MATH 136: Complex Variables
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***** Adapted from E. B. Saff & A. D. Snider, *Fundamentals of Complex Analysis* (2003) and SP25 lectures.

1 Complex Numbers

We'll start with the absolute basics of complex numbers.

Definition: Complex number

A complex number is an expression of the form $a + bi$, with $a, b \in \mathbb{R}$ and $i^2 = -1$; the set of complex numbers is denoted $\mathbb{C}$. The real part $\mathrm{Re}(z) = a$ and the imaginary part $\mathrm{Im}(z) = b$.

Adding and subtracting complex numbers looks exactly how we'd expect—if $z = a + bi$ and $w = c + di$, then $z \pm w = (a + c) + i(b + d)$. Multiplication and division look like

$$zw = (ac - ba) + i(ad + bc), \qquad \frac{z}{w} = \left(\frac{ac + bd}{c^2 + d^2}\right) + i\left(\frac{bc - ad}{c^2 + d^2}\right),$$

where we require $w \neq 0$ for division. To get a more geometric sense for these, we make a couple more definitions motivated by framing complex numbers as vectors in the real-imaginary plane.

Definition: Norm and argument

Let the number $z = a + bi$ make an angle θ with the positive-real axis. The norm (or magnitude, modulus) and argument of z are, respectively,

$$|z| = \sqrt{a^2 + b^2}, \arg(z) = \{\theta + 2\pi k \mid k \in \mathbb{Z}\},$$

where $\arg(0)$ is undefined. The principal argument $\mathrm{Arg}(z)$ is the unique value of $\arg(z)$ in $(-\pi, \pi]$.

We can therefore express complex numbers in polar form via

$$z = r\cos\theta + ir\sin\theta = r\operatorname{cis}\theta,$$

and we could use some trigonometric identities to show that multiplication should be interpreted as multiplying magnitudes and adding arguments. (Note that arguments add, but Arguments might not.) We can thus state de Moivre's formula

$$(\cos\theta + i\sin\theta)^n = \cos(n\theta) + i\sin(n\theta), \quad n \in \mathbb{N},$$

which has a straightforward proof by induction. This also allows us to reason about complex roots! Suppose $z = r\operatorname{cis}\theta$; we seek $w = \rho\operatorname{cis}\phi$ such that $w^n = z$ for some $n \in \mathbb{N}$, that is,

$$\rho^n \operatorname{cis} n\phi = r\operatorname{cis}\theta.$$

We can see that $\rho = r^{1/n}$, the unique positive root of $r > 0$, and $\phi = (\theta + 2\pi k)/n = (\theta/n) + (2\pi/n)k$ for $k \in \mathbb{Z}$. These roots are distinct for $k \in \{0, 1, \ldots, n-1\}$, and together they define our nth roots! (Note that an nth root is called primitive if there is no $k \in \{1, \ldots, n-1\}$ for which $w^k = 1$.)

Now we'll introduce a new, distinctly complex feature of this number system.

Definition: Complex conjugate

The complex conjugate of $z = a + bi$ is

$$\overline{z} = a - bi.$$

It is clear that $\overline{z \pm w} = \overline{z} \pm \overline{w}$ and $z\overline{z} = |z|^2$, and if we interpret conjugation as reflection over the real axis, we get

$$|\overline{z}| = |z|, \quad \overline{zw} = \overline{z}\overline{w}, \quad \overline{\left(\frac{z}{w}\right)} = \frac{\overline{z}}{\overline{w}}, \quad \overline{\overline{z}} = z.$$

We also have a few handy identities:

$$\mathrm{Re}(z) = \frac{z + \overline{z}}{2}, \qquad \mathrm{Im}(z) = \frac{z - \overline{z}}{2i}, \qquad \frac{z}{w} = \frac{z\overline{w}}{|w|^2}.$$

Finally, we can use the notion of conjugation to arrive at a familiar result!

Theorem 1.1: Triangle inequality

For any $z, w \in \mathbb{C}$,

$$|z + w| \leq |z| + |w|.$$

Proof. We could prove this by interpreting complex numbers as vectors, but alternatively, we can simply write

$$\begin{aligned} |z + w| &= (z + w) \cdot \overline{z + w} \\ &= z\overline{z} + z\overline{w} + w\overline{z} + w\overline{w} \\ &= |z|^2 + |w|^2 + 2\,\mathrm{Re}(z\overline{w}). \end{aligned}$$

By the triangle inequality in $\mathbb{R}$,

$$\begin{aligned} &\leq |z|^2 + |w|^2 + |z\overline{w}| \\ &= (|z| + |w|)^2, \end{aligned}$$

as desired. □

We'll end by constructing an interesting model of the complex plane. Let Σ be a diameter-1 sphere whose south pole is at the origin. If the xy-plane is identified with $\mathbb{C}$ and the u-axis is vertical, the sphere is then described by

$$x^2 + y^2 + \left(u - \frac{1}{2}\right)^2 = \frac{1}{4}.$$

Now, for each $z \in \mathbb{C}$ we draw a line segment to the north pole N and notice that the segment intersects with the sphere at exactly one other point. This defines a bijection from $C \to \Sigma \setminus \{N\}$, and to map to all of Σ we add a point ∞ defined by $\sigma \leftrightarrow N$. We therefore define the extended complex plane $\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$, and we have $\hat{\mathbb{C}} \simeq \Sigma$.

To make this bijection explicit we first parametrize the line via

$$\ell(t) = (x, y, 0) + t(-x, -y, 1), \quad t \in [0, 1]$$

and notice that it intersects with the sphere when

$$x^2(1 - t)^2 + y^2(1 - t)^2 + \left(t - \frac{1}{2}\right)^2 = \frac{1}{4}.$$

This equation has roots

$$t = 1, \qquad t = \frac{x^2 + y^2}{1 + x^2 + y^2},$$

and so the nontrivial intersection is at

$$\left(\frac{x}{1 + x^2 + y^2}, \frac{y}{1 + x^2 + y^2}, \frac{x^2 + y^2}{1 + x^2 + y^2}\right) \in \Sigma.$$

This characterization of Σ is called the Riemann sphere. (In topological terms, we call $\hat{\mathbb{C}}$ a one-point compactification of $\mathbb{C}$.)

2 Analytic Functions

2.1 Limits and Differentiability

Now we'll begin our study of functions $f : \mathbb{C} \to \mathbb{C}$, with the aim of mimicking the key results from calculus.

Definition: Limit

We say that $\lim_{z \to z_0} f(z) = w_0$ if, for all $\varepsilon > 0$, there exists a $\delta > 0$ such that

$$0 < |z - z_0| < \delta \implies |f(z) - w_0| < \varepsilon.$$

That is, for every radius-ε neighborhood around w_0, there is a radius-δ neighborhood around z_0 such that all z in the z_0-neighborhood have $f(z)$ in the w_0-neighborhood. We'll take all the basic limit properties as given, as their proofs are analogous to those in multivariable calculus.

Definition: Continuity

We say $f(z)$ is continuous at z_0 if

$$\lim_{z \to z_0} f(z) = f(z_0).$$

f is continuous on a set Ω if f is continuous at each $z_0 \in \Omega$.

Now, since the output of f has real and imaginary parts, we must be able to write f in the form

$$f(x + iy) = u(x, y) + i\,v(x, y),$$

where $u, v : \mathbb{R}^2 \to \mathbb{R}$; in this case we write $u = \mathrm{Re}(f)$ and $v = \mathrm{Im}(f)$. Conveniently, f is continuous if and only if u and v are continuous at (x_0, y_0)! We'll once again take all the basic continuity properties as given.

Definition: Derivative

Suppose $f(z)$ is defined in a neighborhood of $z_0 \in \mathbb{C}$. The derivative of f at z_0 is

$$f'(z_0) = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0} = \lim_{h \to 0} \frac{f(z_0 + h) - f(z_0)}{h},$$

provided this limit exists. Such an f is said to be differentiable at z_0.

Once again, all rules for differentiation in $\mathbb{R}$ apply to differentiation in $\mathbb{C}$. Note that f is not necessarily differentiable if $\mathrm{Re}(f)$ and $\mathrm{Im}(f)$ are differentiable!

Definition: Analytic

A function f is analytic (or holomorphic) on an open set Ω if f is differentiable at each $z_0 \in \Omega$. f is analytic at z_0 if there exists a neighborhood of z_0 on which f is analytic, and if f is analytic on $\mathbb{C}$ we say f is entire.

Analytic functions have a nice relationship between the transformations they represent and their derivatives. From the definition we have the approximation

$$f(z_0 + h) = f(z_0) + f'(z_0)h + \mathcal{O}(h),$$

meaning analytic functions are locally affine and $f'(z_0)$ encodes the information needed to take an small "step" away from z_0. Such a step involves three kinds of transformations: a rotation, a dilation, and a translation.

Analytic functions are foundational to the study of the complex numbers. They're characterized by a very particular relationship between their real and imaginary parts.

Lemma 2.1

If $f = u + iv$ is differentiable at z_0 then

$$f'(z_0) = u_x + iv_x = v_y - iu_y.$$

Proof. Suppose $f(x + iy) = u(x, y) + i\,v(x, y)$ is differentiable at $z_0 \in \mathbb{C}$, so the limit

$$f'(z_0) = \lim_{h\to 0} \frac{f(z_0 + h) - f(z_0)}{h}$$

exists. If we approach along the horizontal axis (so $\mathrm{Im}(h) = 0$) and the vertical axis (so $\mathrm{Re}(h) = 0$) the limit becomes

$$f'(z_0) = \frac{\partial u}{\partial x}(x_0, y_0) + i\frac{\partial v}{\partial x}(x_0, y_0), \qquad f'(z_0) = -i\frac{\partial u}{\partial y}(x_0, y_0) + \frac{\partial v}{\partial y}(x_0, y_0)$$

respectively. These expressions are equivalent. □

Theorem 2.2: Cauchy-Riemann equations

If $f = u + iv$ is differentiable at z_0 then

$$\begin{aligned} u_x &= v_y, \\ u_y &= -v_x. \end{aligned}$$

Consequently, if f is analytic on an open set Ω then f satisfies these equations everywhere in Ω.

Notice that if the Cauchy-Riemann equations hold then we have the derivative matrix

$$D\mathbf{f} = \begin{bmatrix} u_x & u_y \\ v_x & v_y \end{bmatrix} = \begin{bmatrix} u_x & -v_x \\ v_x & u_x \end{bmatrix},$$

which we recognize as a rotation-dilation matrix! Now, we can see that satisfying the Cauchy-Riemann equations is a necessary condition for differentiability, but they are only one part of a sufficient condition.

Theorem 2.3: Sufficient condition for differentiability

Let $f = u + iv$ be defined in an open set Ω containing $z_0 = x_0 + iy_0$. If the first partials of u and v exist on Ω, are continuous at (x_0, y_0), and satisfy the Cauchy-Riemann equations at (x_0, y_0), then f is differentiable at z_0.

Consequently, if these conditions hold everywhere in Ω then f is analytic on Ω.

Proof. Let $h = s + it$. We first break the difference quotient into real and imaginary parts,

$$\frac{f(z_0 + h) - f(z_0)}{h} = \left(\frac{u(z_0 + h) - u(z_0)}{h}\right) + i\left(\frac{v(z_0 + h) - v(z_0)}{h}\right),$$

and look specifically at its real component:

$$\frac{u(z_0 + h) - u(z_0)}{h} = \frac{1}{h}\left[\Big(u(x_0 + s,\, y_0 + y) - u(x_0 + s,\, y_0)\Big) + \Big(u(x_0 + s,\, y_0) - u(x_0, y_0)\Big)\right].$$

By the mean value theorem in $\mathbb{R}$, there exist $0 < \theta_1, \theta_2 < 1$ such that

$$\begin{aligned} &= \frac{1}{h}\left[\frac{\partial u}{\partial y}(x_0 + s,\, y_0 + \theta_1 t)t + \frac{\partial u}{\partial x}(x_0 + \theta_2 s,\, y_0)s\right] \\ &= \frac{t}{h}\frac{\partial u}{\partial y}(x_0 + s,\, y_0 + \theta_1 t) + \frac{s}{h}\frac{\partial u}{\partial x}(x_0 + \theta_2 s,\, y_0). \end{aligned}$$

We could go through an analogous line of reasoning to get a similar expression for the imaginary component. Adding the results gives

$$\begin{aligned} \frac{f(z_0+h) - f(z_0)}{h} = \frac{t}{h}&\left[\frac{\partial u}{\partial y}(x_0 + s,\, y_0 + \theta_1 t) + i\frac{\partial v}{\partial y}(x_0 + s,\, y_0 + \theta_3 t)\right] \\ &+ \frac{s}{h}\left[\frac{\partial u}{\partial x}(x_0 + \theta_2 s,\, y_0) + i\frac{\partial v}{\partial x}(x_0 + \theta_4 s,\, y_0)\right]. \end{aligned}$$

We'd like to show that this expression approaches $f'(z_0)$ as $h \to 0$. To this end we subtract $\frac{s+it}{h}(u_x + iv_x)$ and show that the result goes to zero. Call the difference δ, so

$$\begin{aligned} \delta = \frac{t}{h}&\Big[u_y(x_0 + s,\, y_0 + \theta_1 t) + v_x(x_0, y_0) + i\Big(v_y(x_0 + s,\, y_0 + \theta_3 t) - u_x(x_0, y_0)\Big)\Big] \\ &+ \frac{s}{h}\Big[u_x(x_0 + \theta_2 s,\, y_0) - u_x(x_0, y_0) + i\Big(v_x(x_0 + \theta_4 s,\, y_0) - v_x(x_0, y_0)\Big)\Big]. \end{aligned}$$

Applying the Cauchy-Riemann equations demonstrates that, if u and v are continuous, then $\delta \to 0$. Thus $f'(z_0)$ exists. □

2.2 Harmonic Functions

It turns out that analytic functions are deeply tied to the harmonic functions central to the study of PDEs! We'll state the following a fun fact for now, deferring a proof to later.

