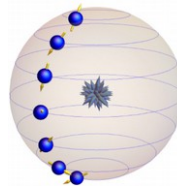


Quantization by LLL Projection, Fuzzy Hemisphere, and Stuff



Mykola Dedushenko



SIMIS

Based on [2407.15948](#) and some old material
(c.f. also [2407.15914](#) by Zhou-Zou)

QFT and Beyond, November 28, Southeast University

Plan

- Formal Story

- ◆ Geometric quantization + Landau quantization = Berezin-Toeplitz (BT) quantization
- ◆ Bosonic particle on LLL \rightarrow scalar BT
- ◆ Superparticle on LLL \rightarrow spinor BT
- ◆ Fuzzy sphere and hemisphere

- Numerical Story

- ◆ Model on fuzzy sphere
- ◆ Boundary criticality in 3d Ising

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Define the physical states as $\psi \in \mathcal{H} \subset \Gamma(M, L)$, such that $\nabla_X \psi = 0$, $X \in P$

(like saying that ψ only depends on positions)

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Let me give a physical derivation of this,
leading to two versions of the BT quantization [MD'10]

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$$\int Dx e^{i \int (\alpha_i \dot{x}^i - H) dt} \quad \text{Where locally, } d\alpha = \omega$$

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- A nicer measure in the phase space path integral is built from:

Symplectic volume form on M, Pf $\omega = \frac{1}{n!} \omega \wedge \dots \wedge \omega = dx^1 \wedge \dots \wedge dx^{2n} \int d\psi^1 \dots d\psi^{2n} e^{\frac{1}{2} \omega_{ij} \psi^i \psi^j}$

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It has supersymmetry: $\delta x^i = i\hbar \epsilon \psi^i, \quad \delta \psi^i = -\epsilon \dot{x}^i$ [Morozov-Niemi-Palo'91]

Every quantum-mechanical system has this hidden SUSY!

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- Again, introduce the μ mass term, while preserving SUSY.

SUSY extension is unique: $\int Dx D\psi e^{\frac{i}{\hbar} \int \mu g(\dot{x}, \dot{x}) dt + \alpha + i\mu \hbar g_{ij} \psi^i (\dot{\psi}^j + \Gamma_{pq}^j \dot{x}^p \psi^q) dt - \frac{i\hbar}{2} \omega_{ij} \psi^i \psi^j dt}$

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$\mu \rightarrow 0$ projects on $\ker Q$

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= supersymmetric ground states of the superparticle on M [MD'11]

= stationary zero-energy one-particle states of the complex spinor field on $M \times \mathbb{R}$

(because $D_{M \times \mathbb{R}} = D_M + \Gamma \partial_t$ and on stationary zero-energy states, $\partial_t = 0$)

Fuzzy Sphere

Applying this to $M = S^2$ with symplectic form: $\omega = n \times \text{Vol} = n \times \frac{i dz \wedge d\bar{z}}{(1+|z|^2)^2}$

i.e., the magnetic flux is n : $L = O(n)$ over $S^2 = \mathbb{C}P^1$

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i.e., the magnetic flux is n : $L = O(n)$ over $S^2 = \mathbb{C}P^1$

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quantization

Bosonic
BT (Landau)

Spinorial
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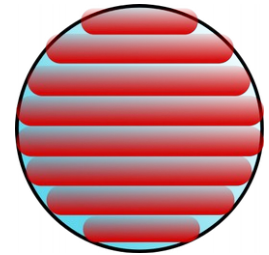
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Writing $z_0 = \cos \frac{\theta}{2} e^{i\varphi/2}$, $z_1 = \sin \frac{\theta}{2} e^{-i\varphi/2}$, the fuzzy orbitals (monopole sphere harmonics):

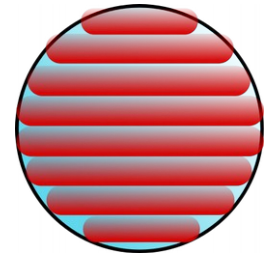
$$z_0^k z_1^{n-k} = \cos^k \frac{\theta}{2} \sin^{n-k} \frac{\theta}{2} e^{i(k-n/2)\varphi} \text{ localized around } \tan^2 \frac{\theta}{2} = \frac{n-k}{k}$$



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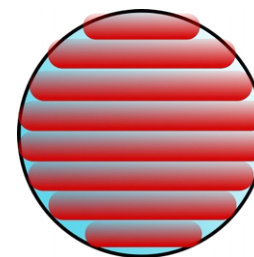


Functions on the fuzzy sphere are $(n+1) \times (n+1)$ matrices acting on these.

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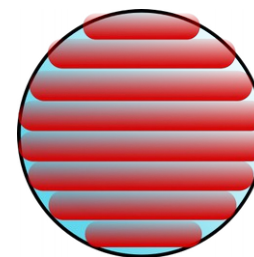
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Also, a fantastic toy model for studying the classical limit of quantum mechanics.
(Topic for a separate discussion)

Fuzzy Hemisphere

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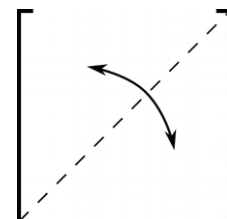
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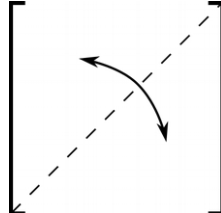
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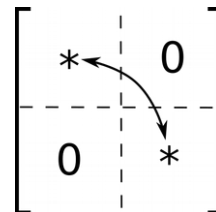
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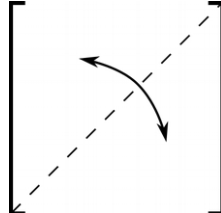
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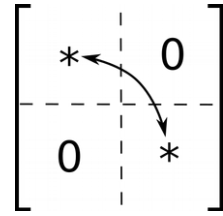
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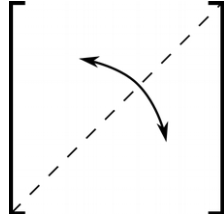
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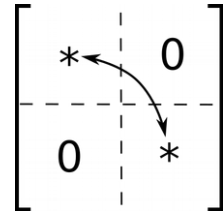
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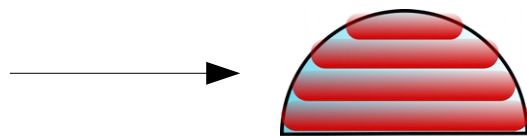
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Now $[n/2]+1$ fuzzy orbitals.

