MATH 82: Differential Equations

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* Adapted from SU23 lectures.

1 Introduction

Definition: Ordinary differential equation

An ordinary differential equation (ODE) is an equation involving a function of one independent variable and its derivatives. A function y is a solution to an ordinary differential equation if, when substituted into the ODE, both sides of the equation are equal for all values of the independent variable.

This is in contrast with partial differential equations (PDEs), which involve functions with more than one independent variable and their partial derivatives. We will not discuss PDEs in any detail here, so the initialisms "ODE" and "DE" will be interchangeable in these notes.

There are many different ways we can classify differential equations. We list four of them below.

Definition: Classifications of an ODE

- The order of a DE is that of the highest-order derivative it contains.
- An autonomous DE is one in which the independent variable does not appear explicitly in the DE. A DE that is not autonomous is called nonautonomous.
- A linear DE in y is one of the form

$$a_n(x)y^{(n)} + \cdots + a_1(x)y' + a_0(x)y = b(x),$$

  where $a_0, a_1, \ldots, a_n, b$ are differentiable functions of the independent variable x. A DE that is not linear is called nonlinear.
- A driven (or forced, non-homogeneous) DE includes a nonzero term that does not contain the dependent variable. A DE that is not driven is called undriven (or unforced, homogeneous).

  Note: The terms "driven" and "undriven" are usually reserved for linear DEs.

When we solve a differential equation, there are two ways we can express the solution. Which we use depends on the complexity of the solution.

Definition: Explicit or implicit solution to an ODE

An explicit solution to an ODE consists of an isolated dependent variable written as a function of the independent variable. An implicit solution is not solved for the independent variable.

The solution to an ODE is usually a class of related equations rather than one particular equation. If we want to pinpoint a specific solution with specific parameters, we might impose some conditions on what points the solution crosses.

Definition: Initial- or boundary-value problem

An initial-value problem (IVP) is a pairing of a differential equation with conditions all specified at some value of the independent variable. A boundary-value problem (BVP) is a pairing of a differential equation with conditions specified at the extremes of the independent variable.

In general, after we solve a DE and obtain a family of solutions, we can impose on the solutions whatever condition we are given to yield a specific solution to the DE. Sometimes, however, we aren't concerned with any particular initial condition or boundary condition and just want a general solution.

2 First-Order Differential Equations

2.1 Separable Differential Equations

The first class of differential equations we'll discuss involves those first-order differential equations that can, broadly, be "separated" into its independent and dependent variables.

Definition: Separable differential equation

An ordinary differential equation is separable if it can be written in the form

$$\frac{dy}{dt} = G(y) \cdot H(t).$$

Example: Solving a separable ODE

Suppose we are given the separable ODE

$$\frac{dy}{dt} = G(y) \cdot H(t). \tag{2.1}$$

To solve this differential equation, we begin by moving the y-dependent stuff to the left side:

$$\frac{1}{G(y)} \cdot \frac{dy}{dt} = H(t). \tag{2.2}$$

Now we integrate both sides with respect to t:

$$\int \frac{1}{G(y)} \cdot \frac{dy}{dt}\, dt = \int H(t)\, dt \tag{2.3}$$

For the integral on the left we can do a change of variabes from t to y to get

$$\int \frac{dy}{G(y)} = \int H(t)\, dt. \tag{2.4}$$

From here, all that's left to do is make the resulting equation as simple as possible. This may or may not involve solving for y, depending on how complex the solution is.

Note: In practice, we normally skip from Equation (2.1) directly to Equation (2.4). I only included some extra steps here to show why we can sort of treat the dy/dt as a fraction and move the dt to the right side of the equation.

2.2 First-Order Linear Differential Equations

Here, we provide a way to solve any first-order linear differential equation. Though this method will be less obvious than separation of variables, what will be more obvious is when we can actually apply the method.

Example: Solving a first-order linear ODE

All first-order linear ODEs can be written in the form

$$a(x)y' + b(x)y = c(x).$$

However, this DE will be more useful to us when it is in normal form:

$$y' + p(x)y = q(x)$$

We multiply by some function $\mu(x)$ that satisfies $\mu'(x) = \mu(x)p(x)$:

$$\begin{aligned} \mu(x)y' + \mu(x)p(x)y &= \mu(x)q(x) \\ \mu(x)y' + \mu'(x)y &= \mu(x)q(x) \end{aligned}$$

By the product rule,

$$\frac{d}{dx}\left[\mu(x) \cdot y\right] = \mu(x)q(x)$$

Integrating with respect to x:

$$\begin{aligned} \mu(x) \cdot y &= \int \mu(x)q(x)\, dx \\ y &= \frac{1}{\mu(x)} \int \mu(x)q(x)\, dx \end{aligned}$$

This gives an explicit solution to the DE. But what is the function $\mu(x)$? Recall that we have the separable DE

$$\frac{d\mu}{dx} = \mu(x)p(x).$$

The solution to this DE turns out to be

$$\mu(x) = e^{\int p(x)\, dx}.$$

For this reason, $\mu(x)$ is called the integrating factor for first-order linear ODEs.

We summarize this example with a theorem.

Theorem 2.1: Integrating factor method for solving first-order linear ODEs

Suppose we have a first-order linear ODE in its normal form

$$y' + p(x)y = q(x).$$

If we define an integrating factor $\mu(x) = e^{\int p(x)\, dx}$, then the general solution to this DE is

$$y = \frac{1}{\mu(x)} \int \mu(x)q(x)\, dx.$$

2.3 Numerical Methods

We'll discuss two numerical methods for visualizing and approximating solutions to initial-value problems. The first of these is below.

Definition: Slope field

Consider the first-order differential equation

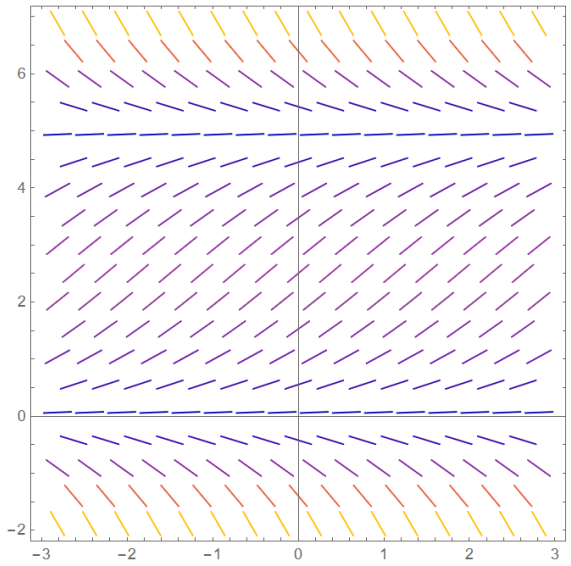
$$\frac{dy}{dx} = f(x, y).$$

The slope field of this DE encodes the value of dy/dx at each point (x, y) as line segments that lay tangent to a solution curve $y(x)$ at each point on a grid.

Slope fields help us visualize the general trend that solutions to a differential equation follow.

Example: Slope field of a logistic DE

The slope field of the logistic DE $dy/dx = y\,(1 - y/5)$ is shown below.



