Integrals of Motion and ODE/IM correspondence

Katsushi Ito (Institute of Science Tokyo)

International Workshop on QFT and Beyond
@Shing-Tung Yau Center of Southeast University
November 27-29,2025

arXiv:2408.12917, 2305.03283 (w/ Mingshuo Zhu)
Ongoing Work with Daichi Ide and Wataru Kono (Science Tokyo)

Introduction

Integrals of motion in CFT

- 2-dim Integrable QFT has an infinite number of integrals of motion (IM) [Zamolodchikov]
- In 2d CFT +integrable perturbation, the IMs I_s can be constructed as a quantum version of soliton hierarchy [Zamolodchikov,Sasaki-Yamanaka, Eguchi-Yang, Kupershmidt-Mathieu,...]

$$[I_s, I_{s'}] = 0, \quad I_s = \oint P_s(z) dz$$

- Integrable structure (Yang-Baxter relations) in 2d CFT [Bazhanov-Lukyanov-Zamolochikov]
  T-function $\Longrightarrow$ IMs on a cylinder
- Application to thermal properties of CFT [Dymarsky-Pavlenko,Maloney-Ng-Ross-Tsiares, Ashok-Parihar-Sengupta-Sudhakar-Tateo, Downing-Karimi-Sengupta-Sudhakar-Watts]

ODE/IM correspondence

a relation between spectral analysis approach of ordinary differential equation (ODE), and the "functional relations" approach to 2d quantum integrable model (IM)

[Dorey-Tateo 9812211, Bazhanov-Lukyanov-Zamolodchikov 9812247, ...]

ODE/IM correspondence (2)

the Schrödinger equation with centrifugal potential term $(2M > 0)$

$$\left[-\frac{d^2}{dx^2} + \frac{\ell(\ell+1)}{x^2} + x^{2M} - E\right]\psi(x, E, \ell) = 0$$

XXZ spin chain in the continuum limit $N \to \infty$

$$H = -\frac{1}{2}\sum_{i=1}^{N}(\sigma_i^x\sigma_{i+1}^x + \sigma_i^y\sigma_{i+1}^y - \cos 2\eta\sigma_i^z\sigma_{i+1}^z)$$

$$\sigma_{N+1}^z = \sigma_1^z, \quad (\sigma_{N+1}^x \pm i\sigma_{N+1}^y) = e^{\pm 2i\phi}(\sigma_1^x \pm i\sigma_1^y)$$

CFT(minimal model $\mathcal{M}_{2,2M+2}$): $c = 1 - 6(\beta - \frac{1}{\beta})^2$, $\Delta_0 = \left(\frac{p}{\beta}\right)^2 + \frac{c-1}{24}$

ODE	XXZ spin chain
order of the potential $2M$	anisotropy parameter $\eta = \frac{\pi}{2}(1-\beta^2) = \frac{\pi M}{2M+2}$
angular momentum ℓ	twist parameter $\phi = 2\pi p = \frac{\pi(2\ell+1)}{2M+2}$
energy E	spectral parameter $\theta = \frac{M+1}{2M}\log E$

ODE/IM correspondence (3)

$$\left[-\frac{d^2}{dx^2} + \frac{\ell(\ell+1)}{x^2} + x^{2M} - E\right]\psi(x, E, \ell) = 0$$

- $x = 0$: regular singularity $\chi_+ = x^{\ell+1} + \cdots$, $\chi_- = x^{-\ell} + \cdots$
- $x = \infty$: irregular singularity $y^{\pm} \sim \exp\left(\pm \frac{x^{M+1}}{M+1}\right)$
- $y^- = Q_+\chi_+ + Q_-\chi_-$ connection coefficients $Q_\pm$
- Wronskian $W[y_k, y_{k'}]$ of two aysmpt. sols. $y_k, y_{k'}$: Stokes coefficients

ODE	IM
Connection coeffs between 0 and ∞	Q-functions (T-Q relation)
Stokes coefficients	T-functions
Voros symbols (exact WKB periods)	Y-functions

