MATH 131: Mathematical Analysis I

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1 Basic Topology

1.1 Cardinality

In this first part of the course we'll talk about sets, structure for sets, and the consequences of this structure. Let's start with the natural numbers $\mathbb{N} = \{1, 2, 3, \ldots\}$; the rules that precisely define the natural numbers are specified by the Peano axioms.

Definition: Peano axioms

Let $\mathbb{N}$ be a set containing the element 1. Define a successor function $S : \mathbb{N} \to \mathbb{N}$, with the following:

- $S(x) \neq 1$ for all $x \in \mathbb{N}$
- If $S(x) = S(y)$, then $x = y$ for all $x, y \in \mathbb{N}$.
- Let A be any subset of $\mathbb{N}$ which contains 1 and is closed under S, so $S(x) \in A$ for all $x \in A$.

Then $A = \mathbb{N}$.

These axioms together ensure that there is a first element in $\mathbb{N}$, and that for each element in $\mathbb{N}$ there is a "next" element. This implies all of the basic properties of the natural numbers! We give one without proof.

Theorem 1.1: Well-ordering principle

Every nonempty subset of $\mathbb{N}$ has a smallest element.

The Peano axioms also form the basis of mathematical induction. Specifically, let $P(n)$ be a set of statements indexed by $n \in \mathbb{N}$, and define $S = \{n : P(n) \text{ is true}\}$. To show that $P(n)$ is true for all n, it suffices to show that S satisfies the Peano axioms! Specifically, we must show that

- $P(1)$ is true, meaning the first element of S is 1 (the base case); and that
- $P(k) \implies P(k+1)$, so each element in S has a successor (the inductive step).

If we can do both of these things, then $S = \mathbb{N}$ by the Peano axioms, and $P(n)$ is true for all n. One common variant on this principle is called strong induction, in which the inductive step assumes the truth of all $P(1), P(2), \ldots, P(k)$ rather than just $P(k)$.

We can leverage our definition of the natural numbers to rigorously define the size of a set.

Definition: Cardinality

Suppose $J_n = \{1, 2, \ldots, n\} \subset \mathbb{N}$. If A is in one-to-one correspondence with J_n, then A has cardinality (or size) n and we write $|A| = n$. If $A = \emptyset$, we define $|A| = 0$.

Most interesting sets, however, have far too many elements to have such a correspondence. This is what we usually mean by "infinity".

Definition: Finite and infinite

We say A is finite if there is a bijection between A and J_n. Otherwise, A is infinite.

To begin our characterization of infinity, we stick with what we know: sets that are structurally similar to the natural numbers.

Definition: Countable

A set A is countable if A is in one-to-one correspondence with $\mathbb{N}$.

We have two powerful theorems that we can use to determine the cardinalities of sets built using countable sets. They're both pretty unintuitive at first glance.

Theorem 1.2

Every infinite subset of a countable set is countable.

Proof. Suppose A is a countable set, so $A = \{x_1, x_2, \ldots\}$.

Consider a subset $E \subseteq A$. By the well-ordering principle, E has a smallest element; index this element as n_1. The next element receives index n_2, then n_3, and so on, so $E = \{x_{n_1}, x_{n_2}, \ldots\}$. This process creates the bijection $f(k) = x_{n_k}$ between E and $\mathbb{N}$, so E is countable. $\square$

Theorem 1.3: Hilbert's hotel

The union of a countable set of countable sets is countable.

Proof. Suppose $E_n = \{x_{n_1}, x_{n_2}, \ldots\}$ is an uncountable set, so

$$S = \bigcup_{i=1}^{\infty} E_i$$

is a union of a countable set of countable sets. We can list out the elements in S as an infinite grid:

$$\begin{matrix} x_{11} & x_{12} & x_{13} & \cdots \\ x_{21} & x_{22} & x_{23} & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{matrix}$$

We can create a bijection with $\mathbb{N}$ just by slithering across the diagonals of this grid! For example, the first six instances of this bijection are

$$\begin{aligned} x_{11} &\mapsto 1 & x_{31} &\mapsto 4 \\ x_{12} &\mapsto 2 & x_{22} &\mapsto 5 \\ x_{21} &\mapsto 3 & x_{13} &\mapsto 6 \end{aligned}$$

Therefore, there is a bijection between S and $\mathbb{N}$, and S is countable. $\square$

We can use these to show, for example, that the rational numbers are countable, even if it seems like there should be many more of them. This might suggest that we can't get any bigger than countable, but we can!

Theorem 1.4: Cantor's diagonalization argument

The set $(0, 1) \subset \mathbb{R}$ is uncountable.

Proof. Construct an arbitrary mapping $f : \mathbb{N} \to (0, 1)$:

$$\begin{aligned} 1 &\mapsto x_1, \quad x_1 = 0.x_{11}x_{12}x_{13}\cdots \\ 2 &\mapsto x_2, \quad x_2 = 0.x_{21}x_{22}x_{23}\cdots \\ &\vdots \end{aligned}$$

Now construct the real number $r = 0.r_1r_2r_3\cdots$ by assigning each decimal place as follows:

$$r_i = \begin{cases} 1 & \text{if } x_{ii} \neq 1 \\ 2 & \text{if } x_{ii} = 1 \end{cases}$$

This construction guarantees that r differs from x_1 in its first decimal place, from x_2 in its second decimal place, and so on. So r is not in the range of f for any $f : \mathbb{N} \to (0,1)$, meaning there is no bijection between $(0,1)$ and $\mathbb{N}$. Therefore, $(0,1)$ is uncountable. □

This shows that the set of real numbers between 0 and 1 is somehow larger than the set of all natural numbers. We can go even larger than this!

Definition: Power set

The power set $P(A)$ of a set A is the set of all subsets of A.

Theorem 1.5: Cantor's theorem

There is a function $f : A \to P(A)$ that is injective, but there is no such surjective function.

Proof. It is easy to show that there is an injection $f : A \to P(A)$—one example is $f(x) = \{x\}$.

Now, for contradiction, suppose that there is a surjection $g : A \to P(A)$. So by the definition of $P(A)$, for any $B \subseteq A$, there exists a $y \in A$ such that $g(y) = B$. In this case, $y \in B$ if and only if $y \in g(y)$.

But we can construct a B that is not in the range of g! Specifically, consider the subset of all elements in A that, under g, do not map to themselves; that is,

$$B = \{x \in A : x \notin g(x)\}.$$

By definition, $y \in B$ if and only if $y \notin g(y)$, but this contradicts our previous conclusion about B. Therefore, there is no such surjection g. □

This proof is really just a slicker version of Cantor's diagonal argument. Imagine we're constructing B by going element by element in A—we include $x \in A$ if it maps to a subset that does not contain itself, and we otherwise exclude it. This new subset B differs from all others in $\mathrm{range}(g)$. Any $x \in A$ satisfying $x \notin g(x)$ mustn't map to B because $x \in B$. We can make the reverse argument for $y \in A$ satisfying $y \in g(y)$.

Cantor's theorem provides a way to construct arbitrarily large infinities by constructing arbitrarily large power sets, so there is no "biggest" infinity. Another natural question is whether or not there's is a set whose cardinality falls between these infinities. This is called the continuum hypothesis, and it has been shown to be undecidable under today's standard axioms for set theory!

1.2 Order, Fields, and Bounds

Right now our sets are nothing more than collections of things. A reasonable first step in giving them some structure is to assign an order to the elements.

Definition: Order

An order on a set S is a relation $<$ satisfying the following.

- (Trichotomy) If $x, y \in S$, then exactly one is true: $x < y$, $x = y$, or $y < x$.
- (Transitivity) If $x, y, z \in S$ and if $x < y$ and $y < z$, then $x < z$.

Note: Writing $x > y$ is equivalent to writing $y < x$. Also, $x \leq y$ is equivalent to "$x < y$ or $x = y$", and $x \geq y$ is defined similarly.

We can also slap on a couple of reasonable operations so we can start doing some algebra.

Definition: Field

A field F is a set with two operations $+, \times$ that each satisfies the following axioms.

- (Closure) $x + y \in F$ and $xy \in F$.
- (Commutativity) $x + y = y + x$ and $xy = yx$.
- (Associativity) $(x + y) + z = x + (y + z)$ and $(xy)z = x(yz)$
- (Identity) F contains 0 such that $0 + x = x$ and contains 1 such that $1x = x$.
- (Inverse) For every $x \in F$ there is a $-x \in F$ such that $x + (-x) = 0$, and if $x \neq 0$ then there is a $x^{-1} \in F$ such that $xx^{-1} = 1$.

The operations also satisfy the distributive law, $x(y + z) = xy + xz$.

Both of these ideas fit together in such a way that they complement each other very nicely.

Definition: Ordered field

Suppose F is a field and has an order. Let $x, y, z \in F$; we say F is an ordered field if the following hold.

- If $y < z$, then $y + x < z + x$.
- If $y < z$ and $x > 0$, then $xy < xz$.

With this, we can talk about what it means for a set to be restricted in the context of some larger set.

Definition: Upper and lower bound

Let $E \subseteq S$ be an ordered set. If there is an element $\beta \in S$ such that for all $x \in E$ we have $x \leq \beta$, then β is an upper bound for E. In this case we say E is bounded above. We can similarly define lower bound, in which case E is bounded below.

Definition: Supremum and infimum

Again consider $E \subseteq S$. If there exists an $\alpha \in S$ that satisfies

- α is an upper bound of E, and
- if $\gamma < \alpha$ then γ is not an upper bound of E,

then α is the least upper bound or supremum for E. We write $\alpha = \sup E$. We can similarly define the greatest lower bound or infimum of E.

