Graphs/Graph Algorithms

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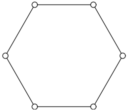

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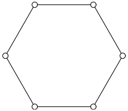

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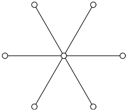

Star

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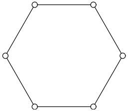

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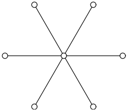

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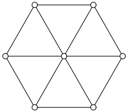

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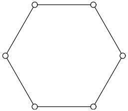

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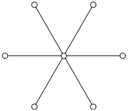

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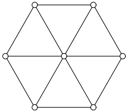

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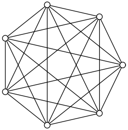

Complete Graph

CompSci 161—Fall 2018—©M. B. Dillencourt—University of California, Irvine

Directed vs. Undirected graphs

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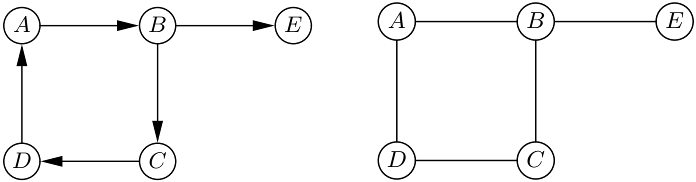

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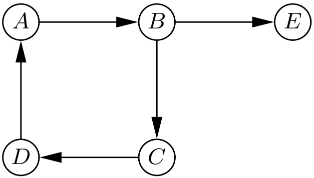

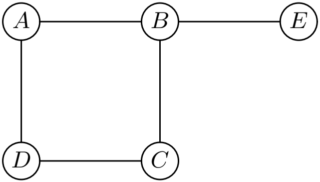

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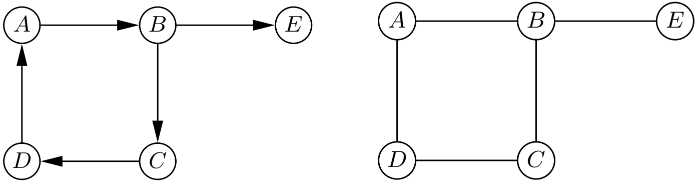
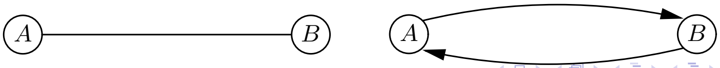


- It is possible to define a mixed graph with some edges undirected, others directed.
- For some purposes, we can replace an edge in an undirected graph by a pair of antiparallel directed edges



Examples of graphs (from [GT, Section 13.1])

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Loops and Multiedges

Loops and Multiedges

- A loop (sometimes called a self-loop) is an edge with both endpoints the same

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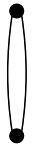
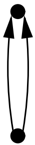
- A loop (sometimes called a self-loop) is an edge with both endpoints the same

- Multiedges (also called multiple edges, or parallel edges)
  - A multiedge in an undirected graph is a pair of edges with the same endpoints.
  - A multiedge in a directed graph is a pair of edges with the same endpoints and the same direction.



Multiedge



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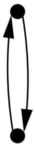

Not a Multiedge

Simple graphs

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- A simple graph is a graph that has no loops and no multiedges.

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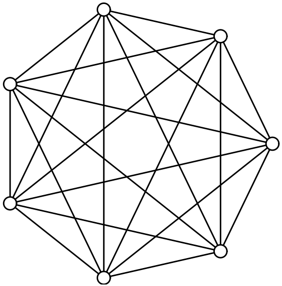
Complete Graphs

Complete Graphs

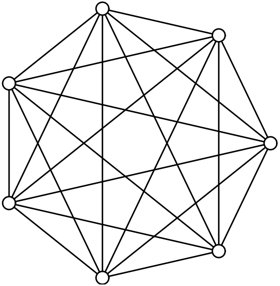
- A complete graph is an undirected graph with an edge between every pair of vertices.



Complete Graph on 7 vertices

Complete Graphs

- A complete graph is an undirected graph with an edge between every pair of vertices.
- A complete graph on n vertices has $m = \binom{n}{2}$ edges



Complete Graph on 7 vertices
$(n = 7, m = 21)$

Some graph terminology

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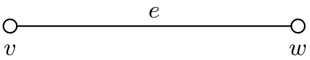

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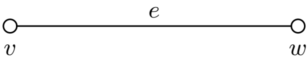

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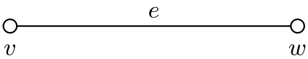

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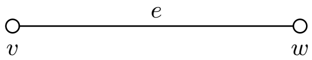

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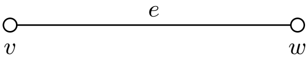

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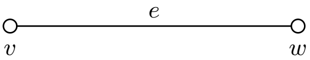

- The degree of a vertex v is the number of edges incident on v.

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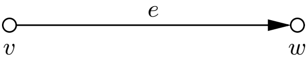

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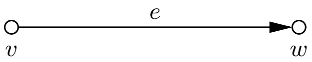

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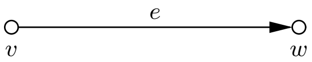

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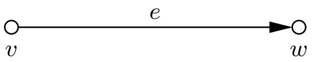

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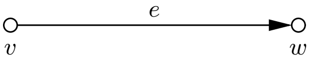

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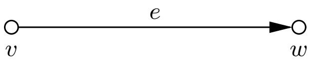

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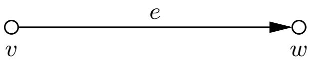

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Paths

- A path from vertex v to vertex w is a sequence of edges

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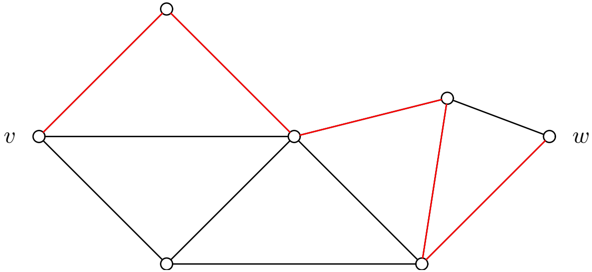

Cycles

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- In an undirected graph, a cycle is a set of edges

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- In a directed graph, the definition is similar, but we can omit the requirement that $k \geq 3$, and we can omit the word "simple" in the definition of acyclic.

Cycles, continued

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Usually when we write a cycle we just list the vertices, since the edges are implicit:

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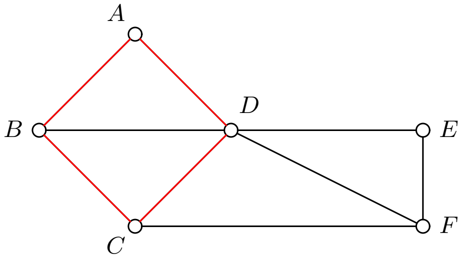

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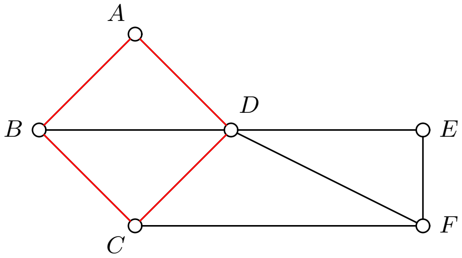

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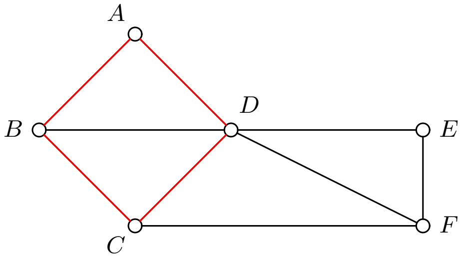

So we can write: *AB*,*BC*,*CD*,*DA*
or *ABCDA*

Cycles, continued

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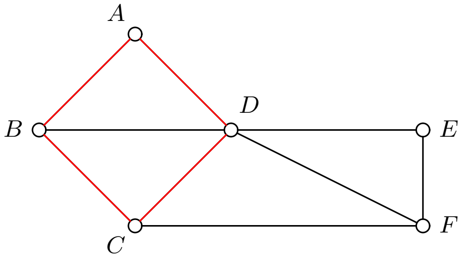

