CompSci 161

Design and Analysis of Algorithms
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CompSci 161

- Course Web page:
  http://www.ics.uci.edu/~dillenco/compsci161/
- Course Notes:
  http://www.ics.uci.edu/~dillenco/compsci161/notes/
- User name is UCI ID (ALL CAPS)
- Password is student number

Books

Text book:

- Goodrich and Tamassia, *Algorithm design and Applications*, Wiley.

A few other books worth knowing about:

- Baase and van Geldin, *Computer Algorithms*, Addison-Wesley.
- Kleinberg and Tardos, *Algorithm Design*, Addison Wesley.
- Cormen, Leiserson, Rivest, and Stein, *Introduction to Algorithms*, MIT Press.
- Dasgupta, Papadimitriou and Vazirani, *Algorithms*,McGraw-Hill.

Analysis of Algorithms

Why bother analyzing algorithms? Why not just implement them and run them?

- Predict behavior before implementation.
- Helps choose among different solutions.
- Experimentation is limited by the test cases on which experimentation is performed.
- Experimentation may be biased by choice of hardware.
- Provides insight into possible improvements.

Analysis of Algorithms

What do we analyze?

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- Correctness

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  - It is very easy to develop fast, incorrect algorithms.
  - However, this is not very useful.

Analysis of Algorithms

What do we analyze?

- Correctness
  - It is very easy to develop fast, incorrect algorithms.
  - However, this is not very useful.
- Performance
  - Qualitative: work, storage, disk accesses, communication, . . . ?
    - Most commonly: running time.
  - Quantitative: what do we count?
    - Need a methodology, quantifiable model.
- Scalability: What happens as our inputs get large?

Analysis of Algorithms

We need. . .

- Methodology for algorithm analysis
  - Way of describing algorithms
  - A computational model
  - A metric for measuring running times
  - An approach for characterizing running times
- Mathematical tools for performing the analysis and expressing the results
  - Terminology for expressing scalability (Asymptotic notation)
  - Some basic functions
  - Techniques for reasoning, justification
  - Probability

Describing Algorithms: Pseudocode

- Intended for humans, not machines
- Not computer programs, but more structured than normal language
- Focus on high-level ideas, not low-level implementation details
- Captures important steps
- [GT] enumerate some reasonable constructs
- Requires finding the right balance

Example 1: Finding Maximum, version 1

Algorithm Maximum1(A,n)
Input: An array $A[n]$, where $n \geq 1$
Output: A maximum element in A

$v \leftarrow A[0]$;
for $i \leftarrow 1$ to $n-1$ do
    if $A[i] > v$ then
        $v \leftarrow A[i]$;
return v;

- Correctness: At start of iteration i, v is "leftmost maximum" of first i array entries.

Example 2: Finding Maximum, version 2

Algorithm Maximum2(A,n)
Input: An array $A[n]$, where $n \geq 1$
Output: A maximum element in A

$v \leftarrow -\infty$;
for $i \leftarrow 0$ to $n-1$ do
    if $A[i] \geq v$ then
        $v \leftarrow A[i]$;
return v;

- Correctness: At start of iteration i, v is "rightmost maximum" of first i array entries.
- What is the maximum of an empty collection?

Example 3: Sequential Search

Determine whether an array contains a particular item and, if so, the index at which the item is stored.

Algorithm Search(A,n)
Input: An array $A[n]$, where $n \geq 1$; an item x
Output: Index where x occurs in A, or -1

for $i \leftarrow 0$ to $n-1$ do
    if $A[i] = x$ then return(i);
return(-1);

- Correctness: At start of iteration i, either we have returned a correct value or x is not one of the first i entries.

