1 Unsorted Notes

1.1 Lecture 3

So far we've been talking about dynamics—forces, conserved quantities, and what drives motion. Now we'll go into kinematics, which focuses on how to describe motion—think energy and momentum. A rapid review of special relativity is in order!

The energy and momentum of a particle are given by $E = \gamma mc^2$ and $\mathbf{p} = \gamma m\mathbf{v}$, and these two can be combined into the four-vector

$$p^\mu = \left(\frac{E}{c}, \mathbf{p}\right).$$

Now, instead of working primarily with four-vectors as we have in the past, here we'll try to think about relativity primarily in terms of invariants. For example, we often work with the Lorentz-invariant scalar product

$$p^2 = p \cdot p = \left(\frac{E}{c}\right)^2 - |\mathbf{p}|^2.$$

It's generally a fruitful problem-solving strategy in a simple frame before moving into whatever frame we care about. For example, in a particle's rest frame we will always have $p^\mu = (mc, \mathbf{0})$, meaning $p^2 = m^2c^2$ (in any frame, since p^2 is invariant!) and so

$$\frac{E^2}{c^2} - |\mathbf{p}|^2 = m^2c^2$$

in any frame.

A four-vector is defined by how it transforms when changing frames to an observer S' with $\mathbf{V} = V\hat{z}$: if $\gamma = 1/\sqrt{1-\beta^2}$ with $\beta = V/c$, we have

((TODO: SEE SLIDES.))

We define a general "upstairs" (contravariant) four-vector A^μ, of which p^μ is an example. Their "dot product" is given by

$$A \cdot B = A^0B^0 = \mathbf{A} \cdot \mathbf{B} = \sum_{\mu,\nu} g_{\mu\nu}A^\mu B^\nu,$$

where $g_{\mu\nu}$ is the metric

$$g_{\mu\nu} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}.$$

Such a metric actually has profound implications for the geometry of spacetime (see GR!), but for the purposes of this class we may view this as a handy bookkeeping technique.

We also define a "downstairs" (covariant) version of a vector $V_\mu = (V^0, -\mathbf{V})$ so that

$$A \cdot B = \sum_{\mu,\nu} g_{\mu\nu}A^\mu B^\nu = \sum_\mu A_\mu B^\mu = \sum_\mu A^\mu B_\mu$$

A common shorthand is to write $A \cdot B = g_{\mu\nu}A^\mu A^\nu$ as an implicit sum over $\mu, \nu = 0, 1, 2, 3$.

Now suppose I have a bunch of μ^+, μ^-—how could we tell if they came from X-decay? By conservation of momentum,

$$\begin{aligned} p_x^\alpha &= p_\mu^\alpha + p_{\overline{\mu}} \\ p_x \cdot p_x &= (p_\mu + p_{\overline{\mu}}) \cdot (p_\mu + p_{\overline{\mu}}) \\ m_x^2 c^2 &= \frac{(E_\mu + E_{\overline{\mu}})^2}{c^2} - |\mathbf{p}_\mu + \mathbf{p}_{\overline{\mu}}|^2 \equiv m(\mu, \overline{\mu})c^2. \end{aligned}$$

We call $m(\mu, \overline{\mu})$ the "invariant mass" of this pair of particles.

Note that virtual particles do not have to obey the relativistic mass-energy relationship. We refer to such particles as off-shell (as in, "off the mass shell") and real particles on-shell.

There are two kinds of interactions we generally deal with in particle physics. Decay is quantified by some decay rate in the parent's rest frame, which is an intrinsic particle property. Scattering is quantified by a scattering cross section, which depends on the kinematics of the scenario.

Let's look at decay. Because elementary particles don't age (they're all identical), the decay probability is independent of the particle's history. With N particles we define the width Γ:

$$dN = \frac{-\Gamma}{\hbar} N\, dt \implies N(t) = N(0)e^{-\Gamma t/\hbar}, \quad \tau \equiv \frac{\hbar}{\Gamma}.$$

We call τ the particle's lifetime. We prefer to work with Γ over τ because the relationship between a coupling constant and Γ is proportional, while that with τ is inverse. (We can thus interpret Γ as a proxy for the decay rate.)