So we can write: *AB,BC,CD,DA*

or *ABCDA*

or just *ABCD*

Subgraphs

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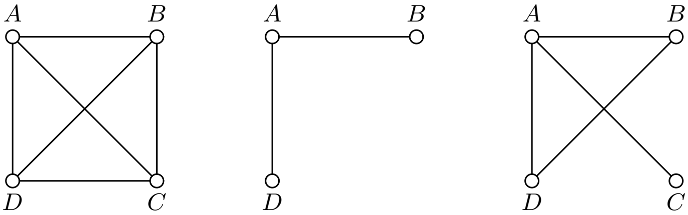
- A subgraph of $G = (V, E)$ is a graph $H = (V', E')$ such that

  $V' \subseteq V$ and $E' \subseteq E$



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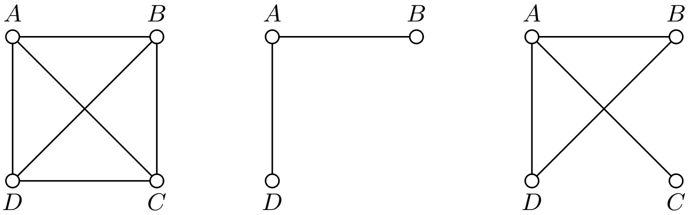
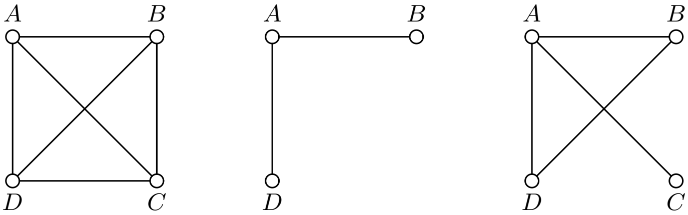


- H must be a valid graph. So every endpoint of an edge in E' must belong to V'.

Subgraphs

- A subgraph of $G = (V, E)$ is a graph $H = (V', E')$ such that

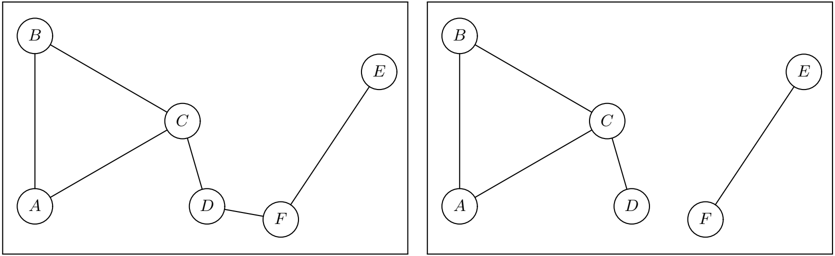
  $$V' \subseteq V \text{ and } E' \subseteq E$$



- H must be a valid graph. So every endpoint of an edge in E' must belong to V'.
- A spanning subgraph of G is a subgraph that contains all vertices of G.

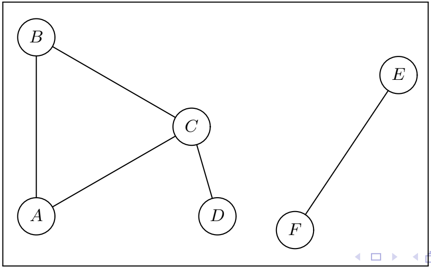
Connected Graphs

- An undirected graph is connected if there is a path between any pair of vertices.



Connected   Not connected

Connected Components



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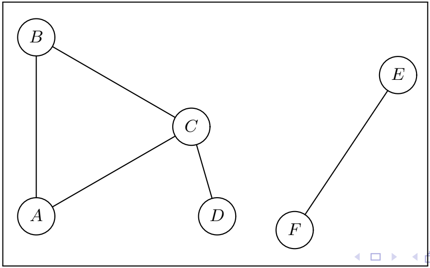
- If a graph G is not connected, a connected component of G is a maximal connected subgraph of G.



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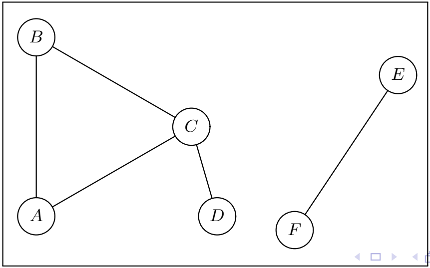

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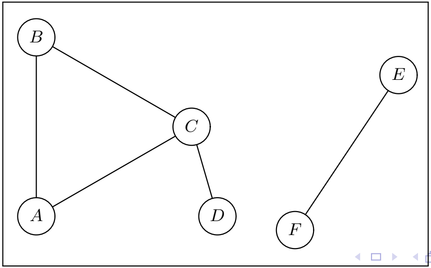

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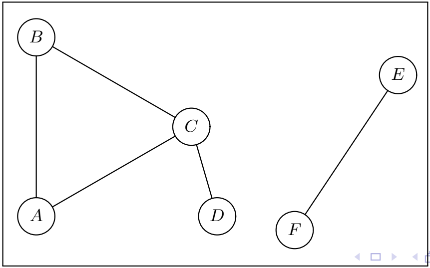

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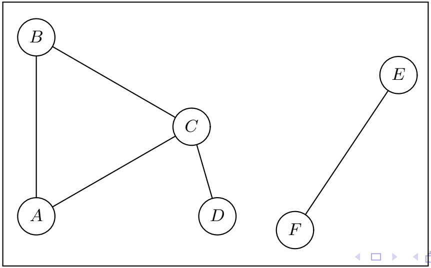
  - The connected components are the equivalence classes of vertices, together with the edges connecting vertices in the equivalence class.



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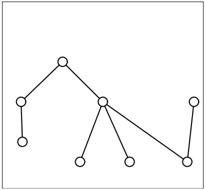

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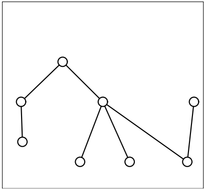

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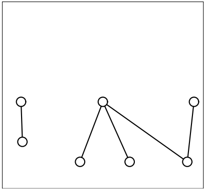

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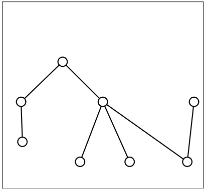

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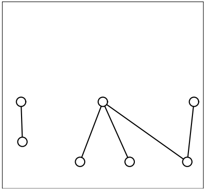

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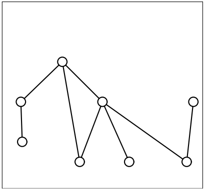

Connected, but not acyclic

Representations of graphs

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1. Graph drawing

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2. Edge List

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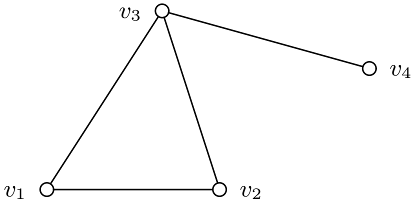
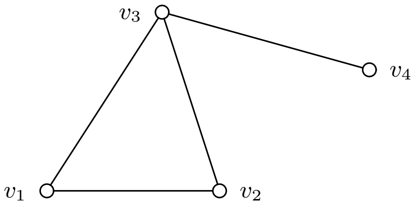


- Good for reasoning on paper, or for a GUI.

2. Edge List

- List of vertices, edges.
- Simple description for input, output.



$$V = (v_1, v_2, v_3, v_4)$$

$$E = (v_1 v_2, v_1 v_3, v_2 v_3, v_3 v_4)$$

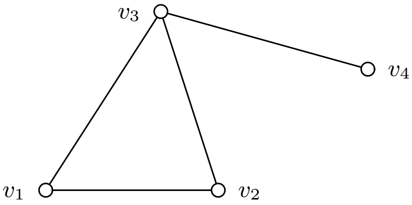
3. Adjacency matrix

- Represent G with a matrix (2D array):

$$a_{i,j} = \begin{cases} 1 & \text{if } v_i v_j \text{ is an edge of } G \\ 0 & \text{otherwise} \end{cases}$$

- Space requirement $= \Theta(n^2)$
- Can modify to handle:
  - Directed graphs
  - Weighted graphs

	1	2	3	4
1	0	1	1	0
2	1	0	1	0
3	1	1	0	1
4	0	0	1	0



4. Adjacency list

- Vertices are stored in an array
- For each vertex, there is a pointer to a linked list describing its neighbors
- Space requirement $= \Theta(n + m)$
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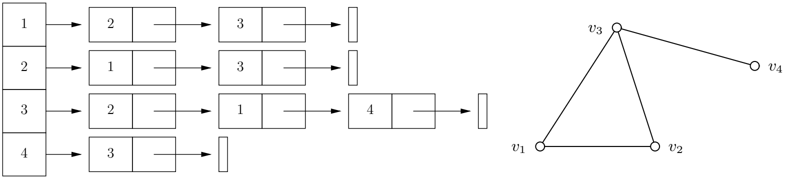

