$O = \{2n + 1 : n \in Z\}$

1.7 ALGEBRA OF SETS

Properties of Set (Laws of Sets)

1. **Commutative Laws**

$$A \cup B = B \cup A$$
$$A \cap B = B \cap A$$

2. **Associative laws**

$$A \cap (B \cap C) = (A \cap B) \cap C$$
$$A \cup (B \cup C) = (A \cup B) \cup C$$

3. **Distributive laws**

$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$$
$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$$

4. **Identity laws**

$$A \cup \phi = A$$
$$A \cup U = U$$

5. **Complement laws**

$$A \cup \bar{A} = U$$
$$A \cap \bar{A} = \phi$$

6. **Idempotent laws**

$$A \cup A = A$$
$$A \cap A = A$$

7. **Absorption laws**

$$A \cup (A \cap B) = A$$
$$A \cap (A \cup B) = A$$

8. **Law of Double complementation**

$$\overline{(\bar{A})} = A$$

9. **De – Morgan's laws**

$$\overline{A \cup B} = \bar{A} \cap \bar{B}$$
$$\overline{A \cap B} = \bar{A} \cup \bar{B}$$

It can be restated as "let $A, B \subseteq X$

then $X \sim (A \cup B) = (X \sim A) \cap (X \sim B)$

$X \sim (A \cap B) = (X \sim A) \cup (X \sim B)$

This law can be extended for n subsets of X

If $A_1, A_2 \dots A_n \subseteq X$ then

$X \sim (A_1 \cup A_2 \cup \cdots \cup A_n) = (X \sim A_1) \cap (X \sim A_2) \cap \cdots \cap (X \sim A_n)$

or $X \sim \bigcup_{i=1}^{n} A_i = \bigcap_{i=1}^{n} (X \sim A_i)$

and $X \sim (A_1 \cap A_2 \cap \dots \cap A_n) = (X \sim A_1) \cup (X \sim A_2) \dots \cup (X \sim A_n)$

1.9 CARTESIAN PRODUCT

Ordered Pair

An element of the form (x, y) is called an *Ordered pair*.

Here the element x is called the *First element* and y is called the *second element*.

Equality of Ordered Pairs

Two ordered pairs (a, b) and (a', b') are said to equal if $a = a'$ and $b = b'$

Cartesian Product

Given two sets A, B their cartesian product $A \times B$ is the set of all ordered pairs (a, b) such that $a \in A$ and $b \in B$

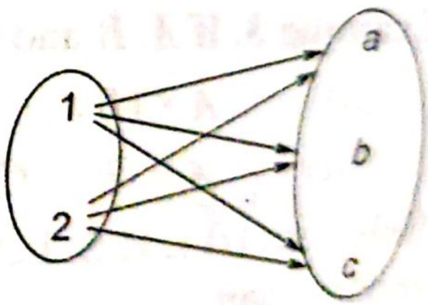

Fig. 1.12

$$\therefore \quad A \times B = \{(a, b) : a \in A \text{ and } b \in B\}$$

e.g., $A = \{1, 2\}$ $B = \{a, b, c\}$

$$\therefore \quad A \times B = \{(1, a), (1, b), (1, c)\ (2, a)\ (2, b)\ (2, c)\}$$

Analogously we can define triples or 3-tuples (a_1, a_2, a_3), 4-triples (a_1, a_2, a_3, a_4) ... n triples $(a_1, a_2, \ldots a_n)$

$\therefore A_1 \times A_2 \times \ldots \times A_n = \{(a_1, a_2, \ldots a_n): a_1 \in A_1 \text{ and } a_2 \in A_2 \ldots a_n \in A_n\}$

More over we can define

$$A_1^2 = A \times A, A^3 = A \times A \times A \text{ etc}$$

and

$$A^n = A \times A \times \ldots \times A \ (n \text{ times})$$

Note: If $|A| = m$ and $|B| = n$ then $|A \times B| = mn$

Note: If either A or B is null set then $A \times B$ a null set.

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Example 8. If $A = \{n^7 : n \text{ is a positive integer}\}$

$B = \{n^5 : n \text{ is a positive integer}\}$ then find $\boldsymbol{A \cap B}$.

Example 11. Prove that

$$|P[P(P(\phi))]| = 4$$

Solution: Since ϕ represents an empty set, so

$$|\phi| = 0$$

$\therefore$ $$|P(\phi)| = 2^0 = 1$$

$\therefore$ $$|P(P(\phi))| = 2^1 = 2$$

and $$|P[P(P(\phi))]| = 2^2 = 4$$

Example 12. If $a \in N$ such that

$$aN = \{ax : x \in N\}.$$

Describe the set $3N \cap 7N$.