Given a boundary (or in another case, initial) condition $y(x_0) = y_0$, we could use this slope field to sketch a solution curve, using the sloped line segments to "push" the curve in the right direction.

Notice the behavior of the slope field at $y = 0$ and $y = 5$. At each of these locations we have a series of horizontal line segments. If a solution curve $y(x)$ ever reaches either of these y-values, it will stay there for the rest of its existence because there $dy/dx = 0$. Despite this similarity, the two "equilibria" are fundamentally different: while solutions tend to converge at $y = 5$, they tend to diverge at $y = 0$. We summarize these observations below.

Definition: Equilibrium solution of a first-order DE

A solution $y = y(x)$ to a first-order differential equation is called an equilibrium solution to the DE if its derivative $dy/dx = 0$ for all x. If other solutions tend to converge to an equilibrium, the equilibrium is stable; if other solutions tend to diverge, the equilibrium is unstable.

Next, we give a way to actually compute approximations for initial-value problems.

Definition: Euler's method

Consider the initial-value problem

$$\frac{dy}{dx} = f(x, y), \quad y(x_0) = y_0.$$

To approximate the value of y at a different x-value, we iterate through the following algorithm a sufficient number of times.

Let $v_0 = y(x_0) = y_0$, and let Δx be some small "step." Then v_n is an approximation for $y(x_n)$, where

$$x_{n+1} = x_n + \Delta x$$
$$v_{n+1} = v_n + \Delta x \cdot f(x, y)$$

Euler's method is a very bad method for approximating solutions to differential equations. But it is very simple, based only on a series of tangent line approximations. Smaller step sizes lead to better approximations.

2.4 Existence and Uniqueness of Solutions

We give two theorems relating to whether or not a sole solution exists for a given DE. There is a trade-off here: the first theorem gives a stronger result but works on a smaller class of functions.

Theorem 2.2: Existence and uniqueness of solutions to a first-order linear IVP

Consider this IVP for a first-order linear DE in normal form:

$$y' + p(t)y = q(t) \quad \text{with} \quad y(t_0) = y_0.$$

If the coefficients $p(t)$ and $q(t)$ (a, b) containing t_0, then the solution of the IVP exists and is unique on the whole interval (a, b).

Theorem 2.3: Existence and uniqueness of solutions to a first-order IVP

Consider the IVP for the first-order DE:

$$y'(t) = f(t, y) \quad \text{with} \quad y(t_0) = y_0.$$

If $f(t, y)$ and $\frac{\partial f}{\partial y}(t, y)$ are continuous on the rectangle $t \in (\alpha, \beta)$ and $y \in (\gamma, \delta)$ containing (t_0, y_0), then the solution to the IVP exists and is unique on some interval $t \in (t_0 - h, t_0 + h)$ for some $h > 0$.

Both of these boil down to the functions in the DE being "well-behaved." It's reasonable that continuity should be a condition for both of these. The $\partial f/\partial y$ condition in Theorem 2.3 ensures a smooth transition between two solution curves in the same neighborhood. In fact, this theorem allows us to state something significant about the solution curves of a DE.

Corollary 2.4: Solution curves do not intersect

Under the same hypothesis as Theorem 2.3, two solution curves in the region cannot intersect.

3 Second-Order Differential Equations

3.1 Theory of Higher-Order Differential Equations

We begin this new chapter by extending our discussion of existence and uniqueness.

Theorem 3.1: Existence and uniqueness of solutions to a linear DE

Consider this IVP for an n-th order linear DE in normal form:

$$y^{(n)} + p_{n-1}(t)y^{(n-1)} + \cdots + p_1(t)y' + p_0(t)y = q(t)$$
$$\text{with } \; y(t_0),\; y'(t_0),\; \ldots,\; y^{(n-1)}(t_0)$$

If the coefficients $p_0(t),\; \ldots,\; p_{n-1}(t),\; q(t)$ (a, b) containing t_0, then the solution of the IVP exists and is unique on the whole interval (a, b).

Note that Theorem 3.1 is simply Theorem 2.3 recast in terms of order n instead of order 1. This gives one example of how results for relatively simple differential equations can often be generalized quite easily to higher-order DEs. In a similar vein, the remainder of this section will focus on the theory of linear second-order differential equations, but all of the definitions and results can be easily generalized to the n-th order.

Theorem 3.2: Solution spaces as vector spaces

Let the coefficients a_0, a_1, and a_2 be continuous on an interval I. Then solutions to the homogeneous linear DE $a_2(t)y'' + a_1(t)y' + a_0(t)y = 0$ comprise a two-dimensional vector space.

It is easy to show that the solution space S of a homogeneous DE is a vector space: S is a subset of the set of twice-differentiable functions, which is known to be a vector space; also, it is easy to verify that S is closed under linear combinations. In order to determine $\dim(S)$, however, we must develop the notion of linear independence. Since we can think of functions as elements of a vector space, the definition of linear independence is natural.

Definition: Linear dependence

Two functions f and g are called linearly dependent if there exist constants c_1 and c_2, not both zero, such that

$$c_1 f(t) + c_2 g(t) = 0.$$

A pair of functions that is not linearly dependent is called linearly independent.

Depending on the functions at hand, determining linear dependence by solving for c_1 and c_2 can be cumbersome. In a step toward an easier test, suppose that f and g are linearly dependent on an interval I. Rewrite the linear combination as $\dfrac{g(t)}{f(t)} = -\dfrac{c_1}{c_2}$ and differentiate to get

$$\frac{f(t)g'(t) - g(t)f'(t)}{f(t)^2} = 0.$$

Therefore, if f and g are linearly dependent, then $f(t)g'(t) - g(t)f'(t) = 0$ for all $t \in I$. We'll give this difference a special name and state the result's contrapositive as a theorem.

Definition: Wronskian

The Wronskian W of two functions y_1 and y_2 is the determinant

$$W[y_1, y_2] = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}.$$

Theorem 3.3: Condition for linear independence

If $W[y_1, y_2](t) \neq 0$ for some $t \in I$, then y_1 and y_2 are linearly independent on I.

Note that the converse is not true: if $W[y_1, y_2] = 0$ for all $t \in I$, the test is inconclusive. Things are simpler, however, when the functions are solutions to a linear DE.

Suppose y_1 and y_2 solve the DE $y'' + py' + qy = 0$. When we differentiate the Wronskian $W[y_1, y_2]$, we find that $dW/dt = -pW$, which is a separable DE. Solving this DE gives the following.

Theorem 3.4: Abel's theorem

Let p and q be continuous on an interval I and let y_1 and y_2 solve the DE $y'' + py' + qy = 0$. Then, for some constant C, $W[y_1, y_2] = Ce^{-\int p(t)dt}$.

Since either $C = 0$ or $C \neq 0$, we have a corollary.

Corollary 3.5: $W(t_0) = 0$ implies identially zero

If y_1 and y_2 are any two solutions to $y'' + py' + qy = 0$ on an interval I, then either $W[y_1, y_2](t) = 0$ or $W[y_1, y_2](t) \neq 0$ for all $t \in I$.

