

Surgeries and branched covers for 3d theories

量子场论及其拓展国际研讨会

SEU YC

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Out line

0. Introduction & Motivation
1. Surgery construction
2. Branched cover construction

Ref :

[CCV] Cecotti, Córdova, Vafa, "Braids, Walls, and Mirrors", 1110.2115

[GGP] Gukov, Putrov, "Fivebranes and 4-manifolds", 1306.4320

[SC] 2410.03852, 2310.07624,

Some work in progress with B. Haghighat, M. Romo

Some work in progress with Jiahua Tian

Introduction & Motivation

- Compactify M5-branes on 2-mfds & 3-mfds.

2d / 4d correspondence \rightsquigarrow 3d / 3d correspondence

$$2+4=6$$

$$3+3=6$$

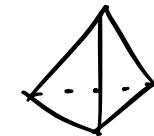
class S, $T[\Sigma_{g,n}]$

class H, $T[M_3]$

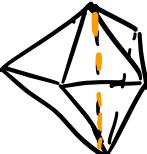
- We will only consider 3d $T[M_3]$. The 3d/3d correspondence is another thing.

- The first construction is DGG construction, [1108.4389]

- Decompose hyperbolic 3-mfds into tetrahedrons



dictionary :  $\xleftrightarrow{1:1}$ Φ , chiral multiplet

internal edge  $\xleftrightarrow{1:1}$ $W=XYZ$ cubic superpotential

- A sign of success : geometrically realize SQED - XYZ duality.

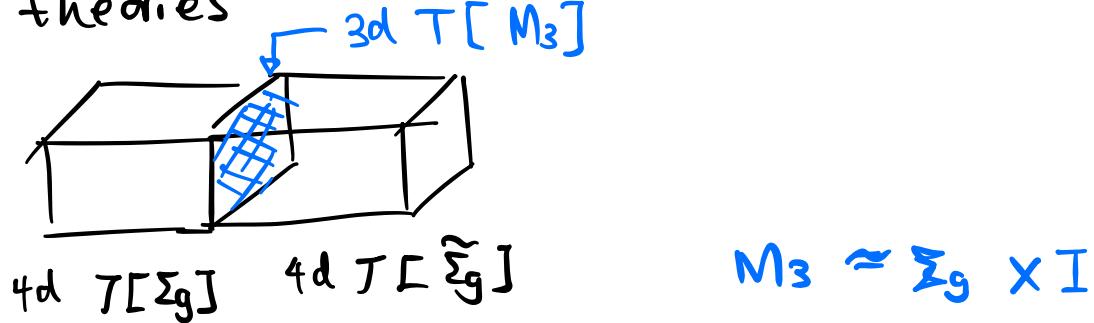
Remark : DGG construction is not complete.

- For instance:
- $\triangle \leftrightarrow \square$ is more like a conjecture
 - Gauge group is not very obvious.

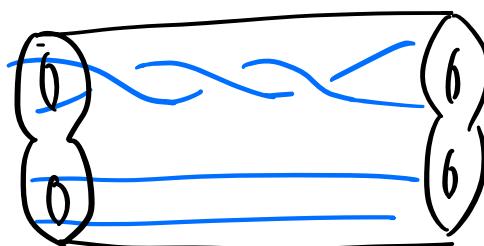
Almost at the same time, [CCV] construction emerges

[CCV] construction

- 3d $T[M_3]$ is considered as the domain wall theory of class S theories



- The time evolution of branched loci of 4d theories gives a braid



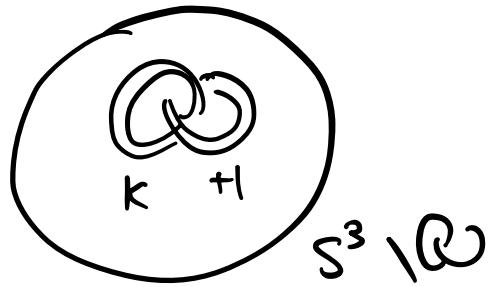
- [CCV] should be more generic than [DGG], because the correspondence between $\begin{array}{c} \text{diamond} \\ \leftrightarrow \\ \Phi \end{array}$ is physically interpreted. The SQED-XYZ duality can also be interpreted.
- However, [CCV] construction (branched cover construction) is also not complete, because the 3-mfds have boundaries, and hence not compact.
 - The braids should be lifted to knots by capping boundaries.
- Not that, these braids & knots are not those for knot complement $S^3 \setminus K$, The knot complement is used for surgeries. This reminds us another construction :

[GGP] "Five branes & 4-mfds"

↳ but many parts of this paper are about 3-mfd.

