(G_g, \mathbb{R}/\mathbb{Z}) \right] \oplus \\
&\quad \left[\oplus_{p=1}^{d-1} \mathrm{Tor}_1^{\mathbb{Z}}[\mathcal{H}^{d-p+1}(G_s, \mathbb{Z}), \mathcal{H}^p(G_g, \mathbb{R}/\mathbb{Z})] \right] \\
&= \mathcal{H}^d(G_s, \mathbb{R}/\mathbb{Z}) \oplus \mathcal{H}^d(G_g, \mathbb{R}/\mathbb{Z}) \oplus \\
&\quad \left[\oplus_{p=1}^{d-1} \mathcal{H}^{d-p}[G_s, \mathcal{H}^p(G_g, \mathbb{R}/\mathbb{Z})] \right] \\
&= \oplus_{p=0}^{d} \mathcal{H}^{d-p}[G_s, \mathcal{H}^p(G_g, \mathbb{R}/\mathbb{Z})],
\end{aligned}
\tag{49}
$$

Then we apply the last equality to the symmetry group $G \times U(1)$. As we are considering 1D case, we take $d = 2$ and

$$
\begin{aligned}
\mathcal{H}^2[G \times U(1), U(1)] &= \mathcal{H}^2[G, \mathcal{H}^0[U(1), U(1)]] \oplus \mathcal{H}^1[G, \mathcal{H}^1[U(1), U(1)]] \oplus \mathcal{H}^0[G, \mathcal{H}^2[U(1), U(1)]] \\
&= \mathcal{H}^2[G, U(1)] \oplus \mathcal{H}^1[G, Z] \oplus \mathcal{H}^0[G, Z_1]
\end{aligned}
\tag{50}
$$

In the last line of the above expression, the first term corresponds to the group cohomology of the symmetry group G only. The second term corresponds to the $U(1)$ charges and indeed the labels n_{ij} we introduce on the bosonic operators belongs to this group cohomology. i.e. $n_{ij} \in \mathcal{H}^1[G, Z]$ and t corresponds to the number of $U(1)$ charges created or annihilated. The last term is just trivial. Now we see the Kunneth formula has a physical interpretation that is consistent with the decoration picture.

5 Conclusion

In this thesis, we review the the classification of bosonic SPT phases by the theory of group cohomology and it turns out in d-spatial dimension the bosonic SPT phases can be characterized by the group elements in $\mathcal{H}^{d+1}[G, U_T(1)]$. Then we focus on the construction of fixed point wavefunctions and exactly solvable Hamiltonian for 1D bosonic SPT phases with a discrete symmetry group G. Since there is no such construction for a discrete group tensor product(for semi-product the case will be more complicated) with a Lie group, we also try to find the corresponding Hamiltonian by the decoration scheme and explain its relation to the Kunneth formula. Although we use $U(1)$ group as an example, we note that it can be an arbitrary Lie group such as $SO(3)$ or $SU(2)$.

Also, for simplicity we only consider the 1D case but indeed this is possible to generalize to higher dimension. A case worth mentioning is the construction of $U(1)$ symmetry group only(without tensor product with other discrete groups). In this situation, the 1D and 3D classification is trivial but it is non-trivial in 2D. Indeed, Professor Wen Xiao Gang has a paper[4] regarding this case. In that paper, they create an exactly soluble path integral on spacetime lattice for

$U(1)$ SPT phases in (2+1)D with a non-zero Hall conductance. Then they converted the path integral on a (2+1)D spacetime lattice into a 2D Hamiltonian lattice model, and show that the Hamiltonian consists of mutually commuting local projectors.

6   Acknowledgement

7   References

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