Numerical Story

Ising-like model

Let's study a 3d Ising-like model constructed by [\[Zhu-Han-Huffman-Hofmann-He'22\]](#)

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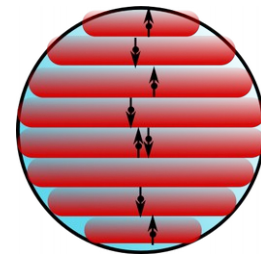
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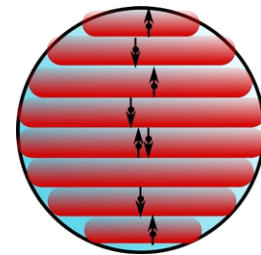
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→ $2N$ available slots. The full Hilbert space of this field on the FS is $\mathbb{C}^{2^{2N}}$

We will focus on the half-filled subspace: $\mathbb{C}^{\binom{2N}{N}} \subset \mathbb{C}^{2^{2N}}$

Ising-like model

Fields decompose into the fuzzy harmonics:

$$\Phi_m \propto e^{im\varphi} \cos^{\frac{n}{2}+m} \frac{\theta}{2} \sin^{\frac{n}{2}-m} \frac{\theta}{2}$$

$$\psi_{\uparrow\downarrow} = \frac{1}{\sqrt{N}} \sum_{m=-n/2}^{n/2} \Phi_m^* c_{m,\uparrow\downarrow},$$

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$$V_{m_1,m_2,m_3,m_4} = \sum_{l=0,1} V_l (2n-2l+1) \begin{pmatrix} n/2 & n/2 & n-l \\ m_1 & m_2 & -m_1-m_2 \end{pmatrix} \begin{pmatrix} n/2 & n/2 & n-l \\ m_4 & m_3 & -m_3-m_4 \end{pmatrix}$$

Where $V_1=1$, V_0 , h – spin-spin interaction strength and transverse magnetic field.

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→ The model is defined on the half-filled Hilbert subspace $\mathbb{C}^{\binom{2N}{N}} \subset \mathbb{C}^{2^{2N}}$

In the (V_0, h) plane, a line of quantum critical points described by 3D Ising CFT.

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- Symmetries:

- ◆ $SO(3)$ rotations of the fuzzy sphere (the main advantage of the fuzzy sphere as a UV regulator)

- ◆ Ising \mathbb{Z}_2 $|\rangle \rightarrow |\rangle, |\uparrow\rangle \rightarrow |\downarrow\rangle, |\downarrow\rangle \rightarrow |\uparrow\rangle, |\uparrow\downarrow\rangle \rightarrow -|\uparrow\downarrow\rangle$

- ◆ Particle-hole P $|\rangle \rightarrow -|\uparrow\downarrow\rangle, |\uparrow\downarrow\rangle \rightarrow |\rangle, |\uparrow\rangle \rightarrow |\uparrow\rangle, |\downarrow\rangle \rightarrow |\downarrow\rangle$

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These symmetries are extremely useful for solving CFT

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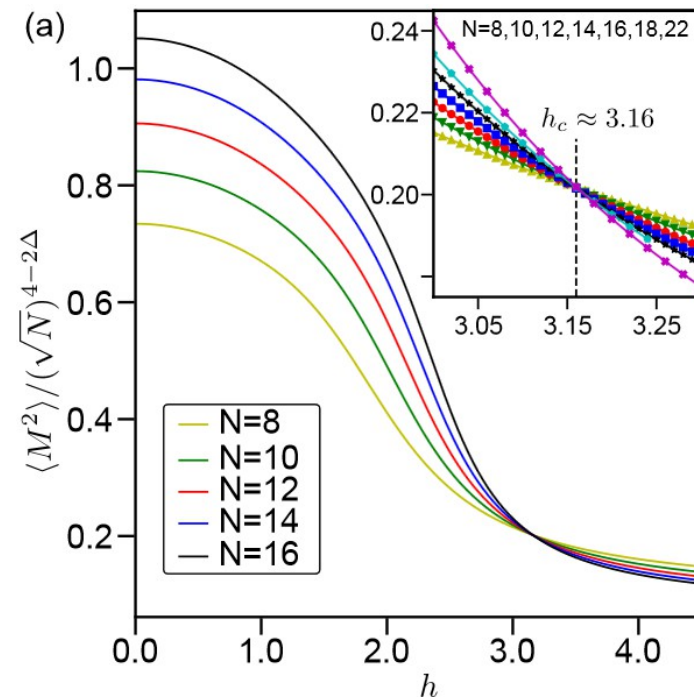
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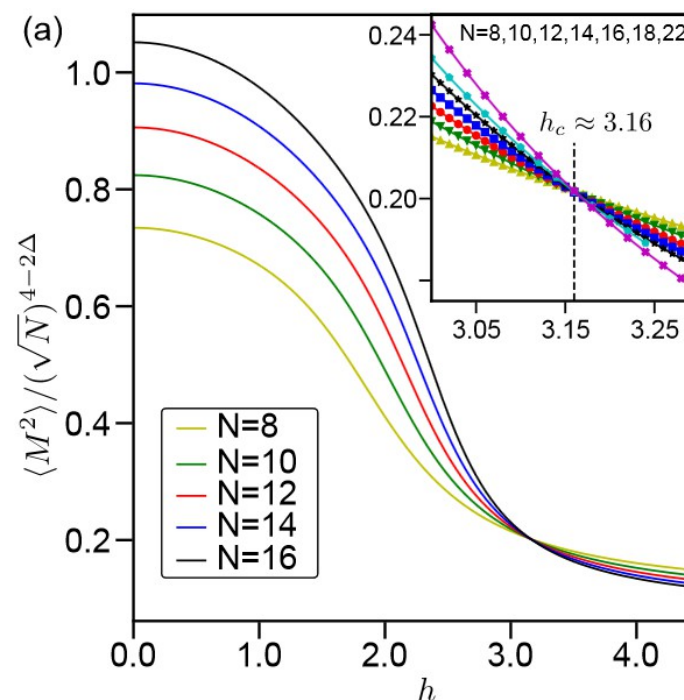
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We will do something similar on the hemisphere...

Shifted Hamiltonian

- Notice that if we replace: $H \rightarrow H + a L^2, \quad a > 0$

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- (Also shifts by irrelevant operators, but at least those are $O(1/N^*)$)

Back to Hemisphere

- Symmetries on the hemisphere:
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What are the possible boundary conditions?

