MATH 180: Intro. to PDEs

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* Adapted from FA24 lectures.

1 The Advection Equation

1.1 Derivation: Traffic Flow

Consider a one-dimensional freeway with no entrances or exits, and let $\rho(x,t)$ denote the mass density of cars at a given position and time. We'll build a model that describes the evolution of ρ on a stretch of freeway with endpoints $x=a$ and $x=b$. First note that the total mass $M(t)$ of cars on this stretch and its derivative are given by

$$M(t) = \int_a^b \rho(x,t)\,dx, \qquad \frac{dM}{dt} = \int_a^b \frac{\partial \rho}{\partial t}\,dx,$$

where by a theorem of Leibniz the time derivative can be brought inside of the integral when ρ and ρ_t are both continuous. Now, we can get another independent expression for dM/dt by looking at the boundaries of our stretch of freeway—define $q(x,t)$ to be the rightward mass flux through a given point, so

$$\frac{dM}{dt} = q(a,t) - q(b,t) = -\int_a^b \frac{\partial q}{\partial x}\,dx.$$

By equating our two expressions we get the equation

$$\int_a^b \left[\frac{\partial \rho}{\partial t} + \frac{\partial q}{\partial x}\right] dx = 0;$$

since this works for arbitrary a, b, the integrand must be zero and

$$\rho_t + q_x = 0.$$

This is what we call the transport equation (or conservation law) for our setup. It does a great job at conveying what's physically happening on the freeway at each point in space and time, but it's an equation in two potentially unrelated functions ρ and q. This isn't super conducive to an analytic solution.

1.2 The Method of Characteristics

Conveniently, we're often able to write the flux q in terms of the density ρ. For example, if the cars are traveling at a constant speed c we have $q(x,t) = c\,\rho(x,t)$ and

$$\rho_t + c\,\rho_x = 0.$$

This is called the advection (or convection) equation, and it's an important one. But we'll focus our attention on the more general case in which c may be a function of position and time—in particular, we will solve

$$u_t + c(x,t)\,u_x = 0, \qquad u(x,0) = f(x)$$

for $-\infty < x < \infty$ and $t > 0$, where we've tacked on an initial condition for completion's sake. We'll approach the problem using the method of characteristics.

The idea here is to find curves, called characteristics, on which the above PDE reduces to an ODE. Specifically, we seek characteristics $x = x(t)$ on which $u(x(t),t)$ is constant in t. We will exploit the fact that, under this condition,

$$0 = \frac{du}{dt} = \frac{\partial u}{\partial t} + \frac{\partial u}{\partial x}\frac{dx}{dt} = u_t + \frac{dx}{dt}u_x.$$

Comparing this with the PDE we started with reveals that $dx/dt = c(x,t)$, the solutions of which are the $x(t)$ we desire. It turns out that substituting $\xi = x(0)$ and $\tau = t$ into our PDE reduces it to an ODE; solving it and undoing the substitution gives us the $u(x,t)$ we desired. This is best illustrated via an example.

Example: Characteristics on an inhomogeneous PDE

Consider the PDE

$$u_t + c\,u_x = -u, \qquad u(x,0) = f(x)$$

on $-\infty < x < \infty$ and $t > 0$. Despite this being an inhomogeneous equation, we can still solve it using characteristics! Using the chain rule to match the left-hand side of the PDE reveals that $dx/dt = c$, meaning our characteristics are

$$x(t) = ct + \xi$$

for constant ξ. We'll substitute $\xi = x - ct$, $\tau = t$, and $U(\xi, \tau) = u(x,t)$; the relevant derivatives are

$$\begin{aligned}\frac{\partial u}{\partial x} &= \frac{\partial U}{\partial \xi}\frac{\partial \xi}{\partial x} + \frac{\partial U}{\partial \tau}\frac{\partial \tau}{\partial x} \\ &= \frac{\partial U}{\partial \xi},\end{aligned} \qquad \begin{aligned}\frac{\partial u}{\partial t} &= \frac{\partial U}{\partial \xi}\frac{\partial \xi}{\partial t} + \frac{\partial U}{\partial \tau}\frac{\partial \tau}{\partial t} \\ &= -c\frac{\partial U}{\partial \xi} + \frac{\partial U}{\partial \tau}.\end{aligned}$$

Substituting these into our PDE gives

$$\frac{\partial U}{\partial \tau} = -U,$$

and we conclude that $U(\xi, \tau) = A(\xi)\,e^{-\tau}$ for some function A. After the substitution is undone we're left with $u(x,t) = A(x - ct)\,e^{-t}$, and applying the initial condition gives

$$u(x,t) = f(x - ct)\,e^{-t}.$$

Note that, despite our method for solving inhomogeneous PDEs being very similar to the one we have for homogeneous ones, in general there's a pretty big difference between the two. Namely, the solutions to a homogeneous PDE comprise an infinite-dimensional vector space (so any linear combination of solutions is itself a solution) while those to an inhomogeneous PDE do not.

There's a couple of ways we might visualize the behavior of the solutions we get to a PDE of this form. We could plot the characteristics in the x-t plane for several different choices of ξ, or we could plot several different "snapshots" of our solution curve in the x-u plane. Both of these are simple in principle, but can be quite complicated to pull off for nonlinear PDEs. More on that soon.

1.3 Characteristics and Nonlinear PDEs

Going back to our traffic model, it would make the most sense if speed depended on density. In this case the flux looks like $q(\rho) = c(\rho)\rho$, and from the transport equation we get

$$\rho_t + [c(\rho)\rho]_x = 0.$$

For our purposes, we'll say an individual car travels with phase velocity $c(\rho) = c_\star(1 - \rho/\rho_\star)$. Substituting into the above equation gives

$$\rho_t + c_\star\left(1 - \frac{2\rho}{\rho_\star}\right)\rho_x = 0.$$

The coefficient on the ρ_x is called the group velocity. We might interpret it as the speed at which the cars, as a whole, tend to move through space.

