Selection and Related problems

We will examine several related problems:

- Finding the maximum: How often do we update the "candidate maximum"?
- Finding the second largest element ("tournament algorithm")
- Finding the maximum and minimum simultaneously
- Finding the kth smallest (selection problem).
  - Special case of selection: finding the median
  - Median: kth smallest element, where $k = \lceil \frac{n}{2} \rceil$
  - Note: with this definition:
    - Median of 5 items is the 3rd smallest
    - Median of 6 items is the 3rd smallest

These problems can be solved in $O(n \log n)$ time by sorting, but we can do better.

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for i = 0 to n-1:
   if A[i] > v:
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  - What about the average case? ...

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$$X = \sum_{i=0}^{n-1} X_i$$

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If there are 3,000,000,000 elements in the list, the expected update count is about 22.4

Finding the second-largest element

Obvious algorithm:

1. Compute the maximum ($n - 1$ comparisons)
2. Delete the maximum
3. Compute the maximum of the remaining elements ($n - 2$ comparisons)

This requires $2n - 3$ comparisons. We can do better.

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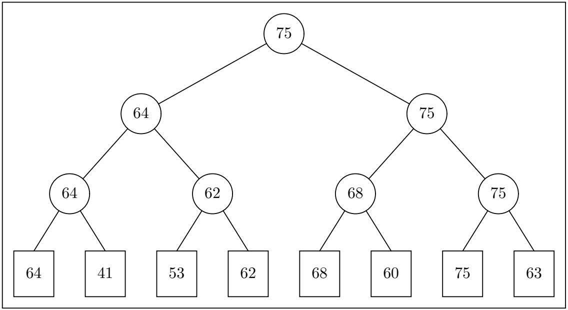

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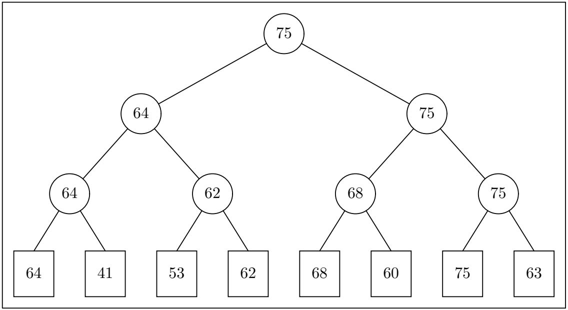

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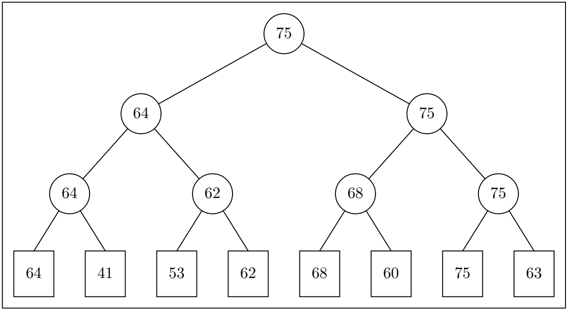

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Total number of comparisons required is $7 + 2 = 9$

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This is optimal. Proof can be found in [Baase, chapter 3].

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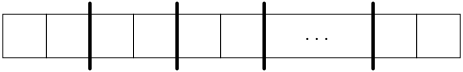

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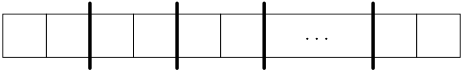

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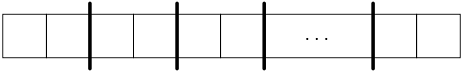

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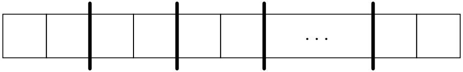

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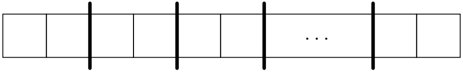