ODE/IM correspondence and Integrals of Motion

- T-function: (generating function of IMs) $\Longleftrightarrow$ Wronskian of the solutions at infinity
- WKB solutions of the ODE are applied to calculate the Wronskians
- Schrödinger Eq. $\Longleftrightarrow$ Virasoro minimal model [BLZ]
  3rd order ODE $\Longleftrightarrow$ W_3 minimal model
  [Bazhanov-Hibberd-Koroshkin,Ashok et al., Kudrna-Prochazka]
- Exact relation between CFT data $(c, \Delta_2, \cdots)$ and ODE data (potential, monodromy)

ODE/IM correspondence

$\hat{\mathfrak{g}}^{\vee}$ linear differential system $\Longleftrightarrow$ $W\hat{\mathfrak{g}}$ minimal model

Linear Problem and WKB analysis

Linear Problem

- $\mathfrak{g}$: a (simply-laced) Lie algebra of rank r
- $\{H^i, E_\alpha\}$; the Cartan-Weyl basis of $\mathfrak{g}$ ($i = 1, \ldots, r$, $\alpha \in \Delta$)
- $\alpha_1, \ldots, \alpha_r$: simple roots
- extended root $\alpha_0 = -\theta$, θ is the highest root

  the first order linear differential system [Sun]

$$[\epsilon \partial_x + A(x)]\, \Psi(x) = 0$$

$$A(x) = \epsilon \frac{1}{x}(l \cdot H)\Psi + \sum_{i=1}^{r} E_{\alpha_i} + p(x) E_{\alpha_0}$$

- $l \cdot H = \sum_{i=1}^{r} l_i \alpha_i^\vee \cdot H$, l_i "angular momentum" parameter
- $p(x) = x^{h^\vee M} - E$, "potential" term $(E = 1)$
- $\epsilon = \hbar$ Planck constant
- Ψ take values in the representation of $\mathfrak{g}$

Comments

- The linear system is obtained from the Lax system $[L_z, L_{\bar{z}}] = 0$ of the affine Toda field equation by taking the light-cone ($\bar{z} \to 0$) and the conformal limit. [Lukyanov-Zamolodchikov,KI-Locke, KI-Shu,...]
- For $A_1^{(1)}$, the linear system is equivalent to the Schrödinger equation with potential $p(x)$ with centrifugal term [Dorey-Tateo, BLZ]

$$\left[-\hbar^2 \frac{d^2}{dx^2} + \hbar^2 \frac{\ell(\ell+1)}{x^2} + x^{2M} - E\right] y(x, E, \ell) = 0$$

- The ODE can be regarded as a quantum Seiberg-Witten curve for (G, A_{hM-1})-type Argyres-Douglas theory [KI-Shu] when $l_i = 0$. The central charge coincides with that obtained from the $4d/2d$ correspondence of Beem et al.

WKB analysis of $A_r^{(1)}$ type Linear problem

$(r+1)$-dim representation of A_r, $E_{\alpha_i} = E_{i,i+1}$, $E_{\alpha_0} = E_{r+1,1}$
$\psi(x)$: the top component of Ψ

$$\left[\epsilon^{r+1}\left(\partial_x - \frac{l_1}{x}\right)\left(\partial_x - \frac{l_2}{x}\right)\dots\left(\partial_x - \frac{l_r}{x}\right)\left(\partial_x - \frac{l_{r+1}}{x}\right) + (-1)^r p(x)\right]\psi(x,\epsilon) = 0$$

[Dorey-Dunning-Tateo]

$$\left[(\epsilon\partial_x)^h + \sum_{i=2}^{h}\epsilon^i \frac{L_i}{x^i}(\epsilon\partial_x)^{h-i} + (-1)^r p(x)\right]\psi(x,\epsilon) = 0$$

WKB expansion

$$\psi(x,\epsilon) = \exp\left(\frac{1}{\epsilon}\int^x dx\, S(x,\epsilon)\right),$$

Riccati eq. for S

$$(\epsilon\partial_x + S)^r S + \sum_{i=2}^{r+1}\frac{L_i}{x^i}(\epsilon\partial_x + S)^{r-i} S + (-1)^r p(x) = 0,$$

