Moreover, the truth values of a statement is T when it is true and F when it is false. A statement can be **simple** or **compound. A simple statement consists of one and only one declarative statement** (one subject and one predicate).

For example:

$p$: Earth is round, is a simple statement.

**A compound statement consists of two or more than two statements.**

For example:

$p$: $2 + 2 = 4$ and $5 + 7 < 13$, is a compound statement consisting of two simple statements

$q$: $2 + 2 = 4$

$r$: $5 + 7 < 13$.

### 5.3.1 Negation of a Statement

Associated with every statement is another statement called its *negation*. In other words, the statement having meaning contradictory to the given statement $p$ is called the negation of $p$ and is denoted by $\sim p$ or $\neg p$.

$\therefore$ $\sim p$ or $\neg p$ is a statement "not $p$". It can also be expressed as

$\neg p$ = It is not the case that $p$ is

OR

= It is false that $p$ is

Thus, if $p$ is true then $\sim p$ is false and if $p$ is false then $\sim p$ is true.

For example:

Consider

$p$: $2 + 2 = 4$

then

$\neg p$: $2 + 2 \neq 4$

OR

$\neg p$: It is not the case that $2 + 2 = 4$

OR

$\neg p$: It is false that $2 + 2 = 4$.

The truth values of a statement can be enlisted in a table called **Truth** table. So, the Truth table of $p$ and $\neg p$ is given below:

| $p$ | $\neg p$ |
|---|---|
| T | F |
| F | T |

### 5.3.2 Conjunction ($\wedge$)

If $p$ and $q$ are statements, the conjunction of $p$ and $q$ is the compound statement "$p$ and $q$", denoted by $p \wedge q$. The connective **and** is denoted by the symbol $\wedge$.

The truth value of $p \wedge q$ is True when both $p$ and $q$ are true otherwise it is false. The truth table of $p \wedge q$ is given below.

**Truth Table**

| $p$ | $q$ | $p \wedge q$ |
|---|---|---|
| T | T | T |
| T | F | F |
| F | T | F |
| F | F | F |

For example: $p$: $2 + 2 = 4$

$q$: God exists

So the conjunction of $p$ and $q$ is

$p \wedge q$: $2 + 2 = 4$ and God exists.

### 5.3.3 Disjunction ($\vee$)

If $p$ and $q$ are statements, the disjunction of $p$ and $q$ is the compound statement "$p$ or $q$", denoted by $p \vee q$. The connective **OR** is denoted by symbol $p \vee q$.

The truth value of $p \vee q$ is True (T) when atleast one of $p$ or $q$ is true; otherwise false (F). The truth table for $p \vee q$ is given below:

**Truth Table**

| $p$ | $q$ | $p \vee q$ |
|---|---|---|
| T | T | T |
| T | F | T |
| F | T | T |
| F | F | F |

For example:

$p$: $2 + 2 = 5$

$q$: God exists

$\therefore$ $p \vee q$: $2 + 2 = 5$ or God exists.

### 5.3.4 "Exclusisve OR"

Let $p$ and $q$ be propositions. The exclusive OR of $p$ and $q$, denoted by $p \oplus q$, is the proposition that is true when exactly one of $p$ and $q$ is true and is false otherwise. The truth table for exclusive or is given below:

**Truth Table**

| $p$ | $q$ | $p \oplus q$ |
|---|---|---|
| T | T | F |
| T | F | T |
| F | T | T |
| F | F | F |

### 5.3.5 Implication or Conditional Statement

If $p$ and $q$ be two statements, the implication of $p$ and $q$, denoted by $p \Rightarrow q$, is a statement "If $p$ then $q$". The symbol "$\Rightarrow$" is used for "If, then".

For example:

Consider $p$: $2 + 2 = 4$

$q$: pigs can fly.

Therefore, the implication $p \Rightarrow q$ is "If $2 + 2 = 4$, then pigs can fly".

Here the statement $p$ is called **Hypothesis** or **antecedent** or **Assumption** whereas $q$ is called **Conclusion** or **Consequent**.

The truth value of $p \Rightarrow q$ is F only when $p$ is true and $q$ is false; otherwise it is always True (T). Thus the conditional statement $p \Rightarrow q$ is false only when Hypothesis is True and the conclusion is false.

The truth table of $p \Rightarrow q$ is given as follows:

**Truth Table**

| $p$ | $q$ | $p \Rightarrow q$ |
|---|---|---|
| T | T | T |
| T | F | F |
| F | T | T |
| F | F | T |

The proposition $p \Rightarrow q$ may be expressed as:

*p implies q*

*if p then q*

*p only if q*

$p$ is a sufficient condition for $q$,

$q$ is a necessary condition for $p$,

$q$ if $p$, $q$ follows from $p$,

$q$ provided $p$,

$q$ whenever $p$.