[GGP] construction

- This construction is inspired by Witten's work 0307041
"SL(2, \mathbb{Z}) action on 3d gauge theory"
- Abelian and pure 3d $N=2$ gauge theories are geometrically realized by surgery properties of 3-mfd.
- $SL(2, \mathbb{Z})$ transformation is identical to Kirby moves.
- Benefit; [GGP] construction could address compact 3-mfd.
- However, [GGP] is also not complete, because matter fields are not considered and introduced.



surgery is an operation of drilling out and gluing back solid tori.

What we are doing ?

- We try to compare and complete surgery construction [GGP] and branched cover construction. [CCV]
- We try to unify or identify [GGP] and [CCV].

What we can use ?

- Mathematical ideas from 3-mfds
- string dualities and brane webs.

Surgery construction

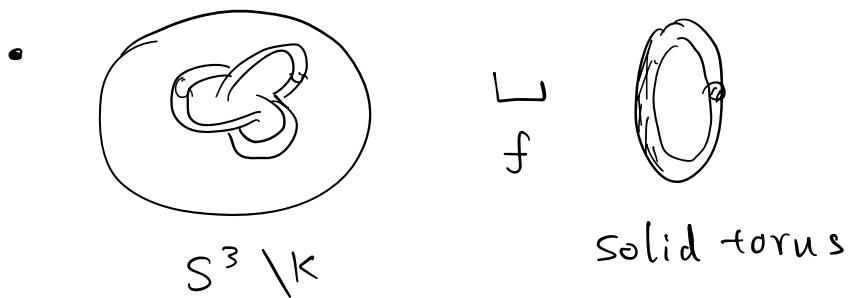
try to complete [GGP]

- How to complete surgery construction ?

At least, we should find a way to add chiral multiplets.

- How to encode $SL(2, \mathbb{Z})$ & superpotentials of SQED-XYZ in the presence of chiral multiplets ?

surgery (introduction)



$$S^3 \setminus K$$

solid torus

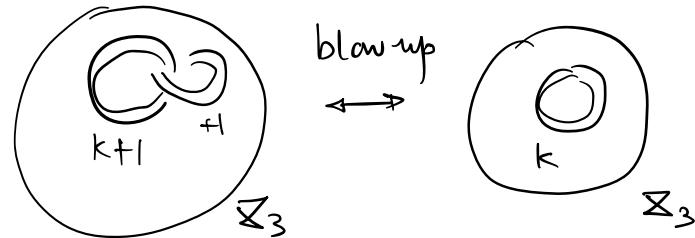
- LW theorem : Any compact 3-mfd can be obtained by surgeries on a link.

Surgeries on a knot can be turned into surgeries on a link by Kirby moves. So, we only need to consider links.

Kirby moves (introduction)

- For a given 3-manifolds, surgeries are not unique, there are infinite many equivalent surgeries related by Kirby moves.

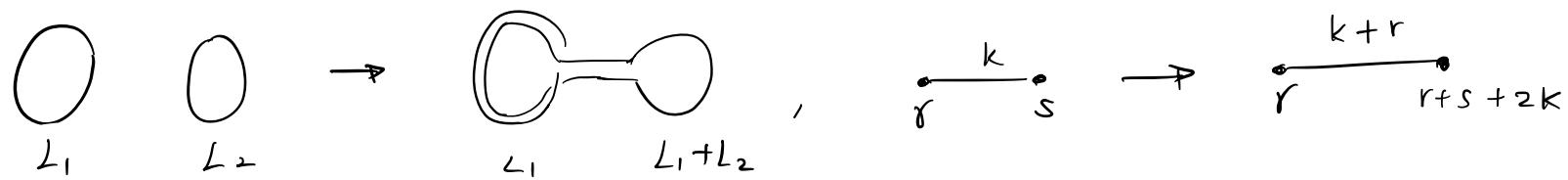
- blow-up/down



plumbed graphs :