We can now show that the solution space of a linear second-order homogeneous DE has dimension two.

Consider the DE $y'' + py' + qy = 0$. For some $t_0 \in I$, let y_1 be a solution satisfying the initial conditions $y(t_0) = 1$, $y'(t_0) = 0$ and let y_2 be a solution satisfying $y(t_0) = 0$, $y'(t_0) = 1$. Since $W[y_1, y_2](t_0) = 1 \neq 0$, y_1 and y_2 are linearly independent on I. Therefore, the dimension of the DE's solution space is at least 2.

On the other hand, consider the same DE under the initial conditions $y(t_0) = \alpha$, $y'(t_0) = \beta$. We can show that the solution to this IVP is $y = \alpha y_1 + \beta y_2$. Therefore, a solution y to any IVP involving the above DE must necessarily lay in the span of y_1 and y_2, meaning the dimension of the DE's solution space is at most 2.

All of this means that the DE's solution space is of dimension 2, and any pair of linearly independent solutions forms a basis for the space. This basis has a special name.

Definition: Fundamental solution set

A pair of functions y_1, y_2 solving the DE $y'' + py' + qy = 0$ form a fundamental solution set on the interval I if $W[y_1, y_2] \neq 0$ for some $t \in I$.

Now, what if our DE is instead inhomogeneous? Including a forcing term $f(t)$ complicates things slightly. When we solve, we still include the general solution to the corresponding homogeneous DE (since the DE just maps it to zero), but we have to add on some other particular solution that accounts for the forcing term.

Theorem 3.6: General solution to an inhomogenous linear DE

Let y_p be a particular solution to the inhomogeneous DE $y'' + py' + qy = f(t)$ and let y_h be the general solution to the associated homogeneous DE. Then the general solution to the inhomogeneous DE is

$$y(t) = y_p(t) + y_h(t).$$

3.2 Homogeneous, Constant-Coefficient Differential Equations

Arguably the simplest second-order ODEs are those homogeneous ones with constant coefficients. Below, we derive a general solution to DEs of this form.

Example: Solving a constant-coefficient DE

We are concerned with equations of the form

$$ay'' + by' + cy = 0,$$

where a, b, and c are constants. Let $\lambda_{1,2}$ solve the characteristic equation $a\lambda^2 + b\lambda + c = 0$, meaning $b = -a(\lambda_1 + \lambda_2)$ and $c = a\lambda_1\lambda_2$. Substituting these into the DE:

$$\begin{aligned} ay'' - a(\lambda_1 + \lambda_2)y' + a\lambda_1\lambda_2 &= 0 \\ y'' - \lambda_1 y' - \lambda_2(y' - \lambda_1 y) &= 0 \\ (y' - \lambda_1 y)' - \lambda_2(y' - \lambda_1 y) &= 0 \end{aligned}$$

Define $u = y' - \lambda_1 y$ to get the DE $u' - \lambda_2 u = 0$, which has the solution $u = Ae^{\lambda_2 x}$. This in turn yields $y' - \lambda_1 y = Ae^{\lambda_2 x}$, the solution of which is $y = e^{\lambda_1 x} \int Ae^{(\lambda_2 - \lambda_1)x} dx$. We're left with two possibilities.

$$\begin{aligned} \lambda_1 \neq \lambda_2: &\quad y(x) = C_1 e^{\lambda_1 x} + C_2 e^{\lambda_2 x} \\ \lambda_1 = \lambda_2: &\quad y(x) = C_1 e^{\lambda_1 x} + C_2 x e^{\lambda_1 x} \end{aligned}$$

Theorem 3.7: General solution to a homogenenous constant-coefficient DE

Consider the DE $ay'' + by' + cy = 0$, where a, b, and c are constants. Let $\lambda_{1,2}$ solve the equation $a\lambda^2 + b\lambda + c = 0$. If $\lambda_1 \neq \lambda_2$, then the general solution to the DE is

$$y(x) = C_1 e^{\lambda_1 x} + C_2 e^{\lambda_2 x}.$$

If $\lambda_1 = \lambda_2$, then the general solution is instead

$$y(x) = C_1 e^{\lambda_1 x} + C_2 x e^{\lambda_1 x}.$$

3.3 Inhomogeneous Differential Equations

When we introduce a forcing term into the equation, the path to solution splits into two branches: first we determine the general solution to the associated homogeneous equation (perhaps using the method described previously), then we use some other method to find a particular solution to the inhomogeneous DE. We'll describe two of these methods here, the first of which simply boils down to a guessing game.

Definition: Undetermined coefficients

Consider the DE $p_2(x)y'' + p_1(x)y' + p_0(x)y = f(x)$. The method of undetermined coefficients may be used to find a particular solution to this DE; the procedure is as follows.

- Use $f(t)$ to guess a particular solution as a linear combination of some related functions.
- Plug the guess (a.k.a. ansatz) into the DE and determine the unknown constants.
- If the guess doesn't work, go back to step one and try something different. If it does, then the guess (with the unknown coefficients now determined) is a particular solution to the DE.

If we have initial conditions, we substitute them into the general solution to the DE (see Theorem 3.6).

Remarks. The ansatz should take into account any of the functions that the forcing term cycles through under

differentiation; for example, if $f(x)$ is sinusoidal, then we might guess that the solution is a linear combination of a sine and a cosine. The ansatz also depends on the DE's homogeneous solution. We want a guess that is linearly independent of the corresponding homogeneous DE's fundamental solution set, because otherwise the DE will map the ansatz to zero and no progress will have been made. Based on the repeated roots case of the previous solution method, if $f(x)$ shows up in the homogeneous solution, we might tack an extra x onto the guess and try again.

An advantage of this solution method is that it can involve minimal work to find an answer. However, this relies on the forcing function being nice to work with. If this is not the case, we might appeal to the other solution method in this section, described below.

Example: Solving a second-order linear DE via variation of parameters

We aim to find a particular solution to the DE

$$y'' + p_1(t)y' + p_2(t)y = f(t).$$

Suppose that y_1 and y_2 are linearly independent solutions to the associated homogeneous DE. We'll assume that a particular solution to the inhomogeneous DE is of the form

$$y(t) = u_1(t)y_1 + u_2(t)y_2.$$

Our aim now is to find two functions u_1 and u_2 that satisfy this condition. Let's differentiate to get

$$y'(t) = u_1'y_1 + u_1y_1' + u_2'y_2 + u_2y_2'.$$

The second derivative is unwieldy. To avoid this, let's put another condition on u_1 and u_2: they must satisfy $u_1'y_1 + u_2'y_2 = 0$. This gives

$$y'(t) = u_1y_1' + u_2y_2'$$

and the second derivative

$$y''(t) = u_1'y_1' + u_1y_1'' + u_2'y_2' + u_2y_2''.$$

So, when we substitute y and its derivatives into the DE we get

$$\begin{aligned}(u_1'y_1' + u_1y_1'' + u_2'y_2' + u_2y_2'') + p_1 \cdot (u_1y_1' + u_2y_2') + p_2 \cdot (u_1y_1 + u_2y_2) &= f(t)\\ u_1'y_1' + u_1 \cdot (y_1'' + p_1y_1' + p_2y_1) + u_2'y_2' + u_2 \cdot (y_2'' + p_1y_2' + p_2y_2) &= f(t)\end{aligned}$$

