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- Greedy algorithms that actually obtain a global optimum often do not exist.
- Greedy algorithms are usually simple, but the proof of correctness may not be so simple.
- We will give several examples of greedy algorithms.

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- For now, we will consider just the fractional version.

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- Total weight = 100, total value = 125

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  - Algorithm runs in $O(n + k \log n)$ time, where $k$ is the number of objects fully or partially added to the knapsack.

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- Find the minimum number of required machines.

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Example: 8 tasks:
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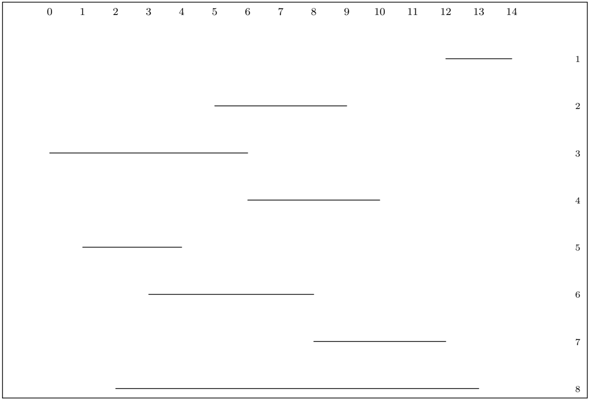

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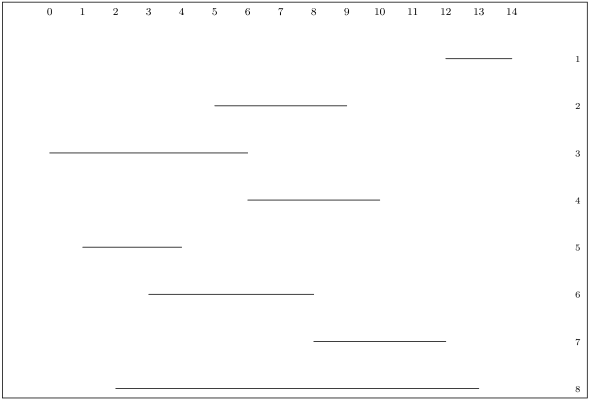

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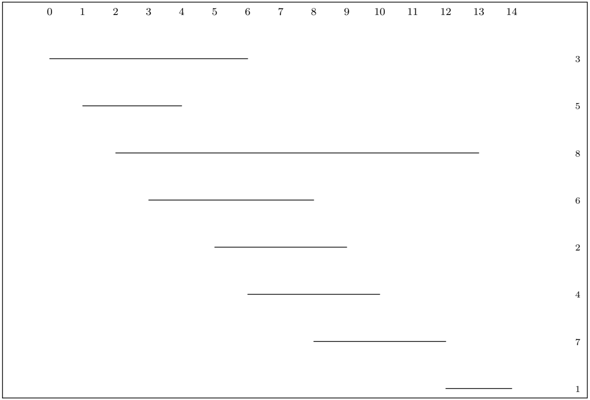

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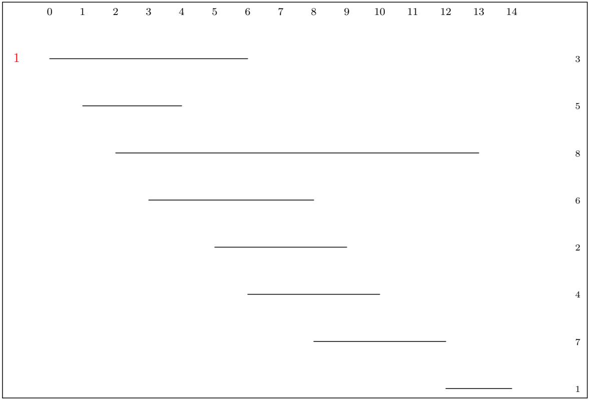

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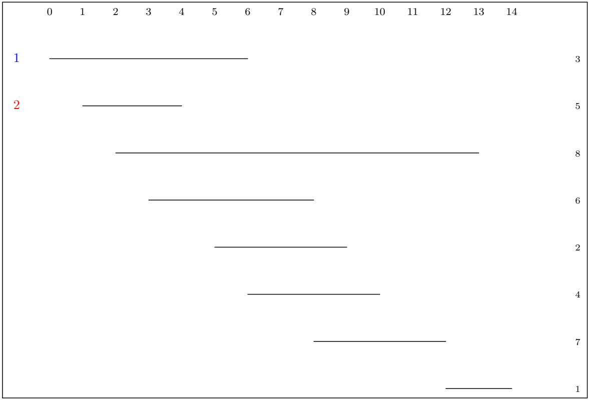