We'll take $x \in \mathbb{R}$, $t > 0$, and $\rho(x,0) = f(x)$. Applying the method of characteristics gives the ODE

$$\frac{dx}{dt} = c_\star\left(1 - \frac{2\rho}{\rho_\star}\right);$$

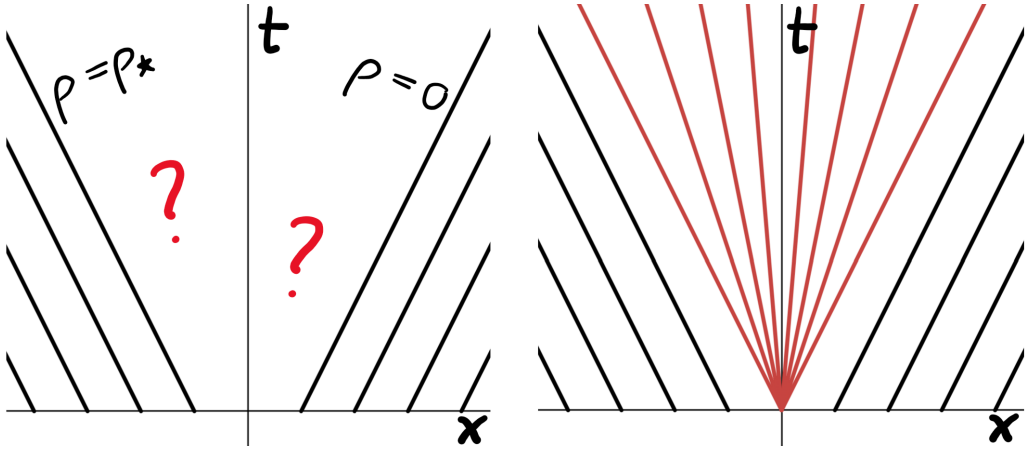
we'll call the right-hand side $v(\rho)$. Solving gives the characteristics $x(t) = v(\rho)\,t + \xi$. Substituting, solving, and un-substituting would give the solutions

$$\rho(x,t) = f\big(x - v(\rho)\,t\big).$$

This would be great if it weren't for one glaring problem: ρ is now given in terms of ρ. To see how we might deal with this, let's look at a very simple initial condition,

$$\rho(x,0) = \begin{cases} \rho_\star & x < 0, \\ 0 & x > 0. \end{cases}$$

The characteristics rooted at $x \neq 0$ are sketched below, on the left. This clarifies that $\rho(x,t) = \rho_\star$ for $x < -c_\star t$ and $\rho(x,t) = 0$ for $x > c_\star t$, but we don't yet have any information about what happens in between.



To resolve this we might imagine that our initial condition is actually smooth, our two pieces connected by a very thin logistic curve. In this region ρ varies from $\rho_\star$ to 0, and the slopes of our characteristics

$$x(t) = c_\star \left(1 - \frac{2\rho}{\rho_\star}\right)$$

vary from $-c_\star$ to $c_\star$. These new characteristics are plotted above in red. Solving for ρ gives the solution to the PDE in the in-between region.

1.4 Finite Difference Methods

Now we'll look a little bit at numerical methods, in particular finite difference methods. Our goal is to sketch a program to simulate solutions to the advection equation

$$u_t + cu_x = 0, \qquad u(x,0) = f(x)$$

for $x \in (0, L)$ and $t \in (0, T)$. Our domain will be split into chunks using fixed step sizes Δx and Δt. In this way, a position in space and time can be specified by two integer coordinates j, n—we'll use the notation

$$u_j^n = u(j\Delta x,\, n\Delta t).$$

We'll often pack all of the position data for a given time into a vector $\mathbf{u}^n$. This will become useful to us very soon! But for now we need to determine how we should approximate derivatives in this discretized space. We have a few options for finite difference methods.

- Forward difference. We start at $\mathbf{u}^n$ and jump ahead to $\mathbf{u}^{n+1}$. Here,

$$\frac{\partial u}{\partial t} \approx \frac{u_j^{n+1} - u_j^n}{\Delta t}.$$

- Backward difference. We start at $\mathbf{u}^n$ and jump backward to $\mathbf{u}^{n-1}$.

$$\frac{\partial u}{\partial t} \approx \frac{u_j^n - u_j^{n-1}}{\Delta t}.$$

- Centered difference. We jump between $\mathbf{u}^{n-1}$ and $\mathbf{u}^{n+1}$.

$$\frac{\partial u}{\partial t} \approx \frac{u_j^{n+1} - u_j^{n-1}}{2\Delta t}.$$

If we work with forward differences in both space and time then the advection equation becomes

$$\frac{u_j^{n+1} - u_j^n}{\Delta t} + c\,\frac{u_{j+1}^n - u_j^n}{\Delta x} = 0$$

with initial conditions $u_0^0, \ldots, u_k^0$. Solving for u_j^{n+1}:

$$u_j^{n+1} = u_j^n - \frac{c\,\Delta t}{\Delta x}\left(u_{j+1}^n - u_j^n\right).$$

The vector equation

$$\begin{aligned}\mathbf{u}^{n+1} &= \mathbf{u}^n - c\frac{\Delta t}{\Delta x}A\mathbf{u}^n \\ &= \left(I - \frac{c\Delta t}{\Delta x}A\right)\mathbf{u}^n\end{aligned}$$

describes this for all valid j, where

$$A = \begin{bmatrix} -1 & 1 & 0 & \cdots & 1 \\ 0 & -1 & 1 & \cdots & 0 \\ 0 & 0 & -1 & \cdots & \vdots \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ 0 & 0 & 0 & \cdots & -1 \end{bmatrix}.$$

Note that the 1 in the top-right corner encodes a periodic boundary condition $u(0,t) = u(L,t)$. Now, A has the sole eigenvalue $\lambda = -1$, meaning those of $I - \frac{c\Delta t}{\Delta x}$ are $\lambda = 1 + \frac{c\Delta t}{\Delta x}$. Our solution is stable if $|\lambda| \leq 1$, so in our case the criterion is

$$-2\Delta x < c\Delta t < 0.$$

Since Δx and Δt are both positive, this method is never stable. Other combinations may work, though!

2 The Heat Equation

2.1 Derivation: Bacteria Movements

Let's go back to modeling with the end of deriving a new differential equation. Suppose we have a long, one-dimensional tube filled with bacteria—these bacteria can run (propel themselves to the left or right) and tumble (randomly change its direction). If at $t = 0$ we start all of the bacteria at $x = 0$ in the tube, we'd like to determine the density of bacteria as functions of position and time.

We will assume the bacteria don't interact with one another in any way and that tumbles are instantaneous with probability $1/2$. Also, each run moves a bacterium a distance Δx in a time Δt. To start, let $P(x, t)$ denote a time-varying probability density function for the bacterium's position; with how we've discretized the tube, we can immediately see that

$$P(x,\ t + \Delta t) = \frac{1}{2}P(x - \Delta x,\ t) + \frac{1}{2}P(x + \Delta x,\ t).$$

To get something more familiar, we may subtract $P(x, t)$ and do some other manipulation to get

$$\frac{P(x,\ t + \Delta t) - P(x, t)}{\Delta t} = \frac{\Delta x^2}{2\Delta t}\frac{P(x - \Delta x,\ t) - 2P(x, t) + P(x + \Delta x,\ t)}{\Delta x^2}.$$

If we take $\Delta t,\ \Delta x \to 0$ in such a way that $\Delta x^2/2\Delta t$ is constant, this becomes

$$\frac{\partial P}{\partial t} = D\frac{\partial^2 P}{\partial x^2}.$$

The D here, called the diffusion coefficient, roughly quantifies how easily the bacteria spread throughout the tube. (More generally, it's a measure of the amount of diffusion a material facilitates.) From here we can get the actual density of bacteria by multiplying $\rho(x, t) = N\,P(x, t)$. where N is the number of bacteria we're looking at. So we get

$$\frac{\partial \rho}{\partial t} = D\frac{\partial^2 \rho}{\partial x^2}.$$

This is called the heat (or diffusion) equation. Note that we can draw a connection to conservation laws by writing

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}\left[-D\frac{\partial \rho}{\partial x}\right] = 0$$

and recognizing the flux $q(x, t) = -D\,(\partial\rho/\partial x)$. (So the flux moves down gradients, reflecting how bacteria move from high concentrations to low ones.)