Theorem 2.4: Harmonic conjugates

If $u + iv$ is analytic on a domain (an open and connected set) Ω then $\Delta u = \Delta v = 0$ on Ω. In other words, u and v are harmonic functions. (In particular, because they are the real and imaginary parts of an analytic function, u and v are called harmonic conjugates.)

Theorem 2.5

Harmonic conjugates have orthogonal level curves.

Proof. Consider a level curve $u(x, y) = C$ with parametrization $\mathbf{x}(t) = \big(x(t), y(t)\big)$. By the chain rule,

$$0 = \frac{d}{dt}u\big(x(t), y(t)\big) = \frac{\partial u}{\partial x}x'(t) + \frac{\partial u}{\partial y}y'(t) = \nabla u \cdot \mathbf{x}'(t),$$

meaning ∇u is orthogonal to the level curves of u. Now, if u, v are harmonic conjugates then by the Cauchy-Riemann equations $\nabla u \cdot \nabla v = (u_x, u_y) \cdot (v_x, v_y) = 0$. So when $\nabla u \neq 0 \neq \nabla v$, the level curves of u and v are orthogonal. □

Now for a nice application of harmonic functions that isn't explicitly related to complex variables, but is interesting nonetheless.

Theorem 2.6: Mean-value property of harmonic functions

Let $B(z_0, r)$ denote be the closed ball of radius r about z_0. If $\phi(x, y)$ is harmonic on the open set $\Omega \subseteq \mathbb{R}^2$ then

$$\phi(x_0, y_0) = \frac{1}{2\pi r}\int_{\partial B(z_0,r)} \phi(x, y)\, ds$$

for any $r > 0$ such that $B(z_0, r) \subseteq \Omega$.

Proof. The average value of ϕ over the boundary of $B(z_0, r)$ is

$$\frac{1}{2\pi r}\int_{\partial B(z_0,r)} \phi(x, y)\, ds = \frac{1}{2\pi r}\int_0^{2\pi} \phi(x_0 + r\cos t,\, y_0 + rt) r\, dt.$$

Differentiating:

$$\begin{aligned}\frac{dm}{dr} &= \frac{1}{2\pi r}\int_0^{2\pi}\left(\frac{\partial\phi}{\partial x}(x, y)\cdot r\cos t + \frac{\partial\phi}{\partial y}(x, y)\cdot r\sin t\right) dt \\ &= \frac{1}{2\pi r}\int_{\partial B(z_0,r)} \nabla\phi\cdot\mathbf{n}\, ds.\end{aligned}$$

By the divergence theorem,

$$\begin{aligned}&= \frac{1}{2\pi r}\iint_{B(z_0,r)} \nabla\cdot(\nabla\phi)\, dA \\ &= \frac{1}{2\pi r}\iint_{B(z_0,r)} \Delta\phi\, dA = 0.\end{aligned}$$

Thus $m(r)$ is independent of by r, and by continuity $m(r) = \lim_{r\to 0} m(r) = \phi(x_0, y_0)$, as desired. $\square$

2.3 Exponential and Trigonometric Functions

Now we'll look at a few particularly important analytic functions called elementary functions.

Definition: Complex exponential

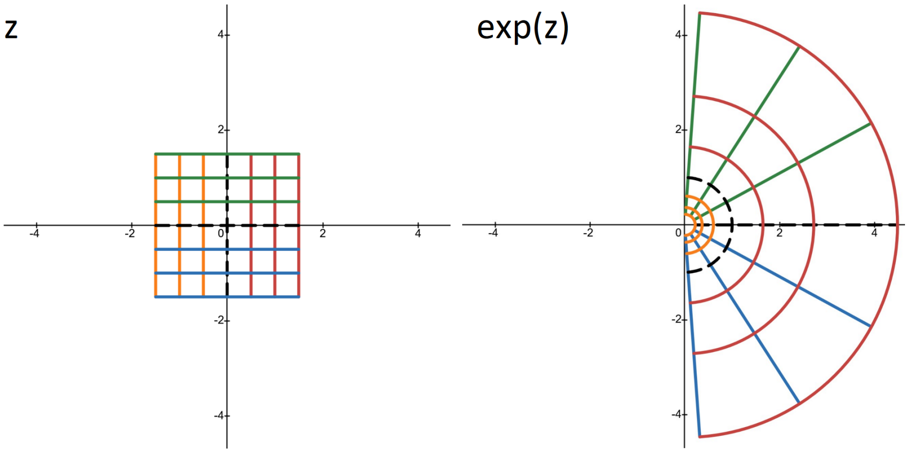
If $z = x + iy$ then

$$e^z = e^x\cos y + ie^x\sin y.$$

We can immediately make a few observations.

- The complex exponential agrees with the familiar real-valued exponential when evaluated on the real line, as we'd expect from a generalization to $\mathbb{C}$.
- e^z is entire with derivative e^z, as we'd hope.
- The familiar $e^{x+iy} = e^x e^{iy}$ and $e^{z+w} = e^z e^w$ hold, meaning $e^{nz} = (e^z)^n$ and $(e^{i\theta})^n = e^{in\theta}$ for $n \in \mathbb{Z}$.
- The magnitude of e^z is determined entirely by the real part of its argument: $|e^z| = |e^{\operatorname{Re} z}|$.
- If $z = iy$ then $e^{iy} = \cos y + i\sin y$, meaning $r\operatorname{cis}\theta = re^{i\theta}$! It follows that e^z is 2π-periodic parallel to the imaginary axis.
- e^z and its components are harmonic.

From the definition we can also see that exponentiation maps vertical lines $z = x_0 + iy$ to radius-e^{x_0} circles, and horizontal lines $z = x + iy_0$ to rays going off in the direction of e^{iy_0}.



Now we'll use Euler's formula $e^{i\theta} = \cos\theta + i\sin\theta$ to define the complex sine and cosine.

Definition: Complex sine and cosine

If $z \in \mathbb{C}$ then
$$\cos z = \frac{e^{iz} + e^{-iz}}{2}, \qquad \sin z = \frac{e^{iz} - e^{-iz}}{2i}.$$

We'll once again make some observations.

- Both sine and cosine agree with their real-valued counterparts when evaluated on the real line.
- $\sin z$ and $\cos z$ are entire with derivatives $\cos z$ and $-\sin z$, respectively.
- The familiar trigonometric identities hold: $\sin(-z) = -\sin z$, $\cos(-z) = \cos z$, $\sin^2 z + \cos^2 z = 1$,
$$\sin(z+w) = \sin z\cos w + \cos z\sin w,$$
$$\cos(z+w) = \cos z\cos w - \sin z\sin z.$$
- The real and imaginary parts of each function are
$$\begin{aligned} \operatorname{Re}[\sin z] &= \sin x\cosh y, & \operatorname{Re}[\cos z] &= \cos x\cosh y, \\ \operatorname{Im}[\sin z] &= \cos x\sinh y, & \operatorname{Im}[\cos z] &= -\sin x\sinh y. \end{aligned}$$

Let's take a quick look at sine. If we define $u = \sin x\cosh y$ and $v = \cos x\sinh y$ then we can write
$$\frac{u^2}{\cosh^2 y} + \frac{v^2}{\sinh^2 y} = 1, \qquad \frac{u^2}{\sin^2 x} - \frac{v^2}{\cos^2 x} = 1.$$

Thus horizontal lines at y_0 map to ellipses and vertical lines at x_0 map to half-hyperbolas. (For $0 < x_0 < \pi$ we get a right-hyperbola, while for $\pi < x_0 < 2\pi$ we get a left-hyperbola.)

Finally, note that we define the complex hyperbolic trig functions in the exact same way as the real ones.

2.4 Logarithms and Power Functions

In defining the complex logarithm we must keep in mind that we should have $\log z = w$ if and only if $e^w = z$. We can see that if $w = u + iv$ then
$$e^u = |z|, \qquad v = \arg z.$$

Thus the logarithm must be a multi-valued function—although we can guarantee $e^{\log z} = z$ for all admissible z, in general we must write $\log e^z = z + 2\pi ik$ for all $k \in \mathbb{Z}$. The definition follows.

Definition: Complex logarithm

If $z \in \mathbb{C} \setminus \{0\}$, define the set- and single-valued functions

$$\log z = \ln|z| + i\arg(z),$$
$$\operatorname{Log} z = \ln|z| + i\operatorname{Arg}(z).$$

We will continue to write $\ln x$ for the real function $\ln x = \int_1^x dt/t$.

Theorem 2.7: Logarithms are analytic

The function $\operatorname{Log} z$ is analytic on all of $\mathbb{C}$, excluding the non-positive real axis; for such z,

$$\frac{d}{dz}\operatorname{Log} z = \frac{1}{z}.$$

It follows that the real part $\frac{1}{2}\ln(x^2+y^2)$ is harmonic.

Proof. Let $w = \operatorname{Log} z$ and $w_0 = \operatorname{Log} z_0$. We know, from properties of the exponential function, that

$$\lim_{w\to w_0}\frac{z-z_0}{w-w_0} = \frac{dz}{dw}\bigg|_{w=w_0} = e^{w_0} = z_0.$$

Note that $w \to w_0$ as $z \to z_0$ and that $w \neq w_0$ as long as $z \neq z_0$. Thus

$$\lim_{z\to z_0}\frac{w-w_0}{z-z_0} = \lim_{w\to w_0}\frac{1}{\frac{z-z_0}{w-w_0}} = \frac{1}{z_0},$$

as desired. □

We can see that we must treat the non-positive real axis carefully while working with the principal logarithm. But really, there's nothing special about this axis—we could easily have defined the single-valued logarithm a different way and gotten a different ray to be careful of.

Definition: Branch

$F(z)$ is a branch of the multi-valued $f(z)$ on a domain D if $F(z)$ is single-valued and continuous on Ω, and for each $z \in \Omega$, $F(z)$ is one of the values $f(z)$. The set of points at which $F(z)$ is continuous is called the branch cut of F.

With this in mind, we call $\operatorname{Log} z$ the principal branch of the logarithm; the non-positive real axis is its branch cut. A branch with analytic range $(\theta, \theta + 2\pi]$ is denoted by $\operatorname{Log}_\theta z$ (with argument $\operatorname{Arg}_\theta z$), and its branch cut is in the direction of $e^{i\theta}$. The branch we choose to work with depends primarily on where we want the branch cut to be—or, rather, where we don't want it to be.

We can use logarithms to define power functions involving complex numbers!

Definition: Power function

Let $z \in \mathbb{C} \setminus \{0\}$ and $\alpha \in \mathbb{C}$. Then
$$z^\alpha = e^{\alpha \log z}.$$

Power functions are generally multi-valued, but not always—note that

$$z^\alpha = e^{\alpha\ln|z| + i\alpha\operatorname{Arg} z}e^{i\cdot 2\pi\alpha k},$$

so for real α we have three cases. If $\alpha \in \mathbb{Z}$ then $e^{i\cdot 2\pi\alpha k} = 1$ for all k and z^α is single-valued; if $\alpha \in \mathbb{Q}$ then the $e^{i\cdot 2\pi\alpha k}$ are evenly spaced around the unit circle; and for irrational α the exponential $e^{i\cdot 2\pi\alpha k}$ fills the unit circle densely.

We choose a branch of z^α by picking out a branch of the logarithm, so the principal branch is

$$z^\alpha = e^{\alpha \operatorname{Log} z}$$

with the branch cut along the non-positive real axis. Off the branch cut we have

$$\frac{d}{dz} z^\alpha = \frac{d}{dz} e^{\alpha \operatorname{Log} z} = \frac{\alpha}{z} e^{\alpha \operatorname{Log} z} = \alpha z^{\alpha - 1}.$$

A couple more notes. First, the familiar laws of exponents hold—multiplication turns into addition, and division into subtraction. We can also use out brief analysis of power functions to determine when we're allowed to split (principal) square roots like $\sqrt{z^2} = \sqrt{z}\sqrt{z}$! Following the definition, we can see that

$$\sqrt{z^2} = e^{\ln|z|} e^{i \frac{1}{2} \operatorname{Arg}(z^2)}, \qquad \sqrt{z}\sqrt{z} = e^{\ln|z|} e^{i \operatorname{Arg} z}.$$

For equality we require $\operatorname{Arg} z^2 = 2 \operatorname{Arg} z$, so we must have $\operatorname{Arg} z \in (-\pi/2, \pi/2]$. In other words, these are the z for which squaring doesn't move its argument outside the principal branch!