Since y_1 and y_2 are homogeneous solutions, this simplifies to $u_1'y_1' + u_2'y_2' = f(t)$, giving us the system

$$\begin{aligned}u_1'y_1 + u_2'y_2 &= 0\\ u_1'y_1' + u_2'y_2' &= f(t)\end{aligned}$$

This can be written as a matrix equation:

$$\begin{bmatrix}y_1 & y_2\\ y_1' & y_2'\end{bmatrix}\begin{bmatrix}u_1'\\ u_2'\end{bmatrix} = \begin{bmatrix}0\\ f(t)\end{bmatrix}$$

If $W[y_1, y_2] \neq 0$ then we can write

$$\begin{bmatrix}u_1'\\ u_2'\end{bmatrix} = \frac{1}{W(t)}\begin{bmatrix}y_2' & -y_2\\ -y_1' & y_1\end{bmatrix}\begin{bmatrix}0\\ f(t)\end{bmatrix}.$$

Therefore,

$$\begin{aligned}u_1'(x) = -\frac{y_2(x)f(x)}{W(x)} &\implies u_1(t) = -\int \frac{y_2(x)f(x)}{W(x)}dx\\ u_2'(x) = \frac{y_1(x)f(x)}{W(x)} &\implies u_2(t) = \int \frac{y_1(x)f(x)}{W(x)}dx\end{aligned}$$

This means that a solution to the DE is

$$y(x) = -y_1 \int \frac{y_2(x)f(x)}{W(x)}dx + y_2 \int \frac{y_1(x)f(x)}{W(x)}dx.$$

In fact, this is the general solution to the DE—notice how, because each integral comes with a constant of integration, the homogeneous solution is already packed into $y(x)$.

Theorem 3.8: Variation of parameters

Consider the DE $y'' + p_1(t)y' + p_2(t)y = f(t)$, and let y_1 and y_2 be linearly independent solutions to the associated homogeneous DE. Then the general solution to the inhomogeneous DE is

$$y(t) = -y_1 \int \frac{y_2(t)f(t)}{W[y_1, y_2](t)}dx + y_2 \int \frac{y_1(t)f(t)}{W[y_1y_2](t)}dx.$$

3.4 Series Methods

When the general solution to a differential equation cannot be expressed in terms of elementary functions (think exponentials, trig functions, etc), we might resort to expressing the solution as an infinite series. We'll discuss two general methods here; before we do, though, we must define some new properties of functions and differential equations.

Definition: Analytic function

A function f is analytic at a point x_0 if there exists $R > 0$ such that $f(x)$ equals its Taylor series centered at x_0 for $|x - x_0| < R$.

To simplify things, we will only consider Taylor (power) series centered at $x = 0$. Further, all of the basic functions we're familiar with are analytic on any open interval of their domain, and these functions will be the only ones we'll deal with in this course. So for now, in order to determine whether a function is analytic at a point, we need only check that a function is continuous at that point. This will be very useful for our next definition, which gives us the two cases that we'll be working with in this section.

Definition: Ordinary and singular points

Consider the second-order linear homogeneous DE in normal form:

$$y'' + p(x)y' + q(x)y = 0.$$

The point x_0 is an ordinary point of the DE if both $p(x)$ and $q(x)$ are analytic at $x = 0$. Otherwise, x_0 is a singular point.

Generally, the case in which x_0 is an ordinary point is easier to deal with than the singular point case. They're both relatively straightforward, however. First, the case of the ordinary point is summarized below.

Theorem 3.9: Analytic solution theorem

If x_0 is an ordinary point, then the general solution of $y'' + p(x)y' + q(x)y = 0$ has the form

$$y(x) = \sum_{n=0}^{\infty} a_n x^n = a_0 y_1(x) + a_1 y_2(x)$$

where $a_0 = y(0)$ and $a_1 = y'(0)$ are arbitrary constants and y_1, y_2 are linearly independent analytic functions. The radius of convergence of y_1 and y_2 is at least the minimum radius of convergence for $p(x)$ and $q(x)$.

When we apply this theorem to a problem, the solution boils down to a few key steps.

1. Assume the solution can be represented by a power series centered at zero.
2. Differentiate the solution term-by-term and substitute it into the DE.
3. Simplify any terms and reindex series to have common powers.
4. Determine a recurrence relation for the coefficients a_n.

Usually, we will obtain two linearly independent power series solutions, each with an infinite number of terms. There are some notable exceptions, however, one of which is given below.

Example: Legendre's equation of order λ

Consider the DE below, called Legendre's equation of order λ:

$$(1-x^2)y'' - 2xy' + \lambda(\lambda+1)y'' = 0.$$

When we go through the motions of solving this DE using power series, we obtain the recurrence relation

$$a_{n+2} = \frac{(n-\lambda)(n+\lambda+1)}{(n+2)(n+1)}.$$

If λ is a natural number N, then we will eventually come across a term a_N that is zero. This sparks a kind of domino effect—every subsequent term whose index has the same parity as N will also be zero, cutting part of the sequence off at a_{N-2}.

In this case, then, one of the linearly independent solutions to the DE is a polynomial rather than a power series. This solution is called the Legendre polynomial of degree N, and they have some neat properties making them useful in applications.

This all handles the ordinary point case, but how about singular points? The method does not change much. For the purposes of this course, if we must find a solution about a singular point, then a particular solution to the DE is of the form

$$y(x) = x^r \sum_{n=0}^{\infty} a_n x^n,$$

where $a_0 \neq 0$ and $r \in \mathbb{R}$. When we solve, we go through all the same steps as we did before; this is called the method of Frobenius.

Example: Bessel's equation of order λ

Consider the DE below, called Bessel's equation of order λ.

$$x^2y'' + xy' + (x^2 - \lambda^2)y = 0.$$

Since $x_0 = 0$ is singular, we seek constants r and a_n such that

$$y(x) = \sum_{n=0}^{\infty} a_n x^{n+r}$$

is a solution. When we go through the motions of solving, assuming $a_0 \neq 0$ we obtain three conditions. If all of these are satisfied, then $y(x)$ satisfies the DE.

(a) $r = \pm\lambda$

(b) $(2r+1)a_1 = 0$

(c) $a_n = \dfrac{-a_{n-2}}{n(2r+n)}$, $n \geq 2$

4 Systems of Differential Equations

4.1 Introduction

Definition: System of differential equations

A system of differential equations has the general form

$$\mathbf{x}'(t) = \mathbf{f}(t, \mathbf{x}(t)).$$

In terms of components, this is

$$\begin{aligned} x_1'(t) &= f_1(t, x_1(t), x_2(t), \ldots, x_n(t)) \\ &\vdots \\ x_n'(t) &= f_n(t, x_1(t), x_2(t), \ldots, x_n(t)) \end{aligned}$$

Here, the state vector $\mathbf{x}(t)$ denotes the states (or quantities) of interest, $\mathbf{x}'(t)$ denotes the rates of change of the states, and $\mathbf{f}(t, \mathbf{x}(t))$ represents the governing dynamics that describe how the states change over time in response to interactions among the states or external forces.