For a finite tube with length L there's a few especially common choices of boundary conditions.

- (Dirichlet) We might fix the values of ρ at the caps, specifying $\rho(0, t)$ and $\rho(L, t)$.
- (Neumann) We might also fix the flux through the caps, specifying $\rho_x(0, t)$ and $\rho_x(L, t)$.
- (Robin) The flux might depend on external conditions. If these conditions are $f(t)$ on the far side of the tube then we can write
$$-Du_x(L, t) = H\big(f(t) - u(L, t)\big)$$
where the constant H is called the transfer coefficient.

2.2 Separation of Variables

We'll start our study of the heat equation with the Dirichlet problem with homogeneous boundary conditions:

$$\begin{aligned} u_t = Du_{xx} \qquad & 0 < x < \pi, \quad t > 0, \\ u(0, t) = u(\pi, t) = 0 \qquad & t > 0, \\ u(x, 0) = f(x) \qquad & 0 < x < \pi. \end{aligned}$$

We'll solve this equation using a method called separation of variables—substituting the ansatz $u(x,t) = X(x)T(t)$ into the differential equation gives

$$X(x)T'(t) = DT(t)X''(x) \implies \frac{T'(t)}{DT(t)} = \frac{X''(x)}{X(x)}.$$

The only way for this to possibly be true for all x, t is if both sides are constant at λ. So we've turned our PDE into two disjoint ODE subproblems with solutions

$$T(t) = Ce^{\lambda Dt}, \qquad X(x) = C_1 e^{\sqrt{\lambda}\,x} + C_2 e^{-\sqrt{\lambda}\,x}.$$

Now, our boundary conditions imply $X(0) = 0$ and $X(\pi) = 0$. For the latter condition we get no nontrivial solutions for $\lambda \geq 0$, so we require that $\lambda < 0$ and get

$$X(x) = A\cos\left(\sqrt{-\lambda}\,x\right) + B\sin\left(\sqrt{-\lambda}\,x\right).$$

The condition $X(0) = 0$ implies $A = 0$, while $X(\pi) = 0$ gives the condition $\sqrt{-\lambda} = n$ for $n = 0, 1, \ldots$. So we can index

$$X_n(x) = B\sin(nx).$$

Since $T(t)$ depends on our choice of λ, our quantization of λ affects this other part of the solution, too. So we have the countable set of solutions

$$u_n(x,t) = X_n(x)T_n(t) = b_n e^{-n^2 Dt}\sin nx, \quad n = 0, 1, \ldots$$

The heat equation is homogeneous, meaning its solution space is closed under linear combinations and we have the solution

$$u(x,t) = \sum_{n=0}^{\infty} b_n e^{-n^2 Dt}\sin nx.$$

The coefficients b_n are determined by the initial condition $u(x,0) = f(x)$; specifically, we require that

$$f(x) = \sum_{n=0}^{\infty} b_n \sin nx.$$

Such a sum is called a Fourier sine series, and the b_n are called the Fourier coefficients. The coefficients are found by projecting $f(x)$ onto each of the $\sin nx$:

$$b_n = \frac{\langle f(x),\, \sin nx\rangle}{\|\sin nx\|^2} = \frac{2}{\pi}\int_0^{\pi} f(x)\sin nx\, dx.,$$

where we've employed the $L^2[0,\pi]$ inner product

$$\langle u, v\rangle_{L^2[0,\pi]} = \int_0^{\pi} u(x)v(x)\, dx.$$

This kind of projection is valid whenever we have a countable "basis" of functions, not just with sines. (In particular, note that we'd end up with the same kind of result using Neumann boundary conditions, just with cosines instead of sines.)

2.3 The Fourier Eigenvalue Problem

While solving the Dirichlet problem we've stumbled upon another problem that's much more general, namely,

$$-u''(x) = \lambda u(x), \qquad 0 \in (0,\pi), \quad u(a) = u(b) = 0.$$

Noting that differentiation is linear, we call this the Fourier eigenvalue problem. A solution so such an equation is a nonzero eigenpair (λ, u). For this discussion we'll restrict ourselves to the vector space $\mathcal{V}_D$ of functions on $[a,b]$ that are twice-differentiable with continuous derivatives, are square-integrable, and satisfy our boundary conditions. Formally,

$$\mathcal{V}_D = \left\{u \in \mathcal{C}^2[a,b] \cap \mathcal{L}^2[a,b] : u(a) = u(b) = 0\right\}.$$

We require square-integrability so that the L^2 inner product is well defined.

Definition: Self-adjoint operator

Let $\mathcal{L}$ be a linear operator acting on an inner product space $\mathcal{U}$. We say $\mathcal{L}$ is self-adjoint if for any $u, v \in \mathcal{U}$ it satisfies

$$\langle u, \mathcal{L}v \rangle = \langle \mathcal{L}u, v \rangle .$$

We might view this as a generalization of symmetric matrices, noting that if $A = A^T$ then

$$u \cdot Av = u^\intercal Av (Au)^\intercal v = Au \cdot v.$$

Notice that our operator $D = -d^2/dx^2$ is self-adjoint. We could prove this directly using integration by parts. note that the proof also works for homogeneous Neumann boundary conditions and for periodic boundary conditions, in which cases we denote our vector spaces $\mathcal{V}_N$ and $\mathcal{V}_P$, respectively.

Theorem 2.1

Let $\mathcal{L}$ be self-adjoint on $\mathcal{U}$ and let $u = p + iq$, where $p, q \in \mathcal{U}$. Then the eigenvalue problem $\mathcal{L}u = \lambda u$ has only real λ.

Proof. We can use the fact that $\mathcal{L}$ is self-adjoint to write

$$\langle \overline{u}, \mathcal{L}u \rangle = \langle p - iq, \mathcal{L}p + i\mathcal{L}q \rangle = \langle p, \mathcal{L}p \rangle + \langle q, \mathcal{L}q \rangle .$$

On the other hand,

$$\begin{aligned} \langle \overline{u}, \mathcal{L}u \rangle &= \langle \overline{u}, \lambda u \rangle \\ &= \lambda \big[\langle p, p \rangle + \langle q, q \rangle + i \big(\langle p, q \rangle - \langle q, p \rangle \big) \big] \\ &= \lambda \left[\|p\|^2 + \|q\|^2 \right] \end{aligned}$$

By transitivity

$$\lambda = \frac{\langle p, \mathcal{L}p \rangle + \langle q, \mathcal{L}q \rangle}{\|p\|^2 + \|q\|^2},$$

which is real. □

Theorem 2.2

Let $\mathcal{L}$ be self-adjoint on $\mathcal{U}$. If $u_n, u_m \in \mathcal{U}$ are eigenfunctions with distinct eigenvalues λ_n, λ_m for the eigenvalue problem $\mathcal{L}u = \lambda u$ then u_n, u_m are orthogonal.