But a particle might have several decay modes $A \to X_1 Y_1$ ... $A \to X_\ell Y_\ell$. In this case we must include

$$dN_A = -\frac{1}{\hbar} \sum_{i=1}^{\ell} \Gamma_{A \to X_i Y_i} N_A\, dt$$

and we define the branching fraction

$$\mathrm{Br}_i = \frac{\Gamma_{A \to X_i Y_i}}{\sum_i \Gamma_{A \to X_i Y_i}}.$$

Finally, the meaning of Γ. By energy-time uncertainty, $\Delta E \geq \Gamma/2$. Roughly speaking, Γ is the width $2\Delta E$ of a decay distribution (plotting E against σ).

Now onto scattering. We can experimentally determine the rate at which a process occurs (as a function of incident particle energy) and the angular/kinematic distributions of the final-state particles. The scattering cross section bridges the gap between theory and experiment!

The luminosity $\mathcal{L}$ of a beam is the number of interacting particle pairs per unit area per unit time. Our unit of area will be the barn (b), so the units of luminosity are $b^{-1}s^{-1}$.

1.2 Lecture 4

The scattering cross section is defined as

$$\sigma_{AB \to CD} = \frac{1}{\mathcal{L}} \frac{dN_{AB \to CD}}{dt}.$$

This "divides out" the number of collisions to get a quantity that's independent of the beam intensity. We'll find it useful to define the integrated luminosity

$$\mathcal{L}_{\text{int}} = \int \mathcal{L}\, dt,$$

so that we can determine the number of events expected in an experiment:

$$N_{AB \to CD} = \sigma_{AB \to CD} \int \mathcal{L}\, dt = \sigma_{AB \to CD} \mathcal{L}_{\text{int}}.$$

The differential cross section

$$\frac{d\sigma}{d\Omega} = \frac{\#\text{ of products within } d\Omega \text{ of } (\theta,\phi)}{\mathcal{L}_{\text{int}} d\Omega}$$

encodes the angular distribution of the final particles.

At this point we introduce natural units, where we set $\hbar = c = 1$. It's similar to saying "Im about ten minutes away"—we have a generally agreed-upon speed, either by walking or by car. We can express "10 minutes" as an energy:

$$10 \text{ min} = \frac{600 \text{ s}}{\hbar} = \frac{600 \text{ s}}{6.58 \times 10^{-25} \text{ GeV s}} \approx 10^{27} \text{ GeV}^{-1}.$$

Similarly, we would say that the proton mass is 938 MeV/$c^2 = 938$ MeV. The takeaway here is that, in natural units, all units are energy! (The distinction doesn't really matter for class, but keep it in mind when reading the literature.)

Now, given a calculation of a quantum amplitude $\mathcal{M}$ from Feynman diagrams, we can calculate the decay rate or cross section via Fermi's Golden Role:

$$\frac{dN_{i\to f}}{dt} = \sum_{\text{configs}} \frac{2\pi}{\hbar} |\mathcal{M}(i \to f)|^2.$$

This is a result from time-dependent perturbation theory—take Jedi QM!

For calculating decays we use the relativistic Fermi's Golden Rule for Decays in the rest frame. Look at the integral in the Calculating Rates slide. Here're the pieces:

- The $\delta^4(p_A - p_B - p_C)$ is there to conserve momentum. This factor is zero if momentum isn't conserved for all four components of the four-momentum.
- The other two deltas impose the mass-shell condition for external particles. The scalar $p^2 = mc^2$ in the rest frame, so this equality should hold in all frames.
- The step functions θ are there to ensure that we have positive energy.
- All of the 2π factors come from Fourier transforms! We have one for each δ and an inverse for each d.
- The $\mathcal{S}$ is a correction factor—it is $1/2$ when B and C are identical and 1 otherwise.