Boundary criticality in 3d Ising

- The standard description

- ◆ Classical Ising model on a 3D lattice with boundary:

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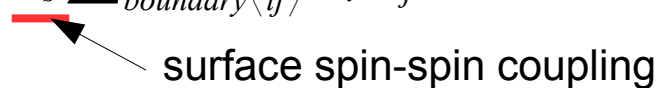
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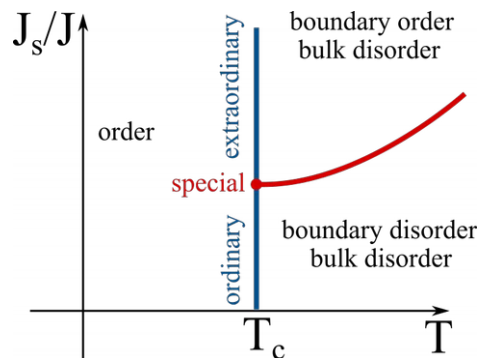
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Extraordinary = normal \oplus normal

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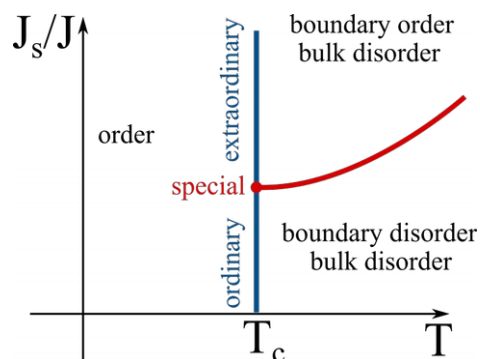
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[Binder'83, Cardy'96, Deng-Blote-Nightingale'05, Diehl'98, Liendo-Rastelli-van Rees'12, Gliozzi-Liendo-Meineri-Rago'15, Metlitski'20, Padayasi-Krishnan-Metlitski-Gruzberg-Meineri'21, Toldin-Metlitski, Trepanier'21]

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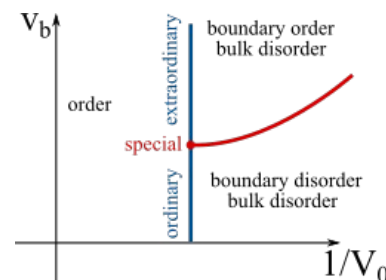
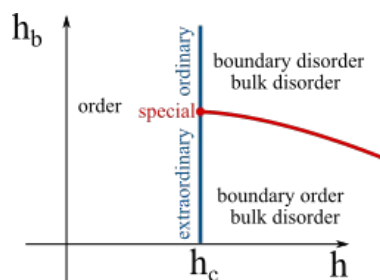
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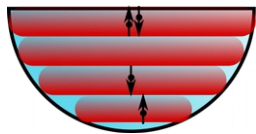
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 $\hat{\sigma}^2$ induces RG flow to ordinary or extraordinary.
 - ◆ Expectation: BCFT as a quantum critical point on the fuzzy hemisphere.



Boundary criticality in 3d Ising

- ◆ How to work with the fuzzy hemisphere:
(always half-filled)

+ boundary interaction
(may break \mathbb{Z}_2 or not)



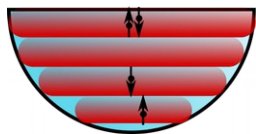
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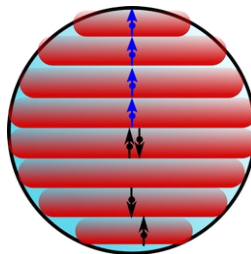
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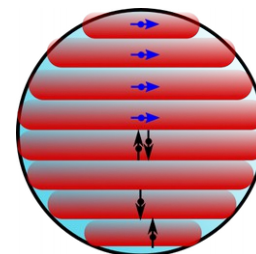


Breaks P

Explicitly breaks \mathbb{Z}_2
(normal b.c. expected)



Preserves \mathbb{Z}_2



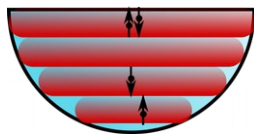
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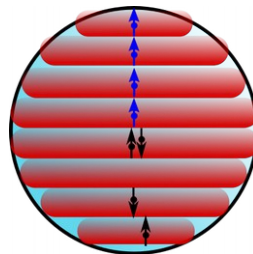
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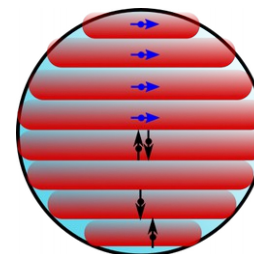
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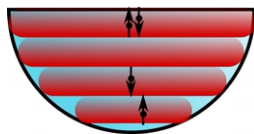
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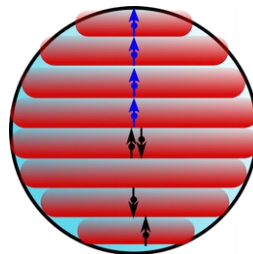
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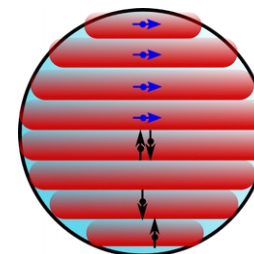
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Preserve P

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+ may add interaction
along the equator

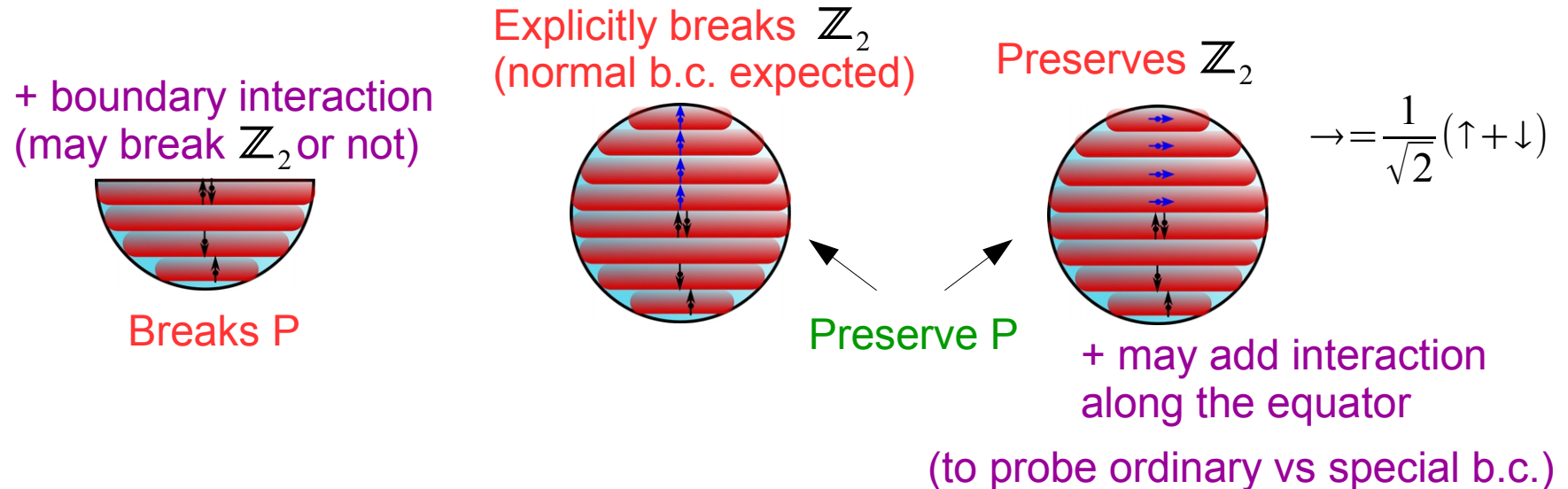
(to probe ordinary vs special b.c.)