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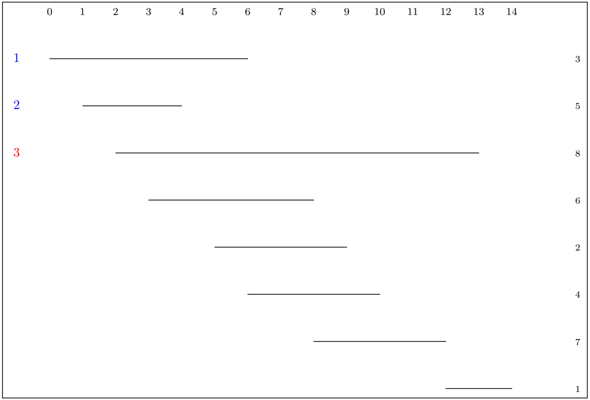

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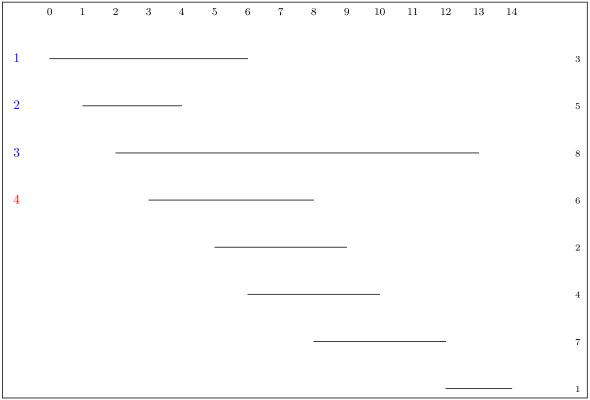

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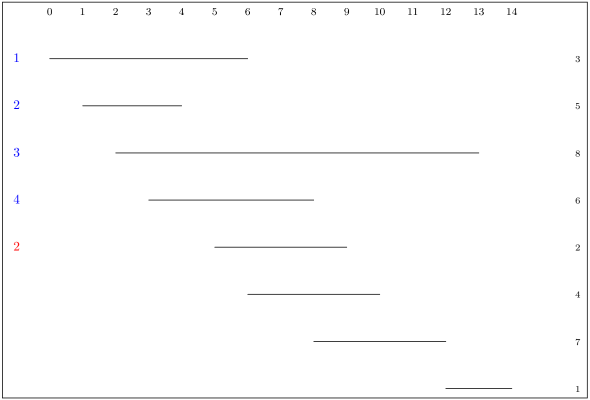

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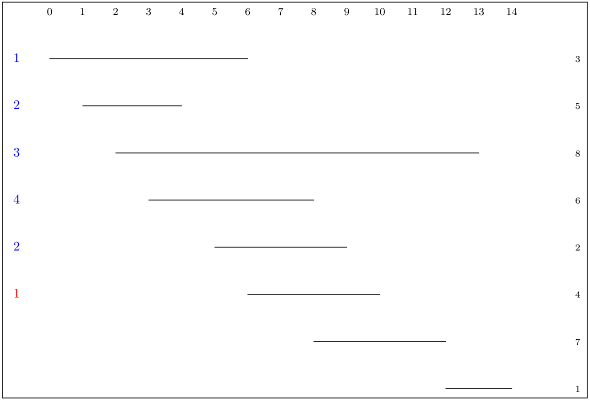

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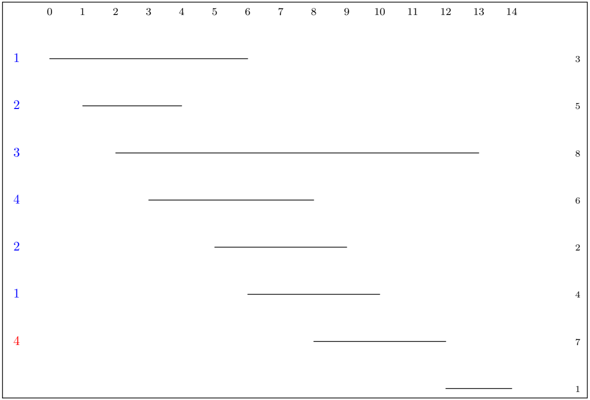

