ASTR 62: Intro. to Astrophysics

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* Adapted from SP24 lectures.

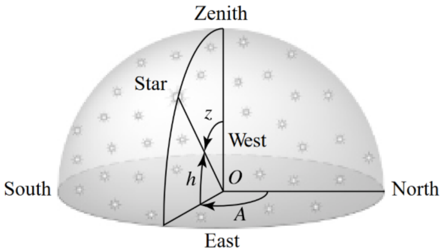
1 The Tools of Astronomy

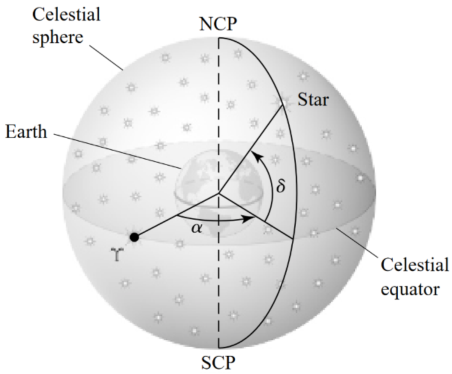
1.1 Positions in Space

We'll begin by studying how we can measure the position of an astronomical object in space. There are two dimensions to consider: where the object appears in the night sky, and how far away it is from Earth. We'll tackle each of these, starting with the first.

Celestial Coordinate Systems

For now, it will be useful for us to imagine the sky as a "**celestial sphere**" that revolves around us, the radius of which is of no concern. To specify the position of an object on this sphere we will need two coordinates: one for the horizontal angle at which the object can be found, and another for its height on the sphere. -pp There's a couple of natural ways in which we can assign these coordinates. One is the **altitude-azimuth coordinate system**, which is defined with respect to the observer's local horizon—below are drawn the horizontal **azimuth** A and the vertical **altitude** h (or **zenith distance** z). All angles are measured in degrees.





This coordinate system is great for casual observations, but not so much for cataloging—an object's altitude-azimuth coordinates will change over time due to movements of the Earth! To combat this we can define a coordinate system with respect to Earth's equator, with quantities that are analogous tow the familiar geographic coordinate system. We can extend the equator outward onto the celestial sphere to create a **celestial equator**; similarly extending the poles gives the **celestial poles**. The equivalent of longitude is called **right ascension** α, and latitude becomes **declination** δ. This arrangement is called the **equatorial coordinate system**.

Declination is easy. Objects that are above the celestial equator get a positive δ while those below get a negative δ. We usually write declinations using degrees, arcminutes, and arcseconds: there are sixty arcminutes in a degree ($1^\circ = 60'$) and sixty arcseconds in a arcminute ($1' = 60''$).

Right ascension is a little frustrating. We write it using hours, minutes, and seconds—which are distinct from arcseconds! The celestial sphere is divided into 24 hours, so $1^\mathrm{h} = 15^\circ$. From there we again divide by sixty for each subdivision, so $1^\mathrm{m} = 15'$ and $1^\mathrm{s} = 15''$. The logic here is to mirror the way we measure time as Earth rotates, but we just end up with needlessly confusing units.

Now, as Earth rotates, the Sun traces out a "**great circle**" around the celestial sphere called the **ecliptic**. The highest and lowest points on the ecliptic are called the summer and winter solstices, respectively. The points at which it crosses the celestial equator are called the vernal and autumnal equinoxes; the former occurs when the Sun is traveling upward, while the latter occurs while traveling downward. We arbitrarily define the **vernal equinox** Υ to be our $\alpha = 0$ reference point for right ascension.

Alas, not even this equatorial coordinate system is perfect since the Earth really acts like a giant gyroscope with a precession period on the order of tens of thousands of years. A better reference plane would be one defined by Earth's orbit around the Sun—we only stick with the equator because of institutional inertia. For consistency, we usually record positions as they were at noon on January 1, 2000. (We call this the **J2000.0 epoch**, using J for the Julian calendar which astronomers use for its simplicity.)

Angular Separation

With this understanding of the equatorial coordinate system, we can do some trigonometry to calculate the **angular separation** $\Delta\theta$ between two objects in the night sky. Spherical trigonometry is hard, but if the objects we're considering are close enough together, we can approximate the triangle formed by $\Delta\theta$, $\Delta\delta$, and $\Delta\alpha$ as a planar right triangle that we can do trig with.

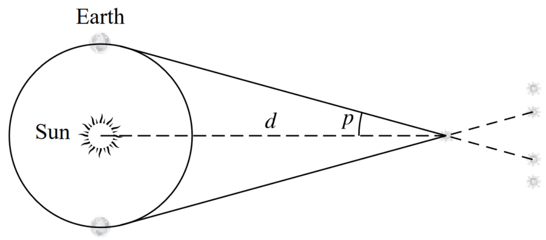
Suppose the celestial sphere has radius R. (The choice of R doesn't matter since we hope it'll cancel out in the end anyway.) Obviously, the hypotenuse of our triangle is $R\Delta\theta$ and one of the legs is $R\Delta\delta$. The leg due to the right ascension, however, is not $R\Delta\alpha$ because the rings of right ascension get smaller as they get closer to the poles. The radius of this ring is $R\cos\delta$, where δ is any declination between the two objects we're considering. (The precise value is unimportant in our approximation.) So the leg we really want is $R\cos\delta\,\Delta\alpha$, and by the Pythagorean theorem we can write

$$(R\Delta\theta)^2 = (R\Delta\delta)^2 + (R\cos\delta\,\Delta\alpha)^2$$
$$\boxed{(\Delta\theta)^2 = (\Delta\delta)^2 + (\cos\delta\,\Delta\alpha)^2},$$

where δ and α have the same units (usually degrees or arcseconds). This gives the angular separation!

Trigonometric Parallax

We'll now turn to the second question we posed at the beginning of this section, the question of distance. We can measure distances to relatively nearby stars using the method of **stellar parallax**. As the Earth orbits around the Sun, these stars appear to move back and forth very slightly relative to background stars.



The angle p is called the **parallax angle**. The distance between the Earth and the Sun is known to be one astronomical unit (AU), so we can use trigonometry to determine the distance d to the star:

$$\tan p = \frac{1\text{ AU}}{d}.$$

Applying the small-angle approximation $\tan p \approx p$ gives

$$\boxed{d = \frac{1}{p}\text{ AU}},\ p \text{ in radians.}$$

However, p is often given in arcseconds, which is a little unwieldy here. To make things simpler, we can define the **parsec** (pc), which is the distance to a star whose parallax angle is $1''$. So the conversion from radians to arcseconds gives

$$\boxed{d = \frac{1}{p}\text{ pc}},\ p \text{ in arcseconds.}$$

1.2 The Two-Body Problem

The Planar Restricted Two-Body Problem

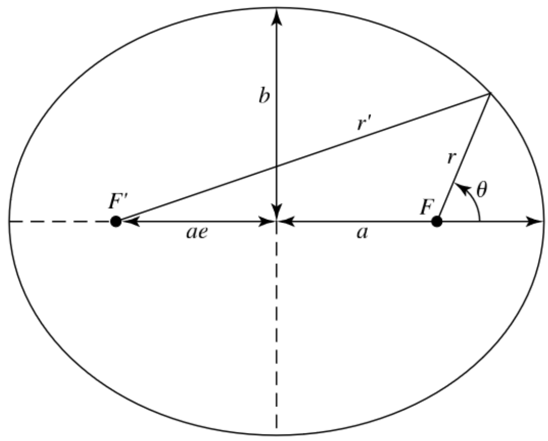
Consider a star with mass $M_\star$ about which a planet with mass m orbits. The star is positioned at the origin, and the planet has position $\mathbf{r}$ and velocity $\mathbf{v}$. Notice that all of the interactions between the star and planet occur in the same plane, so treating this as a planar problem does not sacrifice any generality. However, for now we will restrict ourselves to the case in which m is negligibly small relative to $M_\star$; that is, we'll take $m \to 0$. This arrangement is called the **planar restricted 2-body problem**.

The acceleration of the star is proportional to the planet's mass, so our restriction makes it negligible. The planet's acceleration, however, yields the equation of motion

$$\ddot{\mathbf{r}} = -\frac{GM_\star}{r^2}\hat{r},$$

which takes the initial conditions $\mathbf{r}(0) = \mathbf{r}_0$ and $\dot{\mathbf{r}}(0) = \mathbf{v}_0$. We could go into mathematical detail about what kind of orbit this equation leads to, but it isn't super useful for the purposes of this class. Instead, we'll jump straight to the result.

Kepler's first law states that the orbit of our planet is an ellipse, with its star at one of its foci. (The other focus is just empty space.) The planet's point of closest approach to the star is called **pericenter**, and the opposite point is called **apocenter**.



Geometrically, the ellipse is characterized by its **semimajor axis** a, **semiminor axis** b, and **eccentricity** e. There's a pretty simple relationship between these three quantities! Let r be the distance from the planet to the star; by the Pythagorean theorem, at a point on the semiminor axis,

$$r^2 = b^2 + a^2e^2.$$

Recall that an ellipse is defined by the equation $r + r' = 2a$, where r' is the distance from a point to the other (empty) focus. In this case, $r = r'$, so $r = a$ and

$$\boxed{b^2 = a^2(1 - e^2)}.$$

There's another relationship that we can derive, one that gives the planet's distance from the sun given any θ on the ellipse (as defined in the figure). Again applying the Pythagorean theorem, this time at the point draw above:

$$\begin{aligned} r'^2 &= (r\sin\theta)^2 + (2ae + r\cos\theta)^2 \\ r'^2 &= r^2 + 4ae(ae + r\cos\theta) \end{aligned}$$

Substituting the definition of an ellipse $r' = 2a - r$ and solving for r gives

$$\boxed{r = \frac{a(1-e^2)}{1 + e\cos\theta}, \quad 0 \le e < 1}.$$

This equation actually applies to all of the different conic sections, just with different values of e! In the special case $e = 0$ we get a circle, which reveals that the circle is just a special type of ellipse in which the foci coincide. (In fact, the area of an ellipse can be expressed as $A = \pi ab$, which easily reduces to that of a circle by taking $a = b = r$.) Setting $e = 1$ and $e > 1$ gives parabolas and hyperbolas, respectively. Each of these four curves describes a different type of celestial motion (we'll return to this soon).

Now, the orbital elements a, e, and θ (also called f) are enough to characterize an ellipse and an object's position on it. However, it says nothing about the ellipse's orientation in space. So we introduce three more orbital elements to describe it.

- The inclination i is the angle between the orbital plane and a reference plane, like the equatorial plane.
- The longitude of ascending node Ω is the horizontal angle (longitude) from a point in the reference plane, like the vernal equinox, to the point at which the orbit intersects the reference point going northward (the ascending node).
- The argument of pericenter ω is the angle (argument) from the ascending node to pericenter.

Unsurprisingly, there is a one-to-one mapping between the initial conditions $\mathbf{r}_0, \mathbf{v}_0$ and the six orbital elements we've defined; we won't describe that mapping here. What we *will* describe, however, is the connection between some of these orbital elements and the system's main conserved quantities.

Since gravity exerts no torque on the planet, the angular momentum $\mathbf{L} = m\mathbf{r} \times \mathbf{v}$ is conserved. It will be useful for us to define the **specific angular momentum**—that is, angular momentum per unit mass—as

$$\mathbf{h} = \mathbf{r} \times \mathbf{v}.$$

It turns out that, by solving the initial-value problem that governs the plane's dynamics, we can write

$$r = \frac{h^2/GM_\star}{1 + e\cos\theta}.$$

(This is another one of those things that we'll take for granted.) So we have the relationship

$$\frac{h^2}{GM_\star} = a(1 - e^2) \implies \boxed{h^2 = GM_\star a(1 - e^2)}.$$

Similarly, the energy $\mathcal{E} = v^2/2 - GM_\star/r$ is conserved since gravity is the only force acting on the planet, and we can define the **specific energy**

$$\mathcal{E} = \frac{1}{2}v^2 - \frac{GM_\star}{r}.$$

We can derive a relationship for this one on our own. Consider the state of the planet at pericenter—it has position $a(1 - e)$ and velocity v_p. We can find an expression for v_p^2 using the specific angular momentum, noting that the position and velocity vectors are orthogonal at pericenter:

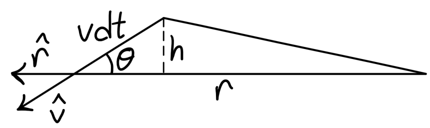
$$h = a(1 - e) \cdot v_p \implies v_p = \frac{h}{a(1 - e)}.$$

Using this to calculate the specific energy:

$$\begin{aligned}
\mathcal{E} &= \frac{1}{2}\frac{h^2}{a^2(1 - e)^2} - \frac{GM_\star}{r} \\
&= \frac{GM_\star a(1 - e^2)}{2a^2(1 - e)^2} - \frac{GM_\star}{a(1 - e)} \\
&\boxed{\mathcal{E} = -\frac{GM_\star}{2a}}.
\end{aligned}$$

We're now in a position to derive Kepler's other laws with ease.

Kepler's second law states that the rate at which a planet's orbit sweeps out area is constant. To show this, consider the triangle that the planet sweeps out in an infinitesimal time dt.



If the area of this triangle is dA, then

$$dA = \frac{1}{2}r(v\,dt\,\sin\theta) \implies \frac{dA}{dt} = \frac{rv\sin\theta}{2}.$$

So $\boxed{dA/dt = h/2}$, which is constant!

Kepler's third law gives a relationship between a planet's orbital period and its semimajor axis. For this one, we simply use the fact that

$$P = \frac{A_{\text{ellipse}}}{dA/dt} = \frac{\pi ab}{h/2} = \frac{2\pi a(a\sqrt{1-e^2})}{\sqrt{GM_\star a(1-e^2)}} = \frac{2\pi}{\sqrt{GM}} a^{\frac{3}{2}}.$$

Squaring gives

$$\boxed{P^2 = \frac{4\pi^2}{GM_\star} a^3}.$$

That's it for Kepler's laws! These are valid for bound orbits, which are ones that keep the planet (or other astronomical object) within the grasp of a parent star. Another important property of energy in bound systems is the **virial theorem**. Recall how, for circular orbits,

$$\frac{GM_\star}{r^2} = \frac{v^2}{r},$$

so $v^2 = GM/r$. This gives the kinetic energy

$$K = \frac{GM_\star m}{2r} = -\frac{U}{2}.$$

In a slightly more general sense, we can make the following statement for any gravitationally bound system in equilibrium:

$$\boxed{\langle K \rangle = -\frac{\langle U \rangle}{2}},$$

where quantities in the angle brackets are time averages (usually over one period).

To determine if an orbit is bound or unbound we look at the amount of energy an object possesses. If its energy is positive (or zero) then it has enough energy to escape the star's potential energy well, so its orbit is unbound. Negative energies, on the other hand, correspond to bound orbits. More specifically:

Shape	Bound?	Energy	Semimajor Axis	Eccentricity
Hyperbola	No	$E > 0$	$a < 0$	$e > 1$
Parabola	No	$E = 0$	$a = \infty$	$e = 1$
Ellipse	Yes	$E < 0$	$a > 0$	$0 \leq e < 1$
Circle	Yes	$E < 0$	$a > 0$	$e = 0$

The General Two-Body Problem

Now we can tackle the general two-body problem—that is, the two-body problem with two finite masses m_1 and m_2 Because the system's total momentum is conserved, our analysis will be easiest in the center-of-mass reference frame.