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Boundary criticality in 3d Ising

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What kind of boundary/equator interactions are useful?

- ◆ Boundary/equator spin-spin interaction v , e.g. $V_0 \rightarrow V_0 + v$
- ◆ Boundary/equator transverse magnetic field h_b

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Shifting $V_0 \rightarrow V_0 + v$ only along the boundary. $\mathbb{Z}_2 \text{ OK, no P}$

$$v \int_{S^1} n_{\uparrow}(\varphi) n_{\downarrow}(\varphi) d\varphi = \sum_{\substack{m_1+m_2 \\ m_3+m_4}} W_{m_1} W_{m_2} W_{m_3} W_{m_4} c_{m_1,\uparrow}^+ c_{m_4,\uparrow} c_{m_2,\downarrow}^+ c_{m_3,\downarrow}$$

$$v \int_{S^1} n^z(\varphi) n^z(\varphi) d\varphi \quad W_m = N_m / 2^s$$

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$$v \int_{S^1} n_\uparrow(\varphi) n_\downarrow(\varphi) d\varphi = \sum_{\substack{m_1+m_2 \\ m_3+m_4}} W_{m_1} W_{m_2} W_{m_3} W_{m_4} c_{m_1,\uparrow}^+ c_{m_4,\uparrow} c_{m_2,\downarrow}^+ c_{m_3,\downarrow}$$

$$v \int_{S^1} n^z(\varphi) n^z(\varphi) d\varphi \quad \mathbb{Z}_2 \text{ OK, P OK} \quad W_m = N_m / 2^s$$

Seeking critical point

- Introduce the boundary order parameter:

$$b = \frac{1}{2} (c_{-1/2, \uparrow}^+ c_{-1/2, \uparrow} - c_{-1/2, \downarrow}^+ c_{-1/2, \downarrow})$$

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- Conformal perturbation theory like in [Lao-Rychkov'23] (I don't)

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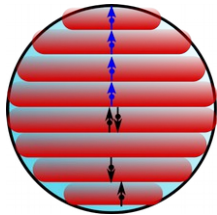
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What we always do: (1) shift the energy spectrum to start at 0;

(2) rescale the energy spectrum to ensure that $D \propto T_{\perp\perp}$ is a dimension 3 scalar.
displacement operator

Normal, or \mathbb{Z}_2 -breaking

- For normal boundary conditions, the best results were achieved via:

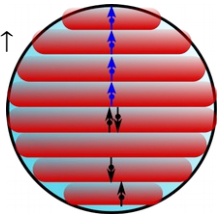


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$$\Pi_{\uparrow} : \mathcal{H} \rightarrow \mathcal{H}_{\uparrow}$$

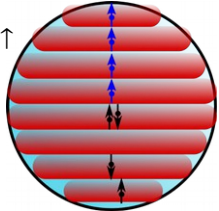
$$H_{hemi} = \Pi_{\uparrow} H$$



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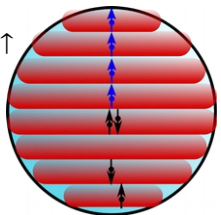
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(additional shift of spectrum at each value of L_z^2)

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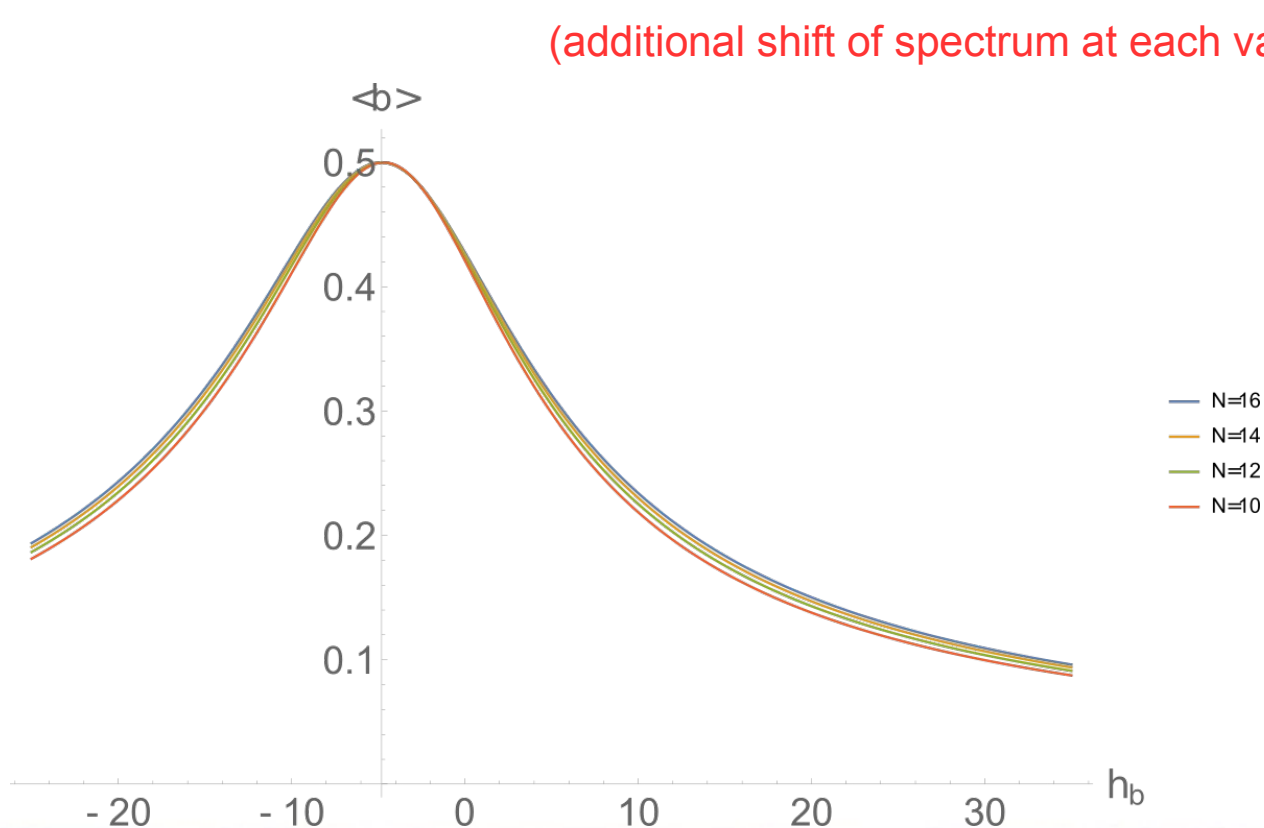
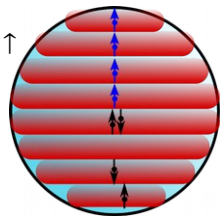
(P acts by $L_z \rightarrow -L_z$, so can only shift by even powers of L_z or else, we break P)

