MATH 136: Complex Variables

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* Adapted from SP25 lectures.

1 Complex Numbers

We'll start with the absolute basics of complex numbers.

Definition: Complex number

A complex number is an expression of the form $a + bi$, with $a, b \in \mathbb{R}$ and $i^2 = -1$; the set of complex numbers is denoted $\mathbb{C}$. The real part $\mathrm{Re}(z) = a$ and the imaginary part $\mathrm{Im}(z) = b$.

Adding and subtracting complex numbers looks exactly how we'd expect—if $z = a + bi$ and $w = c + di$, then $z \pm w = (a + c) + i(b + d)$. Multiplication and division look like

$$zw = (ac - ba) + i(ad + bc), \qquad \frac{z}{w} = \left(\frac{ac + bd}{c^2 + d^2}\right) + i\left(\frac{bc - ad}{c^2 + d^2}\right),$$

where we require $w \neq 0$ for division. To get a more geometric sense for these, we make a couple more definitions motivated by framing complex numbers as vectors in the real-imaginary plane.

Definition: Norm and argument

Let the number $z = a + bi$ make an angle θ with the positive-real axis. The norm (or magnitude, modulus) and argument of z are, respectively,

$$|z| = \sqrt{a^2 + b^2}, \arg(z) = \{\theta + 2\pi k \mid k \in \mathbb{Z}\},$$

where $\arg(0)$ is undefined. The principal argument $\mathrm{Arg}(z)$ is the unique value of $\arg(z)$ in $(-\pi, \pi]$.

We can therefore express complex numbers in polar form via

$$z = r\cos\theta + ir\sin\theta = r\,\mathrm{cis}\,\theta,$$

and we could use some trigonometric identities to show that multiplication should be interpreted as multiplying magnitudes and adding arguments. (Note that arguments add, but Arguments might not.) We can thus state de Moivre's formula

$$(\cos\theta + i\sin\theta)^n = \cos(n\theta) + i\sin(n\theta), \quad n \in \mathbb{N},$$

which has a straightforward proof by induction. This also allows us to reason about complex roots! Suppose $z = r\,\mathrm{cis}\,\theta$; we seek $w = \rho\,\mathrm{cis}\,\phi$ such that $w^n = z$ for some $n \in \mathbb{N}$, that is,

$$\rho^n\,\mathrm{cis}\,n\phi = r\,\mathrm{cis}\,\theta.$$

We can see that $\rho = r^{1/n}$, the unique positive root of $r > 0$, and $\phi = (\theta + 2\pi k)/n = (\theta/n) + (2\pi/n)k$ for $k \in \mathbb{Z}$. These roots are distinct for $k \in \{0, 1, \ldots, n-1\}$, and together they define our nth roots! (Note that an nth root is called primitive if there is no $k \in \{1, \ldots, n-1\}$ for which $w^k = 1$.)

Now we'll introduce a new, distinctly complex feature of this number system.

Definition: Complex conjugate

The complex conjugate of $z = a + bi$ is

$$\overline{z} = a - bi.$$

It is clear that $\overline{z \pm w} = \overline{z} \pm \overline{w}$ and $z\overline{z} = |z|^2$, and if we interpret conjugation as reflection over the real axis, we get

$$|\overline{z}| = |z|, \quad \overline{zw} = \overline{z}\overline{w}, \quad \overline{\left(\frac{z}{w}\right)} = \frac{\overline{z}}{\overline{w}}, \quad \overline{\overline{z}} = z.$$

We also have a few handy identities:

$$\mathrm{Re}(z) = \frac{z+\overline{z}}{2}, \qquad \mathrm{Im}(z) = \frac{z-\overline{z}}{2i}, \qquad \frac{z}{w} = \frac{z\overline{w}}{|w|^2}.$$

Finally, we can use the notion of conjugation to arrive at a familiar result!

Theorem 1.1: Triangle inequality

For any $z, w \in \mathbb{C}$,

$$|z+w| \leq |z| + |w|.$$

Proof. We could prove this by interpreting complex numbers as vectors, but alternatively, we can simply write

$$\begin{aligned} |z+w| &= (z+w) \cdot \overline{z+w} \\ &= z\overline{z} + z\overline{w} + w\overline{z} + w\overline{w} \\ &= |z|^2 + |w|^2 + 2\,\mathrm{Re}(z\overline{w}). \end{aligned}$$

By the triangle inequality in $\mathbb{R}$,

$$\begin{aligned} &\leq |z|^2 + |w|^2 + |z\overline{w}| \\ &= (|z| + |w|)^2, \end{aligned}$$

as desired. $\square$

We'll end by constructing an interesting model of the complex plane. Let Σ be a diameter-1 sphere whose south pole is at the origin. If the xy-plane is identified with $\mathbb{C}$ and the u-axis is vertical, the sphere is then described by

$$x^2 + y^2 + \left(u - \frac{1}{2}\right)^2 = \frac{1}{4}.$$

Now, for each $z \in \mathbb{C}$ we draw a line segment to the north pole N and notice that the segment intersects with the sphere at exactly one other point. This defines a bijection from $C \to \Sigma \setminus \{N\}$, and to map to all of Σ we add a point ∞ defined by $\sigma \leftrightarrow N$. We therefore define the extended complex plane $\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$, and we have $\hat{\mathbb{C}} \simeq \Sigma$.

To make this bijection explicit we first parametrize the line via

$$\ell(t) = (x, y, 0) + t(-x, -y, 1), \quad t \in [0, 1]$$

and notice that it intersects with the sphere when

$$x^2(1-t)^2 + y^2(1-t)^2 + \left(t - \frac{1}{2}\right)^2 = \frac{1}{4}.$$

This equation has roots

$$t = 1, \qquad t = \frac{x^2+y^2}{1+x^2+y^2},$$

and so the nontrivial intersection is at

$$\left(\frac{x}{1+x^2+y^2}, \frac{y}{1+x^2+y^2}, \frac{x^2+y^2}{1+x^2+y^2}\right) \in \Sigma.$$

This characterization of Σ is called the Riemann sphere. (In topological terms, we call $\hat{\mathbb{C}}$ a one-point compactification of $\mathbb{C}$.)

2 Analytic Functions

2.1 Limits and Differentiability

Now we'll begin our study of functions $f : \mathbb{C} \to \mathbb{C}$, with the aim of mimicking the key results from calculus.