Place the center of mass at the origin of our coordinate system. The position vectors of the masses are $\mathbf{r}_1$ and $\mathbf{r}_2$, respectively. But the dynamics of the two-mass system really only depend on the displacement $\mathbf{r} = \mathbf{r}_2 - \mathbf{r}_1$, so it will be useful to relate each position vector to the displacement. This is done pretty easily by substituting the proper information into the equation $(m_1\mathbf{r}_1 + m_2\mathbf{r}_2)/M = 0$, where $M = m_1 + m_2$.

$$\mathbf{r}_1 = \frac{m_2}{M}(-\mathbf{r}), \quad \mathbf{r}_2 = \frac{m_1}{M}\mathbf{r}.$$

We can actually use this to greatly simplify our problem! First, we write two equivalent expressions for the quantity $m_2\ddot{\mathbf{r}}_2 - m_1\ddot{\mathbf{r}}_1$.

$$\begin{aligned} m_2\ddot{\mathbf{r}}_2 - m_1\ddot{\mathbf{r}}_1 &= m_2\left(\frac{m_1}{M}\ddot{\mathbf{r}}\right) - m_1\left(\frac{m_2}{M}(-\ddot{\mathbf{r}})\right) \\ &= \frac{2m_1m_2}{M}\ddot{\mathbf{r}} \end{aligned} \qquad \begin{aligned} m_2\ddot{\mathbf{r}}_2 - m_1\ddot{\mathbf{r}}_1 &= -m_2\frac{Gm_1}{r^2}\hat{r} - m_1\frac{Gm_2}{r^2}\hat{r} \\ &= -\frac{2Gm_1m_2}{r^2}\hat{r} \end{aligned}$$

Therefore,

$$\frac{m_1 m_2}{M}\ddot{\mathbf{r}} = -\frac{G m_1 m_2}{r^2}\hat{r}.$$

To write this a little more suggestively, we can define the **reduced mass** $\mu = \frac{m_1 m_2}{m_1+m_2}$ to get

$$\boxed{\mu\ddot{\mathbf{r}} = -\frac{GM\mu}{r^2}\hat{r}}.$$

Notice how this is the equation of motion for a body with mass μ orbiting around a fixed mass M at a distance r. We've just taken our general two-body problem and reduced it to an equivalent one-body problem! So all of the equations that we've derived for the restricted two-body problem also apply here, after a few substitutions.

We don't need to change anything about the eccentricity of the orbit. It should make sense, for example, that a highly eccentric two-body system would stay highly eccentric in our equivalent reduced model. To see what happens to the semimajor axis, we exploit the fact that the distance r between μ and M is the sum of the objects' distances to the center of mass. At pericenter we have

$$\begin{aligned} r &= r_1 + r_2 \\ a(1-e) &= a_1(1-e) + a_2(1-e) \\ a &= a_1 + a_2, \end{aligned}$$

Finally, we replace all instances of $M_\star$ with M. This gives us the set of substitutions

$$\boxed{\begin{aligned} r &\to r_1 + r_2 \\ a &\to a_1 + a_2 \\ M_\star &\to M \end{aligned}}$$

As a corollary, we can also derive a more direct pair of relationships for the semimajor axes (using the fact that $r_1 = a_1$ and $r_2 = a_2$ on the semiminor axis):

$$a_1 = \frac{m_2}{M}a, \quad a_2 = \frac{m_1}{M}a.$$

We also have simple equations for the kinetic energy and angular momentum, both in the center-of-mass frame.

$$\boxed{\begin{aligned} &E = \frac{1}{2}\mu v^2 - \frac{GM - \mu}{r^2} \\ &\mathbf{L} = \mu(\mathbf{r}\times\mathbf{v}), \end{aligned}}$$

where $\mathbf{v} \equiv d\mathbf{r}/dt$. Lastly, we give an alternative relation between $\mathbf{r}$ and the position vectors.

$$\boxed{\mathbf{r}_1 = \frac{\mu}{m_1}(-\mathbf{r}), \quad \mathbf{r}_2 = \frac{\mu}{m_2}\mathbf{r}}.$$

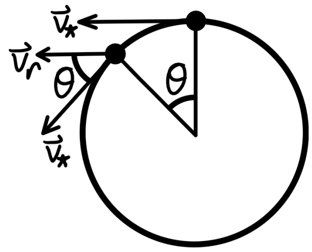
Detection of Exoplanets

As an application, let's discuss the detection of planets in extrasolar systems. The most obvious way of doing this is via direct imaging. We cover up the light coming from a star so that its brightness doesn't overpower that of any orbiting bodies, and we see what we can see. (This method isn't perfect since some light diffracts around the blockage, but good software will be able to filter this out.) This generally only works with very large planets with very large orbits.

Alternatively, we can use our newfound knowledge of orbital mechanics! Suppose we have a planet-star system orbiting around a common center of mass; let $\mathbf{v}_\star$ be the star's orbital velocity and let $\mathbf{v}_r$ be the component of this velocity that is in the observer's direction.

As the star moves toward or away from us, the light it emits experiences a Doppler shift—it is blueshifted and redshifted, respectively. For $v_r \ll c$, this shift is quantified by

$$\frac{\lambda_{\text{obs}} - \lambda_{\text{em}}}{\lambda_{\text{em}}} = \frac{\Delta\lambda}{\lambda_{\text{em}}} = \frac{v_r}{c}.$$



So clearly v_r is an important quantity. We can relate it to known values using some simple trigonometry! If θ is the angle between $\mathbf{v}_\star$ and $\mathbf{v}_r$, then we have $v_r = v_\star \cos\theta$. If we define a time $t = 0$ at the angle $\theta = 0$, then the relationship is

$$v_r = v_\star \cos\left(\frac{2\pi t}{P}\right),$$

where P is the orbital period of the star. But in general, we won't be looking at planetary systems edge-on, but rather at some inclination i. If $i = 90°$ corresponds to the edge-on view, then we have

$$\boxed{v_r = (v_\star \sin i)\cos\left(\frac{2\pi t}{P}\right)}.$$

From here we can do a couple of things. For one, we could observe the change in the star's spectrum over time to determine its mass; we'll do more of that later on. But assuming we have the result of this analysis, we could also determine the planet's mass m_p with the knowledge we have now!

Let's take a look at our observables. We have the maximum radial velocity $v_{\max} = v_\star \sin i$, the orbital period P, and the mass $M_\star$ of the star. We also have the simple $d = rt$ relationship

$$v_\star = \frac{2\pi a_\star}{P}.$$

Since $a_\star = (m_p/M_{\text{tot}})a$, where a is the reduced-mass semimajor axis, this is equivalent to

$$v_\star = \frac{2\pi}{P}\frac{m_p}{M_{\text{tot}}}a = \frac{2\pi}{P}\frac{m_p}{M_{\text{tot}}}\left(\frac{P^2 G M_{\text{tot}}}{4\pi^2}\right)^{1/3},$$

where the last step follows from Kepler's third law. When we set this equal to $v_{\max}/\sin i$ and take $m_p \ll M_\star$, we get

$$\boxed{m_p \sin i = v_{\max}\left(\frac{P M_\star^2}{2\pi G}\right)}.$$

Everything on the right side of this equation is an observable quantity! Since $\sin i$ never exceeds 1, this expression gives us a lower bound for the mass of the exoplanet.

1.3 Measures of Brightness

We've spent plenty of time analyzing the positions and motions of objects in the sky. Now we'll turn to the question of how bright something is, in both relative terms and absolute ones.

Luminosity, Flux, and Magnitude

The most fundamental measure of brightness we can define is the **luminosity** $L = dE/dt$, which is simply the energy emitted per unit time. We could call this "power" like we do elsewhere in physics, but we normally don't to distinguish the production of radiative energy from other phenomena.

When objects like stars give off energy, they do so in a spherically symmetric manner. So it will be useful to define the **flux**

$$\boxed{F_{\text{obs}} = \frac{dE}{dt\,dA} = \frac{L}{4\pi d^2}},$$

which changes as a function of the distance d from the source. This flux is not the same as the quantity we defined in E&M! We can interpret F as a measure of "power" per unit area on some radius-d sphere. If we take the radius of this sphere to simply be that of the star we're looking at, R, then we get another absolute quantity: the **emitted flux**

$$F_{\text{em}} = \frac{L}{4\pi R^2}.$$

The relationship between the observed and emitted flux is simply $F_{\text{obs}} = F_{\text{em}}(R/d)^2$.

These two quantities we've defined are relatively modern ones in the broader context of astronomy. In ancient times observers had only their logarithmic eyes to go off of, and all of their observations were relative rather than absolute. The **magnitude scale**, devised in ancient Greece, is defined in modern terms as follows. If we have two objects with fluxes F_1 and F_2, then their magnitudes m_1 and m_2 are

$$\boxed{m_1 - m_2 = 2.5 \log_{10} \frac{F_2}{F_1}}.$$

A few important things to note about this scale. Also:

- This definition is inherently relative—in isolation it is only meaningful when we compare the magnitudes of two stars. So if we want to assign each object its own unique magnitude we must agree on a standard value of F that corresponds to $m = 0$; Vega is the usual choice for this.
- This definition is also relative in another sense: it depends on distance, just like F. To distinguish it from the absolute magnitude (which we'll define shortly), we call m an apparent magnitude.
- Finally, the magnitude scale runs opposite of how we might expect it to. Brighter stars have lower magnitudes than dimmer ones. This feels weird, but its origins are logical—it's a consequence of using zero as a base line for the brightest stars in the sky and going from there.

Now, if apparent magnitude is defined in terms of flux, we might expect that the absolute magnitude of an object is related to its luminosity. Specifically, if we define the **absolute magnitude** M as the apparent magnitude of an object seen at a distance of $d = 10$ pc, we get

$$\begin{aligned} M_1 - M_2 &= 2.5 \log_{10} \left(\frac{F_2(d)}{F_1(d)} \right) \\ &= 2.5 \log_{10} \left(\frac{L_2}{4\pi d^2} \cdot \frac{4\pi d^2}{L_1} \right) \\ \boxed{M_1 - M_2 = 2.5 \log_{10} \frac{L_2}{L_1}} & \end{aligned}$$

Again, we usually use Vega as our standard $M = 0$. We can take advantage of our two definitions here to approximate the distance to an object! We know how to measure m directly, and we can often make a pretty good guess for M based on other characteristics (we'll see how later). The distance d to the object is related to these quantities by the equation

$$\begin{aligned} m - M &= 2.5 \log_{10} \left(\frac{F(10 \text{ pc})}{F(d)} \right) \\ &= 2.5 \log_{10} \left(\frac{L}{4\pi (10 \text{ pc})^2} \cdot \frac{4\pi d^2}{L} \right) \\ \boxed{m - M = 5 \log_{10} \frac{d}{10 \text{ pc}}} & \end{aligned}$$

This is often called the **distance modulus**.

Solid Angles and Intensity

All of our definitions here are great for point sources that emit light isotropically in all directions. But what if this isn't the case? As a step toward accounting for this, we'll define a quantity that, in a sense, indicates direction: the **solid angle** Ω. A solid angle is just an angle in three dimensions—just like how we have $d\theta = ds/r$ in two dimensions, in 3D we have

$$\boxed{d\Omega = \frac{dA}{r^2} = d\theta \sin\theta \, d\varphi},$$

where $dA = r\, d\theta \cdot r \sin\theta \, d\phi$ is a differential area on the celestial sphere. (φ is the angle in the xy-plane while θ is the angle from the z-axis.) Solid angles are measured in steradians (sr) which, like radians, *technically* aren't real units but we usually write them down for bookkeeping purposes. We could show, by integrating $d\Omega$ over the entire celestial sphere, that there are 4π sr in a full sphere.

Now we're ready to define the "directional brightness" that we initially desired. Taking into account that an observer may detect incoming light at an angle θ (where $\theta = 0$ corresponds to observing the light straight-on), we define the **intensity** as

$$\boxed{I = \frac{dE}{dt\, d\Omega\, dA \cos\theta}},$$

where $dA\cos\theta$ is the differential normal area of some detector. This is the energy detected (or emitted!) per unit time, per unit solid angle, per unit normal area.

There's a pretty straightforward relationship between F and I that we can determine just by inspecting their definitions:

$$F = \int_\Omega I \cos\theta \, d\Omega.$$

The extra factor of $\cos\theta$ ensures that light incident at a relatively large θ contributes less to the overall flux than light from other directions. There are a couple of assumptions that we can often take to make the integral much easier. First, if the observer is looking directly at the object then $\theta \approx 0$ for all light, so $\cos\theta \approx 1$ and

$$F \approx \int_\Omega I \, d\Omega = I\Omega.$$

If the sky is uniformly bright, then I is constant and the flux from all light is

$$F = \int_0^{\pi/2} I \cos\theta(\sin\theta \, d\theta) \int_0^{2\pi} d\varphi = \pi I.$$

This result is useful in determining the total (observed) emitted flux from an isotropic emitter, like we'd see on the surface of a star!

Specific Brightness

We can split up each of our quantities in one more way. We're often only concerned with the brightness of an object in one wavelength or frequency. For example, we can define the specific luminosity

$$L_\lambda = \frac{dL}{d\lambda} \text{ or } L_\nu = \frac{dL}{d\nu},$$

which are not equivalent but communicate the same sort of thing. They're characterized by the integrals

$$L = \int_0^\infty L_\lambda \, d\lambda = \int_0^\infty L_\nu \, d_\nu,$$

respectively, which suggests the relationship

$$L_\lambda |d\lambda| = L_\nu |d\nu|$$

and thus

$$\boxed{L_\nu = \left|\frac{d\lambda}{d\nu}\right| L_\lambda} \implies \boxed{\nu L_\nu = \lambda L_\lambda}.$$

We can do something similar with specific flux and luminosity, which is often useful.

1.4 Blackbody Radiation

Blackbody Radiation

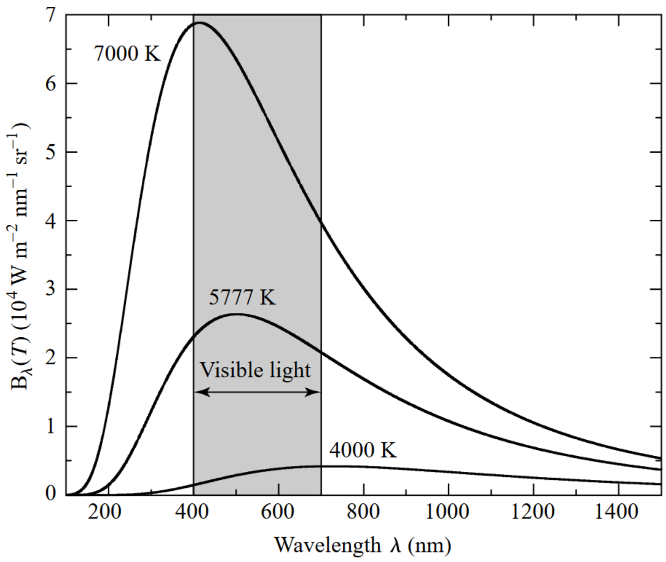
Armed with these definitions, we can get back into some astrophysics! We'll start by talking about blackbodies, which are objects that are completely opaque, nonreflective, and in thermal equilibrium. Of course, such a body does not exist, but this model approximates the properties of objects like stars to a surprising degree.