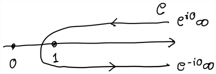
WKB solutions and period integral

$S(x,\epsilon) = \sum_{i=0}^{\infty} \epsilon^i S_i(x)$

$$S_0(x) = -p(x)^{\frac{1}{h}}, \quad S_1(x) = \frac{h-1}{2h}\partial_x \log(p),$$

$$S_2(x) = \frac{L_2}{h} p^{-\frac{1}{h}} x^{-2} - \frac{(h-1)(h+1)M(hM-1)}{12} p^{-\frac{1}{h}-1} x^{hM-2} + \frac{(h-1)(h+1)(2h+1)M^2}{24} p^{-\frac{1}{h}-2} x^{2hM-2}, \dots$$

period integral $Q_i = \oint_{\mathcal{C}} dx\, S_i(x)$



$$J(a,b) \equiv \int_{\mathcal{C}} \left(x^{hM} - 1\right)^a x^b dx = -\frac{e^{\pi i a} 2\pi i}{hM} \frac{\Gamma\left(-a - \frac{b+1}{hM}\right)}{\Gamma(-a)\Gamma\left(1 - \frac{b+1}{hM}\right)},$$

WKB periods

$$Q_2 = \frac{1}{h} J_{1,2}\Big(L_2 + \frac{1}{24}(h-1)h(hM-1)\Big),$$

$$Q_3 = \frac{1}{h} J_{2,3}\Big(-L_3 - (h-2)L_2\Big), \dots$$

$S_{1+hk} = \partial(*)$, $Q_{1+hk} = 0$ ($k \in \mathbb{Z}$)

WKB analysis of $D_r^{(1)}$ type Linear problem

$2r$-dim vector representation [Dorey-Dunning-Masoero-Suzuki-Tateo]

$$\epsilon^{2r-2}\left(\partial_x - \frac{l_1}{x}\right)\cdots\left(\partial_x - \frac{l_r}{x}\right)\partial_x^{-1}\left(\partial_x + \frac{l_r}{x}\right)\cdots\left(\partial_x + \frac{l_1}{x}\right)\psi = 4\sqrt{p(x)}\partial_x\sqrt{p(x)}\psi,$$

- WKB analysis of the ODE including a pseudo-differential operator ∂_x^{-1}
- Diagonalization (abelianization)

WKB analysis of pseudo-ODE

- Set $l_r \to 0$, $D_r^{(1)} \to B_{r-1}^{(1)}$

$$\epsilon^{2r-2}\left(\partial_x - \frac{l_1}{x}\right)\cdots\left(\partial_x - \frac{l_{r-1}}{x}\right)\partial_x\left(\partial_x + \frac{l_{r-1}}{x}\right)\cdots\left(\partial_x + \frac{l_1}{x}\right)\psi(x) \\ - 4\sqrt{p(x)}\partial_x\sqrt{p(x)}\psi(x) = 0.$$

- WKB expansion of $B_{r-1}^{(1)}$-type ODE

$$(\epsilon\partial_x + S)^{2r-2}S + \sum_{i=2}^{2r-1}\frac{L_i'}{x^i}(\epsilon\partial_x + S)^{2r-2-i}S - 4\sqrt{p(z)}\partial_z\sqrt{p(z)} = 0.$$

  L_i' are elementary symmetric polynomials in $l_1^2, \ldots, l_{r-1}^2$
  We can solve the Riccati eq.

- Replace L_i' by symmetric polynomials in $l_1^2, \ldots, l_{r-1}^2, l_r^2$ ($B_{r-1}^{(1)} \to D_r^{(1)}$)
  $\Longrightarrow$ WKB expansion of $D_r^{(1)}$-type ODE
- This Riccati eq. does not cover the full WKB solutions.

Diagonalization of the Lax operator

Consider d-dimensional representation.

$$A_{\mathsf{diag}}(z) = T^{-1}(z)A(z)T(z) + \epsilon T^{-1}(z)\partial_z T(z).$$

$$\Psi(z) = diag\left[\exp\left(-\frac{1}{\epsilon}\int dz A_{\mathsf{diag}}(z)\right)\right].$$

step-by-step diagonalization

$$T(z) = T_d = \begin{pmatrix} 1 & & & & \\ & & \ddots & & \\ g_1 & g_2 & \dots & g_{d-1} & 1 \end{pmatrix}$$

$$\mathbf{Gau}_{T_d}[A(z)] = \begin{pmatrix} & & & \\ & & & \\ \color{red}{\mathbf{Gau}_{T_d}[A(z)]_{d,1}} & \dots & \color{red}{\mathbf{Gau}_{T_d}[A(z)]_{d,d-1}} & \mathbf{Gau}_{T_d}[A(z)]_{d,d} \end{pmatrix}$$