3 Integration

3.1 Contour Integrals

The integral in $\mathbb{C}$ is defined in pretty much the same way as it is in $\mathbb{R}$, and evaluating them is reminiscent of line integrals in multivariable calculus.

Definition: Integral

Let γ be a directed smooth curve in $\mathbb{C}$ that is split into several pieces γ_k, each with endpoints z_k, z_{k+1} and a point c_k between them. Then

$$\int_\gamma f(z)\,dz = \lim_{\substack{N\to\infty \\ \Delta z_k \to 0}} \sum_{k=0}^{N-1} f(c_k)\Delta z_k,$$

provided the limit exists and is independent of the choice of partition.

Theorem 3.1

Let f be a function continuous on the directed smooth curve γ. Then if $z = z(t)$, $t \in [a, b]$ is a parametrization of γ, we have

$$\int_\gamma f(z)\,dz = \int_a^b f(z(t))\,z'(t)\,dt.$$

Proof. Suppose γ has a parametrization $z(t) = x(t) + i\,y(t)$, $t \in [a, b]$; the path is traversed once and $z'(t) \neq 0$ for all t. Defining $\Delta t_k = t_{k+1} - t_k$, we can write

$$\int_\gamma f(z)\,dz = \lim_{\substack{N\to\infty \\ \Delta t_k \to 0}} \sum_{k=0}^{N-1} f(z(c_k))\frac{z(t_{k+1}) - z(t_k)}{\Delta t_k}\Delta t_k,$$

where $c_k \in t_k, t_{k+1}$. There is no mean value theorem in $\mathbb{C}$, but we can split this difference quotient into real and imaginary parts to apply the MVT in $\mathbb{R}$: for some $\alpha_k, \beta_k \in (t_k, t_{k+1})$, we can write

$$\begin{aligned} &= \lim_{\substack{N\to\infty \\ \Delta t_k \to 0}} \sum_{k=0}^{N-1} f(z(c_k))\big(x'(\alpha_k) + i\,y'(\beta_k)\big)\Delta t_k \\ &= \int_a^b f(z(t))\,z'(t)\,dt, \end{aligned}$$

as desired. □

We could use this to show that

$$\oint_\gamma (z - z_0)^n\,dz = \begin{cases} 0 & n \neq -1, \\ 2\pi i & n = -1, \end{cases}$$

where γ is a once-traversed circle centered at z_0, oriented counterclockwise; this particular integral will prove to be very useful later. We should expect that this result, like any other integral, is independent of how we choose to parametrize γ, and this is the case! Proving it is a simple application of the above theorem.

Corollary 3.2

Let $\tilde{\gamma}$ be a reparametrization of γ—that is, suppose there is a $\phi(t)$ satisfying $\phi'(t) > 0$ such that $\gamma(t) = \tilde{\gamma}(\phi(t))$. Then

$$\int_\gamma f(z)\,dz = \int_{\tilde{\gamma}} f(z)\,dz.$$

Given a path γ, we will use $-\gamma$ to denote γ as traversed in the opposite direction, and we could show that

$$\int_{-\gamma} f(z)\,dz = -\int_\gamma f(z)\,dz.$$

In general we will integrate over contours, which are finite sequences of directed smooth curves that connect one after another. We will not always need to evaluate such contour integrals however; an upper bound is often good enough, and we can get one pretty easily.

Lemma 3.3: ML lemma

Let γ be a contour with length $\ell(\lambda)$. If f is continuous on γ and if $|f(z)| \le M$ for all z on γ, then

$$\left|\int_\gamma f(z)\,dz\right| \le M \cdot \ell(\gamma).$$

Proof. Suppose $\int_\gamma f(z)\,dz = r_0 e^{i\theta_0}$. Then

$$\begin{aligned}\left|\int_\gamma f(z)\,dz\right| &= e^{-i\theta_0} \int_a^b f(z(t))\,z'(t)\,dt \\ &= \int_a^b \operatorname{Re}\left[e^{-i\theta_0} f(z(t))\,z'(t)\right] dt \\ &\le \int_a^b M\,|z'(t)|\,dt = M \cdot \ell(\gamma),\end{aligned}$$

as desired. □

Note that the "lower triangle inequality" $\big||a| - |b|\big| \le |a+b|$ will often be useful in using this inequality.

3.2 Path Independence

Now we're ready for some of the bigger theorems in integral calculus, starting with an old friend.

Theorem 3.4: Fundamental theorem

Suppose f is continuous on a domain Ω and has a primitive (antiderivative) $F(z)$ throughout Ω. Then for any contour γ in Ω with endpoints z and w we have

$$\int_\gamma f(z)\,dz = F(w) - F(z).$$

Proof. Let $\gamma : [a, b] \to \mathbb{C}$ be a smooth contour. Then

$$\int_\gamma f(z)\,dz = \int_a^b f(\gamma(t))\gamma'(t)\,dt = F(\gamma(b)) - F(\gamma(a)).$$

For any other contour, we can break it up into smooth contours and sum them to get the same. □

Note that this implies that F is analytic, and so also continuous, in Ω. Next we have a less immediately applicable set of equivalent statements, but one whose use will soon become clearer.

Theorem 3.5: ABC theorem

Let f be continuous in a domain Ω. Then the following statements are equivalent.

(a) f has an antiderivative in Ω.

(b) If γ is a closed loop in Ω then $\int_\gamma f(z)\,dz = 0$.

(c) The contour integrals of f are path-independent in Ω.

Proof. It is easy to see that (a) implies both (b) by the fundamental theorem. Now let γ_1, γ_2 be any two contours in Ω with the same endpoints and define $\Gamma = \gamma_1 - \gamma_2$. If we assume (b), then

$$0 = \int_{\gamma_1} f(z)\,dz + \int_{-\gamma_2} f(z)\,dz = \int_{\gamma_1} f(z)\,dz - \int_{\gamma_2} f(z)\,dz$$

and we get (c). Now, assuming (c), fix $z_0 \in \Omega$ and let $\gamma(z)$ be a path from z_0 to some $z \in \Omega$. Define

$$F(z) = \int_{\gamma(z)} f(w)\,dw, \quad z \neq z_0.$$

This function is well-defined because $F(z)$ is independent of the specific choice of path γ, and we could apply the definition of the derivative to show that $F' = f$, so we get (a). $\square$

3.3 Cauchy's Theorem

The ABC theorem might seem difficult to apply at present, but we're about to derive another result that gives a sufficient condition for (b). We need to lay a bit of topological groundwork first, though.

Definition: Homotopy

Let $\gamma_0, \gamma_1 : [0,1] \to \Omega$ be closed curves in a domain Ω. γ_1 is homotopic to γ_0 if there exists a continuous homotopy $H : [0,1] \times [0,1] \to \Omega$ such that

$$H(0,t) = \gamma_0(t), \qquad H(1,t) = \gamma_1(t), \qquad H(s,0) = H(s,1).$$

(We include the third condition because we're interested in closed curves.)

Definition: Simply connected domain

A domain Ω is simply connected if every closed path in Ω is homotopic to a point in Ω—in other words, the domain has no "holes".

We could do some straightforward calculus to show that if

$$I(s) = \int_{\gamma_s} f(z)\,dz = \int_0^1 f(H(s,t))\frac{\partial H}{\partial t}(s,t)\,dt,$$

then $I'(s) = 0$ so long as $f \in C^1$ (to bring the $\partial/\partial s$ into the integral) and $H \in C^2$ (for the equality of mixed partials). Thus if f has a continuous derivative, its integral is the same over any pair of homotopic curves.

A couple more clarifications. A simple closed curve is one that does not self-intersect, and it is positively oriented if the enclosed region is on the left of the curve. (This is how we generalize counterclockwise-ness.)

Now we'll give a couple of proofs for the same theorem—one using this idea of contour deformation we just arrived at, and another using vector calculus.

Theorem 3.6: Cauchy's theorem (preliminary)

If f is analytic with a continuous derivative on a simply connected domain Ω and γ is a positively oriented simple closed curve in Ω, then

$$\oint_\gamma f(z)\,dz = 0.$$

Proof. On a simply connected domain Ω, a closed loop γ in Ω is homotopic to a point $z_0 \in \Omega$. Thus

$$\oint_\gamma f(z)\,dz = \int_{\{z_0\}} f(z)\,dz = 0$$

as desired. □

Proof. Let γ be parametrized by $z(t)$, $t \in [a, b]$. If $z = x + iy$ and $f = u + iv$, then we can write

$$\begin{aligned}\oint_\gamma f\,dz &= \int_a^b f(z(t))z'(t)\,dt \\ &= \int_a^b (ux' - vy') + i(vx' + uy')\,dt \\ &= \oint_\gamma u\,dx - v\,dy + i\oint_\gamma v\,dx + u\,dy,\end{aligned}$$

and by Green's theorem

$$= \iint_\Omega (-v_x - u_y)\,dA + i\iint_\Omega (u_x = v_y)\,dA;$$

by the Cauchy-Riemann equations, this goes to zero. □

These proofs are simple, but we'd like to remove the condition of having a continuous derivative. We begin with a special case.

Lemma 3.7

Let $R = [a, b] \times [c, d]$ be a rectangle in Ω. If f is analytic on Ω then

$$\oint_{\partial R} f\,dz = 0.$$

Proof. Suppose a rectangle R has diagonal and perimeter lengths D and P respectively. Divide this rectangle into four sub-rectangles and suppose, without loss of generality, that R_{11} is the sub-rectangle for which $\left|\oint_{\partial R_{i1}} f\,dz\right|$ is maximized. Now split this maximal rectangle into four and let R_{12} be the piece that maximizes the line integral. Continuing on this way for k subdivisions reveals that

$$\left|\oint_{\partial R} f\,dz\right| \le 4^k \left|\oint_{\partial R_{1k}} f\,dz\right|.$$

From analysis we know that there exists a $z_0 \in \bigcap_{k=1}^{\infty} R_{1k}$. f is analytic at this point, meaning

$$f(z) = f(z_0) + f'(z_0)(z - z_0) + o(z - z_0), \qquad \lim_{z \to z_0} \frac{o(z - z_0)}{z - z_0} = 0.$$

Thus for every $\varepsilon > 0$ there is a $k > 1$ such that $|o(h)/(z - z_0)| < \varepsilon$ for all $z \in R_{1k}$. By the ABC theorem and ML lemma,

$$\left| \oint_{\partial R_{1k}} f(z)\, dz \right| = \left| \oint_{\partial R_{1k}} o(h)\, dz \right| \leq (\varepsilon \cdot 2^{-k} D) \cdot 2^{-k} P.$$

The original integral therefore becomes

$$\left| \oint_{\partial R} f\, dz \right| \leq 4^k \left| \oint_{\partial R_{1k}} f\, dz \right| \leq 4^k \cdot (\varepsilon \cdot 2^{-k} D \cdot 2^{-k} P) = \varepsilon DP.$$

Since our $\varepsilon > 0$ was arbitrary, this integral is zero. ☐

Now we'll take another step by proving Cauchy's theorem, but only in a disk.

Lemma 3.8: Cauchy's theorem (in a disk)

Let Ω be an open disk with radius R centered at z_0. If f is analytic in Ω and γ is a closed contour in Ω then

$$\oint_\gamma f(z)\, dz = 0.$$

Proof. Given $w \in \Omega$ define γ_w as the path from z_0 to w traversed in two purely horizontal and vertical components γ_1, γ_2. The function

$$F(w) = \int_{\gamma_w} f(z)\, dz$$

is well-defined for all $w \in \Omega$, and we'd like to prove that $F'(w) = f(w)$.

We first determine $F(w + h) - F(w)$—given $\varepsilon > 0$, there exists a $\delta > 0$ not only such that $|z - w| < \delta$ implies $|f(z) - f(w)| < \varepsilon$, but also such that $\{|z - w| < \delta\} \subseteq \Omega$. For $|h| < \delta$ we similarly define γ_{w+h} in terms of horizontal and vertical components and notice that

$$\int_{\gamma_{w+h}} f\, dz = \int_{\gamma_w} f\, dz \pm \oint_{\partial R} f\, dz + \int_{w \to w+h} f\, dz,$$

where R is the rectangle formed by connecting γ_w and γ_{w+h} and the $\pm$ is whichever sign is needed to make things cancel nicely. The integral over ∂R is zero by the previous lemma, meaning

$$F(w + h) - F(w) = \int_{w \to w+h} f\, dz.$$

Now by the ML lemma,

$$\left| \frac{F(w + h) - F(w)}{h} - f(w) \right| = \frac{1}{|h|} \left| \int_{w \to w+h} (f(z) - f(w))\, dz \right| < \frac{1}{|h|} (\varepsilon \cdot 2|h|) = 2\varepsilon,$$

and since $\varepsilon > 0$ was arbitrary we arrive at $F'(w) = f(w)$ for all $w \in \Omega$. Thus by the ABC theorem $\oint_\gamma f\, dz = 0$, as desired. ☐

Finally, we can make a contour deformation argument to get what we want!