For a blackbody at a temperature T, one might derive its specific intensity at different wavelengths (or frequencies). That's a problem for statistical mechanics, though, so we'll just quote the result, which is called the **Planck function**.

$$\boxed{I_\lambda = B_\lambda = \frac{2hc^2/\lambda^5}{e^{hc/kT\lambda} - 1} \qquad I_\nu = B_\nu = \frac{2h\nu^3/c^2}{e^{h\nu/kT} - 1}}$$

Here $k = k_B$ is the Boltzmann constant, h is the Planck constant, and c is the speed of light in a vacuum. Like with luminosity, we can convert between these quantities using the equation $I_\lambda |d\lambda| = I_\nu |d\nu|$.

Notice that an object's Planck function depends *only* on its temperature. Objects at higher temperatures have higher intensities at all wavelengths, but their peak wavelengths are lower than in cooler counterparts. The plot below illustrates this at three different temperatures (including the Sun's, at 5777 K).



To determine what this peak wavelength is, one could solve the equation $dB_\lambda/d\lambda = 0$ to find that

$$\boxed{\lambda_{\text{peak}} = \frac{b}{T}}, \text{ with } b = 0.29 \text{ cm K}.$$

This is **Wien's displacement law**, and it explains why bodies with different temperatures are different colors!

When we integrate across all wavelengths, we get the total intensity

$$\begin{aligned} I = B &= \int_0^\infty B_\lambda \, d\lambda \\ &= \int_0^\infty \frac{2hc^2 d\lambda}{\lambda^5 (e^{hc/kT\lambda} - 1)} \\ I = B &= \frac{\sigma T^4}{\pi}, \end{aligned}$$

where $\sigma = (2\pi^5 k^4)/(15h^3c^2)$ is the Stefan-Boltzmann constant. Since $F = \pi I$ for an isotropic sky, the emitted flux from a blackbody is

$$\boxed{F_{\text{em}} = \sigma T^4}.$$

This gives the total luminosity

$$\boxed{L = 4\pi R^2 \sigma T^4},$$

where R is the radius of the blackbody. This equation is particularly useful when dealing with thermal equilibria, which are characterized by $L_{\text{abs}} = L_{\text{em}}$.

Albedo and Color Magnitudes

Now, what happens when we relax one of our assumptions by allowing our bodies to be reflective? We'll define **albedo** a as the fraction of incident light that is reflected rather than absorbed. This fraction is generally a function of wavelength. Assuming the body is still in thermal equilibrium, emitted light must exactly match absorbed light at all wavelengths. Expressed in terms of wavelength,

$$\begin{aligned} I_{\text{abs}}(\lambda) &= [1 - a(\lambda)]\, I_{\text{inc}} \\ I_{\text{em}}(\lambda) &= [1 - a(\lambda)]\, B_\lambda \end{aligned}$$

We might use this information to estimate the equilibrium temperature of a planet by setting $I_{\text{abs}} = I_{\text{em}}$ and considering only the wavelengths of maximum absorption and emission. This approximation, however, does not account for atmospheres and so may be cooler than reality.

We now have several different ways of estimating temperature from a blackbody spectrum, but they all require information spanning a wave range of wavelengths, which can be quite challenging to obtain. We'll give one much cruder, but still effective, method for measuring these temperatures.

Thus far we've been talking about bolometric (total) magnitudes, that is, magnitudes across all wavelengths. However, telescopes make observations through filters for very small bands of wavelengths (like Ultraviolet, Blue, Visual, Red, and Infrared). The flux through a filter X is given by

$$F_X = \int_0^\infty F_\lambda S_X(\lambda)\, d\lambda,$$

where $S_X(\lambda)$ is the probability that a photon of wavelength λ passes through filter X. We use this to define the **color magnitude** for X:

$$m_{X,1} - m_{X,2} = 2.5 \log_{10}\left(\frac{F_{X,2}}{F_{X,1}}\right).$$

But as long as the specific flux F_λ doesn't change too abruptly in X, we can write this as

$$\boxed{X_1 - X_2 = 2.5 \log_{10}\left(\frac{F_{\lambda,2}(\lambda_X)}{F_{\lambda,1}(\lambda_X)}\right)},$$

where λ_X is the maximum wavelength in X and the magnitudes are denoted by X_1 and X_2.

We can take advantage of this definition to characterize the temperatures of stars! Start with the color magnitudes in the V and B bands:

$$\begin{aligned} V - V_0 &= 2.5 \log_{10}\left(\frac{F_{\lambda_0}(\lambda_V)}{F_\lambda(\lambda_V)}\right) \\ B - B_0 &= 2.5 \log_{10}\left(\frac{F_{\lambda_0}(\lambda_B)}{F_\lambda(\lambda_B)}\right) \end{aligned}$$

Subtracting these equations,

$$\begin{aligned} B - V &= 2.5 \log_{10}\left(\frac{F_{\lambda_0}(\lambda_B)}{F_{\lambda_0}(\lambda_V)} \cdot \frac{F_\lambda(\lambda_V)}{F_\lambda(\lambda_B)}\right) \\ &= 2.5 \log_{10}\left(\frac{F_\lambda(\lambda_V)}{F_\lambda(\lambda_B)}\right) + 2.5 \log_{10}\left(\frac{F_{\lambda_0}(\lambda_B)}{F_{\lambda_0}(\lambda_V)}\right) \end{aligned}$$

But the second is the B-V of Vega, which we define to be zero. This gives us the **color index**

$$\boxed{B - V = 2.5 \log_{10}\left(\frac{F_\lambda(\lambda_V)}{F_\lambda(\lambda_B)}\right)}.$$

Notice that the color magnitude is a function only of temperature, not distance! Also note that a larger $F_\lambda(\lambda_B)$ compared to $F_\lambda(\lambda_V)$ corresponds to a higher temperature, so B-V decreases with increasing temperature.

The Hertzsprung-Russel (H-R) diagram is a plot of luminosity against color index. There's a very prominent negative correlation of stars called the main sequence. As it turns out, the location of a given star on the main sequence is entirely determined by its mass—the higher the mass, the higher the position on the main sequence (brighter but cooler). There are some pretty tight empirical correlations between the main stellar parameters on the main sequence. Specifically:

$$\frac{L}{L_\odot} \approx \left(\frac{M}{M_\odot}\right)^{3.5} \qquad \frac{R}{R_\odot} \approx \left(\frac{M}{M_\odot}\right)^{0.7} \qquad \frac{T}{T_\odot} \approx \left(\frac{M}{M_\odot}\right)^{0.56}$$

We'll spend some time building stellar models in an attempt to understand the origins of empirical relations like these.

2 Stellar Properties

2.1 Spectral Formation

If we were to zoom into the blackbody spectrum of a star like the Sun we'd notice some deviations from the ideal blackbody spectrum. Specifically, there are many wavelengths where the emission is smaller than we'd expect from a blackbody. These features are called **absorption lines**. But some objects like planetary nebulae are characterized more by **emission lines**, which are spikes of emission in a sea of none. In this section we'll try to explain these phenomena!

Quantized Energy States

We'll start with **Kirchhoff's laws**, which is a set of empirical laws that seem to predict spectral formation.

0. Each chemical element produces a unique pattern of spectral lines that have the same wavelengths in emission and absorption.
1. A hot, dense gas or solid object produces a continuous spectrum with no dark spectral lines. (This is a blackbody spectrum!)
2. A hot, diffuse gas produces bright spectral lines (emission lines).
3. A cold, diffuse gas in front of a hotter continuous-spectrum source produces dark spectral lines (absorption lines) in the continuous spectrum.

The first solar spectrum was published by Fraunhofer in the early nineteenth century. A few decades later, Balmer discovered an empirical formula for hydrogen absorption lines, now called **Balmer lines:**

$$\frac{1}{\lambda} = R_H \left(\frac{1}{4} - \frac{1}{n^2} \right), \quad n = 3, 4, \ldots,$$

where R_H is called the Rydberg constant.

The explanation for all these empirical observations is quantum mechanical. The energy in electromagnetic radiation comes in discrete packets called photons, each with energy $E_\gamma = h\nu = hc/\lambda$. Electrons can only occupy predetermined, discrete energy levels within atoms. When an electron absorbs or emits a photon it transitions to a different energy state; the resulting change in the electron's energy is equal to the photon's energy, meaning electrons can only interact with photons of certain wavelengths.

In the early twentieth century Bohr devised the first model of the hydrogen atom that correctly predicted the quantized energy levels for an electron. We now know this model to be completely wrong, but it remains a useful tool for our analysis. He proposed that the electron orbits around the proton (due to the electrostatic force) at radii r_n that are multiples of its de Broglie wavelength, $\lambda_{\mathrm{dB}} = h/m_e v$. This led to the energies

$$\boxed{E_n = -\frac{m_e e^4}{8\varepsilon_0^2 h^2 n^2} = -\frac{E_1}{n^2}},$$

where e is the charge of an electron and $E_1 = 2.18 \times 10^{-18}\ \mathrm{J} = 13.6\ \mathrm{eV}$.

This can be used to recover a more general version of Balmer's equation! Suppose that a photon with energy E_γ is emitted as an electron moves between its kth and nth energy states. So $E_\gamma = hc/\lambda = E_n - E_k$ and

$$\frac{1}{\lambda} = \frac{m_e e^4}{8\varepsilon_0^2 h^3 c} \left(\frac{1}{k^2} - \frac{1}{n^2} \right).$$

The constant in front evaluates to the Rydberg constant, and setting $k = 2$ gives the Balmer equation!

Originally stars were classified by letter (A, B, C, etc.) based on the strength of their hydrogen lines. A few decades later, the spectra were re-understood as a temperature sequence: from hottest to coolest, O B A F G K M

Nowadays each spectral type is subdivided into ten subtypes (e.g., B0, B1, ..., B9). Each type corresponds to a very specific pattern of spectral lines, their relative strengths, and thus a variety of elements and molecules!

Excitation and Ionization

So a star's spectrum is a function of its temperature. Now, in order for a spectral line to form, there must be a large enough number of atoms in both the proper excitation and ionization states—that is, it must have an electron at the proper energy level, and it must also have the proper amount of electrons. How can we quantify these things in terms of temperature?

Generally, excitations and ionizations are caused by high-speed collisions between atoms. These velocities are related to temperature by the equation

$$\left\langle \frac{1}{2}mv^2 \right\rangle = \frac{3}{2}kT,$$

where m is the mass of an emitter. So whatever we end up with should give us higher excitation and ionization states with increasing temperature.

Let's start with excitation states. Consider two energy levels a and b inside an atom such that $E_B > E_a$. Let $P(a)$ is the probability that an electron occupies level a, and let N_a be the number of electrons in this level. Using statistical mechanics one could show that

$$\frac{N_b}{N_a} = \frac{P(b)}{P(a)} = \frac{e^{-E_b/kT}}{e^{-E_a/kT}} = e^{-\Delta E/kT},$$

where $\Delta E = E_b - E_a$. Notice that we have $N_a > N_b$ for all $\Delta E > 0$, meaning higher energy states will never have a higher probability that lower ones.

But this doesn't quite tell the whole story because energy levels can be degenerate—that is, there can be several states with the same energy. For example, hydrogen has two ground-state levels (2s) and eight first excited-state levels (2s + 6p). Let g_a be the number of states with energy E_a (for hydrogen $g_n = 2n^2$); making a slight modification to our equation to account for degeneracy gives the **Boltzmann equation**:

$$\boxed{\frac{N_b}{N_a} = \frac{P(b)}{P(a)} = \frac{g_b e^{-E_b/kT}}{g_a e^{-E_a/kT}} = \frac{g_b}{g_a} e^{-\Delta E/kT}}.$$

Now let's talk ionization. We represent ionization states using Roman numerals—for example, the first three ionization states for calcium are Ca I, Ca II, and Ca III, in which zero, one, and two electrons have been removed, respectively. Let N_i be the number of atoms in their ith ionization state and χ_i the energy required to bring an atom from its ith ionization state to its $(i+1)$th state; we might expect that $N_{i+1}/N_i \propto e^{-\chi_i/kT}$. This will actually be the case! To account for degeneracy, we define the partition function

$$Z_i = \sum_{k=1}^{\infty} g_k e^{-(E_k - E_1)/kT}.$$

This is a weighted average of all the (infinitely many) degeneracies, with the Boltzmann factor there to give more weight to the higher-probability energies. Using statistical mechanics magic we get the **Saha equation**,

$$\boxed{\frac{N_{i+1}}{N_i} = \frac{2}{n_e} \left(\frac{2\pi m_e kT}{h^2} \right)^{\frac{3}{2}} \frac{Z_{i+1}}{Z_i} e^{-\chi_i/kT}},$$

where n_e is the number density of electrons in space. Note that a larger T corresponds to more collisions and thus a higher N_{i+1}, and that a larger n_e implies more recombinations and thus a lower N_{i+1}.

The Boltzmann and Saha equations together describe why spectral lines appear and then disappear again as temperature increases! Take the Hα line for example, which corresponds to a transition between the first and second excited states of a neutral hydrogen atom. At low temperatures there are no excited hydrogens, so the line is invisible; as temperature increases, more hydrogens become excited and Hα becomes stronger; but as temperature increases further the hydrogens begin to ionize and the line fades.

2.2 Radiative Transfer

Let's get to work on a physical model for stars, starting with **radiative transfer**—the production of radiation in the stellar atmosphere and its path to Earth. We'll focus on absorption for now.

Absorption

Suppose a photon passes through a cloud on its way to an observer. If we want to determine the probability that this photon gets absorbed along the way, we need only consider the electrons in the cloud—only high-energy gamma rays can interact directly with atomic nuclei and other elementary particles. Let's say our cloud contains n_e electrons per unit volume, and a photon gets absorbed if it comes within the **cross section** σ_λ (the "target area") of an electron. This cross section is a function of the photon's wavelength.

We can simplify this scenario slightly by thinking of the electrons as point particles and of the photon as having an effective cross section σ_λ. In a time Δt, the photon sweeps out a cylinder that contains $N = n_e \sigma_\lambda c \Delta t$ electrons, so the collision rate is

$$R = \frac{N}{\Delta t} = \sigma c n_e.$$

We can use this to define the **mean free time** and **mean free path** between collisions:

$$t_{\mathrm{mf}} = \frac{1}{R} = \frac{1}{\sigma_\lambda c n_e} \qquad l_{\mathrm{mf}} = c t_{\mathrm{mf}} = \frac{1}{n\sigma_\lambda}.$$

In practice things will not be so simple. The medium in which the photon travels may have many absorbing species, each with their own number densities n. So rather than deal with the cross section σ_λ (an effective area per absorber), we'll often work with the **opacity**

$$\boxed{\kappa_\lambda = \sigma_\lambda \frac{n}{\rho}},$$

which is an effective area per unit mass. (In this equation ρ is the mass density of absorbers.) A particularly convenient property of this quantity is that, when working with several absorbing species, we can determine the total opacity just by summing up the individual opacities!

We're now in a position to determine the effect of absorption on light as it propagates through space. We choose intensity as our measure of brightness because, according to **Liouville's theorem**, it remains constant as radiation propagates from source to observer. To show this, consider a source that creates a solid angle $\Omega = A/d^2$, where A is the area that "fits" inside the solid angle at a distance d from the observer. For a small source, $I_{\mathrm{obs}} = F/\Omega = Fd^2/A$; but flux is proportional to A/d^2, so I_{obs} is constant over d!