Note that the supremum and the maximum are not the same thing—maxima have the additional requirement that they must actually be in the set they're describing. The same goes for infima and minima. We'll focus on suprema from here on, but all of the statements we make for suprema have analogs for infima. The supremum has some intuitive properties, many of which we can state without proof.

Theorem 1.6: Properties of suprema

(a) γ is an upper bound for A if and only if $\sup A \leq \gamma$.

(b) $a \leq \gamma$ for all $a \in A$ if and only if $\sup A \leq \gamma$.

(c) $a < \gamma$ for all $a \in A$ if and only if $\sup A \leq \gamma$.

(d) If $\gamma < \sup A$, then there exists $a \in A$ such that $\gamma < a \leq \sup A$.

(e) If $\sup A \leq \sup B$ and $\sup B \leq \sup A$, then $\sup A = \sup B$.

Theorem 1.7

If $A \subseteq B$ and both suprema exist, then $\sup A \leq \sup B$.

Proof. Because $A \subseteq B$, all $a \in A$ satisfy $a \in B$. So by definition, $a \leq \sup B$. From Theorem 1.6(b), we get $\sup A \leq \sup B$. $\square$

Not every set that has an upper bound has a least upper bound. It is, however, very nice when they do!

Definition: Axiom of completeness

A set S satisfies the axiom of completeness (or the least upper bound property) if every nonempty subset of S that has an upper bound has a supremum in S.

As it turns out, this property is the defining feature of $\mathbb{R}$. We state this without proof for now, but we'll come back to it later.

Theorem 1.8: Completeness of $\mathbb{R}$

There exists an ordered field $\mathbb{R}$ which both satisfies the completeness axiom and contains $\mathbb{Q}$ as a subfield.

Quite a few interesting facts follow from this! For example, it is the reason why every real number has a decimal expansion, and why every real number has an nth root. There are a couple of other consequences which we'll find useful.

Theorem 1.9: Archimedean property

If $x, y \in \mathbb{R}$ and $x > 0$, then there exists a positive integer n such that $nx > y$. (Equivalently, if $x > 0$, then there exists $n \in \mathbb{N}$ such that $1/n < x$.)

Proof. Consider a set $A = \{nx : n \in N\}$. For contradiction, suppose A were bounded above by y, that is, $nx < y$ for all n. But we know $A \subset \mathbb{R}$ has a supremum by the axiom of completeness.

Call this least upper bound $\sup A = \alpha$. Then $\alpha - x$ is not an upper bound for A. But then there exists an $m \in \mathbb{N}$ such that $\alpha - x < mx$, which implies that $\alpha < mx + x = (m+1)x$. Thus, α is not an upper bound for A, a contradiction. Therefore, $nx > y$, as desired. $\square$

Theorem 1.10: $\mathbb{Q}$ is dense in $\mathbb{R}$

Suppose we have $x, y \in \mathbb{R}$ with $x < y$. Then there exists at $q \in \mathbb{Q}$ such that $x < q < y$.

Proof. Choose $n \in \mathbb{N}$ satisfying $1/n < y - x$; such an n exists by the Archimedean property. Consider multiples of $1/n$; again, by the Archimedean property, these multiples are unbounded so we can choose the first multiple such that $m/n > x$.

Now we show that $m/n < y$. Suppose not: then $(m-1)/n < x$ and $m/n > y$. But combining these implies that $1/n > y - x$, a contradiction. $\square$

The first theorem here roughly states that there are no "infinitely large" or "infinitely small" real numbers to the point of being inaccessible. This second theorem hints at one of three equivalent characteristics of what we call dense sets, which we'll look more into later.

1.3 Metric Spaces, Open Sets, and Closed Sets

Let's keep adding structure to our sets by introducing distance.

Definition: Metric space

A set X is a metric space if there exists a metric $d : X \times X \to \mathbb{R}$ such that the following are satisfied for all $p, q, r \in X$.

- (Nonnegativity) $d(p, q) \geq 0$, with $d = 0 \iff p = q$.
- (Symmetry) $d(p, q) = d(q, p)$.
- (Triangle inequality) $d(p, q) \leq d(p, r) + d(r, q)$.

We can use this to define the notion of a point's "closeness" to another point!

Definition: Neighborhood

A neighborhood of a point x with radius r is the set $N_r(x) = \{y : d(x, y) < r\}$. This is also known as an open ball of radius r; if we allow $d = r$ then we have a closed ball, denoted by $\overline{N_r(x)}$.

Neighborhoods are useful in classifying the different types of points that arise when we construct subsets.

Definition: Limit, interior, and isolated point

Let X be a metric space and $E \subseteq X$. Consider a point $p \in X$.

- p is a limit point of E if every neighborhood of p contains a point $p \in E$ where $p \neq q$.
- p is an interior point of E if there exists a neighborhood $N(p)$ such that $N \subseteq E$.
- p is an isolated point of E if $p \in E$ and p is not a limit point of E.

Note that a limit point need not be in the subset it is characterized by! It just provides a definite "edge" to the subset. Interestingly, the neighborhoods around these limit points are infinite.

Theorem 1.11: Neighborhoods of limit points

If p is a limit point of E, then every neighborhood of p contains infinitely many points of E.

Proof. Consider a limit point p of a set E. For contradiction, suppose there exists a neighborhood N of p that contains only finitely many points $e_1, e_2, \ldots, e_N$.

Let r be the minimum nonzero distance between p and one of the points in the neighborhood. We know this minimum exists because the set of distances is finite. We can then construct a new neighborhood of size r, which contains no points except p, which is a contradiction since p is a limit point. □

A key motivator behind classifying points like we have is the definition of two very important types of sets.

Definition: Open and closed sets

Suppose X is a metric space and $E \subseteq X$.

- E is open if every point in E is an interior point of E.
- E is closed if E contains all of its limit points.

Importantly, open and closed sets are not opposites! It is possible to have a set that falls into neither category, or even both of them. Not all is lost though—there is still a nice relationship between open and closed sets.

Lemma 1.12: Neighborhoods are open sets

A neighborhood $N(p)$ of a point p is an open set.

Proof. Recall that $N(p)$ is defined by some radius r. Therefore, if a point $q \in N(p)$, then $d(p,q) < r$. Let $r' - d(p,q)$ be the distance from q to the edge of the neighborhood, and construct a smaller neighborhood $N(q)$ with this radius. By the triangle inequality, any point $x \in N(q)$ must satisfy

$$d(x,y) \leq d(x,q) + d(q,p) < r' + d(q,p) = r.$$

So $d(x,p) < r$ and q is an interior point of $N(p)$. Therefore, $N(p)$ is open. □

Theorem 1.13: Complement of an open set is closed

E is open if and only if its complement E^c is closed.

Proof. We provide a chain of equivalent statements.

$$\begin{aligned} E \text{ is open} &\iff \text{any } x \in E \text{ is an interior point} \\ &\iff \text{for all } x \in E \text{ there exists a neighborhood of } x \text{ that is disjoint from } E^c \\ &\iff x \in E \text{ is not a limit point of } E^c \\ &\iff E^c \text{ contains all of its limit points} \\ &\iff E^c \text{ is closed,} \end{aligned}$$

as desired. □

We'll consider what happens when we take unions and intersections of these sets. Finite unions and intersections lead to fairly intuitive results; infinite sets, as usual, are a little strange!

Lemma 1.14: De Morgan's Law

Suppose we have a (potentially uncountable) collection of sets $\{E_\alpha\}$ (where α labels the sets). Then

$$\left(\bigcup_\alpha E_\alpha\right)^c = \bigcap_\alpha E_\alpha^c.$$

Proof. Consider an element $y \in (\bigcup_\alpha E_\alpha)^c$, so $y \notin \bigcup_\alpha E_\alpha$. Then y is not in E_α for any a, meaning $y \in E_\alpha$ for all a and $y \in \bigcap_\alpha E_\alpha^c$. This argument holds in reverse. □

Theorem 1.15: Unions and intersections of open and closed sets

(a) Arbitrary unions of open sets are open.

(b) Finite intersections of open sets are open.

(c) Finite unions of closed sets are closed.

(d) Arbitrary intersections of closed sets are closed.

Proof. We'll prove each part in turn.

(a) Suppose $x \in \bigcup_{i=1}^\infty A_i$, where all A_i are open. Then $x \in A_i$ for some i and x has a neighborhood $N(x) \subset A_i$. But $A_i \subseteq \bigcup_{i=1}^\infty A_i$, meaning x is an interior point of the union and the union is open.

(b) Suppose $x \in \bigcup_{i=1}^n A_i$, where all A_i are open. For each A_i, there exists a neighborhood $N_{r_i}(x) \subseteq A_i$. Let r be the minimum radius of these neighborhoods; then $N_r(x)$ is contained in the intersection and the intersection is open.

(c) Consider a collection of closed sets $B_1, \ldots, B_n$; by (b), the intersection of their complements is open. The complement of this intersection is closed, so by De Morgan's Law, $\bigcup_{i=1}^n B_i$ is closed.

(d) Consider a collection of closed sets $B_1, B_2 \ldots$; by (a), the union of their complements is open. The complement of this union is closed, so by De Morgan's law, $\bigcap_{i=1}^{\infty}$ is closed.

This completes the proof. $\square$

Let's look a little more closely at closed sets by defining the closure of a set.

Definition: Closure

The closure of a set A is $\overline{A} \equiv A \cup A'$, where A' is the set of limit points of A.