Eliminate $g_1, \dots, g_{d-2}$ and $p(x)$ by solving

$\mathbf{Gau}_{T_d}[A(z)]_{d,1} = \cdots = \mathbf{Gau}_{T_d}[A(z)]_{d,d-1} = 0$ (Riccati Eqs.)

$\mathbf{Gau}_{T_d}[A(z)]_{d,d} = \epsilon\left(\frac{l\cdot H}{x}\right)_{d,d} + g_{d-1} = S(z,\epsilon)$

same as the construction of classical conserved charges in the Drinfeld-Sokolov reduction of the soliton hierarchy.

WKB periods for $D_r^{(1)}$-type linear problem

WKB expansion

$$S_0(x) = 2^{\frac{2}{h}} p(x)^{\frac{1}{h}}, \quad S_1(x) = -\frac{1}{2}\partial_x \log(p),$$

$$S_2(x) = -4^{-1/h}\frac{L_2'}{h}p^{-\frac{1}{h}}x^{-2} - 2^{-\frac{2}{h}-3}\frac{(h+1)(h+2)(2h+1)M^2}{3}p^{-\frac{1}{h}-2}x^{-2+2hM}$$
$$+\frac{4^{-\frac{h+1}{h}}(h+1)(h+2)(hM-1)M}{3h}p^{-\frac{1}{h}-1}x^{-2+hM},$$

$$S_r' = 2p^{-\frac{1}{2}}x^{-r}K_r, \quad K_r := x^r\left(\partial_x - \frac{l_1}{x}\right)\cdots\left(\partial_x - \frac{l_r}{x}\right)\cdot 1$$

WKB period integrals

$$Q_2 = -\frac{2^{-\frac{2}{h}}}{h}J_{1,2}\left(L_2' + \frac{1}{24}h(h+2)(hM-1)\right),$$

$$Q_4 = -\frac{2^{-\frac{6}{h}}}{h}J_{3,4}\Bigg[L_4' - \frac{h-3}{2h}L_2'^2 + \frac{3(h-2)}{2}L_3' - \frac{1}{8}\left[(M-4)h^2 - (6M-9)h - 6\right]L_2'$$
$$+\frac{1}{1920}h(h-6)(h+2)(Mh-3)(Mh-1)(2(M+1)h-1)\Bigg],$$

$$Q_r' = 2J_{-\frac{1}{2},-r}(\epsilon_1, l+\rho)\cdots(\epsilon_r, l+\rho),$$

$Q_{\mathrm{odd}} = 0$, Q_r' is found only in the diagonalization method.

Integrals of motion in W CFT

W minimal model

$W\hat{\mathfrak{g}}$ minimal model: spin $i+1$ currents (i: exponent of $\mathfrak{g}$)

$WA_r^{(1)}$: $W_2, W_3, \ldots, W_{r+1}$
$WD_r^{(1)}$: $W_2, W_4, \ldots, W_{2r-2}$ and W'_r
$WE_6^{(1)}$: $W_2, W_5, W_6, W_8, W_9, W_{12}$

- OPE closure

$$A(z)B(w) = \sum_{1\leq k\leq \Delta_A+\Delta_b} \frac{\{AB\}_k(w)}{(z-w)^k} + \,:\!AB\!:(w) + O(z-w).$$

- quantum Miura transformaltion [Fateev-Lukyanov] $A_r^{(1)}$ and $D_r^{(1)}$
- Quantum Drinfeld-Sokolov reduction of affne Lie algebra $\hat{\mathfrak{g}}$ [Feigin-Frenkel]
- WE_6: WA_5+ a free boson [Keller-Makareeya-Song-Tachikawa]