Solution: We have

$$aN = \{ax : x \in N\}$$

$\therefore$ $$3N = \{3x, x \in N\} = \{3, 6, 9, 12 \cdots\}$$

and $$7N = \{7x, x \in N\} = \{7, 14, 21, 28 \cdots\}$$

$\therefore$ $$3N \cap 7N = \{21, 42, \cdots\}$$

$\therefore$ $$3N \cap 7N = \{21, 42, \cdots\} = 21\,N.$$

Example 23. Two finite sets A and B have a and b number of elements respectively. The total number of subsets of the first set is 8 times the total number of subsets of the second set. Find the value of $a - b$.

Solution: Here $|A| = a$ and $|B| = b$

$\therefore$ The number of subsets of A and B are 2^a and 2^b respectively.

Now Number of subsets of A = 8 (Number of subsets of B)

$$2^a = 8.2^b$$

i.e., $$2^a = 2^3 \cdot 2^b$$

i.e., $$2^a = 2^{3+b}$$

$\Rightarrow$ $$a = 3 + b$$

$\Rightarrow$ $$a - b = 3$$

WORKED OUT EXAMPLES : LOTS

Example 1. For any two sets A and B

if $n(A \cup B) = 50, n(A) = 30, n(B) = 24$

Find (*i*) $n(A \cap B)$ (*ii*) $n(A - B)$ (*iii*) $n(B - A)$.

Solution: We know that

$$n(A \cup B) = n(A) + n(B) - n(A \cap B)$$

$$50 = 30 + 24 - n(A \cap B)$$

$\Rightarrow$ $n(A \cap B) = 54 - 50 = 4$

(*ii*) $n(A - B) = n(A) - n(A \cap B) = 30 - 4 = 26$

(*iii*) $n(B - A) = n(B) - n(A \cap B) = 24 - 4 = 20$

Example 2. In a class of 100 students, 60 speak French, 55 speak Spanish and 7 of them do not speak any of these two languages. There are some students who speak both the languages. Find the number of students who speak

(*i*) Both languages

(*ii*) French but not Spanish

(*iii*) Spanish but not French.

Solution: Let us define the sets as

F: Set of students who speak French

S: Set of students who speak Spanish

$\therefore$ $|F| = 60, |S| = 55, |\bar{F} \cap \bar{S}| = 7, |U| = 100$

Now $|\bar{F} \cap \bar{S}| = |\overline{F \cup S}| = |U| - |(F \cup S)|$

$$7 = 100 - |F \cup S|$$

$\Rightarrow$ $|F \cup S| = 93$

Now $|F \cup S| = |F| + |S| - |F \cap S|$

$\Rightarrow$ $93 = 60 + 55 - |F \cap S|$

$\Rightarrow$ $|F \cap S| = 115 - 93 = 22$

(*ii*) Number of students who speak French but not Spanish

$$= |F - S| = |F \cap \bar{S}|$$

$$= |F| - |F \cap S| = 60 - 22 = 38$$

(*iii*) Number of students who speak Spanish but not French

$$= |S - F| = |S \cap \bar{F}|$$

$$= |S| - |S \cap F| = 55 - 22 = 33$$

Example 3. A bag contains tickets bearing numbers 1 to 200. Find the number of tickets which bear a multiple of 3 or 4 or 5.

Solution: Let $S = \{1, 2, 3, \ldots 200\}$

and $A = \{x : x \in S \text{ and divisible by } 3\}$

$B = \{x : x \in S \text{ and divisible by } 4\}$

$C = \{x : x \in S \text{ and divisible by } 5\}$

$\therefore$ $$|A| = \left\lfloor \frac{200}{3} \right\rfloor = 66$$

where $\lfloor x \rfloor$ represents greatest integer or floor function of x.

$$|B| = \left\lfloor \frac{200}{4} \right\rfloor = 50$$

$$|C| = \left\lfloor \frac{200}{5} \right\rfloor = 40$$

$$|A \cap B| = \left\lfloor \frac{200}{l.c.m\,(3,4)} \right\rfloor = 16$$

$$|B \cap C| = \left\lfloor \frac{200}{l.c.m\,(4,5)} \right\rfloor = \left\lfloor \frac{200}{20} \right\rfloor = 10$$

$$|A \cap C| = \frac{200}{l.c.m\,(3,5)} = \frac{200}{15} = 13$$

$$|A \cap B \cap C| = \left\lfloor \frac{200}{l.c.m\,(3,4,5)} \right\rfloor = \left\lfloor \frac{200}{60} \right\rfloor = 3$$