Not only do systems of DEs allow us to consider different types of problems than those we've seen so far, but they allow us to look at familiar problems through a different lens: any "scalar" differential equation can be written as a system of first-order differential equations. For example, consider the nth-order DE

$$\frac{d^n y}{dt^n} = f(t, y, y', \ldots, y^{(n-1)}).$$

If we define state variables $x_1 = y,\ x_2 = y',\ \ldots,\ x_n = y^{(n-1)}$, we can recast the DE as the system

$$\begin{aligned} x_1' &= x_2 \\ x_2' &= x_3 \\ &\vdots \\ x_{n-1}' &= x_n \\ x_n' &= f(t, x_1, x_2, \ldots, x_n) \end{aligned}$$

Now, we'll extend some familiar terminology to these new problems.

Definition: Linear system of DEs

A system of differential equations $\mathbf{x}' = \mathbf{f}(t, \mathbf{x})$ is linear if each component of $\mathbf{f}$ depends linearly on the state variables. Otherwise, it is nonlinear.

Definition: Autonomous system of DEs

A system of differential equations $\mathbf{x}' = \mathbf{f}(t, \mathbf{x})$ is autonomous if $\mathbf{f}$ does not depend explicitly on the variable t, that is, if $\partial f / \partial t = \mathbf{0}$. Otherwise, it is nonautonomous.

Note that any nonautonomous system can be recast as an autonomous one by defining a new state variable for the independent variable. We thus restrict our attention to autonomous systems without loss of generality.

Definition: Equilibrium points of a system of DEs

A point $\mathbf{x}_0$ is an equilibrium point of the system $\mathbf{x}' = \mathbf{f}(\mathbf{x})$ if $\mathbf{f}(\mathbf{x}_0) = \mathbf{0}$.

Finally, we give a condition for whether a solution to a system exists. It is an extension of Theorem 2.3.

Theorem 4.1: Existence and uniqueness of solutions

Let $\mathbf{f}(x)$ have components $f_1, f_2, \ldots, f_n$. If f_i and $\partial f_i/\partial x_j$ $(1 \leq i, j \leq n)$ are continuous in a neighborhood of $\mathbf{x}_0$ then there exists $\varepsilon > 0$ such that the initial-value problem

$$\mathbf{x}' = \mathbf{f}(\mathbf{x}), \quad \mathbf{x}(t_0) = \mathbf{x}_0.$$

has a unique solution $\mathbf{x}(t)$, defined for $t \in (t_0 - \varepsilon, t_0 + \varepsilon)$.

4.2 Constant-Coefficient Linear Systems

4.2.1 Fundamental Matrices and the Flow

For now, we'll focus our attention on homogeneous constant-coefficient linear systems. These can be written in the form $\mathbf{x}' = A\mathbf{x}$, where A is an $n \times n$ matrix and $\mathbf{x} : \mathbb{R} \to \mathbb{R}^n$. We first verify that this problem can be solved for any initial condition.

Theorem 4.2: Global solution theorem

Let A be an $n \times n$ matrix with real entries. For each $\mathbf{x}_0 \in \mathbb{R}^n$ the initial-value problem

$$\mathbf{x}' = A\mathbf{x}, \quad \mathbf{x}(t_0) = \mathbf{x}_0.$$

has a unique solution $\mathbf{x}(t)$ which is defined for all $t \in (-\infty, \infty)$.

This gives us some information about the general solution to a differential equation of this form.

Corollary 4.3: Solutions as a vector space

The set of solutions to $\mathbf{x}' = A\mathbf{x}$ is an n-dimensional vector space.

Therefore, the general solution of $\mathbf{x}' = A\mathbf{x}$ has the form

$$\mathbf{x}(t) = c_1\mathbf{x}_1(t) + c_2\mathbf{x}_2 + \cdots + c_n\mathbf{x}_n(t) = \begin{bmatrix} | & | & & | \\ \mathbf{x}_1(x) & \mathbf{x}_2(x) & \cdots & \mathbf{x}_n(t) \\ | & | & & | \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix}$$

where $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n$ are linearly independent solutions and c_i are arbitrary constants.

The matrix of linearly independent solution vectors is important enough to warrant a definition.

Definition: Fundamental matrix

A fundamental matrix $\Psi(t)$ of the system $\mathbf{x}' = A\mathbf{x}$ is one whose columns are linearly independent solutions of the system.

There's a couple of ways we can go about finding a fundamental matrix. The first involves making an ansatz: based on results from Section 3.2, we might guess that a particular solution to the system is some vector of exponentials, say $\mathbf{x}(t) = e^{\lambda t}\mathbf{v}$ for a constant vector $\mathbf{v}$. To determine λ and $\mathbf{v}$, consider how $\mathbf{x}$ is a solution

to the system of DEs if and only if

$$\begin{aligned}\mathbf{x}' &= A\mathbf{x}\\ \lambda e^{\lambda t}\mathbf{v} &= A(e^{\lambda t}\mathbf{v})\\ \lambda\mathbf{v} &= A\mathbf{v}\end{aligned}$$

We can see, now, that λ and $\mathbf{v}$ comprise an eigenvalue-eigenvector pair (an "eigenpair") of A. We formalize this in a theorem.

Theorem 4.4: Solution to a diagonalizable linear system

Let A be a diagonalizable $n \times n$ matrix with eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$ and corresponding eigenvectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$. The general solution to the system $\mathbf{x}' = A\mathbf{x}$ is

$$\mathbf{x}(t) = c_1 e^{\lambda_1 t}\mathbf{v}_1 + c_2 e^{\lambda_2 t}\mathbf{v}_2 + \cdots + c_n e^{\lambda_n t}\mathbf{v}_n,$$

where $c_1, c_2, \ldots, c_n$ are constants.

Thus, so long as A is diagonalizable (has n distinct eigenvectors), computing the eigendata of A gives us sufficient information to determine n linearly independent solutions to the system, giving us the general solution $\mathbf{x}(t) = \Psi(t)\mathbf{c}$. It doesn't matter if we get repeated eigenvalues—linearly independent eigenvectors give linearly independent solutions.

As a side note, if we have complex eigenvalues, we can use the following shortcut to get two linearly independent solutions.