Theorem 3.9: Cauchy's theorem

If f is analytic in a simply connected domain Ω and γ is any closed contour in Ω, then

$$\oint_\gamma f(z)\,dz = 0.$$

Proof. If γ_0 is a closed contour in Ω then it is homotopic to a circle γ_1 interior to γ_0. If we break the domain of the homotopy H into a bunch of very small chunks R_{ij} and define $\Gamma_{ij} = H(\partial R_{ij})$ then by the previous theorem

$$\int_\Gamma f(z)\,dz = \sum_{i,j} \int_{\Gamma_{ij}} f(z)\,dz = \sum_{i,j} 0 = 0,$$

where Γ is the contour produced by traversing γ_0, moving inward to traverse γ_1, and then back out again. If $\gamma_0 = H(0,t)$ and $\gamma_1 = H(1,t)$ then we can write

$$0 = \int_\Gamma f(z)\,dz = \int_{-H(0,t)} f(z)\,dz + \int_{H(1,t)} f(z)\,dz + \int_{H(s,0)} f(z)\,dz + \int_{-H(s,1)} f(z)\,dz,$$

meaning $\oint_{\gamma_0} f(z)\,dz = \oint_{\gamma_1} f(z)\,dz$. Thus integrals in Ω are invariant under homotopy and for some $z_0 \in \Omega$

$$\oint_{\gamma_0} f(z)\,dz = \int_{\{z_0\}} f(z)\,dz = 0,$$

as desired. □

3.4 Cauchy's Integral Formula

Contour deformation also provides motivation for our next key result.

Theorem 3.10: Cauchy's integral formula

Let γ be a simple closed positively-oriented curve. If f is analytic in a simply connected domain Ω containing γ and z_0 is interior to γ, then

$$f(z_0) = \frac{1}{2\pi i} \oint_\gamma \frac{f(z)}{z - z_0}\,dz.$$

Proof. f is continuous at z_0, so for every $\varepsilon > 0$ there is a $\delta > 0$ such that $|z - z_0| < \delta$ implies $|f(z) - f(z_0)| < \varepsilon$. Let $r \in (0, \delta)$ be the radius of a positively-oriented circle γ_r centered at z_0; then, because $2\pi i = \oint_{\gamma_r} dz/(z - z_0)$,

$$\left| \oint_{\gamma_r} \frac{f(z)}{z - z_0}\,dz - 2\pi i\, f(z_0) \right| = \left| \oint_{\gamma_r} \frac{f(z) - f(z_0)}{z - z_0} \right| \le \oint_{\gamma_r} \frac{|f(z) - f(z_0)|}{|z - z_0|}\,dz < \frac{\varepsilon}{r} \cdot 2\pi r.$$

Since $\varepsilon > 0$ was arbitrary we conclude, by contour deformation, that

$$\oint_\gamma \frac{f(z)}{z - z_0}\,dz = \oint_{\gamma_r} \frac{f(z)}{z - z_0} = 2\pi i\, f(z_0),$$

as desired. □

Now, if we treat this integral as a function (rather than evaluating it only at z_0), its derivatives turn out to behave nicely.

Theorem 3.11

Let g be continuous on the contour γ, and for each $z \notin \gamma$ set

$$G(z) = \int_\gamma \frac{g(s)}{s-z}\,ds.$$

Then G is analytic and

$$G'(z) = \int_\gamma \frac{g(s)}{(s-z)^2}\,ds.$$

Proof. We could do some algebra and then apply the ML lemma to get

$$\begin{aligned}\left|\frac{G(z+h)-G(z)}{h} - \int_\gamma \frac{g(s)}{(s-z)^2}\,dz\right| &= \left|\int_\gamma g(s)\left[\frac{h}{(s-z-h)(s-z)^2}\right]\right| \\ &= |h|M\int_\gamma \frac{1}{|s-z-h||s-z|^2}\,ds,\end{aligned}$$

where $M = \max_{s\in\gamma}|g(s)|$. Now let $d = \min_{s\in\gamma}|s-z|$; then $|s-z-h| \geq |s-z| - |h| \geq d - d/2$, so

$$\leq |h|M\int_\gamma \frac{1}{(d/2)d^2}\,ds,$$

which goes to zero as $h \to 0$. □

In showing this, we've also shown that the expression in Cauchy's formula is analytic with

$$f'(z) = \frac{1}{2\pi i}\oint_\gamma \frac{f(s)}{(s-z)^2}\,ds.$$

It turns out that we can go through a similar line of reasoning to show that this f' is analytic with a similar integral expression; a simple proof by induction gives the following.

Theorem 3.12: Cauchy's integral formula (for the kth derivative)

If f is analytic on and inside a positively-oriented simple closed curve γ and z lies inside γ, then

$$f^{(k)}(z) = \frac{k!}{2\pi i}\oint_\gamma \frac{f(s)}{(s-z)^{k+1}}\,ds, \qquad k \in \mathbb{N}.$$

Corollary 3.13

If f is analytic in a domain Ω then all of its derivatives exist and are analytic on Ω.

Because differentiability implies continuity, $f \in C^\infty$ and the continuous-derivative assumption we made earlier on was true all along! Now we have a couple more theorems that tie some loose ends.

Corollary 3.14

If $f = u + iv$ is analytic then u and v are harmonic functions.

Proof. Since $f \in C^\infty$, we have $u, v \in C^\infty$. The Cauchy-Riemann equations imply $u_{xx} = v_{yx}$ and $v_{yy} = -v_{xy}$, and since $v \in C^2$ its mixed partials are equal and so $\Delta u = \Delta v = 0$. □

Theorem 3.15: Morera's theorem

If f is continuous on Ω and $\oint_\gamma f\,dz = 0$ for all closed contours γ in Ω, then f is analytic.

Proof. Suppose $\oint_\gamma f\,dz = 0$ for all γ. Then by the ABC theorem there exists an F such that $F = f'$. Thus F is analytic, and so is its derivative f. □

3.5 Bounds for Analytic Functions

Now we"ll use these results to place restrictions on the behavior of analytic functions.

Theorem 3.16: Cauchy's estimate

Let f be analytic on a closed ball $B(z_0, R)$. If $|f(z)| \leq M$ for all $z \in B$, then

$$\left|f^{(k)}(z_0)\right| \leq \frac{k!M}{R^k}, \qquad k \in \mathbb{N}.$$

Proof. The integrand in the Cauchy integral formula (for the kth derivative) is bounded by M/R^{n+1}, so by the ML lemma

$$\left|f^{(k)}(z_0)\right| \leq \frac{k!}{2\pi}\left(\frac{M}{R^{k+1}} \cdot 2\pi R\right),$$

as desired. □

Notice that if a function is entire and bounded over all of $\mathbb{C}$, then it satisfies the conditions of the previous theorem for any ball $B(z_0, R)$. Thus if we take $R \to \infty$ its derivatives must vanish and we get the following.

Theorem 3.17: Liouville's theorem

If $f(z)$ is bounded and entire, then f is constant.

We can use Liouville's theorem to prove a familiar statement, but an unexpected one for this context!

Theorem 3.18: Fundamental theorem of algebra

Every non-constant polynomial with complex coefficients has at least one zero.

Proof. Suppose $P(z) = a_n z^n + \cdots + a_0$, $a_n \neq 0$, has no zeroes, so $H(z) = 1/P(z)$ is entire. For sufficiently large $|z|$ we have $|P(z)/z^n| \geq |a_n|/2$, meaning there exists an $R \gg 1$ such that

$$|H(z)| = \frac{1}{|P(z)|} \leq \frac{2}{R^n|a_n|}, \qquad |z| \geq R.$$

Also, by continuity (and the extreme value theorem) there exists a finite M such that $|H(z)| \leq M$ for all $|z| \leq R$, meaning

$$|H(z)| \leq \max\left\{M, \frac{2}{R^n|a_n|}\right\}, \qquad z \in \mathbb{C}.$$

Thus $H(z)$ is constant, and so is $P(z)$—a contradiction. Thus $P(z)$ has at least one root. □

In fact, all of these polynomials' roots lie in $\mathbb{C}$—any polynomial can be factored ito $P(z) = (z - z_1)Q_1(z)$, where Q_1 is another polynomial that can be factored in the same way. (This process has a finite number of iteration.) We therefore call $\mathbb{C}$ "algebraically closed". Now for one last theorem!

Theorem 3.19: Maximum-modulus principle

If f is analytic on a domain Ω and $|f(z)|$ has a local maximum at $z_0 \in \Omega$, then f is constant on a neighborhood of z_0.

Proof. Suppose $z_0 \in \Omega$ with $|f(z_0)| \geq |f(z)|$ for all $z \in \Omega$', where the domain $\Omega' \subset \Omega$. Choose $R > 0$ such that $B(z_0, R) \subset \Omega'$; by Cauchy's integral formula

$$f(z_0) = \frac{1}{2\pi i} \oint_{\partial B} \frac{f(z)}{z - z_0}\, dz = \frac{1}{2\pi} \int_0^{2\pi} f(z_0 + Re^{it})\, dt,$$

and so

$$|f(z_0)| \leq \frac{1}{2\pi} \int_0^{2\pi} |f(z + Re^{it})|\, dt \leq \frac{1}{2\pi} \int_0^{2\pi} |f(z_0)|\, dt = |f(z_0)|.$$

It follows that the two intermediate expressions are equal, so

$$\frac{1}{2\pi} \int_0^{2\pi} \Big(|f(z_0)| - |f(z_0 + Re^{it})| \Big) dt = 0$$

and $|f(z_0)| = |f(z_0 + Re^{it})|$. Thus $|f|$ is constant on a $\partial B(z_0, R)$ of any radius $0 < r < R$, and so $|f|$ is constant on $B(z_0, R)$.

Now, if $f = u + iv$ then $u^2 + v^2$ is constant, meaning both of its partial derivatives are equal to zero. An application of the Cauchy-Riemann equations gives the matrix equation

$$\begin{bmatrix} u & -v \\ v & u \end{bmatrix} \begin{bmatrix} u_x \\ u_y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad (x_0, y_0) \in B.$$

If this coefficient matrix is singular somewhere in B then $u = v = 0$ there, and we're done. Otherwise the vector above must be the zero vector, meaning $u_x = v_y = 0$ and $v_x = -u_y = 0$ on B. Thus u and v are constant on B, and so is f. $\square$

This may feel like a relatively weak statement, but we can turn this local statement into a global one!

Corollary 3.20

If f is analytic in a domain Ω and $|f(z)|$ achieves its maximum value at a point $z_0 \in \Omega$, then f is constant in Ω.

Proof. Suppose $|f(z)|$ is not constant. Then there is a $z_1 \in \Omega$ such that $|f(z_1)| < |f(z_0)|$. Let γ be a path in Ω running from z_0 to z_1, and let w be the first point along this path where $|f(z)|$ first starts to decrease. Thus $|f(w)| = |f(z_0)|$.

Now, since w is an interior point, there must be a disk centered at w that lies within Ω. But $|f|$ is constant in this disk by the previous theorem, meaning the points that come immediately after w along γ do not have decreasing modulus, a contradiction. Thus $|f|$ is constant and so is f. $\square$

Corollary 3.21

If f is analytic on a bounded domain Ω and continuous on the closure $\overline{\Omega} = \Omega \cup \partial\Omega$, then $|f|$ achieves its maximum on $\partial\Omega$.

4 Series Representations

4.1 Taylor Series

We'll begin our discussion of series representations of functions with a few familiar definitions.

Definition: Sequence and series

Let $\{a_n\}_{n=0}^{\infty}$ be a sequence of complex numbers. We say a_n converges to L if for every $\varepsilon > 0$ there exists an $N \in \mathbb{N}$ such that

$$m > N \implies |a_m - L| < \varepsilon.$$

In this case we say $a_n \to L$ and $\lim_{n\to\infty} a_n = L$. We also say the series

$$\sum_{n=0}^{\infty} a_n = L \text{ if } \lim_{N\to\infty} \sum_{n=0}^{N} a_n = L.$$

Definition: Uniform convergence

The series $\sum_{k=0}^{\infty} f_k(z)$ converges uniformly to $f(z)$ on a set K if for every $\varepsilon > 0$ there exists an $n \in N$ such that

$$m \geq n \implies \left| f(z) - \sum_{k=0}^{m} f_k(z) \right| < \varepsilon.$$

An important example of a series that converges is the geometric series. Let $z \in \mathbb{C}$ with $|z| < 1$; then

$$\sum_{k=0}^{\infty} z^k = \frac{1}{1-z}.$$

This will be useful in our analysis of Taylor series, the key (patently complex) result about which is below.