Example 4. In a colony of 10000 people, 3250 like brand A of a particular product, 2630 like brand B, 2710 like brand C, 560 like brand A and B, 315 like B and C, 220 like brand A and C and 50 are there who like all the three brands. Find the number of people who like

(*i*) Brand A only

(*ii*) Brand B only

(*iii*) Brand A and B but not C

(*iv*) At least one of three Brands A, B

(*v*) None of the three Brands.

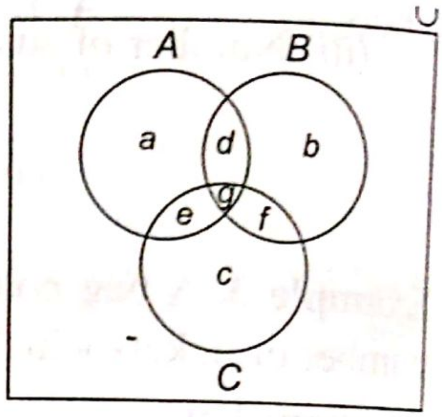

Fig. 1.10

Solution:

Here

$$|A| = 3250 = a + d + e + g$$
$$|B| = 2630 = b + d + g + f$$
$$|C| = 2710 = c + e + g + f$$
$$|A \cap B| = 560 = d + g$$
$$|B \cap C| = 315 = g + f$$
$$|A \cap C| = 220 = g + e$$
$$|A \cap B \cap C| = 50 = g$$

$\Rightarrow$ $$g = 50$$

On substituting the value of 'g' in above equations we obtain

$$e = 170, f = 265, d = 510$$
$$c = 2225, b = 1805, a = 2520$$

(*i*) Number of people who like brand A only $= a = 2520$

(*ii*) Number of people who like brand B only $= b = 1805$

(*iii*) Number of people who brand A and B but not C

$$= |A \cap B \cap \bar{C}| = d = 510$$

(*iv*) Number of people who like at least one of the three brands

$$= |A \cup B \cup C|$$
$$= a + b + c + d + e + f + g$$
$$= 2520 + 1805 + 2225 + 510 + 170 + 265 + 50$$
$$= 7545$$

Example 22. Let D_n denote the set of natural numbers that divide n exactly. Write down the sets D_{84} and $D_{84} \cap D_{60}$. Find a number m such that

$$D_m = D_{60} \cap D_{84}.$$

gcd

Solution: Here D_n represents the set of natural numbers that divide n.

$\therefore \quad D_{60} = \{1, 2, 3, 4, 5, 6, 10, 12, 15, 20, 30, 60\}$

and $\quad D_{84} = \{1, 2, 3, 4, 6, 7, 12, 14, 21, 42, 84\}$

$\therefore \quad D_{60} \cap D_{84} = \{1, 2, 3, 4, 6, 12\}$

$D_{60} \cap D_{84} =$ Set of all the divisor of 12

$D_{60} \cap D_{84} = D_m = D_{12}$

$\Rightarrow \quad m = 12.$

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Example 14 Each set X_r contains 5 elements and each set Y_r contains 2 elements and $\bigcup_{r=1}^{20} X_r = S = \bigcup_{r=1}^{n} Y_r$. If each element of S belong to exactly 10 of the X_r's and to exactly 4 of the Y_r's, then n is

(A) 10 (B) 20 (C) 100 (D) 50

Solution The correct answer is (B)

Since, $n(X_r) = 5$, $\bigcup_{r=1}^{20} X_r = S$, we get $n(S) = 100$

But each element of S belong to exactly 10 of the X_r's

So, $\frac{100}{10} = 10$ are the number of distinct elements in S.

Also each element of S belong to exactly 4 of the Y_r's and each Y_r contain 2 elements. If S has n number of Y_r in it. Then

$$\frac{2n}{4} = 10$$

which gives

$$n = 20$$

Example 15 Two finite sets have m and n elements respectively. The total number of subsets of first set is 56 more than the total number of subsets of the second set. The values of m and n respectively are.

(A) 7, 6 (B) 5, 1 (C) 6, 3 (D) 8, 7

Solution The correct answer is (C).