Theorem 1.16

A set is closed if and only if it is equal to its closure.

Proof. Consider a set E and the set E' of its limit points. $\overline{E} = E \cup E'$ is the closure of E.

($\Rightarrow$) Assume E is closed. By definition, $E' \subset E$, so $\overline{E} \subset E$. Also, since $E \subset \overline{E}$, we conclude $\overline{E} = E$.

($\Leftarrow$) Assume $E = \overline{E}$. Then $E = E \cup E'$, and so it contains all of its limit points and is closed. $\square$

Another natural property of the closure is that, when a "parent set" is known to be closed, then any of its subsets remains in the parent set after being closed.

Theorem 1.17

If $E \subseteq F$, where F is closed, then $\overline{E} \subseteq F$.

Proof. If F is closed then it contains all of its limit points. In particular, it must contain all limit points of a subset of itself. So $F \subseteq E'$, meaning $\overline{E} \subseteq F$. $\square$

A nice corollary of this theorem is that the smallest closed superset of E is its closure $\overline{E}$. Finally, we can put all this together to get a pretty unintuitive theorem—our first result related to sequences.

Theorem 1.18: Nested interval property of $\mathbb{R}$

For each $n \in \mathbb{N}$, consider $I_n = \{x \in \mathbb{R} : a_n \leq x \leq b_n\}$. If each $I_{n+1} \subseteq I_n$, then $\bigcap_{i=1}^{n} I_i$ is nonempty.

Proof. Let $A = \{a_1, a_2, \ldots\}$ be the left-endpoints of $\{I_n\}$. $A \in \mathbb{R}$ is bounded above by b_1, so $x = \sup A$ exists. So $a_n \leq x \leq b_n$ for all n. Therefore, $x \in \bigcup_{n=1}^{\infty}$, and the intersection is nonempty. $\square$

This can, in fact, be generalized to closed sets in $\mathbb{R}^n$.

1.4 Dense Sets

We are now ready to formally define density, as we alluded to before, and to explore its properties.

Definition: Dense set

Consider a metric space X. A set $E \subseteq X$ is dense in X if

- every point of X is in E or is a limit point of E,
- $\overline{E} = X$, or
- every open set of X contains a point of E.

We'll kick things off with a direct application of the nested interval theorem.

Lemma 1.19

A countable intersection of dense, open sets in $\mathbb{R}$ is not empty.

Proof. Let $\{G_1, G_2, \ldots\}$ be a collection of dense, open sets in $\mathbb{R}$. First, by definition, every open set in $\mathbb{R}$ contains a point in G_1. Consider an arbitrary nonempty closed interval $I_1 = [a_1, b_1]$; this is the closure of the open interval $A_1 = (a_1, b_1)$, which must contain a point in G_1.

We continue on to construct a closed interval $I_2 \subseteq I_1$, which is the closure of the open interval A_2. A_2 must contain a point in G_2, and also one in G_1. Continuing on, we get a countable sequence of nested, closed intervals, all of which we know to be nonempty by the nested interval property of $\mathbb{R}$.

Each I_n is the closure of the open interval A_n, so we have also constructed a countable nested sequence of nonempty open intervals. Thus the intersection of these open sets is nonempty and contains a point in every G_n, so the intersection of all G_n is nonempty. □

Now consider, for a moment, the set of positive rationals. This set is obviously not dense in $\mathbb{R}$, but we might say that it's at least dense *somewhere* in $\mathbb{R}$, particularly in the positive reals. The set of integers, on the other hand, isn't dense anywhere in $\mathbb{R}$. We'll find this characterization useful in proving our next theorem.

Definition: Nowhere-dense set

A set E is nowhere-dense in a metric space X if $\overline{E}$ contains no nonempty open intervals.

We might re-interpret this definition to mean that there are no open sets of X contained in $\overline{E}$, or that E is not dense in any nonempty open subset of X. Intuitively, nowhere-dense sets are very "thin", so countable unions of nowhere-dense subsets represent a very small type of infinite set (called meager sets). This allows for another perspective on the cardinality of $\mathbb{R}$—given what we know already, we expect that $\mathbb{R}$ is not a meager set, and does turn out to be the case!

Theorem 1.20: Baire's theorem

$\mathbb{R}$ cannot be written as the countable union of nowhere-dense sets.

Proof. Let $S = \{S_1, S_2, \ldots\}$ be a countable set of nowhere-dense (in $\mathbb{R}$) sets, so each $\overline{S_n}$ contains no nonempty open intervals. For contradiction, suppose $\bigcup_{n=1}^{\infty} \overline{S_n} = \mathbb{R}$. By de Morgan's law, $\bigcap_{n=1}^{\infty} \overline{S_n}^c = \emptyset$. Now, each $\overline{S_n}^c$ is dense and open, and their countable intersection is empty—which contradicts the previous lemma. So $\bigcup_{n=1}^{\infty} \overline{S_n} \neq \mathbb{R}$, and since $S_n \subset \overline{S_n}$, we also have $\bigcup_{n=1}^{\infty} S_n \neq \mathbb{R}$, as desired. □

1.5 Compact Sets

Now we'll turn our attention to a different kind of set, one that has much in common with finite sets. We need to make a couple of other definitions first.

Definition: Cover

Let X be a metric space. An open cover (or, simply, cover) of E in X is a collection of open sets $\{G_\alpha\}$ whose union contains ("covers") E. If there is a subcollection of $\{G_\alpha\}$ that still covers E, we call this subcollection a subcover.

Definition: Compact set

We say a set K is compact in X if every open cover of K contains a finite subcover.

Theorem 1.21

Finite sets are compact.

Proof. Consider a finite set $\{x_1, \dots, x_n\}$ and some arbitrary open cover $\{G_\alpha\}$. For each x_i, choose a G_{α_i} that contains x_i. Then these G_{α_i} cover the set and comprise a finite subcover of $\{G_\alpha\}$. $\square$

The definition we've provided for compact sets isn't exactly abstract, but it isn't super well-motivated either. We'll dedicate the remainder of this section to characterizing these sets, starting with a couple of theorems.

Definition: Bounded set

Let K be a set in a metric space X. K is bounded if $K \subset N_r(x)$ for some $x \in X$.

Theorem 1.22: Compact sets are bounded

If K is compact then K is bounded.

Proof. Let K be a compact set and $B(x)$ a unit ball about a point x. Then $\{B(x)\}_{x\in K}$ is a cover of K, and there exists a finite subcover $\{B(x_i)\}_{i=1}^N$. Take one of these unit balls, say $B(x_1)$, and define $M = \max\{d(x_1, x_i)\}_{i=1}^N$ to be the maximum distance to the center of another unit ball. Then the neighborhood $N_{M+2}(x_1)$ contains the subcover $\{B(x_i)\}_{i=1}^N$ and thus contains K. So K is bounded. $\square$

Theorem 1.23: Compact sets are closed

If K is compact then K is closed.

Proof. Let K be compact and consider $p \notin K$; we will show that there exists a $N(p) \subseteq K^c$.

For all $q \in K$ let $V_q = N_r(q)$ and $U_q = N_r(p)$, where $r = \frac{1}{2}d(p,q)$ so that $V_q \cap U_q = \emptyset$. Notice that $\{V_q\}$ is a cover of K, so there exists a finite subcover $\{V_{q_1}, \dots, V_{q_n}\}$. Also, the intersection of all U_{q_i} is open because it's the intersection of finitely many open sets; in fact, W is a ball of radius $\hat{r} = \min\{r_1, \dots, r_n\}$.

Now, $W \cap V_{q_i} = \emptyset$ for all i since $W \subset U_{q_i}$. Thus K^c contains W, K^c is open, and K is closed. $\square$

Sufficient Conditions for Compactness

Now we'll move on from necessary conditions for compactness to find sufficient conditions.

Theorem 1.24

A closed subset of a compact set is compact.

Proof. Let K be a compact set and $B \subseteq K$ be a closed subset. Consider an open cover $\{U_\alpha\}$ of B. The set B^c is open, so $\{U_\alpha\}$ is also open. This union is a cover of K.

By the compactness of K, there exists a finite subcover $\{U_{\alpha_1}, \dots, U_{\alpha_n}, B^c\}$ of K. But B^c is not necessary to cover B, so $\{U_{\alpha_1}, \dots, U_{\alpha_n}\}$ is a finite subcover for any cover and B is compact. $\square$

Corollary 1.25

Suppose F is closed and K is compact. Then $F \cap K$ is compact.

Proof. K is compact, so it is closed. Hence the intersection $F \cap K \subseteq K$ is closed and thus compact. □

Continuing in our quest for sufficient conditions, we prove compactness for a particular type of set: closed intervals in $\mathbb{R}$. This will prove to be very useful in our final theorems of this section.

Theorem 1.26

The closed interval $[a, b]$ is compact in $\mathbb{R}$.

Proof. Suppose, for contradiction, that $[a, b]$ is not compact, so there exists an open cover $\{G_\alpha\}$ that has no finite subcover. Divide this interval at the halfway point; then $\{G_\alpha\}$ covers both subintervals and at least one of them, say I_1 contains no finite subcover.

Then divide I_1 in half in a similar manner. One of the resulting halves I_2 has no finite subcover. Continue doing this to obtain a sequence of nested close intervals where each interval is smaller than the last and has no finite subcover.

By the nested closed interval theorem, there is an $x \in I_n$ for all n. So x is in a set $G_{\hat{\alpha}} \in \{G_\alpha\}$ and there is a neighborhood $N(x) \subseteq G_{\hat{\alpha}}$. Since the intervals can get arbitrarily small, there is an $I_n \subseteq N(x)$. Hence $G_{\hat{\alpha}}$ is a finite subcover for I_n, a contradiction. □

Now for what is perhaps the best characterization of compact sets in $\mathbb{R}$!