Definition: Limit

We say that $\lim_{z \to z_0} f(z) = w_0$ if, for all $\epsilon > 0$, there exists a $\delta > 0$ such that

$$0 < |z - z_0| < \delta \implies |f(z) - w_0| < \epsilon.$$

That is, for every radius-ϵ neighborhood around w_0, there is a radius-δ neighborhood around z_0 such that all z in the z_0-neighborhood have $f(z)$ in the w_0-neighborhood. We'll take all the basic limit properties as given, as their proofs are analogous to those in multivariable calculus.

Definition: Continuity

We say $f(z)$ is continuous at z_0 if

$$\lim_{z \to z_0} f(z) = f(z_0).$$

f is continuous on a set Ω if f is continuous at each $z_0 \in \Omega$.

Now, since the output of f has real and imaginary parts, we must be able to write f in the form

$$f(x + iy) = u(x, y) + i\, v(x, y),$$

where $u, v : \mathbb{R}^2 \to \mathbb{R}$; in this case we write $u = \mathrm{Re}(f)$ and $v = \mathrm{Im}(f)$. Conveniently, f is continuous if and only if u and v are continuous at (x_0, y_0)! We'll once again take all the basic continuity properties as given.

Definition: Derivative

Suppose $f(z)$ is defined in a neighborhood of $z_0 \in \mathbb{C}$. The derivative of f at z_0 is

$$f'(z_0) = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0} = \lim_{h \to 0} \frac{f(z_0 + h) - f(z_0)}{h},$$

provided this limit exists. Such an f is said to be differentiable at z_0.

Once again, all rules for differentiation in $\mathbb{R}$ apply to differentiation in $\mathbb{C}$. Note that f is not necessarily differentiable if $\mathrm{Re}(f)$ and $\mathrm{Im}(f)$ are differentiable!

Definition: Analytic

A function f is analytic (or holomorphic) on an open set Ω if f is differentiable at each $z_0 \in \Omega$. f is analytic at z_0 if there exists a neighborhood of z_0 on which f is analytic, and if f is analytic on $\mathbb{C}$ we say f is entire.

Analytic functions have a nice relationship between the transformations they represent and their derivatives. From the definition we have the approximation

$$f(z_0 + h) = f(z_0) + f'(z_0)h + \mathcal{O}(h),$$

meaning analytic functions are locally affine and $f'(z_0)$ encodes the information needed to take an small "step" away from z_0. Such a step involves three kinds of transformations: a rotation, a dilation, and a translation.

Analytic functions are foundational to the study of the complex numbers. They're characterized by a very particular relationship between their real and imaginary parts.

Lemma 2.1

If $f = u + iv$ is differentiable at z_0 then

$$f'(z_0) = u_x + iv_x = v_y - iu_y.$$

Proof. Suppose $f(x + iy) = u(x, y) + i\,v(x, y)$ is differentiable at $z_0 \in \mathbb{C}$, so the limit

$$f'(z_0) = \lim_{h \to 0} \frac{f(z_0 + h) - f(z_0)}{h}$$

exists. If we approach along the horizontal axis (so $\mathrm{Im}(h) = 0$) and the vertical axis (so $\mathrm{Re}(h) = 0$) the limit becomes

$$f'(z_0) = \frac{\partial u}{\partial x}(x_0, y_0) + i\frac{\partial v}{\partial x}(x_0, y_0), \qquad f'(z_0) = -i\frac{\partial u}{\partial y}(x_0, y_0) + \frac{\partial v}{\partial y}(x_0, y_0)$$

respectively. These expressions are equivalent. □

Theorem 2.2: Cauchy-Riemann equations

If $f = u + iv$ is differentiable at z_0 then

$$\begin{aligned} u_x &= v_y, \\ u_y &= -v_x. \end{aligned}$$

Consequently, if f is analytic on an open set Ω then f satisfies these equations everywhere in Ω.

Notice that if the Cauchy-Riemann equations hold then we have the derivative matrix

$$D\mathbf{f} = \begin{bmatrix} u_x & u_y \\ v_x & v_y \end{bmatrix} = \begin{bmatrix} u_x & -v_x \\ v_x & u_x \end{bmatrix},$$

which we recognize as a rotation-dilation matrix! Now, we can see that satisfying the Cauchy-Riemann equations is a necessary condition for differentiability, but they are only one part of a sufficient condition.

Theorem 2.3: Sufficient condition for differentiability

Let $f = u + iv$ be defined in an open set Ω containing $z_0 = x_0 + iy_0$. If the first partials of u and v exist on Ω, are continuous at (x_0, y_0), and satisfy the Cauchy-Riemann equations at (x_0, y_0), then f is differentiable at z_0.

Consequently, if these conditions hold everywhere in Ω then f is analytic on Ω.

Proof. Let $h = s + it$. We first break the difference quotient into real and imaginary parts,

$$\frac{f(z_0 + h) - f(z_0)}{h} = \left(\frac{u(z_0 + h) - u(z_0)}{h}\right) + i\left(\frac{v(z_0 + h) - v(z_0)}{h}\right),$$

and look specifically at its real component:

$$\frac{u(z_0 + h) - u(z_0)}{h} = \frac{1}{h}\left[\Big(u(x_0 + s,\, y_0 + y) - u(x_0 + s,\, y_0)\Big) + \Big(u(x_0 + s,\, y_0) - u(x_0, y_0)\Big)\right].$$

By the mean value theorem in $\mathbb{R}$, there exist $0 < \theta_1, \theta_2 < 1$ such that

$$= \frac{1}{h}\left[\frac{\partial u}{\partial y}(x_0 + s,\, y_0 + \theta_1 t)t + \frac{\partial u}{\partial x}(x_0 + \theta_2 s,\, y_0)s\right]$$
$$= \frac{t}{h}\frac{\partial u}{\partial y}(x_0 + s,\, y_0 + \theta_1 t) + \frac{s}{h}\frac{\partial u}{\partial x}(x_0 + \theta_2 s,\, y_0).$$

We could go through an analogous line of reasoning to get a similar expression for the imaginary component. Adding the results gives

$$\frac{f(z_0+h) - f(z_0)}{h} = \frac{t}{h}\left[\frac{\partial u}{\partial y}(x_0 + s,\, y_0 + \theta_1 t) + i\frac{\partial v}{\partial y}(x_0 + s,\, y_0 + \theta_3 t)\right]$$
$$+ \frac{s}{h}\left[\frac{\partial u}{\partial x}(x_0 + \theta_2 s,\, y_0) + i\frac{\partial v}{\partial x}(x_0 + \theta_4 s,\, y_0)\right].$$