Proof. Because $\mathcal{L}$ is self-adjoint, $\langle u_n, \mathcal{L}u_m \rangle = \langle \mathcal{L}u_n, u_m \rangle$ and $\lambda_m \langle u_n, u_m \rangle = \lambda_n \langle u_n, u_m \rangle$. This is satisfied if and only if $\langle u_n, u_m \rangle = 0$. □

Conveniently for us, it turns out that the eigenvalues associated with our $Du = \lambda u$ are strictly non-negative. This result combined with orthogonality go a long way in justifying some of the choices we made in solving the Dirichlet problem, and even in streamlining the process for other problems.

2.4 Real Fourier Series

Going back to the heat equation for a moment, we might also use the periodic boundary conditions $u(-\ell, t) = u(\ell, t)$ and $u_x(-\ell, t) = u_x(\ell, t)$. The Fourier eigenvalue problem has solutions

$$X(x) = A\cos(\sqrt{\lambda}\, x) + B\sin(\sqrt{\lambda}\, x),$$

but applying the boundary conditions puts no restrictions on the coefficients A, B, but rather just $\lambda = n^2\pi^2/\ell$. So we get the solutions

$$X(x) = a_0 + \sum_{n=1}^{\infty} \left[a_n \cos\left(\frac{n\pi x}{\ell}\right) + b_n \sin\left(\frac{n\pi x}{\ell}\right) \right]$$

with coefficients determined by projection of any initial conditions onto the eigenfunctions. This "Fourier projection" is more than just a way to write down solutions to PDEs, though—we can also use it to approximate functions in general, even non-periodic ones! (The $X(x)$ above is simply the Fourier expansion of whatever initial condition we're working with.)

Definition: Periodic extension

Suppose f is defined on $[-\ell, \ell]$. The periodic extension of f is the unique $\tilde{f} : \mathbb{R} \to \mathbb{R}$ such that

- $\tilde{f}(x) = f(x)$ for all $x \in [-\ell, \ell]$ and
- $\tilde{f}(x + 2\ell) = \tilde{f}(x)$ for all $x \in \mathbb{R}$.

We could similarly define the odd extension f_o as the unique function satisfying $f_o(-x) = f(x)$ for $x \in [-\ell, \ell]$, and the even extension f_e is defined analogously. (For both of these we keep $x \geq 0$ and mirror on $x < 0$.)

Now, when we go to write functions in terms of sines and cosines, infinite sums aren't very practical. So we'll have to settle with approximations—if we want to approximate $f \in L^2[0, \pi]$ as a linear combination of N "Fourier modes" $\{e_1, \ldots, e_N\}$, we simply write

$$f \approx \sum_{j=1}^{n} c_j e_j, \quad c_m \approx \frac{\langle f, e_m \rangle}{\|e_m\|^2}.$$

We have a way to quantify the "goodness" of this approximation.

Theorem 2.3

Suppose f is an element of an inner product space V and $\{e_i\}$ is an orthogonal set. The approximation of f by a linear combination

$$\hat{f}_n = \sum_{i=1}^{N} c_i e_i$$

that minimizes the error $E_n = \|f - \hat{f}_N\|$ is to choose $c_i = a_i$ with $a_i = \langle f, e_i \rangle / \|e_i\|^2$.

Proof. The expression for squared error is

$$\begin{aligned}
E_N^2 &= \|f - \hat{f}_N\|^2 \\
&= \langle f, f \rangle + \left\langle \hat{f}_N, \hat{f}_N \right\rangle - 2 \left\langle f, \hat{f}_N \right\rangle \\
&= \|f\|^2 + \sum_{i=1}^{N} \sum_{j=1}^{N} c_i c_j \langle e_i, e_j \rangle - 2 \sum_{i=1}^{N} c_i \langle f, e_i \rangle \\
&= \|f\|^2 + \sum_{i=1}^{N} c_i^2 \|e_i\|^2 - 2 \sum_{i=1}^{N} c_i a_i \|e_i\|^2, \\
&= \|f\|^2 + \sum_{i=1}^{N} \left((c_i - a_i)^2 - a_i^2 \right) \|e_i\|^2,
\end{aligned}$$

where $a_i = \langle e_i, f \rangle / \|e_i\|^2$. E_N^2 is minimized (and so is E_N) when $c_i - a_i = 0$, as desired. □

All this means the error satisfies

$$E_N^2 = \|f\|^2 - \sum_{i=1}^{N} c_i^2 \|e_i\|^2.$$

Since $E_N^2 \geq 0$ and $\lim_{N \to \infty} E_N = 0$ we have Bessel's inequality and Parseval's inequality, respectively:

$$\|f\|^2 \geq \sum_{i=1}^{N} c_i^2 \|e_i\|^2, \qquad \|f\|^2 = \sum_{i=1}^{\infty} c_i^2 \|e_i\|^2.$$

Of course, as we add more terms to our approximation, our series should converge to the function we're approximating. There are three different ways we might describe this behavior.

Theorem 2.4: Criteria for uniform convergence

The Fourier series $\hat{f}$ converges to f uniformly on $[a, b]$ provided the following are true:

- $f \in C^2[a, b]$. That is, f and its first two derivatives exist and are continuous.
- f satisfies any given boundary conditions.

Theorem 2.5: Criterion for pointwise convergence

The Fourier series $\hat{f}$ converges to f pointwise on $[a, b]$ provided that f and f' are piecewise continuous.

The third notion of convergence might be unfamiliar.

Definition: L^2 convergence

A series S_n converges in the L^2 (mean square) sense to $f(x)$ in (a, b) if

$$\lim_{n\to\infty} \|f(x) - S_n\| = 0.$$

Theorem 2.6: Criterion for L^2 convergence

The Fourier series $\hat{f}$ converges in (a, b) to f in the mean square sense provided that $f \in L^2(a, b)$.