Consider a very short distance ds in the absorptive medium. The probability that a photon gets absorbed in this length is ds/l_{mf}, so the expected change in intensity

$$dI_\lambda = -I_\lambda(s)\frac{ds}{l_{\mathrm{mf}}}.$$

This is a separable differential equation whose solution is

$$\boxed{I_\lambda(s) = I_\lambda(0)e^{-\int_0^s ds/l_{\mathrm{mf}}} = I_\lambda(0)e^{-s/l_{\mathrm{mf}}}},$$

where the last expression represents the special case in which the mean free length is constant. The dimensionless quantity in the exponent is of huge importance—we call it the **optical depth** $\tau_\lambda = s/l_{\mathrm{mf}}$, and we might interpret it as the number of collisions a photon will experience if it travels in a straight line. We usually define $\tau_\lambda = 0$ on the side of the medium facing the observer. So we can repackage this intensity equation as

$$d\tau_\lambda = -\frac{ds}{l_{\mathrm{mf}}} \implies \boxed{I_\lambda(0) = I_\lambda(\tau_{\lambda,c})e^{-\tau_\lambda}},$$

where $\tau_{\lambda,c}$ is the optical depth at the side of the medium opposite the observer. In the optically thick limit $\tau_\lambda \gg 1$ most of the light is absorbed so $I_\lambda(0) \approx 0$, but in the optically thin limit $\tau_\lambda \ll 1$ a Taylor approximation gives $I_\lambda(0) \approx I_\lambda(\tau_{\lambda,c})(1 - \tau_\lambda)$.

The Equation for Radiative Transfer

We take a similar path to account for emission. Define the **emission coefficient** j_λ as the energy emitted per unit time, per unit solid angle, per unit mass, per unit wavelength. In order to go from an emission coefficient to an intensity, we'll have to multiply by a mass and divide by an area to get the right units:

$$dI_\lambda = j_\lambda \frac{\rho A\, ds}{A} = j_\lambda \rho\, ds.$$

Combined with absorption , we get the differential equation

$$dI_\lambda = -I_\lambda \frac{ds}{l_{\mathrm{mf}}} + j_\lambda \rho ds \implies dI_\lambda = I_\lambda d\tau_\lambda - \frac{j_\lambda}{\kappa_\lambda} d\tau_\lambda$$

The local quantity $S_\lambda = j_\lambda/\kappa_\lambda$, called the **source function**, describes the removal and replacement of traveling photons. So we can write

$$\boxed{\frac{dI_\lambda}{d\tau_\lambda} = I_\lambda - S_\lambda}.$$

One could solve this differential equation via multiplication by the integrating factor $e^{-\tau_\lambda}$. This gives the observed intensity

$$\boxed{\begin{aligned} I_\lambda(0) &= I_\lambda(\tau_\lambda)e^{-\tau_\lambda} + \int_0^{\tau_\lambda} S_\lambda e^{-\tau_\lambda'} d\tau_\lambda' \\ &= I_\lambda(\tau_\lambda)e^{-\tau_\lambda} - S_\lambda\left(e^{-\tau_\lambda} - 1\right), \end{aligned}}$$

where the last expression is the special case in which S_λ is constant. The first term in each accounts for the absorption of some initial intensity, while the second accounts for emission and the potential for re-absorption.

In the optically thick limit we can see that $I_\lambda(0) = S_\lambda$. But if the cloud is in thermal equilibrium then it acts like a blackbody, meaning $S_\lambda = B_\lambda(T)$. (In fact, since S_λ is a local property, this relationship holds for *any* cloud in thermal equilibrium!) In the optically thin limit, a Taylor expansion gives

$$I_\lambda(0) = I_\lambda(\tau_\lambda)(1 - \tau_\lambda) + S_\lambda(\tau_\lambda)\tau_\lambda.$$

Kirchhoff's Laws and Line Widths

We can use this to prove Kirchhoff's second and third laws quite easily. Suppose we have an optically thick star sitting behind an optically thin cloud at temperatures T_B and T_C, respectively. The observed intensity is

$$\begin{aligned} I_\lambda(0) &= B_\lambda(T_B)(1-\tau_\lambda) + B_\lambda(T_C)\tau_\lambda \\ &= B_\lambda(T_B) + [B_\lambda(T_C) - B_\lambda(T_B)]\tau_\lambda. \end{aligned}$$

If $T_C > T_B$ then the stuff in the brackets is positive. Also, τ_λ peaks at atomic transition wavelengths, so overall we get an intensity spectrum with spikes at particular wavelengths, which we recognize as emission lines. If $T_B < T_C$, then by an analogous argument we get absorption lines.

It is important to note that these spectral lines are not infinitely thin as one might expect. In fact, they are roughly Gaussian in shape. Three big reasons for this are given below.

- **Natural broadening**. Excited states have finite lifetimes Δt, and there is an uncertainty relation in quantum mechanics which relates this to the uncertainty in energy ΔE (which is easily related to $\Delta\lambda$):

$$\Delta E \Delta t \geq \frac{\hbar}{2}.$$

- **Doppler broadening**. If an object's velocity has a radial component with respect to an observer then the light it gives off will be redshifted or blueshifted. Specifically,

$$\frac{\Delta\lambda}{\lambda} \approx \frac{\langle v_r \rangle}{c} = \frac{2}{c}\sqrt{\frac{2kT\ln 2}{m}}.$$

- **Pressure broadening**. As atoms run into each other, they distort each others' electron orbitals and thus their energy eigenvalues. The effects of these collisions are captured by the equation

$$\Delta\lambda \approx \frac{\lambda^2}{c}\frac{n\sigma}{\pi}\sqrt{\frac{2kT}{m}}.$$

In a group of stars that all have the same surface temperatures, larger stars have lower densities near their photospheres and thus narrower lines. So we can use a star's stellar spectrum not only to determine its spectral type, but also to determine its luminosity class! (Some common classes are V for the main sequence, III for giants, and I for supergiants; the Sun is a G2V star.)

The Photosphere

All stellar spectra have absorption lines, which might suggest that stars have some emitting surface upon which there is a cooler atmosphere. But reality is much more complicated—mainly, stars have no well-defined surface. So where does their light come from?

Let's imagine the simplest possible source function that is monotonically increasing: $S_\lambda = a + b\tau_\lambda$. (We might think of this as a Taylor approximation for a more complex source function.) Let τ_λ^* be some point deep inside the star, so we have

$$I_\lambda(0) = I_\lambda(\tau_\lambda^*)e^{-\tau_\lambda^*} + \int_0^{\tau_\tau^*}(a + b\tau_\lambda)e^{-\tau_\lambda}d\tau_\lambda.$$

But stars are opaque, so τ_λ^* is a very good approximation. This gives

$$I_\lambda(0) = \int_0^\infty (a + b\tau_\lambda)e^{-\tau_\lambda}d\tau_\lambda = a + b.$$

So the observed intensity is $I_\lambda(0) = S_\lambda(\tau_\lambda = 1)$. We can, therefore, interpret τ_λ as the "surface" of the star, called the **photosphere**.

The precise configuration of the photosphere is relative to the observer. Consider our view of the Sun as a disk in the sky. If we pick a point near the center of this circle and go τ_λ inward, we'll end up deeper within the Sun than if we had picked a point near the disk's edge. Thus the photosphere actually gets more shallow as the distance from the center of the solar disk increases, and the solar temperature and intensity of emitted light decrease similarly. This phenomenon in which the solar disk gets redder and darker near its edges is called **limb darkening**.

2.3 Stellar Interiors

All this describes what goes on in the "atmosphere" of a star, and it's pretty easy to observe since this is precisely where the light reaching us comes from. We cannot, however, see into the inside of a star, so we'll have to construct a reasonable model to describe it.

Hydrostatic Equilibrium

Let's start with the fairly reasonable requirement that the star does not collapse in on itself due to gravity. The only thing that stops this from happening is pressure—to quantify it, consider a cylindrical chunk of mass whose axis is aligned with the star's radius. The chunk is at a distance r from the center of the star and has mass

$$dm = A\,dr\,\rho(r),$$

where ρ is the mass density of the star and A and dr are the base area and height of the chunk, respectively. If P is the pressure acting on the cylinder, then the equation of the cylinder's radial motion is

$$F_{\text{net}} = P(r)A - [P(r + dr)A + dm\,g(r)],$$

where g is the local acceleration due to gravity. (We need not concern ourselves with the perpendicular directions because those forces cancel each other out.) In equilibrium $F_{\text{net}} = 0$, and we can rearrange to get the **hydrostatic equilibrium equation**

$$\boxed{\frac{dP}{dr} = -\rho(r)g(r)}.$$

Note that we can use Newton's shell theorem to write $g(r) = GM_r(r)/r^2$, where $M_r(r)$ is the mass contained within a distance r of the star's center. But either way we have three unknowns and only one equation. We can add another, the **conservation of mass equation**

$$\boxed{\frac{dM_r}{dr} = 4\pi r^2 \rho(r)},$$

but we're still one short. We'll take some time to discuss other stellar behaviors with the aim of writing down more equations that, hopefully, will comprise a complete model for the interiors of stars.

Let's start with pressure. We know that pressure can arise from the thermal motions of gas molecules, and that their behavior is (with few exceptions) described by the ideal gas law $PV = NkT$, where N is the number of gas molecules. Written in terms of number density, this is

$$P_{\text{gas}} = nkT.$$

To avoid introducing n as a new variable, we can write $n = \rho/\overline{m}$, with the average mass per particle $\overline{m} = \mu m_p$. (m_p is the proton mass and μ is the mean molecular mass). So we have

$$\boxed{P_{\text{gas}}(r) = \frac{\rho(r) kT(r)}{\mu m_p}}.$$

This is the equation of state that's most relevant to main sequence stars. But photons, too, carry momentum and can thus produce pressures that, in the brightest and most massive stars, can be even greater than that produced by gas molecules. As a slight simplification, suppose N photons are incident upon a reflective area A in the star, each with initial momentum p_γ and collision time Δt. The total pressure on the area is

$$P_{\text{rad}} = \frac{\text{force}}{A} = \frac{N \cdot 2p_\gamma}{A\,\Delta t}.$$

Now, by the energy-momentum relation $p_\gamma = E_\gamma/c$ and the definition of flux $NE_\gamma = FA\,\Delta t$,

$$P_{\text{rad}} = \frac{N \cdot 2E_\gamma}{cA\,\Delta t} = \frac{2F}{c}.$$

In terms of temperature, this is $P_{\text{rad}} = 2\sigma T^4/c$. If we were to take into account the different directions from which a photon may collide with the surface, we would instead get

$$\boxed{P_{\text{rad}} = \frac{4}{3}\frac{\sigma T^4(r)}{c}}.$$

Which of these **equations of state** we use depends on the scenario. For some objects, the effects of gas pressure and radiation pressure are comparable enough to use both equations (by adding them).

Energy Generation

We now have a new equation to work with, but we've also introduced a new unknown: temperature. A reasonable next step, then, is to work out where this kinetic energy comes from.

For a while the leading theory was **gravitational contraction**. Stars would start out large and over time shrink to smaller radii, converting gravitational potential energy to kinetic energy in the process. It can be shown that the potential energy of a uniform sphere of mass M and radius R is given by

$$U(R) = -\frac{3}{5}\frac{GM^2}{R},$$

and by the virial theorem the total mechanical energy is

$$E(R) = -\frac{3}{10}\frac{GM^2}{R}.$$

So if a star shrinks from radius R_i to R_f (where $R_i \gg R_f$), then the amount of energy liberated is $\Delta E_g = -E(R_f)$. But there's a problem with this: assuming a roughly constant luminosity over the Sun's lifetime, it implies that the estimated age of the Sun is

$$t_{\text{KH}} = \frac{\Delta E_g}{L_\odot} \sim 10^7 \text{ yr}.$$

This is called the Kelvin-Helmholtz timescale, and it's at least four hundred times smaller than even the estimated age of Earth. So clearly gravitational contraction cannot play a major role in energy production in main-sequence stars, though it is important in earlier and later stages of life.

Today we know that the main source of energy is **nuclear fusion**. Specifically, in the nuclear process

$$4\,\mathrm{H} + 2\,\mathrm{e}^- \longrightarrow \mathrm{He}$$

some amount Δm of it is converted to energy according to $\Delta E = \Delta m\, c^2$. We define the **efficiency parameter** $\eta = \Delta m / m_i$ to quantify this conversion.

Since only about ten percent of the Sun is hot enough to do fusion, we have the nuclear timescale

$$t_{\text{nuc}} = \frac{E_{\text{em}}}{L_\odot} = \frac{\Delta m}{4m_p}\frac{0.1 M_\odot c^2}{L_\odot} \sim 10^{10}\ \text{yr},$$

ignoring the negligible mass of the electrons. This lines up much more nicely with observations on Earth—nuclear fusion, unlike gravitational contraction, works as a mechanism for energy production!

The fusion is not a simple as four protons smashing together to create a helium nucleus. The main reaction chain in Sun-like stars is called the **proton-proton chain**, and it is described below.

$$\begin{aligned}
{}^1_1\mathrm{H} + {}^1_1\mathrm{H} &\longrightarrow {}^2_1\mathrm{H} + \mathrm{e}^+ + \nu_\mathrm{e} \\
{}^2_1\mathrm{H} + {}^1_1\mathrm{H} &\longrightarrow {}^3_2\mathrm{H} + \gamma \\
{}^3_2\mathrm{He} + {}^3_2\mathrm{He} &\longrightarrow {}^4_2\mathrm{He} + 2\,{}^1_1\mathrm{H}
\end{aligned}$$

Here γ is a photon, e^+ is a positron (the antiparticle of an electron), and ν_e is called an electron neutrino. Putting all these together gives

$$4\,{}^1_1\mathrm{H} \longrightarrow {}^4_2\mathrm{He} + 2\,\mathrm{e}^+ + 2\,\nu_\mathrm{e} + 2\,\gamma\,.$$

Note that the positrons will be annihilated with electrons, creating more photons. Reactions like these follow five key conservation laws: conservation of nucleons, charge, lepton number, momentum, and energy. (Neutrinos have a lepton number of 1, and positrons -1.)

How hot must it be for this fusion to occur? The rate-limiting step is the first one, in which an incoming proton must overcome the Coulomb potential of another to enter the strong force's domain of influence. In particular, the kinetic energy of the incoming proton must satisfy

$$K \geq \frac{e^2}{4\pi\varepsilon_0 r_0},$$

where the strong force dominates for $r < r_0$. But $\langle K \rangle = (3/2)kT$, so on average we have

$$T \geq \frac{e^2}{6\pi\varepsilon_0 r_0} \sim 10^{10}\ \mathrm{K}.$$

But it's known that the core temperature of the Sun is closer to 10^7 K, so what gives? Quantum tunneling! The incoming proton is likely to tunnel through the Coulomb potential at temperatures above $T \sim 10^6$ K, which the Sun satisfies.

All this leads us to another equation for our stellar model. Define ε as the energy generated by fusion per unit mass per unit time. In general ε has a very complex form, but for Sun-like cores we can approximate $\varepsilon_{\text{ppc}} \propto \rho T^4$. So we get the **energy conservation** (or **luminosity gradient**) **equation**

$$\boxed{\frac{dL_r}{dr} = 4\pi r^2 \rho(r) \varepsilon(\rho, T)},$$

where $L_r(r)$ is the net energy flow through a sphere of radius r.

Energy Transport

Let's consider one last aspect of stars that will yield an equation that introduces no new unknowns. There are three broad mechanisms by which energy can be transported:

- conduction, in which energy is transferred via particle-particle interactions;
- radiation, in which energy is transferred via photon absorption and emission; and
- convection, which is facilitated by the motion of macroscopic parts of the system.