- handle slides : Linear recombinations of surgical circles



- An exception : identical surgery



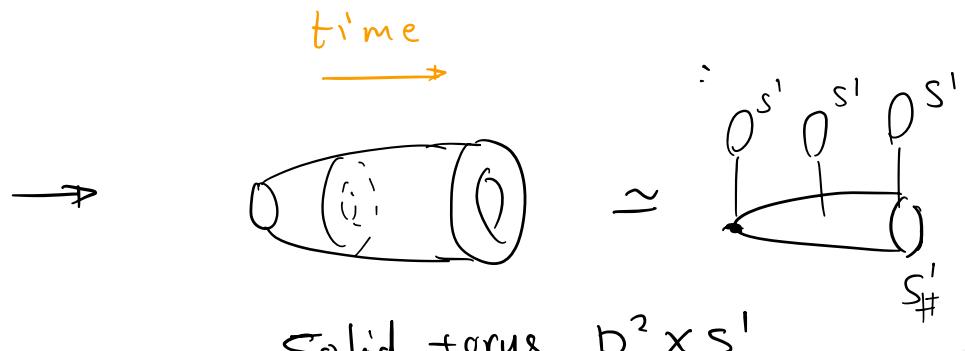
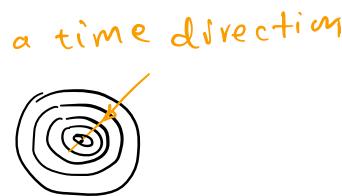
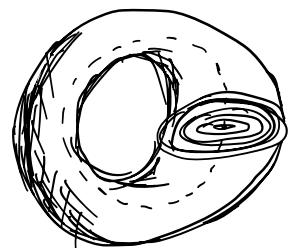
Dictionary between surgery and gauge theory : [QGV]

surgery circle	O_K	\leftrightarrow	$U(1)_K$	gauge group
			\uparrow	Chern-Simons level
blow-up/down	$\begin{array}{c} \bullet \\[-1ex] k+l \end{array} \xrightarrow{\quad} \begin{array}{c} \bullet \\[-1ex] +l \\[-1ex] k \end{array}$	\leftrightarrow	$U(1)_{k+l} \times U(1)$	$\rightarrow U(1)_k$
			\uparrow	Integrate in/out gauge fields.
linking number	$L_i \cdot L_j$	\leftrightarrow	K_{ij}	
			\uparrow	Mixed Chern-Simons levels

- How to understand this dictionary ?

Answer : use M-theory / IIB duality, and foliation.

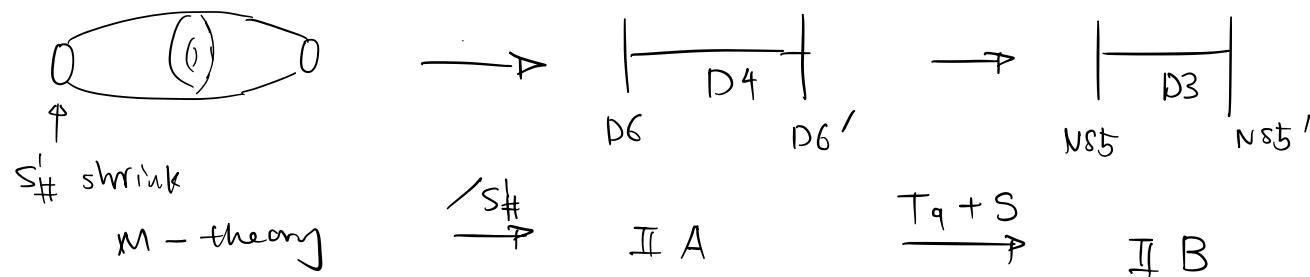
Foliation : A solid torus can be foliated by giving it a time direction

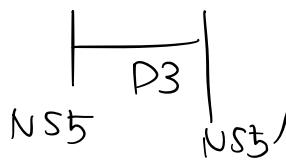


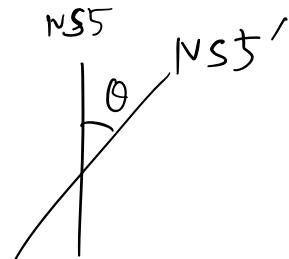
solid torus

- At the tip of the cigar, a circle S^1 shrinks.
- If we put a solid torus in M-theory, shrinking a $S^1_\#$ implies a D0-brane which is EM dual to a D6-brane.

ex1: Lens space $L(K, 1)$, We put a M5-brane on it



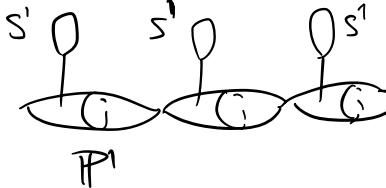
-  is the standard brane web for $3d \ N=2, U(1)_K$, $k = \tan \theta$, θ is the relative angle between NS5 and NS5'.