3 Laplace's Equation

3.1 Derivation: Electrostatics

Back to modeling once again. Consider a two-dimensional region D in space with closed boundary ∂D, and let $u(x, y)$ denote the electric potential at a point in D. (We'll seek stationary solutions, so we assume u has no time dependence). The charge density $\rho(x, y) = 0$ everywhere in D and on ∂D. We begin with Gauss's law, which states that

$$\int_{\partial D} \mathbf{E} \cdot \mathbf{n}\, dS = \iint_D \frac{\rho(x,y)}{\varepsilon_0}\, dA,$$

where ε_0 is a constant called the permittivity of free space. By the divergence theorem this is equivalent to

$$\begin{aligned} \iint_D \nabla \cdot \mathbf{E}\, dA &= \iint_D \frac{\rho(x,y)}{\varepsilon_0}\, dA \\ \nabla \cdot \mathbf{E} &= \frac{\rho(x,y)}{\varepsilon_0}. \end{aligned}$$

In the free space we've assumed, this becomes $\nabla \cdot \mathbf{E} = 0$. We also have Faraday's law,

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t};$$

since we seek stationary solutions this becomes $\nabla \times \mathbf{E} = 0$. This implies that $\mathbf{E}$ is conservative, so there is a potential $u(x, y)$ such that $\nabla u = \mathbf{E}$. Substituting this into the differential form of Gauss's law gives

$$\nabla^2 u(x, y) = 0.$$

This is Laplace's equation. Note that the operator

$$\begin{aligned} \nabla^2 &= \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \\ &= \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2} \end{aligned}$$

is called the Laplacian, and it generalizes to any number of dimensions.

3.2 Solution on a Disk

Suppose our region D is a radius-a disk centered at the origin. This is conducive to using polar coordinates, in which case our PDE becomes

$$\begin{aligned} \frac{1}{r^2}u_{\theta\theta} + \frac{1}{r}u_r + u_{rr} = 0 \qquad & r \in (0, a),\ \theta \in \mathbb{R}, \\ u(a, \theta) = f(\theta) \qquad & \theta \in \mathbb{R}. \end{aligned}$$

We'll once again proceed via separation of variables. Substituting $u(r, \theta) = R(r)\Theta(\theta)$ gives

$$\frac{r^2 R'' + rR'}{R} = -\frac{\Theta''}{\Theta},$$

and both of these are equal to the same constant λ. In θ we get $\Theta'' = -\lambda\Theta$, the Fourier eigenvalue problem—the solutions are

$$\Theta(\theta) = c_1 \cos\left(\sqrt{\lambda}\,\theta\right) + c_2 \sin\left(\sqrt{\lambda}\,\theta\right),$$

where Θ must be 2π-periodic meaning $\lambda_n^2 = n$ for $n = 1, 2, \ldots$. The eigenfunctions are $\{1, \cos n\theta, \sin n\theta\}$.

With this choice of λ our equation in r becomes

$$r^2R'' + rR' - n^2R = 0,$$

which we call a Cauchy-Euler eigenvalue problem. For $n = 0$ we get $R_0(r) = c_0 + c_1 \ln r$. For $n > 0$ we proceed by making and substituting the ansatz $R(r) = r^\alpha$; this would give

$$R_n(r) = k_0 r^n + k_1 r^{-n}$$

for some constants k_0, k_1. But solutions should be well-defined as $r \to 0$, so we take $c_1 = k_1 = 0$ and

$$R(r) = \begin{cases} c_0 & n = 0 \\ r^n & n = 1, 2, \dots \end{cases}$$

In all, solutions look like

$$u(r, \theta) = c_0 + \sum_{n=1}^{\infty} r^n \big(a_n \cos(n\theta) + b_n \sin(n\theta)\big).$$

The coefficients can be found in the same way as before.

3.3 Inhomogeneous Problems

Now we'll look at a couple of different ways of tackling problems with inhomogeneous differential equations or boundary conditions. For the first we'll take an inhomogeneous heat equation $u_t = Du_{xx} + \alpha$ with homogeneous boundary conditions and zero initial condition on $x \in (0, 1)$, $t > 0$. Separation of variables won't work here, so we need to be a little clever.

Let's focus our attention on finding a steady-state solution u_s, one that's constant in time. This reduces the equation to $u_s''(x) = -\alpha/D$; integrating twice gives

$$u_s(x) = -\frac{\alpha}{2D}x^2 + Ax + B.$$

Applying $u_s(0) = 0$ gives $B = 0$, while $u_s(1) = 0$ gives $A = \alpha/2D$. Thus

$$u_s(x) = \frac{\alpha}{2D}x(1 - x).$$

Now suppose this u_s is just one component of some larger solution—that is, $u(x, t) = u_s(x) + v(x, t)$ for some v. Since u solves the heat equation we can write

$$\begin{aligned} [u_s(x) + v(x,t)]_t &= D\left[u_s(x) + v(x,t)\right], \\ v_t &= Dv_{xx} + \left[Du_s''(x) + \alpha\right], \\ v_t &= Dv_{xx}. \end{aligned}$$

Solving this equation with homogeneous boundary conditions and the initial condition $v(x, 0) = -u(x)$ would give us what we need to write down a solution to the original problem.

For our other method, let's go back to Laplace's equation $\nabla^2 u(x, y) = 0$ with rectangular boundary conditions

$$u(x, 0) = f_1(x), \qquad u(x, H) = f_2(x), \qquad u(0, y) = g_1(y), \qquad u(L, y) = g_2(y).$$

Things get a bit dicey here if multiple sides are inhomogeneous. To remedy this we might split our problem into four—we'll take

$$u(x, y) = u_1 + y_2 + u_3 + u_4$$

where u_k is subject to the kth boundary condition listed above with all the others homogeneous. In practice we would need to solve the corresponding problem for each of these four pieces; here we'll focus on the second:

$$\nabla^2 u_2(x, y) = 0, \qquad u(x.H) = f_2(x)$$

with the rest of the boundaries zero. Substituting the ansatz $u(x, y) = X(x)Y(y)$ gives

$$\frac{X''(x)}{X(x)} = -\frac{Y''(y)}{Y(y)} = \lambda.$$

For convenience we'll start with the side subject to homogeneous boundary conditions. From $X''(x) = \lambda X(x)$ we get

$$X_n(x) = b_n \sin\left(\frac{n\pi x}{L}\right), \quad \lambda_n = -\left(\frac{n\pi}{L}\right)^2.$$

The other equation at hand is

$$Y''(y) = \left(\frac{n\pi}{L}\right)^2 Y(y),$$

and the solutions look like

$$\begin{aligned} Y(y) &= \alpha e^{(n\pi/L)y} + \beta e^{-(n\pi/L)y} \\ &= c_1 \cosh\left(\frac{n\pi}{L}y\right) + c_2 \sinh\left(\frac{n\pi}{L}y\right). \end{aligned}$$

Using hyperbolic functions makes the boundary conditions a lot nicer to work with! Substituting $Y(0) = 0$ gives $c_1 = 0$ and, we can combine our two solutions to get

$$u_2(x, y) = \sum_{n=1}^{\infty} c_n \sin\left(\frac{n\pi}{L}x\right) \sinh\left(\frac{n\pi}{L}y\right).$$

Finally, substituting the inhomogeneous $u(x.H) = f_2(x)$ turns our sum into a Fourier sine series, which we know how to work with by now. The other three components of the problem follow very similarly.

3.4 Well-Posed Problems

Now we'll take a step back again and see what it means for a problem to be well-posed, and under what conditions it happens.

Definition: Well-posed problem

A problem is well-posed if a solution exists, is unique, and is stable (that is, if a small perturbation made to the initial state or boundary condition only changes the solution by a small amount).