Now we'd like to transform the integral into one over the 3-momenta rather than the 4-momenta. In doing this we use the property

$$\delta(f(x)) = \sum_{x_0} \frac{1}{|f'(x_0)|}\delta(x - x_0), \qquad f(x_0) = 0.$$

Note that $\delta(p^2 - m^2c^2) = \delta((p^0)^2 - |\mathbf{p}|^2 - m^2c^2)$, so in this case $f(p^0) = (p^0)^2 - |\mathbf{p}|^2 - m^2c^2$ and $f'(p^0) = 2p^0$. We can see that p^0 is a root of f if $p^0 = \pm\sqrt{|\mathbf{p}|^2 + m^2c^2} = \pm E(\mathbf{p})/c$. We can therefore write

$$\int 2\pi\, \delta(p_B^2 - m_B^2 c^2)\theta(p_B^0)\frac{d^4 p_B}{(2\pi)^4} = \int 2\pi \left[\frac{1}{2E_B/c}\,\delta(p^0 - E_B/c) + \frac{1}{2E_B/c}\,\delta(p_B^0 + E_B/c)\right]\theta(p_B^0)\frac{d^4 p_B}{(2\pi)^4}.$$

Because $p_B^0 > 0$ the second delta vanishes and $\theta(p_B^0) = 1$. One of the 2π's also goes away, so

$$= \int \frac{d^3 p_B}{(2\pi)^3}\frac{1}{2E_B/c}.$$

Doing this simplification in the larger integral gives the integral on the Simplifying Decay Rate slide. But we still have four delta functions to go.

Let's hone in on the $A \to BB$, $m_B = 0$ case:

$$\Gamma_{A\to BB} = \frac{Sc^2}{2M_A}\int |\mathcal{M}|^2 (2\pi)^4 \delta(p_A^0 - p_B^0 - p_C^0)\delta^3(\mathbf{p}_A - \mathbf{p}_B - \mathbf{p}_C)\frac{d^3 p_B}{2E_B(2\pi)^3}\frac{d^3 p_C}{2E_C(2\pi)^3}$$

By integrating over the latter $d^3 p_C$ in the A rest frame $\mathbf{p}_A = 0$, $\mathbf{p_B} = -\mathbf{p}_C$:

$$= \frac{Sc^2}{2M_A}\frac{1}{(2\pi)^2}\int |\mathcal{M}|^2 \delta(p_A^0 - p_B^0 - p_B^0)\frac{d^3 p_B}{1}\frac{1}{2|\mathbf{p}_B|^2 c^2}\cdot\frac{1}{2}$$

Since B is massless, $E_B = |\mathbf{p}_B|c$ and the delta becomes $\delta(m_A c - 2|\mathbf{p}_B|) = \frac{1}{2}\delta(m_A c/2 - |\mathbf{p}_B|)$. Replacing the $d^3 p_B$ with a solid angle differential:

$$= \frac{S}{8\pi^2 m_A} \int |\mathcal{M}|^2 \frac{1}{2} \delta\left(\frac{m_A c}{2} - |\mathbf{p}_B|\right) \frac{|\mathbf{p}_B|^2 d|\mathbf{p}_B| d\Omega}{4|\mathbf{p}_B|^2}$$

The integral over all solid angles is 4π, so

$$\begin{aligned} &= \frac{S \cdot 4\pi}{8\pi^2 m_A} \frac{|\mathcal{M}|^2}{2} \cdot \frac{1}{4} \\ &= \frac{S}{16\pi m_A} |\mathcal{M}|^2, \end{aligned}$$

and $S = 1/2$ for identical particles. So we end with the simpler expression on the Simplifying Decay Rate slide (in which $B = C$).

1.3 Lecture 5

The axis on which it makes the most sense to measure spin is the direction of a particle's momentum. So we define the helicity

$$\hat{h} = \frac{\mathbf{p} \cdot \hat{\mathbf{S}}}{|\mathbf{p} \cdot \mathbf{S}|},$$

whose eigenvalue is $+1$ if the spin is aligned with the momentum and -1 if it is anti-aligned. For a massive particle (like an electron) it's possible to change the sign of $\mathbf{p} \cdot \hat{\mathbf{S}}$ by boosting into another frame, but for a massless particle (like a neutrino) it isn't. So for massless particles the quantity $\mathbf{p} \cdot \hat{\mathbf{S}}$ is Lorentz invariant.