Free Field realization of W algebras

$\phi_i(z)$: r free bosons with OPE $\phi_i(z)\phi_j(w) \sim -\delta_{ij} \log(z-w)$

energy-momentum tensor

$$T(z) = -\frac{1}{2} :(\partial_z\phi)^2(z): - i\alpha_0\rho \cdot \partial_z^2\phi(z).$$

The central charge

$$c = r - 12\alpha_0^2\rho^2,$$

Quantum Miura transformation WA_r [Fateev-Lukyanov]

$$(\alpha_0\partial_z)^{r+1} - \sum_{k=2}^{r+1} \tilde{W}_k(z)\,(\alpha_0\partial_z)^{r+1-k} =: (\alpha_0\partial_z - \epsilon_1 \cdot i\partial_z\phi)\ldots(\alpha_0\partial_z - \epsilon_{r+1} \cdot i\partial_z\phi):(z)\ .$$

primary field $V_\Lambda(z) =: e^{i\Lambda\cdot\phi(z)}:$

$$\tilde{\Delta}_2 = \Delta_\Lambda = \frac{1}{2}\Lambda(\Lambda+2\rho) = -\sigma_2 - \frac{1}{4}\binom{r+2}{3}\alpha_0^2$$

$$\tilde{\Delta}_3 = \sigma_3 + (r-1)\alpha_0\sigma_2 + \binom{r+2}{4}\alpha_0^3$$

symmetric polynomials in $p_i = (\epsilon_i, \Lambda + \alpha_0\rho)$

$$\sigma_k = \sum_{1\le i_1<i_2<\cdots<i_k\le r+1} p_{i_1}p_{i_2}\ldots p_{i_k}.$$

Free field realization of the W-algebras (2)

WD_r algebra : $W_2 \ldots, W_{2r-2}, R_r,\ c = r - r(2r-1)(2r-2)\alpha_0^2$

$$R_r(z) =\!\!: (\alpha_0\partial_z - i\epsilon_1 \cdot \partial_z\phi(z)) \ldots (\alpha_0\partial_z - i\epsilon_n \cdot \partial_z\phi(z)) :\!\cdot 1.$$

$$R_r(z)R_r(w) = \frac{A_r}{(z-w)^{2r}} + \sum_{k=1}^{r-1} \frac{A_{r-k}}{(z-w)^{2(r-k)}} \left(\tilde{W}_{2k}(z) + \tilde{W}_{2k}(w)\right),$$

$WD_r = WD_{r-1} \oplus [\tilde{\phi}_r]$

$$\begin{aligned}
\tilde{\Delta}_2 &= \frac{1}{2}\sigma_1 - \frac{r(2r-2)(2r-1)}{24}\alpha_0^2, \\
\tilde{\Delta}_4 &= \frac{1}{2}\sigma_2 - \left(\frac{1}{12}(r-7)(r-2)(2r-3)\alpha_0^2 + \frac{1}{4}\right)\sigma_1 + \frac{r(2r-1)(r-1)}{24}\alpha_0^2 \\
&\quad + \frac{1}{720}(r-2)(r-1)r(2r-3)(2r-1)(5r-71)\alpha_0^4,
\end{aligned}$$

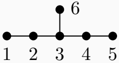
$\sigma_k = \sum_{1\le i_1 < i_2 < \cdots < i_k \le r} p_{i_1}^2 p_{i_2}^2 \ldots p_{i_k}^2, \quad p_i = (\epsilon_i, \Lambda + \alpha_0\rho)$

Free field realization of the W-algebras (3)

WE_6 algebra [Keller-Makareeya-Song-Tachikawa]

- WA_5 $(\tilde{U}_2, \tilde{U}_3, \ldots, \tilde{U}_6)$+a free boson ϕ

  Z_2-automorphism $\tilde{U}_k \to (-1)^k \tilde{U}_k$,



- W-currents

$$T = U_2 - \frac{1}{2}(\partial\phi)^2 - \frac{11}{\sqrt{2}} a i \partial^2 \phi$$

$$W_5 = \tilde{U}_5 - \frac{1}{2}\tilde{U}_3(\partial\phi)^2 + \frac{3}{2}a^2\partial^2\tilde{U}_3 + \frac{3}{2\sqrt{2}}a\partial\tilde{U}_2 i\partial\phi + \sqrt{2}a\tilde{U}_3 i\partial^2\phi$$