Systematic traversal of graphs and digraphs

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Two basic approaches:

Systematic traversal of graphs and digraphs

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- Depth-first search

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- Computing the connected components of G
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- Computing a cycle in an undirected graph G (or reporting that G is acyclic)

Depth-first search, Breadth-first search

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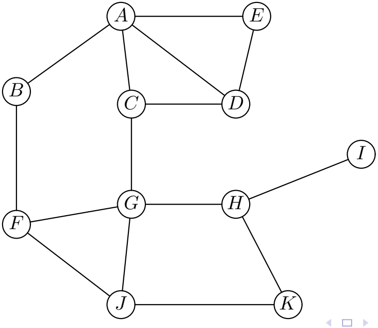
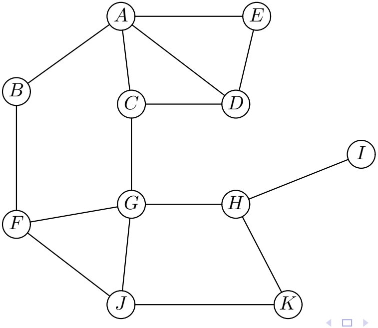


Depth-first search, Breadth-first search

- String/paint analogy



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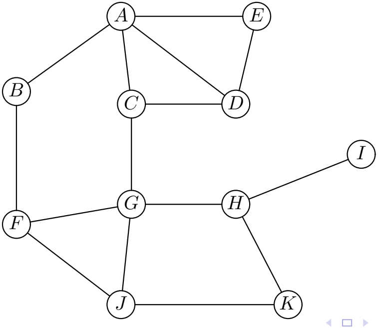

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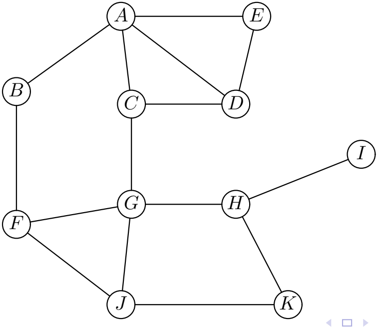

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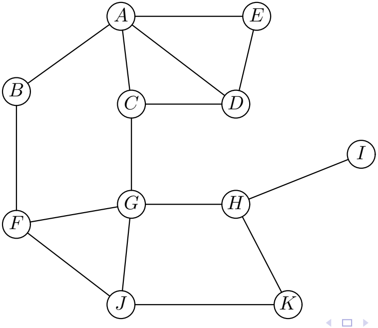

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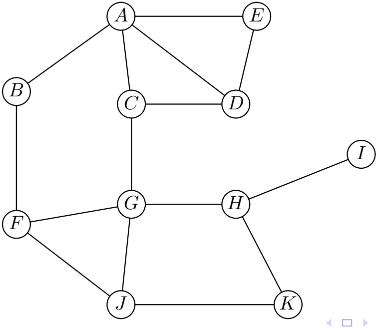
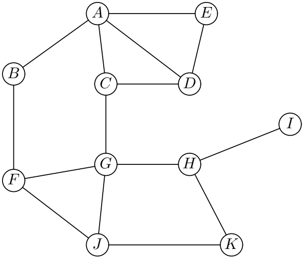


Depth First Search (DFS) in an undirected graph



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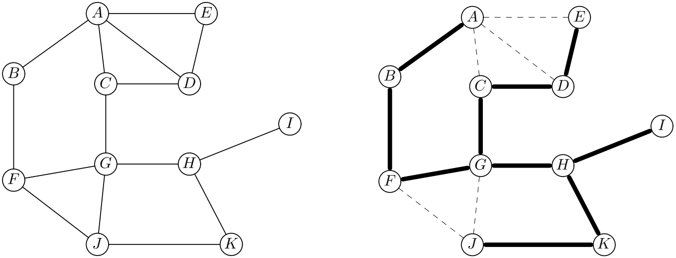

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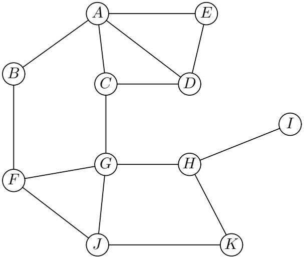

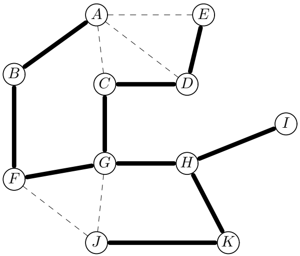

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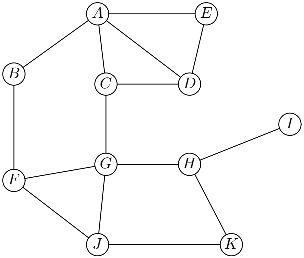

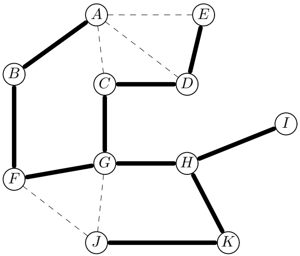

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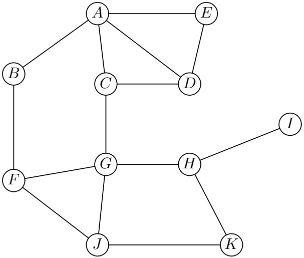

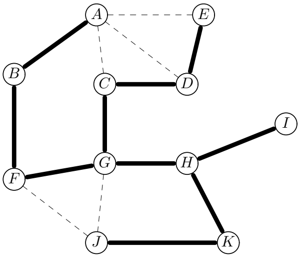

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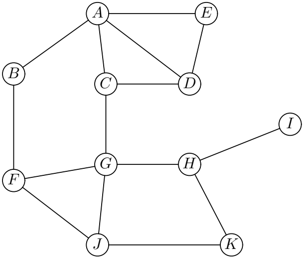

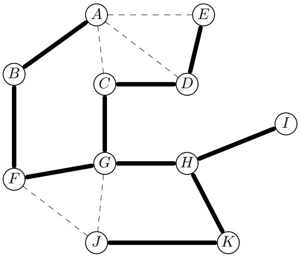

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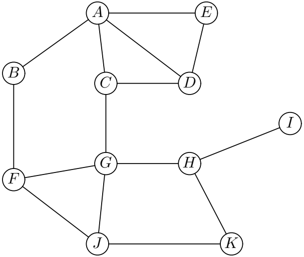
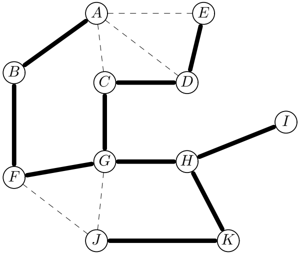
- Algorithms based on DFS return correct result for every result of these choices





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```
def DFS(G,v):
  label v as explored
  for all edges e incident on v:
    if edge e is unexplored:
      w ← opposite(v,e) // opposite endpoint of e from v
      if vertex w is unexplored:
        label e as a discovery edge
        recursively call DFS(G,w)
      else:
        label e as a back edge
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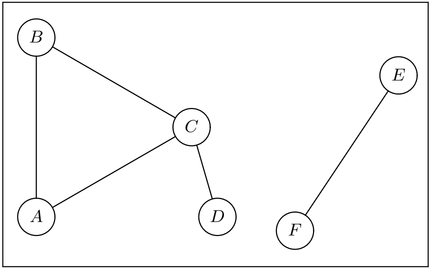

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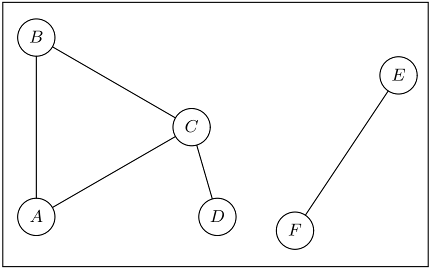

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Simple application of depth-first search in undirected graphs

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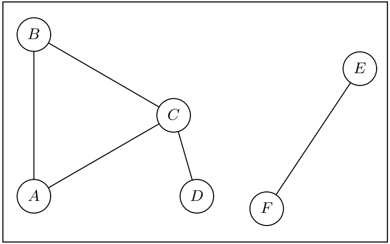