Theorem 4.1: Taylor series

Let f be analytic on a domain Ω. If $z_0 \in \Omega$ and $B(z_0, R) \subset \Omega$, then for all z interior to $B(z_0, R)$

$$f(z) = \sum_{k=0}^{\infty} \frac{f^{(k)}(z_0)}{k!}(z - z_0)^k.$$

Proof. We'll prove the statement for $z_0 = 0$, noticing that we can generalize our result to any z_0 by translation. (The chain rule doesn't introduce anything new into the picture.)

Suppose $0 \in \Omega$ and $B(0, R) \subset \Omega$, and let C be a circle defined by $|z| = R_0 < R$. For z inside C,

$$\begin{aligned} f(z) &= \frac{1}{2\pi i} \oint_C \frac{f(s)}{s - z}\, ds \\ &= \frac{1}{2\pi i} \oint_C \frac{f(s)}{s} \left[\frac{1}{1 - z/s} \right] ds \\ &= \frac{1}{2\pi i} \oint_C \frac{f(s)}{s} \left[\sum_{k=0}^{N-1} \left(\frac{z}{s}\right)^k + \frac{(z/s)^N}{1 - z/s} \right], \end{aligned}$$

since $|z| < |s|$. This effectively allows us to "swap" the order of the integral and the sum:

$$= \sum_{k=0}^{N-1} \left(\frac{1}{2\pi i} \oint_C \frac{f(s)}{s^{k+1}}\, ds \right) z^k + \frac{z^N}{2\pi i} \oint_C \frac{f(s)}{s^N (s-z)}\, ds$$
$$= \sum_{k=0}^{N-1} \frac{f^{(k)}(0)}{k!} z^k + \rho_N(z),$$

where we've defined the last term to be $\rho_N(z)$. We need only show that $|\rho_N(z)| \to 0$. Let $M = \max_{s\in C} |f(s)| < \infty$, which we know exists because f is continuous. Then by the ML lemma

$$|\rho_N(z)| \le \frac{|z|^N}{2\pi} \frac{M}{R_0^N (R_0 - |z|)} \cdot 2\pi R_0 = \frac{M R_0}{R_0 - |z|} \left(\frac{|z|}{R} \right)^N,$$

and since N can be arbitrarily large, $|\rho_N(z)| < \varepsilon$. Thus the Taylor series of f converges uniformly to f for all closed "sub-disks" of $B(z_0, R)$, meaning the convergence is uniform on $B(z_0, R)$. □

Note that this theorem says nothing about convergence *on* the disk. Also note that the Taylor series of, say, f' is equivalent to term-by-term derivative of the Taylor series for f. Finally, we can generate new Taylor series from known ones using the Cauchy product terms

$$\frac{(fg)^{(k)}(z_0)}{k!} = \frac{1}{k!} \sum_{\ell=0}^{k} \binom{k}{\ell} f^{(k-\ell)}(z_0) g^{(\ell)}(z_0) = \sum_{\ell=0}^{k} \frac{f^{(k-\ell)}(z_0)}{(k-\ell)!} \frac{g^{(\ell)}(z_0)}{\ell!} = \sum_{\ell=0}^{k} a_{k-\ell} b_\ell = c_k,$$

where a_k and b_k are the coefficients of two Taylor series and c_k are the coefficients of the product.

4.2 Power Series

Now we'll discuss the convergence (and divergence) of power series more generally, starting with the following.

Lemma 4.2: Convergence lemma

Given $R_0 > 0$, if there exists an $M > 0$ such that

$$|a_k| \le \frac{M}{R_0^k}$$

for all k, then $\sum_{k=0}^{\infty} a_k (z - z_0)^k$ converges absolutely and uniformly for all z satisfying $|z - z_0| < R_0$.

Proof. Let $z \in \mathbb{C}$ such that $|z - z_0| \le r_\star < R_0$ then for all k,

$$\sum_{k=0}^{\infty} |a_k (z - z_0)^k| \le \sum_{k=0}^{\infty} M \left(\frac{r_\star}{R_0} \right)^k = \frac{1}{M(1 - r_\star / R_0)},$$

which is bounded because $|r_\star / R_0| < 1$. □

This leads to a couple of nice corollaries—roughly speaking, if a series converges at a point then it also converges everywhere "interior" to that point, and if it diverges at a point then it diverges everywhere "exterior" to that point.

Corollary 4.3

If $\sum_{k=0}^{\infty} a_k(z-z_0)^k$ converges at $z_1 \neq z_0$, then it converges uniformly and absolutely for all

$$|z-z_0| < R_0 = |z_1 - z_0|.$$

Proof. Let $R_0 = |z_1 - z_0|$ and pick an $r_\star < R_0$. Since $\sum_{k=0}^{\infty} a_k(z_1-z_0)^k$ converges, the terms $a_k(z_1-z_0)^k \to 0$, so given $\varepsilon > 0$ there is an N such that $m \geq N$ implies

$$|a_m(z_1-z_0)^m| = |a_m|R_0^m < \varepsilon.$$

With this in mind, let $M = \max\left\{|a_0|, |a_1|R_0, |a_2|R_0^2, \dots, |a_{N-1}|R_0^{N-1}, \varepsilon\right\}$. It follows from the existence of such an M that $|a_k| \leq M/R_0^k$ for all k, and by the previous lemma $\sum_{k=0}^{\infty} a_k(z-z_0)^k$ converges absolutely and uniformly. □

Corollary 4.4

If $\sum_{k=0}^{\infty} a_k(z-z_0)^k$ diverges at $z_1 \neq z_0$, then it also diverges for all z satisfying

$$|z-z_0| > |z_1 - z_0|.$$

Now, given z_0 we define the radius of convergence

$$R = \sup\left(|z-z_0| \;\middle|\; \sum_{k=0}^{\infty} a_k(z-z_0)^k \text{ converges at } z\right).$$

The circle $|z-z_0| = R$ is called the circle of convergence. Everything inside converges, but on this circle the behavior is not definite. We can say a couple of things, though:

- if a point on the circle converges absolutely then everything on the circle converges absolutely, and consequentially
- if a point on the circle diverges then nothing on the circle converges absolutely.

When determining convergence we generally use the ratio test, though sometimes we also resort to the root test. Now we'll look at a series of lemmas that together reveal a deep relationship between power series and analytic functions. (The first is a standard result from analysis—see Math 131 for proof.)

Lemma 4.5

If $f_k \to f$ uniformly on a set K and the f_k are continuous then f is continuous on K.

Lemma 4.6

Suppose f_k are continuous on a set K and Γ is a contour in K. If $f_k \to f$ uniformly on K then

$$\int_\Gamma f_k(z)\,dz \longrightarrow \int_\Gamma f(z)\,dz.$$

Proof. Given $\varepsilon > 0$ there is an $N \in \mathbb{N}$ such that $m \geq N$ implies $|f_m(z) - f(z)| < \varepsilon/\ell$ for $z \in K$, where ℓ is the length of Γ. Then

$$\left|\int_\Gamma f_m(z)\,dz - \int_\Gamma f(z)\,dz\right| \leq \ell \cdot \max_{z\in\Gamma} |f_m(z) - f(z)| < \ell \cdot \frac{\varepsilon}{\ell},$$

as desired. □

Lemma 4.7

Suppose $f_k \to f$ uniformly on a simply connected domain Ω. If the f_k are analytic on Ω then f is analytic on Ω.

Proof. From the first lemma, f is continuous. Since Ω is simply connected and f_k analytic, $\oint_\Gamma f_k(z)\,dz = 0$ for any closed loop Γ in Ω. It follows from the previous lemma that $\oint_\Gamma f(z)\,dz = 0$, and by Morera's theorem f is analytic. □

Now suppose

$$S(z) = \sum_{k=0}^{\infty} a_k (z - z_0)^k$$

converges on the set Ω defined by $|z - z_0| < R$. This final lemma shows that $S(z)$ is analytic on Ω! But then it is also equal to its Taylor series, which is a potentially different representation of $S(z)$. To reconcile the two, we can do some partial-sum manipulation with the second lemma to get

$$\sum_{k=0}^{\infty} \oint_\Gamma a_k (z - z_0)^k\,dz = \oint_\Gamma \left(\sum_{k=0}^{\infty} a_k (z - z_0)^k \right) dz.$$

But by Cauchy's integral formula, for some integral C within the circle of convergence we also have

$$\begin{aligned}
S^{(k)}(z_0) &= \frac{k!}{2\pi i} \oint_C \frac{S(z)}{(z - z_0)^{k+1}}\,dz \\
&= \frac{k!}{2\pi i} \oint_C \frac{1}{(z - z_0)^{k+1}} \left(\sum_{\ell=0}^{\infty} a_\ell (z - z_0)^\ell \right) dz \\
&= \frac{k!}{2\pi i} \sum_{\ell=0}^{\infty} \oint_C a_\ell (z - z_0)^{\ell-k-1} dz \\
&= \frac{k!}{2\pi i} \cdot a_k \cdot 2\pi i.
\end{aligned}$$

Thus $a_k = S^{(k)}(z_0)/k!$, meaning our power-series representations are the same! So in $\mathbb{C}$ there is a precise equivalence between analytic functions (defined in terms of complex derivatives) and convergent power series.

As a side note, all this also gives us a new proof of Liouville's theorem. If an analytic function $f(z)$ is bounded (by M) and entire then by Cauchy's estimate

$$\frac{1}{k!} \left| f^{(k)}(z_0) \right| \le \frac{M}{R^k},$$

meaning the higher-degree coefficients of the Taylor series of f go to zero as $R \to \infty$. This leaves $f(z) = a_0$ as the only possibilities for f.

4.3 Laurent Series

Many of the functions we care about have singularities, points about which there is no faithful power series expansion. This is not to say, however, that there's no way to represent these functions as series! Take, for example, the function

$$f(z) = \frac{1}{z(z-1)}.$$

We can use our knowledge of power series to expand $f(z)$ in the regions $|z| > 1$ and $0 < |z| < 1$, respectively:

$$\begin{aligned}
f(z) &= \frac{1}{z^2} \frac{1}{1 - 1/z} = \frac{1}{z^2} \sum_{k=0}^{\infty} \left(\frac{1}{z} \right)^k = \frac{1}{z^2} + \frac{1}{z^3} + \frac{1}{z^4} + \cdots, \qquad |z| > 1, \\
f(z) &= -\frac{1}{z} \frac{1}{1 - z} = -\frac{1}{z} \sum_{k=0}^{\infty} z^k = -\frac{1}{z} - 1 - z - z^2 \cdots, \qquad 0 < |z| < 1.
\end{aligned}$$

These are examples of Laurent series—they look like Taylor series, just with possibly negative powers. Notice how we define these series on an annulus rather than a neighborhood, reflecting how the function is potentially ill-defined at the center of the expansion.

Theorem 4.8: Laurent series

If f is analytic on an annular domain $A = \{z \mid r < |z - z_0| < R\}$ then

$$f(z) = \sum_{k=-\infty}^{\infty} c_k (z - z_0)^k, \qquad c_k = \frac{1}{2\pi i} \oint_\Gamma \frac{f(s)}{(s - z_0)^{k+1}}\, ds, \qquad z \in A, \quad k \in \mathbb{Z}$$

for any simple closed positively oriented curve Γ about z_0 in A. The convergence is uniform on closed sub-annuli, and the series representation is unique.