In regular stars conduction is not important, but the other two play major roles. Let's start with radiative transport. Consider a cylindrical parcel of gas somewhere inside the star with its axis aligned in the radial direction. For energy to be transported outward, there is a net flux through the caps that changes as a function of photon absorption. (Emission is not a factor because it is isotropic.) In particular,

$$F_{\rm net}(r+dr) - F_{\rm net}(r) = F_{\rm net}(r)\left(e^{-d\tau} - 1\right),$$

where $d\tau = dr/l_{\rm mf}$ is the optical depth through the parcel. A first-order Taylor expansion gives

$$\begin{aligned} F_{\rm net}(r+dr) - F_{\rm net}(r) &= -F_{\rm net}(r)\frac{dr}{l_{\rm mf}} \\ \frac{dF_{\rm net}}{dr} &= -F_{\rm net}(r)\kappa\rho, \end{aligned}$$

where κ is the opacity. Now, the radiation pressure on an absorbing surface is $P_{\rm rad} = F_{\rm net}/c$; substituting the derivative gives

$$\begin{aligned} \frac{dP_{\rm rad}}{dr} &= -\frac{\kappa\rho}{c}F_{\rm net}(r) \\ \frac{dP_{\rm rad}}{dr} &= -\frac{\kappa\rho}{c}\frac{L_r(r)}{4\pi r^2} \end{aligned}$$

Finally, substituting the derivative of $P = (4/3)(\sigma T^4/c)$ gives

$$\boxed{\frac{dT}{dr} = -\frac{3}{16}\frac{\kappa\rho}{\sigma T^3}\frac{L_r(r)}{4\pi r^2}}$$

This is the **radiative energy transport equation**. The relationships between all these variables should make sense: a large $L_r(r)$ causes a large dT/dr to drive more radiation outward, and a large opacity requires a large dT/dr to drive the same L_r. (Opacity acts as the "friction" for radiation.)

Let's move on to convection. Consider another cylindrical bubble of gas with density and temperature ρ_B, T_B; the surroundings have ρ_S, T_S. Our gaseous equation of state says that $P \propto \rho T$, so in order for the bubble to maintain a constant pressure relative to its surroundings, any change in temperature must be matched with a multiplicatively equal and opposite change in density. If $T_B > T_S$, then $\rho_B < \rho_S$ and the bubble rises.

As the bubble rises, we have three scenarios for the relative changes in temperature and density.

- Vigorous convection. The bubble's temperature drops more slowly than that of its surroundings, so it continues to rise with increasing speed.
- Limited convection. The bubble's temperature drops more quickly than that of its surroundings, so its rise eventually slows to a stop.
- Marginal convection. The bubble's temperature drops at the same rate as that of its surroundings.

But all forms of convection move heat around in such a way that promotes marginal convection, so we can take

$$\frac{dT_B}{dr} = \frac{dT_s}{dr}.$$

Now, from thermodynamics it is known that $P_B = aT_B^{\gamma/(\gamma-1)}$, where a is some constant and γ is called the adiabatic exponent of the gas, which depends only on its composition—no heat is exchanged with its surroundings. (For a monatomic ideal gas, $\gamma = 5/3$.) We can differentiate this expression to get

$$\frac{dP_B}{dr} = a\left(\frac{\gamma}{\gamma-1}\right)\frac{T_B^{\gamma/(\gamma-1)}}{T_B}\frac{dT_B}{dr} = \frac{P_B}{T_B}\frac{\gamma}{\gamma-1}\frac{dT_B}{dr}.$$

Since the bubble is always in pressure equilibrium with its surroundings, we can define $P = P_B = P_S$. Also, in practice our bubble is only going to be slightly hotter than its surroundings, so we can approximate $T = T_B \approx T_S$ to get

$$\boxed{\frac{dP}{dr} = \frac{P}{T}\frac{\gamma}{\gamma - 1}\frac{dT}{dr}}.$$

This is the **convection energy transport equation**, and it's the last one we need to get a complete set of differential equations.

Convection, of course, will not always occur—the temperature gradient in the star must be greater than that in a bubble. In particular,

$$\begin{aligned} \frac{dT}{dr} &< \frac{\gamma - 1}{\gamma}\frac{T}{P}\frac{dP}{dr} \\ \frac{dT}{T} &< \frac{\gamma - 1}{\gamma}\frac{dP}{P} \\ d(\ln T) &< \frac{\gamma - 1}{\gamma} d(\ln P) \end{aligned}$$

Since both differentials are negative,

$$\boxed{\frac{d(\ln P)}{d(\ln T)} < \frac{\gamma}{\gamma - 1}}$$

Convection occurs if and only if this condition is satisfied. In this case, since convection is much more efficient than radiation, we may use only the convection equation. Otherwise, we'll have to stick to radiation.

2.4 Building Stellar Models

We now have a complete set of equations that can be solved to obtain functional forms for the key properties of a star. At $r = 0$ we have the initial conditions $M_r = L_r = 0$ and at $r = R$ we have $\rho = T = 0$.

According to the **Vogt-Russel theorem**, the mass and composition structure of a star completely determines its radius, luminosity, internal structure, and subsequent evolution. For example, the mass determines the energy transport structure of a star.

- Smaller and cooler stars tend to have more neutral atoms and thus higher opacities in their outer layers. This leads to a steeper temperature gradient and thus convection. The coolest stars exhibit full convection over its entire radius.
- More massive stars tend to have higher luminosities and thus higher temperatures in their cores. This once again leads to a steeper temperature gradient and convection in the inner layers.

Mass also determines the nature of the nuclear fusion in stellar cores. High temperatures facilitate the carbon-nitrogen-oxygen cycle, which fuses hydrogen into helium with a much higher energy generation rate.

We can even predict the lifetime of a star using its mass! For main-sequence stars we have the relations

$$\frac{L}{L_\odot} = \left(\frac{M}{M_\odot}\right)^{3.5} \quad \text{and} \quad t_{\text{nuc}} = \frac{0.1\eta M c^2}{L},$$

where η is the efficiency of nuclear fusion in a mass-M star with luminosity L. If we take the star's nuclear lifetime t_{nuc} and divide by that of the Sun we get

$$\frac{t_{\text{nuc}}}{t_{\text{nuc},\,\odot}} = \frac{M}{M_\odot}\frac{L_\odot}{L} = \left(\frac{M}{M_\odot}\right)^{-2.5}.$$

So the star's fusion lifetime is inversely related to its mass!

It will be fruitful to put bounds on what a star's mass can physically be. On the lower end the limiting factor is temperature—a main-sequence star must be hot enough to facilitate nuclear fusion. This depends a bit on composition. Let Z be the mass proportion of "metals" in a star (so X and Y are the hydrogen and helium proportions); a metal-rich star with $Z = 0.02$ has $M_{\rm min} \approx 0.072 M_\odot$, while a metal-poor one with $Z = 0.0$ has $M_{\rm min} \approx 0.09 M_\odot$.

The upper bound is set by radiation pressure. Consider another cylindrical parcel of gas that contains only N hydrogen atoms. For a very massive star, temperatures are high enough for hydrogen gas to be completely ionized at all radii. The gravitational force on the parcel, significant only for the protons, is

$$F_{\rm g} = \frac{GM(Nm_p)}{R^2},$$

and the force due to radiation pressure, applicable only to the electrons, is

$$F_{\rm rad} = \frac{4}{3}\frac{F}{c}\sigma_{\rm tot} = \frac{4}{3}\frac{L}{4\pi R^2 c}N\sigma_T,$$

where the Thompson cross section σ_T is that for the interaction between electrons and photons. Though these two forces act on different sets of particles, the protons and electrons are bound together by the Coulomb force, so they move as a unit.

For the star to stay together we must have $F_{\rm rad} < F_{\rm g}$, so

$$\frac{L}{3\pi R^2 c}N\sigma_T < \frac{GM_\star N m_p}{R^2} \implies L < \frac{3\pi G c m_p M}{\sigma_T}.$$

This is called the **Eddington luminosity**. Any brighter star will blow material off from its surface. Skipping many details, it follows that $M_{\rm max} \sim 70 M_\odot$.

3 Stellar Evolution

3.1 Pre-main-sequence Evolution

Now we'll turn our attention to stellar evolution before the main-sequence stage. The most basic prerequisite for these processes is the presence of matter that is not already bound up in stars. This matter comprises the **interstellar medium**, and it is mostly made up of hydrogen and helium with a very small amount of metals.

The Interstellar Medium

Studying the properties of the interstellar medium is a first step toward understanding stellar formation. We can do this by focusing on the obscuration of light, produced by interstellar dust particles. These particles, usually on the order of 100 nanometers, are composed mostly of silicon and carbon. These particles absorb starlight; the amount of absorption is quantified by the **extinction**

$$A_\lambda = m_\lambda - m_{\lambda,0},$$

where m_λ is the observed magnitude and $m_{\lambda,0}$ is the magnitude that would've been observed without absorption. We refer to extinction in magnitudes—for example, $A_\lambda = 10$ corresponds to "extinction of ten magnitudes". By definition,

$$A_\lambda = 2.5 \log_{10}\left(\frac{F_{\lambda,0}}{F_\lambda}\right).$$

The cloud is cold, so we can ignore emission and approximate $I_\lambda = I_{\lambda,0}e^{-\tau_\lambda}$. Now, if nothing special is happening at each solid angle when we also have $F_\lambda = F_{\lambda,0}e^{-\tau_\lambda}$; substituting gives

$$A_\lambda = \tau_\lambda(2.5\log_{10} e) \simeq 1.086\tau_\lambda.$$

We'll just say that $A_\lambda \simeq \tau_\lambda$. But $\tau_\lambda = \int_0^{s_c} n_d(s')\sigma_\lambda ds'$, where n_d is the number density of dust particles and σ_λ is the cross-section of one dust particle; assuming constant σ, this gives $A_\lambda \simeq \sigma_\lambda N_d$, where N_d is the "column density" of dust grains.

Since we can't directly measure the hypothetical $m_{\lambda,0}$, to measure extinction we must rely on the fact that it depends on λ. In particular, the cross section of a size-a dust particle looks like

$$\sigma_\lambda \propto \begin{cases} a^2 & \lambda \ll a \\ a^2\left(\frac{a}{\lambda}\right) & \lambda \sim a \\ 0 & \lambda \gg a \end{cases}$$

Given the size of a typical interstellar dust particle, σ_λ varies as $1/\lambda$ in the visible and near-IR range. There's a couple of notable consequences of this fact. First, blue light is more readily scattered than red light, so stars behind dust clouds look reddened. Second, dust clouds next to bright sources look blue since they redirect scattered light toward us; such clouds are called reflection nebulae.

Aside from thermal emission from dust and atomic emission lines from hot regions around stars, there are three key ways emission can occur in the interstellar medium. All have their origins in quantum mechanics.

- Molecular bonds can be modeled using a quantum harmonic oscillator. Each energy level here is separated by an amount $\hbar\omega_0$, where $\omega_0 = \sqrt{k/\mu}$ is a physical property of the potential well. So this is the energy of a photon emitted after a fall between adjacent states, called a **vibrational transition**.
- A molecule's angular momentum is quantized, meaning its rotational energy is also quantized with

  $$E = \frac{j(j+1)\hbar^2}{2I},$$

  where $j = 0, 1, \ldots$ and I is the rotational inertia. A fall between adjacent states (called a **rotational transition**) involves the emission of a photon with energy $E_\gamma = \hbar^2(j+1)/I$, where j indexes the lower state. Note that rotational transitions can only occur in asymmetric molecules—that is, molecules with a permanent electric dipole moment.

- The energy of an individual hydrogen atom is influenced by its spin configuration. A state in which the spins of the proton and electron match has a higher energy than one in which the spins differ. The excited state has a lifetime of $\sim 10^6$ years, but when a transition does occur it emits a 21-cm photon.

Combining extinction and emission information allows us to determine the ratio of gas and dust in the galaxy. In particular, the ratio of hydrogen atoms to dust grains is on the order of 10^{11}. Note that this is a number ratio—a mass ratio would be much smaller.

Clouds in the interstellar medium can be broken up by radius into three relatively fuzzy zones.

- In the inner zone, matter is shielded from starlight and cosmic rays and molecules are allowed to form—this is where density is the highest and temperature is the lowest. Emission here is dominated by molecular lines.
- The outer zone is dominated by HII, fully ionized hydrogen. Here density is at its lowest, temperature at its highest, and atomic emission lines at their strongest.
- In the middle we see mostly HI and dust. Emission is dominated by the 21-cm line and blackbody emission from warm dust.

The formation rate of carbon monoxide is a very sensitive function of density and temperature, so we can infer the total mass of a cloud's core based on CO observations.

Gravitational Collapse

New stars form from these interstellar clouds via **gravitational collapse**. Consider a uniform cloud with mass M and radius R; in the spirit of the virial theorem, we say that a cloud will collapse in on itself if

$$\langle K \rangle < -\frac{\langle U \rangle}{2}.$$

In other words, gravity wins out over the thermal motions of the cloud's molecules. We can rewrite this:

$$\begin{aligned} \frac{3}{2}NkT &< \frac{3}{5}\frac{GM^2}{2R} \\ \frac{M}{\mu m_p}kT &< \frac{1}{5}\frac{GM^2}{R} \end{aligned}$$

But this R secretly has an M in it. Specifically , $R^3 = 3M/(4\pi\rho)$, so

$$5\left(\frac{3M}{4\pi\rho}\right)^{\frac{1}{3}}\frac{kT}{\mu m_p G} < M$$

Solving for M,

$$M^{\frac{2}{3}} > \frac{5kT}{\mu m_p G}\left(\frac{3}{4\pi\rho}\right)^{\frac{1}{3}}$$

$$\boxed{M > \left(\frac{5kT}{\mu m_p G}\right)^{\frac{3}{2}}\left(\frac{3}{4\pi\rho}\right)^{\frac{1}{2}}}$$

This is called the **Jeans mass** M_J. Any higher and we expect to see gravitational collapse. But not all the collapsing mass goes into the same star—instead, **fragmentation** occurs. As gravitational collapse takes its course, density increases and the Jeans mass decreases. Thus smaller pockets of matter begin to collapse in on themselves individually, which themselves spawn smaller collapsing pockets, and so on. Many of these pockets eventually become hot enough to facilitate nuclear fusion and become main-sequence stars!

3.2 Post-main-sequence Evolution

We've qualitatively described how stellar formation works, and we've spent plenty of time talking about main-sequence stars. Now it's time to get into stellar evolution after a star leaves the main sequence! We'll see that this evolution is driven almost entirely by stellar composition.

The big changes begin in the core. At some point, it runs out of hydrogen and is no longer able to facilitate nuclear burning. But the core is still hotter than it surroundings and so loses energy to the outside; to maintain equilibrium, gravitational contraction ensues. By the virial theorem, about half of the liberated energy goes into the kinetic energy of the core and the rest gets absorbed by the envelope, making it less dense and cooler.

The gravitational contraction of the core leads to increased temperatures, which facilitates new nuclear reactions that produce heavier elements. For example, we have the triple-α process:

$$
\begin{aligned}
{}^4_2\text{He} + {}^4_2\text{He} &\longleftrightarrow {}^8_4\text{Be} \\
{}^8_4\text{Be} + {}^4_2\text{He} &\longrightarrow {}^{12}_6\text{C} + \gamma
\end{aligned}
$$

This beryllium is incredibly unstable and it decays very quickly, so this reaction essentially requires three things to collide with each other in very short succession. Thus energy is released according to $\varepsilon \propto \rho^2 T^{41}$. But after this things are much simpler—once we obtain ${}^{12}_6\text{C}$ we can keep adding ${}^4_2\text{He}$ nuclei to make heavier elements like oxygen-16 and neon-20.