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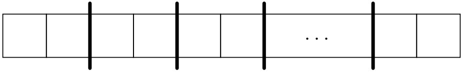

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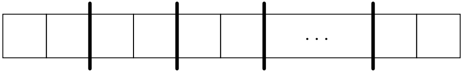

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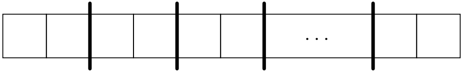

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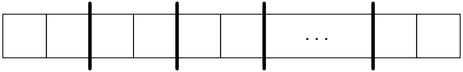

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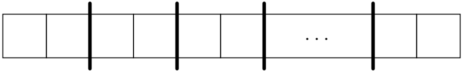

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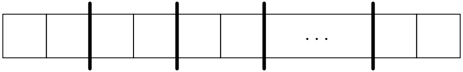

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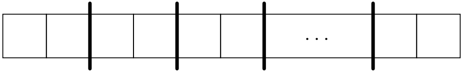

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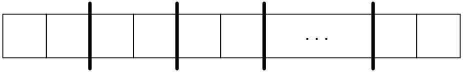

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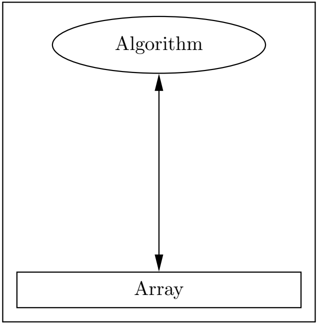
The proof is based on an adversary argument

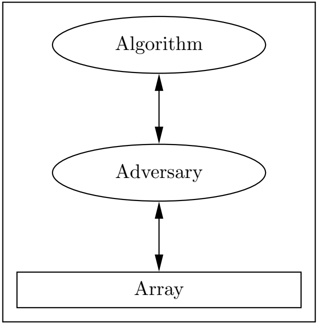
Adversary argument

Given any algorithm for the problem, we need to show that there is an input array of size n that forces the algorithm to either

- Do at least $\frac{3n}{2} - 2$ comparisons; or
- Give an incorrect answer.

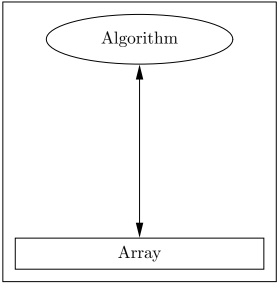
We construct such an array using an adversary.

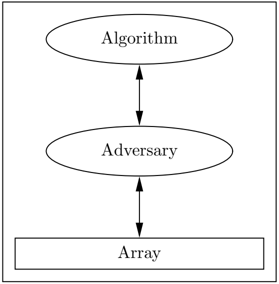




Adversary argument

- The adversary is allowed to and make up answers and set/modify array values
- The array must be consistent with all answers to comparison queries that the adversary has given
- If the algorithm were to run again on the final array, it would behave exactly the same.
- We will describe an adversary that forces any deterministic algorithm that computes maximum and minimum to perform $\frac{3n}{2} - 2$ comparisons.





Adversary strategy: the basics

- Terminology: Suppose we compare two array entries $A[i]$ and $A[j]$. If $A[i] > A[j]$, we say:
  - $A[i]$ wins the comparison
  - $A[j]$ loses the comparison
- A unit of information is the knowledge that a particular entry has won (or lost) at least one comparison.
- To correctly conclude that the maximum value is $A[r]$ and the minimum value is $A[s]$ is the minimum value, the algorithm must have determined that:
  - All elements other than $A[r]$ have lost at least one comparison. ($n - 1$ units of information)
  - All elements other than $A[s]$ have won at least one comparison. ($n - 1$ units of information)

  So the algorithm must collect $2n - 2$ units of information
- We will see that the adversary can force the algorithm to perform $\frac{3n}{2} - 2$ comparisons in order to collect these $2n - 2$ units of information.

