Universal Relation

A relation R from set A to set B is said to be universal if

$$R = A \times B$$

e.g., $A = \{a, b\}, B = \{c, d\}$

then $R = \{(a, c), (a, d), (b, c), (b, d)\}$ is a universal relation from set A to set B.

Void, Null or Empty Relation

Any relation R is called *empty* or void relation from set A to B

if $\qquad R = \phi$

Inverse Relation

Let R be a relation from A to B. Then the relation R^{-1} from B to A is called the *inverse relation* of R if

$$R^{-1} = \{(b, a) : (a, b) \in R\}$$

e.g., $\quad A = \{a, b\}, B = \{1, 2, 3\}$ and R is relation from A to $B \ni$

$$R = \{(a, 1), (b, 2), (b, 3), (a, 3)\}$$

then $\quad R^{-1} = \{(1, a), (3, a), (2, b), (3, b)\}$

Reflexive Relation

A Relation R defined on a set A is said to be reflexive if

$$\underline{a R a} \;\; \forall \; a \in A \qquad A \times A$$

i.e., $\quad (a, a) \in R \;\; \forall \; a \in A$

Let $A = \{1, 2, 3\}$ let R be a relation defined on A. If R is Reflexive then it must contain ordered pairs (1, 1), (2, 2) and (3, 3).

Example 1. Let $A = \{1, 2, 3, 4\}$ let R be a relation on set A defined as $R = \{(1, 1), (1, 2), (2, 2)\ (3, 2), (3, 2), (3, 3), (4, 1)\ (4, 4)\}$

Clearly R is reflexive because it contains every ordered (1, 1) (2, 2),(3, 3) and (4, 4)

i.e., $\quad a R a \;\; \forall \; a \in A.$

Consider, R_1 defined on $A \ni$

$$R_1 = \{(1, 1), (1, 2), (3, 2). (4, 4)\ (3, 3)\}$$

Clearly R_1 is not Reflexive because for $2 \in A$, $(2, 2) \notin R_1$.

Symmetric Relation

A relation R defined on set A is said to be *symmetric* if

$$a\,R\,b \Rightarrow b\,R\,a, \text{ where } a, b \in A$$

i.e., for any $a, b \in A$

$$(a, b) \in R \Rightarrow (b, a) \in R$$

e.g., Consider $R = \{(1, 2), (2, 1), (2, 2), (3, 1)\ (1, 3)\}$

defined on $A = \{1, 2, 3\}$

Clearly R is symmetric

$\therefore$

$$(1, 2) \in R \Rightarrow (2, 1) \in R$$

$$(2, 2) \in R \Rightarrow (2, 2) \in R$$

$$(1, 3) \in R \Rightarrow (3, 1) \in R$$

Whereas $R_1 = \{(1, 2), (1, 3), (2, 2)\ (3, 1)\}$

defined on $A = \{1, 2, 3\}$ is not symmetric because

$$(1, 2) \in R \Rightarrow (2, 1) \notin R.$$

Transitive

A relation *R* on a set *A* is called *transitive* if for $a, b, c \in A$

i.e., If $(a, b) \in R$ and $(b, c) \in R$ then $(a, c) \in R$.

e.g., The relation "*is parallel*" to on the set of lines in a plane is transitive because if a line l_1 is parallel to l_2 and if a l_2 is parallel to line l_3 then l_1 is parallel to l_3. A relation *R* on *A* is not transitive only when $(a, b) \in R$, $(b, c) \in R$ but $(a, c) \notin R$, otherwise it is always transitive.

Compatible Relation

A binary relation *R* on a set *A* is called *compatible* if it is reflexive and symmetric.

Less than Relation

A relation *R* from set *A* to *B* is said to be *less than relation* if

$$R = \{(a, b) \mid a < b, a \in A, b \in B\}$$

e.g., Let $A = \{1, 3\}$. $B = \{2, 5\}$. Let *R* be a 'less than relation' from set *A* and set *B* then.

$$R = \{(1, 2), (1, 5), (3, 5)\}$$

It clearly says that *R* contains all those ordered pairs of $A \times B$ whose domain elements are less than that of range elements.

Greater than Relation

A relation from set *A* to set *B* is said to be "*greater than relation*" if

$$R = \{(a, b) \mid a > b, a \in A, b \in B\}$$

It clearly says that *R* contains all those ordered pairs of $A \times B$ whose domain elements are greater than that of range elements.

So if *R* is a greater than relation from set $A = \{1, 3\}$ to set $B = \{2, 5\}$ then $R = \{(3, 2)\}$

Identity Relation

A relation R defined from set A to set B is called *identify* relation if

$$R = \{(a, b) \mid a = b, a \in A, b \in B\}$$

It implies that all Domain set of R = Range set of R.

e.g., $A = \{1, 2, 3\}, B = \{1, 3, 5\}$

Let R be an identity relation from set A to set B then

$$R = \{(1, 1), (5, 5)\}$$

Example 4. Following relations are defined on $A \times A$ where $A = \{1, 2, 3\}$. Which of them are (*i*) Reflexive, (*ii*) Symmetric, (*iii*) Transitive.