We'll illustrate these ideas via a case study of the heat equation's Dirichlet problem on $x \in (0, 1)$ with homogeneous boundary conditions and $u(x, 0) = f(x)$.

Example: Existence

The most obvious way to show existence is to simply find an explicit solution to the problem; more generally, though, we can show that there is a series that converges to a solution. Let

$$u_N(x, t) = \sum_{n=1}^{N} a_n e^{-n^2\pi^2 Dt} \sin(n\pi x), \quad a_n = 2\int_0^1 f(x)\sin(n\pi x)dx,$$

and define $u(x, t) = \lim_{N\to\infty} u_N(x, t)$. To show that $u(x, t)$ is a solution to our problem, start by defining $M/2 = \max_{x\in(0,1)} |f(x)|$ so that $|a_n| \leq M$ for all n. By the triangle inequality,

$$|u_N(x, t)| \leq \sum_{n=1}^{N} |a_n| e^{-n^2\pi^2 Dt} \leq M \sum_{n=1}^{N} e^{-n^2\pi^2 Dt}$$

This converges absolutely as $N \to \infty$, and thus so does $u_N(x, t)$. So $u(x, t)$ satisfies the differential equation and boundary conditions, and since the Fourier sine series

$$\hat{f}(x) = \sum_{n=1}^{\infty} a_n \sin(n\pi x)$$

converges to $f(x)$, our initial condition is also satisfied.

Example: Uniqueness

One way to show uniqueness is via an energy method. We begin by multiplying both sides of the differential equation by u to get $uu_t = Duu_{xx}$, and integrating with respect to x gives

$$\int_0^1 uu_t dx = D\int_0^1 uu_{xx}dx$$
$$\frac{1}{2}\int_0^1 \frac{d}{dt}(u^2)dx = D\left[uu_x\Big|_0^1 - \int_0^1 u_x^2 dx\right]$$

We'll define the energy $E(t) = \frac{1}{2}\int_0^1 u^2 dx$ to turn this into

$$\frac{d}{dt}E(t) = -D\int_0^1 u_x^2 dx$$
$$\frac{d}{dt}E(t) \leq 0.$$

Now suppose, for contradiction, that there are two distinct solutions u_1, u_2 to our Dirichlet problem. Define $v(x,t) = u_2(x,t) = u_1(x,t)$; by linearity v satisfies the heat equation with homogeneous boundary conditions and a zero initial condition. Thus v also satisfies

$$\frac{d}{dt}\left(\frac{1}{2}\int_0^1 v^2(x,t)\,dx\right) \leq 0.$$

Note that the parenthetical $E(t)$ is strictly non-negative and that $E(0) = 0$. Combined with the fact that $E(t)$ is non-increasing we conclude that the integral is zero for all $t \geq 0$, meaning $v(x,t) = 0$ and $u_1(x,t) = u_2(x,t)$. Solutions to our problem are unique!

Example: Stability

Suppose u_1, u_2 both solve our Dirichlet problem with respective initial conditions $f_1(x), f_2(x)$. We'd like to show that if u_1, u_2 start close together then they stay close together—specifically, that

$$\|u_1(x,t) - u_2(x,t)\| \leq \|f_1(x) - f_2(x)\|, \quad t > 0.$$

Define $v(x,t) = u_2(x,t) - u_1(x,t)$ so, like before, v satisfies the Dirichlet problem with initial condition $v(x,0) = f_2(x) - f_1(x)$. We have the energy

$$E[v] = \frac{1}{2}\int_0^1 v^2(x,t)dx = \frac{1}{2}\|v\|^2 = \frac{1}{2}\|u_2(x,t) - u_1(x,t)\|^2.$$

At $t = 0$ we have

$$E[v(x,0)] = \frac{1}{2}\|f_2(x) - f_1(x)\|^2.$$

As before, $E[v]$ is non-negative and non-increasing, so we have $E[v(x,t)] \leq E[v(x,0)]$ and

$$\frac{1}{2}\|u_1(x,t) - u_2(x,t)\|^2 \leq \frac{1}{2}\|f_1(x) - f_2(x)\|^2$$

for all $t \geq 0$.

As an example of an ill-posed problem, consider a "backward heat equation" $u_t = -u_{xx}$. Separation of variables would yield

$$u_n(x,t) = \frac{1}{n}e^{n^2\pi^2 t}\sin(n\pi x).$$

At $t = 0$, $|u_n(x,0)|$ attains a maximum value of $1/n$ for $x \in (0,1)$. But at an arbitrarily small $t = 1/n$,

$$\max_{x\in(0,1)} |u_n(x,1/n)| = e^{n\pi^2}/n.$$

So small initial conditions become very large at very small times, meaning the problem is not stable.

3.5 Maximum Principles

Remaining in the realm of theory, now we'll look at where solutions to problems are extremized. We'll focus our attention on the heat equation for convenience.

Theorem 3.1: Maximum principle

Let R be a closed region with $0 \leq x \leq L$ and $0 \leq t \leq T$, and let $u(x,t)$ be a solution to the heat equation $u_t = Du_{xx}$ which is continuous on R. (That is, $u \in C_x^2[0,L] \cap C_t[0,T]$.) Then $u(x,t)$ achieves its maximum either initially or on one of the boundaries.

Proof. Suppose, for contradiction, that there is a maximum on the interior of R. At this maximum we have $u_t(x_1,t_1)$, $u_x(x_1,t_1)$, and $u_{xx}(x_1,t_1) \leq 0$. If $u_{xx} < 0$ then we already have a contradiction because u wouldn't satisfy the heat equation, so we'll focus on the case in which $u_{xx} = 0$.

We'll address this by defining a continuous "subfunction" $v(x,t)$ that satisfies $u \equiv v$ at $x = 0$, $x = L$, and $t = 0$ with $v(x,t) < u(x,t)$ everywhere else:

$$v(x,t) = u(x,t) - \varepsilon tx(L-x), \quad \varepsilon > 0.$$

Note that $u(x,t) - v(x,t)$ tends uniformly to zero as $\varepsilon \to 0$, and that the maximum of $v(x,t)$ in R tends toward that of $u(x,t)$ as $\varepsilon \to 0$. Also, since u satisfies the heat equation, v satisfies

$$\begin{aligned} [v(x,t) + \varepsilon tx(L-x)]_t &= D[v(x,t) + \varepsilon tx(L-x)]_{xx}, \\ v_t + \varepsilon x(L-x) &= Dv_{xx} - 2D\varepsilon t, \\ v_t &= Dv_{xx} - \varepsilon[x(L-x) + 2Dt]. \end{aligned}$$

We have three cases. If v has its maximum on the interior of R then $v_t(x_1,t_1) = 0$ and here we must have

$$Dv_{xx} = \varepsilon[x(L-x) + 2Dt] > 0,$$

a contradiction. So v doesn't attain its maximum on the interior of R. If v has its maximum at $t = T$ then $v_x(x_1,T) = 0$ and $v_{xx} \leq 0$ there. Also,

$$v_t = Dv_{xx} - \varepsilon[x(L-x) + 2Dt] < 0,$$

a contradiction. This leaves only the maximum occurring at $x = 0$, $x = L$, or $t = 0$, in which cases we have $v \equiv u$. The maximum of v must occur on one of these.