  $W_6 = (W_5W_5)_4$, $W_8 = (W_5W_5)_2$, $W_9 = (W_5W_6)_2$, $W_{12} = (W_6W_8)_2$

- central charge $c = 6 - 936a^2 = r - 12\rho^2 a^2$
- W-charges $\tilde{q} = q + a\rho$

$$\Delta_2 = \frac{1}{6}\mathrm{tr}(\tilde{q}\cdot H)^2 - 39a^2, \quad \Delta_5 = \frac{1}{60}\mathrm{tr}(\tilde{q}\cdot H)^5$$

CFT on a cylinder

- On the complex plane, the integrals of motion is constructed as $I_s = \oint dz J_s(T, W, \dots) dz$ with $[I_s, I_{s'}] = 0$.
- the IMs $\hat{J}_s(u)$ on the cylinder is obtained by the conformal map $z = e^u$ ($u = \tau + i\sigma$) from $J_s(z)$
- $J_s(z)$ is made of normal ordered products of W-currents and is not primary field in general.
- A (non-primary) field $A(z) = \sum_n A_n z^{-n-\Delta_A}$ on the complex plane transforms as ($\frac{dz}{du} = z$)

$$\hat{A}(u) = \sum_n \hat{A}_n e^{-nu} = z^{\Delta_A} A(z) (+\delta A(z)) =: z^{\Delta_A} A_R(z)$$

- normal-ordered product on the cylinder

$$:\hat{A}\hat{B}:(v) = \frac{1}{2\pi i} \oint_v du \frac{T(\hat{A}(u)\hat{B}(v))}{u - v} = \frac{1}{2\pi i} \oint_w \frac{dz}{z} \frac{z^{\Delta_A} w^{\Delta_B} R(A_R(z) B_R(w))}{\log \frac{z}{w}}.$$

Normal ordered product on the cylinder

Bernoulli polynomial $\psi_n(x)$: $\frac{(z+1)^x}{\log(1+z)} = \sum_{n=0}^{\infty} \psi_n(x) z^{n-1}$

$$A_R(z)B_R(w) = \sum_{k=1}^{\Delta_A+\Delta_B} \frac{\{A_R B_R\}_k(w)}{(z-w)^k} + \{A_R B_R\}_0(w) + \cdots ,$$

[Dymarksy-Pavlenko-Solovyev,Novaes]

$$:\hat{A}\hat{B}:(v) = \hat{A}_-(v)\hat{B}(v) + \hat{B}(v)\hat{A}_+(v) + \sum_{k=1}^{\Delta_A+\Delta_B} f_k(\Delta_A - 1)\{A_R B_R\}_k(w) w^{\Delta_A+\Delta_B-k},$$

$f_k(x) = \psi_k(x) - \frac{(x)_k}{k!}$

$$\hat{A}_-(v)\hat{B}(v) + \hat{B}(v)\hat{A}_+(v) = w^{\Delta_A+\Delta_B}\{A_R B_R\}_0(w) + \sum_{n=1}^{\Delta_A-1}\sum_{k=1}^{\Delta_A+\Delta_B} w^{\Delta_A+\Delta_B-k}\frac{(\Delta_A-n-1)_{k-1}}{(k-1)!}\{A_R B_R\}_k(w).$$

Example [BLZ]: $\hat{T}(u) = z^2 T(z) - \frac{c}{24}$

$$:\hat{T}\hat{T}:(v) = w^4\{TT\}_0(w) - \frac{c-10}{12}w^2 T(w) + \frac{3}{2}w^3\partial T(w) + \frac{22c+5c^2}{2880}.$$

$$(:\hat{T}\hat{T}:)_0 = 2\sum_{n=1}^{\infty} L_{-n}L_n + L_0^2 - \frac{c+2}{12}L_0 + \frac{5c^2+22c}{2880}.$$