Adversary strategy

- The adversary assigns each array entry value and a status.
- The initial value for each array entry is arbitrary. The adversary modifies the value for each array entry as the algorithm proceeds.
- The status is used to keep track of whether the entry has won or lost any comparisons.
- There are 4 possible status values:
  - **W**: The array entry has won at least one comparison and has not lost any
  - **L**: The array entry has lost at least one comparison and has not won any
  - **WL**: The array entry has won at least once and lost at least once
  - **N**: The array entry has not yet participated in any comparisons
- When the algorithm compares two array entries, the adversary:
  1. Examines statuses of the two entries
  2. Based on the statuses, decides who it will say won the comparison
  3. Modifies array entries if necessary to ensure that the array values are consistent with the returned result for this comparison and all previous comparisons.

Adversary strategy (Part 1 of 2)

Status $A[i]$	Status $A[j]$	Comparison Result	New Status $A[i]$	New Status $A[j]$	Units of Info.	Value Updates
N	N	$A[i] > A[j]$	W	L	2	Set $A[i]$, $A[j]$
N	W	$A[i] < A[j]$	L	W	1	Set $A[i]$
N	L	$A[i] > A[j]$	W	L	1	Set $A[i]$
N	WL	$A[i] > A[j]$	W	WL	1	Set $A[i]$
W	N	$A[i] > A[j]$	W	L	1	Set $A[j]$
W	W	$A[i] > A[j]$	W	WL	1	Increase $A[i]^*$
W	L	$A[i] > A[j]$	W	L	0	Increase $A[i]^*$
W	WL	$A[i] > A[j]$	W	WL	0	Increase $A[i]^*$
L	N	$A[i] < A[j]$	L	W	1	Set $A[j]$
L	W	$A[i] < A[j]$	L	W	0	Decrease $A[i]^*$
L	L	$A[i] < A[j]$	WL	L	1	Decrease $A[i]^*$
L	WL	$A[i] < A[j]$	L	WL	0	Decrease $A[i]^*$

* if necessary

Adversary strategy (Part 2 of 2)

Status $A[i]$	Status $A[j]$	Comparison Result	New Status $A[i]$	New Status $A[j]$	Units of Info.	value updates
WL	N	$A[i] < A[j]$	WL	W	1	Set $A[j]$
WL	W	$A[i] < A[j]$	WL	W	0	Increase $A[j]^*$
WL	L	$A[i] > A[j]$	WL	L	0	Decrease $A[j]^*$
WL	WL	consistent with values	WL	WL	0	—

* if necessary

Example of adversary strategy

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Initially

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	0	1	2	3	4	5	6
Status	N	N	N	N	N	N	N
Value							

Example of adversary strategy

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Final steps in proof

Let c_1 = number of comparisons that yield 1 new unit of information

c_2 = number of comparisons that yield 2 new units of informatio n

c = total number of comparisons ($= c_1 + c_2$)

Facts:

1. Total units of information gleaned by algorithm $c_1 + 2c_2$
2. Total units of information must be at least $2n - 2$
3. At most $n/2$ comparisons (the NN comparisons) give the algorithm 2 units of information (so $c_2 \leq n/2$)

So

$$\begin{array}{rcll} c_1 + 2c_2 & \geq & 2n - 2 & \text{By Facts 1 and 2} \\ -c_2 & \geq & -\dfrac{n}{2} & \text{By Fact 3} \\ c = c_1 + c_2 & \geq & \dfrac{3n}{2} - 2 & \text{Add} \end{array}$$

Hence the total number of comparisons required is $\frac{3n}{2} - 2$.

Conclusions

The problem of finding the maximum and minimum of n elements

...