The amount of energy involved in a nuclear reaction is determined by the **binding energy**

$$E_B = [Zm_p + (A - Z)m_n - m_{\text{nuc}}]c^2.$$

As we move up the chain of nuclear fusion, the amount of binding energy gained per nucleon decreases until we get to iron-56. This is where E_B/A reaches a maximum—there is nothing else the star can do to release energy, so this is where fusion really stops.

Low-mass Evolution

With this big picture out of the way, we can track the specifics of post-main-sequence evolution in low-mass stars (ones with masses smaller than $8M_\odot$). We'll focus in particular on a star with mass $5M_\odot$.

After hydrogen burning in the core ceases, gravitational contraction takes over as the primary means of energy release. This energy goes into increasing the temperature of the surrounding envelope, so there is a shell of hydrogen burning surrounding the core. This increases the mass of the inert helium core.

At this point there is no fusion supplying a temperature gradient, so the only thing supporting the core against itself is a density gradient. This is a precarious situation to be in, and eventually the core becomes massive enough to begin collapsing in on itself and the star enters the **subgiant branch**. At first the entire star contracts—density and temperature increase together, which causes an abrupt increase in fusion energy output. But this energy can't be transported to the exterior of the star quickly enough, so it goes back into envelope expansion. This results in a much larger, much redder star.

Eventually the stellar envelope becomes convective again, expansion slows, and the star enters the **red giant branch**. Most things continue as they have—the inert helium core continues to contract and hydrogen continues to burn in a surrounding shell. The envelope continues to expand in such a way that its temperature remains basically constant, and the increased energy transport increases the luminosity of the star.

The core eventually reaches a high enough temperature and density that helium fusion can begin. The threshold is accompanied by a spectacular flash of light lasting for only a few seconds. So the core begins expanding once again, the hydrogen-burning shell cools, and the star's luminosity decreases in response to envelope contraction. The star then enters the **horizontal branch**, which is essentially the hydrogen-burning analog of the main sequence.

But eventually the core becomes exhausted of helium, too, and the remaining carbon-oxygen core begins to contract. The helium-burning shell expands; so does the hydrogen-burning shell, but fusion there eventually stops altogether.

So the star enters the **asymptotic giant branch**, which is full of activity. As the envelope temperature rises, the hydrogen burning shell eventually reignites. The narrowing helium shell, however, turns on and off somewhat periodically. Ignitions are accompanied by shell flashes and the temporary turning-off of the hydrogen shell. The resulting thermal pulses also cause convection to reach deep into the star, dredging up carbon-rich material toward the surface.

As this occurs, the core continues to contract and the envelope continues to expand. Eventually the envelope is lost completely and becomes a **planetary nebula**, a beautiful cloud of (optically thin) glowing gas named for its visual similarity to giant gaseous planets when viewed through small telescopes. The carbon-oxygen core is left to cool off and become a **white dwarf**—more on that later.

High-mass Evolution

Evolution in higher-mass stars looks very similar until the carbon-oxygen core forms. Rather than decay into a white dwarf, a high-mass core is hot and dense enough to facilitate more fusion. The subsequent evolution is much more dramatic!

In very late stages of life these stars develop an onion-like structure. The deeper layers harbor less efficient nucleosynthesis as the binding curve gets more shallow for heavier elements. As a result, the timescales for each burning stage decrease rapidly—for a $20M_\odot$ star, the main-sequence timescale is 10^7 years while the silicon-burning timescale is just two days!

As this fusion declines, pressure is no longer enough to hold the star up against its own gravity and **electron degeneracy pressure** takes over. Qualitatively, as the core continues to heat and shrink as energy leaks out to the rest of the star, all the electrons become much more tightly packed together; this reduces the uncertainty in a given electron's position, thus increasing that in its momentum. We'll revisit this in more detail later.

While all this is happening, the silicon-burning shell is depositing more mass onto the iron core. Once it grows to around $2M_\odot$, not even degeneracy can keep it stable. Two important processes take over.

- **Photodisintegration**. When an iron or helium atom absorbs a high-energy γ-ray, one of the following two decay reactions occurs.

$$\begin{aligned} {}^{56}_{26}\mathrm{Fe} + \gamma &\longrightarrow 13\,{}^{4}_{2}\mathrm{He} + 2\,\mathrm{n} \\ {}^{4}_{2}\mathrm{He} + \gamma &\longrightarrow 2\,\mathrm{p} + 2\,\mathrm{n} \end{aligned}$$

  In this way, millions of years of nucleosynthesis are undone in an instant.

- **Neutronization**. At high densities, a proton may "capture" an electron and fuse according to

$$\mathrm{p} + \mathrm{e} \longrightarrow \mathrm{n} + \nu_\mathrm{e},$$

  where ν_e is an electron neutrino.

All of these reactions are endothermic, but most of the converted energy escapes the star via neutrinos. Also note that neutronization rids the core of all of its electrons, meaning there's nothing left to support the core against itself—the result is an immediate collapse of the core over the course of several seconds.

Neutron degeneracy provides one final line of defense. If the core's mass is large enough to overcome this degeneracy, collapse continues until we have a black hole. (More on that later.) If degeneracy does its job, though, we end up with a **neutron star**. This is an extremely high-density object comprised entirely of neutrons (with a smattering of protons and neutrons) held together by gravity. This neutron star is very "stiff", though, so the infalling layers of the dying star bounce back. The collapse becomes an expansion. Further, the near-core layers are dense enough to capture a significant amount of neutrino energy as kinetic energy, turning the expansion into a shockwave that completely disrupts the rest of the star. This is a **supernova**!

This process describes the formation of type II, Ic, and Ib supernovae. The difference resides in whether its progenitor retained its hydrogen envelope before exploding. Type Ia supernovae are completely different—they're formed via the collision of stars in a binary system, one of which is a white dwarf.

Once a supernova forms, its magnitude decreases linearly over time. In particular,

$$\log_{10} L \propto -\lambda t \implies L \propto e^{-\lambda t}.$$

It turns out that radioactive decay is what keeps supernovae alight for years to come. We can even use the lightcurve of the supernova to determine what element dominates the decay!

3.3 Tests of Stellar Evolution

Our discussion of stellar evolution so far has been based primarily in theory. There's a variety of ways in which we might test our theories against observations!

Take, for example, **spectroscopic binaries**. We've already discussed how to get the oscillation period, orbital velocities, masses, and semimajor axes from spectroscopic data. But if we observe eclipses, too, that means we're observing the binary basically edge-on and there is no longer any ambiguity in the stars' masses! From there we can get stellar radii and temperatures to check against our models.

We also have neutrinos, whose characteristic lack of significant interaction with matter provides a nice glimpse into the central regions of stars. Neutrinos are produced whenever a proton is converted into a neutron or vice versa, in order to converse lepton number. They are highly relativistic particles whose energies vary depending on the reaction. The lower-energy neutrinos ($E_\nu \leq 0.4$ MeV) produced by the proton-proton chain can be detected using gallium or chlorine, but higher-energy ones (~ 10 MeV) produced by less likely reaction pathways can interact directly with electrons, so we can simply use water! But no matter what the detector is made out of, it needs to be huge—otherwise we might miss out on the very few neutrinos that reach Earth after, say, a supernova.

Observations of solar neutrinos turn out to detect roughly half of the expected number. This resulted in a decade-long fight between astronomers and particle physicists that culminated in the discovery of **neutrino oscillations**. As it turns out, while neutrinos travel they are in a quantum superposition of three flavors that individually evolve on scales of thousands of kilometers. The cross sections of muon and tau neutrinos are each smaller than that of an electron neutrino, there is an observed deficit of the first two flavors.

Moving along, it turns out that stars expand and contract periodically as a whole. These **solar oscillations** come in many modes, the simplest of which is the fundamental mode: a very simple pattern of expansion and contraction. The period of this oscillation is $t_{\text{osc}} = 2R_\odot / v_s$, where v_s is the gaseous speed of sound. Now,

$$\frac{1}{2}m_p v_s^2 \sim \frac{3}{2}kT \implies v_s \sim \sqrt{\frac{3kT}{m_p}}.$$

For the Sun, the frequency of this oscillation is on the order of 1 mHz. (Higher-order oscillations have higher frequencies.) In general, oscillations can be decomposed into spherical harmonics Y_{l,m_l}, where l represents the number of nodes in latitude and m represents that in latitude.

Finally, we can consider the abundance of elements in the solar photosphere. (These are by numbers of atoms, but by mass.) Elements with masses below nickel are produced through nuclear fusion in stellar cores, mostly in massive stars—even atomic numbers are emphasized since fusion involves the capture of a helium nucleus. Heavier elements are produced in the envelopes of asymptotic giants and during supernovae via neutron capture and β-decay. This process can be visualized particularly well using an isotope table, which has proton number on the vertical axis and neutron number on the horizontal. A nuclear process is represented by a jump between entries on the table—neutron capture causes horizontal movement while β decay causes diagonal movement.

3.4 White Dwarfs and Neutron Stars

Immediately after collapse a star may turn into a white dwarf, a neutron star, or a black hole.

White Dwarfs

White dwarfs are hot and incredibly dense objects. We'll try to understand the electron degeneracy pressure supporting them against gravity. Let's start by computing the density at the center of the star using the equation

$$\frac{dP}{dt} = -\rho g = -\rho \frac{GM_r}{r^2}.$$

Assuming constant density gives $M_r = \rho(4/3)\pi r^3$. Substituting this and solving with $P(R) = 0$ gives

$$\boxed{P(0) = \frac{2}{3}\pi\rho^2 GR^2}.$$

Applying the ideal gas law $\rho kT/\mu m_p$ gives $T \sim 10^9$ K, which is way too big for a carbon-oxygen interior. This confirms that gas pressure cannot be the source of pressure here.

Now we'll determine the energy associated with this degeneracy. The average value of an electron's momentum p_x can't be much larger than the uncertainty, so by the Heisenberg uncertainty principle $p_x \sim \hbar/2\Delta x$. Now, we may interpret the Pauli exclusion principle as saying that the electrons' wave functions cannot overlap and that Δx is on the order of the separation between an electron's nearest neighbors. Thus

$$\Delta x \sim \sqrt[3]{\frac{L^3}{N_e}} = n_e^{-1/3},$$

where n_e is the number density of electrons, and the total electron momentum is on average $p \sim \hbar n_e^{1/3}$. Thus the average energy $\overline{\mathcal{E}} = p^2/2m$

$$\boxed{\overline{\mathcal{E}} \sim \frac{\hbar^2}{2m} n_e^{2/3}}.$$

A more careful quantum mechanical calculation gives the maximum and average energies

$$\mathcal{E}_F = \frac{\hbar^2}{2m}\left(3\pi^2 n_e\right)^{2/3}, \quad \overline{\mathcal{E}} = \frac{3}{5}\mathcal{E}_F.$$

$\mathcal{E}_F$ is called the **Fermi energy**. But $n_e = (Z/A)(\rho/m_p)$, where Z/A is the ratio of atomic number to mass number, so

$$\mathcal{E}_F = \frac{\hbar^2}{2m_e}\left(3\pi^2 \frac{Z}{A}\frac{\rho}{m_p}\right)^{2/3}.$$

(For a carbon-oxygen interior, $Z/A = 1/2$.) If the Fermi energy is much larger than the the thermal energy of an electron—that is, if $\mathcal{E}_F \gg (3/2)kT$—then the electron cannot transition into an unoccupied quantum state and most of the gas's pressure comes from degeneracy. In this case, it is called a **degenerate gas**.

The pressure in this gas is proportional to $\overline{\mathcal{E}} n_e \propto n_e^{5/3} \propto \rho^{5/3}$. An exact calculation would give

$$\boxed{P_{\text{deg,e}} = \frac{(3\pi^2)^{2/3}}{5}\frac{\hbar^2}{m_e}\left(\frac{Z}{A}\frac{\rho}{m_p}\right)^{5/3}}.$$

We can use this result to get some more information about the structure of the white dwarf! Substituting it into the equation of hydrostatic equilibrium gives a differential equation that has no analytic solution, but if we assume constant $\rho = M/(\frac{4}{3}\pi R^3)$ we can take $P_{\text{deg,e}} = P(0)$, so

$$\frac{(3\pi^2)^{2/3}}{5}\frac{\hbar^2}{m_e}\left(\frac{Z}{A}\frac{\rho}{m_p}\right)^{5/3} = \frac{2}{3}\pi\rho^2 G R^2.$$

Substituting the density and solving for R gives the **mass-radius relation**

$$\boxed{R_{\text{WD}} = \frac{(18\pi)^{2/3}}{10}\frac{\hbar^2}{m_e G}\left(\frac{Z}{A}\frac{1}{m_p}\right)^{5/3} M_{\text{WD}}^{-1/3}}.$$

So counterintuitively, there is an inverse relationship between the mass and radius of a white dwarf! This prediction seems to fit with the data pretty well, but white dwarfs also seem to have a maximum mass above which none have been observed.

In high-mass white dwarfs ($\sim 1M_\odot$) the average energy of an electron is comparable to the the electron mass energy, meaning relativistic effects must be accounted for. Unfortunately this is difficult to do unless the electron is ultra-relativistic, in which case the relativistic mass-energy relation gives $\mathcal{E} = pc \sim \hbar c n_e^{1/3}$. Thus $P_{\text{deg,e}} \sim \hbar c n_e^{4/3}$. Another careful calculation would have given

$$P_{\text{deg,e}} = \frac{(3\pi^2)^{1/3}}{5}\hbar c\left(\frac{Z}{A}\frac{\rho}{m_p}\right)^{4/3},$$

which we could use with the equation of hydrostatic equilibrium to get

$$M_{\text{ch}} = \frac{3\sqrt{2\pi}}{8}\left(\frac{\hbar c}{G}\right)^{3/2}\left(\frac{Z}{A m_p}\right)^2 \approx 0.44 M_\odot.$$

An even better calculation would have taken into account variable density to get a different constant out front, and from that the **Chandrasekhar mass** $M_{\text{ch}} = 1.4M_\odot$. This is the maximum allowed mass for the white dwarf. The fact that the radius is absent here indicates that any perturbation toward a smaller radius will not result in sufficient restoring force to counteract the perturbation, so the star begins to collapse.

Neutron Stars

During this collapse, neutronization occurs and we're left with a neutron star held up by neutron degeneracy. The resulting degeneracy pressure is analogous to what we saw with electrons:

$$P_{\text{deg,n}} = \frac{(3\pi^2)^{2/3}}{5}\frac{\hbar^2}{m_n}\left(\frac{\rho}{m_n}\right)^{5/3}.$$

The mass-radius relation, too, is similar:

$$R_{\text{NS}} = \frac{(18\pi)^{2/3}}{10}\frac{\hbar^2}{Gm_n^{8/3}}M_{\text{NS}}^{-1/3}.$$

For $M_{\text{NS}} \sim M_{\text{ch}}$ we get $R_{\text{NS}} \sim 4.4$ km, which is again too small due to the constant-density approximation. Current theories estimate radii around 10 or 20 km, which still gives $\overline{\rho}_{\text{NS}} \sim 7\times10^{17}$ kg/m^3. This is an absurdly high density comparable to that of an atomic nucleus, just with 10^{19} the radius. But there's once again an upper limit to the mass. Beyond this, nothing can stop the collapse and we get a black hole, which we'll discuss later.