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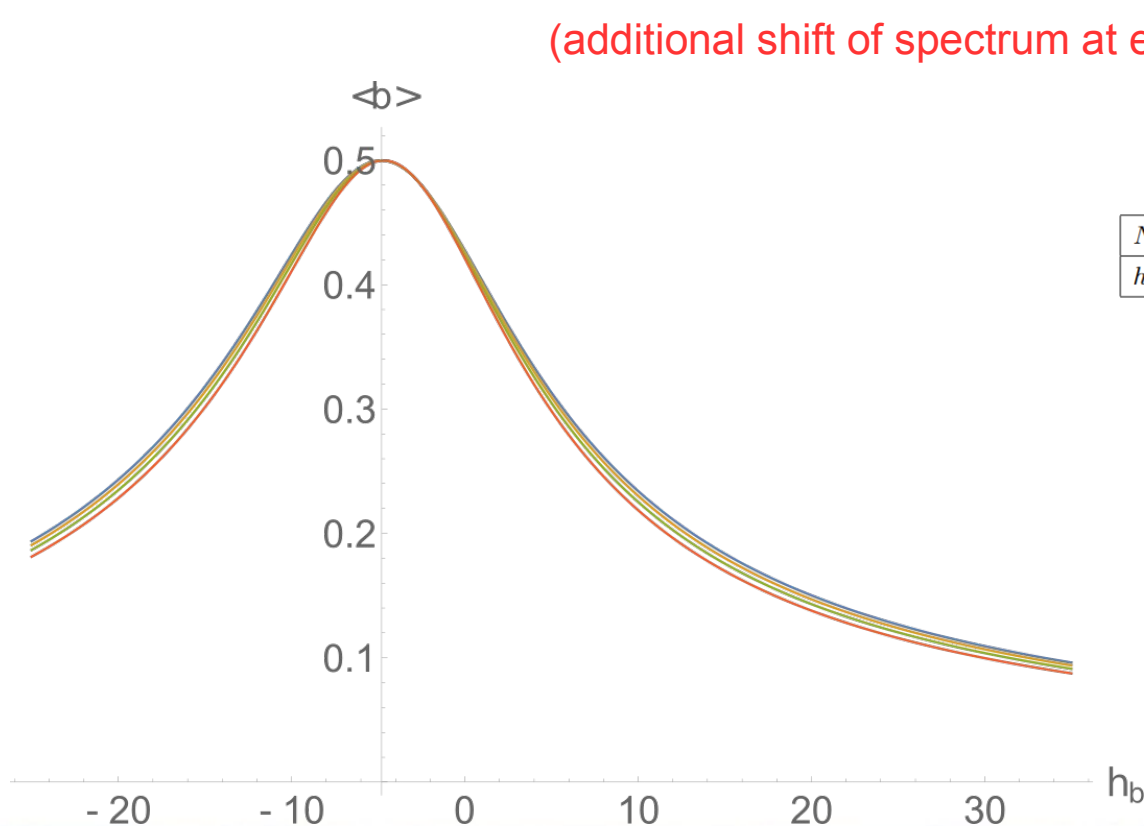
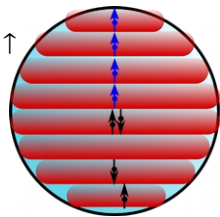
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(additional shift of spectrum at each value of L_z^2)



Maximum at:

N	10	12	14	16	18	20	22
h_b	-4.6775	-4.7470	-4.7985	-4.8386	-4.8709	-4.8978	-4.9204

$N=16$
 $N=14$
 $N=12$
 $N=10$

Normal, or \mathbb{Z}_2 -breaking

- No relevant boundary operators. The spectrum of scalars starts with:

$$D \propto T_{\perp\perp} \quad \text{Whose dimension is 3.}$$

Normal, or \mathbb{Z}_2 -breaking

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N=22

L_z	Dimensions
0	0, 3, 5.088, 5.858, 7.032, 7.819, 8.239, 8.498, 9.075, 9.328, 9.496, 9.860, 10.075, 10.139, 10.358, 10.379, 10.433, 10.514, 10.605, 10.664, 10.684, 10.964, 11.202, 11.400, 11.631, 11.810, 11.927, 12.081, 12.097, 12.146
1	4, 6.225, 6.788, 8.014, 8.783, 9.000, 9.085, 9.445, 9.721, 9.986, 10.168, 10.204, 10.302, 10.361, 10.451, 10.509, 10.862, 10.983, 11.312, 11.533, 11.549, 11.622, 11.855, 11.974, 12.023, 12.159, 12.221, 12.237, 12.402, 12.535
2	5, 7.148, 7.589, 7.816, 8.498, 9.265, 9.501, 9.682, 9.888, 9.908, 10.010, 10.036, 10.126, 10.182, 10.385, 10.662, 10.719, 11.101, 11.227, 11.334, 11.656, 11.760, 11.904, 12.057, 12.077, 12.161, 12.186, 12.203, 12.255, 12.440
3	6, 7.781, 8.400, 8.729, 8.815, 9.275, 9.517, 9.653, 9.744, 9.799, 9.985, 10.526, 10.681, 10.824, 10.987, 11.196, 11.238, 11.287, 11.527, 11.580, 11.681, 11.692, 11.811, 11.887, 12.188, 12.298, 12.428, 12.545, 12.605, 12.736

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$L_z = 0 : \Delta = 3, \quad 5.088, \quad 7.032, \quad 9.075,$ correspond to $D, \partial\bar{\partial}D, (\partial\bar{\partial})^2D, (\partial\bar{\partial})^3D$

$L_z = 1 : \Delta = 4, \quad 6.225, \quad 8.014$ must be $\partial D, \partial(\partial\bar{\partial})D, \partial(\partial\bar{\partial})^2D$

$L_z = 2 : \Delta = 5, \quad 7.148$ must be $\partial^2D, \partial^2(\partial\bar{\partial})D$

$L_z = 3 : \Delta = 6, \quad 7.781$ must be $\partial^3D, \partial^3(\partial\bar{\partial})D.$

$L_z = 0 : \Delta = 5.858, \quad 7.819, \quad 9.860$ correspond to $\mathcal{O}, \partial\bar{\partial}\mathcal{O}, (\partial\bar{\partial})^2\mathcal{O}$

Next scalar primary.