Computational Model: RAM

Random Access Machine (RAM)

- Define primitive operations
- Operations include:
  - Assigning a value to a variable
  - Calling a function (method)
  - Performing an arithmetic operation
  - Comparing two numbers
  - Indexing into an array
  - Following an object reference (Dereferencing a pointer)
  - Returning from a function
- To measure running time: count operations

Example: Finding Maximum

$v \leftarrow A[0]$;	2
// for $i \leftarrow 1$ to $n-1$ do	
$i \leftarrow 1$;	1
Loop: if $i \leq n-1$ then	$2n$
if $A[i] > v$ then	$2n-2$
$v \leftarrow A[i]$;	between 0 and $2n-2$
$i = i+1$	$2n-2$
go to Loop	
return v;	1

- Best case: cost $= 6n$
- Worst case: cost $= 8n-2$

Best case vs. Worst case vs. Average Case

- Algorithms run faster on some inputs than others
- What about average case (taken over all inputs)?
  - Often requires heavy mathematics, probability
  - Requires knowing the probability distribution on the set of inputs. This can be hard to determine.
- We will focus on worst-case analysis
  - Gives us a guarantee.
  - Murphy's law: "If anything can go wrong, it will."
  - If we design for worst case, sometimes we get a better algorithm
- Another type of average case analysis: algorithm makes random decisions.

Recursive Algorithms
Alternative to Iterative Algorithms

Example: Finding maximum

recursiveMax(A,n) // returns maximum of A[0]...A[n-1]	
if $n = 1$ then;	1
return A[0];	2
return max(recursiveMax(A,n-1),A[n-1])	$T(n-1)+6$

- Let $T(n)$ be cost for input of size n. Equation:

$$T(n) = \begin{cases} 3 & \text{if } n = 1 \\ T(n-1)+7 & \text{otherwise} \end{cases}$$

- Solution: $T(n) = 7n - 4$

Asymptotic notation: O, o, Ω, Θ

- Measures growth rates of functions as $n \to \infty$.
- Treats two functions as being roughly the same if they are roughly constant multiples of each other.
- Function may represent description of algorithm behavior. (E.g., $f(n)$ could be the worst-case running time of a given algorithm on an input of size n).
- Allows us to focus on the most important considerations when analyzing an algorithm, and to ignore fine-grain details.
- Allows us to compare two algorithms easily.
- (Allows us to be "imprecise in a precise way")
- Read the book! ([GT] 1.2; also [CLRS] 3.1)

O ("big oh")

Informally:

- $g \in O(f)$ if g is bounded above by a constant multiple of f (for sufficiently large values of n).
- $g \in O(f)$ if "g grows no faster than (a constant multiple of) f."
- $g \in O(f)$ if the ratio g/f is bounded above by a constant (for sufficiently values of n).

O ("big oh")

Formally:

- $g \in O(f)$ if and only if:

$$\exists_{C>0}\,\exists_{n_0>0}\,\forall_{n>n_0}\; g(n) \leq C \cdot f(n).$$

- Equivalently: $g \in O(f)$ if and only if:

$$\exists_{C>0}\,\exists_{n_0>0}\,\forall_{n>n_0}\; \frac{g(n)}{f(n)} \leq C.$$

- Sometimes we write: $g = O(f)$ rather than $g \in O(f)$

Examples of O-notation:

Example 1: $f(n) = n$, $g(n) = 1000n$: $g \in O(f)$.

Proof: Let $C = 1000$. Then $g(n) \leq C \cdot f(n)$ for all n.

Examples of O-notation:

Example 2: $f(n) = n^2$, $g(n) = n^{3/2}$: $g \in O(f)$.

Proof: $\lim_{n\to\infty} \frac{g(n)}{f(n)} = 0$. Hence for any $C > 0$ the ratio is less than C as long as n is sufficiently large. (Of course, how large n must be to be"sufficiently large" depends on C).

Alternate Proof: If $n \geq 1$, $n^{1/2} \geq 1$, so $n^{3/2} \leq n^2$. Hence we can choose $C = 1$ and $n_0 = 1$.