Theorem 4.5: Linearly independent solutions from complex eigendata

Let $\mathbf{x}(t) = e^{\lambda t}\mathbf{v}$, $\lambda \in \mathbb{C}$ solve the system $\mathbf{x}' = A\mathbf{x}$. Define $\lambda = \alpha + i\beta$ and $\mathbf{v} = \mathbf{p} + i\mathbf{q}$. Then $\mathrm{Re}(\mathbf{x}(t))$ and $\mathrm{Im}(\mathbf{x}(t))$ are linearly independent solutions to the system, with

$$\begin{aligned}\mathrm{Re}(\mathbf{x}(t)) &= e^{\alpha t}(\cos\beta t\ \mathbf{p} - \sin\beta t\ \mathbf{q})\\ \mathrm{Im}(\mathbf{x}(t)) &= e^{\alpha t}(\sin\beta t\ \mathbf{p} + \cos\beta t\ \mathbf{q})\end{aligned}$$

Now, if we are given initial conditions, we can use Ψ to find the solution to the IVP without going through all the trouble of solving a system of equations. If $\mathbf{x}(0) = \mathbf{x}_0$, then we must have $\Psi(0)\mathbf{c} = \mathbf{x}_0$. Solving for $\mathbf{c}$ gives $\mathbf{c} = \Psi^{-1}(0)\mathbf{x}_0$. Therefore, the solution to the IVP is

$$\mathbf{x}(t) = \Psi(t)\Psi^{-1}(0)\mathbf{x}_0.$$

The matrix $\Psi(t)\Psi^{-1}(0)$ is the key to understanding linear systems and the global behavior of their solutions. It is so important that it gets its own name.

Definition: Flow of a linear system

The flow of the linear system $\mathbf{x}' = A\mathbf{x}$ is the matrix $\Psi(t)\Psi^{-1}(0)$, where Ψ is any fundamental matrix.

All this information is great, but again, we can only get it if A is diagonalizable. If this is not the case, we must resort to other means.

4.2.2 Matrix Exponentials

First, notice that the initial-value problem $\mathbf{x}' = \mathbf{f}(\mathbf{x})$, $\mathbf{x}(0) = \mathbf{x}_0$, is equivalent to the integral equation

$$\mathbf{x}(t) = \mathbf{x}_0 + \int_0^t \mathbf{f}(\mathbf{x}(s))\, ds.$$

Like many differential equations, it will not always be possible to solve this integral equation directly, but we can approximate a solution. One method of doing so is defined below.

Theorem 4.6: Picard iteration

Consider the IVP $\mathbf{x}' = \mathbf{f}(\mathbf{x})$, $\mathbf{x}(0) = \mathbf{x}_0$. Let $\mathbf{x}_0(t) = \mathbf{x}_0$ and for $n \in \mathbb{N}$ define $\mathbf{x}_n$ by

$$\mathbf{x}_n(t) = \mathbf{x}_0 + \int_0^t \mathbf{f}(\mathbf{x}_{n-1}(s))\, ds.$$

If the approximate solutions $\mathbf{x}_n$ converge to some function $\mathbf{x}$, then the limit function $\mathbf{x}$ is the unique solution to the IVP.

Picard iteration gives us a concrete way to approximate solutions to first-order systems. Further, if we can determine the limit of $\mathbf{x}_n$, we can use it to solve the first-order system outright. We'll do this below.

Example: Solving a linear system using Picard iteration

Our aim is to solve the initial-value problem $\mathbf{x}' = A\mathbf{x}$, $\mathbf{x}(0) = \mathbf{x}_0$. The Picard iterates are defined by

$$\mathbf{x}_n(t) = \mathbf{x}_0 + \int_0^t A\mathbf{x}_{n-1}(s)\, ds$$
$$\mathbf{x}_n(t) = \mathbf{x}_0 + A\int_0^t \mathbf{x}_{n-1}(s)\, ds$$

After we go through some iterations, we find that the iterates converge to the exponential power series

$$\mathbf{x}(t) = \left[I + At + \frac{A^2t^2}{2!} + \cdots + \frac{A^nt^n}{n!} + \cdots\right]\mathbf{x_0}.$$

So, this is the solution to the IVP and we will use it to define a new matrix operation.

Definition: Matrix exponential

If A is a square matrix, we define its matrix exponential as

$$e^A = \sum_{k=0}^{\infty} \frac{A^k}{k!} = I + A + \frac{A^2}{2!} + \frac{A^3}{3!} + \cdots.$$

Theorem 4.7: Solution to a homogeneous linear system

The solution to the IVP $\mathbf{x}' = A\mathbf{x}$, $\mathbf{x}(0) = \mathbf{x}_0$ is

$$\mathbf{x}(t) = e^{At}\mathbf{x}_0.$$

We can compute some matrix exponentials simply using the power series definition, but this can get a bit tedious. We'll give a few useful properties that can make computing matrix exponentials a little easier.

Theorem 4.8: Selected matrix exponentials

(a) Let A be a diagonal matrix with diagonal entries $\lambda_1, \lambda_2, \ldots, \lambda_n$. Then its matrix exponential is also diagonal, with

$$e^A = \begin{bmatrix} e^{\lambda_1} & 0 & \cdots & 0 \\ 0 & e^{\lambda_2} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & e^{\lambda_n} \end{bmatrix}.$$

(b) Let $M = \begin{bmatrix} 0 & q \\ -q & 0 \end{bmatrix}$ where $q > 0$ is a constant. Then its matrix exponential is

$$e^{Mt} = \begin{bmatrix} \cos qt & \sin qt \\ -\sin qt & \cos qt \end{bmatrix}.$$

Theorem 4.9: Condition for commutative exponentials

Let A and B be square matrices of the same size. Then the following statements are equivalent.

(a) A and B commute.

(b) $e^{A+B} = e^A e^B$.

(c) $e^A e^B = e^B e^A$.

So we've found two ways to represent the solution to the IVP: $\mathbf{x}(t) = \Psi(t)\Psi^{-1}(0)\mathbf{x}_0$ and $\mathbf{x}(t) = e^{At}\mathbf{x}_0$. However, we've seen that the solution to an IVP is unique, giving us the following.

Theorem 4.10: Characterization of e^{At}

Let Ψ be any fundamental matrix of the linear system $\mathbf{x}' = A\mathbf{x}$. Then

$$e^{At} = \Psi(t)\Psi^{-1}(0).$$

We now have two ways to compute the flow of $\mathbf{x}' = A\mathbf{x}$: using the fundamental matrix and via the power series definition of the matrix exponential. One final way involves the diagonalization of A, made practical by the following theorem.

Theorem 4.11: Similarity of matrix exponentials

If $B = P^{-1}AP$, then $e^{Bt} = P^{-1}e^{At}P$.

So, similar to how we compute powers of matrices like A^{10}, we can compute matrix exponentials by diagonalizing A, computing the exponential, and then undoing the diagonalization to find e^{At}.

4.2.3 Inhomogeneous Systems

The matrix exponential gives us an important insight into how to solve linear systems with a forcing term. Key to this insight is that the derivative of a matrix exponential is analogous to that of a scalar exponential.

Lemma 4.12: Derivative of a matrix exponential

If $A \in M_n(\mathbb{R})$, then $\dfrac{d}{dt}\left[e^{At}\right] = Ae^{At} = e^{At}A$.

Now we can solve the general constant-coefficient linear system $\mathbf{x}' = A\mathbf{x} + \mathbf{F}(t)$.