Since there's no way to frame boost to turn a "right-handed" $(+1)$ neutrino into a "left-handed" (-1) one, we call these two kinds of neutrinos different particles altogether. Every observer will agree that a RH neutrino is RH, and same for LH. We have never observed a RH neutrino.

We can think of massive fermions as a superposition of the two chiral states: a left-handed particle and a right-handed particles. These particles are distinct—they interact differently with the fundamental forces, for example.

Let's do some math to justify this. We'll try to find a relativistic wave equation that makes sense. The Schrödinger equation doesn't seem to work because here $\hat{H} = \hat{p}^2/2m$, but in relativity we have $E = \sqrt{p^2c^2 + m^2c^4}$. So maybe we can take $\hat{H} = \sqrt{\hat{p}^2 c + m^2 c^4}$ and, applying $\hat{H}$ twice to avoid square roots,

$$\hat{H}^2 \psi = \left(i\hbar \frac{d}{dt}\right)^2 \psi \implies (\hat{p}^2 c^2 + m^2 c^4)\psi = -\hbar^2 \frac{d^2\psi}{dt^2}.$$

This is the Klein-Gordon equation. Alternatively, we can write

$$\partial_\mu \partial^\mu \psi + \frac{M^2 c^2}{\hbar^2} \psi = 0, \qquad \partial_\mu \equiv \frac{\partial}{\partial x^\mu} = \left(\frac{1}{c}\frac{\partial}{\partial t}, \nabla\right).$$

The spacial wave functions might look like $\psi \sim e^{i\mathbf{p}\cdot\mathbf{r}/\hbar - iEt/\hbar}$ with $E = \cdot$ (see slide).

The issue with the KG equation is that if we supplement our wave function with a spin part (a two-dimensional spinor) with components ψ_+, ψ_-, we get two completely disjoint equations in ψ_+ and ψ_-. The root of the problem here is that $\hat{P}^2$ commutes with the spin operators (? so there's no relationship between the spin and the momentum ?). We want the Hamiltonian to treat spin states differently depending on the momentum, so we need some $\gamma^\mu \hat{P}_\mu$ in there.

So let's look at a first-order relativistic wave equation, the Dirac equation:

$$i\hbar\gamma^\mu \partial_\mu \psi - Mc\psi = 0, \qquad \gamma^\mu = (\gamma^0, \vec{\gamma}), \quad \gamma^\mu\gamma^\nu + \gamma^v\gamma^\mu = 2g^{\mu\nu}.$$

Importantly, the smallest-dimension matrices satisfying this equations are 4×4. These matrices are

$$\gamma^0 = \begin{bmatrix} 0 & \mathbb{I}_2 \\ \mathbb{I}_2 & 0 \end{bmatrix}, \qquad \gamma^j = \begin{bmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{bmatrix},$$

where the σ^i are the Pauli spin matrices. (This is us using the Weyl basis.) These matrices encode the behavior of two spin-1/2 particles—a particle and an antiparticle!

The energy eigenstates of the Dirac equation are

$$\psi(\mathbf{r},t) = u(\mathbf{p})e^{i\mathbf{p}\cdot\mathbf{r}/\hbar - iEt/\hbar} = u(\mathbf{p})e^{-ip\cdot x}.$$

To find E, $\mathbf{p}$, and $u(\mathbf{p})$ that satisfy this equation, we substitute and get

$$\begin{aligned}\left(i\hbar\gamma^\mu \frac{\partial}{\partial x^\mu} - Mc\right) u(\mathbf{p})e^{-ip_\mu x^\mu/\hbar} &= 0,\\ (\gamma^\mu p_\mu - Mc)u(\mathbf{p}) &= 0.\end{aligned}$$

This is the equation we'll focus on.