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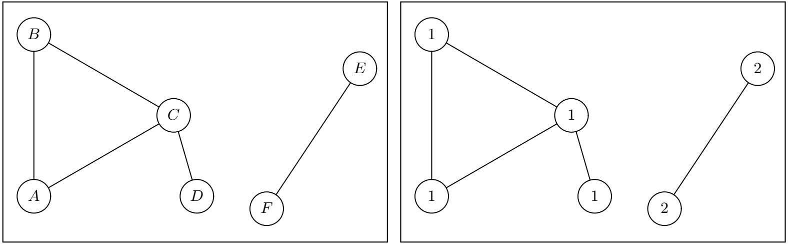

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      mark e as explored
      if (w.label = null):  DFSLabel(G,w,k)
```

Analysis: Runs in $O(m+n)$ time.

One application of connected-component labeling: Image processing

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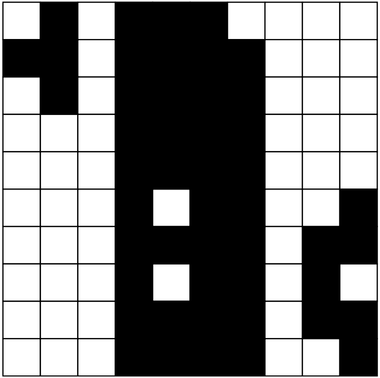

Binary image (black pixels are "on", white pixels are "off")

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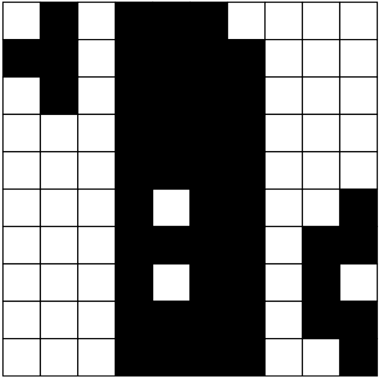

Binary image (black pixels are "on", white pixels are "off")

	1		2	2	2				
1	1		2	2	2	2			
	1		2	2	2	2			
			2	2	2	2			
			2	2	2	2			
			2		2	2			3
			2	2	2	2		3	3
			2		2	2		3	
			2	2	2	2		3	3
			2	2	2	2			3

Connected components of pixels in image that are "on"

Biconnected components,separation edges, separation vertices

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(Material from [GT] Section 13.5. Read it!)

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- A separation vertex is a vertex whose removal causes G to become disconnected.

Biconnected components,separation edges, separation vertices

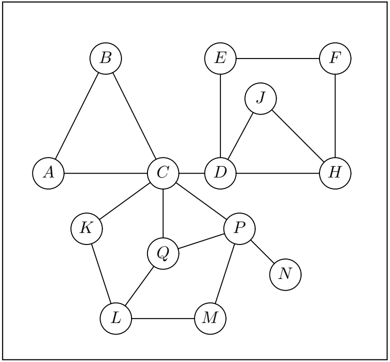
(Material from [GT] Section 13.5. Read it!)

Let G be a connected graph.

- A separation edge is an edge whose removal causes G to become disconnected.
- A separation vertex is a vertex whose removal causes G to become disconnected.
- G is biconnected (or 2-connected) if for any two vertices $u, v \in V(G)$, there are at least two disjoint paths between u and v (i.e., two different paths that have no edges or vertices in common except for u and v).

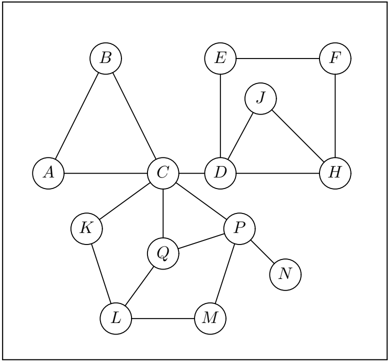
Biconnected components

- Let G be a connected graph.



Biconnected components

- Let G be a connected graph. A biconnected component (or bicomponent of G is a subgraph G' such that either:



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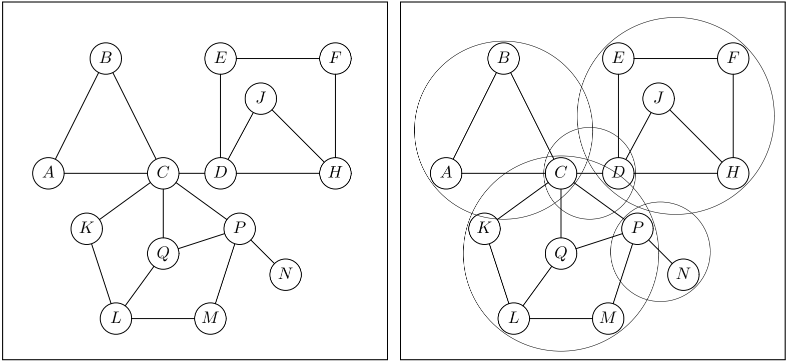

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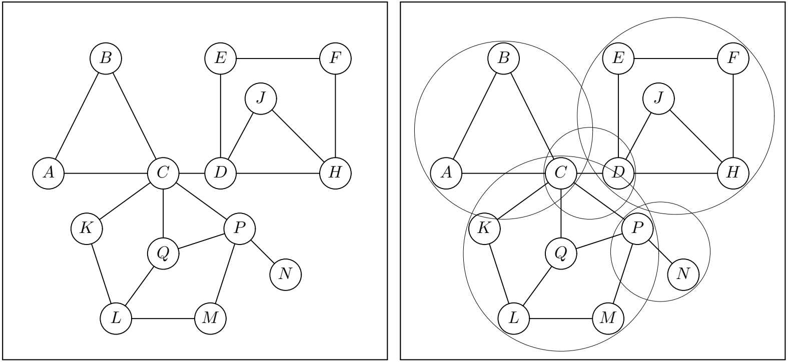
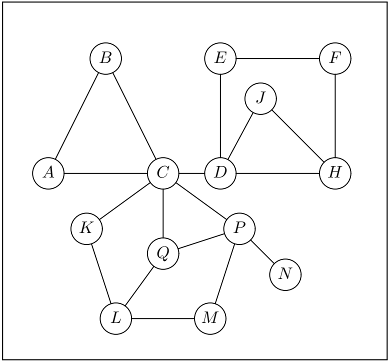
  - G' consists of a single separation edge and its two endpoints

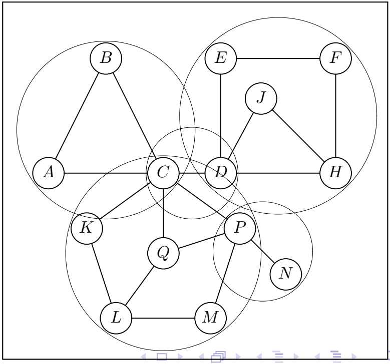


Characterization of Biconnectivity

Lemma: Let G be a connected graph. The following are equivalent:

1. G is biconnected.
2. For any two vertices of G, there is a simple cycle containing them.
3. G does not have any separating vertices or separating edges.





Equivalence relations

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Every element of C is in exactly one equivalence class.

Define a link relation on the edges of a graph G

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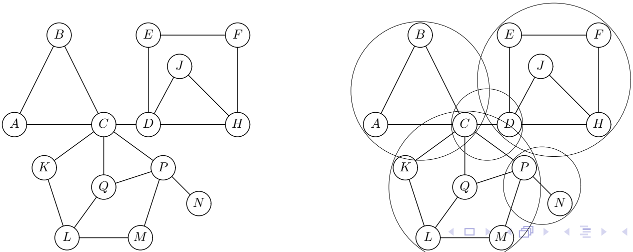
Two edges e and f in $E(G)$ are linked if $e = f$ or if G has a simple cycle containing e and f.

Lemma: Let G be a connected graph. Then

1. The link relation forms an equivalence relation on the edges of G.
2. A bicomponent is the subgraph induced by the edges of an equivalence class of linked edges.
3. Edge e is a separation edge if and only if it is in a single-element equivalence class
4. Vertex v is a separation vertex if and only if it has incident edges in two different equivalence classes.



Preliminary biconnected components algorithm

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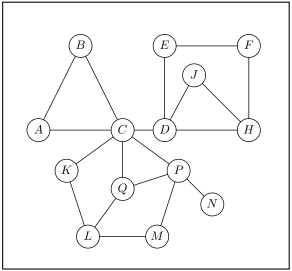
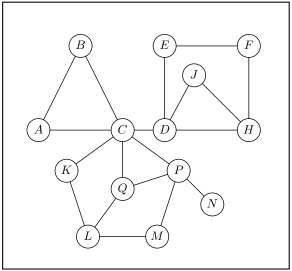
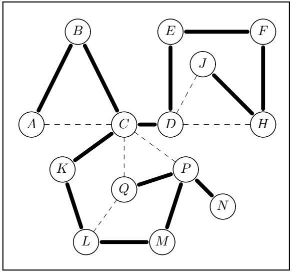