1. Can be solved using $\leq \frac{3n}{2} - 2$ comparisons in the worst case
   - Because there is an algorithm that solves the problem using $\frac{3n}{2} - 2$ in the worst case.
2. Requires $\geq \frac{3n}{2} - 2$ comparisons to solve in the worst case
   - Because as we just showed, for any algorithm that solves the problem an adversary can construct an input that forces the algorithm to perform

So we can conclude

- The algorithm we presented is optimal (by #2 above and the fact that our algorithm performs $\frac{3n}{2} - 2$ comparisons in the worst case)
- The lower bound of $\frac{3n}{2} - 2$ comparisons cannot be improved (by #1 above)

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  - However, if $k = \Theta(n)$ (e.g., if k is close to $n/2$), this takes $O(n \log n)$ time.

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We will describe how to do selection in $O(n)$ time, irrespective of the value of k.

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  2. Deterministic algorithm: runs in $O(n)$ worst-case time.
- Both algorithms have the same high-level strategy.

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L	E	G
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```
if k <= |L|:  return select(L,k)
else if k <= |L|+|E|:  then return m*
else:  return select(G,k-|L|-|E|)
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Quickselect and Determinisic Selection choose m^* differently.

QuickSelect

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```
def quickSelect(S,k):
    if |S| = 1:  return the (unique) element in S
    choose a random element m* from S
    split S into three sequences:
        L = all elements in S less than m*
        E = all elements in S equal to m*
        G = all elements in S greater than m*
    if k <= |L| then return quickSelect(L,k)
    else if k <= |L|+|E| then return m*
    else return quickSelect(G,k-|L|-|E|)
```

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$$ET(n) \leq 2b\left(n + \left(\frac{3}{4}\right)n + \left(\frac{3}{4}\right)^2 n + \cdots\right) = O(n).$$

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    Here, b is a constant that accounts for the overhead of each recursive call.
  - See [GT], Section 9.2 for details
- Because the algorithm is randomized, no single input elicits the worst-case behavior.

Deterministic Selection

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- The precise condition is:

Neither L nor G has more than $\frac{7n}{10} + 3$ elements.

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There are two recursive calls: One in Step 3 and one in Step 5.

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$$51, 24, 32, 78, 66, 47, 61, 76, 82, 70, 85, 73, 15, 99, 86, 13, 88]$$

$n = 35,\ k = 18$

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3. Compute m^* using recursive call:

Example

$$S = [\,29, 90, 93, 14, 11, 38, 22, 81, 91, 17, 62, 83, 97, 64, 66, 26, 55, 89,$$
$$51, 24, 32, 78, 66, 47, 61, 76, 82, 70, 85, 73, 15, 99, 86, 13, 88]$$

$n = 35,\ k = 18$

1. Divide S into groups of 5:

29	38	62	26	32	76	15
90	22	83	55	78	82	99
93	81	97	89	66	70	86
14	91	64	51	47	85	13
11	17	66	24	61	73	88

2. Compute median of each group using brute force:

$\boxed{29}$	$\boxed{38}$	62	26	32	$\boxed{76}$	15
90	22	83	55	78	82	99
93	81	97	89	66	70	$\boxed{86}$
14	91	64	$\boxed{51}$	47	85	13
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3. Compute m^* using recursive call:
`Dselect([29,38,66,61,61,76,86],7,4)`

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3. Compute m^* using recursive call:
`Dselect([29,38,66,61,61,76,86],7,4)` returns 61

Example, continued

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Example, continued

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4. Partition into L, E, and G:

Example, continued

$S = [\ 29, 90, 93, 14, 11, 38, 22, 81, 91, 17, 62, 83, 97, 64, 66, 26, 55, 89,$
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$n = 35,\ k = 18$
From previous slide, $m^* = 61$.

4. Partition into L, E, and G:
   $L = [\ 29, 14, 11, 38, 22, 17, 26, 55, 51, 24, 32, 47, 15, 13]$

Example, continued

$S = [\,29, 90, 93, 14, 11, 38, 22, 81, 91, 17, 62, 83, 97, 64, 66, 26, 55, 89,$
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   $|L| = 14,\ |E| = 1, |G| = 20$

Example, continued

$S = [\,29, 90, 93, 14, 11, 38, 22, 81, 91, 17, 62, 83, 97, 64, 66, 26, 55, 89,$
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   $|L| = 14,\ |E| = 1, |G| = 20$
5. Compute the result using a recursive call:
   `DSelect(G,20,3)`

Example, continued

$S = [\,29, 90, 93, 14, 11, 38, 22, 81, 91, 17, 62, 83, 97, 64, 66, 26, 55, 89,$
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4. Partition into L, E, and G:

   $L = [\,29, 14, 11, 38, 22, 17, 26, 55, 51, 24, 32, 47, 15, 13]$

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   (Because $k - |L| - |E| = 18 - 14 - 1 = 3$)

Example, continued

$S = [\,29, 90, 93, 14, 11, 38, 22, 81, 91, 17, 62, 83, 97, 64, 66, 26, 55, 89,$
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   $L = [\,29, 14, 11, 38, 22, 17, 26, 55, 51, 24, 32, 47, 15, 13]$

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5. Compute the result using a recursive call:

   `DSelect(G,20,3)`

   (Because $k - |L| - |E| = 18 - 14 - 1 = 3$)

Again, note that there are two recursive calls: one in Step 3, and one in step 5

Pseudocode for Deterministic Selection