### 5.3.6 Converse and Inverse

If $p \Rightarrow q$ is an implication, then the converse of $p \Rightarrow q$ is the implication $q \Rightarrow p$ and the inverse of $p \Rightarrow q$ is $\neg p \Rightarrow \neg q$ whereas the contrapositive of $p \Rightarrow q$ is the implication $\sim q \Rightarrow \sim p$.

**Double Implication or Bi-conditional**

If $p$ and $q$ be statements, then bi-conditional of $p$ and $q$, denoted by $p \Leftrightarrow q$, is a statement

"$p$ If and only if $q$"

OR

"If $p$ then $q$ and if $q$ then $p$".

The truth value of $p \Leftrightarrow q$ is T when either both $p$ and $q$ are True or both are false. The truth table of $p \Leftrightarrow q$ is given as follows:

**Truth Table**

| $p$ | $q$ | $p \Rightarrow q$ | $q \Rightarrow p$ | $p \Leftrightarrow q$ |
|---|---|---|---|---|
| T | T | T | T | T |
| T | F | F | T | F |
| F | T | T | F | F |
| F | F | T | T | T |

## 5.4 LOGICALLY EQUIVALENCE

Two statements are said to be *logically equivalent if* both have the identical truth values.

## WORKED OUT EXAMPLE: LOTS

**Example 2.** Which of the following sentences are statements? Give reasons for your answer.

(*i*) $2 + 3 = 5$.

(*ii*) The sum of all interior angles of a triangle is 180°.

(*iii*) The sun will come out tomorrow.

(*iv*) Delhi is capital of India.

(*v*) $\sqrt{2}$ is an irrational number.

**Solution:**

(*i*) is a statement that happens to be true.

(*ii*) is a statement; it has already been mathematically established that sum of the interior angles of a triangle is 180°.

(*iii*) is a statement since it is either true or false but not both, although we would have to wait until tomorrow to find out if it is true or false.

(*iv*) is a statement.

(*v*) is a statement.

**Example 3.** Negate the following statements:

(*i*) $2+5>7$.

(*ii*) Pigs can fly.

(*iii*) Both the diagonals of a rectangle have the same length.

(*iv*) $\sqrt{2}$ is rational.

**Solution:** The Negation of given statements are as follows:

(*i*) $2+5 \ngtr 7$.

(*ii*) Pigs cannot fly.

(*iii*) It is false that both the diagonals of a rectangle have same length.

OR

It is not the case that both the diagonals of a rectangle have same length.

(*iv*) $\sqrt{2}$ is not rational.

**Example 4.** Combine the given below statements using conjunction:

(*i*) $p$: $2+2=4$ $\quad$ $q$: God exists

(*ii*) $p$: It is snowing $\quad$ $q$: $5+7<9$

(*iii*) $p$: Pigs can fly $\quad$ $q$: It is raining.

**Solution:**

(i) $p \wedge q$: $2+2=4$ and God exists.

(*ii*) $p \wedge q$: It is snowing and $5+7<9$

(*iii*) $p \wedge q$: Pigs can fly and it is raining.

**Example 5.** Combine the given below statements using disjunction.

(*i*) $p$: Delhi is capital of India $\quad$ $q$: $\sqrt{2}$ is rational

(*ii*) $p$: It will rain tomorrow $\quad$ $q$: It will snow tomorrow

(*iii*) $p$: I will drive my car $\quad$ $q$: I will be late.

**Solution:**

(*i*) $p \vee q$: Delhi is capital of India or $\sqrt{2}$ is rational.

(*ii*) $p \vee q$: It will rain tomorrow or It will snow tomorrow.

(*iii*) $p \vee q$: I will drive my car or I will be late.

**Example 6.** Write the truth value of the following propositions:

(*i*) $2 + 3 = 7$

(*ii*) $7 + 8 < 10$

(*iii*) Punjab is capital of USA

(*iv*) God exists and $2 + 2 = 4$

(*v*) 2 is a positive integer or $\frac{5}{7}$ is a rational number.

**Solution:**

(*i*) False

(*ii*) False

(*iii*) False

(*iv*) Let $p$: God exists $\quad q$: $2 + 2 = 4$

So the given statement can be written in symbolic form as $p \wedge q$.

Here $p$ is True and $q$ is also True so $p \wedge q$ is also True.

(*v*) Let $p$: 2 is a positive integer

$q$: $\frac{5}{7}$ is a rational number.

So the given statement can be written in symbolic form as $p \vee q$.