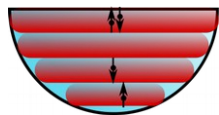
$L_z = 1 : \Delta = 6.788, \quad 8.783$ must be $\partial\mathcal{O}, \partial(\partial\bar{\partial})\mathcal{O}$

Ordinary \mathbb{Z}_2 -preserving

- In this case, real hemisphere and half-frozen sphere both work decently well:

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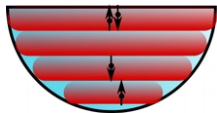


$$\Pi_S: \mathcal{H} \rightarrow \mathcal{H}_S,$$

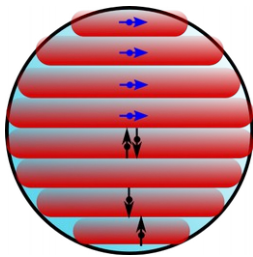
$$H_{hemi} = \Pi_S H \quad (\mathcal{H}_S - \text{states with half-filled Southern and empty Northern hemisphere})$$

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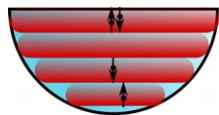


$$\Pi_{\rightarrow}: \mathcal{H} \rightarrow \mathcal{H}_{\rightarrow}, \quad H_{hemi} = \Pi_{\rightarrow} H$$

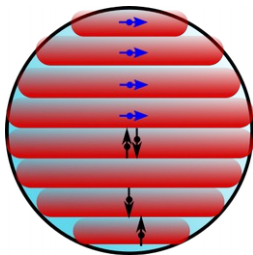
$$\rightarrow = \frac{1}{\sqrt{2}}(\uparrow + \downarrow)$$

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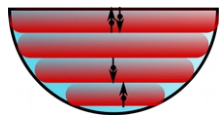


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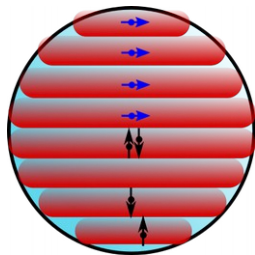
With boundary/equator magnetic field: $-h_b(c_{-1/2,\uparrow}^+ c_{-1/2,\downarrow} + c_{-1/2,\downarrow}^+ c_{-1/2,\uparrow})$

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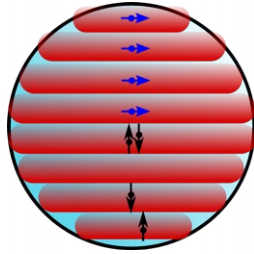
And one of the following spin-spin couplings: $v(c_{-1/2,\uparrow}^+ c_{-1/2,\uparrow} - c_{-1/2,\downarrow}^+ c_{-1/2,\downarrow})^2$

$$v \int_{S^1} n^z(\varphi) n^z(\varphi) d\varphi$$

$$v \int_{S^1} n_{\uparrow}(\varphi) n_{\downarrow}(\varphi) d\varphi$$

Ordinary \mathbb{Z}_2 -preserving

- Main working case:

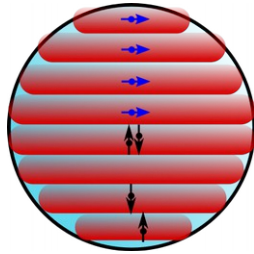


$$-h_b(c_{-1/2,\uparrow}^+ c_{-1/2,\downarrow} + c_{-1/2,\downarrow}^+ c_{-1/2,\uparrow})$$

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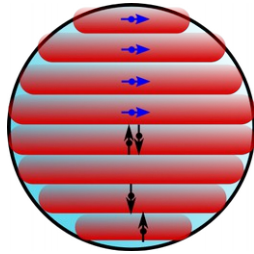
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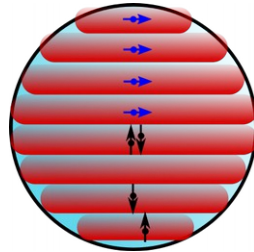
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Also scan over the space of (h_b, v) to make spectrum “more conformal” (minimize cost function)

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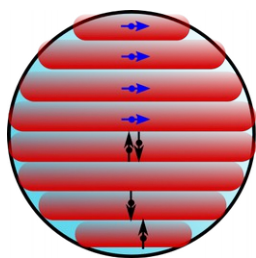
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N=22

L_z	\mathbb{Z}_2	Dimensions
0	+1	0, 3, 4.89, 5.31, 6.28, 6.66, 6.84 , 7.17, 7.42, 7.51
	-1	1.26, 3.26, 4.81, 5.07, 6.10, 6.86, 7.10, 7.35, 7.53, 7.59
1	+1	4, 5.57, 5.77, 6.27, 7.00, 7.09, 7.49, 7.76, 7.93, 8.05
	-1	2.26, 4.26, 5.64, 6.05, 6.67, 7.29, 7.46, 7.47, 7.49, 7.58
2	+1	5, 5.36, 6.76, 6.93, 7.14, 7.46, 7.88, 8.20, 8.51, 8.60
	-1	3.26, 4.91, 6.07, 6.67, 6.88, 6.95, 7.29, 7.33, 7.34, 7.41
3	+1	6, 6.12, 7.25, 7.49, 7.80, 8.00, 8.15, 8.28, 8.61, 8.75
	-1	4.26, 5.50, 6.63, 7.15, 7.21, 7.33, 7.34, 7.40, 7.50, 7.55

Ordinary \mathbb{Z}_2 -preserving

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	-1	4.26, 5.50, 6.63, 7.15, 7.21, 7.33, 7.34, 7.40, 7.50, 7.55

One relevant boundary primary:

$$\hat{\sigma}, \Delta_{\hat{\sigma}} \approx 1.26$$

Descendants:

$$L_z = 0, \quad \Delta = 3.26, 4.81 : \quad \partial \bar{\partial} \hat{\sigma}, \quad (\partial \bar{\partial})^2 \hat{\sigma},$$