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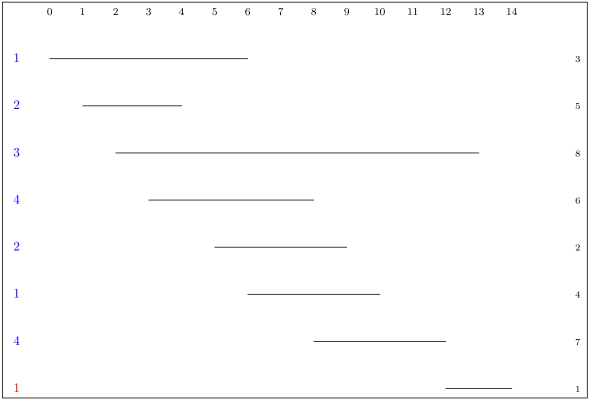

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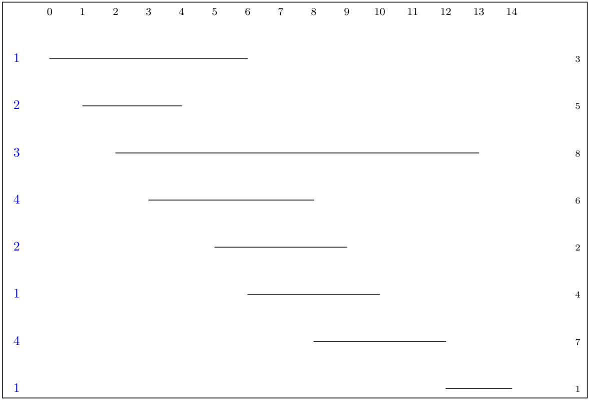

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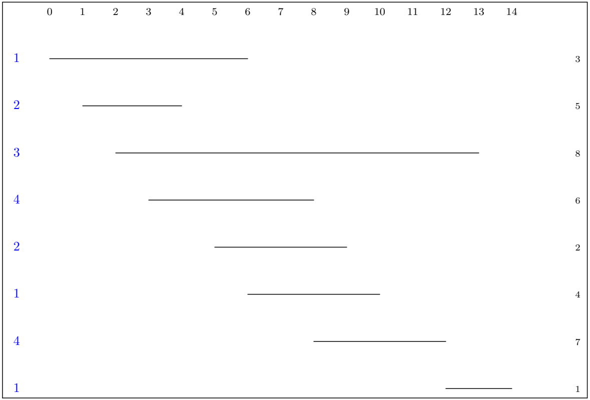


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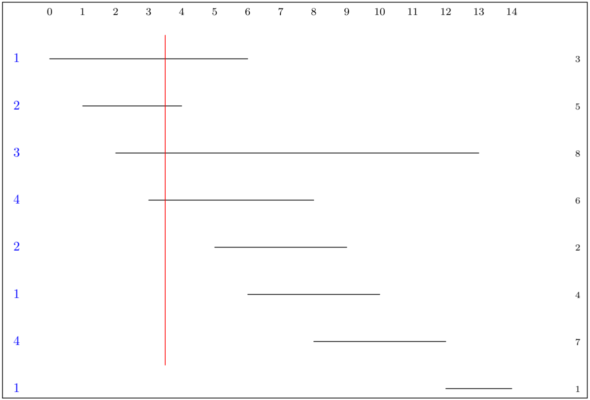


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4 machines are necessary (so solution is optimal)

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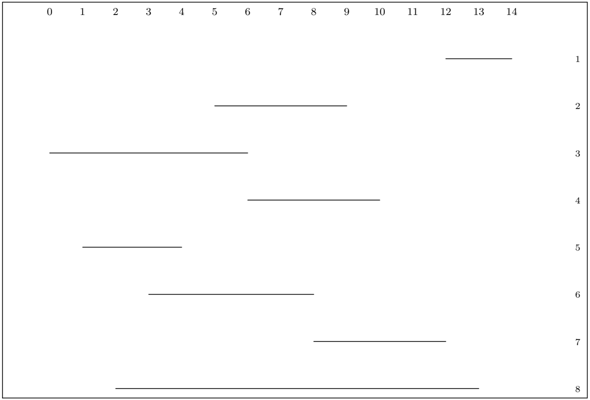

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[Kleinberg-Tardos] Jon Kleinberg and Éva Tardos, Algorithm Design, Addison-Wesley, 2006.