Examples of O-notation:

Example 3: $f(n) = n^3$, $g(n) = n^4$: $g \notin O(f)$.

Proof: $\lim_{n\to\infty} \frac{g(n)}{f(n)} = \infty$. Hence there is no $C > 0$ such that $g(n) \leq C \cdot f(n)$ for sufficiently large n.

Examples of O-notation:

Example 4: $f(n) = n^2$, $g(n) = 5n^2 + 23n + 2$: $g \in O(f)$.

Proof: If $n \geq 1$, then $n \leq n^2$ and $1 \leq n^2$. Hence:

$$\begin{aligned} g(n) &= 5n^2 + 23n + 2 \\ &\leq 5n^2 + 23n^2 + 2n^2 \\ &\leq 30n^2 \\ &= 30f(n) \end{aligned}$$

So we can take $C = 30$, $n_0 = 1$.

More asymptotic notation: o ("little oh"), Ω ("big Omega")

- o ('little oh"):

$$g \in o(f) \quad \text{if and only if } \lim_{n\to\infty} \frac{g(n)}{f(n)} = 0.$$

- Ω ("big Omega") (or just "Omega")

$$g \in \Omega(f) \quad \text{if and only if} \quad \exists_{C>0}\, \exists_{n_0>0}\, \forall_{n>n_0}\, g(n) \geq C \cdot f(n).$$

Equivalently:

$$g \in \Omega(f) \quad \text{if and only if} \quad \exists_{C>0}\, \exists_{n_0>0}\, \forall_{n>n_0}\, \frac{g(n)}{f(n)} \geq C.$$

One more definition: Θ ("Theta")

- Θ ("Theta"):

$$g \in \Theta(f) \quad \text{if and only if} \quad g \in O(f) \text{ and } g \in \Omega(f).$$

- Equivalently: $g \in \Theta(f)$ if and only if:

$$\exists_{C_1>0}\, \exists_{C_2>0}\, \exists_{n_0>0}\, \forall_{n>n_0}\; C_1 \leq \frac{g(n)}{f(n)} \leq C_2.$$

Examples of Asymptotic notation

Example 1: $f(n) = n$, $g(n) = 1000n$:
$g \in \Omega(f)$, $g \in \Theta(f)$

To see that $g \in \Omega(f)$, take $C = 1000$.

To see that $g \in \Theta(f)$, take $C_1 = C_2 = 1000$.

Examples of Asymptotic notation

Example 2: $f(n) = n^2$, $g(n) = n^{3/2}$: $g \in o(f)$

Because $\lim_{n\to\infty} \frac{g(n)}{f(n)} = 0$.

Examples of Asymptotic notation

Example 3: $f(n) = n^3$, $g(n) = n^4$:
$g \in \Omega(f)$

Because $\lim_{n\to\infty} \frac{g(n)}{f(n)} = \infty$, so we can choose any C we want.

Examples of Asymptotic notation

Example 4: $f(n) = n^2$, $g(n) = 5n^2 - 23n + 2$: $g \in \Omega(f)$.

Proof: If $n \geq 23$, then $23n \leq n^2$. Hence if $n \geq 23$,

$$\begin{aligned} g(n) &= 5n^2 - 23n + 2 \\ &\geq 5n^2 - n^2 \\ &\geq 4n^2 \\ &= 4f(n) \end{aligned}$$

So we can take $C = 4$, $n_0 = 23$.

Another Example

Example 5: $\ln n = o(n)$
Proof: Examine the ratio $\frac{\ln n}{n}$ as $n \longrightarrow \infty$. If we try to evaluate the limit directly, we obtain the "indeterminate form" $\frac{\infty}{\infty}$. We need to apply L'Hôpital's rule (from calculus).