So v satisfies the maximum principle, and since v (and its maximum) converges uniformly to u as $\varepsilon \to 0$, meaning u also satisfies the maximum principle. □

There is a completely analogous minimum principle, and we may combine it with the maximum principle to get another, stronger result.

Theorem 3.2: Minimum principle

Under the same conditions as the previous theorem, $u(x,t)$ achieves its minimum either initially or on one of the boundaries.

Theorem 3.3: Strong maximum principle

Let $u(x,t)$ be a solution to the heat equation in the rectangle $R : [0,L] \times [0,T]$. If u achieves its maximum on the interior of R then u is constant.

Put together, all this gives us a new tool for studying many problems' uniqueness and stability.

4 The Wave Equation

4.1 Derivation: Guitar String

Modeling once more! This derivation's a quick one. Consider a guitar string with cross-sectional area A and density ρ held under a tension τ. When the string is displaced a distance $u(x,t)$, it tries to bring itself back to a straight configuration with restoring force $\mathcal{F} = \tau u_{xx} A\Delta x$. By Newton's second law $\mathcal{F} = (\rho A \Delta x) u_{tt}$, so $\tau u_{xx} = \rho u_{tt}$ or, written a different way,

$$u_{tt} = c^2 u_{xx}$$

where c denotes the speed of the wave. Unsurprisingly, we call this the wave equation.

4.2 d'Alembert's Solution

When we're working on a finite domain the wave equation is solvable via separation of variables. With homogeneous boundary conditions and $u(x,0) = f(x)$ and $u_t(x,0) = g(x)$, the solutions are

$$u(x,t) = \sum_{n=1}^{\infty} \sin(k_n x)\big[A_n \cos(\omega_n t) + B_n \sin(\omega_n t)\big], \qquad \omega_n = \frac{n\pi c}{L}, \quad k_n = \frac{n\pi}{L}$$

with coefficients

$$A_n = \frac{2}{L}\int_0^L f(x)\sin(k_n x)\,dx, \qquad B_n = \frac{2}{\omega_n L}\int_0^L g(x)\sin(k_n x)\,dx.$$

If we wanted to consider solutions on an infinite domain, however, we ditch the boundary conditions and separation of variables doesn't quite work anymore. (We call such a problem a Cauchy problem.) To inch toward a solution we might write the wave equation in operator form,

$$\left(\frac{\partial^2}{\partial t^2} - c^2\frac{\partial^2}{\partial x^2}\right) u = 0,$$

and recognize a difference of squares to write

$$\left(\frac{\partial}{\partial t} - c\frac{\partial}{\partial x}\right)\left(\frac{\partial}{\partial t} + c\frac{\partial}{\partial x}\right) u = 0 \quad \text{or} \quad \left(\frac{\partial}{\partial t} + c\frac{\partial}{\partial x}\right)\left(\frac{\partial}{\partial t} - c\frac{\partial}{\partial x}\right) u = 0.$$

We can see, now, that some version of the advection equation is hiding in here. The solutions to these equations are $u(x,t) = A(x-ct)$ and $u(x,t) = B(x+ct)$ for some functions A, B, respectively; we hypothesize that solutions to the wave equation are linear combinations of these.

We'll make the change of variables $\xi = x - ct$, $\eta = x + ct$, and $U(\xi,\eta)$. We could show that the derivative operators are

$$\frac{\partial}{\partial t} = -c\frac{\partial}{\partial \xi} + c\frac{\partial}{\partial \eta}, \qquad \frac{\partial}{\partial x} = \frac{\partial}{\partial \xi} + \frac{\partial}{\partial \eta},$$

so our wave equation operator turns out to be

$$\frac{\partial^2}{\partial t^2} - c^2\frac{\partial^2}{\partial x^2} = -4c^2\frac{\partial}{\partial \xi}\frac{\partial}{\partial \eta}.$$

So we have the equation

$$\frac{\partial^2 U}{\partial \xi \partial \eta} = 0,$$

and we can see that $U(\xi,\eta) = A(\xi) + B(\eta)$ for some functions A, B. Applying the initial conditions gives

$$\begin{aligned} f(x) &= A(x) + B(x), \\ g(x) &= -cA'(x) + cB'(x). \end{aligned}$$

Now we just need to solve for these unknown functions. We have

$$\int_0^x g(z)dz = c\big(B(x) - A(x) - B(0) + A(0)\big),$$
$$\frac{1}{c}\int_0^x g(z)dz = f(x) - 2A(x) - B(0) + A(0),$$
$$A(x) = \frac{1}{2}f(x) - \frac{1}{2c}\int_0^x g(z)dz - \frac{1}{2}\big(B(0) - A(0)\big).$$

Similarly,

$$B(x) = \frac{1}{2}f(x) + \frac{1}{2c}\int_0^x g(z)dx + \frac{1}{2}\big(B(0) - A(0)\big).$$

We can substitute these into our general solution and combine the integrals together to get

$$u(x,t) = \frac{1}{2}\big[f(x-ct) + f(x+ct)\big] + \frac{1}{2c}\int_{x-ct}^{x+ct} g(z)dz.$$

We call this d'Alembert's solution to the wave equation.

4.3 Fourier Series and Galerkin Methods

We'll find it useful, in future discussion, to generalize some of the theory we've developed from $\mathbb{R}$ to $\mathbb{C}$. We must first modify the properties that define an inner product—if u, v, w are elements of a vector space over the field $\mathbb{C}$, and if $\alpha, \beta \in \mathbb{C}$, then the following are true about the inner product $\langle \cdot, \cdot \rangle$.

- (Hermitian symmetry) $\langle u, v\rangle = \overline{\langle v, u\rangle}$.
- (Second-argument linearity) $\langle u,\, \alpha v + \beta w\rangle = \alpha \langle u, v\rangle + \beta \langle u, w\rangle$.
- (Positive definite) $\langle u, u\rangle \geq 0$, with equality if and only if $u = 0$.

The norm is still defined by $\|u\|^2 = \langle u, u\rangle$. Now, for complex-valued functions $u, v : [a, b] \to \mathbb{C}$ the L^2 inner product becomes

$$\langle u, v\rangle = \int_a^b \overline{u} v\, dx.$$

For our purposes, one particularly important set of complex-valued functions is defined by $e_n = e^{i(n\pi x/\ell)}$ for $n \in \mathbb{Z}$. It isn't difficult to show that this set is orthogonal with the $L^2[-\ell, \ell]$ inner product, and we can exploit this fact to expand complex functions (defined on $x \in [-\ell, \ell]$) in terms of these functions:

$$f(x) = \sum_{z\in\mathbb{Z}} c_n e^{i(n\pi x/\ell)}, \qquad c_m = \frac{\langle e_m,\, f(x)\rangle}{\langle e_m,\, e_m\rangle} = \frac{1}{2\ell}\int_{-\ell}^{\ell} e^{-i(n\pi x/\ell)} f(x)\, dx.$$

This is a complex Fourier series, and we could show that the given choice of coefficients still minimizes the L^2 error $E_n = \|f - S_n\|$. In fact, use of Euler's formula shows that the above series is exactly the same as the real Fourier series we worked with earlier, just with different basis functions!