Example: Solving a forced constant-coefficient linear system

We aim to find the general solution to the initial-value problem

$$\mathbf{x}' = A\mathbf{x} + \mathbf{F}, \quad \mathbf{x}(0) = \mathbf{x}_0.$$

Taking inspiration from the integrating factor method discussed early in the course, we rewrite the system as

$$\mathbf{x}' - A\mathbf{x} = \mathbf{F}$$

and multiply by the integrating factor $\mu(t) = e^{\int(-A)dt} = e^{-At}$ to get

$$\begin{aligned} e^{-At}\mathbf{x}' - e^{-At}A\mathbf{x} &= e^{-At}\mathbf{F} \\ \frac{d}{dt}\left[e^{-At}\mathbf{x}\right] &= e^{-At}\mathbf{F} \\ e^{-At}\mathbf{x} - \mathbf{x}_0 &= \int_0^t e^{-As}\mathbf{F}\, ds \\ \mathbf{x}(t) &= e^{At}\mathbf{x}_0 + e^{At}\int_0^t e^{-As}\mathbf{F}\, ds \end{aligned}$$

This is the solution to the IVP.

Theorem 4.13: Solution to an inhomogeneous linear system

The unique solution of the inhomogeneous system $\mathbf{x}' = A\mathbf{x} + F$ with initial value $\mathbf{x}(0) = \mathbf{x}_0$ is

$$\mathbf{x}(t) = e^{At}\mathbf{x}_0 + e^{At}\int_0^t e^{-As}\mathbf{F}(s)\, ds.$$

Written in terms of a fundamental matrix Ψ,

$$\mathbf{x}(t) = \Psi(t)\Psi^{-1}(0)\mathbf{x}_0 + \Psi(t)\int_0^t \Psi^{-1}(s)\mathbf{F}(s)\, ds.$$

Notice how this solution can be interpreted as the sum of a homogeneous solution $\mathbf{x}_h$ and a particular one $\mathbf{x}_p$. $\mathbf{x}_h$ is the system's response to the initial conditions, and $\mathbf{x}_p$ is the cumulative response to the forcing.

4.2.4 Phase Portraits

Here, we'll develop a way to visualize the dynamics of two-dimensional systems. This can be directly extended to three dimensions and, more abstractly, higher dimensions. We begin with a connection between the eigendata of A and e^{At}.

Theorem 4.14: Eigendata of a matrix exponential

If $(\lambda, \mathbf{v})$ is an eigenpair of A, then $(e^{\lambda t}, \mathbf{v})$ is an eigenpair of e^{At}.

If A is diagonalizable, then any initial condition $\mathbf{x}_0$ can be written as a linear combination of the eigenvectors of A. So when e^{At} acts on $\mathbf{x}_0$, the matrix has the effect of scaling the eigen-components of $\mathbf{x}_0$ exponentially in t. The nature of this scaling, determined by the eigenvalues of A, entirely determines the path a solution curve will take with increasing time. When several of these paths are compiled together, we get a good picture of how the flow acts on state space.

Definition: Phase portrait

A phase portrait is a diagram in state space that indicates several solution curves (also called orbits or trajectories). The solution curves are flow lines of the vector field $\mathbf{f}(\mathbf{x})$ since $\mathbf{f}'(t) = \mathbf{f}(\mathbf{x}()t)$.

Key to predicting what the orbits will look like is the distinction between the real and complex parts of an eigenvalue λ.

The real part of λ controls the exponential growth or decay of a system.

- If $\mathrm{Re}(\lambda) > 0$, then the system exhibits growth.
- If $\mathrm{Re}(\lambda) < 0$, then the system exhibits decay.
- If $\mathrm{Re}(\lambda) = 0$, then we get neither growth nor decay, so the "magnitude" of the system remains constant.

If the system has two real eigenvalues, then the scaling described here happens in the direction of each eigenvalue's corresponding eigenvector.

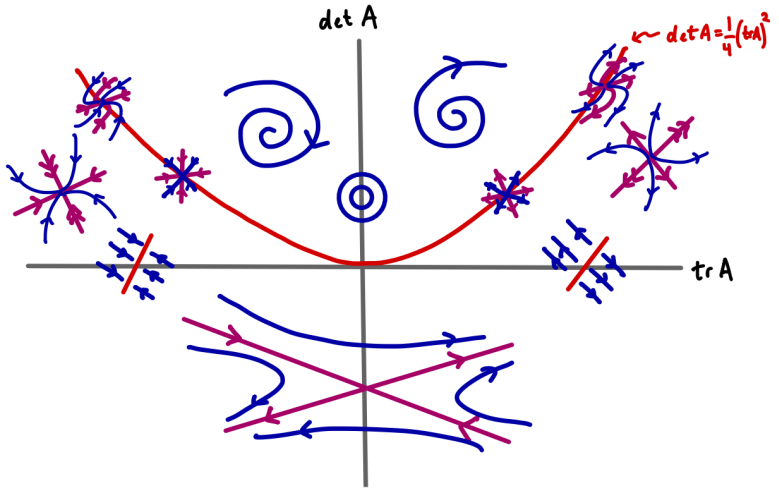
The imaginary part of λ controls the rotation of a system.

- If $\mathrm{Im}(\lambda) > 0$, then the system rotates counterclockwise.
- If $\mathrm{Im}(\lambda) < 0$, then the system rotates clockwise.
- If $\mathrm{Im}(\lambda) = 0$, then the system does not rotate.

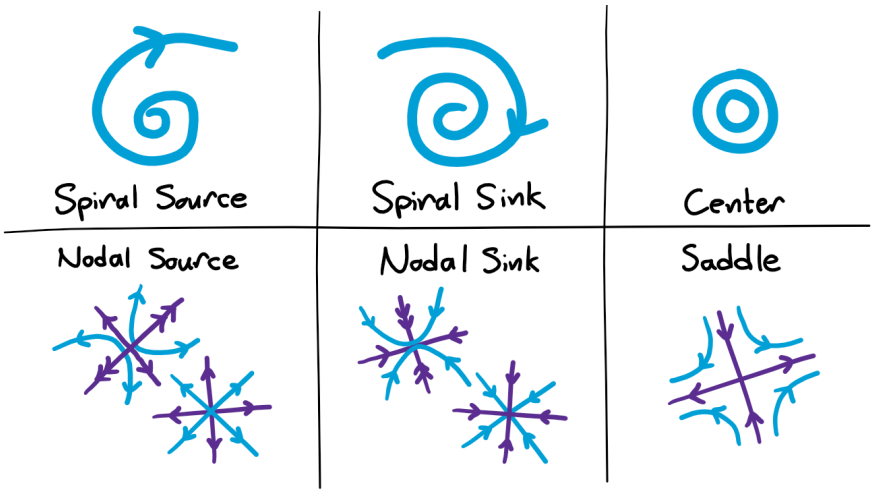
Now, using the fact that

$$\lambda_{1,2} = \frac{\mathrm{tr}A \pm \sqrt{(\mathrm{tr}A)^2 - 4\det A}}{2},$$

we can graphically represent all possible phase portraits. The figure below is courtesy of Prof. Jacobsen.



Each portion of the diagram above represents a different class of phase portrait. A few of these have special names, which are given in the chart below.



There is a variety of definitive statements we can make about the behavior of initial conditions under the influence of each of these flows. We'll give one of these to finish off our discussion of linear systems.