Example 5, continued: $\ln n = o(n)$

L'Hôpital's rule: If the ratio of limits

$$\frac{\lim_{n\to\infty} g(n)}{\lim_{n\to\infty} f(n)}$$

is an indeterminate form (i.e., ∞/∞ or $0/0$), then

$$\lim_{n\to\infty} \frac{g(n)}{f(n)} = \lim_{n\to\infty} \frac{g'(n)}{f'(n)}$$

where f' and g' are, respectively, the derivatives of f and g.

Example 5, continued: $\ln n = o(n)$

So let $f(n) = n$, $g(n) = \ln n$. Then $f'(n) = 1$, $g'(n) = 1/n$. By L'Hôpital's rule:

$$\begin{aligned}
\lim_{n\to\infty} \frac{g(n)}{f(n)} &= \lim_{n\to\infty} \frac{g'(n)}{f'(n)} \\
&= \lim_{n\to\infty} \frac{1/n}{1} \\
&= \lim_{n\to\infty} \frac{1}{n} \\
&= 0.
\end{aligned}$$

Hence $g(n) = o(f(n))$.

Math background—review!?

- Sums, Summations
- Logarithms, Exponents Floors, Ceilings, Harmonic Numbers
- Proof Techniques
- Basic Probability

Sums, Summations

- Summation notation:

$$\sum_{i=a}^{b} f(i) = f(a) + f(a+1) + \cdots + f(b).$$

- Special cases: What if $a = b$? What if $a > b$?
- If $S = \{s_1, \ldots, s_n\}$ is a finite set

$$\sum_{x \in S} f(x) = f(s_1) + f(s_2) + \cdots + f(s_n).$$

Geometric sum

- Geometric sum:

$$\sum_{i=0}^{n} a^i = 1 + a^1 + a^2 + \cdots + a^n = \frac{1 - a^{n+1}}{1 - a},$$

provided that $a \neq 0, a \neq 1$. (Recall that $a^0 = 1$ if $a \neq 0$.)

- Special case of geometric sum:

$$\sum_{i=0}^{n} 2^i = 1 + 2 + 4 + 8 + \cdots + 2^n = 2^{n+1} - 1.$$

Infinite Geometric sum

- If $|a| < 1$, can take limit as $n \to \infty$:

$$\sum_{i=0}^{\infty} a^i = 1 + a^1 + a^2 + \cdots = \frac{1}{1-a},$$

- Special case of infinite geometric sum:

$$\sum_{i=0}^{n} \frac{1}{2^i} = 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots = 2.$$

Other Summations

- Sum of first n integers

$$\sum_{i=1}^{n} i = 1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2}.$$

- Sum of first n squares

$$\sum_{i=1}^{n} i^2 = 1 + 4 + 9 + 16 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}$$

- In general, for any fixed positive integer k:

$$\sum_{i=1}^{n} i^k = 1 + 2^k + 3^k + \cdots + n^k = \Theta(n^{k+1})$$

Logarithms

Definition: $\log_b x = y$ if and only if $b^y = x$.

Some useful properties:

1. $\log_b 1 = 0$.
2. $\log_b b^a = a$.
3. $\log_b(xy) = \log_b x + \log_b y$.
4. $\log_b(x^a) = a \log_b x$.
5. $x^{\log_b y} = y^{\log_b x}$.
6. $\log_x b = \frac{1}{\log_b x}$.
7. $\log_a x = \frac{\log_b x}{\log_b a}$.
8. $\log_a x = (\log_b x)(\log_a b)$.

Exercise: Prove the above properties.

Logarithms

Example (#2): Prove $\log_b b^a = a$.

Solution: Let

$$\begin{aligned} y &= \log_b b^a \\ b^y &= b^a \quad \text{[by definition of log]} \\ y &= a \end{aligned}$$

Logarithms

Special Notations:

- $\ln x = \log_e x$ ($e = 2.71828\ldots$)
- $\lg x = \log_2 x$

Some conversions (from Rules #7 and #8 on previous slides):

- $\ln x = (\log_2 x)(\log_e 2) = 0.693 \lg x$.
- $\lg x = \frac{\log_e x}{\log_e 2} = \frac{\ln x}{0.693} = 1.44 \ln x$.