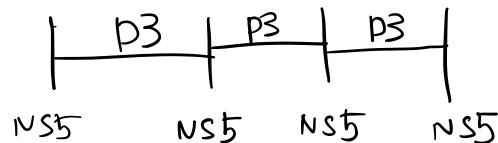
ex 2 : ALE space : a chain of lens space $S^2 \times S^1 = L(0,1)$



\sim



$\downarrow M/\mathbb{Z}B$



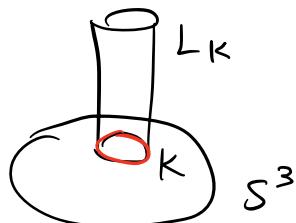
\leftarrow a linear quiver theory $U(1) \times U(1) \times U(1)$
3d $N=4$

Now, we can see that some 3-mfds could realize abelian theories.
Then, how to realize chiral multiplets?

Matter

- Let us use Ooguri - Vafa construction, since it is hard to find other choices.

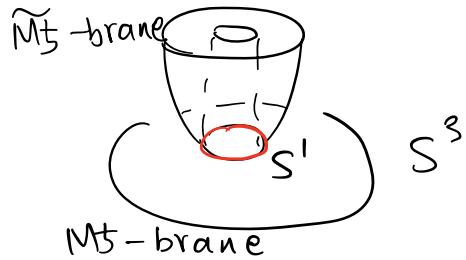
OV-construction : use for engineering open top. string & Wilson loops.



$L_K \cap S^3 = K$, $L_K \subset T^*S^3$ is a Lag. submfld.

- In our cases, $K = \text{unknot}$.

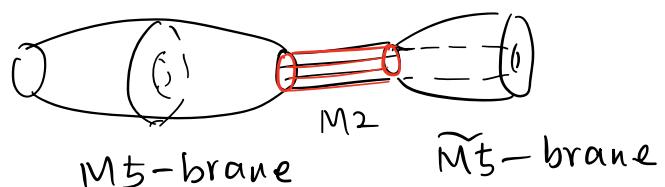
$L_K = D^2 \times S^1$ is a solid torus. So we can draw more precisely:



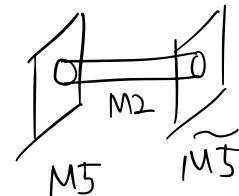
- A question emerges :

Does this unknot unknot link to the surgical circle for $S^3 = L(\pm 1, 1)$?

- To answer this question, let us foliate the three-sphere, then

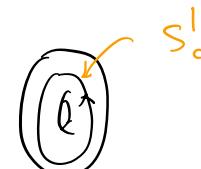


similarly



- M2-branes should stretch between these M5-branes, and give bi fundamental hypermultiplets.

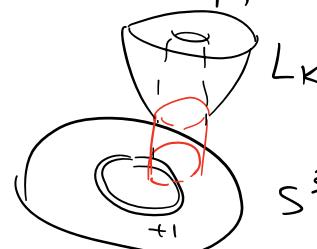
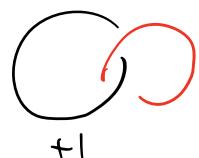
- the unknot unknot $\rightarrow S_q^1$ longitude of



- surgery circle $\rightarrow S_\#^1$ meridian of



Hence, these two circles should form a Hopf link :



More precise
picture for
OV construction

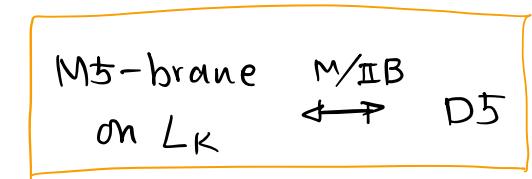
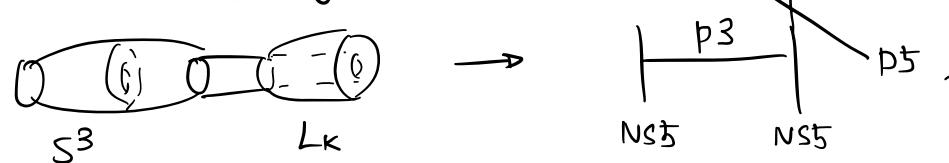
- We can use plumbed graphs to represent