Proof. Let $z \in A$ with $r < r' \le |z - z_0| \le R' < R$ with the circles $C_{r'}, C_{R'}$. If Γ is a small circle about z then we can deform Γ to get

$$f(z) = \frac{1}{2\pi i} \oint_\Gamma \frac{f(s)}{s - z}\, ds = \frac{1}{2\pi i} \oint_{C_{R'}} \frac{f(s)}{s - z}\, ds - \frac{1}{2\pi i} \oint_{C_{r'}} \frac{f(s)}{s - z}\, ds.$$

Call these integrals I_1 and I_2, respectively. Noting that $|(z - z_0)/(s - z_0)| < 1$ for $s \in C_{R'}$, we could do some algebra and exploit uniform convergence to get

$$\begin{aligned} I_1 &= \frac{1}{2\pi i} \oint_{C_{R'}} \frac{f(s)}{s - z_0} \sum_{k=0}^{\infty} \left(\frac{z - z_0}{s - z_0} \right)^k ds \\ &= \sum_{k=0}^{\infty} \left(\frac{1}{2\pi i} \oint_{C_{R'}} \frac{f(s)}{(s - z_0)^{k+1}}\, ds \right) (z - z_0)^k. \end{aligned}$$

This sum converges uniformly on $|z - z_0| \le R' < R$. Now, $|(s - z_0)/(z - z_0)| < 1$ for $s \in C_{r'}$, so using similar methods

$$\begin{aligned} I_2 &= \frac{1}{2\pi i} \oint_{C_{r'}} \frac{f(s)}{z - z_0} \sum_{k=0}^{\infty} \left(\frac{s - z_0}{z - z_0} \right)^k ds \\ &= \sum_{k=0}^{\infty} \left(\frac{1}{2\pi i} \oint_{C_{r'}} \frac{f(s)}{(s - z_0)^{-k}}\, ds \right) (z - z_0)^{-k-1} \\ &= \sum_{k=1}^{\infty} \left(\frac{1}{2\pi i} \oint_{C_{r'}} \frac{f(s)}{(s - z_0)^{-k+1}}\, ds \right) (z - z_0)^{-k}. \end{aligned}$$

This sum converges uniformly on $|z - z_0| \ge r' > r$. Thus if we define c_k as in the theorem statement we get

$$f(z) = \sum_{k=0}^{\infty} c_k (z - z_0)^k + \sum_{k=1}^{\infty} \frac{c_{-k}}{(z - z_0)^k} = \sum_{k=-\infty}^{\infty} c_k (z - z_0)^k,$$

which converges uniformly on $r' \le |z - z_0| \le R'$. We can deform $C_{R'}$ and $C_{r'}$ into any contour Γ in A. To prove uniqueness we consider another Laurent series with coefficients d_k; in this case,

$$\oint_\Gamma \frac{f(s)}{(s - z_0)^{n+1}}\, ds = \sum_{k=-\infty}^{\infty} \oint_\Gamma d_k \frac{(s - z_0)^k}{(s - z_0)^{n+1}}\, ds = d_n \cdot 2\pi i,$$

meaning

$$d_n = \frac{1}{2\pi i} \oint_\Gamma \frac{f(s)}{(s - z_0)^{n+1}}\, ds = c_n.$$

The Laurent series is therefore unique. □

Note that if f is analytic on the entire disk $|z - z_0| < R$, then by uniqueness $c_k = 0$ for negative k and the remaining coefficients are the Taylor coefficients. Typically, however, f is not analytic at z_0, so the positive-indexed terms are not generally the Taylor coefficients (because the derivatives at z_0 do not exist). Also, this analysis doesn't work on branch cuts, since in that case no sufficient annulus exists.

4.4 Zeros and Singularities

We'll begin by quickly classifying the zeroes of a function.

Definition: Classifying zeroes

A point z_0 is called a zero of order m for the function f if f is analytic at z_0 and it, along with its first $m-1$ derivatives, vanish at z_0 but $f^{(m)}(0) \neq 0$. A zero of order 1 is called a simple zero.

In this case the Taylor series for f about z_0 looks like

$$\begin{aligned} f(z) &= a_m(z-z_0)^m + a_{m+1}(z-z_0)^{m+1} + a_{m+2}(z-z_0)^{m+2} + \cdots \\ &= (z-z_0)^m \left[a_m + a_{m+1}(z-z_0) + a_{m+2}(z-z_0)^2 + \cdots\right], \qquad a_m \neq 0. \end{aligned}$$

The bracketed series defines an analytic function in a neighborhood of z_0, so we deduce the following.

Theorem 4.9

Let f be analytic at z_0. Then z_0 is a zero of order m if and only if

$$f(z) = (z-z_0)^m g(z)$$

for some g that is analytic at z_0 and satisfies $g(z_0) \neq 0$.

By continuity there exists an $\varepsilon > 0$ for which $|z-z_0| < \varepsilon$ implies $g(z) \neq 0$. Thus z_0 is the only zero of f in this neighborhood, and in general the zeroes of an analytic function are isolated!

Definition: Classifying singularities

Let f have the Laurent expansion

$$f(z) = \sum_{k=-\infty}^{\infty} a_k(z-z_0)^k.$$

If f is not analytic at z_0 but is analytic on some punctured disk $0 < |z-z_0| < \varepsilon$, then we say f has an isolated singularity at z_0. This singularity is classified as follows.

- If $a_k = 0$ for all $k < 0$ then z_0 is a removable singularity.
- If $a_{-m} \neq 0$ for some positive integer m but $a_k = 0$ for all $j < -m$ then z_0 is a pole of order m. (If $m = 1$ then we call z_0 a simple pole.)
- If $a_k \neq 0$ for an infinite number of negative k then z_0 is an essential singularity of f.

Now we'll look a bit more closely at the behavior of a function f at each of these kinds of singularities.

- If f has a removable singularity at z_0 it has a Laurent series

  $$f(z) = c_0 + c_1(z-z_0) + c_2(z-z_0)^2 + \cdots, \qquad 0 < |z-z_0| < \varepsilon.$$

  Strictly speaking this series is defined a punctured neighborhood of z_0, but it's analytic at z_0, too! So we can simply "remove" the singularity at z_0 by defining $f(z_0) = c_0$. We can also see that f is bounded on a punctured neighborhood around z_0, and the extended f is analytic on some potentially larger neighborhood.

- If f has a pole of order N at z_0 then there exists an analytic g satisfying $g(z_0) \neq 0$ such that

$$f(z) = \frac{g(z)}{(z - z_0)^N}.$$

Limits approaching z_0 are now unbounded as f is not bounded around z_0, but f is analytic around z_0.

- If f has an essential singularity at z_0 then near z_0 the output $f(z)$ can be made close to any point in the complex plane! This is formalized by the following.

Theorem 4.10: Casorati-Weirerstrass

If f has an essential singularity at z_0, then for every $w \in \mathbb{C}$ there exists a sequence $\{z_k\}$ such that $z_k \to z_0$ and $f(z_k) \to w$. (In other words, $f(B(z_0, \varepsilon) \setminus \{z_0\})$ is dense in $\mathbb{C}$.)

Proof. If not, then there exist $\varepsilon > 0$ and $\delta > 0$ such that $|f(z) - w| > \varepsilon$ for $0 < |z - z_0| < \delta$. Then

$$R(z) = \frac{1}{f(z) - w}$$

is analytic on the δ-ball, and $|R(z)| < 1/\varepsilon$ there. Also, if $R(z)$ has a Laurent expansion with coefficients d_k then

$$|d_k| = \left| \frac{1}{2\pi i} \oint_\gamma \frac{R(s)}{(s - z_0)^{k+1}} \, ds \right| \leq \frac{1}{2\pi} \frac{1/\varepsilon}{r^{k+1}} \cdot 2\pi r = \frac{r^{-k}}{\varepsilon}.$$

for a radius-r circle γ around z_0. Thus as $r \to 0$ we have $|d_k| \to 0$ for $k < 0$, meaning $R(z)$ has a removable singularity at z_0. But then

$$f(z) = w + \frac{1}{R(z)}, \qquad 0 < |z - z_0| < \delta$$

either has a pole of finite order at z_0 or $R(z_0) \neq 0$, both of which disallow f from having an essential singularity there. □

A stronger statement, called the Picard theorem, shows that the image of f around z_0 includes the entire complex plane barring at most one point.

Now let's bring our discussion into the extended complex plane, which includes a point at ∞. If f has a pole at z_0 then $\lim_{z \to z_0} |f(z)| = +\infty$, in which case we say that $f(z_0) = \infty$. We can, however, also study the behavior of f at $z = \infty$ by looking at how $\tilde{f}(w) = f(1/w)$ behaves at $w = 0$. In particular, if f is analytic in a neighborhood $|z| > R$ of $z = \infty$ then $\tilde{f}$ is analytic in a punctured neighborhood $0 < |w| < R$ of $w = 0$, and so we can use the familiar classification of isolated singularities at $w = 0$ to see what's going on at $z = \infty$.

If f is entire then

$$f(z) = a_0 + a_1 z + a_2 z^2 + \cdots, \qquad z \in \mathbb{C}$$

is valid in every neighborhood of ∞, meaning this is the Laurent series for f at ∞! We have three cases.

- If $f(z) = a_0$ then f is analytic at ∞.
- If $f(z)$ is a degree-N polynomial then f has a pole of order N at ∞.
- If $f(z)$ is an infinite power series then f has an essential singularity at ∞.

5 Residue Theory

5.1 Computing Residues

We'll spend the next while seeing how the theory of contour integrals can be leveraged to evaluate integrals over the real numbers. Our discussion will center around the following definition and theorem.

Definition: Residue

If $f(z)$ has an isolated singularity at $z_0 \in \mathbb{C}$ then the coefficient c_{-1} of $(z-z_0)^{-1}$ in its Laurent series about z_0 is called the residue of f at z_0; it is denoted $\mathrm{Res}(f; z_0)$.

Theorem 5.1: The residue theorem

Let Γ be a simple closed positively-oriented curve, and let f be analytic on Γ. If Γ is also analytic inside Γ, except potentially at isolated singularities $z_1, \ldots, z_N$, then

$$\oint_\Gamma f(z)\,dz = 2\pi i \sum_{k=1}^{N} \mathrm{Res}(f; z_k).$$

Proof. We can deform Γ into a collection of small circles γ_k surrounding each singularity, all connected by twice-traversed lines. Thus

$$\oint_\Gamma f(z)\,dz = \sum_{k=1}^{N} \oint_{\gamma_k} f(z)\,dz = 2\pi i \sum_{k=1}^{N} \mathrm{Res}(f; z_k),$$

as desired. $\square$

Thankfully, we don't need to actually compute the Laurent series of a function in order to determine its residues. If f has a simple pole at z_0 then we can simply write

$$(z-z_0)f(z) = c_{-1} + c_0(z-z_0) + \cdots,$$

where c_k are the terms of the Laurent series of f about z_0. We can make all of the non-c_{-1} terms vanish by taking $z \to z_0$.

Lemma 5.2: Useful lemma

If f has a simple pole at z_0 then

$$\mathrm{Res}(f; z_0) = \lim_{z \to z_0} (z-z_0)f(z).$$

Lemma 5.3: Very useful lemma

If $f(z) = P(z)/Q(z)$ with P, Q analytic, $Q(z)$ has a simple pole at z_0, and $P(z_0) \neq 0$, then

$$\mathrm{Res}(f; z_0) = \frac{P(z_0)}{Q'(z_0)}.$$

Proof. By the previous lemma,

$$\begin{aligned}\operatorname{Res}(f; z_0) &= \lim_{z\to z_0} \frac{P(z)}{Q(z) - 0} \\ &= \lim_{z\to 0} \frac{P(z)}{[Q(z) - Q(z_0)]/(z - z_0)} \\ &= \frac{P(z_0)}{Q'(z_0)},\end{aligned}$$

as desired. □

If we have a pole of order N we can take a similar approach by first multiplying by $(z - z_0)^N$ to get

$$(z - z_0)^N f(z) = c_{-N} + \cdots + c_{-1}(z - z_0)^{N-1} + \cdots,$$

and then we can take a bunch of derivatives to peel off the c_{-1} coefficient.

Theorem 5.4

If $f(z)$ has a pole of order N at z_0 then

$$\operatorname{Res}(f; z_0) = \frac{1}{(N-1)!} \lim_{z\to z_0} \frac{d^{N-1}}{dz^{N-1}} \left[(z - z_0)^N f(z)\right].$$

For an essential singularity there is, unfortunately, no nice formula—we just need to “massage out” the sum. Now we'll go through a bunch of examples, each of which illustrates a different technique for evaluating real integrals using residues.

Example: Parametrization

We'd like to evaluate

$$I = \int_0^{2\pi} \frac{1}{1 + a^2 - 2a\cos\theta}\, d\theta, \qquad a > 0, \quad a \neq 1.$$

The bounds here suggest that we make the substitution $z = e^{i\theta}$, $\theta \in [0, 2\pi]$:

$$\begin{aligned}I &= \oint_{|z|=1} \frac{1}{1 + a^2 + 2a[(z + 1/z)/2]} \frac{dz}{iz} \\ &= i \oint_{|z|=1} \frac{1}{(az - 1)(z - a)}\, dz.\end{aligned}$$

The integrand has simple poles at $z = a$ and $z = 1/a$, so technically speaking we have two cases: $0 < a < 1$ and $a > 1$. In both cases, though, the residues are

$$\operatorname{Res}(f; a) = \frac{1}{a^2 - 1},$$

and since exactly one of these singularities is in the contour we get

$$I = \frac{2\pi}{|1 - a^2|}.$$

We should quickly note that, strictly speaking, integrals over singularities do not exist. What we've actually done here is determine the “principal value” of the integral—if f has an isolated singularity at $x_0 \in (a, b)$, then

$$\text{p.v.} \int_a^b f(x)\, dx = \lim_{\varepsilon\to 0} \left[\int_a^{x_0-\varepsilon} f(x)\, dx + \int_{x_0+\varepsilon}^b f(x)\, dx\right].$$

We're only concerned with the principal values of our integrals here, so we'll continue to omit the p.v.

5.2 Improper Integrals via Contour Closure

A very common technique for evaluating improper integrals in $\mathbb{R}$ is to imagine the line we're integrating over as one piece of a very large, closed contour. The next few examples will show how we choose such a contour.

Example: A semicircle

We'd like to confirm that

$$\int_{-\infty}^{\infty} \frac{1}{1+x^2}\, dx = \pi.$$

To do this we construct a closed contour $\Gamma_C = \Gamma_1 + \Gamma_R$, where Γ_1 is the segment of the real line $(-R, R)$ with $R > 1$, and Γ_R is the upper semicircle which closes the contour.