Theorem 1.27: Heine-Borel theorem

A set $K \subset \mathbb{R}$ is compact if and only if K is closed and bounded.

Proof. ($\Rightarrow$) We already have two individual theorems showing that, if K is compact, then it is closed and bounded.

($\Leftarrow$) Let $K \subset \mathbb{R}$ be closed and bounded. So there exists an $r > 0$ such that $K \subseteq [-r, r]$. Hence K is a closed subset of a compact set and K is compact. □

In fact, this result applies to any Euclidean space $\mathbb{R}^n$. The more general proof has a very similar structure to the one above. We have another theorem that applies to all metric spaces, but it is weaker.

Theorem 1.28

K is compact if and only if every infinite subset E of K has a limit point in K.

Proof. ($\Rightarrow$) Assume K is compact. Suppose, for contradiction, that no point of K is a limit point of some infinite $E \subseteq K$. Then each $q \in K$ has a neighborhood $N(q)$ containing at most one point of E (namely, q, if $q \in E$). But then, $\{N(q)\}$ is a cover of E with no finite subcover, a contradiction.

($\Leftarrow$) Exercise. □

Finally, we can use this to prove something that we probably already knew intuitively, but it's nice to get some closure.

Corollary 1.29: Bolzano-Weierstrass theorem

Every bounded infinite subset of $\mathbb{R}$ has a limit point in $\mathbb{R}$.

Proof. Consider an infinite, bounded set $E \subset \mathbb{R}$. E is contained in some compact interval $[a, b]$, so it has a limit point in $[a, b]$ and thus in $\mathbb{R}$. □

This theorem also generalizes to $\mathbb{R}^k$.

2 Sequences and Series

2.1 Sequences

Our attention, now, shifts to sequences of elements. We'll begin with a rigorous definition for these objects.

Definition: Sequence

A sequence $\{p_n\}$ in a metric space X is a function $f : \mathbb{N} \to X$ which maps $n \mapsto p_n$.

Oftentimes we'll encounter sequences that seem to end up arbitrarily close to a particular point, whereas others never seem to settle anywhere. We can capture this intuition with the following definition.

Definition: Convergence

A sequence $\{p_n\}$ converges if there is a $p \in X$ such that for any $\epsilon > 0$ there exists an $N \in \mathbb{N}$ such that $n \geq N$ implies $d(p_n, p) < \epsilon$.

In this case we say "p_n converges to p" or "p is the limit of the sequence", and we write $p_n \to p$ or $\lim_{n\to\infty} p_n = p$.

If we're trying to prove the convergence of a given sequence, we may do so directly by determining a value for N (usually as a function of ϵ) for which the sequence becomes constrained to a radius-ϵ neighborhood. Some of the following properties might also be useful.

Theorem 2.1: Convergence properties

(a) If $p_n \to p$ and $p_n \to q$ then $p = q$.

(b) If p_n converges then p_n is bounded.

(c) If p is a limit point of $E \subset X$ then there exists a sequence $\{p_n\}$ in E such that $p_n \to p$.

(d) $p_n \to p$ if and only if every neighborhood of p contains all but finitely many p_n.

Proof. We prove each part in turn.

(a) Suppose $p_n \to p$ and $p_n \to q$ and let $\epsilon = d(p, q)$. Assume, for contradiction, that $p \neq q$, so $\epsilon > 0$. By the definition of convergence, there exist N_p, N_q such that $n \geq N_p \implies d(p_n, p) < \epsilon/2$ and $n \geq N_q \implies d(p_n, q) < \epsilon/2$. Let $N = \max\{N_p, N_q\}$; by the triangle inequality,

$$\epsilon = d(p, q) \leq d(p_n, p) + d(p_n, q) < \epsilon,$$

a contradiction.

(b) Let $\epsilon = 1$. Then there exists an N such that $n > N \implies d(p_n, p) < 1$. All p_n are contained in the ball $B_{R+1}(p)$, where $R = \max\{1, d(p_1, p), \ldots, d(p_N, p)\}$, so p_n is bounded.

(c) If p is a limit point of E then every neighborhood contains a point $p_n \neq p$ such that $p_n \in E$. So we can construct a convergent sequence $\{p_n\}$ where $p_n \in B_{1/n}(p)$.

(d) ($\Rightarrow$) Consider $\epsilon > 0$. Then there exists an N such that $n > N \implies d(p_n, p) < \epsilon$. There are finitely many natural numbers satisfying $n \leq N$, so there are finite p_n outside this neighborhood.

($\Leftarrow$) Consider an $N_\epsilon(p)$ that contains all but M points. Then $d(p_n, p) < \epsilon$ for all $n \geq M + 1$.

This completes the proof. □

Notably, bounded sequences do not necessarily converge, and points of convergence are not necessarily limit points of a sequence's range.

Convergent sequences in $\mathbb{C}$ also play nicely with the field operations, giving us a few more useful rules.

Theorem 2.2: Convergence properties in $\mathbb{C}$

Suppose we have two sequences $\{s_n\}, \{t_n\} \in \mathbb{C}$, where $s_n \to s$ and $t_n \to t$. Then:

(a) $\lim_{n\to\infty}(s_n + t_n) = s + t$.

(b) $\lim_{n\to\infty}(cs_n) = cs$ for scalar c.

(c) $\lim_{n\to\infty}(c + s_n) = c + s$ for scalar c.

(d) $\lim_{n\to\infty}(s_n t_n) = st$.

Proof. For each of the proofs below we consider $\{s_n\} \to s$ and $\{t_n\} \to t$. Suppose c is a scalar.

(a) Fix $\epsilon > 0$. There exist N_1 such that $n > N_1 \implies |s_n - s| < \epsilon/2$ and N_2 such that $n > N_2 \implies |t_n - t| < \epsilon/2$. Let $N = \max\{N_1, N_2\}$; then for $n > N$ we have, by the triangle inequality,

$$|(s_n + t_n) - (s + t)| \le |s_n - s| + |t_n - t|,$$

which is less than ϵ, as desired.

(b) Fix $\epsilon > 0$. There exists an N such that $n > N \implies |s_n - s| < \epsilon/c$. So

$$|cs_n - cs| = |c||s_n - s|,$$

which is less than ϵ, as desired.

(c) This is a simple combination of the previous two arguments.

(d) Fix $\epsilon > 0$ and let $k = \max\{s, t, 1, \epsilon\}$. There exist N_1 such that $n > N_1 \implies |s_n - s| < \epsilon/2$ and N_2 such that $n > N_2 \implies |t_n - t| < \epsilon/2$. Let $N = \max\{N_1, N_2\}$; then for $n > N$ we have

$$|s_n t_n - st| = |(s_n - s)(t_n - t) + s(t_n - t) + t(s_n - s)|,$$

and by the triangle inequality,

$$\begin{aligned} &\le |(s_n - s)(t_n - t)| + |s(t_n - t) + t(s_n - s)| \\ &< \frac{\epsilon}{3k} \cdot \frac{\epsilon}{3k} + |s| \cdot \frac{\epsilon}{3k} + |t| \cdot \frac{\epsilon}{3k} \end{aligned}$$

Since $k \ge 1$, replacing it with a 1 will not make the expression smaller, meaning

$$\le \frac{\epsilon^2}{9} + \frac{\epsilon}{3} + \frac{\epsilon}{3},$$

which is less than ϵ, as desired.

This completes the proof. □

Subsequences, Bounded Sequences, and Cauchy Sequences

Just as we can have subsets embedded within potentially larger sets, we can have subsequences embedded within potentially larger sequences.

Definition: Subsequence

Consider a sequence $\{p_n\}$. If there is an increasing sequence $n_1 < n_2 < \cdots$ in $\mathbb{N}$, then $\{p_{n_i}\}$ is a subsequence.

Theorem 2.3: Subsequences of convergent sequences

If $p_n \to p$ then every subsequence $p_{n_i} \to p$.

Proof. Every neighborhood of p contains all but finitely many points of p_n, so it also contains all but finitely many points of p_{n_i}. □

Of course, if a sequence does not converge, then its subsequences may not all converge to the same value. We'll introduce some vocabulary that describes this.

Definition: Subsequential limit

If a subsequence of $\{p_n\}$ converges to p then p is a subsequential limit.

Definition: Upper and lower limits

Let E be the set of all subsequential limits of $\{s_n\}$, including possibly $\pm\infty$. Then the upper and lower limits are, respectively,

$$S^* = \sup E = \limsup_{n\to\infty} s_n \quad \text{and} \quad S_* = \inf E = \liminf_{n\to\infty} s_n.$$

We can also talk about bounded sequences, that is, sequences whose ranges are bounded. In particular, we can make a nice statement about bounded sequences that are everywhere-increasing or -decreasing.

Definition: Monotonic sequence

A sequence $\{s_n\}$ is monotonically increasing if $s_n \leq s_{n+1}$ for all n. It is monotonically decreasing if $s_n \geq s_{n+1}$. Either could be called monotonic.

Theorem 2.4

A monotonic sequence in $\mathbb{R}$ is bounded if and only if it converges.

Proof. ($\Rightarrow$) Suppose $\{s_n\}$ is bounded and monotonically increasing. Let $s = \sup(\text{range}\,\{s_n\})$. (Note that s exists because $\{s_n\}$ is a bounded, nonempty set in the reals.)