Theorem 4.15: Asymptotic stability theorem

If $\mathrm{Re}(\lambda) < 0$ for all eigenvalues λ of A, then all solutions of $\mathbf{x}' = A\mathbf{x}$ satisfy $\mathbf{x}(t) \to \mathbf{0}$ as $t \to \infty$.

4.3 Nonlinear Systems

4.3.1 Local Linearization

In general, nonlinear systems are much more difficult to work with than linear ones. None of the nice linear structure we developed in the previous section applies anymore, and in fact exact analytic formulas for solutions rarely exist.

A quick examination of a nonlinear phase portrait shows that nonlinear systems can have multiple isolated equilibrium points. This is in direct contrast to linear phase portraits, which have exactly one equilibrium point at the origin. These equilibria, however, are not too different form those found in linear systems; in fact, one important method of analyzing nonlinear systems is to exploit the fact that equilibria are locally linear.

To linearize an equilibrium, we apply the change of coordinates $\mathbf{u}(t) = \mathbf{x}(t) - \mathbf{x}_0$ and analyze the dynamics of $\mathbf{u}$ when it is sufficiently small. We know that

$$\begin{aligned}\mathbf{u}' = \mathbf{x}' &= \mathbf{f}(\mathbf{x}) \\ &= \mathbf{f}(\mathbf{x}_0 + \mathbf{u}) \\ &= \mathbf{f}(\mathbf{x}_0) + D\mathbf{f}(\mathbf{x}_0)\mathbf{u} + \mathcal{O}(\|\mathbf{u}\|^2)\end{aligned}$$

where $D\mathbf{f}(\mathbf{x}_0)$ is the matrix whose (i, j) entry is the partial derivative of f_i with respect to x_j. (This is called the Jacobian matrix.) The higher-order terms disappear, and since $\mathbf{x}_0$ is an equilibrium point we know that $\mathbf{f}(\mathbf{x}_0)$. This gives us the following result.

Theorem 4.16: Local linearization

The linearization of $\mathbf{x}' = \mathbf{f}(\mathbf{x})$ at the equilibrium point $\mathbf{x}_0$ is the linear system

$$\mathbf{u}' = A\mathbf{u}$$

where $\mathbf{u}(t) = \mathbf{x}(t) - \mathbf{x}_0$ and

$$A = D\mathbf{f}(\mathbf{x}_0),$$

the derivative matrix of $\mathbf{f}$ evaluated at $\mathbf{x}_0$.

Using this, we can classify each equilibrium point of a system by stability.

Definition: Equilibrium stability

An equilibrium point is stable if $\mathrm{Re}(\lambda) \leq 0$ for all eigenvalues λ of $A = D\mathbf{f}(\mathbf{x}_0)$. An equilibrium point is asymptotically stable if $\mathrm{Re}(\lambda) < 0$ for all eigenvalues λ of $A = D\mathbf{f}(\mathbf{x}_0)$. Otherwise we say the equilibrium point is unstable.

Informally:

- A stable equilibrium point is one where starts that start close will stay close for all future time.
- An asymptotically stable equilibrium point is on where states that start close will asymptotically converge back to the equilibrium point as time increases.
- An unstable equilibrium point is where where there are states that start close but do not stay close.

In most cases, a system's linearization gives us an accurate understanding of how the system will behave near an equilibrium point. However, this won't always be true; a condition for this is below.

Theorem 4.17: Hartman-Grobman theorem

If $\mathbf{x}_0$ is an equilibrium point of $\mathbf{x}' = \mathbf{f}(\mathbf{x})$ and $\mathrm{Re}(\lambda) \neq 0$ for all eigenvalues λ of $D\mathbf{f}(\mathbf{x}_0)$, then the linearization at $\mathbf{x}_0$ is "faithful."

This theorem can be used to prove a version of Theorem 4.15 for nonlinear systems.

Theorem 4.18: Local stability theorem

If $\mathbf{x}_0$ is an equilibrium point of $\mathbf{x}' = \mathbf{f}(\mathbf{x})$ and $\mathrm{Re} < 0$ for all eigenvalues λ of $D\mathbf{f}(\mathbf{x}_0)$, then all solutions that start sufficiently close to $\mathbf{x}_0$ will satisfy $\mathbf{x}(t) \to \mathbf{x}_0$ as $t \to \infty$.

4.3.2 Nullclines and Energy

Linearization is not the only tool we can use to analyze nonlinear systems.

Definition: Nullclines

Consider the general system

$$\dot{x} = f(x, y)$$
$$\dot{y} = g(x, y)$$

The x-nullclines are defined by the equation $f(x, y) = 0$. The y-nullclines are defined by the equation $g(x, y) = 0$.

Plotting the nullclines and drawing arrows on them to indicate the direction of the flow can be a striking way to roughly visualize the system's phase portrait, mainly rotation.

We can also make the observation that phase portraits look like level curve plots, with each solution curve representing a different level curve of some function. In this way, each solution corresponds to some "energy," and this energy is constant (conserved) over the entire solution curve.

Definition: Conserved quantity

A function $E : \mathbb{R}^n \to \mathbb{R}$ is a conserved quantity for $\mathbf{x}' = \mathbf{f}(\mathbf{x})$ if

$$\frac{d}{dt} E(\mathbf{x}(t)) = 0$$

for all solutions $\mathbf{x}(t)$. (We also call E an energy or constant of motion.) A system with a conserved quantity is called a conservative system.

Note: We also assume E is non-constant on any ball of radius r, because otherwise damped systems could have conserved quantities.

To show that a given quantity E is conserved, we may simply compute the time derivative of E and show that it is identically zero. To find a conserved quantity, however, is an inverse problem—we may solve the DE $\frac{dy}{dx} = \frac{\dot{y}}{\dot{x}}$ to determine the level curves, rely on physical intuition, or use some other insights.

If we determine that the energy of a system is not conserved, it can give us important information about the stability of an equilibrium.

We can also use this energy view to warp state space in different ways that clarify which solution curves correspond to higher energies.

4.3.3 Limit Cycles

We'll finish off our discussion of systems by discussing a new type of solution that only appears in nonlinear systems.

Definition: Limit cycle

A limit cycle is an isolated periodic solution to a system of DEs.

The key word here is "isolated." This means that, given some limit cycle, there are no other limit cycles in its neighborhood. This is in contrast with centers, in which there is a continuum of periodic solutions.

Given some linear system $\mathbf{x}' = \mathbf{f}(\mathbf{x})$, in order to show that a limit cycle exists we must show that f has a trapping region. This is defined below.

Definition: Trapping region

A region of a vector field Ω is called a trapping region if the vector field points inward everywhere along the boundary $\partial\Omega$.

If a solution curve enters a trapping region, there is no way to move radially inward or outward, because it will simply be deflected back. However, there is no guarantee that it will lead to a limit cycle; we give a condition for this below.

Theorem 4.19: Poincaré-Bendixon theorem

If Ω is a trapping region for the planar system $\mathbf{x}' = \mathbf{f}(\mathbf{x})$, $\mathbf{x} \in \mathbb{R}^2$ and Ω contains no equilibrium points, then Ω contains at least one periodic solution.

If the periodic solutions in this trapping region are isolated, then they are limit cycles.