Floors and ceilings

- $\lfloor x \rfloor$ = largest integer $\leq x$. (Read as Floor of x)
- $\lceil x \rceil$= smallest integer $\geq x$ (Read as Ceiling of x)

Factorials

- $n! = 1 \cdot 2 \cdots n$
- $n!$ represents the number of distinct permutations of n objects.

1	2	3
1	3	2
2	1	3
2	3	1
3	1	2
3	2	1

Combinations

$\binom{n}{k}$ = The number of different ways of choosing k objects from a collection of n objects. (Pronounced "n choose k".)

Example: $\binom{5}{2} = 10$

$$\begin{array}{ccccc} \{1,2\} & \{1,3\} & \{1,4\} & \{1,5\} & \{2,3\} \\ \{2,4\} & \{2,5\} & \{3,4\} & \{3,5\} & \{4,5\} \end{array}$$

Formula: $\binom{n}{k} = \frac{n!}{k!(n-k)!}$

Special cases: $\binom{n}{0} = 1$, $\binom{n}{1} = n$, $\binom{n}{2} = \frac{n(n-1)}{2}$

Harmonic Numbers

The *n*th Harmonic number is the sum:

$$H_n = \sum_{i=1}^{n} \frac{1}{i}$$

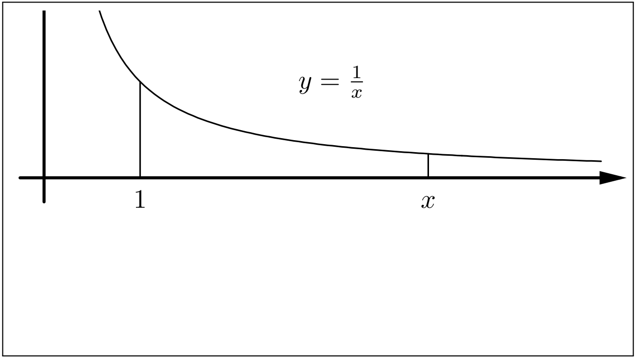
These numbers go to infinity:

$$\lim_{n\to\infty} H_n = \sum_{i=1}^{\infty} \frac{1}{i} = \infty$$

Harmonic Numbers

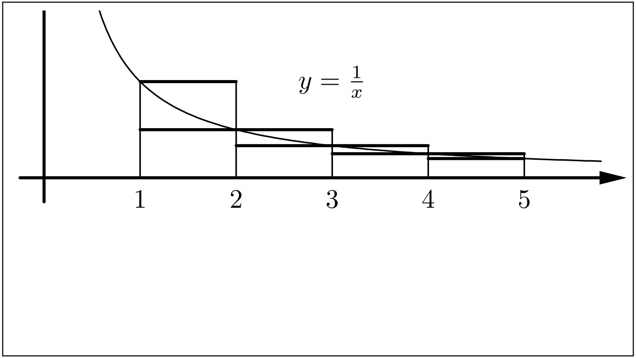
The harmonic numbers are closely related to logs. Recall:

$$\ln x = \int_1^x \frac{1}{x}\, dx$$



We will show that $H_n = \Theta(\log n)$.

Harmonic Numbers



$$
\begin{array}{lcccl}
\frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n} & < & \ln n & < & 1 + \frac{1}{2} + \ldots + \frac{1}{n-1} \\
H_n - 1 & < & \ln n & < & H_n - \frac{1}{n}
\end{array}
$$

Hence $\ln n + \frac{1}{n} < H_n < \ln n + 1$, so $H_n = \Theta(\log n)$.

Proof/Justification Techniques

- Proof by Example: Can be used to prove
  - A statement of the form "There exists..." is true.
  - A statement of the form "For all..." is false.
  - A statement of the form "If P then Q" is false.
- Illustration: Consider the statement:

  All numbers of the form $2^k - 1$ are prime.