Since, let A and B be such sets, i.e., $n(A) = m$, $n(B) = n$

So $n(P(A)) = 2^m$, $n(P(B)) = 2^n$

Thus $n(P(A)) - n(P(B)) = 56$, i.e., $2^m - 2^n = 56$

$\Rightarrow$ $2^n(2^{m-n} - 1) = 2^3 7$

$\Rightarrow$ $n = 3$, $2^{m-n} - 1 = 7$

$\Rightarrow$ $m = 6$

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(D) $R = \{(x, y) : 0 < x < a, \ldots$

In a class of 60 students, 25 students play cricket and 20 students play tennis, and 10 students play both the games. Then, the number of students who play neither is

(A) 0 (B) 25 (C) 35 (D) 45

In a town of 840 persons, 450 persons read Hindi, 300 read English and 200 read both. Then the number of persons who read neither is

(A) 210 (B) 290 (C) 180 (D) 260

If $X = \{8^n - 7n - 1 \mid n \in \mathbf{N}\}$ and $Y = \{49n - 49 \mid n \in \mathbf{N}\}$. Then

(A) $X \subset Y$ (B) $Y \subset X$ (C) $X = Y$ (D) $X \cap Y = \phi$

A survey shows that 63% of the people watch a News Channel whereas 76% watch another channel. If x% of the people watch both channel, then

(A) $x = 35$ (B) $x = 63$ (C) $39 \leq x \leq 63$ (D) $x = 39$

If sets A and B are defined as

$A = \{(x, y) \mid y = \frac{1}{x}, 0 \neq x \in \mathbf{R}\}$ $B = \{(x, y) \mid y = -x, x \in \mathbf{R}\}$, then

(A) $A \cap B = A$ (B) $A \cap B = B$ (C) $A \cap B = \phi$ (D) $A \cup B = A$

Example 32 A market research group conducted a survey of 1000 consumers and reported that 720 consumers like product A and 450 consumers like product B, what is the least number that must have liked both products?

Solution Let U be the set of consumers questioned, S be the set of consumers who liked the product A and T be the set of consumers who like the product B. Given that

$$n(U) = 1000,\ n(S) = 720,\ n(T) = 450$$

So $\qquad n(S \cup T) = n(S) + n(T) - n(S \cap T)$

$$= 720 + 450 - n(S \cap T) = 1170 - n(S \cap T)$$

Therefore, $n(S \cup T)$ is maximum when $n(S \cap T)$ is least. But $S \cup T \subset U$ implies $n(S \cup T) \leq n(U) = 1000$. So, maximum values of $n(S \cup T)$ is 1000. Thus, the least value of $n(S \cap T)$ is 170. Hence, the least number of consumers who liked both products is 170.

Example 33 Out of 500 car owners investigated, 400 owned car A and 200 owned car B, 50 owned both A and B cars. Is this data correct?

Solution Let U be the set of car owners investigated, M be the set of persons who owned car A and S be the set of persons who owned car B.

Given that $\qquad n(U) = 500,\ n(M) = 400,\ n(S) = 200$ and $n(S \cap M) = 50$.

Then $\quad n(S \cup M) = n(S) + n(M) - n(S \cap M) = 200 + 400 - 50 = 550$

But $S \cup M \subset U$ implies $n(S \cup M) \leq n(U)$.

This is a contradiction. So, the given data is incorrect.

5. In a survey of 60 people, it was found that 25 people read newspaper H, 26 read newspaper T, 26 read newspaper I, 9 read both H and I, 11 read both H and T, 8 read both T and I, 3 read all three newspapers. Find:

(i) the number of people who read at least one of the newspapers.

(ii) the number of people who read exactly one newspaper.

6. In a survey it was found that 21 people liked product A, 26 liked product B and 29 liked product C. If 14 people liked products A and B, 12 people liked products C and A, 14 people liked products B and C and 8 liked all the three products. Find how many liked product C only.

Example 6. Determine which of the following are true and which are false. Justify.

(*i*) $1 \in \{1, 2\}$ (*ii*) $3 \subset \{1, 3, 5\}$ (*iii*) $\{3\} \subset \{4, 3, 2\}$

(*iv*) $\phi \subseteq \{\phi\}$ (*v*) $\{a, b\} \subseteq \{a, b, c, \{a, b, c\}\}$

(*vi*) $\{a, b\} \in P\{a, b\}$

Solution:

(*i*) True: 1 is an element of $\{1, 2\}$

So $1 \in \{1, 2\}$.

(*ii*) False, as 3 is an element of $\{1, 3, 5\}$ but not a proper subset.