In the case of an electron at rest we have $\mathbf{p} = 0$, so $\gamma^\mu p_\mu = \gamma^0 p_0 = \gamma^0 E/c$. Our equation is thus

$$\left(\frac{\hbar\omega}{c}\gamma^0 - Mc\,I_4\right)\begin{bmatrix}u_1\\u_2\\u_3\\u_4\end{bmatrix} = 0,$$

and we can multiply these matrices "block-wise" to get

$$\frac{\hbar\omega}{c}\begin{bmatrix}u_3\\u_4\\u_1\\u_2\end{bmatrix} = Mc\begin{bmatrix}u_1\\u_2\\u_3\\u_4\end{bmatrix}.$$

Solving yields $\hbar\omega = \pm Mc^2$, meaning $E = \pm Mc^2$. Positive- and negative-energy solutions have different pairs of eigenvectors—each pair has one spin-up vector and one spin-down (see slides).

The negative energies are troubling because they seem to violate conservation of energy (see reading). But if we write out the corresponding solution...

$$\ldots e^{i\mathbf{k}\cdot\mathbf{r}+iEt/k}\ldots$$

... we might alternatively interpret this as a negative-*time* solution. Specifically, the positive-energy eigenvectors (wave functions) correspond to the amplitude of destroying an electron of the corresponding spin, while the negative-energy ones are the amplitude of creating a positron of the corresponding spin.

This brings us to quantum field theory. Each particle has an operator associated with it called a quantum field: see the $\hat{\Psi}(\mathbf{r},t)$ in the slides. Here, the electron annihilation and positron creation operators are $\hat{a}_{e^-,s}(\mathbf{p})$ and $\hat{a}^\dagger_{e^+,s}(\mathbf{p})$. For more, take Jedi Quantum.

Now, massless solutions to the Dirac equation. Let $\mathbf{p} = p\,\mathbf{z}$, without loss of generality. Going through the computations yields two independent solutions, $E = \pm pc$:

$$\begin{bmatrix}\frac{E}{c}+p & 0\\ 0 & \frac{E}{c}-p\end{bmatrix}\begin{bmatrix}u_1\\u_2\end{bmatrix} = 0, \qquad \begin{bmatrix}\frac{E}{c}-p & 0\\ 0 & \frac{E}{c}+p\end{bmatrix}\begin{bmatrix}u_1\\u_2\end{bmatrix} = 0.$$

The nontrivial solutions for electrons and positrons are on the slides. Note that the electron and positron solutions are left- and right-handed, respectively. Thus suggests that we have the antiparticle pairs $e^-_L \leftrightarrow e^+_R$ and $e^-_R \leftrightarrow e^+_L$. The nomenclature we use is:

- u_L: "electron"
- w_R: "anti-electron"
- u_R: "anti-positron"
- w_L: "positron"

It turns out that the W boson only interacts with the first two of these particles. Also, as a side note, the processes $e^-_L\overline{\nu}_{eR} \to W^-$ and $e^+_R\nu_{eL} \to W^+$ are the ones that are allowed.

1.4 Lecture 6

We have the Dirac equation

$$i\hbar\gamma^\mu\partial_\mu\psi - Mc\psi = 0.$$

The general solution is complicated, but we can get an understanding of massive states in highly relativistic cases by using a perturbative approach. We write the solution as

$$u(p) = u_0(p) + u_1(p),$$

where u_0 is the right-handed solution at $M = 0$ and u_1 is a first-order correction in M. We solve for u_0, substitute into the Dirac equation, and keep only the $\mathcal{O}(M)$ terms. As a result, if we have a ± 1 in one chiral state, we have a small $\pm Mc^2/2E$ in the other chiral state in the small-M (or highly relativistic) limit.

MAIN TAKEAWAYS SLIDE! Now let's get into photons.

We aim to get a photon wave equation. Thankfully, we already know these "wave equations"—they are Maxwell's equations! Now we just need to figure out where the photon is in them. (We will use Gaussian units in this discussion.)

We begin by writing these fields in terms of potentials:

$$\nabla\cdot\mathbf{B} = 0 \;\longrightarrow\; \mathbf{B} = \nabla\times\mathbf{A},$$
$$\nabla\times\left(\mathbf{E} + \frac{1}{c}\frac{\partial\mathbf{A}}{\partial t}\right) = 0 \;\longrightarrow\; E + \frac{1}{c}\frac{\partial\mathbf{A}}{\partial t} = -\nabla V.$$

So instead of speaking in terms of $(\mathbf{E}, \mathbf{B})$ we can use $(V, \mathbf{A})$, meaning we only (at most) four degrees of freedom! This is also how we do Lagrangian and Hamiltonian mechanics with E&M.