Solution:

(*a*) $R_1 = \{(1, 1), (2, 3), (3, 3)\}$

(*b*) $R_2 = \{(1, 2), (2, 3), (2, 1)\}$

(*c*) $R_3 = \phi$

(*d*) $R_4 = \{(1, 2), (2, 3), (4, 3)\}$

Solution:

(*a*) Since $(1, 1) \in R_1$ and $(2, 2) \notin R_1$, so R_1 is neither reflexive nor irreflexive. Therefore, R_1 is non-reflexive.
Moreover $(2, 3) \in R_1$ but $(3, 2) \notin R_1$, so R_1 is not symmetric. Also R_1 is transitive.

(*b*) Clearly in R_1, $(a, a) \notin R_1 \ \forall \ a \in A$ it implies R is non-reflexive.
Also $(2, 3) \in R_1$, but $(3, 2) \notin R_2$ therefore R_2 is not symmetric.
Also $(1, 2) \in R_2$ and $(2, 1) \in R_2$
But $(1, 1) \notin R_2$
So, R_2 is not transitive.

(*c*) $R_3 = \phi$ is non-reflexive but symmetric and transitive.

Example 5. Consider the following relation on

$$A = \{1, 2, 3, 4, 5, 6\}$$
$$R = \{(a, b) : |a - b| = 2\}$$

Is 'R' transitive? Is R reflexive? Is R symmetric?

Solution: Here

$$R = \{(1, 3), (3, 1), (2, 4), (4, 2), (3, 5), (5, 3), (4, 6), (6, 4)\}$$

(*i*) Clearly R is not reflexive as $(a, a) \notin R \ \forall \ a \in A$

(*ii*) Clearly R is not transitive because $(1, 3) \in R$, and $(3, 1) \in R$ but $(1, 1) \notin R$

(*i.e.*, $(a, b) \in R$ and $(b, c) \in R$ but $(a, c) \notin R$, $a, b, c \in A$)

(*iii*) Clearly R is symmetric because

$$(a, b) \in R \Rightarrow (b, a) \in R, a, b \in A.$$

Example 6. Show that R^{-1} is reflexive if R is Reflexive, where R is a relation on any let A.

Solution: Since R is a relation on A and R is reflexive.

So $\quad (a, a) \in R \ \forall \ a \in A$

$\Rightarrow \quad (a, a) \in R^{-1} \ \forall \ a \in A$

$\Rightarrow R^{-1}$ is reflexive.

Example 7. Let $A = \{4, 5, 6\}$ and R be relation on A defined as $R = \{(4, 3), (5, 6), (6, 4)$. Is R symmetric or asymmetric?

Solution: Here $\quad R = \{(4, 3), (5, 6), (6, 4)\}$

R is not symmetric as for every pair $(a, b) \in R$, there dose not exist $(b, a) \in R$.

i.e., $\quad (5, 6) \in R$ but $(6, 5) \notin R$.

Here R is asymmetric because for every pair $(a, b) \in R$, $(b, a) \notin R$.

Example 8. Let $A = \{1, 2, 3\}$ and $R = \{(1, 2), (2, 1), (3, 2), (3, 1)\}$

Is the relation R transitive?

Solution: For a relation to be transitive

$$(a, b) \in R, (b, c) \in R \Rightarrow (a, c) \in R$$

Hence $\quad (1, 2) \in R$ and $(2, 1) \in R$.

But $\quad (1, 1) \notin R$

So R is not transitive.

Example 9. Let $A = \{1, 2, 3, 4\}$. Determine whether the following relations are reflexive, symmetric or transitive:

(*i*) $R_1 = \{(1, 1), (2, 2), (3, 3), (4, 4)\}$

(*ii*) $R_2 = \{(1, 1), (2, 2)\}$

(*iii*) $R_3 = \phi$

Solution:

(*i*) $R_1 = \{(1, 1), (2, 2), (3, 3), (4, 4)\}$

Reflexive: Here R_1 is reflexive as for every $a \in A$, $(a, a) \in R_1$

i.e., $(1, 1), (2, 2), (3, 3), (4, 4) \in R_1$

Symmetric: Here R_1 is symmetric as

$$(a, b) \in R_1 \Rightarrow (b, a) \in R_1, a, b \in A$$

Transitive: R_1 is Transitive because there is no pair for which

$$(a, b) \in R_1, (b, c) \in R_1 \text{ and } (a, c) \notin R_1$$

(*ii*) $R_2 = \{(1, 1), (2, 2)\}$

Reflexive: Here R_2 is non-reflexive as for every $a \in A$, $(a, a) \notin R_2$

i.e., $(1, 1) \in R_2$, $(2, 2) \in R_2$ but $(3, 3) \notin R_2$

Symmetric: Here R_2 is symmetric as

$$(a, b) \in R_2 \Rightarrow (b, a) \in R_2, a, b \in A$$

Transitive: R_2 is transitive because there is no pair for which $(a, b) \in R_2$, $(b, c) \in R_2$ and $(a, c) \notin R_2$.