By definition, $s_N < s$ for any N because s is the supremum. Also, for any $\epsilon > 0$ there exists an N such that $s - \epsilon < S_N$. Since the sequence is increasing, $s_n \geq s_N$ for all $n > N$, so $\{s_n\}$ converges so its supremum s. (We could make the same for the infimum of a monotonically decreasing sequence.)

($\Leftarrow$) Suppose $\{s_n\}$ converges. Then it is bounded. □

If a sequence is not bounded, we say it is unbounded. We can be a little more specific with the behavior of this unboundedness, though.

Definition: Unbounded sequence

If for all $M \in \mathbb{R}$ there exists an $N \in \mathbb{N}$ such that $n > N$ implies...

- ... $s_n > M$, then $s \to \infty$.
- ... $s_n < M$, then $s \to -\infty$.

Finally, our intuition for convergence seems to imply that, after a point, a sequence's terms are all really "close" to each other. Unfortunately this is not the same as convergence, but the two ideas are related.

Definition: Cauchy sequence

A sequence $\{p_n\}$ is Cauchy if for all $\epsilon > 0$ there exists an N such that $m, n \geq N$ implies $d(p_n, p_m) < \epsilon$.

Theorem 2.5

If $\{p_n\}$ converges then $\{p_n\}$ is Cauchy.

Proof. Let $\{p_n\}$ be a convergent sequence. Given $\epsilon > 0$, there exists N such that $n > N$ implies $d(p_n, p), \epsilon/2$. In this case $n, m > N$ implies

$$d(p_n, p_m) \leq d(p_n, p) + d(p_m, p)$$

by the triangle inequality. So $d(p_m, p_n) < \epsilon$, as desired. □

2.2 Compactness and Completeness

We can draw a surprisingly tight connection between subsequences and compact sets, starting with a theorem that we may use to restate something familiar,

Theorem 2.6

Every sequence in a compact metric space X has a subsequence converging to a point in X.

Proof. Let $R = \text{range}\,\{p_n\}$. If R is finite, then some $p \in \{p_n\}$ appears infinitely many times and there is a subsequence that converges to p.

If R is infinite, then it is an infinite subset of a compact set and thus has a limit point p in the broader metric space X. Take a point p_i within a neighborhood $N_r(p)$; then, reduce $r < d(p_i, p)$ and pick another point with index greater than i. Continuing this process indefinitely constructs a sequence converging to p. □

Corollary 2.7: Bolzano-Weierstrass theorem

Every bounded sequence in $\mathbb{R}^k$ contains a convergent subsequence.

But we can do better! Without proof, we'll use an equivalent notion of compactness to generalize further.

Definition: Sequentially compact

A metric space X is sequentially compact if every sequence has a convergent subsequence in X.

Theorem 2.8

A metric space is compact if and only if it is sequentially compact.

Corollary 2.9: Bolzano-Weierstrass theorem

A subset of $\mathbb{R}^k$ is sequentially compact if and only if it is closed and bounded.

But this is just the Heine-Borel theorem! We can use this development to list a few equivalent properties of subsets of $\mathbb{R}^k$.

Theorem 2.10: Big exciting theorem about $\mathbb{R}$

Let $E \subset \mathbb{R}^k$. The following are equivalent.

- E is closed and bounded.
- E is compact.
- E is sequentially compact.
- Every sequence in E has a convergent subsequence.

We've seen that another important property of $\mathbb{R}$ involves completion, which we can define using sequences.

Definition: Complete

A metric space X is complete if every Cauchy sequence in X converges to a point in X.

Theorem 2.11: Completion

Every metric space (X, d) has a completion (X^*, d). That is, there is a way to define $X^* \supseteq X$ such that X^* is complete and has the same metric as X.

We'll start with the particular case of $\mathbb{Q}$ to outline how we might construct such a completion. This will give us a new way to think about the completeness of the reals: $\mathbb{R}$ is the completion of $\mathbb{Q}$, meaning it's a metric space in which every Cauchy sequence converges!

2.3 Constructing the Reals

Up to this point we've essentially been treating the set of real numbers as given. But we're now in a position to rigorously construct $\mathbb{R}$ from the ground up as the completion of the rationals! First, let's make a slight revision to our definitions of Cauchy sequences and convergence.

Definition: Cauchy and convergence in $\mathbb{Q}$

A sequence $\{a_n\}$ is a Cauchy sequence if for any positive $\omega \in \mathbb{Q}$ there exists an $n \in \mathbb{N}$ such that $n, m > N$ implies $|a_n - a_m| < \omega$.

A sequence $\{a_n\}$ converges to $a \in \mathbb{Q}$ if for any positive $\omega \in \mathbb{Q}$ there exists an $n \in \mathbb{N}$ such that $n > N$ implies $|a_n - a| < \omega$.

The only significant change we've made is ensuring that the ϵ we've given (now called ω) is rational.

The idea is to define a real number as a Cauchy sequence of rational numbers. But for any Cauchy sequence, there are infinitely many others that "converge" to the same number. To avoid this ambiguity, we instead define each real number as an equivalence class of Cauchy sequences.

Definition: Real numbers

Let $\{a_n\}$ and $\{b_n\}$ be Cauchy sequences in $\mathbb{Q}$. We say they are equivalent if $a_n - b_n \to 0$.

The real numbers $\mathbb{R}$ is the set of all equivalence classes of these Cauchy sequences.

Showing that the above relation is an equivalence relation involves a relatively straightforward application of the definition of convergence and the triangle inequality. Now let's add some structure to our set!

Definition: Algebraic operations of $\mathbb{R}$

Let $a, b \in \mathbb{R}$, so there are Cauchy sequences of rational numbers $\{a_n\}, \{b_n\}$ that represent the equivalence classes for a and b.

- (Addition) Define $a + b$ to be the equivalence class of $\{a_n + b_n\}$.
- (Multiplication) Define $a \cdot b$ to be the equivalence class of $\{a_n \cdot b_n\}$.

We have to make sure these operations are well-defined—that is, the result of performing them shouldn't depend on which representatives of the equivalence classes we choose. We'll prove that our definition for addition is well-defined; we'll skip multiplication for brevity's sake.

Theorem 2.12: Addition is well-defined

Addition in $\mathbb{R}$ is well-defined.

Proof. Choose two members $\{a_n\}, \{\alpha_n\}$ of the same equivalence class, so $a_n - \alpha_n \to 0$. Also choose two members $\{b_n\}, \{\beta_n\}$ of a different equivalence class, so $b_n - \beta_n \to 0$. We want to show that $\{a_n + b_n\}$ is equivalent to $\{\alpha_n + \beta_n\}$.

Consider $(a_n + b_n) - (\alpha_n + \beta_n)$. We can rewrite ths as $(a_n - \alpha_n) + (b_n - \beta_n)$. By our definition of Cauchy equivalence, for rational $\omega > \mathbb{Q}$ there exist N_1, N_2 such that $n > N_1$ implies $|(a_n - \alpha_n) - 0| < \omega/2$ and $n > N_2$ implies $|(b_n - \beta_n) - 0| < \omega/2$. So for $N = \max\{N_1, N_2\}$ we have, for $n > N$,

$$|(a_n - \alpha_n) + (b_n - \beta_n)| \le \frac{\omega}{2} + \frac{\omega}{2} = \omega.$$

So $(a_n + b_n) - (\alpha_n + \beta_n) \to 0$, as desired. □

The next step would be to show that $\mathbb{R}$ with these operations is a field, but this is a relatively painful and unenlightening process so we'll omit that, too. Now let's add an order!

Definition: Order of $\mathbb{R}$

A number $a \in \mathbb{R}$ is positive if $a \neq 0$ and it is represented by a Cauchy sequence such that for some N, $n > N$ implies $a_n > 0$. We say that $a > b$ (for $b \in \mathbb{R}$) if $a - b$ is positive.

To prove that this is an order we'd need to show that it satisfies trichotomy and transitivity. We must also show that the order is well-defined! Again, we'll omit the details of these arguments.

Lastly, we must show that we've created an ordered field. Again, we'll only show this for the additive axiom.

Theorem 2.13: Additive ordered field axiom

Let $a, b, c \in \mathbb{R}$ with $b < c$. Then $a + b < a + c$.

Proof. Let $a = [\{a_n\}]$, $b = [\{b_n\}]$, $c = [\{c_n\}]$, where the brackets denote equivalence classes. Suppose $c > b$, so by the definition of order there exists an N so that $n > N$ implies $c_n - b_n > 0$. Then $c_n > b_n$, and we can add another rational number a_n to both sides of the inequality and rearrange to get

$$(a_n + c_n) - (a_n + b_n) > 0$$

for any $n > N$. Thus $(a + c) - (a + b) > 0$ and $a + c > a + b$, as desired. □

From here, we could prove all the other cool stuff we know to be true about real numbers! We'll list three big properties here, leaving the proofs as exercises.

Theorem 2.14: Properties of $\mathbb{R}$

(a) $\mathbb{R}$ has the Archimedean property.

(b) $\mathbb{Q}$ is dense in $\mathbb{R}$.

(c) $\mathbb{R}$ has the least upper bound property.

We could use this construction as a template for proving Theorem 2.11. The distance between two elements in a completion X^*, which are equivalence classes of Cauchy sequences in X, is given by the limit of the distance between their terms. This ensures that X is truly embedded within X^*—in particular, there exists a distance-preserving mapping $\varphi : X \to X^*$; we call such a mapping an isometry. (We may interpret $\varphi(X)$ as the set X equipped with a new metric Δ.) It turns out that X^* with this metric is complete! Two important corollaries are that $\varphi(X)$ is dense in X^* and $\varphi(X) = X^*$ for complete X.