Now $p$ is True and $q$ is also True. So the truth value of $p \vee q$ is True.

**Example 7.** Write each of the following in terms of $p$, $q$, $r$ and logical connectives.

(*i*) He is tall and handsome.

(*ii*) It is raining and he is not driving.

(*iii*) I am not in good mood and I am not going to movie.

(*iv*) It is false to say that pigs can fly.

(*v*) He is eating either apple or banana.

(*vi*) If he drives 60 m.p.h then he will reach on time.

**Solution:**

(*i*) Let $p$: He is tall.

$q$: He is handsome

Thus, he is tall and handsome can be written as $p \wedge q$.

(*ii*) Let $p$: It is raining.

$q$: He is not driving.

$\therefore$ It is raining and he is not driving can be expressed as $p \wedge q$

(*iii*) Let $p$: I am in good mood

$q$: I am going to movie

Therefore $\neg p$: I am not in good mood

$\neg q$: I am not going to movie

$\therefore \neg p \wedge \neg q$ = I am not in good mood and I am not going to movie

(*iv*) Let $p$: pigs can fly

$\therefore$ It is false to say that pigs can fly = $\neg p$

(*v*) Let $p$: He is eating apple

$q$: He is eating banana

$\therefore$ He is eating apple or banana = $p \vee q$

(*vi*) Let $p$: He drives 60 m.p.h.

$q$: He will reach on time.

So the given statement "If he drives 60 m.p.h. then he will reach on time" can be expressed in symbolic form as "If $p$ then $q$" so the symbolic form is $p \Rightarrow q$.

**Example 8.** Write the truth value of the following statements:

(*i*) If $2 + 1 = 3$ then pigs can fly.

(*ii*) If God exists then $12 + 5 > 8$.

(*iii*) If earth is not round, then honesty is not the best policy.

**Solution:**

(*i*) Let $p$: $2 + 1 = 3$

$q$: pigs can fly

So the given statement can be expressed as $p \Rightarrow q$.

Here $p$ is True and $q$ is False. We know that the truh value of $p \Rightarrow q$ is F when $p$ is true and $q$ is false. Hence, the statement "If $2 + 1 = 3$ then pigs can fly" has truth F.

(*ii*) Let $p$: God exists

$q$: $12 + 5 > 8$.

So, the given statement can be expressed as $p \Rightarrow q$.

Here $p$ is True and $q$ is also True. We know that the truth value of $p \Rightarrow q$ is T when both $p$ and $q$ are true. Hence, the truth value of "If God exists, then $12 + 5 > 8$" has truth value T.

## 5.6 ALGEBRA OF PROPOSITIONS

### Laws of Logic

1. Idempotent Laws

   $p \vee p \equiv p$ $\qquad$ $p \wedge p \equiv p$

2. Commutative Laws

   $p \vee q \equiv q \vee p$ $\qquad$ $p \wedge q \equiv q \wedge p$

3. Associative Laws

   $(p \vee q) \vee r \equiv p \vee (q \vee r)$ $\qquad$ $(p \wedge q) \wedge r \equiv p \wedge (q \wedge r)$

4. Distributive Laws

   $p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$ $\qquad$ $p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$

5. Law of Double Complimentation (Negation)

   $\neg(\neg p) \equiv p$

6. De Morgan's Laws

   $\neg(p \wedge q) \equiv \neg p \vee \neg q$ $\qquad$ $\neg(p \vee q) \equiv \neg p \wedge \neg q$

7. Negation Laws

   $p \vee \neg p \equiv T$ $\qquad$ $p \wedge \neg p \equiv F$

   $\neg T \equiv F$ $\qquad$ $\neg F \equiv T$

8. Identity Laws

   $p \wedge T \equiv p$ $\qquad$ $p \vee F \equiv p$

9. Absorption Laws

   $p \vee (p \wedge q) \equiv p$ $\qquad$ $p \wedge (p \vee q) \equiv p$

10. Law of contrapositive

    $p \Rightarrow q \equiv \neg q \Rightarrow \neg p$

11. Others

    $p \vee T = T$ $\qquad$ $p \wedge c = c$

**Example 24.** Find a logical expression equivalent to $\neg(p \Rightarrow q)$ which is free from implication.

**Solution:** We already have established in earlier example

$$p \Rightarrow q \equiv \neg p \vee q$$

$$\therefore \quad \neg(p \Rightarrow q) \equiv \neg(\neg p \vee q)$$

$$\equiv \neg(\neg p) \wedge \neg q \qquad \text{(By De-Morgan's law)}$$

$$\equiv p \wedge \neg q$$

Since R.H.S. is free from implication, therefore $\neg(p \Rightarrow q) \equiv p \wedge \neg q$.