(*iii*) **Reflexive:** R_3 is irreflexive but not reflexive as $(a, a) \notin R_3$ $\forall$ $a \in A$

Symmetric: R_3 is symmetric because there is no pair $(a, b) \in R_3$ for which $(b, a) \notin R_3$

Transitive: R_3 is transitive because there is no pair $(a, b) \in R_3$, $(b, c) \in R_3$ for which $(a, c) \notin R_3$

HOTS

Example 10. In a set $X = \{a_1, a_2, a_3, a_4\}$ of four men, a_1, is younger to other three, a_2 is younger to a_3 and a_4 only and a_3 is younger to a_4 only. Is the relation "is younger to"

(*i*) Reflexive

(*ii*) Symmetric

(*iii*) Transitive

Solution: Let us define relation R as ' < ' on X as follows:

$a \, R \, b$ *i.e.*, $\quad a < b \Leftrightarrow a$ is younger to b

It is being given that

$$a_1 < a_2,\ a_1 < a_3,\ a_1 < a_4$$

$$a_2 < a_3,\ a_2 < a_4 \text{ and } a_3 < a_4$$

$$\Rightarrow \quad a_1 < a_2 < a_3 < a_4$$

(*i*) Clearly R is not Reflexive as no element can be younger to itself.

(*ii*) R is not symmetric because if a is younger to b then b can never be younger to a.

(*iii*) R is Transitive.

Example 11. Is it true to say that the intersection of two symmetric relations is symmetric?

Solution: Let R_1 and R_2 be two symmetric relation on A.

Let $\quad (a, b) \in R_1 \cap R_2 \qquad$ where $a, b \in A$

$\Rightarrow \quad (a, b) \in R_1$, and $(a, b) \in R_2$

Since R_1 is symmetric, So

$$(a, b) \in R_1 \Rightarrow (b, a) \in R_1,\ a, b \in A$$

Similarly $\quad (a, b) \in R_2 \Rightarrow (b, a) \in R_2,\ a, b \in A$

$\Rightarrow \quad (b, a) \in R_1 \cap R_2,\ a, b \in A$

$\Rightarrow R_1 \cap R_2$ is symmetric.

Example 12. For two Reflexive relations R and S on any set A, show that $R \cap S$ is also reflexive.

Solution: R is a reflexive relation

Since

$\therefore \quad (a, a) \in R \ \forall \ a \in A$

Also S is reflexive relation

$\therefore \quad (a, a) \in S \ \forall \ a \in A$

$\Rightarrow \quad (a, a) \in R \cap S \ \forall \ a \in A$

$\Rightarrow R \cap S$ is reflexive.

Example 3. Suppose $f : A \to B$ is any function and R is a relation on A such that

$R = \{(x, y) : f(x) = f(y), x, y \in A\}$. Show that R is an equivalence relation.

Solution: Here

$$x\,R\,y \Leftrightarrow f(x) = f(y) \text{ where } x, y \in A$$

(*i*) **Reflexive:** Since

$$f(x) = f(x)$$

$$\Rightarrow \quad x\,R\,x \;\; \forall\; x \in A$$

Hence, R is reflexive.

(*ii*) **Symmetric:** Let $x\,R\,y$, $x, y \in A$

Since $x\,R\,y$ therefore

$$f(x) = f(y)$$

$$\Rightarrow \quad f(y) = f(x)$$

$$\Rightarrow \quad y\,R\,x, \;\; \forall\; x, y \in A$$

Hence, R is symmetric.

(*iii*) **Transitive:** Let $x\,R\,y$ and $y\,R\,z$, $\quad x, y, z \in A$

$$x\,R\,y \Rightarrow f(x) = f(y)$$

$$y\,R\,z \Rightarrow f(y) = f(z)$$

$$\Rightarrow f(x) = f(z) \Rightarrow x\,R\,z$$

Therefore, R is transitive.

Since R is reflexive, symmetric and transitive, therefore R is an equivalence relation.

Example 4. Let A be a set of integers and R be a relation on A defined as $a\,R\,b$ if n divides $a - b$ where $a, b \in A$, $n \in N$. Show that R is an equivalence relation.