Quantum IM for $WA_r^{(1)}$

charge densities

$$\hat{J}_2(v) = \hat{T}(v), \quad \hat{J}_3(v) = \hat{W}_3(v),$$

$$\hat{J}_4(v) = \hat{W}_4 + a_1 :\hat{T}\hat{T}:(v), \quad a_1 = \frac{3(r-4)\left[(r+2)c + 8r^2 - 18r + 4\right]}{2(5c+22)(r-2)(r-1)r}$$

integrals of motion

$$\mathbf{I}_s = \int_0^{2\pi} \frac{d\sigma}{2\pi} \hat{J}_s(i\sigma).$$

$(\hat{W}_s)_0|\{\Delta_s\}\rangle = \Delta_s|\{\Delta_s\}\rangle.$

$$\mathbf{I}_s|\{\Delta_s\}\rangle = I_s|\{\Delta_s\}\rangle.$$

$$I_2 = \Delta_2 - \frac{c}{24}, \quad I_3 = \Delta_3,$$

$$I_4 = \Delta_4 + a_1\left(\Delta_2^2 - \frac{c+2}{12}\Delta_2 + \frac{5c^2+22c}{2880}\right),$$

$$I_5 = \Delta_5 + b_1\left(\Delta_3\Delta_2 - \frac{6+c}{24}\Delta_3\right).$$

Status: WA_r Lukyanov-Fateev up to 3rd order , I-Zhu, up to 5th order
WA_2: Ashok et al. up to 11th order

Quantum IM for $WD_r^{(1)}$

$$\hat{J}_2(v) = \hat{T}(v),$$

$$\hat{J}_4(v) = \hat{W}_4 + a_1 : \hat{T}\hat{T} : (v), \quad a_1 = -\frac{3(r-4)\left(2cr + c + 16r^2 - 10r\right)}{2(5c+22)(r-1)r(2r-1)}$$

$$\hat{J}_6(v) = \hat{W}_6 + x_1 : T(: TT :) + x_2 : \partial T \partial T :,$$

$$I_2 = \Delta_1 - \frac{c}{24},$$

$$I_4 = \Delta_4 + a_1 \left(\Delta_2^2 - \frac{c+2}{12}\Delta_2 + \frac{5c^2+22c}{2880}\right).$$

WD_4

$$I_6 = \Delta_6 + \frac{(656+11c)(23+52c)}{2646(2c-1)(68+7c)}\Delta_2^3 - \frac{(c+4)(656+11c)(23+52c)}{21168(2c-1)(68+7c)}\Delta_2^2$$
$$+ \frac{(656+11c)(-96+364c+231c^2+26c^3)}{254016(2c-1)(68+7c)}\Delta_2 - \frac{c(656+11c)(60+13c)}{128024064}.$$

ODE/IM correspondence

- ODE : $Q_2 = -h(M+1)J_{1,2}\left[-\frac{1}{h^2(M+1)}s_2 - \frac{r}{24}\right]$
  CFT: $I_2 = -\sigma_2 - \frac{r}{24}$
- $Q_3 = \frac{1}{h}J_{2,3}s_3$
  $I_3 = \sigma_3$
- $Q_4 =$
  $J_{3,4}\left[\frac{1}{h}s_4 - (h-3)\left(\frac{1}{2h^2}s_2^2 + \frac{1+M}{8h}s_2 - \frac{(1+M)^2}{1920}h^2(h-1)(\frac{2hM^2}{1+M} - 9)\right)\right]$
  $I_4 = -\sigma_4 + \frac{r-2}{2(r+1)}\sigma_2^2 + \frac{r-2}{8(r+1)}\sigma_2 + \frac{(r-2)r(9-2(r+1)\alpha_0^2)}{1920(1+r)}$.

ODE/IM Dictionary $A_r^{(1)}, D_r^{(1)}$

$$s_k = (h\sqrt{1+M})^k\sigma_k, \quad \Lambda + \alpha_0\rho = \frac{1}{h\sqrt{1+M}}(l+\rho).$$

$$\alpha_0^2 = \frac{1}{h^2(M+1)}, \quad c_{\text{eff}} = r - \frac{12}{h^2(M+1)}(l+\rho^\vee)^2$$

$E_6^{(1)}$

Diagonalization of the $E_6^{(1)}$ linear problem

[Ide-KI-Kono, work in progress]