Preliminary biconnected components algorithm

Run DFS on G.





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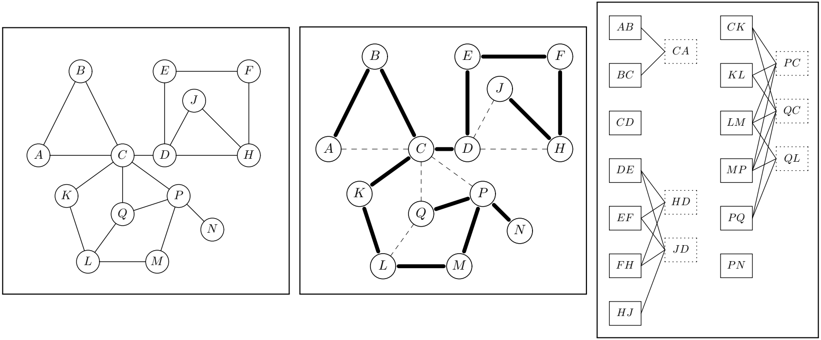

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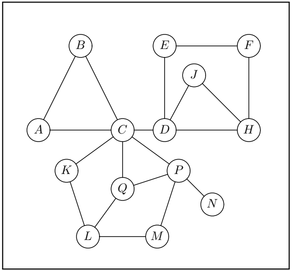

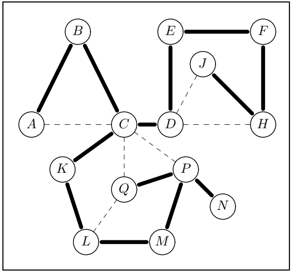

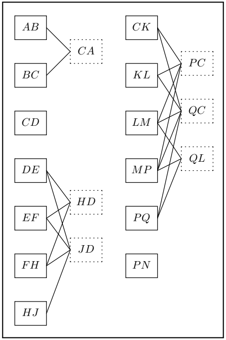

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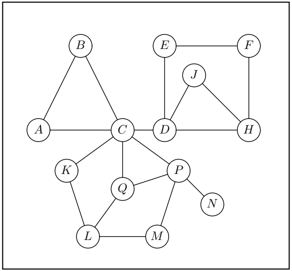
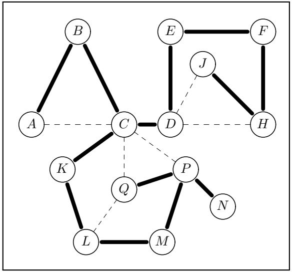
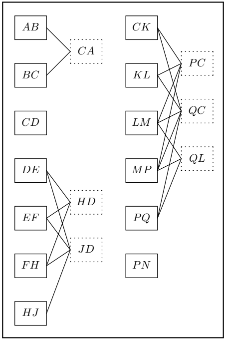
- The vertices of F are the edges of G
- For every back edge e in G, let $f_1, \ldots, f_k$ be the discovery edges of G that form a cycle with e. F contains the edges $(e, f_1), \ldots, (e, f_k)$.







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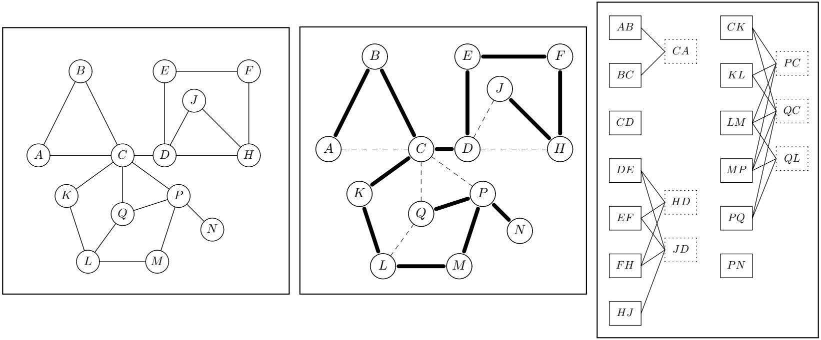
Output the connected components of F. Each of these is a equivalence class of the link relation, and hence corresponds to a bicomponent of G.



Improved bicomponent algorithm

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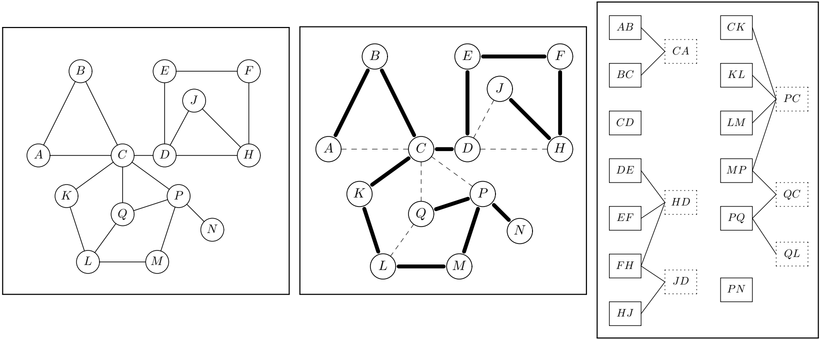
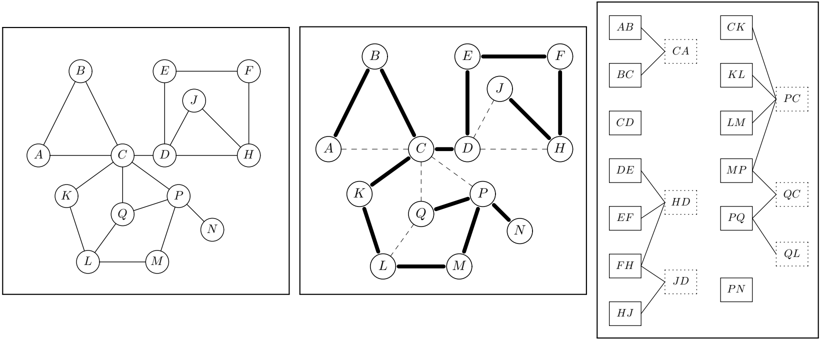


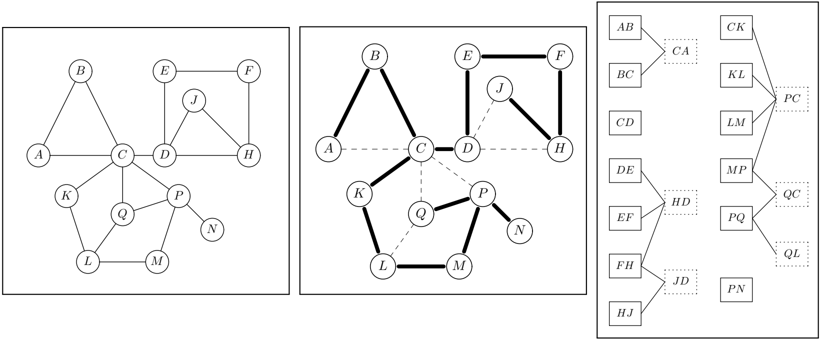
Improved bicomponent algorithm

- The previous algorithm can require $\Omega(m \cdot n)$ time. We can improve this to $O(m + n)$.



Improved bicomponent algorithm

- The previous algorithm can require $\Omega(m \cdot n)$ time. We can improve this to $O(m + n)$.
- Idea behind improved algorithm: We don't need to compute F. We only need the connected components of F. So we build a spanning tree for each connected component of F (a spanning forest of F).



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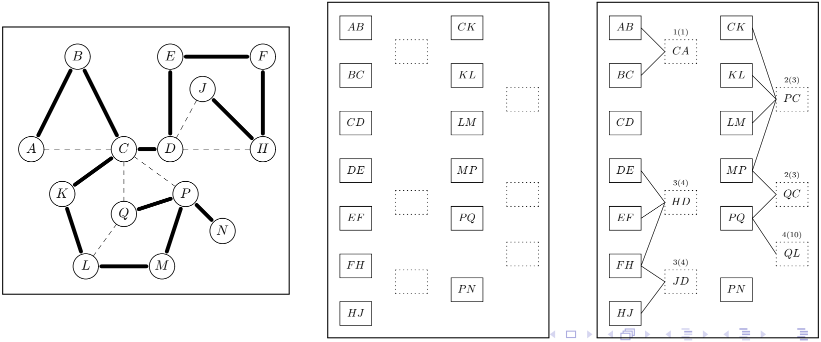
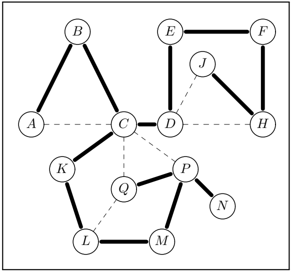
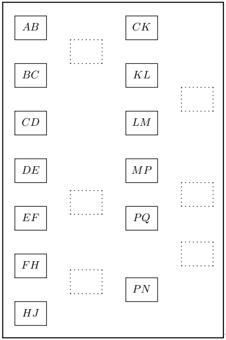
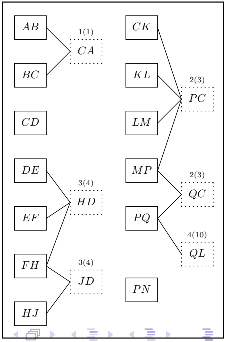


Improved bicomponent algorithm

1. Run DFS on G. Rank nodes of G according to order visited.

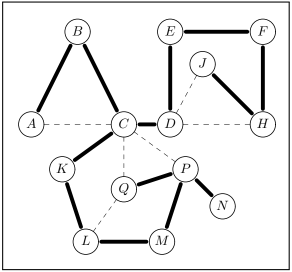


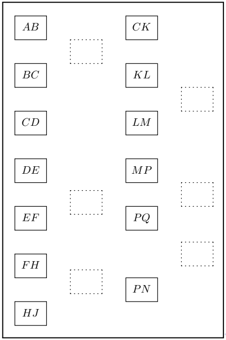


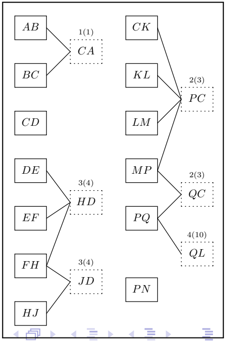


Improved bicomponent algorithm

1. Run DFS on G. Rank nodes of G according to order visited.
2. Add discovery edges of G to F. Mark discovery edges unlinked







Improved bicomponent algorithm

1. Run DFS on G. Rank nodes of G according to order visited.
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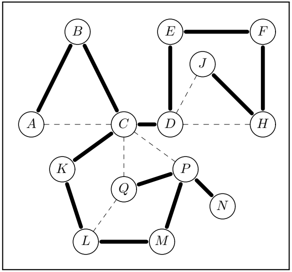
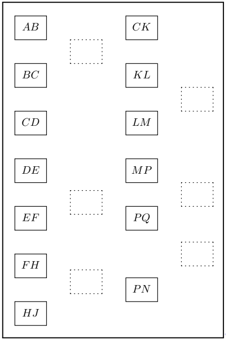
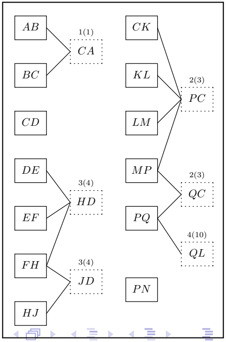
3. Process back edges $e = (u, v)$ in order of rank of v. For each such e, let $f_1, \ldots, f_k$ be the discovery edges of G that form a cycle with e. Add edges (e, f_k), $(e, f_{k-1}), \ldots,$ (e, f_j) to F, stopping at first linked edge. Mark edges f_k, $f_{k-1}, \ldots,$ f_{j+1} linked







Pseudocode for Biconnected Component Algorithm

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```
F ← an initially empty auxiliary graph
perform DFS traversal of G, starting at some vertex s
add each discovery edge f as a vertex in F, mark f "unlinked"
for each vertex v, in increasing rank order as visited in the
      DFS traversal
  for each back edge e=(u,v) with destination v
    add e as a vertex of the graph
    while u != v do
      Let f be the vertex in F corresponding to the
          discovery edge (parent(u),u)
      add the edge (e,f) to F
      if f is marked "unlinked" then
        mark f as "linked"
        u ← parent(u)
      else
        u ← v //exit while loop
compute the connected components of F
```