As a first application, let's look at the linearized Korteweg-de Vries equation on $-\pi \leq \theta \leq \pi$ and $t > 0$:

$$u_t + u_\theta = u_{\theta\theta\theta}, \qquad u(\theta, 0) = f(\theta).$$

We'll solve this problem using a Galerkin method, which involves breaking it down into more discrete pieces using some function basis. We'll specifically be working with the above exponential basis, so we call this a Fourier spectral method. We begin by writing the solution as a complex Fourier series with time-dependent coefficients,

$$u(\theta, t) = \sum_{n\in\mathbb{Z}} A_n(t) e^{in\theta},$$

and substituting it into the PDE to get

$$\sum_{n\in\mathbb{Z}} A_n'(t) e^{in\theta} + \sum_{n\in\mathbb{Z}} (in) A_n(t) e^{in\theta} = \sum_{n\in\mathbb{Z}} (in)^3 A_n(t) e^{in\theta},$$
$$\sum_{n\in\mathbb{Z}} \left[\frac{dA_n}{dt} + i(n + n^3) A_n(t)\right] e^{in\theta} = 0.$$

All of the coefficients here must be zero, so we get an infinite set of ODEs defined by

$$\frac{dA_n}{dt} = -i(n+n^3)A_n(t), \quad n \in \mathbb{Z}.$$

From our initial condition we get

$$u(\theta,0) = f(\theta) = \sum_{n\in\mathbb{Z}} c_n e^{in\theta}, \qquad c_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-in\theta} f(\theta) d\theta,$$

and we must have $A_n(0) = c_n$. So the ODEs have solutions $A_n(t) = c_n e^{-i(n+n^3)t}$, and for the PDE

$$\begin{aligned} u(\theta,t) &= \sum_{n\in\mathbb{Z}} c_n e^{-i(n+n^3)t} e^{in\theta} \\ &= \sum_{n\in\mathbb{Z}} c_n e^{i(n\theta-(n+n^3)t)}. \end{aligned}$$

4.4 The Fourier Transform

We'll continue to build upon our theory by developing a kind of continuous analog for Fourier series. Define the discrete wavenumber $k_n = n\pi/\ell$ and the scaled Fourier coefficients

$$\hat{f}_\ell(k_n) = 2\ell c_n = \int_{-\ell}^{\ell} e^{-ik_n x} f(x)\, dx.$$

In the $\ell \to \infty$ limit the k_n get infinitesimally close together and $\hat{f}_\ell$ becomes a continuous function, $\hat{f}$. With this, we can define the Fourier transform of a function $f : \mathbb{R} \to \mathbb{C}$:

$$\mathcal{F}\{f(x)\} = \hat{f}(k) = \int_{-\infty}^{\infty} f(x) e^{-ikx} dx,$$

provided the integral exists. This happens when f is absolutely integrable—that is, when $\int_{-\infty}^{\infty} |f(x)| dx$ is finite. So we'll restrict ourselves to the space of functions in which $f(x) \to 0$ as $x \to \pm\infty$. We also have an explicit formula for the inverse Fourier transform:

$$\mathcal{F}^{-1}\{\hat{g}(k)\} = g(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{g}(k) e^{ikx} dk,$$

again provided the integral exists. $\mathcal{F}$ has a few nice properties that we can exploit to more easily compute new, potentially nasty transforms. Most important is the fact that $\mathcal{F}$ is linear (which we can use with the "building blocks" on the left), but we also have a few other general rules on the right.

$$\begin{aligned} H(x)e^{-x} &\xrightarrow{\mathcal{F}} \frac{1}{1+ik} & \qquad f(-x) &\xrightarrow{\mathcal{F}} \hat{f}(-k) \\ e^{-a|x|} &\xrightarrow{\mathcal{F}} \frac{2a}{a^2+k^2} \quad a>0 & \qquad f(ax) &\xrightarrow{\mathcal{F}} \frac{1}{|a|}\hat{f}(k/a) \quad a \neq 0 \\ H(x)e^{-x}\sin x &\xrightarrow{\mathcal{F}} \frac{1}{2-k^2+2ik} & \qquad \frac{df}{dx} &\xrightarrow{\mathcal{F}} ik\,\hat{f}(k) \\ \delta(x-x_0) &\xrightarrow{\mathcal{F}} e^{-ikx_0} & \qquad f(x) * g(x) &\xrightarrow{\mathcal{F}} \hat{f}(k)\hat{g}(k) \end{aligned}$$

Here $H(x)$ is the Heaviside function, which is zero for $x < 0$, one for $x > 0$, and $1/2$ for $x = 0$, abd $*$ denotes convolution, which is defined by

$$f(t) * g(t) = \int_{-\infty}^{\infty} f(s) g(t-s)\, ds.$$

The $\delta(x)$ is the Dirac delta function (or distribution), which can be defined in two ways. We might set

$$\delta_\epsilon(x-x_0) = \begin{cases} 1/2\epsilon & |x-x_0| < \epsilon \\ 0 & \text{otherwise} \end{cases}$$

and say that $\delta_\epsilon \to \delta$ as $\epsilon \to 0$; this is the limit description of δ. But we can also define it through the sampling property: for any function f,

$$f(x_0) = \int_{-\infty}^{\infty} f(x)\delta(x - x_0)\, dx.$$

The response of a linear ODE to a δ-function forcing is called the Green's function for the ODE. These are very important solutions because they act as "building block" solutions to the same equation with different forcing functions! Specifically, if $g(t)$ is the Green's function for an ODE, then for any other forcing function $f(t)$ we get the solution

$$y(t) = f(t) * g(t).$$

For a PDE, the Green's function is the solution with a δ-function initial condition.

When we go to solve equations using the Fourier transform, the aim is simply to turn the problem at hand into a simpler one that we can solve and then apply an inverse transform. We generally go from an ODE to an algebraic equation and from a PDE to an ODE.

We'll often encounter Gaussians in doing this, so we note that

$$\mathcal{F}\left\{e^{-ax^2/2}\right\} = e^{-k^2/2a}\sqrt{\frac{2\pi}{a}}, \qquad \mathcal{F}^{-1}\left\{e^{-ck^2}\right\} = \frac{1}{\sqrt{4\pi c}}e^{-x^2/4c},$$

and that we may sometimes write

$$\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}}\int_0^x e^{-y^2} dy.$$

We have $\operatorname{erf}(0) = 0$ and asymptotes at 1 and -1.