2.4 Series

We'll shift our attention, now, to adding up successive terms in a sequence.

Definition: Series

Suppose $q > p$. Given $\{a_n\}$, the sum

$$\sum_{n=p}^{q} a_n = a_p + a_{p+1} + \cdots + a_q.$$

is called a series, and each a_n is called a term. We may have an infinite number of terms, in which case the sum is called an infinite series.

But we can frame series in such a way that they just become sequences in disguise! To make this more clear, we'll define the terms in such a sequence.

Definition: Partial sum

The nth partial sum of a series is the sum of the first n terms in the series.

So determining the behavior of a series is equivalent to determining the behavior of the sequence of its partial sums. We're particularly interested in the convergence of infinite series; we'll spend some time developing criteria to determine this convergence. Let's start by taking advantage of the completeness of $\mathbb{R}$.

Theorem 2.15: Cauchy criterion

Suppose $a_n \in \mathbb{R}$ and $m > n \in \mathbb{N}$. Then $\sum_{n=1}^{\infty} a_n$ converges if and only if for all $\epsilon > 0$ there exists an N such that $n, m > N$ implies $|\sum_{k=n}^{m} a_k| < \epsilon$.

Proof. Consider the sequence of partial sums $\{s_n\}$ where $s_k = \sum_{n=1}^{k} a_n$ and $a_n \in \mathbb{R}$. The distance between two terms in this sequence is given by

$$d(s_{k-1}, s_m) = \left|\sum_{n=1}^{m} a_n - \sum_{n=1}^{k-1} a_n\right| = \left|\sum_{n=k}^{m} a_n\right|.$$

($\Rightarrow$) Suppose $\sum_{n=1}^{\infty} a_n$ converges. Then the corresponding sequence of partial sums converges and thus is Cauchy. So for all $\epsilon > 0$ there exists an N such that $n, m > N$ implies $|\sum_{n=k}^{m} a_n| < \epsilon$, as desired.

($\Leftarrow$) Suppose there exists an N such that $m, n > N$ implies $|\sum_{n=k}^{m} a_n| < \epsilon$ for any $\epsilon > 0$. Then $\{s_n\}$ is Cauchy, and because the terms in the sequence are real it also converges. Thus $\sum_{n=1}^{\infty} a_n$ converges. $\square$

In the special case $m = n$, then we get a corollary about the terms of a convergent series.

Corollary 2.16: Term test

If $\sum_{n=1}^{\infty} a_n$ converges then $a_n \to 0$.

The converse of this statement is false, but the contrapositive provides a useful sufficient condition for divergence. There's another theorem we can prove that relates a series' terms to its convergence.

Theorem 2.17: Nonnegative test

Let $a_n \in \mathbb{R}$. If $a_n \geq 0$ for all n then $\sum_{n=1}^{\infty} a_n$ converges if and only if its partial sums are bounded.

Proof. Consider an infinite series with nonnegative terms. Then the corresponding sequence of partial sums is monotonic, meaning it is bounded if and only if it converges. □

We're now in a position to make a statement about the convergence of a particular class of series that appears in many applications.

Definition: Geometric series

Let $x \in \mathbb{R}$. The series of the form $\sum_{n=0}^{\infty} x^n$ is called a geometric series.

Theorem 2.18: Geometric series convergence

The sum $\sum_{n=0}^{\infty} x^n$ converges to $1/(1-x)$ if and only if $|x| < 1$.

Proof. If $|x| \geq 1$ then the terms do not approach zero and the series diverges. If $|x| < 1$ then the nth partial sum s_n of the series obeys

$$\begin{aligned} s_n &= 1 + x + \cdots + x^n, \\ (1-x)s_n &= (1 + x + \cdots + x^n) - (x + x^2 + \cdots + x^{n+1}) \\ &= 1 - x^{n+1}. \end{aligned}$$

Since $x \neq 1$ we have $s_n = (1 - x^{n+1})/(1-x)$, which we can show converges to $1/(1-x)$. □

Once we know about the convergence of relatively simple series, we can use them to determine the behavior of somewhat more complex series.

Theorem 2.19: Comparison test

(a) If $|a_n| \leq c_n$ for sufficiently large n and $\sum_{n=1}^{\infty} c_n$ converges, then $\sum_{n=1}^{\infty} a_n$ converges.

(b) If $a \geq d_n \geq 0$ for sufficiently large n and $\sum_{n=1}^{\infty} d_n$ diverges, then $\sum_{n=1}^{\infty} a_n$ diverges.

Proof. The second statement is the contrapositive of the first, so need only prove the first. Fix $\epsilon > 0$; since $\sum_{n=1}^{\infty} c_n$ converges, there exists an N_1 such that $m, n \geq N_1$ implies $|\sum_{k=n}^{m} c_k| < \epsilon$ for $m > n$. Now suppose there exists some N_2 such that $|a_n| \leq c_n$ for $n > N_2$. If $N = \max\{N_1, N_2\}$, then for $m, n > N$ we have

$$\left| \sum_{k=n}^{m} a_k \right| \leq \sum_{k=n}^{m} |a_k| \leq \sum_{k=n}^{m} c_k < \epsilon.$$

Thus $\sum_{k=1}^{\infty} a_k$ converges, as desired. □

We also have two more closely-related tests that once again concern the behavior of the individual terms in their limit.

Theorem 2.20: Root test

Given $\sum_{n=1}^{\infty} a_n$, let $\alpha = \limsup_{n\to\infty} \sqrt[n]{|a_n|}$. Then:

- If $\alpha < 1$, the series converges.
- If $\alpha > 1$, the series diverges.
- If $\alpha = 1$, the test is inconclusive.

Proof. We have three things to prove.

- $(\alpha = 1)$ Consider the divergent $\sum_{n=1}^{\infty} \frac{1}{n}$. By Rudin 3.20 $\sqrt[n]{n} \to 1$, so $\alpha = \limsup_{n\to\infty} \sqrt[n]{1/n} = 1$.

  Now consider the convergent (by Rudin 3.28) $\sum_{n=1}^{\infty} \frac{1}{n^2}$. We have $\alpha = \limsup_{n\to\infty} \sqrt[n]{1/n^2} = 1$. So $\alpha = 1$ cannot give us any information about the convergence of a series!
- $(\alpha < 1)$ There exists a β such that $\alpha < \beta < 1$ and, by the definition of limsup, an N such that $n > N$ implies $\sqrt[n]{|a_n|} < \beta$ for all n. So $|a_n| < \beta^n$. By the comparison test, $\sum_{n=1}^{\infty} a_n$ converges.
- $(\alpha > 1)$ We have $\limsup_{n\to\infty} \sqrt[n]{|a_n|} > 1$, so $\limsup_{n\to\infty} a_n > 1$. Now consider the subsequence $\sqrt[n]{|a_{n_k}|} \to \alpha$; there exists an N such that $k > N$ implies $|a_{n_k}| > 1$. Thus $\sum_{n=1}^{\infty} a_n$ diverges by the term test.

This completes the proof. $\square$

Theorem 2.21: Ratio test

The sum $\sum_{n=1}^{\infty} a_n$ converges if $\limsup_{n\to\infty} |a_{n+1}/a_n| < 1$ and diverges if there is an N such that $n \geq N$ implies $|a_{n+1}/a_n| \geq 1$.

The proof of the ratio test is very similar to that of the root test, so we omit it.

Algebraic Manipulations of Series

Sums of convergent series are relatively simple. If we have $\sum_{n=0}^{\infty} a_n \to A$ and $\sum_{n=0}^{\infty} b_n \to B$, then

$$\sum_{n=0}^{\infty} a_n + \sum_{n=0}^{\infty} b_n = \sum_{n=0}^{\infty} (a_n + b_n) = A + B.$$

Products are a little more interesting! The product of the above series is

$$\sum_{n=0}^{\infty} c_n = (a_0 b_0) + (a_1 b_0 + b_0 a_1) + (a_2 b_0 + a_1 b_1 + a_0 b_2) + \cdots,$$

with terms grouped based on the sums of their indices. So the nth term in the series is the sum of all the products whose terms add to n:

$$c_n = \sum_{k=0}^{n} a_k b_{n-k}.$$

This is called the Cauchy product (and is the discrete analog of convolution). We'll come back to this—first, let's see what we can say about products of terms.

Theorem 2.22: Summation by parts

Let $A_n = \sum_{k=1}^n a_k$ and define $A_0 = 0$. Then

$$\sum_{n=p}^{q} a_n b_n = A_q b_q - A_{p-1} b_p + \sum_{n=p}^{q-1} A_n (b_n - b_{n+1}).$$

There's a straightforward algebraic proof for this, but a geometric picture is much more enlightening. Construct a rectangle with length $a_1 + \cdots + a_q$ and height b_q, and inside it draw smaller $a_n \times b_n$ rectangles. The area covered by the rectangles from p onward is equal to the total area $A_q b_q$ of the big rectangle minus the area not included in the p-q rectangles! This comes with a nice couple of corollaries.

Theorem 2.23

If $A_n = \sum_{k=1}^n a_k$ is bounded for all n, $b_n > 0$ is decreasing, and $b_n \to 0$, then $\sum_{n=1}^\infty a_n b_n$ converges.