Note on the pseudocode

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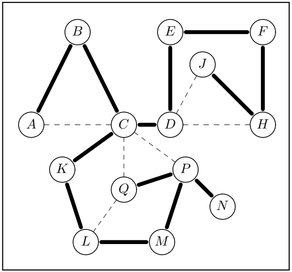
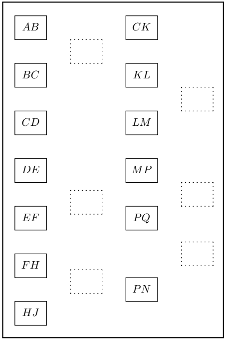
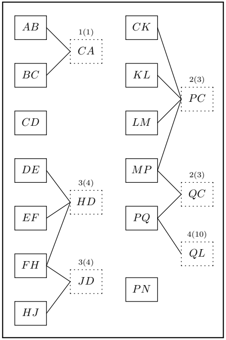


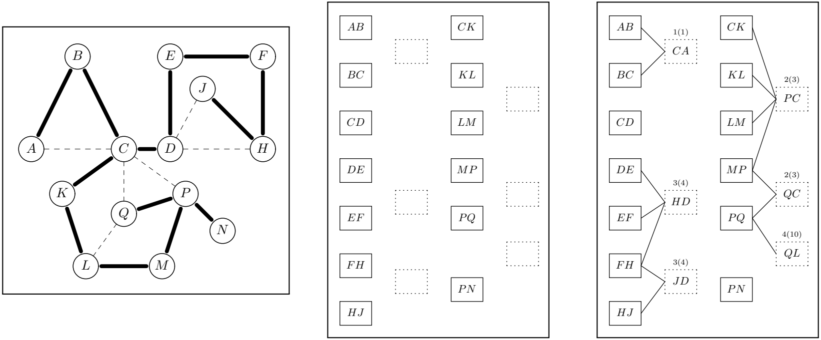




Note on the pseudocode

Note the sequencing:

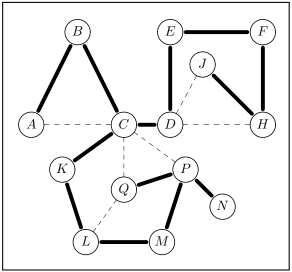
1. Build DFS tree

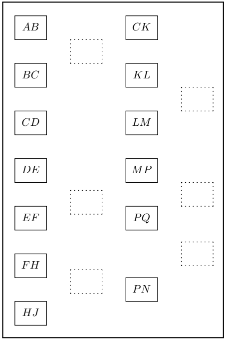


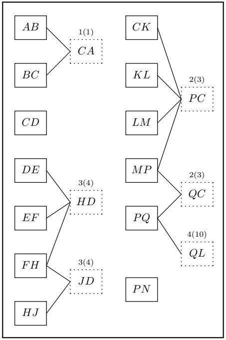
Note on the pseudocode

Note the sequencing:

1. Build DFS tree
2. Add discovery edges to F







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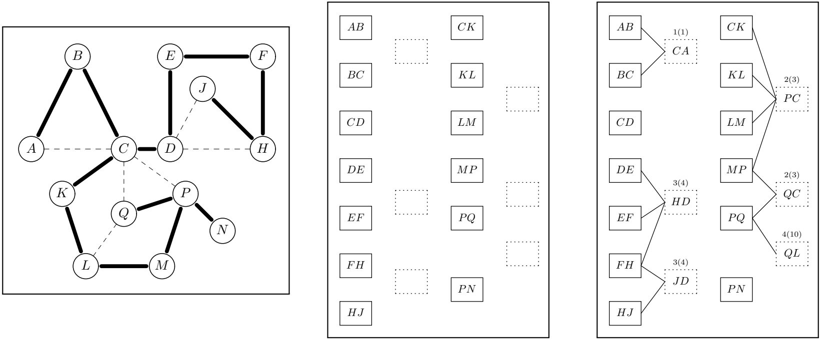

Note on the pseudocode

Note the sequencing:

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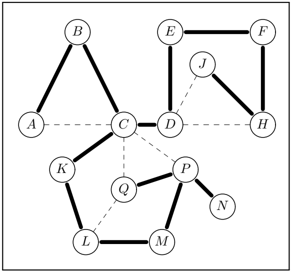
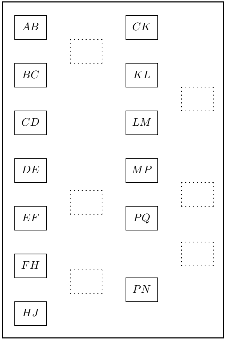
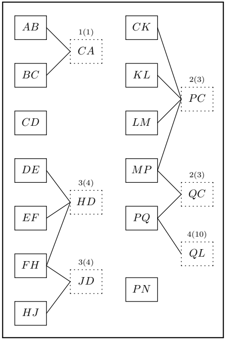
3. Add back edges to F. Back edges are processed in the order in which their destination nodes were visited in DFS traversal.