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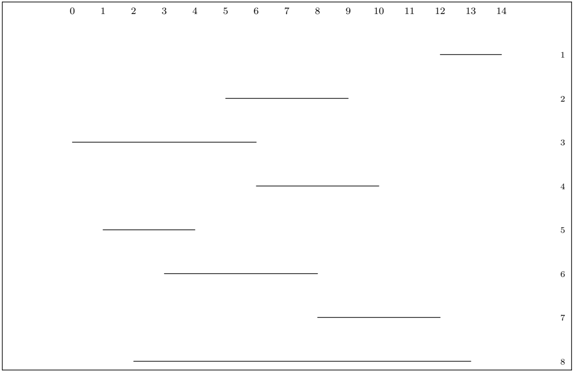

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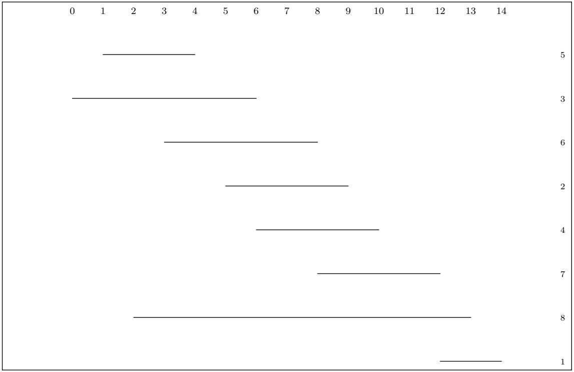

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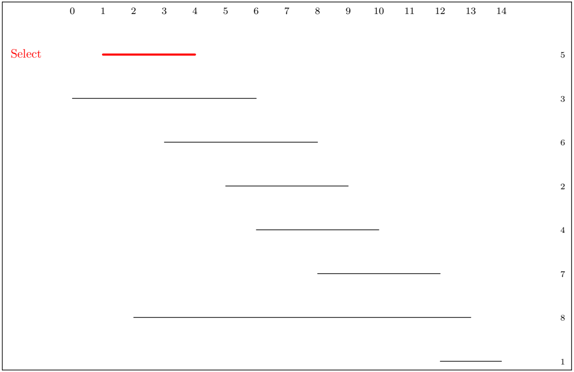

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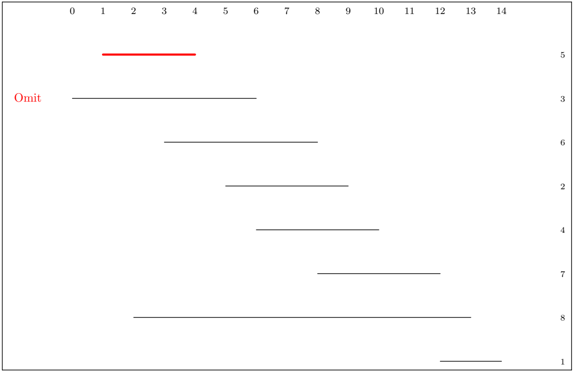

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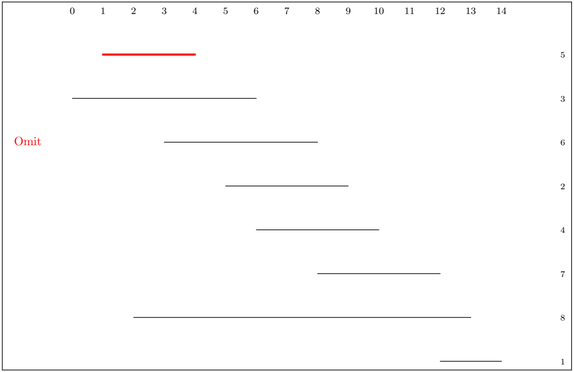

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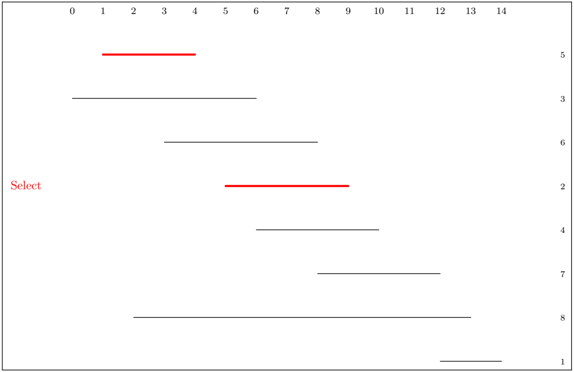