27-dimensional representation

$$
\begin{aligned}
E_{\alpha_1} &= E_{1,2} + E_{12,15} + E_{14,17} + E_{16,19} + E_{18,21} + E_{20,22} \\
E_{\alpha_2} &= E_{2,3} + E_{10,12} + E_{11,14} + E_{13,16} + E_{21,23} + E_{22,24} \\
E_{\alpha_3} &= E_{3,4} + E_{8,10} + E_{9,11} + E_{16,18} + E_{19,21} + E_{24,25} \\
E_{\alpha_4} &= E_{4,5} + E_{6,8} + E_{11,13} + E_{14,16} + E_{17,19} + E_{25,26} \\
E_{\alpha_5} &= E_{5,7} + E_{8,9} + E_{10,11} + E_{12,14} + E_{15,17} + E_{26,27} \\
E_{\alpha_6} &= E_{4,6} + E_{5,8} + E_{7,9} + E_{18,20} + E_{21,22} + E_{23,24} \\
E_{-\theta} &= E_{20,1} + E_{22,2} + E_{24,3} + E_{25,4} + E_{26,5} + E_{27,7}
\end{aligned}
$$

$$
\left(\epsilon\partial_x + \epsilon\frac{l\cdot H}{x} + \sum_{i=1}^{r} E_{\alpha_i} + p(x)E_{\alpha_0}\right)\Psi(x) = 0
$$

No way to write down quantum Miura transformation

gauge transformation $A^T = T_{27}^{-1}AT_{27} + \epsilon T_{27}^{-1}\partial T_{27}$

Riccati Eqs. $(A^T)_{27,1} = \cdots (A^T)_{27,26} = 0$ for gauge parameters $g_1,\ldots,g_{26}$.

WKB analysis

$g_{26}(x) = \sum_{n=0}^{\infty} \epsilon^n g_{26}^{(n)}(x)$

$$g_{26} = s_{26}, \quad p(x) = \frac{-45 - 26\sqrt{3}}{9}(s_{26})^{12} = x^{12M} - 1$$

$$g_{26}^{(1)} = -v_5 + 8\frac{\partial s_{26}}{s_{26}}$$

$$g_{26}^{(2)} = \frac{3+\sqrt{3}}{24}\left(\frac{w_2}{x^2 s_{26}} + 39\frac{\partial^2 s_{26}}{s_{26}^2}\right) + \partial(*)$$

$$g_{26}^{(5)} = -\frac{3+2\sqrt{3}}{6}\frac{w_5}{x^5 s_{26}^4} + \partial(*)$$

$w_2 = (l, l+2\rho)$, w_5: order 5 Casimirs in $l+\rho$.

The relations $Q_2 \propto I_2$, $Q_5 \propto I_5$ agree with the ODE/IM dictionary and the numerical analysis NLIE for the Q-function [I-Kondo-Kuroda-Shu]

Outlook

Outlook

ODE/IM Dictionary for A_r, D_r and E_6 (E_7, E_8)

$$Q_2 = const.\left((l, l+2\rho) - \rho^2 \frac{hM-1}{h+1}\right), \quad I_2 = \frac{1}{2}(\Lambda, \Lambda + 2\alpha_0\rho) - \frac{c}{24}$$

$$\Lambda + \alpha_0\rho = \frac{1}{h\sqrt{1+M}}(l+\rho), \quad \alpha_0^2 = \frac{1}{h^2(M+1)}, \; c = r - 12\alpha_0^2\rho^2$$

- IMs in WE_r minimal model
- WKB analysis for non-simply-laced and twisted affine Lie algebras
  Langlands duality $W\hat{\mathfrak{g}}^\vee$-minimal model $\Longleftrightarrow$ linear problem based on $\hat{\mathfrak{g}}$
  $(B_r^{(1)})^\vee = A_{2r-1}^{(2)}$, $(C_r^{(1)})^\vee = D_{r+1}^{(2)}$, $(F_4^{(1)})^\vee = E_6^{(2)}$, $(G_2^{(1)})^\vee = D_4^{(3)}$
- quantum period in E-type Argyres-Douglas therories
  E-type TBA equation [Al.B. Zamolodchikov]
  $(A_2, A_3) \sim (E_6, A_1)$
- $p(x)$ polynomial potential [KI-Mariño-Shu], IMs in Homogeneous sine-Gordon model (generalized parafermion)