Proof. Choose M such that $|A_n| \leq M$ for all n. Given $\epsilon > 0$, there is an integer N such that $b_N \leq (\epsilon/2M)$. So for $N \leq p \leq q$ we have

$$\begin{aligned} \left| \sum_{n=p}^{q} a_n b_n \right| &= \left| \sum_{n=p}^{q-1} A_n (b_n - b_{n+1}) + A_q b_q - A_{p-1} b_p \right| \\ &\leq M \left| \sum_{n=p}^{q-1} (b_n - b_{n+1}) + b_q - b_p \right| \\ &= 2M b_p \leq 2M b_N \leq \epsilon. \end{aligned}$$

Convergence now follows from the Cauchy criterion. □

Corollary 2.24: Alternating term test

$\sum_{n=1}^\infty c_n$ converges if the $|c_i|$ are decreasing, the c_i are alternating in sign, and $c_i \to 0$.

Proof. In the previous theorem let $a_n = (-1)^{n+1}$ and $b_n = |c_n|$. □

Sometimes we'll have a diverging series that becomes convergent if we allow its terms to alternate signs. To formalize this behavior, we'll define a slightly more restrictive type of convergence.

Definition: Absolute convergence

A series $\sum_{n=0}^\infty a_n$ converges absolutely if $\sum_{n=0}^\infty |a_n|$ converges.

Theorem 2.25

If $\sum_{n=0}^\infty$ converges absolutely, then $\sum_{n=0}^\infty a_n$ converges.

Proof. We know that $|a_n| \leq |a_n|$ and $\sum_{n=1}^\infty |a_n|$ converges, so by comparison $\sum_{n=1}^\infty a_n$ converges. □

As it turns out, this stronger notion of absolute convergence is exactly what we need to guarantee that the product of two series converges! We'll omit the proof here.

Theorem 2.26

If $\sum_{n=0}^{\infty} a_n \to A$ and $\sum_{n=0}^{\infty} b_n \to B$, both converging absolutely, then

$$AB = \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} a_k b_{n-k} \right).$$

We'll finish off with a shocking result about rearrangements of series. We'll omit this proof too.

Theorem 2.27: Riemann rearrangement theorem

Consider $a_n, \alpha, \beta \in \mathbb{R}$ with $\alpha \leq \beta$. If $\sum_{n=0}^{\infty} a_n$ converges, but not absolutely, then there exists a rearrangement of the series such that α and β are the lower and upper limits of the series, meaning the series can converge to any desired real number.

However, if the series converges absolutely then all rearrangements converge to the same limit.

3 Limits and Sequences of Functions

3.1 Limits and Continuity

With all this groundwork about topology and sequences, from here on we'll begin a rigorous construction of all the calculus we've taken for granted in the past. Limits of functions will be at the core of this study, so let's start there!

Definition: Limits of functions

Let X and Y be metric spaces with metrics d_X and d_Y, respectively. Suppose we have $E \subseteq X$, where p is a limit point of E. Consider $f : E \to Y$.

$\lim_{x\to p} f(x) = q$ for some $q \in Y$ if for all $\epsilon > 0$ there exists a $\delta > 0$ such that $0 < d_X(x, p) < \delta$ implies $d_Y(f(x), q) < \epsilon$ for all $x \in E$.

Put more plainly, we must be able to take any ϵ and find a corresponding δ such that any point within a distance δ of p has an image within ϵ of q. This is very similar to our strategy for proving sequence convergence, and in fact we can make the link even stronger!

Theorem 3.1: Sequence characterization of the limit

$\lim_{x\to p} f(x) = q$ if and only if $f(p_n) \to q$ for all convergent sequences $\{p_n\} \in E$ with $p_n \neq p$.

Proof. ($\Rightarrow$) Suppose $\lim_{x\to p} f(x) = q$. Given $\epsilon > 0$, by definition there exists a $\delta > 0$ such that $0 < d(x, p) < \delta$ implies $d(f(x), q) < \epsilon$. Also, for a given sequence $\{p_n\}$ with $p_n \to p$, there exists an N such that $n > N$ implies $d(p_n, p) < \delta$. Thus $n > N$ implies $d(f(p_n), q) < \epsilon$ and $f(p_n) \to q$, as desired.

($\Leftarrow$) Suppose $\lim_{x\to p} f(x) \neq q$. We will prove that there exists a sequence as described such that $f(p_n)$ does not converge to q.

There exists an $\epsilon > 0$ such that, for all > 0, there is an $x \in E$ where $d(x, p) < \delta$ but $d(f(x), q) \geq \epsilon$. If we let $\delta_n = 1/n$, then we can construct a sequence x_n by choosing points that satisfy this condition. Then $x_n \to p$ but $d(f(x_n), q) \geq \epsilon$ by construction, so $f(x_n)$ does not converge to q, which proves the contrapositive. □

With this, we can carry over a lot of the results we proved with sequences.

Corollary 3.2: Limits are unique

If f has a limit at p then this limit is unique.

Corollary 3.3: Algebraic properties of limits

If a function's codomain is $\mathbb{C}$ then the algebraic rules for limits analogous to those for sequences apply.

Now we can use limits to characterize the continuity of a function.

Definition: Continuity

Suppose X and Y are metric spaces, $p \in E \subseteq X$, and $f : E \to Y$. f is continuous at p if for all $\epsilon > 0$ there exists a $\delta > 0$ such that $d(x, p) < \delta$ implies $d(f(x), f(p)) < \epsilon$ for all $x \in E$.

This is quite a natural definition, but it has a strange consequence: any function that maps from a discrete matric space to an arbitrary metric space is vacuously continuous! For any point p in the domain we can draw a small δ-ball that only contains p. More naturally, we also have the following theorem.

Theorem 3.4

(a) If p is a limit point of E, then f is continuous at p if and only if $\lim_{x\to p} f(x) = f(p)$.

(b) If $\{x_n\}$ converges, then f is continuous if and only if $\lim_{n\to\infty} f(x_n) = f(\lim_{n\to\infty} x_n)$.

We have two more corollaries that follow directly from previous results.

Corollary 3.5

Sums and products of continuous functions are continuous.

Corollary 3.6

Suppose $f, g : X \to \mathbb{R}^k$ such that $f = (f_1, \cdots, f_k)$, similarly for g. Then

(a) f is continuous if and only if each f_i is continuous, and

(b) sums $f + g$ and dot products $f \cdot g$ of vectors are continuous if f, g are continuous.

It will be useful to draw one last characterization for continuity, this time in terms of open and closed sets.

Definition

$f : X \to Y$ is continuous if and only if for all open $U \in Y$ the preimage $f^{-1}(U)$ is open in X.

Corollary 3.7

$f : X \to Y$ is continuous if and only if for all closed $K \in Y$ the preimage $f^{-1}(K)$ is closed in X.

The definition comes from the fact that the image U contains an ϵ-neighborhood, so there must also be an open set in the domain (containing a δ-neighborhood). The corollary is an application of previously-shown facts about complements of sets. This gives us a bunch of useful results, some of which are listed below.

Theorem 3.8: Composition of continuous functions

Suppose X, Y, Z are metric spaces. If f, g are continuous functions with $f : X \to Y$ and $g : Y \to Z$, then the composition $g \circ f$ is continuous.

Proof. If $U \in Z$ is open, then $g^{-1}(U)$ is open in Y by the continuity of g. Similarly, $f^{-1}(g^{-1}(U)) = (g \circ f)^{-1}(U) \in X$ is open, meaning $g \circ f$ is continuous. □

Theorem 3.9: Continuity and compactness

Suppose X is a compact metric space.

(a) If $f : X \to Y$ is continuous then $f(X)$ is compact.

(b) If $f : X \to \mathbb{R}^k$ is continuous then $f(X)$ is closed and bounded.

(c) If $f : X \to \mathbb{R}$ is continuous then f achieves its maximum and minimum.

(d) If $f : X \to Y$ is a continuous bijection then f^{-1} is continuous.

Proof. We prove each part in turn.

(a) Let $\{V_\alpha\}$ be a cover of $f(X)$. Define $U_\alpha = f^{-1}(V_\alpha)$; since there exists a finite subcover $\{U_1, \ldots, U_n\}$ of X, $\{V_1, \ldots, V_n\}$ must be a cover for $f(X)$ and is a finite subcover.

(b) Apply the Heine-Borel theorem to the previous result.

(c) $f(X)$ is bounded so it has a supremum and infimum, and by compactness it contains both.

(d) Consider an open set $U \in X$. Then U^C is closed, and as a closed subset of a compact set it is also compact. Then $f(U^C)$ is compact and thus closed, so $f(U)$ is open and f^{-1} is continuous.

This completes the proof. $\square$

There's one last fairly intuitive property that's preserved by continuous functions, and it brings us to one of the most basic results of introductory calculus.

Definition: Connected set

Two sets $A, B \subset X$ are separated if $A \cap \overline{B}$ and $\overline{A} \cap B$ are both empty. A set $E \subset X$ is connected if it is not a union of two nonempty separated sets.

Theorem 3.10

Suppose $f : X \to Y$ is continuous and $E \subset X$ is connected. Then $f(E)$ is also connected.

Theorem 3.11: Intermediate value theorem

If $f : [a, b] \to \mathbb{R}$ is continuous and $f(a) < c < f(b)$, then there exists an $x \in (a, b)$ such that $f(x) = c$.

Proof. $[a, b]$ is connected, so $f([a, b])$ is connected. We have $c \in (f(a), f(b)) \subset f([a, b])$, meaning there exists an $x \in [a, b]$ such that $f(x) = c$. Since $c \neq f(a)$ and $c \neq f(b)$, $x \in (a, b)$. $\square$

3.2 Differentiation

Let's continue with the calculus and introduce another familiar object, one that's closely tied to continuity.