Solution: Here R is defined as

$$R = \{(a, b) : n \text{ divides } (a - b), a, b \in A \text{ and } n \in N\}$$

Reflexivity: Let $a \in A$. We know that $n \mid 0$

i.e., $\quad n \mid (a - a) \Rightarrow a\,R\,a$

$\Rightarrow \qquad (a, a) \in R$

$\Rightarrow R$ is reflexive.

Symmetry: Let $a, b \in A \ni a\,R\,b$. Since $a\,R\,b$

$\Rightarrow \qquad n \mid (a - b) \Rightarrow n \mid -(a - b)$

i.e., $\qquad n \mid b - a \Rightarrow b\,R\,a$

$\Rightarrow R$ is symmetric.

Transitive: Let $a, b, c \in A \ni a\,R\,b$ and $b\,R\,c$

Since $\qquad a\,R\,b \Rightarrow n \mid a - b$

and $\qquad b\,R\,c \Rightarrow n \mid b - c$

We know that if $n \mid x$ and $n \mid y$ then $n \mid (x + y)$ where $x, y \in Z$

So $\qquad n \mid (a - b + b - c)$

i.e., $\qquad n \mid (a - c)$

$\Rightarrow \qquad a\,R\,c$

Thus, R is transitive.

Since R is reflexive, symmetric and transitive. Hence, R is an equivalence relation.

Example 6. A relation R is defined on the set $N \times N$ where N is the set of natural numbers, by setting

$$(a, b)\ R\ (c, d) \Leftrightarrow a + d = b + c,\ a, b, c, d \in N$$

Show that this relation is an equivalence relation.

Solution:

Reflexivity: Let $a, b \in N$

$\therefore \quad (a, b)\ R\ (a, b) \Leftrightarrow a + b = b + a$

which is true. Thus, $(a, b)\ R\ (a, b)\ \forall\ (a, b) \in N \times N$

$\Rightarrow R$ is reflexive.

Symmetry: Let $a, b, c, d \in N \ni (a, b)\ R\ (c, d)$

Since $(a, b)\ R\ (c, d)$

$\therefore \quad (a, b)\ R\ (c, d) \Leftrightarrow a + d = b + c$

i.e., $\quad d + a = c + b \Leftrightarrow (c, d)\ R\ (a, b)$

Thus, R is symmetric.

Transitive: Let $a, b, c, d, e, f \in N \ni$

$(a, b)\ R\ (c, d)$ and $(c, d)\ R\ (e, f)$

To claim: $(a, b)\ R\ (e, f)$

Now $\quad (a, b)\ R\ (c, d) \Leftrightarrow a + d = b + c \quad$ ...(*i*)

and $\quad (c, d)\ R\ (e, f) \Leftrightarrow c + f = d + e \quad$ ...(*ii*)

$\Rightarrow \quad a + d + c + f = b + c + d + e \quad$ (adding (*i*) and (*ii*))

$\Rightarrow \quad a + f = b + e$

$\Rightarrow \quad (a, b)\ R\ (e, f)$

$\therefore$ R is transitive. Hence, R is an equivalence relation.

Example 7. If R is an equivalence relation on set A then show that R^{-1} is also an equivalence relation.

Solution: Here R is given an equivalence relation on set A, so R is Reflexive, symmetric and transitive.

We know that if $(a, b) \in A$ then $(b, a) \in R^{-1}$, $a, b \in A$.

So we are to show that R^{-1} is an equivalence relation.

Reflexivity: Let $a \in A$

Since R is reflexive

So $\quad (a, a) \in R\ \forall\ a \in A$

$\Rightarrow \quad (a, a) \in R^{-1}\ \forall\ a \in A$

$\therefore$ R^{-1} is reflexive.

Symmetry: Let $a, b \in A \ni (a, b) \in R^{-1}$

$\therefore \quad (b, a) \in R \quad$ [$\because$ By definition of inverse relation]

$\Rightarrow \quad (a, b) \in R \quad$ [$\because$ R is symmetric]

$\Rightarrow \quad (b, a) \in R^{-1}$

Thus $\quad (a, b) \in R^{-1}$

$\Rightarrow \quad (b, a) \in R^{-1}$

Thus, R^{-1} is symmetric.

Transitive: Let $a, b, c \in A \ni (a, b) \in R^{-1}$ and $(b, c) \in R^{-1}$

To claim: $\quad (a, c) \in R^{-1}, a, c \in A$

Now $\quad (a, b) \in R^{-1} \Rightarrow (b, a) \in R$

and $\quad (b, c) \in R^{-1} \Rightarrow (c, b) \in R$

Now

$\Rightarrow \quad (c, b) \in R$ and $(b, a) \in R$

$\Rightarrow \quad (c, a) \in R \quad$ [$\because$ R is transitive]

$\Rightarrow \quad (a, c) \in R^{-1}$

Thus, R is transitive.

Hence, R is an equivalence relation.