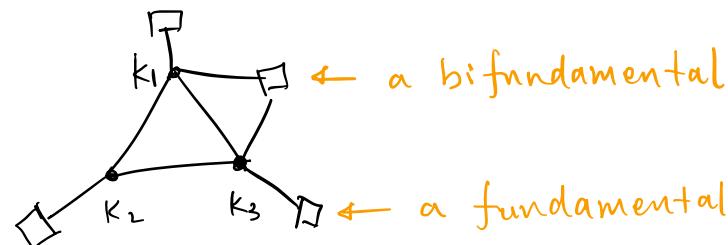
The linking number is the charge of the chiral multiplet. In general, we have

$$\text{we have } \begin{array}{c} \bullet \xrightarrow{q} \square \\ K \end{array}, \quad U(1)_K + \mathbb{Z}_2.$$

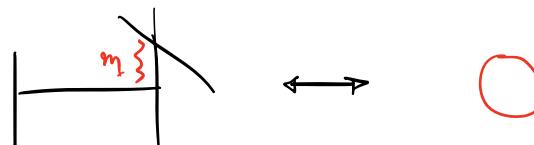
- Use M/IIB-duality again,



- OV construction & 3-mfd are much more generic than 3d brane web, since they realize abelian theories with mixed Chern-Simons levels and charged chiral multiplets. For example,



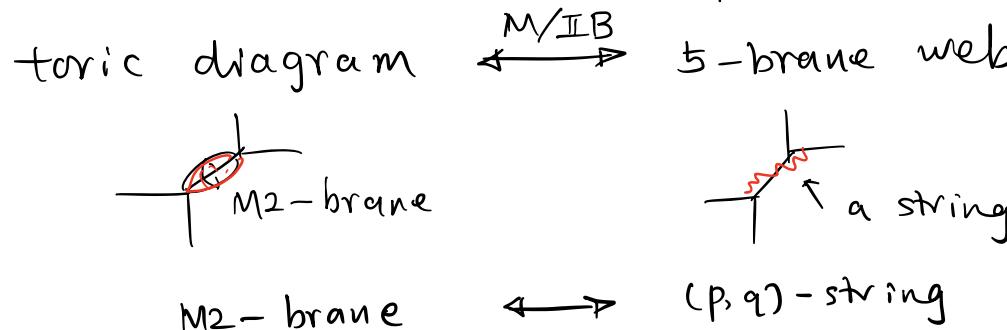
- mass parameter :



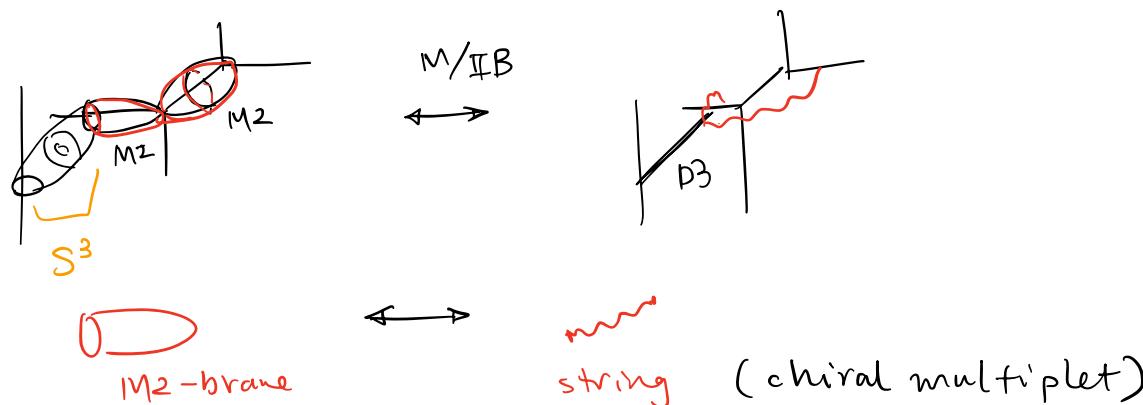
mass parameter is the length of the matter circle.

Decoupling & geometric transition

- A problem left : 1 hyper = 1 chiral + 1 anti-chiral .
How to decouple a chiral multiplet ?
- To begin with, think about the correspondence between toric diagram $\xleftrightarrow{M/IB}$ 5-brane web



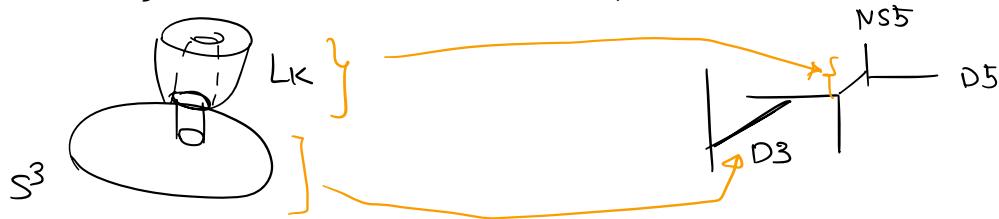
- The geometric transition



- In the toric geometry, the holonomy of O is equivalent to the length of strings, which is the mass parameter.