We'd like to show that this expression approaches $f'(z_0)$ as $h \to 0$. To this end we subtract $\frac{s+it}{h}(u_x + iv_x)$ and show that the result goes to zero. Call the difference δ, so

$$\delta = \frac{t}{h}\Big[u_y(x_0 + s,\, y_0 + \theta_1 t) + v_x(x_0, y_0) + i\Big(v_y(x_0 + s,\, y_0 + \theta_3 t) - u_x(x_0, y_0)\Big)\Big]$$
$$+ \frac{s}{h}\Big[u_x(x_0 + \theta_2 s,\, y_0) - u_x(x_0, y_0) + i\Big(v_x(x_0 + \theta_4 s,\, y_0) - v_x(x_0, y_0)\Big)\Big].$$

Applying the Cauchy-Riemann equations demonstrates that, if u and v are continuous, then $\delta \to 0$. Thus $f'(z_0)$ exists. $\square$

2.2 Harmonic Functions

It turns out that analytic functions are deeply tied to the harmonic functions central to the study of PDEs! We'll state the following a fun fact for now, deferring a proof to later.

Theorem 2.4: Harmonic conjugates

If $u + iv$ is analytic on a domain (an open and connected set) Ω then $\Delta u = \Delta v = 0$ on Ω. In other words, u and v are harmonic functions. (In particular, because they are the real and imaginary parts of an analytic function, u and v are called harmonic conjugates.)

Theorem 2.5

Harmonic conjugates have orthogonal level curves.

Proof. Consider a level curve $u(x, y) = C$ with parametrization $\mathbf{x}(t) = \big(x(t), y(t)\big)$. By the chain rule,

$$0 = \frac{d}{dt}u\big(x(t), y(t)\big) = \frac{\partial u}{\partial x}x'(t) + \frac{\partial u}{\partial y}y'(t) = \nabla u \cdot \mathbf{x}'(t),$$

meaning ∇u is orthogonal to the level curves of u. Now, if u, v are harmonic conjugates then by the Cauchy-Riemann equations $\nabla u \cdot \nabla v = (u_x, u_y) \cdot (v_x, v_y) = 0$. So when $\nabla u \neq 0 \neq \nabla v$, the level curves of u and v are orthogonal. $\square$

Now for a nice application of harmonic functions that isn't explicitly related to complex variables, but is interesting nonetheless.

Theorem 2.6: Mean-value property of harmonic functions

Let $B(z_0, r)$ denote be the closed ball of radius r about z_0. If $\phi(x, y)$ is harmonic on the open set $\Omega \subseteq \mathbb{R}^2$ then

$$\phi(x_0, y_0) = \frac{1}{2\pi r} \int_{\partial B(z_0, r)} \phi(x, y)\, ds$$

for any $r > 0$ such that $B(z_0, r) \subseteq \Omega$.

Proof. The average value of ϕ over the boundary of $B(z_0, r)$ is

$$\frac{1}{2\pi r} \int_{\partial B(z_0, r)} \phi(x, y)\, ds = \frac{1}{2\pi r} \int_0^{2\pi} \phi(x_0 + r\cos t,\, y_0 + rt) r\, dt.$$

Differentiating:

$$\begin{aligned} \frac{dm}{dr} &= \frac{1}{2\pi r} \int_0^{2\pi} \left(\frac{\partial \phi}{\partial x}(x, y) \cdot r\cos t + \frac{\partial \phi}{\partial y}(x, y) \cdot r\sin t \right) dt \\ &= \frac{1}{2\pi r} \int_{\partial B(z_0, r)} \nabla\phi \cdot \mathbf{n}\, ds. \end{aligned}$$

By the divergence theorem,

$$\begin{aligned} &= \frac{1}{2\pi r} \iint_{B(z_0, r)} \nabla \cdot (\nabla\phi)\, dA \\ &= \frac{1}{2\pi r} \iint_{B(z_0, r)} \Delta\phi\, dA = 0. \end{aligned}$$

Thus $m(r)$ is independent of by r, and by continuity $m(r) = \lim_{r \to 0} m(r) = \phi(x_0, y_0)$, as desired. $\square$

2.3 Exponential and Trigonometric Functions

Now we'll look at a few particularly important analytic functions called elementary functions.

Definition: Complex exponential

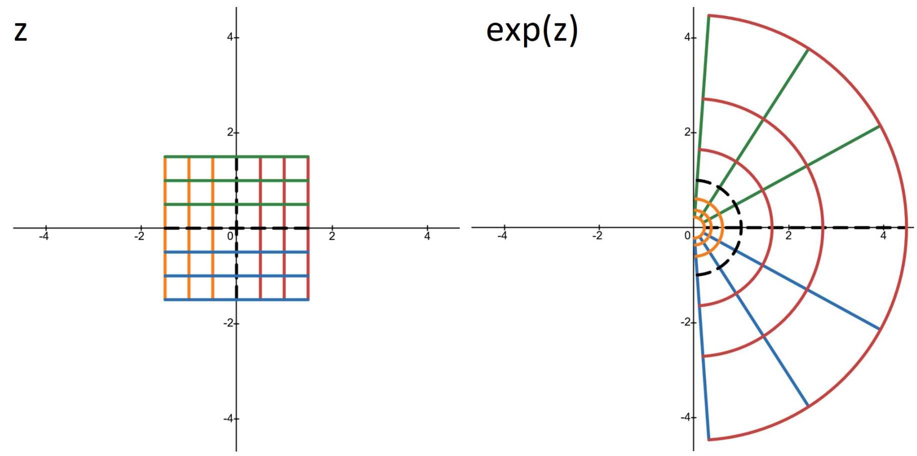
If $z = x + iy$ then

$$e^z = e^x \cos y + ie^x \sin y.$$

We can immediately make a few observations.

- The complex exponential agrees with the familiar real-valued exponential when evaluated on the real line, as we'd expect from a generalization to $\mathbb{C}$.
- e^z is entire with derivative e^z, as we'd hope.
- The familiar $e^{x+iy} = e^x e^{iy}$ and $e^{z+w} = e^z e^w$ hold, meaning $e^{nz} = (e^z)^n$ and $(e^{i\theta})^n = e^{in\theta}$ for $n \in \mathbb{Z}$.
- The magnitude of e^z is determined entirely by the real part of its argument: $|e^z| = |e^{\operatorname{Re} z}|$.
- If $z = iy$ then $e^{iy} = \cos y + i\sin y$, meaning $r\operatorname{cis}\theta = re^{i\theta}$! It follows that e^z is 2π-periodic parallel to the imaginary axis.
- e^z and its components are harmonic.