```
def DSelect(S,k,n):// Parameter n is not really necessary
    if n ≤ 5:  return(bruteForceSelect(S,k,n))
    // Step 1:  Divide into groups of 5
    for i = 0 to n-1:  group[floor(i/5)][mod(i,5)] = S[i])
    // Step 2:  Compute median of each group oof 5 non-recursively
    for i = 1 to ceil(n/5):
        groupMedian[i] = bruteForceSelect({group[i][j]:j=0..4},3,5)
    // Step 3:  Compute m* = median of group medians
    m* = DSelect(groupMedian,ceil(ceil(n/5)/2),ceil(n/5))
    // Step 4:  Partition elements of S in L, E, G
    Allocate three empty lists L, E, and G
    for i = 0 to n-1:
        if S[i] < m*:  L = L + {S[i]}
        else if S[i] = m*:  E = E + {S[i]}
        else:  G = G + {S[i]} // S[i] > m*
    // Step 5:  Return m* or recursively call DSelect in L or G
    if k ≤ |L|:  return DSelect(L,k,|L|)
    else if k ≤ |L| + |E|:  return m*
    else:  return DSelect(G,k-|L|-|E|,|G|)
```

Analysis of Deterministic Selection Algorithm

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We first show that neither L nor G has more than $\frac{7n}{10} + 3$ items.

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To get an upper bound on the size of G, we show a lower bound on the size of the complement of G.

Analysis of Deterministic Selection Algorithm

We first show that neither L nor G has more than $\frac{7n}{10} + 3$ items.We will show this for G.

To get an upper bound on the size of G, we show a lower bound on the size of the complement of G.

When we compute m^*:

Analysis of Deterministic Selection Algorithm

We first show that neither L nor G has more than $\frac{7n}{10} + 3$ items.We will show this for G.

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When we compute m^*:

- There are $\lceil \frac{n}{5} \rceil$ groups

Analysis of Deterministic Selection Algorithm

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To get an upper bound on the size of G, we show a lower bound on the size of the complement of G.

When we compute m^*:

- There are $\left\lceil \frac{n}{5} \right\rceil$ groups
- There are at least $\left\lceil \frac{1}{2} \left\lceil \frac{n}{5} \right\rceil \right\rceil$ groups with median $\leq m^*$.

Analysis of Deterministic Selection Algorithm

We first show that neither L nor G has more than $\frac{7n}{10} + 3$ items.We will show this for G.

To get an upper bound on the size of G, we show a lower bound on the size of the complement of G.

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- There are at least $\left\lceil \frac{1}{2} \left\lceil \frac{n}{5} \right\rceil \right\rceil$ groups with median $\leq m^*$.
- There are at least $\left\lceil \frac{1}{2} \left\lceil \frac{n}{5} \right\rceil \right\rceil - 1$ full groups with median $\leq m^*$.

Analysis of Deterministic Selection Algorithm

We first show that neither L nor G has more than $\frac{7n}{10} + 3$ items.We will show this for G.

To get an upper bound on the size of G, we show a lower bound on the size of the complement of G.

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Analysis of Deterministic Selection Algorithm

We first show that neither L nor G has more than $\frac{7n}{10} + 3$ items.We will show this for G.

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- It follows that the number of items not in G is at least:

Analysis of Deterministic Selection Algorithm

We first show that neither L nor G has more than $\frac{7n}{10}+3$ items.We will show this for G.

To get an upper bound on the size of G, we show a lower bound on the size of the complement of G.