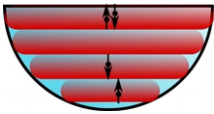
$$L_z = 1, \quad \Delta = 2.26, 4.26 : \quad \partial \hat{\sigma}, \quad \partial (\partial \bar{\partial}) \hat{\sigma},$$

$$L_z = 2, \quad \Delta = 3.26, 4.91 : \quad \partial^2 \hat{\sigma}, \quad \partial^2 (\partial \bar{\partial}) \hat{\sigma},$$

$$L_z = 3, \quad \Delta = 4.26, 5.50 : \quad \partial^3 \hat{\sigma}, \quad \partial^3 (\partial \bar{\partial}) \hat{\sigma}$$

Ordinary \mathbb{Z}_2 -preserving

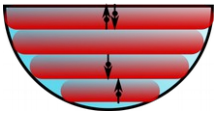
Other possibilities:

• 
$$-h_b(c_{-1/2,\uparrow}^+ c_{-1/2,\downarrow} + c_{-1/2,\downarrow}^+ c_{-1/2,\uparrow}) \quad \text{With:}$$
$$v(c_{-1/2,\uparrow}^+ c_{-1/2,\uparrow} - c_{-1/2,\downarrow}^+ c_{-1/2,\downarrow})^2 \quad \text{or} \quad v \int_{S^1} n_{\uparrow}(\varphi) n_{\downarrow}(\varphi) d\varphi$$

Breaks P, yet, surprisingly, seems to give better precision for the spectrum.

Ordinary \mathbb{Z}_2 -preserving

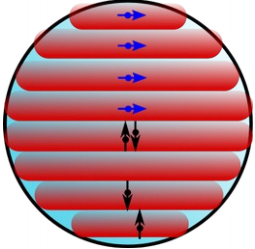
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Breaks P, yet, surprisingly, seems to give better precision for the spectrum.

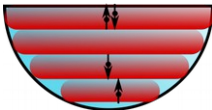
- 

$$\text{With } -h_b(c_{-1/2,\uparrow}^+ c_{-1/2,\downarrow} + c_{-1/2,\downarrow}^+ c_{-1/2,\uparrow}) \quad \text{and} \quad v \int_{S^1} n^z(\varphi) n^z(\varphi) d\varphi$$

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Ordinary \mathbb{Z}_2 -preserving

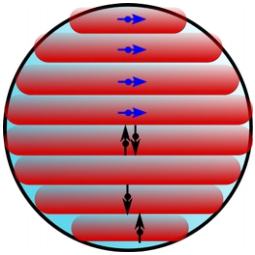
Other possibilities:

- 

$$-h_b(c_{-1/2,\uparrow}^+ c_{-1/2,\downarrow} + c_{-1/2,\downarrow}^+ c_{-1/2,\uparrow}) \quad \text{With:}$$

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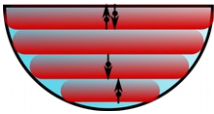
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Δ	0	1.25	3	3.27	4.60	4.95	5.10	5.44	6.04	6.12
\mathbb{Z}_2	+	-	+	-	-	+	-	+	-	+
P	+	+	+	+	+	+	+	+	+	+

Ordinary \mathbb{Z}_2 -preserving

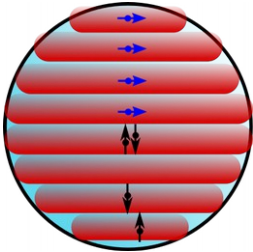
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Need to clean up this mess...

Special \mathbb{Z}_2 -preserving

Partial results only

Special \mathbb{Z}_2 -preserving

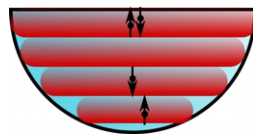
Partial results only

- Strategy: vary the boundary couplings (v, h_b) , identify the special phase transition.

Special \mathbb{Z}_2 -preserving

Partial results only

- Strategy: vary the boundary couplings (v, h_b) , identify the special phase transition.



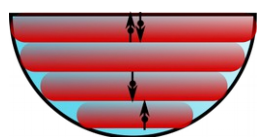
$V_0 \rightarrow V_0 + v$ along the boundary

$$-h_b (c_{-1/2, \uparrow}^+ c_{-1/2, \downarrow} + c_{-1/2, \downarrow}^+ c_{-1/2, \uparrow})$$

Special \mathbb{Z}_2 -preserving

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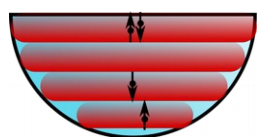
Look at the behavior of the boundary order parameter:

$$\langle b^2 \rangle = \langle 0 | b^2 | 0 \rangle$$

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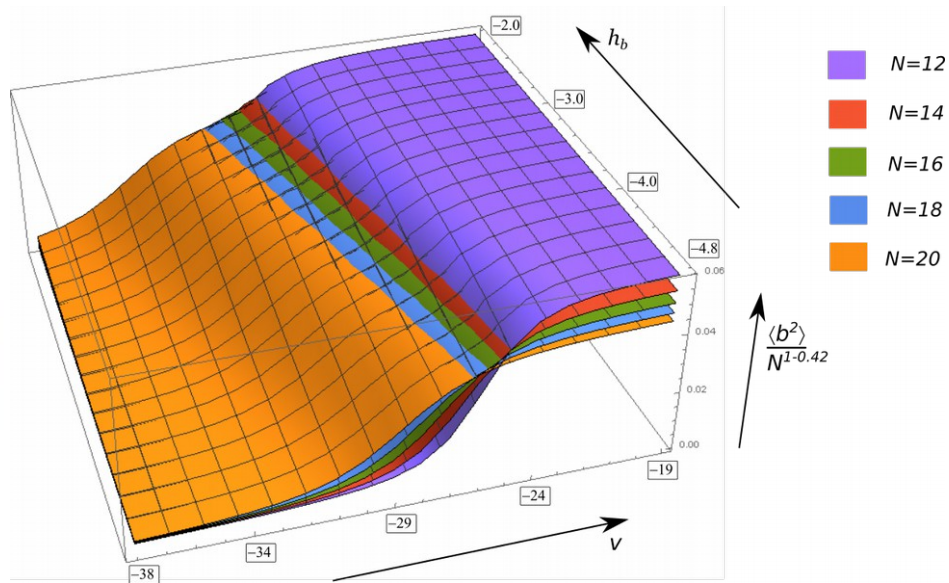
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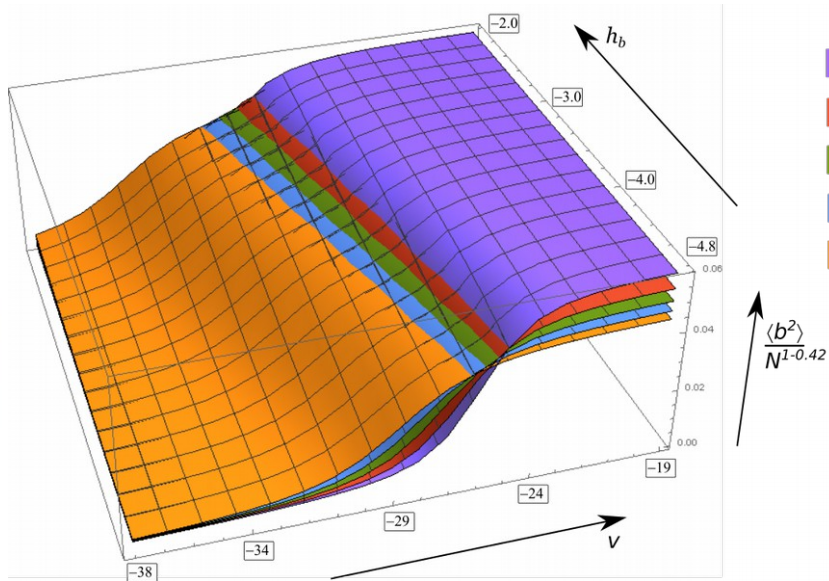
At the *special* BCFT line, we expect: $\langle b^2 \rangle \propto N^{1-\Delta_{\hat{\sigma}}}$, where $\Delta_{\hat{\sigma}} \approx 0.42$
 [Liendo-Rastelli-van Rees'12]