From the definition we can also see that exponentiation maps vertical lines $z = x_0 + iy$ to radius-e^{x_0} circles, and horizontal lines $z = x + iy_0$ to rays going off in the direction of e^{iy_0}.



Now we'll use Euler's formula $e^{i\theta} = \cos\theta + i\sin\theta$ to define the complex sine and cosine.

Definition: Complex sine and cosine

If $z \in \mathbb{C}$ then
$$\cos z = \frac{e^{iz} + e^{-iz}}{2}, \qquad \sin z = \frac{e^{iz} - e^{-iz}}{2i}.$$

We'll once again make some observations.

- Both sine and cosine agree with their real-valued counterparts when evaluated on the real line.
- $\sin z$ and $\cos z$ are entire with derivatives $\cos z$ and $-\sin z$, respectively.
- The familiar trigonometric identities hold: $\sin(-z) = -\sin z$, $\cos(-z) = \cos z$, $\sin^2 z + \cos^2 z = 1$,
$$\begin{aligned}\sin(z+w) &= \sin z\cos w + \cos z \sin w,\\ \cos(z+w) &= \cos z \cos w - \sin z \sin z.\end{aligned}$$
- The real and imaginary parts of each function are
$$\begin{aligned}\mathrm{Re}[\sin z] &= \sin x \cosh y, & \mathrm{Re}[\cos z] &= \cos x \cosh y,\\ \mathrm{Im}[\sin z] &= \cos x \sinh y, & \mathrm{Im}[\cos z] &= -\sin x \sinh y.\end{aligned}$$

Let's take a quick look at sine. If we define $u = \sin x \cosh y$ and $v = \cos x \sinh y$ then we can write
$$\frac{u^2}{\cosh^2 y} + \frac{v^2}{\sinh^2 y} = 1, \qquad \frac{u^2}{\sin^2 x} - \frac{v^2}{\cos^2 x} = 1.$$

Thus horizontal lines at y_0 map to ellipses and vertical lines at x_0 map to half-hyperbolas. (For $0 < x_0 < \pi$ we get a right-hyperbola, while for $\pi < x_0 < 2\pi$ we get a left-hyperbola.)

Finally, note that we define the complex hyperbolic trig functions in the exact same way as the real ones.

2.4 Logarithms and Power Functions

In defining the complex logarithm we must keep in mind that we should have $\log z = w$ if and only if $e^w = z$. We can see that if $w = u + iv$ then
$$e^u = |z|, \qquad v = \arg z.$$

Thus the logarithm must be a multi-valued function—although we can guarantee $e^{\log z} = z$ for all admissible z, in general we must write $\log e^z = z + 2\pi i k$ for all $k \in \mathbb{Z}$. The definition follows.

Definition: Complex logarithm

If $z \in \mathbb{C} \setminus \{0\}$, define the set- and single-valued functions

$$\log z = \ln|z| + i\arg(z),$$
$$\operatorname{Log} z = \ln|z| + i\operatorname{Arg}(z).$$

We will continue to write $\ln x$ for the real function $\ln x = \int_1^x dt/t$.

Theorem 2.7: Logarithms are analytic

The function $\operatorname{Log} z$ is analytic on all of $\mathbb{C}$, excluding the non-positive real axis; for such z,

$$\frac{d}{dz}\operatorname{Log} z = \frac{1}{z}.$$

It follows that the real part $\frac{1}{2}\ln(x^2 + y^2)$ is harmonic.

Proof. Let $w = \operatorname{Log} z$ and $w_0 = \operatorname{Log} z_0$. We know, from properties of the exponential function, that

$$\lim_{w \to w_0} \frac{z - z_0}{w - w_0} = \left.\frac{dz}{dw}\right|_{w=w_0} = e^{w_0} = z_0.$$

Note that $w \to w_0$ as $z \to z_0$ and that $w \neq w_0$ as long as $z \neq z_0$. Thus

$$\lim_{z \to z_0} \frac{w - w_0}{z - z_0} = \lim_{w \to w_0} \frac{1}{\frac{z - z_0}{w - w_0}} = \frac{1}{z_0},$$

as desired. $\square$

We can see that we must treat the non-positive real axis carefully while working with the principal logarithm. But really, there's nothing special about this axis—we could easily have defined the single-valued logarithm a different way and gotten a different ray to be careful of.

Definition: Branch

$F(z)$ is a branch of the multi-valued $f(z)$ on a domain D if $F(z)$ is single-valued and continuous on Ω, and for each $z \in \Omega$, $F(z)$ is one of the values $f(z)$. The set of points at which $F(z)$ is continuous is called the branch cut of F.

With this in mind, we call $\operatorname{Log} z$ the principal branch of the logarithm; the non-positive real axis is its branch cut. A branch with analytic range $(\theta, \theta + 2\pi]$ is denoted by $\operatorname{Log}_\theta z$ (with argument $\operatorname{Arg}_\theta z$), and its branch cut is in the direction of $e^{i\theta}$. The branch we choose to work with depends primarily on where we want the branch cut to be—or, rather, where we don't want it to be.

We can use logarithms to define power functions involving complex numbers!

Definition: Power function

Let $z \in \mathbb{C} \setminus \{0\}$ and $\alpha \in \mathbb{C}$. Then

$$z^\alpha = e^{\alpha \log z}.$$

Power functions are generally multi-valued, but not always—note that

$$z^\alpha = e^{\alpha \ln|z| + i\alpha \operatorname{Arg} z} e^{i \cdot 2\pi\alpha k},$$

so for real α we have three cases. If $\alpha \in \mathbb{Z}$ then $e^{i \cdot 2\pi\alpha k} = 1$ for all k and z^α is single-valued; if $\alpha \in \mathbb{Q}$ then the $e^{i \cdot 2\pi\alpha k}$ are evenly spaced around the unit circle; and for irrational α the exponential $e^{i \cdot 2\pi\alpha k}$ fills the unit circle densely.