When we compute m^*:

- There are $\left\lceil \frac{n}{5} \right\rceil$ groups
- There are at least $\left\lceil \frac{1}{2}\left\lceil \frac{n}{5} \right\rceil \right\rceil$ groups with median $\leq m^*$.
- There are at least $\left\lceil \frac{1}{2}\left\lceil \frac{n}{5} \right\rceil \right\rceil - 1$ full groups with median $\leq m^*$. Each of these groups has at least 3 items $\leq m^*$.
- It follows that the number of items not in G is at least:

$$3\left(\left\lceil \frac{1}{2}\left\lceil \frac{n}{5} \right\rceil \right\rceil - 1\right)$$

Analysis of Deterministic Selection Algorithm

We first show that neither L nor G has more than $\frac{7n}{10} + 3$ items.We will show this for G.

To get an upper bound on the size of G, we show a lower bound on the size of the complement of G.

When we compute m^*:

- There are $\lceil \frac{n}{5} \rceil$ groups
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$$3\left(\left\lceil \frac{1}{2} \left\lceil \frac{n}{5} \right\rceil \right\rceil - 1\right) \geq 3\left(\frac{n}{10} - 1\right)$$

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- It follows that the number of items not in G is at least:

$$3\left(\left\lceil \frac{1}{2} \left\lceil \frac{n}{5} \right\rceil \right\rceil - 1\right) \geq 3\left(\frac{n}{10} - 1\right) = \frac{3n}{10} - 3.$$

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We first show that neither L nor G has more than $\frac{7n}{10} + 3$ items.We will show this for G.

To get an upper bound on the size of G, we show a lower bound on the size of the complement of G.

When we compute m^*:

- There are $\left\lceil \frac{n}{5} \right\rceil$ groups
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- Hence, the number of elements that are in G satisfies the inequality:

Analysis of Deterministic Selection Algorithm

We first show that neither L nor G has more than $\frac{7n}{10} + 3$ items.We will show this for G.

To get an upper bound on the size of G, we show a lower bound on the size of the complement of G.

When we compute m^*:

- There are $\left\lceil \frac{n}{5} \right\rceil$ groups
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- Hence, the number of elements that are in G satisfies the inequality:

$$|G| \leq n - \left(\frac{3n}{10} - 3\right)$$

Analysis of Deterministic Selection Algorithm

We first show that neither L nor G has more than $\frac{7n}{10} + 3$ items.We will show this for G.

To get an upper bound on the size of G, we show a lower bound on the size of the complement of G.

When we compute m^*:

- There are $\left\lceil \frac{n}{5} \right\rceil$ groups
- There are at least $\left\lceil \frac{1}{2} \left\lceil \frac{n}{5} \right\rceil \right\rceil$ groups with median $\leq m^*$.
- There are at least $\left\lceil \frac{1}{2} \left\lceil \frac{n}{5} \right\rceil \right\rceil - 1$ full groups with median $\leq m^*$. Each of these groups has at least 3 items $\leq m^*$.
- It follows that the number of items not in G is at least:
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- Hence, the number of elements that are in G satisfies the inequality:
$$|G| \leq n - \left(\frac{3n}{10} - 3\right) = \frac{7n}{10} + 3.$$

Analysis of Deterministic Selection Algorithm

We first show that neither L nor G has more than $\frac{7n}{10} + 3$ items.We will show this for G.

To get an upper bound on the size of G, we show a lower bound on the size of the complement of G.

When we compute m^*:

- There are $\left\lceil \frac{n}{5} \right\rceil$ groups