$\langle b^2 \rangle / N^{1-\Delta_{\hat{\sigma}}}$ must be N-independent, like in
 [Zhu-Han-Huffman-Hofmann-He'22]

Special \mathbb{Z}_2 -preserving



Special \mathbb{Z}_2 -preserving



$(h_b, v) = (-1.9, -24)$.

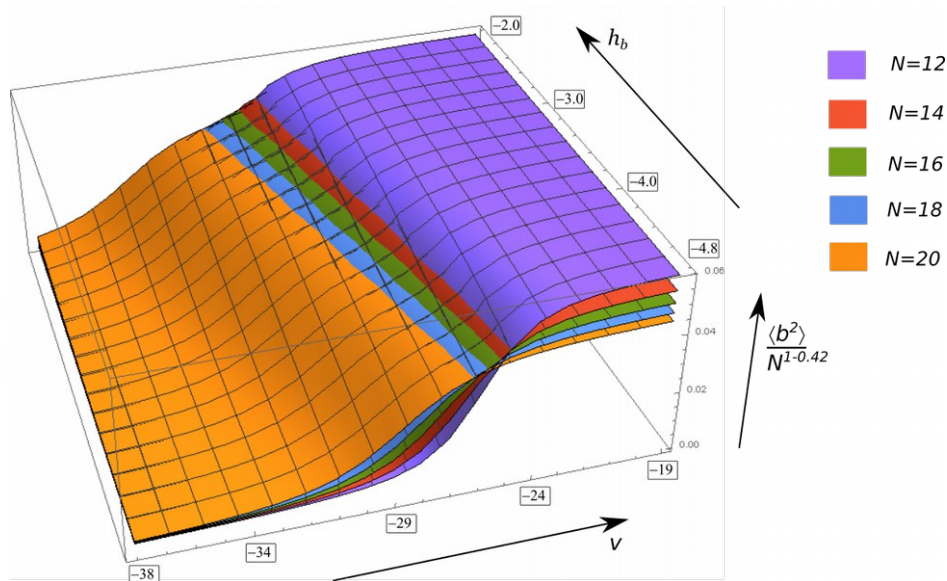
$N=20$

Δ	0	0.42	2.20	2.44	3	3.46	4.28	4.30	4.48	4.90	4.97
\mathbb{Z}_2	+	-	+	-	+	-	-	+	-	-	+

$\hat{\sigma}$ $:\hat{\sigma}^2:$ $\Delta\hat{\sigma}$ D

Four red arrows point from the labels $\hat{\sigma}$, $:\hat{\sigma}^2:$, $\Delta\hat{\sigma}$, and D to the corresponding columns in the table above.

Special \mathbb{Z}_2 -preserving



$(h_b, v) = (-1.9, -24)$ $N=20$

Δ	0	0.42	2.20	2.44	3	3.46	4.28	4.30	4.48	4.90	4.97
\mathbb{Z}_2	+	-	+	-	+	-	-	+	-	-	+

$\hat{\sigma}$ $:\hat{\sigma}^2:$ $\Delta \hat{\sigma}$ D

Tried improving precision by adding :

$$v_1 \int_{S^1} n_{\uparrow}(\varphi) n_{\downarrow}(\varphi) d\varphi$$

$(h_b, v, v_1) = (-2.6, -24, -3.563)$

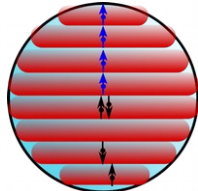
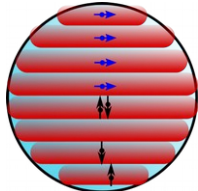
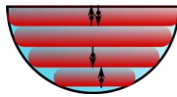
Δ	0	0.42	1.93	2.42	3	3.40	4.16	4.37	4.61	4.92	4.99
\mathbb{Z}_2	+	-	+	-	+	-	+	-	-	-	+

$(h_b, v, v_1) = (-2.6, -26, -2.771)$:

Δ	0	0.42	1.57	2.42	3	3.03	3.78	4.32	4.45	4.64	4.72
\mathbb{Z}_2	+	-	+	-	+	-	+	-	-	+	-

Need a closer look at the boundary couplings...

Take-home messages

- First part:
 - ◆ Quantization by the LLL projection is a powerful and *rigorous* alternative to geometric quantization or brane quantization.
 - ◆ Two versions: particle on M and superparticle on M.
 - ◆ Simplest example: fuzzy sphere.
- Second part:
 - ◆ Ising model on fuzzy hemisphere (or half-frozen sphere) \rightarrow BCFT
 - ◆ Extraordinary/normal b.c. is probed quite well by: 
 - ◆ Ordinary b.c. is probed quite well by:  or  (undecided)
 - ◆ Features of special b.c. are seen, but more analysis is needed.

Take-home messages

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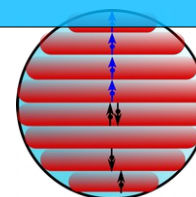
- ◆ Quantization by the LLL projection is a powerful and *rigorous* alternative to geometric quantization or brane quantization.
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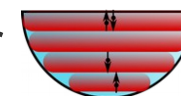
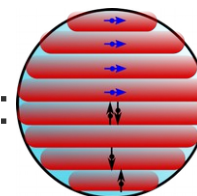
Thank You!

- ◆ Ising model on fuzzy hemisphere (or half-frozen sphere) → BCFT

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