We choose a branch of z^α by picking out a branch of the logarithm, so the principal branch is

$$z^\alpha = e^{\alpha \operatorname{Log} z}$$

with the branch cut along the non-positive real axis. Off the branch cut we have

$$\frac{d}{dz} z^\alpha = \frac{d}{dz} e^{\alpha \operatorname{Log} z} = \frac{\alpha}{z} e^{\alpha \operatorname{Log} z} = \alpha z^{\alpha - 1}.$$

A couple more notes. First, the familiar laws of exponents hold—multiplication turns into addition, and division into subtraction. We can also use out brief analysis of power functions to determine when we're allowed to split (principal) square roots like $\sqrt{z^2} = \sqrt{z}\sqrt{z}$! Following the definition, we can see that

$$\sqrt{z^2} = e^{\ln|z|} e^{i \frac{1}{2} \operatorname{Arg}(z^2)}, \qquad \sqrt{z}\sqrt{z} = e^{\ln|z|} e^{i \operatorname{Arg} z}.$$

For equality we require $\operatorname{Arg} z^2 = 2 \operatorname{Arg} z$, so we must have $\operatorname{Arg} z \in (-\pi/2, \pi/2]$. In other words, these are the z for which squaring doesn't move its argument outside the principal branch!

3 Integration

3.1 Contour Integrals

The integral in $\mathbb{C}$ is defined in pretty much the same way as it is in $\mathbb{R}$, and evaluating them is reminiscent of line integrals in multivariable calculus.

Definition: Integral

Let γ be a directed smooth curve in $\mathbb{C}$ that is split into several pieces γ_k, each with endpoints z_k, z_{k+1} and a point c_k between them. Then

$$\int_\gamma f(z)\,dz = \lim_{\substack{N\to\infty \\ \Delta z_k \to 0}} \sum_{k=0}^{N-1} f(c_k)\Delta z_k,$$

provided the limit exists and is independent of the choice of partition.

Theorem 3.1

Let f be a function continuous on the directed smooth curve γ. Then if $z = z(t)$, $t \in [a,b]$ is a parametrization of γ, we have

$$\int_\gamma f(z)\,dz = \int_a^b f(z(t))\,z'(t)\,dt.$$

Proof. Suppose γ has a parametrization $z(t) = x(t) + i\,y(t)$, $t \in [a,b]$; the path is traversed once and $z'(t) \neq 0$ for all t. Defining $\Delta t_k = t_{k+1} - t_k$, we can write

$$\int_\gamma f(z)\,dz = \lim_{\substack{N\to\infty \\ \Delta t_k \to 0}} \sum_{k=0}^{N-1} f(z(c_k))\frac{z(t_{k+1}) - z(t_k)}{\Delta t_k}\Delta t_k,$$

where $c_k \in t_k, t_{k+1}$. There is no mean value theorem in $\mathbb{C}$, but we can split this difference quotient into real and imaginary parts to apply the MVT in $\mathbb{R}$: for some $\alpha_k, \beta_k \in (t_k, t_{k+1})$, we can write

$$\begin{aligned} &= \lim_{\substack{N\to\infty \\ \Delta t_k \to 0}} \sum_{k=0}^{N-1} f(z(c_k))\big(x'(\alpha_k) + i\,y'(\beta_k)\big)\Delta t_k \\ &= \int_a^b f(z(t))\,z'(t)\,dt, \end{aligned}$$

as desired. □

We could use this to show that

$$\oint_\gamma (z - z_0)^n\,dz = \begin{cases} 0 & n \neq -1, \\ 2\pi i & n = -1, \end{cases}$$

where γ is a once-traversed circle centered at z_0, oriented counterclockwise; this particular integral will prove to be very useful later. We should expect that this result, like any other integral, is independent of how we choose to parametrize γ, and this is the case! Proving it is a simple application of the above theorem.

Corollary 3.2

Let $\tilde{\gamma}$ be a reparametrization of γ—that is, suppose there is a $\phi(t)$ satisfying $\phi'(t) > 0$ such that $\gamma(t) = \tilde{\gamma}(\phi(t))$. Then

$$\int_\gamma f(z)\,dz = \int_{\tilde{\gamma}} f(z)\,dz.$$

Given a path γ, we will use $-\gamma$ to denote γ as traversed in the opposite direction, and we could show that

$$\int_{-\gamma} f(z)\,dz = -\int_\gamma f(z)\,dz.$$

In general we will integrate over contours, which are finite sequences of directed smooth curves that connect one after another. We will not always need to evaluate such contour integrals however; an upper bound is often good enough, and we can get one pretty easily.

Lemma 3.3: ML lemma

Let γ be a contour with length $\ell(\lambda)$. If f is continuous on γ and if $|f(z)| \leq M$ for all z on γ, then

$$\left|\int_\gamma f(z)\,dz\right| \leq M \cdot \ell(\gamma).$$

Proof. Suppose $\int_\gamma f(z)\,dz = r_0 e^{i\theta_0}$. Then

$$\begin{aligned}
\left|\int_\gamma f(z)\,dz\right| &= e^{-i\theta_0}\int_a^b f(z(t))\,z'(t)\,dt \\
&= \int_a^b \operatorname{Re}\left[e^{-i\theta_0} f(z(t))\,z'(t)\right] dt \\
&\leq \int_a^b M\,|z'(t)|\,dt = M \cdot \ell(\gamma),
\end{aligned}$$

as desired. □

Note that the "lower triangle inequality" $\big||a| - |b|\big| \leq |a+b|$ will often be useful in using this inequality.

3.2 Path Independence

Now we're ready for some of the bigger theorems in integral calculus, starting with an old friend.

Theorem 3.4: Fundamental theorem

Suppose f is continuous on a domain Ω and has a primitive (antiderivative) $F(z)$ throughout Ω. Then for any contour γ in Ω with endpoints z and w we have

$$\int_\gamma f(z)\,dz = F(w) - F(z).$$

Proof. Let $\gamma : [a, b] \to \mathbb{C}$ be a smooth contour. Then

$$\int_\gamma f(z)\,dz = \int_a^b f(\gamma(t))\gamma'(t)\,dt = F(\gamma(b)) - F(\gamma(a)).$$

For any other contour, we can break it up into smooth contours and sum them to get the same. □

Note that this implies that F is analytic, and so also continuous, in Ω. Next we have a less immediately applicable set of equivalent statements, but one whose use will soon become clearer.

Theorem 3.5: ABC theorem

Let f be continuous in a domain Ω. Then the following statements are equivalent.

(a) f has an antiderivative in Ω.

(b) If γ is a closed loop in Ω then $\int_\gamma f(z)\,dz = 0$.