- An application of the residue theorem shows that the integral over Γ_C is $I_C = \pi$.
- Rather than precisely determining the Γ_R contribution I_R, we'll just determine an upper bound. By the lower triangle inequality we have $|1 + z^2| \geq \big||1| - |z|^2\big| = |1 - R^2|$, so by the ML lemma

$$|I_R| \leq \frac{1}{R^2 - 1} \cdot \pi R.$$

The integral we seek is therefore

$$\int_{-\infty}^{\infty} \frac{1}{1+x^2}\, dx = \lim_{R\to\infty} (I_C - I_R) = \pi - 0 = \pi.$$

Conveniently, the contribution from the semicircular part of our contour vanishes completely, leaving us just with the π we desired! (The same method works for integrands like $x^2/(1+x^4)$.)

Example: A rectangle

We'd like to evaluate

$$\int_{-\infty}^{\infty} \frac{e^{ax}}{1+e^x}, \qquad a \in (0, 1).$$

Due to the structure of the singularities, rather than using a semicircle (which encloses an increasing number of singularities as it grows), we should use an expanding rectangle with fixed height 2π.

- By the residue theorem, the integral over the closed contour is $-2\pi i e^{a\pi i}$.
- We parametrize the rightmost piece using $z(t) = R + it$, $t \in [0, 2\pi]$ and bound it using the ML lemma. The contribution of this piece vanishes as $R \to \infty$; the same goes for the leftmost piece.
- For the topmost piece, which we call Γ_T we parametrize $z(t) = t + 2\pi i$, $t \in [-R, R]$. After some algebra, this yields

$$\int_{\Gamma_3} \frac{e^{az}}{1+e^z}\, dz = -e^{2\pi a i} \int_{\Gamma_1} f(z)\, dz,$$

  where Γ_1 is the bottommost piece (on the real line).

Putting all this together, the integral turns out to be

$$\int_{-\infty}^{\infty} \frac{e^{ax}}{1+e^x} = \frac{-2\pi i e^{a\pi i}}{1 - e^{2\pi a i}} = -\frac{2\pi i}{e^{-a\pi i} - e^{a\pi i}} = \frac{\pi}{\sin a\pi}.$$

Example: A pizza slice

We'd like to evaluate

$$\int_0^{\infty} \frac{1}{x^3}\, dx.$$

To avoid putting a singularity on our path, we'll integrate over a "pizza slice" contour—a segment on

$[0, R]$ connects with a circular arc of $2\pi/3$ radians, which connects to the origin via another segment.

- By the residue theorem, the integral over the entire slice is $e^{-i(2\pi/3)}/3$.
- For the arc we impose a bound that vanishes as $R \to \infty$.
- For the diagonal ray we parametrize $te^{i(2\pi/3)}$ and do some algebra to get

$$\int_{\Gamma_D} \frac{1}{1+z^3}\, dz = -e^{i(2\pi/3)} \int_0^R \frac{1}{1+z^3} dz.$$

Thus the integral is

$$\int_0^\infty \frac{1}{1+z^3}\, dz = \frac{2\pi i}{3e^{i(2\pi/3)}} \frac{1}{1-e^{i(2\pi/3)}} = \frac{2i\pi/3}{e^{i(2\pi/3)} - e^{-i(2\pi/3)}} = \frac{2\pi}{3\sqrt{3}}.$$

Example: Forcing a semicircle

We'd like to evaluate

$$\int_{-\infty}^\infty \frac{\cos x}{x^2+a^2}\, dx = \mathrm{Re}\left[\int_{-\infty}^\infty \frac{e^{ix}}{x^2+a^2}\, dx\right], \qquad a > 0.$$

We can't immediately use a semicircle since the cosine is exponential on the imaginary axis, but if we frame the integral as the real part of another expression the problem becomes tractable!

- By the residue theorem, the closed integral is $\pi/(ae^a)$.
- On the semicircular bit we put a vanishing bound on the integral.
- Integrating the $i \sin x$ term on the real line gives zero, since it produces an odd function.

Putting it all together, the integral we seek is π/ae^a.

5.3 Some Useful Lemmas

Now we'll detour for a bit and look at some general statements that'll be useful in evaluating more integrals.

Lemma 5.5: Jordan's lemma

If C_R^+ denote the upper half of a radius-R circle centered at 0 then

$$\left|\int_{C_R^+} e^{iz}\, dz\right| < \pi.$$

Proof. By parametrization,

$$\begin{aligned}
\left|\int_{C_R^+} e^{iz}\, dz\right| &= \left|\int_0^\pi e^{iRe^{it}} iRe^{it}\, dt\right| \\
&\le R \int_0^\pi e^{-R\sin t}\, dt \\
&= 2R \int_0^{\pi/2} e^{-R\sin t}\, dt.
\end{aligned}$$

Note that $\sin t \geq (2/\pi)t$ for $t \in [0, \pi]$, meaning

$$\leq 2R \int_0^{\pi/2} e^{-R(2/\pi)t}\, dt < \pi$$

for all $R > 0$, as desired. □

Lemma 5.6: Fancy Jordan lemma

Let $M(r) = \max_{z \in C_R^+} |f(z)|$. If $\lim_{R\to\infty} M(R) = 0$ then for all $m > 0$,

$$\lim_{R\to\infty} \int_{C_R^+} f(z) e^{imz} dz = 0.$$

Proof. Like before, we parametrize:

$$\begin{aligned}
\left| \int_{C_R^+} f(z) e^{imz} dz \right| &\leq \int_0^{\pi} |f(Re^{it})||e^{imRe^{it}}||iRe^{it}|dt \\
&\leq \int_0^{\pi} M(R) e^{-mR\sin t} R\, dt \\
&\leq 2R\, M(R) \int_0^{\pi/2} e^{-mR(2/\pi)t}\, dt \\
&= \frac{\pi M(R)}{m}(1 - e^{-mR}),
\end{aligned}$$

which vanishes as $R \to \infty$. □

Note that we may apply this lemma to any rational integrand whose numerator is of lower degree than its denominator. We have one more lemma before jumping back into examples—note the similarities with the residue theorem.

Lemma 5.7

Let $f(z)$ have a simple pole at z_0 and let Γ_R be the arc $z = z_0 + Re^{it}$, $t \in [0, \alpha]$. Then

$$\lim_{R\to 0} \int_{\Gamma_R} f(z)\, dz = \alpha i \operatorname{Res}(f; z_0).$$

Proof. Since z_0 is a simple pole, there exists an $\varepsilon > 0$ and an analytic $h(z)$ such that

$$f(z) = \frac{c_{-1}}{z - z_0} + h(z).$$

So for $0 < R < \varepsilon$ we have

$$\int_{\Gamma_R} f(z)\, dz = c_{-1} \int_{\Gamma_R} \frac{dz}{z - z_0} + \int_{\Gamma_R} h(z)\, dz,$$

Since $h(z)$ is bounded on $B(z_0, \varepsilon)$, by the ML lemma the second integral vanishes as $R \to 0$. Parametrization reveals that the first term is $c_{-1}(\alpha i)$, meaning the overall integral is $\alpha i \operatorname{Res}(f; z_0)$, as desired. □

5.4 More Examples

Now we'll look at some more instructive examples.

Example: A semicircle with a bump

We'd like to evaluate

$$\int_{-\infty}^{infty} \frac{\cos 2x}{1+x}\,dx = \mathrm{Re}\left[\int_{-\infty}^{\infty} \frac{e^{2ix}}{1+x}\,dx\right].$$

We'll once again use a semicircle, but this time accented with a small semicircular bump at $z = 1$.

- By the fancy Jordan lemma, the integral over the large semicircle vanishes as $R \to \infty$.
- By the previous lemma, the integral over the small semicircle goes to $-\pi i e^{-2i}$ as $\varepsilon \to 0$.

The other two pieces comprise the integral we care about (as $\varepsilon \to 0$), meaning

$$\int_{-\infty}^{\infty} \frac{\cos 2x}{1+x}\,dx = \mathrm{Re}\left[\pi i e^{-2i}\right] = \pi \sin 2.$$

Example: Deriving a sum

The function $f(z) = \pi \cot \pi z$ has simple poles at every $z = k \in \mathbb{Z}$, since it is of the form $h'(z)/h(z)$; defining $g(z) = f(z)/z^2$ turns the singularity at $k = 0$ into a pole of order 3. The residues for $k \neq 0$ are $1/k^2$, and that for $k = 0$ is $-\pi^2/3$.

Consider the contour Γ_k, a square centered at the origin which intersects the axes at $x, y = \pm(k+1/2)$. By expressing $\cot \pi z$ in terms of complex exponentials, we could show that $|\cot \pi z| \leq 1$ and $|\cot \pi z| \leq 2$ on the vertical and horizontal paths, respectively. Using these,

$$\left|\oint_{\Gamma_k} \frac{\pi \cot \pi z}{z^2}\,dz\right| \leq \frac{2\pi}{(k+1/2)^2} \cdot 4(2k+1) \longrightarrow 0,$$

and so

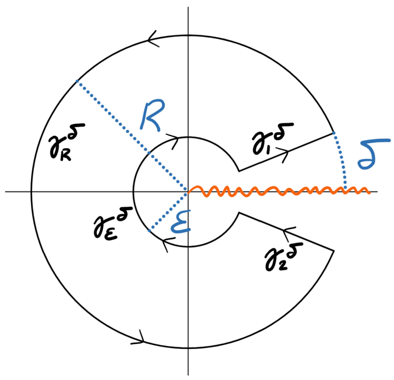
$$0 = 2\pi i \cdot \sum_{k=-\infty}^{\infty} \mathrm{Res}(g; k) = 2\pi i \left(2\sum_{k=1}^{\infty} \frac{1}{k^2} - \frac{\pi^2}{3}\right) \implies \sum_{k=1}^{\infty} = \frac{\pi^2}{6}.$$

But the Laurent expansion of $\pi \cot(\pi z)/z^2$ has a nonzero term for every even power of z, so this is a geneal method for generating values of $\sum_{k=1}^{\infty} k^{-2n}$.

Example: A keyhole

We'd like to evaluate

$$I = \int_0^{\infty} f(x)\,dx, \qquad f(x) = \frac{1}{x^{1/3}(x+2)}.$$



Perhaps contrary to what our intuition would tell us, we'll put the branch cut on the axis we're integrating over so that $\log(z) = \mathrm{Log}_0(z)$. We'll use the "keyhole" contour $\Gamma_\varepsilon^\delta$ drawn at right. It is comprised of four pieces: two almost-circles $\gamma_R^\delta, \gamma_\varepsilon^\delta$ and two segments $\gamma_1^\delta, \gamma_2^\delta$ connecting their loose ends.

There's a simple pole at $z = -2$, so by the residue theorem the integral about $\Gamma_\varepsilon^\delta$ is $2\pi i/(2^{1/3} e^{i(\pi/3)})$. Now for the pieces.

- We put bounds on the integrals over γ_R^δ and $\gamma_\varepsilon^\delta$ which vanish as $R \to \infty$ and $\varepsilon \to 0$.
- γ_1^δ is parametrized by $te^{i\delta}$, $t \in [\varepsilon, R]$, so $z^{1/3} = t^{1/3} e^{i(\delta/3)}$ and as $\delta \to 0$ we get $z^{1/3} \to t^{1/3}$. Similarly, $-\gamma_2^\delta$ is parametrized by $te^{i(2\pi - \delta)}$, $t \in [\varepsilon, R]$. Like before, we get $z^{1/3} \to t^{1/3} e^{i(2\pi/3)}$.

What we've done here is show that

$$\lim_{\delta\to 0^+}\int_{\gamma_1^\delta} f(z)\,dz = \int_\varepsilon^R \frac{dx}{x^{1/3}(x+2)}, \qquad \lim_{\delta\to 0^+}\int_{\gamma_2^\delta} f(z)\,dz = -e^{-i(2\pi/3)}\int_\varepsilon^R \frac{dx}{x^{1/3}(x+2)}.$$

Summing all these pieces' contributions together gives

$$\lim_{\substack{\delta,\varepsilon\to 0^+\\ R\to\infty}} \oint_{\Gamma_\varepsilon^\delta} f(z)\,dz = (1-e^{-i(2\pi/3)})\int_0^\infty \frac{dz}{x^{1/3}(x+2)} = \frac{2\pi i}{2^{1/3}e^{i(\pi/3)}},$$

meaning

$$\int_0^\infty \frac{dz}{x^{1/3}(x+2)} = \frac{2\pi i}{2^{1/3}e^{i(\pi/3)}}\frac{1}{1-e^{-i(2\pi/3)}} = \frac{\pi}{\sqrt[3]{2}}\frac{1}{\sin(\pi/3)} = \frac{2^{2/3}\pi}{\sqrt{3}}.$$

This integral is a special case of a more general integral transform.