Neutron stars provide some of the most extreme conditions in the Universe. For example, they come with very high escape velocities. Consider a neutron star with mass M_{NS} and radius-R_{NS}; the critical velocity v_{esc} required for a mass m to escape the star's gravitational pull satisfies

$$\frac{mv_{\text{esc}}^2}{2} - \frac{GM_{\text{NS}}m}{R_{\text{NS}}} \implies v_{\text{esc}} = \sqrt{\frac{2GM_{\text{NS}}}{R_{\text{NS}}}}.$$

For some neutron stars, this amounts to $v_{\text{esc}} \simeq 0.7c$. By symmetry, this also means that an incoming object with negligible initial velocity is accelerated to this speed by the time it hits the surface! The efficiency parameter of the energy generated upon impact is $K_{\text{imp}}/mc^2 \simeq 0.2$, so if the neutron star is accompanied by another star the two can form a powerful source of x-rays.

Neutron stars can also have some very low rotational periods. We'll first put an upper bound on the translational speed v_{crit} can have before flying off of a mass-M, radius-R body:

$$\frac{GMm}{R^2} = \frac{mv_{\text{crit}}^2}{R} \implies v_{\text{crit}} = \sqrt{\frac{GM}{R}}.$$

Thus the lowest rotation period an object can have without breaking up is

$$P_{\min} = 2\pi\sqrt{\frac{R^3}{GM}}.$$

For the above neutron star this is 0.5 ms; for an equivalent-mass white dwarf, it's more like 5 s. To find a more reasonable estimate of the average neutron star's rotational period, though, we can investigate the conservation of angular momentum during a white dwarf's collapse. This would give

$$\frac{\omega_f}{\omega_i} = \frac{P_{\text{WD}}}{P_{\text{NS}}} = \frac{f_i}{f_f}\frac{M_i}{M_f}\left(\frac{R_i}{R_f}\right)^2,$$

where each f is some mass distribution factor. If the differences between the two factors and the two masses are negligible, then we're just left with the radii. Taking $P_{\text{WD}} = 2.6\times10^3$ s, $R_{\text{WD}} = 5\times10^6$ m, and $R_{\text{NS}} = 10^4$ m gives 10 ms. Still very fast, but a little less extreme.

One final extremity can be seen in magnetic fields. When stellar cores collapse, another conserved quantity is magnetic flux. A combination of the decreased neutron star radius and some other amplification processes can produce magnetic fields on the order of 10^7 T to 10^{11} T. Gamma rays are produced at the magnetic poles, and since these poles aren't quite aligned with the spin axis, the pulse around at a steady frequency.

3.5 Black Holes and General Relativity

Let's return to escape velocity. The radius at which the escape velocity exceeds the speed of light is called the **Schwarzschild radius**, and it is given by

$$c = \sqrt{\frac{2GM}{R_{\text{sch}}^2}} \implies \boxed{R_{\text{sch}} = \frac{2GM}{c^2}}.$$

A more careful calculation would require some general relativity, but it turns out to give us the same numerical factor out front. Note that $R_{\rm sch} = 3~\rm km$ for a solar mass, and that $R_{\rm sch}$ grows linearly in M. When a neutron star collapses into a black hole its mass simply seems to accumulate in a singularity, so we interpret $R_{\rm sch}$ as the radius of the black hole's **event horizon**. No information from the interior can escape.

To understand this better we must discuss general relativity, starting with the **equivalence principle**. The idea is that acceleration is indistinguishable from gravity—a person in a spaceship accelerating at g would feel the exact same thing as if they were sitting at rest on the surface of the Earth. This is a very simple and reasonable idea, but the consequences are stunning!

Consider a length-h rocket that is initially at rest and begins accelerating with magnitude a. At the same time, a frequency-$\nu_{\rm em}$ laser beam is shot from the following end of the rocket toward an observer on the other end. If the rocket is very long, then the beam is detected a time $\Delta t = h/c$ after emission, and by this time the observer has velocity $a\Delta t = ah/c$. But the light, of course, is redshifted according to $\Delta/\lambda \simeq v/c$, so we have

$$\frac{\nu_{\rm em} - \nu_{\rm obs}}{\nu_{\rm em}} = \frac{ah}{c^2} \implies \nu_{\rm obs} = \nu_{\rm em}\left(1 - \frac{ah}{c^2}\right).$$

This is just the first-order approximation of the full picture, though, which is $\nu_{\rm obs} = \nu_{\rm em}\left(1 - 2ah/c^2\right)^{1/2}$. But by the equivalence principle, this is also true at rest in a gravitational field! Thus

$$\boxed{\frac{\nu_\infty}{\nu_0} = \left(1 - \frac{2GM}{R_0 c^2}\right)^{1/2} = \left(1 - \frac{R_{\rm sch}}{R_0}\right)^{1/2}},$$

where R_0 is the radius from the gravity well's center at which the beam is emitted or detected. This phenomenon is known as **gravitational redshift**—light loses energy as it goes up a gravity well.

Now, the oscillations in the laser beam provide a convenient way to measure time! At R_0 the period of oscillation is $\Delta t_0 = 1/\nu_0$, and at $R = \infty$ it's $\Delta t_\infty = 1/\nu_\infty$. Substituting these gives

$$\boxed{\frac{\Delta t_0}{\Delta t_\infty} = \left(1 - \frac{2GM}{R_0 c^2}\right)^{1/2} = \left(1 - \frac{R_{\rm sch}}{R_0}\right)^{1/2}},$$

the expression for **gravitational time dilation**. Since this expression is less than one, cycles higher up along the gravity well are longer than those deeper in the well.

The observer at infinity, looking down at an object deep in the well, would see time for that object pass in slow motion. Once the object reaches the event horizon, its time simply appears to stop. But for the observer at the event horizon nothing special is happening, aside from the tidal forces that would be ripping them to shreds (in the case of low-mass black holes).

It will be fruitful to understand how, exactly, black holes affect spacetime. These objects can be completely characterized by two quantities: mass and angular momentum. Any other properties' effects are contained within the event horizon so they don't influence the rest of the universe.

In the simplest case we assume that there is zero mass and angular momentum. This reduces our problem to special relativity, which uses the **Minkowski metric**

$$ds^2 = -(c\,d\tau)^2 = -(c\,dt)^2 + dx^2 + dy^2 + dz^2,$$

where τ is the proper time between two events separated by a spacetime interval ds. From here we immediately get an equation describing special relativistic time dilation: $d\tau = dt\sqrt{1 - v^2/c^2}$.

If we generalize slightly and only require that there is no angular momentum, then we get the **Schwarzschild metric**

$$ds^2 = -(c\,d\tau)^2 = -\left(1 - \frac{R_{\rm sch}}{r}\right)(c\,dt)^2 + \frac{(dr)^2}{1 - R_{\rm sch}/r} + (r\,d\theta)^2 + (r\sin\theta\,d\phi)^2,$$

this time in spherical coordinates. Note that in this case the proper time τ between two events is not unique because there can be different world lines connecting them. Also, when $dr = d\theta = d\phi = 0$ we recover the equation describing gravitational time dilation (with $\tau = t_0$ and $t = t_\infty$)!

It appears that $r = R_{\rm sch}$ produces a singularity, but this is just a quirk of our coordinate system. It's analogous to how the latitude-longitude system on Earth breaks down at the poles. But when $r < R_{\rm sch}$ the signs on

$c\,dt$ and dr swap, so the roles of space and time effectively switch places! Thus the singularity becomes more of a time than a place, and there's no choice but to move toward it. This explains why black holes collapse to singularities.

All of the above describe **Schwarzschild black holes**, which have no angular momentum. If we add angular momentum to the picture we get a **Kerr black hole** and the metric becomes much more complex. The important pointers are that $L_{\max} = GM^2/c$ is the maximum allowed angular momentum, the horizon radius is smaller, and there is an "ergosphere" in which matter must rotate with the black hole.

Detecting Black Holes

Black holes can be difficult to detect since we can't directly observe them, but there are some indirect ways to study them.

One obvious method is to observe matter that may be orbiting the black hole in an accretion disk. Over time this matter loses gravitational energy and slowly spirals inward. Assuming $r_{\text{in}} \ll r_{\text{out}}$, by the virial theorem this energy loss is quantified by

$$\Delta E = -\frac{GM_{\text{BH}}m}{2r_{\text{in}}}.$$

The highest radius for which orbits around the black hole can be stable is known to be $r_{\text{in}} = 3R_{\text{sch}}$. Now, half of this ΔE goes into heating the disk and the rest is radiated. Thus

$$E_{\text{rad}} = \frac{GM_{\text{BH}}m}{2(3 \cdot 2GM_{\text{BH}}/c^2)} = \frac{mc^2}{12}.$$

So matter ends up radiating about 8% of its mass energy. The matter in the accretion disk thus emits the luminosity

$$L = \frac{\Delta E}{\Delta t} \sim \frac{(\Sigma m)c^2}{12\Delta t} \sim \frac{1}{12}\dot{M}_{\text{disk}}c^2,$$

where $\dot{M}_{\text{disk}}$ is called the mass accretion rate. This rate has a limit—if the accretion disk becomes too massive, then pressure from its luminosity is going to blow off matter. It turns out that we have a similar Eddington limit as with stars:

$$L_{\text{edd}} = (4.2 \times 10^4)\frac{M_{\text{BH}}}{M_\odot}L_\odot.$$

Another method has to do with looking at the gravitational influence black holes have on orbiting stars. The tidal forces distort the shape of the star, so the perceived area A_{obs} of the star changes as it orbits. So the luminosity we detect is

$$L_{\text{obs}} = A_{\text{obs}}\sigma T^4.$$

This allows us to determine some properties of the star and black hole.

Finally, we can take advantage of the measurable ripples that propagate through spacetime when bodies move. These are called **gravitational waves**, and they are especially prominent when two massive objects undergo a merger. This will eventually happen to any black holes that are orbiting each other; upon merging at $a \sim R_s$, we can use Kepler's third law to solve for the orbital frequency. If they have total mass M, we first have

$$P^2 \sim \frac{4\pi^2}{GM}a^3 \implies \nu = \frac{2}{P} = \frac{1}{\pi}\sqrt{\frac{GM}{R_S^3}}.$$

Substituting the Schwarzschild radius gives

$$\nu = \frac{1}{2\pi\sqrt{2}}\frac{c^3}{GM}$$

upon merging.

4 Galaxies and Cosmology

4.1 Galactic Properties

It's time to move up in scale once again and talk about galaxies! The first order of business is to establish a coordinate system that makes sense in this context. For convenience we'll once again define our solar system as the origin; the planar **galactic longitude** l will be measured counterclockwise from the galactic center, and the **galactic latitude** will be the angle an object makes with the galactic disk. Analyzing the coordinates of different bodies in our galaxy can tell us a lot about its structure—for example, the distribution of positions for certain clusters of stars was the first evidence that the Sun is not at the center of the Universe.

Stars in Galaxies

It's relatively easy to determine the galactic longitude and latitude of an object within our galaxy. The harder problem, of course, is determining how far away that object is. The solution is to develop a **galactic distance ladder**—a series of measurement methods that work on progressively larger distances.

- We begin with the tried-and-true trigonometric parallax. If the observed parallax angle of an object p is measured in arcseconds, then the distance to the object in parsecs is given by $d = 1/p$.
- For stars at slightly larger distances we instead use spectroscopic parallax, which makes use of the distance modulus
$$m - M = 5 \log_{10} \frac{d}{10 \text{ pc}},$$
where m and M are the relative and absolute magnitudes of the star. M is inferred from the star's spectral type.
- We can use the results here to infer distances to farther-away groups of stars via **main-sequence fitting**. The idea is to plot both groups on an HR diagram and determine the vertical shift required to fit one main sequence onto the other; this corresponds to the difference $m_1 - m_2$ between the relative magnitudes of "equivalent" stars in each group. We then have
$$m_1 - m_2 = 5 \log_{10} \frac{d_1}{d_2},$$
so if we know the distance to one group we can calculate that to the other.
- The next method up is called the moving cluster method. We won't go into detail here, but it has to do with comparing radial and tangential velocities.
- To measure the farthest objects in our galaxy (and even in other galaxies) we take advantage of pulsating stars, particularly Cepheid variables. The absolute magnitude of such a star is related to its pulsation period by
$$M_V \approx -a \log P_{\text{days}} - b,$$
where the constants depend on the object and the wavelength band of observation. For Polaris we have $a = 2.8$ and $b = 1.43$.

We can readily measure some other stellar properties, like metallicity, which is defined

$$\boxed{\left[\frac{\text{Fe}}{\text{H}}\right] = \log_{10}\left[\frac{N_{\text{Fe}}}{N_{\text{H}}}\right] - \log_{10}\left[\frac{N_{\text{Fe}}}{N_{\text{H}}}\right]_{\odot}}.$$

A star with a positive metallicity is metal-rich. We might use this as a rough proxy for age, as metal-rich stars tend to be younger since their high metallicities derive from supernovae. (We use iron as a proxy for the rest of the metals, since it's relatively abundant and has lots of spectral lines the optical band.)

Holistic Properties

Now we can get into some properties of galaxies as a whole. The distribution of mass in the galaxy is a good place to start, and there are a couple of ways to measure this. For one, we may count up visible emission from matter in the galaxy and use some models to infer a mass of $\lesssim 10^{11} M_\odot$. Alternatively, we can analyze how orbital velocities change as a function of radius.

Consider all of the matter with galactic longitude l and latitude 0. The mass with the highest velocity v is a distance $R = R_0 \sin l$ from the galactic center, where R_0 is the Earth-center distance. Doing this for many different l allows us to relate radii to orbital velocities, and a plot would reveal that $v(R) \propto R$ for $R \lesssim 0.5$ kpc and is constant otherwise! We can relate all this to the mass distribution by assuming circular orbits, so

$$\frac{mv^2}{r} = \frac{GM_r(R)m}{r^2}.$$

Now, let r_0 be the radius after which the velocity plateaus at v_0. Then the cumulative mass before and after the critical radius is given by

$$M_r(r) = \frac{v_0^2 r^2}{G r_0^2}, \qquad M_r(r) = \frac{v_0^2 r}{G},$$

respectively. But if we take $v_0 = 220$ km/s and $r = 50$ kpc, we get $M_r \approx 5 \times 10^{11} M_\odot$, which is five times larger than what we determined previously using visible light! This is the first evidence we have for the existence of dark matter: there must be something that interacts with gravity, but not with light.

This also brings us to some of the key properties distinguishing different types of galaxies. Broadly speaking, we say there are three types: elliptical galaxies, spiral galaxies, and irregular galaxies. When the **Hubble sequence** was first developed, elliptical galaxies were called "early types" and spiral ones "late type"; however, we now believe that elliptical galaxies are the results of interacting spiral galaxies.

- **Elliptical galaxies** generally have mass-light ratios of around 1-100 $M_\odot/L_\odot$, so they tend to be dominated by dark matter. Their total baryonic ("normal") masses fall between $10^7 M_\odot$ and $10^{14} M_\odot$.
- **Spiral galaxies** generally have M/L_B of around 2-50 $M_\odot/L_\odot$, so they fall in a tighter range than ellipticals. The same goes for baryonic mass, which falls between $10^9 M_\odot$ and $10^{12} M_\odot$.
- **Irregular galaxies** have $M/L_B \approx 1$ and baryonic mass between $10^8 M_\odot$ and $10^{10} M_\odot$. So not only do they have less dark matter, but they're much smaller on average.