(c) The contour integrals of f are path-independent in Ω.

Proof. It is easy to see that (a) implies both (b) by the fundamental theorem. Now let γ_1, γ_2 be any two contours in Ω with the same endpoints and define $\Gamma = \gamma_1 - \gamma_2$. If we assume (b), then

$$0 = \int_{\gamma_1} f(z)\,dz + \int_{-\gamma_2} f(z)\,dz = \int_{\gamma_1} f(z)\,dz - \int_{\gamma_2} f(z)\,dz$$

and we get (c). Now, assuming (c), fix $z_0 \in \Omega$ and let $\gamma(z)$ be a path from z_0 to some $z \in \Omega$. Define

$$F(z) = \int_{\gamma(z)} f(w)\,dw, \quad z \neq z_0.$$

This function is well-defined because $F(z)$ is independent of the specific choice of path γ, and we could apply the definition of the derivative to show that $F' = f$, so we get (a). $\square$

3.3 Cauchy's Theorem

The ABC theorem might seem difficult to apply at present, but we're about to derive another result that gives a sufficient condition for (b). We need to lay a bit of topological groundwork first, though.

Definition: Homotopy

Let $\gamma_0, \gamma_1 : [0,1] \to \Omega$ be closed curves in a domain Ω. γ_1 is homotopic to γ_0 if there exists a continuous homotopy $H : [0,1] \times [0,1] \to \Omega$ such that

$$H(0,t) = \gamma_0(t), \qquad H(1,t) = \gamma_1(t), \qquad H(s,0) = H(s,1).$$

(We include the third condition because we're interested in closed curves.)

Definition: Simply connected domain

A domain Ω is simply connected if every closed path in Ω is homotopic to a point in Ω—in other words, the domain has no "holes".

We could do some straight forward calculus to show that if

$$I(s) = \int_{\gamma_s} f(z)\,dz = \int_0^1 f(H(s,t)) \frac{\partial H}{\partial t}(s,t)\,dt,$$

then $I'(s) = 0$ so long as $f \in C^1$ (to bring the $\partial/\partial s$ into the integral) and $H \in C^2$ (for the equality of mixed partials). Thus if f has a continuous derivative, its integral is the same over any pair of homotopic curves.

A couple more clarifications. A simple closed curve is one that does not self-intersect, and it is positively oriented if the enclosed region is on the left of the curve. (This is how we generalize counterclockwise-ness.)

Now we'll give a couple of proofs for the same theorem—one using this idea of contour deformation we just arrived at, and another using vector calculus.

Theorem 3.6: Cauchy's theorem (preliminary)

If f is analytic with a continuous derivative on a simply connected domain Ω and γ is a positively oriented simple closed curve in Ω, then

$$\oint_\gamma f(z)\,dz = 0.$$

Proof. On a simply connected domain Ω, a closed loop γ in Ω is homotopic to a point $z_0 \in \Omega$. Thus

$$\oint_\gamma f(z)\,dz = \int_{\{z_0\}} f(z)\,dz = 0$$

as desired. □

Proof. Let γ be parametrized by $z(t)$, $t \in [a,b]$. If $z = x + iy$ and $f = u + iv$, then we can write

$$\begin{aligned}\oint_\gamma f\,dz &= \int_a^b f(z(t))z'(t)\,dt \\ &= \int_a^b (ux' - vy') + i(vx' + uy')\,dt \\ &= \oint_\gamma u\,dx - v\,dy + i\oint_\gamma v\,dx + u\,dy,\end{aligned}$$

and by Green's theorem

$$= \iint_\Omega (-v_x - u_y)\,dA + i\iint_\Omega (u_x = v_y)\,dA;$$

by the Cauchy-Riemann equations, this goes to zero. □

These proofs are simple, but we'd like to remove the condition of having a continuous derivative. We begin with a special case.

Lemma 3.7

Let $R = [a,b] \times [c,d]$ be a rectangle in Ω. If f is analytic on Ω then

$$\oint_{\partial R} f\,dz = 0.$$

Proof. Suppose a rectangle R has diagonal and perimeter lengths D and P respectively. Divide this rectangle into four sub-rectangles and suppose, without loss of generality, that R_{11} is the sub-rectangle for which $\left|\oint_{\partial R_{i1}} f\,dz\right|$ is maximized. Now split this maximal rectangle into four and let R_{12} be the piece that maximizes the line integral. Continuing on this way for k subdivisions reveals that

$$\left|\oint_{\partial R} f\,dz\right| \le 4^k \left|\oint_{\partial R_{1k}} f\,dz\right|.$$

From analysis we know that there exists a $z_0 \in \bigcap_{k=1}^{\infty} R_{1k}$. f is analytic at this point, meaning

$$f(z) = f(z_0) + f'(z_0)(z - z_0) + o(z - z_0), \qquad \lim_{z \to z_0} \frac{o(z - z_0)}{z - z_0} = 0.$$

Thus for every $\epsilon > 0$ there is a $k > 1$ such that $|o(h)/(z - z_0)| < \epsilon$ for all $z \in R_{1k}$. By the ABC theorem and ML lemma,

$$\left| \oint_{\partial R_{1k}} f(z)\,dz \right| = \left| \oint_{\partial R_{1k}} o(h)\,dz \right| \leq (\epsilon \cdot 2^{-k} D) \cdot 2^{-k} P.$$

The original integral therefore becomes

$$\left| \oint_{\partial R} f\,dz \right| \leq 4^k \left| \oint_{\partial R_{1k}} f\,dz \right| \leq 4^k \cdot (\epsilon \cdot 2^{-k} D \cdot 2^{-k} P) = \epsilon D P.$$

Since our $\epsilon > 0$ was arbitrary, this integral is zero. □

Now we'll take another step by proving Cauchy's theorem, but only in a disk.

Lemma 3.8: Cauchy's theorem (in a disk)

Let Ω be an open disk with radius R centered at z_0. If f is analytic in Ω and γ is a closed contour in Ω then

$$\oint_\gamma f(z)\,dz = 0.$$

Proof. Given $w \in \Omega$ define γ_w as the path from z_0 to w traversed in two purely horizontal and vertical components γ_1, γ_2. The function

$$F(w) = \int_{\gamma_w} f(z)\,dz$$

is well-defined for all $w \in \Omega$, and we'd like to prove that $F'(w) = f(w)$.