  T

  statement is False: $2^4 - 1 = 15 = 3 \cdot 5$
- Note: Above statement can be rewritten:

  If n is an integer of the form $2^k - 1$, then n is prime.

Proof/Justification Techniques

- Suppose we want to prove a statement of the form "If P then Q" is true. There are three approaches:
  1. Direct proof: Assume P is true. Show that Q must be true.
  2. Indirect proof: Assume Q is false. Show that P must be false. This is also known as a proof by contraposition.
  3. Proof by contradiction: Assume P is true and Q is false. Show that there is a contradiction.

  See [GT] Section 1.3.3 for examples.

Proof/Justification Techniques: Induction

- A technique for proving theorems about the positive (or nonnegative) integers.
- Let $P(n)$ be a statement with an integer parameter, n. Mathematical induction is a technique for proving that $P(n)$ is true for all integers $\geq$ some base value b.
- Usually, the base value is 0 or 1.
- To show $P(n)$ holds for all $n \geq b$, we must show two things:
  1. Base Case: $P(b)$ is true (where b is the base value).
  2. Inductive step: If $P(k)$ is true, then $P(k+1)$ is true.

Induction Example

Example: Show that for all $n \geq 1$

$$\sum_{i=1}^{n} i \cdot 2^i = (n-1) \cdot 2^{(n+1)} + 2$$

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Base Case: $(n = 1)$

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LHS

Induction Example

Example: Show that for all $n \geq 1$

$$\sum_{i=1}^{n} i \cdot 2^i = (n-1) \cdot 2^{(n+1)} + 2$$

Base Case: $(n = 1)$

$$\text{LHS} \;=\; \sum_{i=1}^{1} i \cdot 2^i = 1 \cdot 2^1 = 2.$$

Induction Example

Example: Show that for all $n \geq 1$

$$\sum_{i=1}^{n} i \cdot 2^i = (n-1) \cdot 2^{(n+1)} + 2$$

Base Case: $(n = 1)$

$$\text{LHS} = \sum_{i=1}^{1} i \cdot 2^i = 1 \cdot 2^1 = 2.$$

RHS

Induction Example

Example: Show that for all $n \geq 1$

$$\sum_{i=1}^{n} i \cdot 2^i = (n-1) \cdot 2^{(n+1)} + 2$$

Base Case: $(n = 1)$

$$\begin{aligned} \text{LHS} &= \sum_{i=1}^{1} i \cdot 2^i = 1 \cdot 2^1 = 2. \\ \text{RHS} &= (1-1) \cdot 2^{1+1} + 2 = 0 + 2 = 2. \end{aligned}$$

Induction Example

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Induction Example

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Induction Example, continued

Inductive Step:

Induction Example, continued

Inductive Step:

Assume $P(k)$ is true:

$$\sum_{i=1}^{k} i \cdot 2^i = (k-1) \cdot 2^{(k+1)} + 2.$$

Induction Example, continued

Inductive Step:

Assume $P(k)$ is true:

$$\sum_{i=1}^{k} i \cdot 2^i = (k-1) \cdot 2^{(k+1)} + 2.$$

Show $P(k+1)$ is true:

$$\sum_{i=1}^{k+1} i \cdot 2^i = k \cdot 2^{(k+2)} + 2.$$

Induction Example, continued

Show $\displaystyle\sum_{i=1}^{k+1} i \cdot 2^i = k \cdot 2^{(k+2)} + 2$:

Induction Example, continued

Show $\sum_{i=1}^{k+1} i \cdot 2^i = k \cdot 2^{(k+2)} + 2$:

$$\sum_{i=1}^{k+1} i \cdot 2^i$$

Induction Example, continued

$$\text{Show} \quad \sum_{i=1}^{k+1} i \cdot 2^i = k \cdot 2^{(k+2)} + 2 :$$

$$\sum_{i=1}^{k+1} i \cdot 2^i \quad = \quad \sum_{i=1}^{k} i \cdot 2^i + (k+1) \cdot 2^{k+1}$$