(*iii*) True, as $\{3\}$ is a proper subset of $\{4, 3, 2\}$.

(*iv*) True, as ϕ is a subset of $\{\phi\}$.

(*v*) True, as $\{a, b\}$ is contained in $\{a, b, c, \{a, b, c\}\}$.

(*vi*) True as a set is always an element of its power set.

Example 7. Draw Venn diagrams of:

(*a*) B and C are disjoint sets and are subsets of $A \subseteq U$

(*b*) $A \cap B \cap C = \phi$, $A \cap B \neq \phi$, $B \cap C \neq \phi$, $C \cap A \neq \phi$.

Solution: Since B and C are disjoint

$\therefore B \cap C = \phi$. Since $B \subseteq A$ and $C \subseteq A$. It implies that B and C are contained in A.

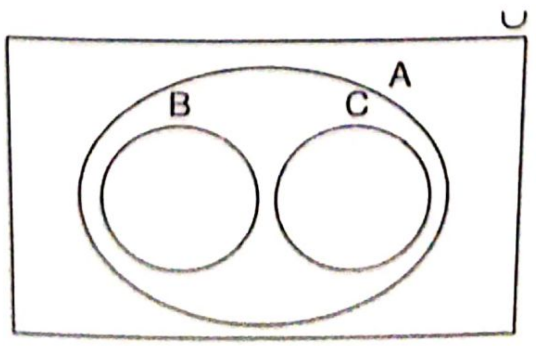

Fig. 1.7

(*b*) $A \cap B \cap C = \phi$, $A \cap B \neq \phi$, $B \cap C \neq \phi$, $C \cap A \neq \phi$

EXERCISE 1.1

1. List the elements of the following sets:

   a. $\{x \mid x \text{ is a real number such that } x^2 = 9\}$

   b. $\{x \mid x \text{ is an integer such that } x^2 - 3 = 0\}$

   c. $\{x \mid x \text{ is a letter in the word NITIN}\}$

2. Write all the subsets of :

   (*i*) $\{1, 2, 3\}$

   (*ii*) $\{\phi, \{\phi\}\}$

   (*iii*) $\{a, b, \{a, b\}\}$

3. If $A = \{1, 5, 6, 9, 2\}$ $B = \{2, 4, 3, 7\}$ $U = \{1, 2, 3, 4, \cdots 9\}$

   $C = \{3, 6, 9\}$ then compute

   (*i*) $A \cup B$

   (*ii*) $(A \cup C) \cap B$

   (*iii*) $A - B$

   (*iv*) $\bar{A} \cap \bar{B}$

4. If $U = \{1, 2, 3, 4, \cdots 9\}$

   $$A = \{2, 3, 5, 7\}$$
   $$B = \{1, 2, 6, 7, 9\}$$

   Then verify that

   (*i*) $\overline{(A \cup B)} = \bar{A} \cap \bar{B}$

   (*ii*) $\overline{(A \cup B)} = \bar{A} \cap \bar{B}$

5. If $A = \{n^2 : n \in N\}$

   $B = \{n^4 : n \in N\}$ then find $A \cap B$.

6. Prove that

   $$|P[P\{P(P(\phi))\}]| = 16.$$

7. If $A = \{x : x^2 - 5x + 6 = 0\}$

   $B = \{x : x^2 - 7x + 12 = 0\}$

   Then, find $A \cap B$ and $A \cup B$.

Example 5. The sports club of GGSIPU awarded 38 medals in Badminton, 15 in Table Tennis and 20 to Chess. If these medals went to a total of 58 students and only three students got medals in all the three sports, how many received medals in exactly two of three sports?

Solution: Let A denotes the set of students who received medals in Badminton, B is the set of students who received medals in Table Tennis and C the set of students received medals in Chess.

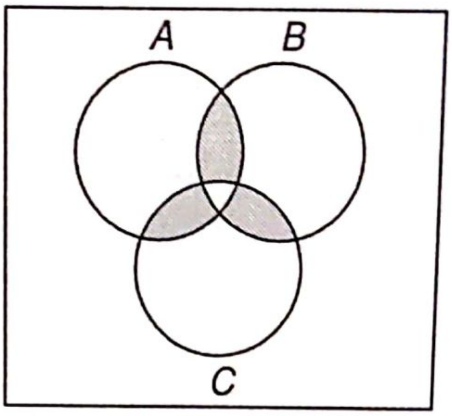

Fig. 1.11

$$\therefore \quad n(A) = 38, n(B) = 15, n(C) = 20$$

$$n(A \cup B \cup C) = 58, n(A \cap B \cap C) = 3$$

Now

$$n(A \cup B \cup C) = n(A) + n(B) + n(C) - n(A \cap B) - n(B \cap C) - n(A \cap C) + n(A \cap B \cap C)$$