- There are at least $\left\lceil \frac{1}{2} \left\lceil \frac{n}{5} \right\rceil \right\rceil$ groups with median $\leq m^*$.
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- It follows that the number of items not in G is at least:
$$3\left(\left\lceil \frac{1}{2} \left\lceil \frac{n}{5} \right\rceil \right\rceil - 1\right) \geq 3\left(\frac{n}{10} - 1\right) = \frac{3n}{10} - 3.$$
- Hence, the number of elements that are in G satisfies the inequality:
$$|G| \leq n - \left(\frac{3n}{10} - 3\right) = \frac{7n}{10} + 3.$$

Analysis of Deterministic Selection Algorithm

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Let $T(n)$ =worst-case time required by `DSelect` on an input of size n.

Analysis of Deterministic Selection Algorithm

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Cost of each step:

Analysis of Deterministic Selection Algorithm

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1. Divide into groups of 5:

Analysis of Deterministic Selection Algorithm

Let $T(n)$ =worst-case time required by `DSelect` on an input of size n.

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Analysis of Deterministic Selection Algorithm

Let $T(n)$ =worst-case time required by `DSelect` on an input of size n.

Cost of each step:

1. Divide into groups of 5: $O(n)$
2. Compute group medians:

Analysis of Deterministic Selection Algorithm

Let $T(n)$ =worst-case time required by `DSelect` on an input of size n.

Cost of each step:

1. Divide into groups of 5: $O(n)$
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Analysis of Deterministic Selection Algorithm

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Cost of each step:

1. Divide into groups of 5: $O(n)$
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3. Recursive call to compute m^*:

Analysis of Deterministic Selection Algorithm

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3. Recursive call to compute m^*: $T\left(\lceil \frac{n}{5} \rceil\right)$
4. Partition into L, E, and G: $O(n)$
5. Final recursive call:

Analysis of Deterministic Selection Algorithm

Let $T(n)$ =worst-case time required by `DSelect` on an input of size n.

Cost of each step:

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3. Recursive call to compute m^*: $T\left(\lceil \frac{n}{5} \rceil\right)$
4. Partition into L, E, and G: $O(n)$
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3. Recursive call to compute m^*: $T\left(\lceil \frac{n}{5} \rceil\right)$
4. Partition into L, E, and G: $O(n)$
5. Final recursive call: $\leq T\left(\frac{7n}{10} + 3\right)$

So we get the recurrence equation:

$$T(n) \leq T\left(\left\lceil \frac{n}{5} \right\rceil\right) + T\left(\frac{7n}{10} + 3\right)$$

Analysis of Deterministic Selection Algorithm

$$T(n) \le T\left(\left\lceil \frac{n}{5} \right\rceil\right) + T\left(\frac{7n}{10} + 3\right) + O(n)$$

Analysis of Deterministic Selection Algorithm

$$T(n) \leq T\left(\left\lceil \frac{n}{5} \right\rceil\right) + T\left(\frac{7n}{10} + 3\right) + O(n)$$

- This is not a standard divide-and-conquer recurrence, and we cannot use the Master Method or the Alternative Method to solve it.

Analysis of Deterministic Selection Algorithm

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- The solution is $T(n) = O(n)$.

Analysis of Deterministic Selection Algorithm

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Analysis of Deterministic Selection Algorithm

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  - Might be useful in situations where there is a time bound that cannot be exceeded