Analysis of the Biconnected Component Algorithm

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- Connected component analysis of F: $O(m)$ time.

Analysis of the Biconnected Component Algorithm

- DFS: $O(m+n)$ time.
- Building the auxiliary graph F: $O(m)$ time. Because...
  - Each iteration of the `while` loop causes an edge to be added to F.
  - When edge (e, f) is added to F:
    - Charge f if f is marked unlinked
    - Charge e if f otherwise
  - Each edge of G gets charged at most once
- Connected component analysis of F: $O(m)$ time.

So biconnected component analysis can be done in the same asymptotic time as connected component analysis.

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These questions can be answered using network flow techniques.

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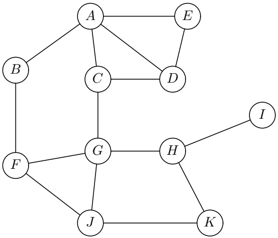

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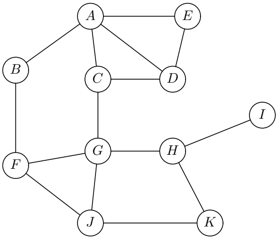

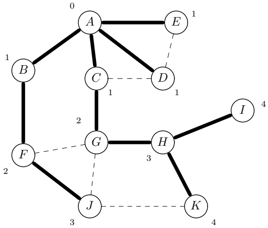

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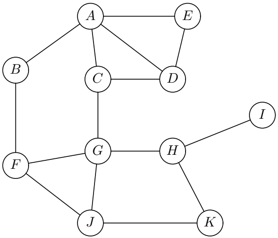

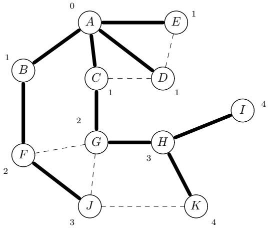

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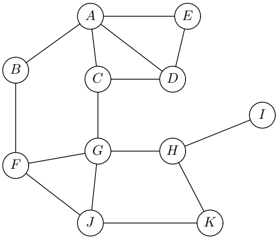

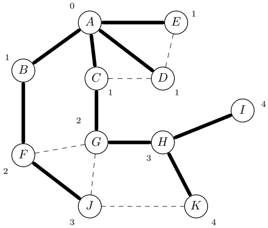

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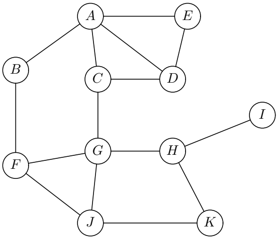
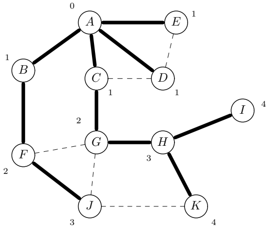
- Process all nodes at one level before moving on to next level (special case: FIFO order)





Pseudocode for BFS in an undirected graph

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```
def BFS(G,s):
  Set all vertices and edges to unexplored
  create an empty Queue Q[0]
  insert s into Q[0]
  i ← 0
  while Q[i] is not empty:
    create an empty Queue Q[i+1]
    for each vertex v in Q[i]:
      for all edges e incident on v:
        if edge e is unexplored:
          w ← opposite(v,e)
          if vertex w is unexplored:
            label e as a discovery edge
            insert w into Q[i+1]
          else:
            label e as a cross edge
    i ← i+1
```

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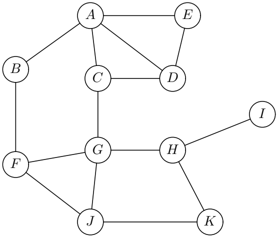
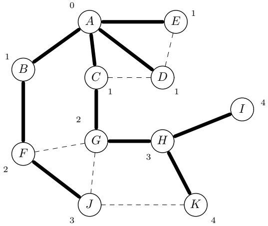
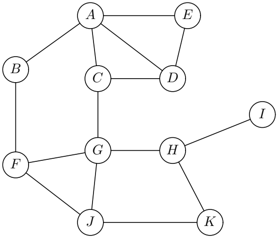
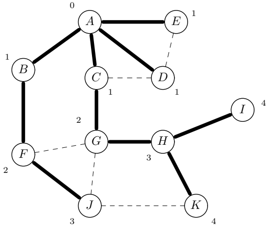




A useful property of BFS

- The level number of vertex v in BFS-tree rooted at s is the smallest number of edges in a path from s to v.





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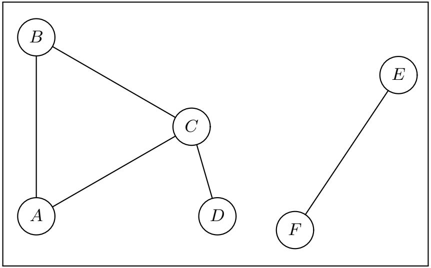
As with DFS, we may have to restart BFS multiple times to visit the entire graph. Running time will still be $O(n + m)$.



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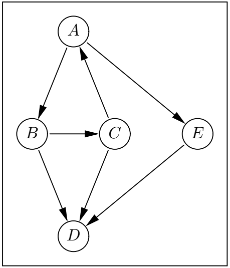

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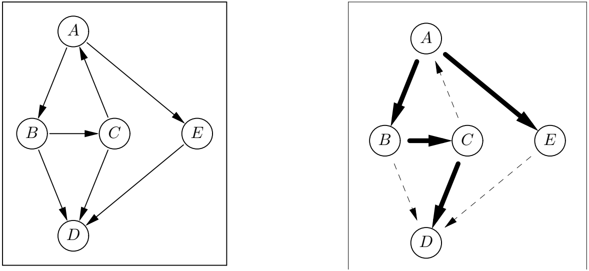

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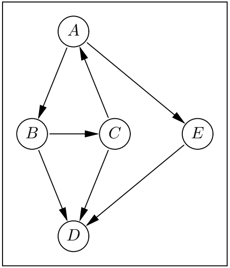

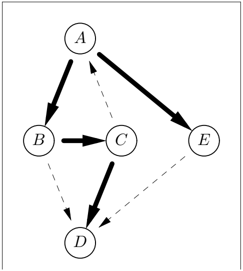

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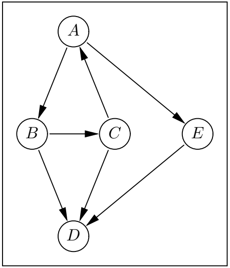

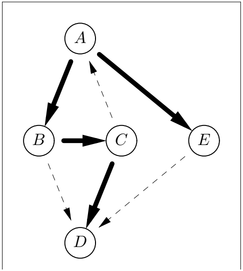

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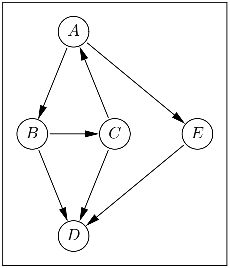

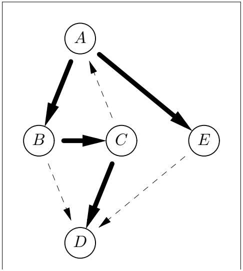

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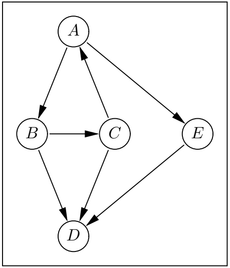

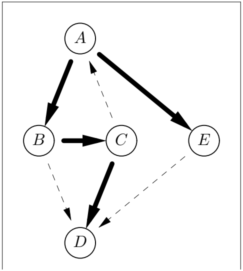

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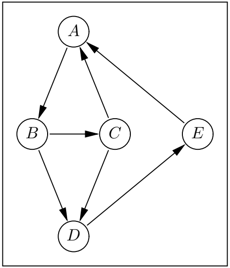

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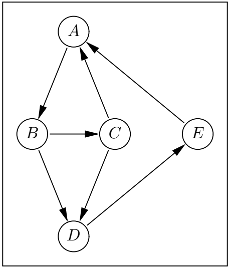

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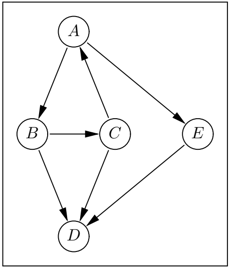

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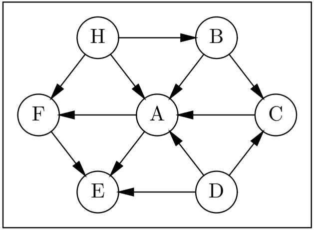

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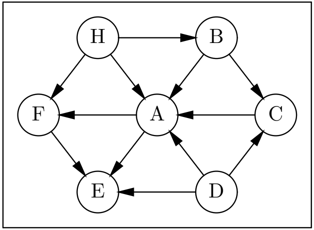

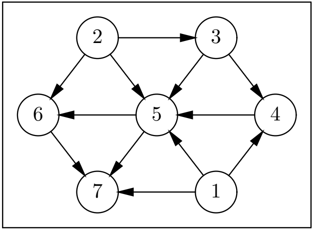

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Proof Sketch:

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  - G must have a vertex with indegree 0.

Topological ordering

Theorem: A directed graph has a topological ordering if and only if it is a DAG

Proof: see [GT], Section 13.4.4.

Proof Sketch:

- $\Rightarrow$: easy
- $\Leftarrow$: Assume G is a DAG.
    - G must have a vertex with indegree 0.
        - Let vertex v_1 be such a vertex.

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- $\Leftarrow$: Assume G is a DAG.
  - G must have a vertex with indegree 0.
    - Let vertex v_1 be such a vertex.
  - Remove v_1 (and all incident edges) from G.
  - Result is a smaller DAG.

Topological Sorting

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- Algorithm repeatedly applies the following steps:
  1. Find a vertex v with indegree 0
  2. Assign v the next available label
  3. Delete v and all its outgoing edges
- Runs in $O(n+m)$ time using $O(n)$ additional space

Topological Sorting Pseudocode

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```
def TopologicalSort(G):
   L ← initially empty list of vertices
   for each vertex v of G:
      incounter(v) = indegree(v)
      if incounter(v) = 0:  add v to L

   i ← 0
   while L is not empty:
      choose a vertex v in L and remove it from L.
      i ← i+1
      v.number ← i
      for each edge e in v.outEdges:
         w = opposite(v,e);
         incounter(w) = incounter(w)-1;
         if incounter(w) = 0:  add w to L

   if i == n:  print the vertices and their numbers
   else:  print("G is not a DAG!!")
```