We first determine $F(w+h) - F(w)$—given $\epsilon > 0$, there exists a $\delta > 0$ not only such that $|z - w| < \delta$ implies $|f(z) - f(w)| < \epsilon$, but also such that $\{|z - w| < \delta\} \subseteq \Omega$. For $|h| < \delta$ we similarly define γ_{w+h} in terms of horizontal and vertical components and notice that

$$\int_{\gamma_{w+h}} f\,dz = \int_{\gamma_w} f\,dz \pm \oint_{\partial R} f\,dz + \int_{w \to w+h} f\,dz,$$

where R is the rectangle formed by connecting γ_w and γ_{w+h} and the $\pm$ is whichever sign is needed to make things cancel nicely. The integral over ∂R is zero by the previous lemma, meaning

$$F(w+h) - F(w) = \int_{w \to w+h} f\,dz.$$

Now by the ML lemma,

$$\left| \frac{F(w+h) - F(w)}{h} - f(w) \right| = \frac{1}{|h|} \left| \int_{w \to w+h} (f(z) - f(w))\,dz \right| < \frac{1}{|h|} (\epsilon \cdot 2|h|) = 2\epsilon,$$

and since $\epsilon > 0$ was arbitrary we arrive at $F'(w) = f(w)$ for all $w \in \Omega$. Thus by the ABC theorem $\oint_\gamma f\,dz = 0$, as desired. □

Finally, we can make a contour deformation argument to get what we want!

Theorem 3.9: Cauchy's theorem

If f is analytic in a simply connected domain Ω and γ is any closed contour in Ω, then

$$\oint_\gamma f(z)\,dz = 0.$$

Proof. If γ_0 is a closed contour in Ω then it is homotopic to a circle γ_1 interior to γ_0. If we break the domain of the homotopy H into a bunch of very small chunks R_{ij} and define $\Gamma_{ij} = H(\partial R_{ij})$ then by the previous theorem

$$\int_\Gamma f(z)\,dz = \sum_{i,j} \int_{\Gamma_{ij}} f(z)\,dz = \sum_{i,j} 0 = 0,$$

where Γ is the contour produced by traversing γ_0, moving inward to traverse γ_1, and then back out again. If $\gamma_0 = H(0,t)$ and $\gamma_1 = H(1,t)$ then we can write

$$0 = \int_\Gamma f(z)\,dz = \int_{-H(0,t)} f(z)\,dz + \int_{H(1,t)} f(z)\,dz + \int_{H(s,0)} f(z)\,dz + \int_{-H(s,1)} f(z)\,dz,$$

meaning $\oint_{\gamma_0} f(z)\,dz = \oint_{\gamma_1} f(z)\,dz$. Thus integrals in Ω are invariant under homotopy and for some $z_0 \in \Omega$

$$\oint_{\gamma_0} f(z)\,dz = \int_{\{z_0\}} f(z)\,dz = 0,$$

as desired. $\square$

3.4 Cauchy's Integral Formula

Contour deformation also provides motivation for our next key result.

Theorem 3.10: Cauchy's integral formula

Let γ be a simple closed positively-oriented curve. If f is analytic in a simply connected domain Ω containing γ and z_0 is interior to γ, then

$$f(z_0) = \frac{1}{2\pi i} \oint_\gamma \frac{f(z)}{z - z_0}\,dz.$$

Proof. f is continuous at z_0, so for every $\epsilon > 0$ there is a $\delta > 0$ such that $|z - z_0| < \delta$ implies $|f(z) - f(z_0)| < \epsilon$. Let $r \in (0, \delta)$ be the radius of a positively-oriented circle γ_r centered at z_0; then, because $2\pi i = \oint_{\gamma_r} dz/(z - z_0)$,

$$\left| \oint_{\gamma_r} \frac{f(z)}{z - z_0}\,dz - 2\pi i\, f(z_0) \right| = \left| \oint_{\gamma_r} \frac{f(z) - f(z_0)}{z - z_0} \right| \le \oint_{\gamma_r} \frac{|f(z) - f(z_0)|}{|z - z_0|}\,dz < \frac{\epsilon}{r} \cdot 2\pi r.$$

Since $\epsilon > 0$ was arbitrary we conclude, by contour deformation, that

$$\oint_\gamma \frac{f(z)}{z - z_0}\,dz = \oint_{\gamma_r} \frac{f(z)}{z - z_0} = 2\pi i\, f(z_0),$$

as desired. $\square$

Now, if we treat this integral as a function (rather than evaluating it only at z_0), its derivatives turn out to behave nicely.

Theorem 3.11

Let g be continuous on the contour γ, and for each $z \notin \gamma$ set

$$G(z) = \int_\gamma \frac{g(s)}{s-z}\, ds.$$

Then G is analytic and

$$G'(z) = \int_\gamma \frac{g(s)}{(s-z)^2}\, ds.$$

Proof. We could do some algebra and then apply the ML lemma to get

$$\begin{aligned} \left|\frac{G(z+h)-G(z)}{h} - \int_\gamma \frac{g(s)}{(s-z)^2}\, dz\right| &= \left|\int_\gamma g(s)\left[\frac{h}{(s-z-h)(s-z)^2}\right]\right| \\ &= |h| M \int_\gamma \frac{1}{|s-z-h||s-z|^2}\, ds, \end{aligned}$$

where $M = \max_{s\in\gamma} |g(s)|$. Now let $d = \min_{s\in\gamma}|s-z|$; then $|s-z-h| \geq |s-z| - |h| \geq d - d/2$, so

$$\leq |h| M \int_\gamma \frac{1}{(d/2)d^2}\, ds,$$

which goes to zero as $h \to 0$. □

In showing this, we've also shown that the expression in Cauchy's formula is analytic with

$$f'(z) = \frac{1}{2\pi i} \oint_\gamma \frac{f(s)}{(s-z)^2}\, ds.$$

It turns out that we can go through a similar line of reasoning to show that this f' is analytic with a similar integral expression; a simple proof by induction gives the following.

Theorem 3.12: Cauchy's integral formula (for the kth derivative)

If f is analytic on and inside a positively-oriented simple closed curve γ and z lies inside γ, then

$$f^{(k)}(z) = \frac{k!}{2\pi i} \oint_\gamma \frac{f(s)}{(s-z)^{k+1}}\, ds, \qquad k \in \mathbb{N}.$$

Corollary 3.13

If f is analytic in a domain Ω then all of its derivatives exist and are analytic on Ω.