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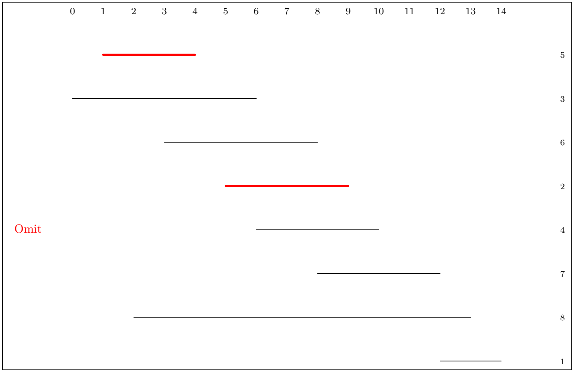

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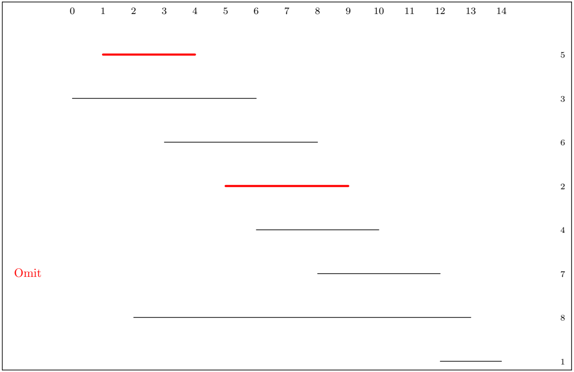

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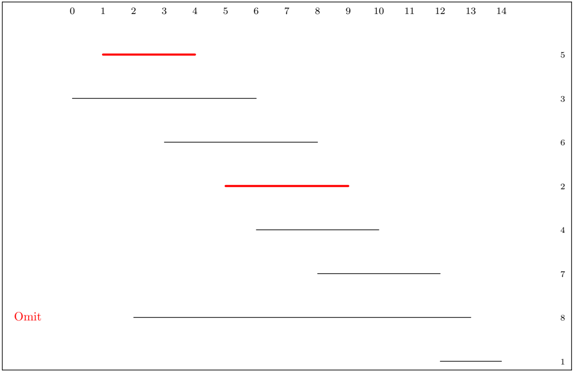

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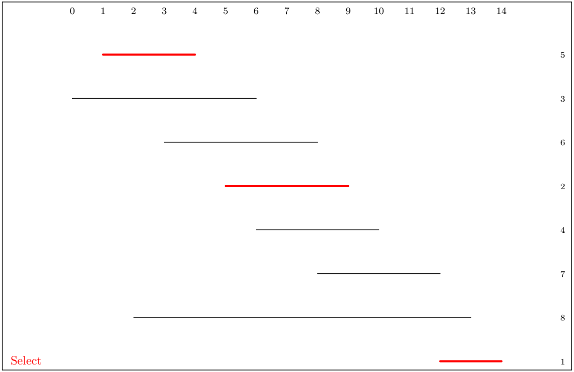

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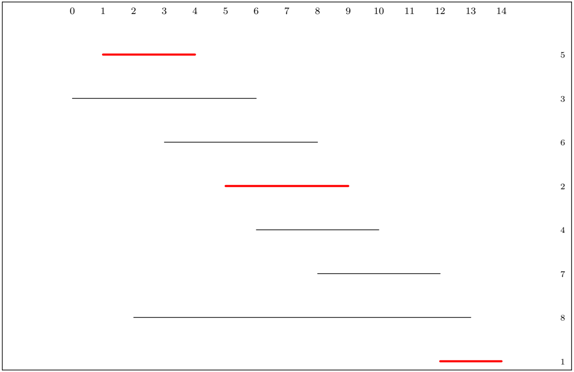

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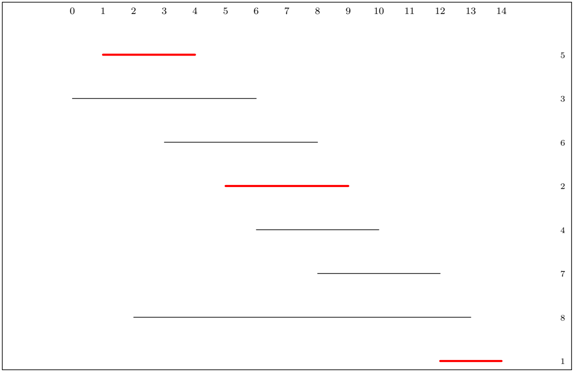


Choose 3 jobs: 5, 2, and 1

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  - If the answer is no (which it often is), seeing why the greedy approach fails may give you some insight into the structure of the problem.