Topological Sorting Example

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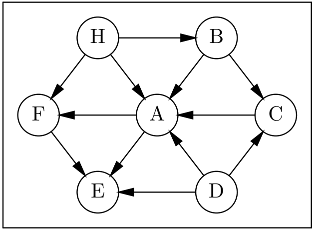
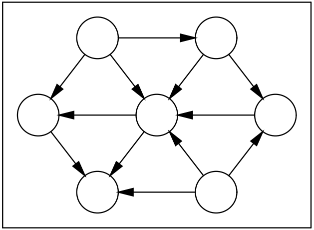
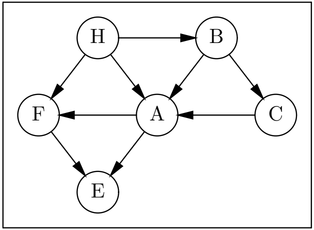


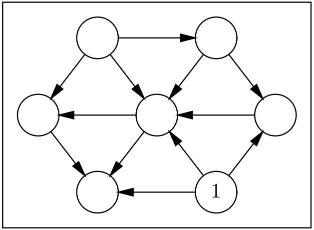


vertex	indegree
A	4
B	1
C	2
D	0
E	3
F	2
H	0

$$L = [D, H]$$

Topological Sorting Example

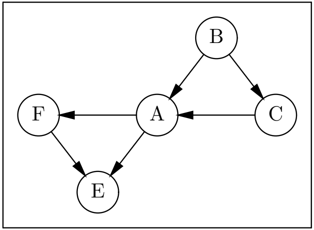


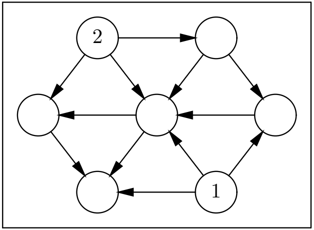


vertex	indegree
A	3
B	1
C	1
D	0
E	2
F	2
H	0

$$L = [H]$$

Topological Sorting Example

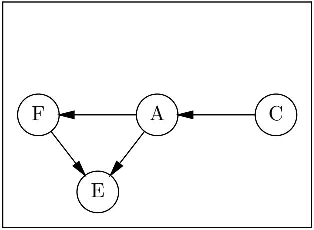


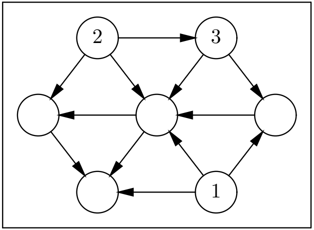


vertex	indegree
A	2
B	0
C	1
D	0
E	2
F	1
H	0

$L = [B]$

Topological Sorting Example

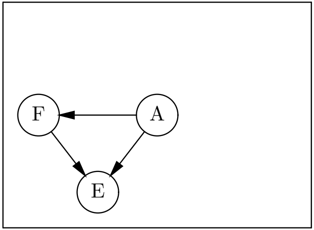


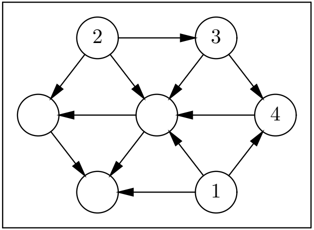


vertex	indegree
A	1
B	0
C	0
D	0
E	2
F	1
H	0

$L = [C]$

Topological Sorting Example

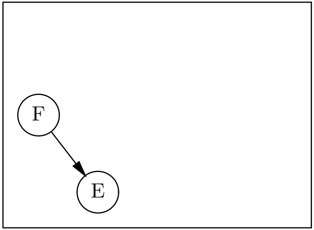


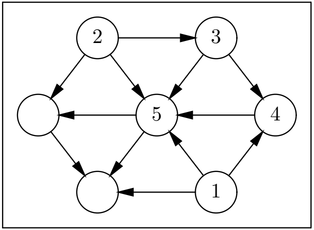


vertex	indegree
A	0
B	0
C	0
D	0
E	2
F	1
H	0

$$L = [A]$$

Topological Sorting Example

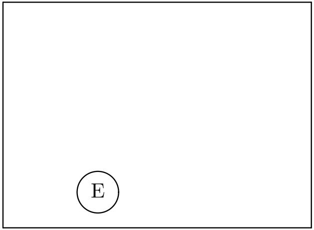


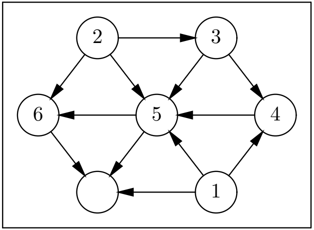


vertex	indegree
A	0
B	0
C	0
D	0
E	1
F	0
H	0

$L = [F]$

Topological Sorting Example

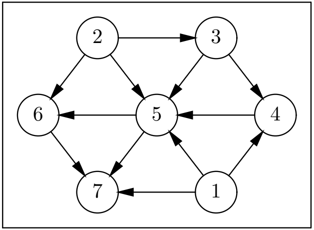




vertex	indegree
A	0
B	0
C	0
D	0
E	0
F	0
H	0

$L = [E]$

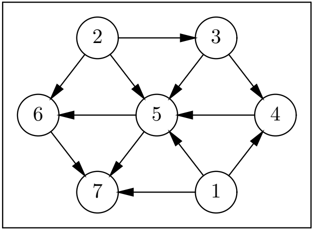
Topological Sorting Example



vertex	indegree
A	0
B	0
C	0
D	0
E	0
F	0
H	0

$L = [\,]$

Topological Sorting Example



vertex	indegree
A	0
B	0
C	0
D	0
E	0
F	0
H	0

$L = [\,]$ Done!

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- At any step, there could be multiple vertices with indegree 0
- Any valid choice will lead to a valid topological ordering
- Note that this implies that a digraph may have many different topological orderings