$$\Rightarrow \quad 58 = 38 + 15 + 20 + - n(A \cap B) - n(B \cap C) - (A \cap C) + 3$$

$$\Rightarrow n(A \cap B) + n(B \cap C) + n(A \cap C) = 18 \qquad \ldots(i)$$

Now we need to find

$$n(A \cap B \cap \bar{C}) + n(A \cap \bar{B} \cap C) + n(\bar{A} \cap B \cap C)$$

So $\quad n(A \cap B \cap \bar{C}) = n(A \cap B) - n(A \cap B \cap C)$

$$n(A \cap \bar{B} \cap C) = n(A \cap C) - n(A \cap B \cap C)$$

and $\quad n(\bar{A} \cap B \cap C) = n(B \cap C) - n(A \cap B \cap C)$

$$\therefore n(A \cap B \cap \bar{C}) + n(A \cap \bar{B} \cap C) + n(\bar{A} \cap B \cap C)$$
$$= n(A \cap B) + n(B \cap C) + n(A \cap C) - 3\, n(A \cap B \cap C)$$

$$\Rightarrow n(A \cap B \cap \bar{C}) + n(A \cap \bar{B} \cap C) + n(\bar{A} \cap B \cap C) = 18 - 3 \times 3 = 9$$

Hence, 9 students received medals in exactly two of the three sports.

Example 4. For any two sets A, B

$$A \times B = B \times \mathrm{A} \Leftrightarrow A = B.$$

Solution: Suppose $A = B$

To show $A \times B = B \times A$

Since $A = B$; therefore

$$A \times B = A \times A$$

and $B \times A = A \times A$

$\Rightarrow$ $A \times B = B \times A$

Conversely, Suppose that $A \times B = B \times A$

To show: $A = B$ *i.e.*, $A \subseteq B$ and $B \subseteq A$

Let $x \in A$

$\Rightarrow$ $(x, y) \in A \times B \; \forall \; y \in B$

$\Rightarrow$ $(x, y) \in B \times A$ $\quad (\because A \times B = B \times A)$

$\Rightarrow$ $x \in B$

$\Rightarrow$ $A \subseteq B$ $\quad$ ...(*i*)

Again $q \in B$

$\Rightarrow$ $(p, q) \in A \times B \; \forall \; p \in A$

$\Rightarrow$ $(p, q) \in B \times A$ $\quad (\because A \times B = B \times A)$

$\Rightarrow$ $q \in A$

$\Rightarrow$ $B \subseteq A$ $\quad$ ...(*ii*)

(*i*) and (*ii*)

$\Rightarrow$ $A = B$

Example 24. Let $S = \{1, 2, 3, 4\}$. Find the class of all subsets of S which contains 2 and two other elements of S.

Solution: Here $S = \{1, 2, 3, 4\}$. Let δ be the class of all subsets of S which contains 2 and two other elements of S

$$\therefore \qquad \delta = \{\{2, 1, 3\}, \{2, 1, 4\}, \{2, 3, 4\}\}.$$

Example 25. Let $N = \{1, 2, 3...\}$ denote the set of all positive integers and

$$A = \{x: x \in N \text{ and } 3 < x < 12\}$$

$$B = \{x: x \in N \text{ and } x \text{ is even, } x < 15\}$$

Find $A \cap B$, $A \cup B$, A^C and B^C.

Solution: Here $N = \{1, 2, 3, ...\}$

$$A = \{x: x \in N, 3 < x < 12\}$$

$$\therefore \qquad A = \{4, 5, 6, 7, 8, 9, 10, 11\}$$

$$B = \{x: x \in N, x \text{ is even and } x < 15\}$$

$$\therefore \qquad B = \{2, 4, 6, 8, 10, 12, 14\}$$

$$\therefore \qquad A \cap B = \{4, 6, 8, 10\}$$

$$A \cup B = \{2, 4, 5, 6, 7, 8, 9, 10, 11, 12, 14\}$$

$$A^C = N - A$$

$$= \{1, 2, 3, 12, 13, 14, ...\}$$

$$B^C = N - B$$

$$= \{1, 3, 5, 7, 9, 11, 13, 15, 16, 17, 18, 19...\}$$