Induction Example, continued

Show $\sum_{i=1}^{k+1} i \cdot 2^i = k \cdot 2^{(k+2)} + 2$:

$$\begin{aligned}
\sum_{i=1}^{k+1} i \cdot 2^i &= \sum_{i=1}^{k} i \cdot 2^i + (k+1) \cdot 2^{k+1} \\
&= (k-1) \cdot 2^{(k+1)} + 2 + (k+1) \cdot 2^{k+1}
\end{aligned}$$

Induction Example, continued

$$\text{Show} \quad \sum_{i=1}^{k+1} i \cdot 2^i = k \cdot 2^{(k+2)} + 2 :$$

$$\begin{aligned}
\sum_{i=1}^{k+1} i \cdot 2^i &= \sum_{i=1}^{k} i \cdot 2^i + (k+1) \cdot 2^{k+1} \\
&= (k-1) \cdot 2^{(k+1)} + 2 + (k+1) \cdot 2^{k+1} \\
&= 2k \cdot 2^{(k+1)} + 2
\end{aligned}$$

Induction Example, continued

$$\text{Show} \quad \sum_{i=1}^{k+1} i \cdot 2^i = k \cdot 2^{(k+2)} + 2 :$$

$$\begin{aligned}
\sum_{i=1}^{k+1} i \cdot 2^i &= \sum_{i=1}^{k} i \cdot 2^i + (k+1) \cdot 2^{k+1} \\
&= (k-1) \cdot 2^{(k+1)} + 2 + (k+1) \cdot 2^{k+1} \\
&= 2k \cdot 2^{(k+1)} + 2 \\
&= k \cdot 2^{(k+2)} + 2 \quad \text{QED}
\end{aligned}$$

Probability

- Defined in terms of a sample space, S.
- Sample space consists of a finite set of outcomes, also called elementary events.
- An event is a subset of the sample space. (So an event is a set of outcomes).
- Sample space can be infinite, even uncountable. In this course, it will generally be finite.

Example: (2-coin example.) Flip two coins. Sample space $S = \{\mathtt{HH}, \mathtt{HT}, \mathtt{TH}, \mathtt{TT}\}$. The event "first coin is heads" is the subset $\{\mathtt{HH}, \mathtt{HT}\}$.

Probability function

- A probability function is a function $P(\cdot)$ that maps events (subsets of the sample space S) to real numbers such that:
  1. $P(\emptyset) = 0$.
  2. $P(S) = 1$.
  3. For every event A, $0 \leq P(A) \leq 1$.
  4. If $A, B \subseteq S$ and $A \cap B = \emptyset$, then $P(A \cup B) = P(A) + P(B)$.
- Note: Property 4 implies that if $A \subseteq B$ then $P(A) \leq P(B)$.
- In the case of a finite sample space $S = \{s_1, \ldots, s_k\}$, this can be simplified. Each outcome S_i is assigned a probability $P(s_i)$, with

$$\sum_{i=1}^{k} P(s_i) = 1.$$

  The probability of an event $E \subseteq S$ is then given by:

$$P(E) = \sum_{s_i \in E} P(s_i).$$

Probability function

- If the sample space is finite, then the probability of an event is the sum of the probabilities of all the outcomes that the event contains.

Example: (2-coin example, continued). Define

$$P(\mathtt{HH}) = P(\mathtt{HT}) = P(\mathtt{TH}) = P(\mathtt{TT}) = \frac{1}{4}.$$

Then

$$P(\text{first coin is heads}) = P(\mathtt{HH}) + P(\mathtt{HT}) = \frac{1}{2}.$$

Random variables

- Intuitive definition: a random variable is a variable whose values depend on the outcome of some experiment.
- Formal definition: a random variable is a function that maps outcomes in a sample space S to real numbers.
- Special case: An Indicator variable is a random variable that is always either 0 or 1.