Definition: Derivative

A function $f : [a, b] \to \mathbb{R}$ is differentiable at $x_0 \in [a, b]$ if the following limit exists:

$$f'(x_0) = \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0}, \quad x \in (a, b),\ x \neq x_0.$$

$f'(x_0)$ is called the derivative of f at x. If f is differentiable at every point of a set $E \subseteq [a, b]$, we simply say that f is differentiable on E.

Theorem 3.12: Differentiability implies continuity

Let f be defined on $[a, b]$. If f is differentiable at a point $x \in [a, b]$ then f is continuous at x.

Proof. Using some properties of limits we have, as $x \to x_0$,

$$f(x) - f(x_0) = \frac{f(x) - f(x_0)}{x - x_0} \cdot (x - x_0) \to f'(x) \cdot 0 = 0.$$

Thus $\lim_{x \to x_0} f(x) = f(x_0)$ and f is continuous, as desired. $\square$

We can use this to get some more familiar results about combinations of functions.

Theorem 3.13

Suppose that f, g are defined on $[a, b]$ and that $f'(x), g'(x)$ exist. Then

(a) $(f+g)'(x) = f'(x) + g'(x)$ and

(b) $(fg)'(x) = f'(x)g(x) + f(x)g'(x)$.

Proof. (a) follows immediately from properties of limits, so we focus on (b). Define $h = fg$, so

$$h(x) - h(x_0) = f(x)[g(x) - g(x_0)] + g(x_0)[f(x) - f(x_0)].$$

Dividing by $x - x_0$ and taking $x \to x_0$ gives the expression we desire. □

Now we arrive at another pair of familiar theorems from calculus: the mean value theorem and its supporting lemma, Rolle's theorem.

Lemma 3.14: Rolle's theorem

Suppose f is continuous on $[a, b]$. If f has a local extremum at $c \in [a, b]$ and $f'(c)$ exists, then $f'(c) = 0$.

Proof. Consider the quotient $[f(x) - f(c)]/(x - c)$. If $f(c)$ is a maximum then $f(x) - f(c) \leq 0$ for all x; if $x > c$ this difference is negative, and if $x < c$ it's positive. Since $f'(c)$ exists, it must be that $f'(c) = 0$. We could make a similar argument for minima, completing the proof. □

Theorem 3.15: Mean value theorem

If f is continuous on $[a, b]$ and differentiable on (a, b), then there exists a $c \in (a, b)$ such that

$$f(b) - f(a) = (b - a)f'(c).$$

Proof. Define $h(x) = [f(b) - f(a)]x - [b - a]f(x)$. Since $h(a) = h(b)$, by Rolle's theorem we have $h'(c) = 0$ for some $c \in (a, b)$. Now, the derivative is

$$h'(x) = [f(b) - f(a)] - [b - a]f'(x).$$

Thus there exists a c such that $f(b) - f(a) = (b - a)f'(c)$, as desired. □

As our last result from differential calculus, we generalize the mean value theorem to get a powerful way of approximating functions near a point.

Theorem 3.16: Taylor's theorem

Let

$$P_{n-1}(x) = \sum_{k=0}^{n-1} \frac{f^{(k)}}{k!}(x - a)^k.$$

If $f^{(n-1)}$ is continuous on $[a, b]$ and $f^{(n)}$ exists on (a, b) then P_{n-1} approximates $f(x)$ and

$$f(x) = P_{n-1}(x) + \frac{f^{(n)}(c)}{n!}(x - a)^n$$

for some $c \in (a, x)$.

Proof. There is an M such that

$$f(b) = P_{n-1}(b) + M(b-a)^n.$$

Define

$$g(x) = f(x) - P_{n-1}(x) - M(x-a)^n,$$

so $g^{(n)}(x) = f^{(n)}(x) - n!M$. Now we must show that there exists a $c \in [a,b]$ such that $g^{(n)}(c) = 0$. Notice that $P^{(k)}(a) = f^{(k)}(a)$ for $k = 0, \ldots, n-1$ by construction, so similarly $g^{(k)}(a) = 0$.

Now we iteratively apply the mean value theorem. With how we've defined M we have $g(b) = 0$, so there exists a $c_1 \in [a,b]$ such that $g'(c_1) = 0$. Since $g'(a) = 0$, $g''(c_2) = 0$ for some $c \in [a, c_1]$. Continuing this way, we conclude that $g^{(n)}(c_n) = 0$ for some $c_n \in [a, c_{n-1}] \subset [a,b]$, as desired. □

3.3 Sequences of Functions

Branching off into a new direction, not only can we use sequence to study functions, but we can study sequences of functions in their own right. The most obvious way we might define convergence for such a sequence is to fix x and immediately leverage the definition we're already familiar with.

Definition: Pointwise convergence

Let $\{f_n\}$ be a sequence of functions defined on a set E. We say that $\{f_n\}$ converges pointwise to f if

$$f(x) = \lim_{n\to\infty} f_n(x)$$

for all $x \in E$.

Unfortunately, this definition isn't particularly useful. It doesn't preserve any of the properties we'd want it to preserve—for example, a sequence can have continuous terms without its limit being continuous. The same goes for derivatives and integrals.

To remedy this we need a stronger notion of convergence. Rather than considering each x-value in isolation, we might imagine a width-ϵ ribbon around a function, and that a sequence converges to that function if all of the terms are eventually contained within that ribbon.

Definition: Uniform convergence

Define the uniform norm (or sup norm) $\|f\| = \sup_{x\in E} |f(x)|$. We say $\{f_n\}$ converges uniformly to f on E if for all $\epsilon > 0$ there exists an N such that $n > N$ implies $\|f_n - f\| < \epsilon$.

We'll give a few reasons as to why this is a useful definition to have in our pocket. First, it preserves continuity, just like we wanted.

Theorem 3.17: Uniform convergence preserves continuity

If $f_n \to f$ uniformly and f_n is continuous for all n, then f is continuous.

Proof. We'll begin with an application of the triangle inequality:

$$|f(x) - f(y)| \leq |f(x) - f_n(x)| + |f_n(x) - f_n(y)| + |f_n(y) - f(y)|.$$

For any $\epsilon > 0$ there exist N, δ such that by uniform convergence $|x-y| < \delta$ implies the first and third terms are less than $\epsilon/3$. By continuity, the middle term is also less than $\epsilon/3$, and by transitivity $|f(x) - f(y)| < \epsilon$. So f is continuous, as desired. □

The supremum norm behind uniform convergence also creates a complete metric space of functions.

Theorem 3.18

Let X be a metric space and consider the set $\mathcal{C}(X)$ of continuous, bounded functions $f : X \to \mathbb{R}$. $\mathcal{C}(X)$ with the supremum norm is a complete metric space.

Proof. Consider an arbitrary Cauchy sequence $\{f_n\} \in \mathcal{C}(X)$. By definition there exists $N_1 \in \mathbb{N}$ such that $m, k > N_1$ implies $\|f_m - f_k\| < \epsilon$, and in turn $|f_m(x) - f_k(x)| < \epsilon$ for all $x \in X$.

If we fix $x \in X$ then $\{f_n(x)\}$ is a Cauchy sequence of real numbers, and so it converges to some $p_x \in \mathbb{R}$ with "cutoff" N_2. Define the function $f : X \to \mathbb{R}$ as $f(x) = p_x$.

Let $N = \max\{N_1, N_2\}$, so $n > N$ implies

$$|f_n(x) - p_x| = |f_n(x) - f(x)| < \epsilon$$

for all $x \in X$. Thus $\|f_n - f\| < \epsilon$ for $n > N$ and $f_n \to f$ uniformly. f is continuous by Theorem 3.12, meaning every Cauchy sequence $\{f_n\} \subset \mathcal{C}(X)$ converges to a function $f \in \mathcal{C}(X)$, as desired. $\square$

This leads us right into a convergence test for functions that are defined via infinite sums.

Theorem 3.19: Weierstrass M-test

Let X be a metric space. For each $n \in \mathbb{N}$, consider $f_n : X \to \mathbb{R}$ where for each n there exists $M_n > 0$ such that

$$|f_n(x)| \leq M_n$$

for all $x \in X$. If the series $\sum_{n=1}^{\infty} M_n$ converges, then the series $\sum_{n=1}^{\infty} f_n$ converges uniformly on X.

Proof. Let $\epsilon > 0$. Since $M_n \in \mathbb{R}$ and $\sum_{n=1}^{\infty}$ converges, we have $|\sum_{k=n}^{m} M_k| < \epsilon$ for all $m, n \geq N$. So

$$\left|\sum_{k=n}^{m} f_k(x)\right| \leq \sum_{k=n}^{m} M_k < \epsilon.$$

So the sequence of partial sums of $\sum_{n=1}^{\infty} f_n$ satisfies the Cauchy criterion for functions and, by completion, the series converges. $\square$

When used in conjunction with our previous finding about continuity, we can prove some strange things about certain functions. We might show, for example, that the infamous Weierstrass function

$$f(x) = \sum_{n=1}^{\infty} b^n \cos(a^n \pi x)$$

is continuous everywhere but differentiable nowhere if $a \in \mathbb{Z}$ is odd, $0 < b < 1$. and $ab > 1 + 3\pi/2$. Stranger, this is not an edge-case function! It turns out that the set $D(X)$ of functions that are differentiable at at least one point is meager, while $\mathcal{C}(X)$ is not. In other words, $D(X)$ is vanishingly small compared to $\mathcal{C}(X)$, meaning it's actually the norm for a continuous function to be differentiable nowhere.