The kinematics of stellar orbits also differ between galactic types. In spiral galaxies the orbits are generally circular—using $v_0 = 200$ km/s we get the characteristic orbital time

$$P_{\text{orb}} \sim \frac{2\pi r}{v_0} \simeq (30 \text{ Myr}) \left(\frac{r}{1 \text{ kpc}}\right) \left(\frac{200 \text{ km/s}}{v_0}\right)$$

at $r = 1$ kpc. This doesn't quite tell the whole story since the non-spherical distribution of mass in the galaxy causes stars to bob up and down as they orbit, but this is close enough. In elliptical galaxies things are even weirder—the general triaxial ellipsoid has no mass symmetries, so orbit shapes take some very exotic forms!

Despite this, there is a constant: $\langle v_r \rangle = 0$ in the rest frame of the galaxy. So the standard deviation of these radial velocities is

$$\sigma_r = \sqrt{\frac{1}{N} \sum_{\text{stars}} (v_r - \langle v_r \rangle)^2} = \sqrt{\langle v_r^2 \rangle - \langle v_r \rangle^2} = \sqrt{\langle v_r^2 \rangle}.$$

Now, in spherical coordinates it happens that $\langle v^2 \rangle = \langle v_r^2 \rangle + \langle v_\theta^2 \rangle + \langle v_\phi^2 \rangle$, and for reasonably isotropic velocity distributions we have $\langle v^2 \rangle \simeq 3 \langle v_r^2 \rangle$; thus $\sigma_r = \sqrt{\langle v^2 \rangle / 3}$.

Recall that the width of a spectral line is proportional to the spread σ_r of velocities, so σ_r can be measured directly from spectra! We can use this with the virial theorem to estimate galactic masses. For a spherical, uniform clump of stars with total mass M we have

$$\langle K \rangle = \frac{M \langle v^2 \rangle}{2}, \qquad \langle U \rangle \approx -\frac{3}{5} \frac{GM^2}{R},$$

and an application of the virial theorem gives

$$M \sim \frac{5}{3} \frac{\langle v^2 \rangle R}{G} = \frac{5 \sigma_r^2 R}{G}.$$

Finally, we can also estimate the mass of the central object of most galaxies: a supermassive black hole. Assuming a circular orbit, we once again get

$$\frac{mv^2}{r} = \frac{GM_{\mathrm{BH}}m}{r^2} \implies M_{\mathrm{BH}} = \frac{v^2 r}{G}.$$

4.2 Intergalactic Properties

Now we'll take another step up and discuss intergalactic space. This necessitates an extension of the galactic distance ladder, and this can be done quite easily simply by looking at brighter objects like type Ia supernovae, which are reliable up to around a gigaparsec.

Hubble's Law

The final rung on the distance ladder has to do **Hubble's law**, the observation that all galaxies (except for Andromeda) are receding from the Milky Way. In particular, a galaxy's recessional velocity is given by

$$\boxed{v = H_0 d},$$

where d is the distance to the galaxy and H_0 is called the **Hubble constant**. The accepted value for this constant has changed drastically overtime, but the values seem to converge to around 70 km/s/Mpc.

It's tempting to interpret Hubble's law as evidence that the Milky Way is at the center of the Universe, but we know better—the Copernican principle guides us away from conclusions suggesting that our place in the Universe is special in any way. Instead, we go with the more sophisticated conclusion that the observed recession is an artifact of the expansion of spacetime itself.

Because the recession has nothing to do with physical movement, there exists a distance past which objects recede faster than the speed of light and can no longer be detected. All of the space contained within this distance is called the Hubble sphere, and it has radius

$$R_H = \frac{c}{H_0} \sim 4.3 \text{ Gpc} = 13.2 \text{ Gly}.$$

So the "lookback time" for the Universe is around 13 billion years.

As light propagates, it gets redshifted by the expansion of the Universe. Cosmologists use this redshift as a proxy for the distance to an object, defining $z = \Delta\lambda/\lambda$. We've seen that at small recession velocities this reduces to v_r/c; substituting $v_r = H_0 d$ and then $H_0/c = 1/R_H$ gives

$$\boxed{z = \frac{d}{R_H}}.$$

This relationship is invalid for very nearby galaxies since they have non-negligible velocities independent of spacetime expansion. It's also invalid for galaxies exceeding $z \approx 2$ since the Hubble constant actually changes over time—the subscript zero indicates that H_0 is the present value—and light will have traveled far enough to take this into consideration.

Redshift surveys have shown that the distribution of galaxies in the Universe is not uniform. About 20% of all galaxies are contained within clusters, which are collections of fifty to thousands of galaxies contained within a region of 2-10 Mpc. We can determine the mass of such a cluster in precisely the same way as we would for an elliptical galaxy—if $\sigma_r^2 = \langle v^2 \rangle /3$ is the dispersion of radial velocities in the rest frame of the cluster, then $M = 5\sigma_r^2 R/G$, where R is the radius of the cluster portion.

Active Galactic Nuclei

The $v \ll c$ approximation is invalid for very faraway objects, but it's generally still okay for gauging the distance's order of magnitude. In this spirit, consider 3C273, a quasar with $z = 0.158$; using our conversion to distance as is gives $d \simeq 680$ Mpc. Despite this great distance, the relative magnitude in the visible band is $V = 12.8$, which corresponds to an absolute magnitude of $M_V = -26.6$. This is about the relative magnitude of the Sun from our perspective—in terms of luminosity,

$$L_V = 10^{(M_\odot - M_V)/2.5} L_\odot = 1.4 \times 10^{39} \text{ J/s}.$$

And this is just in the visible. The bolometric luminosity is $\sim 10^{40}$ J/s, concentrated mostly in the x-ray. Assuming 3C273 is exactly at its Eddington luminosity, we get a lower bound on its mass:

$$L_{\rm bol} \sim 10^4 \left(\frac{M_{\rm crit}}{M_\odot}\right) L_\odot \implies M_{\rm crit} \sim (6 \times 10^8) M_\odot.$$

The Schwarzschild radius provides a lower bound for the object's size:

$$R_{\rm sch} \gtrsim (3 \text{ km}) \left(\frac{M_{\rm crit}}{M_\odot}\right) = 1.8 \times 10^9 \text{ km} = 12 \text{ AU}.$$

But we also see periodic variability in the brightness of the quasar. Assuming this is due to periodic contractions and expansions in its literal size, we also have an upper bound

$$R_{\rm max} \sim c\Delta t \sim 2.6 \times 10^{13} \text{ m} = 173 \text{ AU}.$$

More sophisticated modeling has suggested that $M_{\rm 3C273} \sim 6 \times 10^9 M_\odot$, giving a luminosity that's about 10% of the Eddington. This leaves a surprisingly small amount of wriggle room, given the Schwarzschild bound.

Today, these quasars ("quasi-stars") are thought to derive their luminosities from accretion disks around supermassive black holes. A typical quasar's emission spectrum suggests a huge amount of line broadening, and if this is really due to Doppler broadening then there is matter moving around at 2.5% the speed of light. So the quasar must be a compact object like a black hole.

In order to estimate the mass of a black hole at the center of a quasar we use a clever technique called **reverberation mapping**. Surrounding the black hole are clouds of ionized gas; radiation from the accretion disk interacts with these clouds on a delay, and the amount τ of delay gives us information about the scope of these clouds. In particular,

$$R_{\rm BLR} \simeq f\tau c,$$

where f is a factor containing information about the clouds' geometry. BLR stands for broad-line region, and we can use the width of these broad emission lines to infer the orbital speed v of the clouds. Assuming circular orbits, we have

$$M_{\rm BH} \simeq \frac{v^2 R_{\rm BLR}}{G} = \frac{v^2 f\tau c}{G}.$$

4.3 Newtonian Cosmology

Our understanding of the Universe on the largest of scales is based on the **cosmological principle**: the Universe is homogeneous and isotropic. With this and some sketchy Newtonian mechanics, we can learn a great deal about the past, future, and present of the Universe!

Consider a dust-filled universe with spatially constant density $\rho(t)$, and construct a mass-m shell with radius $r(t)$ that encloses a mass M. Note that we have $M = \rho(t) \cdot (4/3)\pi r(t)^3$. Also, by conservation of energy,

$$\frac{1}{2} m v(t)^2 - \frac{GMm}{r(t)} = E \implies v(t)^2 = \frac{2E}{m} + \frac{2GM}{r(t)},$$

where $v = dr/dt$. Thus we have three broad possibilities for how $r(t)$ behaves as $t \to \infty$.

- Unbound universe ($E > 0$). $v(t) > 0$ always, the universe expands forever, and $v(\infty) = \sqrt{2E/m}$.
- Critical universe ($E = 0$). $v(t) > 0$ always, the universe expands forever, and $v(\infty) = 0$.
- Closed universe ($E < 0$). Gravity wins and the universe eventually recollapses.

The sign of E is determined by M and, by extension, r. The critical density ρ_c is found by substituting $E = 0$ and our expression for M; this gives

$$\boxed{\rho_c(t) = \frac{3}{8\pi G}\left[\frac{v(t)}{r(t)}\right]^2}.$$

Notice that if we define $H(t) = \sqrt{8\pi G \rho_c(t)/3}$, then we recover Hubble's law

$$v(t) = H(t) r(t).$$

Now, we'll find it useful to define the fraction

$$\boxed{\Omega(t) \equiv \frac{\rho(t)}{\rho_c(t)}}.$$

Based on our current understanding of H_0, the present-day critical density is $\rho_{c,0} \simeq 1.3 \times 10^{11} M_\odot/\mathrm{Mpc}$, about one Milky Way per cubic megaparsec. And it turns out we know the present-day value of Ω pretty well! It's around $\Omega = 0.29$, with about 0.24 of that being from dark matter. (This is consistent with our observation that baryonic matter comprises a small fraction of the total matter in the Universe.)

If we throw some general relativity into the mix, we'd expect that different values of Ω correspond to different spacetime geometries.

- If $\Omega = 1$ then the Universe is flat and Euclidean.
- If $\Omega > 1$ then the Universe is open and sphere-like.
- If $\Omega < 1$ then the Universe is closed and saddle-like.

To the best of our abilities, it appears that the Universe is flat. This seriously contradicts our previous conclusion about Ω, so what's going on?

In our search for reconciliation, we begin by considering the time evolution of a flat dust universe. Setting $E = 0$, we immediately get the differential equation

$$\frac{1}{2} m v(t)^2 = \frac{GMm}{r(t)},$$

which can be easily solved via separation of variables to get

$$r(t) = \left(\frac{3}{2}\sqrt{2GM}\, t\right)^{2/3}.$$

This solution is a little unsatisfying, though, since it depends on the details of the shell we're looking at. To generalize, define the **scale factor**

$$\boxed{R(t) \equiv \frac{r(t)}{r(t_0)}},$$

where t_0 is the present time. Substituting our solution $r(t)$ gives

$$R(t) = \left(\frac{3}{2}\sqrt{\frac{2GM}{r(t_0)}}\frac{t}{r(t_0)}\right)^{2/3} = \left(\frac{3}{2}\frac{v(t_0)}{r(t_0)}t\right)^{2/3} = \left(\frac{3}{2}H_0 t\right)^{2/3},$$

where H_0 is the current Hubble constant. We can use this result to determine the age of our universe by looking at $R(t_0)$:

$$1 = \left[\frac{3}{2}H_0 t_0\right]^{2/3} \implies t_0 = \frac{2}{3}\frac{1}{H_0} = \frac{2}{3}t_H,$$

defining the Hubble time $t_H = 1/H_0 \simeq 13.6\ \mathrm{Gyr}$. So now we have another problem—we've observed globular clusters that are much older than t_0, so this cannot be the correct age of the Universe. Worse, even though an open universe would give us a larger age, the value of Ω that gets us closest to the accepted age is $\Omega = 0$. This implies an empty universe—what gives?

Consider, again, the expanding shell of mass. When Einstein was thinking about this problem he figured that the Universe was static, meaning there must be an extra "repulsive" potential U_Λ that stops it from expanding or contracting. In particular,

$$\boxed{U_\Lambda = -\frac{1}{6}\Lambda m c^2 r^2},$$

where Λ is called the **cosmological constant**. Obviously Einstein's intentions were wrong, but it turns out that this is precisely the term we need to predict an older, expanding universe. Noting that

$$\frac{1}{2}mv^2 = \frac{GMm}{r} + \frac{1}{6}\Lambda m c^2 r^2$$

in a flat universe, we notice that the r^2 term dominates at late times, leading to long-term exponential expansion that looks like $R(t) \propto r(t) \propto e^{\Lambda c t/3}$. (As a side note, from this we can also get $\rho_\Lambda = \Lambda c^2/8\pi G$ and an associated Ω_Λ.) This is where the idea of dark energy originates!

4.4 The Big Bang

In an expanding universe, it's reasonable to extrapolate that it started out much smaller and denser than it is now. We have a general idea about some of the first stages it went through.

- In the initial bit of expansion the Universe cooled enough to facilitate the existence of quarks, then protons and neutrons, then electrons and positrons. All these particles lived together in some equilibrium exceeding $T \gtrsim 10^{10}$ K. Of particular importance is the constant creation and destruction of neutrons in this stage due to interactions with other particles.
- Eventually temperatures fall and neutrons can no longer be created. Given their relatively short half-life, the existing neutrons either decay or are incorporated into nuclei with protons and other neutrons. This is primordial nucleosynthesis, and it's where the first hydrogen, helium, and lithium were created.
- All this has happened within about twenty minutes of the Big Bang. Hundreds of thousands of years later, things cooled down enough to facilitate the creation of full-blown atoms. The Universe suddenly became transparent to photons which were then free to propagate forevermore. These photons, once very high-energy, have been redshifted into the microwave, and we call their remnants the **cosmic microwave background** (CMB).

The CMB is remarkably homogeneous, which leads us to wonder—when electrons joined with nuclei (upon "recombination"), how big was the largest region that could have been in contact with itself?

In a radiation-dominated fluid, perturbations propagate at the speed of sound $c/\sqrt{3}$. The horizon distance we desire is given by

$$d_h(t_{\rm rec}) = \int_0^{t_{\rm rec}} \frac{c}{\sqrt{3}} dt\, \frac{R(t_{\rm rec})}{R(t)},$$

where the scale factor fraction is tacked on to account for expansion. Assuming $R(t) = \left(\frac{3}{2}tH_0\right)^{\frac{2}{3}}$ like we derived, we get

$$\frac{R(t_{\rm rec})}{R(t)} = \left(\frac{t_{\rm rec}}{t}\right)^{\frac{2}{3}}$$

and the integral

$$d_h(t_{\text{rec}}) = \frac{c}{\sqrt{3}} t_{\text{rec}}^{2/3} \int_0^{t_{\text{rec}}} t^{-2/3} dt = \sqrt{3}\, c t_{\text{rec}}.$$

Taking $t_{\rm rec} \simeq 380,000$ yr, we get $d_h = 200$ kpc. As an angular distance relative to Earth this is $\theta_h = d_h/D \simeq 0.017 \text{ rad} \simeq 1^\circ$. Portions of the CMB that are more than this angular distance apart could never have talked to one another, so we're still left wondering how it's so consistent throughout the Universe.

There is a simple solution: inflation. When the Universe was around 10^{-36} s old, it expanded by a factor of $\sim 10^{43}$ over a period $\Delta t \sim 10^{-34}$ s. Thus any quantum fluctuations that existed at the smallest of scales were suddenly blown up to the macroscopic scale, and they persisted as the Universe continued to expand. This hypothesis also explains why the Universe appears to be so flat—as an analogy, we might consider a marble that's been blown up to the size of a galaxy; a person standing on this huge marble would say that it's basically flat.