Expectation

- The expected value, or expectation, of a random variable X represents its "average value".
- Formally: if X is a random variable with a finite number of possible values

$$E(X) = \sum_{x \in X} x \cdot P(X = x).$$

Example: (2-coin example, continued). Let X be the number of heads when two coins are thrown. Then

$$\begin{aligned} E(X) &= 0 \cdot P(X=0) + 1 \cdot P(X=1) + 2 \cdot P(X=2) \\ &= 0 \cdot \left(\frac{1}{4}\right) + 1 \cdot \left(\frac{1}{2}\right) + 2 \cdot \left(\frac{1}{4}\right) \\ &= 1 \end{aligned}$$

Expectation

Example: Throw a single six-sided die. Assume the die is fair, so each possible throw has a probability of $1/6$. The expected value of the throw is:

$$1 \cdot \frac{1}{6} + 2 \cdot \frac{1}{6} + 3 \cdot \frac{1}{6} + 4 \cdot \frac{1}{6} + 5 \cdot \frac{1}{6} + 6 \cdot \frac{1}{6} = 3.5$$

Linearity of Expectation

- For any two random variables X and Y,

$$E(X+Y) = E(X) + E(Y).$$

- Proof: see [GT], 1.3.4
- Very useful, because usually it is easier to compute $E(X)$ and $E(Y)$ and apply the formula than to compute $E(X+Y)$ directly.

Example 1: Throw two six-sided dice. Let X be the sum of the values. Then

$$E(X) = E(X_1 + X_2) = E(X_1) + E(X_2) = 3.5 + 3.5 = 7,$$

where X_i is the value on die i ($i = 1, 2$).

Example 2: Throw 100 six-sided dice. Let Y be the sum of the values. Then

$$E(Y) = 100 \cdot 3.5 = 350.$$

Independent events

- Two events A_1 and A_2 are independent iff

$$P(A_1 \cap A_2) = P(A_1) \cdot P(A_2).$$

Example: (2-coin example, continued). Let

$$\begin{array}{rclcl} A_1 & = & \text{coin 1 is heads} & = & \{\mathtt{HH}, \mathtt{HT}\} \\ A_2 & = & \text{coin 2 is tails} & = & \{\mathtt{HT}, \mathtt{TT}\} \end{array}$$

Then $P(A_1) = \frac{1}{2}$, $P(A_2) = \frac{1}{2}$, and

$$P(A_1 \cap A_2) = P(\mathtt{HT}) = \frac{1}{4} = P(A_1) \cdot P(A_2).$$

So A_1 and A_2 are independent.

Independent events

A collection of n events $C = \{A_1, A_2, \ldots, A_n\}$ is mutually independent (or simply independent) if:

For every subset $\{A_{i_1}, A_{i_2}, \ldots A_{i_k}\} \subseteq C$:

$$P(A_{i_1} \cap A_{i_2} \cap \ldots \cap A_{i_k}) = P(A_{i_1}) \cdot P(A_{i_2}) \cdots P(A_{i_k}).$$

Example: Suppose we flip 10 coins. Suppose the flips are fair ($P(\texttt{H}) = P(\texttt{T}) = 1/2$) and independent. Then the probability of any particular sequence of flips (e.g., `HHTTTHTHTH`) is $1/(2^{10})$.

Example: Probability and counting

Suppose we flip a coin 10 times. Suppose the flips are fair and independent. What is the probability of getting exactly 7 heads out of the 10 flips?

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Solution:

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- The probability of each outcome is $1/(2^{10})$.
- The number of successful outcomes is $\binom{10}{7}$.
- Hence the probability of getting exactly 7 heads is:

$$\frac{\binom{10}{7}}{2^{10}} = \frac{120}{1024} = 0.117.$$