Because differentiability implies continuity, $f \in C^\infty$ and the continuous-derivative assumption we made earlier on was true all along! Now we have a couple more theorems that tie some loose ends.

Corollary 3.14

If $f = u + iv$ is analytic then u and v are harmonic functions.

Proof. Since $f \in C^\infty$, we have $u, v \in C^\infty$. The Cauchy-Riemann equations imply $u_{xx} = v_{yx}$ and $v_{yy} = -v_{xy}$, and since $v \in C^2$ its mixed partials are equal and so $\Delta u = \Delta v = 0$. □

Theorem 3.15: Morera's theorem

If f is continuous on Ω and $\oint_\gamma f\,dz = 0$ for all closed contours γ in Ω, then f is analytic.

Proof. Suppose $\oint_\gamma f\,dz = 0$ for all γ. Then by the ABC theorem there exists an F such that $F = f'$. Thus F is analytic, and so is its derivative f. □

3.5 Bounds for Analytic Functions

Now we"ll use these results to place restrictions on the behavior of analytic functions.

Theorem 3.16: Cauchy's estimate

Let f be analytic on a closed ball $B(z_0, R)$. If $|f(z)| \leq M$ for all $z \in B$, then

$$\left|f^{(k)}(z_0)\right| \leq \frac{k!M}{R^k}, \qquad k \in \mathbb{N}.$$

Proof. The integrand in the Cauchy integral formula (for the kth derivative) is bounded by M/R^{n+1}, so by the ML lemma

$$\left|f^{(k)}(z_0)\right| \leq \frac{k!}{2\pi}\left(\frac{M}{R^{k+1}} \cdot 2\pi R\right),$$

as desired. □

Notice that if a function is entire and bounded over all of $\mathbb{C}$, then it satisfies the conditions of the previous theorem for any ball $B(z_0, R)$. Thus if we take $R \to \infty$ its derivatives must vanish and we get the following.

Theorem 3.17: Liouville's theorem

If $f(z)$ is bounded and entire, then f is constant.

We can use Liouville's theorem to prove a familiar statement, but an unexpected one for this context!

Theorem 3.18: Fundamental theorem of algebra

Every non-constant polynomial with complex coefficients has at least one zero.

Proof. Suppose $P(z) = a_n z^n + \cdots + a_0$, $a_n \neq 0$, has no zeroes, so $H(z) = 1/P(z)$ is entire. For sufficiently large $|z|$ we have $|P(z)/z^n| \geq |a_n|/2$, meaning there exists an $R \gg 1$ such that

$$|H(z)| = \frac{1}{|P(z)|} \leq \frac{2}{R^n|a_n|}, \qquad |z| \geq R.$$

Also, by continuity (and the extreme value theorem) there exists a finite M such that $|H(z)| \leq M$ for all $|z| \leq R$, meaning

$$|H(z)| \leq \max\left\{M, \frac{2}{R^n|a_n|}\right\}, \qquad z \in \mathbb{C}.$$

Thus $H(z)$ is constant, and so is $P(z)$—a contradiction. Thus $P(z)$ has at least one root. □

In fact, all of these polynomials' roots lie in $\mathbb{C}$—any polynomial can be factored ito $P(z) = (z - z_1)Q_1(z)$, where Q_1 is another polynomial that can be factored in the same way. (This process has a finite number of iteration.) We therefore call $\mathbb{C}$ "algebraically closed". Now for one last theorem!

Theorem 3.19: Maximum-modulus principle

If f is analytic on a domain Ω and $|f(z)|$ has a local maximum at $z_0 \in \Omega$, then f is constant on a neighborhood of z_0.

Proof. Suppose $z_0 \in \Omega$ with $|f(z_0)| \geq f(z)$ for all $z \in \Omega$', where the domain $\Omega' \subset \Omega$. Choose $R > 0$ such that $B(z_0, R) \subset \Omega'$; by Cauchy's integral formula

$$f(z_0) = \frac{1}{2\pi i} \oint_{\partial B} \frac{f(z)}{z - z_0}\, dz = \frac{1}{2\pi} \int_0^{2\pi} f(z_0 + Re^{it})\, dt,$$

and so

$$|f(z_0)| \leq \frac{1}{2\pi} \int_0^{2\pi} |f(z + Re^{it})|\, dt \leq \frac{1}{2\pi} \int_0^{2\pi} |f(z_0)|\, dt = |f(z_0)|.$$

It follows that the two intermediate expressions are equal, so

$$\frac{1}{2\pi} \int_0^{2\pi} \Big(|f(z_0)| - |f(z_0 + Re^{it})|\Big) dt = 0$$

and $|f(z_0)| = |f(z_0 + Re^{it})|$. Thus $|f|$ is constant on a $\partial B(z_0, R)$ of any radius $0 < r < R$, and so $|f|$ is constant on $B(z_0, R)$.

Now, if $f = u + iv$ then $u^2 + v^2$ is constant, meaning both of its partial derivatives are equal to zero. An application of the Cauchy-Riemann equations gives the matrix equation

$$\begin{bmatrix} u & -v \\ v & u \end{bmatrix} \begin{bmatrix} u_x \\ u_y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad (x_0, y_0) \in B.$$

If this coefficient matrix is singular somewhere in B then $u = v = 0$ there, and we're done. Otherwise the vector above must be the zero vector, meaning $u_x = v_y = 0$ and $v_x = -u_y = 0$ on B. Thus u and v are constant on B, and so is f. □

This may feel like a relatively weak statement, but we can turn this local statement into a global one!

Corollary 3.20

If f is analytic in a domain Ω and $|f(z)|$ achieves its maximum value at a point $z_0 \in \Omega$, then f is constant in Ω.

Proof. Suppose $|f(z)|$ is not constant. Then there is a $z_1 \in \Omega$ such that $|f(z_1)| < |f(z_0)|$. Let γ be a path in Ω running from z_0 to z_1, and let w be the first point along this path where $|f(z)|$ first starts to decrease. Thus $|f(w)| = |f(z_0)|$.

Now, since w is an interior point, there must be a disk centered at w that lies within Ω. But $|f|$ is constant in this disk by the previous theorem, meaning the points that come immediately after w along γ do not have decreasing modulus, a contradiction. Thus $|f|$ is constant and so is f. □

Corollary 3.21

If f is analytic on a bounded domain Ω and continuous on the closure $\overline{\Omega} = \Omega \cup \partial\Omega$, then $|f|$ achieves its maximum on $\partial\Omega$.