Most importantly for us, though, is that the potentials comprise the 4-potential $A^\mu = (V, \mathbf{A})$. But it turns out that the potentials are not unique—not only can we shift each by a constant without affecting the physics, we can also shift

$$V \to V + \frac{1}{c}\frac{\partial\lambda}{\partial t}, \qquad \mathbf{A} \to \mathbf{A} + \nabla\lambda$$

for some scalar function λ while leaving $\mathbf{E}$ and $\mathbf{B}$ unchanged. We call such a shift a gauge transformation, and this phenomenon of leaving $\mathbf{E}, \mathbf{B}$ alone is called gauge invariance. (Some say gauge symmetry, but Prof. Shuve argues that this is a misnomer since gauge invariance isn't a real, physical symmetry. It's just a mathematical artifact that follows behind the physical descriptions of the potentials.)

We've yet to describe the Maxwell's equations that are source-dependent:

$$\partial_\mu(\partial^\mu A^\nu - \partial^\nu A^\mu)\frac{4\pi}{c}J^\nu, \qquad J^\nu = (\rho c, \mathbf{J}).$$

We could perform a gauge transformation on A^ν to show that this structure ensures the gauge invariance of Maxwell's equations.

This gauge invariant combination in Maxwell's equations leads directly to charge conservation:

$$\partial_\mu J^\mu = \frac{\partial\rho}{\partial t} + \nabla\cdot\mathbf{J} = 0.$$

Showing this quickly. In the $\mu = \nu$ case we immediately get $\partial^\mu A^\mu - \partial^\mu A^\mu = 0$, while in the $\mu \neq \nu$ case we notice that we can add, for example, the $(\mu = 0,\ \nu = 1)$ term with the $(\mu = 1,\ \nu = 0)$ term to get zero.

In the absence of sources the wave equation is thus

$$\partial_\mu\partial^\mu A^\nu - \partial^\nu\partial_\mu A^\mu = 0.$$

This is very close to the Klein-Gordon equation with $M = 0$, if we could only throw away that second term.

Notice that this equation doesn't have a unique solution due to the gauge invariance of A^μ. We can choose to work in the Coulomb gauge, which satisfies $\nabla\cdot\mathbf{A} = 0$. We could use this to show that taking Maxwell's equation with $\nu = 0$ gives $\nabla^2 V = 0$, and so $V = 0$. Also, in the absence of sources the Coulomb gauge is equivalent to the Lorenz gauge in which $\partial_\mu A^\mu = 0$:

$$\partial_\mu A^\mu = \frac{1}{c}\frac{\partial V}{\partial t} + \nabla\cdot\mathbf{A}.$$

Thus we can toss the second term and the photon wave function satisfies the Klein-Gordon equation. The wave function is

$$A^\nu(\mathbf{r}, t) = b\varepsilon^\nu(p)e^{i\mathbf{p}\cdot\mathbf{r}-iEt/\hbar}.$$

The $\varepsilon^\nu(p)$, the direction of the vector potential, is called the polarization. It is equivalent to the spinor for spin-1/2 particles, and in the case of spin-1 it contains spin information which is correlated with photon momentum.

We can see that this polarization is constrained by gauge invariance since $A^0 = 0$ and $\nabla \cdot \mathbf{A} = 0$. The polarization is thus transverse, as we saw in Ph51. If the momentum is in the z-direction, then, the polarization only has ε^1 and ε^2 components. Choosing the unit 4-vector norm $\varepsilon^\mu\varepsilon_\mu = -1$, for RCP and LCP we have, as described in Ph116,

$$\varepsilon^\mu_+ = \frac{1}{\sqrt{2}}(0, 1, i, 0), \qquad \varepsilon^\mu_- = (0, 1, -i, 0).$$

Finally, the process $e^-_L e^+_L \to \gamma$ is forbidden by conservation of angular momentum, but $e^-_L e^+_R to \gamma$ is not (though it is relativistically forbidden)! This confirms that $e^-_L.e^+_R$ is a particle-antiparticle pair.