Definition: Mellin transform

The Mellin transform of $f : [0,\infty) \to \mathbb{R}$ is

$$\mathcal{M}[f](a) = \int_0^\infty x^{a-1} f(x)\,dx, \qquad a \in \mathbb{C},$$

provided the limit exists.

Notice that the Mellin transform of $f(t) = e^{-t}$ is the gamma function $\Gamma(z)$! This function is well-defined and analytic for all $\mathrm{Re}(z) > 0$ (since $x-1 > -1$ for integrability), and notably it satisfies $\Gamma(z+1) = z\,\Gamma(z)$ (so $\Gamma(n+1) = n!$).

We can repeatedly use this property to extend the domain of Γ to all of $\mathbb{C}$, barring simple poles at the negative integers. For example, to extend the domain to all $\mathrm{Re}(z) > -m$ we'd write

$$F_m(z) = \frac{\Gamma(z+m)}{z(z+1)(z+2)\cdots(z+m-1)}.$$

With this, we could show that $\mathrm{Res}(-k) = (-1)^k/k!$. (We could do a similar "analytic continuation" with the zeta function $\zeta(z)$, but that involves a messier functional equation.) There's lots of interesting stuff to be seen with the gamma function, but it's time to move on now.

5.5 The Laplace Transform

Now we'll take a closer look at a more familiar transform.

Definition: Laplace transform

The Laplace transform of $f : [0,\infty) \to \mathbb{C}$ is

$$\mathcal{L}[f](z) = F(z) = \int_0^\infty e^{-zt} f(t)\,dt,$$

provided this limit exists.

When working with the Laplace transform we work specifically with the class of exponential functions, those satisfying $|f(t)| < Ae^{Bt}$ for some $A > 0$, $B \in \mathbb{R}$, and $t \geq 0$. We put further restrictions on these parameters:

$$|F(z)| \leq \int_0^\infty |f(t)|e^{-\mathrm{Re}(z)t}\,dt \leq A\int_0^\infty e^{(B-\mathrm{Re}\,z)t}\,dt = \frac{A}{\mathrm{Re}\,z - B}.$$

For this inequality to work out, we require $\mathrm{Re}\,z > B$. Now, usually at this point we'd go through and compute a bunch of sample transforms and maybe use them to solve some differential equations, but we'll note only that the transforms of derivatives look like

$$\mathcal{L}[f'](z) = z\mathcal{L}[f](z) - f(0),$$
$$\mathcal{L}[f''](z) = z^2\mathcal{L}[f](z) - zf(0) - f'(0),$$

and so on, and then jump straight into inverses.

Definition: Inverse Laplace transform

Given $F(z)$ with finitely many isolated singularities, we define

$$\mathcal{L}^{-1}[F(z)](t) = \frac{1}{2\pi i}\int_{\alpha-i\infty}^{\alpha+i\infty} F(z)\,e^{zt}\,dz$$

with $\alpha \in \mathbb{R}$ chosen so that all singularities lie to the left of α.

This can get messy, but there's an incredibly nice formula for evaluating inverse Transforms for many functions.

Theorem 5.8

If $F(z)$ is analytic on $\mathbb{C}$ except at finitely many isolated singularities z_k, if $\lim_{z\to\infty} F(z) = 0$ then

$$\mathcal{L}^{-1}[F(z)](t) = \sum_{z_k} \operatorname{Res}(e^{zt}F(z); z_k).$$

Proof. Our assumptions about F imply that there exists some $\sigma \in \mathbb{R}$ such that $F(z)$ is analytic for all $\operatorname{Re} z > \sigma$. For any $\alpha > \sigma$, by definition

$$\begin{aligned}
\mathcal{L}^{-1}[F(z)](t) &= \frac{1}{2\pi i}\int_{\alpha-i\infty}^{\alpha+i\infty} e^{zt}F(z)\,dz \\
&= \frac{1}{2\pi i}\left[2\pi i \sum_{z_k} \operatorname{Res}\left(e^{zt}F(z); z_k\right)\right] \\
&= \sum_{z_k} \operatorname{Res}\left(e^{zt}F(z); z_k\right),
\end{aligned}$$

which is valid by the fancy Jordan lemma if $|F| \to 0$ as a semicircle expands leftward to infinity. This works, but it would be better to not rely on our definition of $\mathcal{L}^{-1}$ and instead show that

$$\mathcal{L}\left[\sum_k \operatorname{Res}(e^{zt}f(t); z_k)\right] = F(z),$$

so that by uniqueness (which we'll take for granted) $f(t) = \mathcal{L}^{-1}[F]$. To this end, let γ be a rectangular contour including all of the singularities of F and a finite portion of the a-strip; we write

$$f(t) = \sum_{z_k} \operatorname{Res}(e^{zt}F(z); z_k) = \frac{1}{2\pi i}\oint_\gamma e^{zt}F(z)\,dz,$$

so the Laplace transform is

$$\begin{aligned}
\mathcal{L}[f](z) &= \lim_{N\to\infty}\int_0^N \frac{e^{-zt}}{2\pi i}\oint_\gamma e^{st}F(s)\,ds\,dt \\
&= \lim_{N\to\infty}\frac{1}{2\pi i}\oint_\gamma F(s)\int_0^N e^{t(s-z)}\,dt\,ds \\
&= \lim_{N\to\infty}\frac{1}{2\pi i}\oint_\gamma \left(e^{(s-z)N} - 1\right)\frac{F(s)}{s-z}\,ds.
\end{aligned}$$

If z is fixed with $\operatorname{Re} z > \alpha$ then

$$= -\frac{1}{2\pi i}\oint_\gamma \frac{F(s)}{s-z}\,ds.$$

This looks like the Cauchy integral formula, except F is not analytic inside γ and z is not contained in γ. To remedy this we'll add another rectangular path $\tilde{\gamma}$ that's directly adjacent to γ and encloses z. Setting $\Gamma = \gamma + \tilde{\gamma}$:

$$\mathcal{L}[f](z) = \frac{1}{2\pi i}\oint_{\tilde{\gamma}} \frac{F(s)}{s-z}\,ds - \frac{1}{2\pi i}\oint_\Gamma \frac{F(s)}{s-z}\,ds.$$

The first term is $F(z)$ by the Cauchy integral formula. For the second term, note that Γ is homotopic to a large circle and so is bounded by

$$\frac{2\pi R}{R-|z|}\max_{s\in\Gamma_R}|F(s)| \longrightarrow 0.$$

Thus $\mathcal{L}[f](z) = F(z)$, as desired. □

We'll end our discussion of the Laplace transform with a nice connection with how we usually compute inverse transforms. Suppose $F(z) = P(z)/Q(z)$ with $\deg P < \deg Q$; if Q has k simple zeroes α_k then by the residue theorem the inverse transform is

$$f(t) = \sum_{k=1}^n e^{\alpha_K t}\frac{P(z)}{Q'(z)}.$$

We call this the Heaviside expansion formula. But alternatively, we could directly compute

$$f(t) = \mathcal{L}^{-1}[F](t) = \mathcal{L}^{-1}\left[\sum_{k=1}^n \frac{c_k}{z-\alpha_k}\right] = \sum_{k=1}^n c_k e^{\alpha_k t}.$$

It follows that

$$c_k = \frac{P(\alpha_k)}{Q'(\alpha_k)},$$

meaning that the coefficients of the partial fraction decomposition of $F(z)$ are precisely the residues of $F(z)$! This provides a slick way to do these decompositions without resorting to systems of equations.

5.6 Conformal Maps

We'll cap everything off with a miscellaneous discussion of conformal maps.

Definition: Conformal map

We say f is conformal on a domain Ω if f is analytic on Ω and $f'(z) \neq 0$ for all $z \in \Omega$.

These have a nice geometric interpretation. Suppose f is conformal on Ω and $\gamma(t)$ is a path through $z_0 \in \Omega$ with $\gamma(t_0) = z_0$. Then $w(t) = f(\gamma(t))$ is a path through $w_0 = f(z_0)$, and by the chain rule

$$w'(t_0) = f'(z_0)\gamma'(t_0).$$

The tangent vector at $f(z_0)$ is a fixed complex number times the tangent vector at z_0. Geometrically, the old tangent is scaled by $|f'(z_0)|$ and rotated by $\operatorname{Arg}(f'(z_0))$—crucially, both of these actions depend only on the point z_0, not the path γ. So if we have another path $\tilde{\gamma}$ which also passes through z_0; the tangent here experiences the same rotation under f, so the angles between the first two tangents and the last two tangents are the same. Conformal maps preserve angles!

Further, even though $f'(z_0)$ varies as z_0 varies, it does so continuously, so a small neighborhood of a point z_0 "conforms" to its own shape as it is transformed by f. We characterize this more generally below.

Definition: Conformally equivalent

Two sets $\Omega, \tilde{\Omega}$ are conformally equivalent if there exists a bijective conformal map $\Omega \to \tilde{\Omega}$.

Now we have a lovely application to the theory of harmonic functions.

Theorem 5.9

Let $\phi : \tilde{\Omega} \to \mathbb{R}$ be harmonic with $\phi \in C^2$ and $f : \Omega \to \tilde{\Omega}$ analytic. If $\tilde{\Omega}$ is simply connected then $u = \phi \circ f$ is harmonic on Ω.

Proof. Given ϕ, since $\tilde{\Omega}$ is simply connected there exists an analytic function g on $\tilde{\Omega}$ such that $\phi = \mathrm{Re}(g)$. (The full justification of this is technical and uses the ABC theorem.) Then $u = \phi \circ f = \mathrm{Re}(g \circ f)$, and since $g \circ f$ is analytic, $\Delta u = 0$ on Ω. □

In a rough sense, we can "lift" a harmonic function from $\tilde{\Omega}$ to Ω using any analytic $f : \Omega \to \tilde{\Omega}$. So if we want to determine a harmonic function over some poorly-behaved domain $\tilde{\Omega}$, we can find one by first mapping to a better-behaved Ω for which a harmonic function is known and composing.

Now here's an amazing pair of results.

Theorem 5.10: Riemann mapping

If $\Omega \neq \mathbb{C}$ is a simply connected domain then there exists a conformal bijection $f : \Omega \to D = \{|z| < 1\}$.

Corollary 5.11

Any two simply connected domains are conformally equivalent!

Note that we require $\Omega \neq \mathbb{C}$ so that we don't contradict Liouville's theorem. The proof of the Riemann mapping theorem is slightly beyond the scope of this course, but importantly, it is not constructive. There are lots of numerical efforts to determine these mappings.

Now let's look at a particular class of conformal maps.

Definition: Möbius transformation

A Möbius transformation is a map $S : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ of the form

$$S(z) = \frac{az + b}{cz + b}$$

for $a, b, c, d \in \mathbb{C}$ satisfying $ad - bc \neq 0$.

The condition on the coefficients is sufficient for $S(z)$ to be conformal. Now, if we also take $c \neq 0$ we can also write

$$S(z) = \left(b - \frac{ad}{c}\right) \frac{1}{cz + d} + \frac{a}{c},$$

which reveals the geometric nature of S:

1. dilate/rotate by c and then translate by d;
2. invert via $1/z = \overline{z}/|z|^2$;
3. dilate/rotate by $b - ad/c$ and translate by a/c.

This is a set of very well-defined geometric actions. Also, we could do some algebra to show that

$$S^{-1}(w) = \frac{dw - b}{-cw + a}, \qquad ad - bc \neq 0,$$

meaning S is a bijection on $\hat{\mathbb{C}}$. A composition of Möbius transformations is itself a Möbius transformations, so the set of all such transformations is a group—and in fact, it is precisely the group of automorphisms form $\hat{\mathbb{C}}$ to itself! (What's more, this group is isomorphic to the rotations of the Riemann sphere—we can think of Möbius transformations as the projections of these rotations.)

Let's get back to geometry for a moment.

Theorem 5.12

Möbius transformations preserve the set of circles and lines.

Proof. Looking at the steps for geometrically performing a Möbius transformation, we need only verify (2). If we write $z = x + iy$ then we could show that

$$\frac{1}{z} = \left(\frac{x}{x^2+y^2}\right) + i\left(\frac{-y}{x^2+y^2}\right) = u + iv,$$

the inverse of which is

$$(x, y) = \left(\frac{u}{u^2+v^2}, \frac{-v}{u^2+v^2}\right).$$

Now, lines and circles satisfy the equation

$$Ax + By + C(x^2+y^2) = D$$

for some A, B, C, not all zero. Plugging our (x, y) into this equation gives

$$Au - Bv + C = D(u^2+v^2),$$

which is in the class of lines and circles. □

Finally, notice how if $a \neq b \neq c \in \mathbb{C}$ then the transformation

$$f(z) = \left(\frac{z-a}{z-b}\right)\left(\frac{c-b}{c-a}\right) d$$

maps $a \to 0$, $b \to \infty$, and $c \to d$ for any given d. We can therefore find a Möbius transformation that maps any (a, b, c) to any (x, y, z)—we simply perform f to take $(a, b, c) \to (0, \infty, d)$ and then use another transformation to send $(0, \infty, d) \to (x, y, z)$.