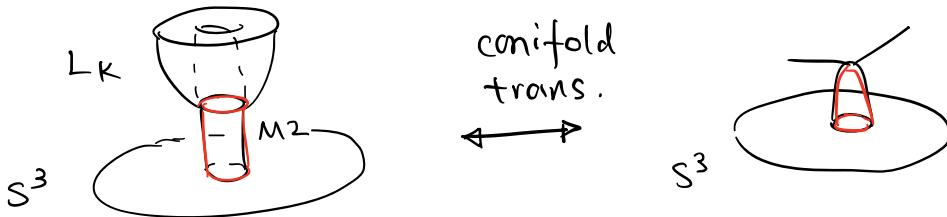
Similarly, the length of surgery circle O_k is the FI parameter.

- There is a fiber-base switch :



- The conifold transition is applied on the L_K , rather than the base S^3 .

So, our 3-mfd & OV construction is consistent with toric diagrams, upto a geometric transition.



- The M2-brane becomes a disc after the transition.

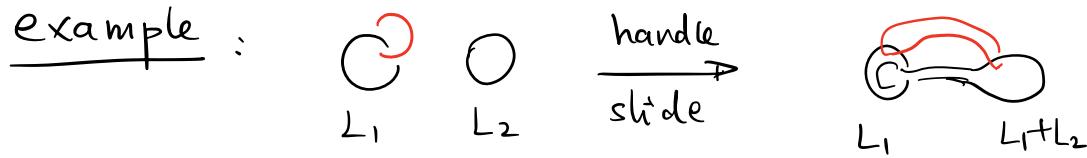
Finally, the problem becomes how the M2-brane ends on a M5-brane.

Kirby move with matter circles

- In [GGP] and previous slides, Kirby moves only apply on surgery circles.

In the presence of matter circles \textcirclearrowleft , Kirby moves still preserve.

\textcirclearrowleft can even extend Kirby moves.



The matter circle could change its locations through Kirby moves.

Extended Kirby moves (Rolfsen twist)

- Recall that



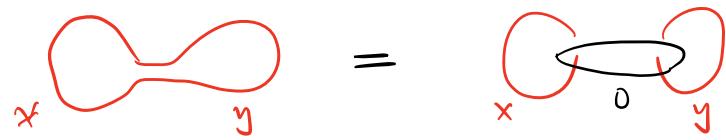
This extended Kirby move is just the 3d duality $1\Phi \xleftrightarrow{ST} U(1)_{\pm 1} + 1\Phi$, which corresponds to the $ST \in SL(2, \mathbb{Z})$.

- This extension is inspired by a trick : on the nbhd of a knot, there is always an identical surgery.

Superpotential & Fusion

- Recall that one success of DGG construction is that it could interpret SQED-XYZ duality, and the superpotential.
- We did not directly find the object for superpotential, but find something more generic, which is the fusion or connected sum of matter circles.

Fusion of matter circles (M2-branes)



The fusion formula : $\frac{(xy, q)_n}{(q, q)_n} = \sum_{k=0}^n \frac{(x, q)_{n-k}}{(q, q)_{n-k}} \cdot \frac{(y, q)_k}{(q, q)_k} \cdot x^k$

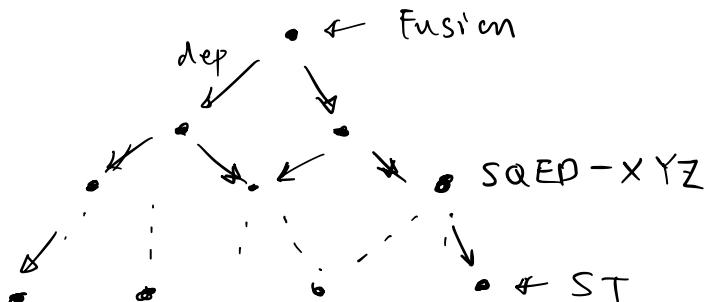
- By decoupling mass parameters x and y in different patterns, one can get the vortex partition functions for SQED-XYZ

Fusion \rightsquigarrow SQED-XYZ.

So, the cubic superpotential $W = XYZ$ is hiding in the fusion.

- Note that partition functions of SQED-XYZ is the pentagon relation, which is derived through quantum relation of Wilson loops, but here everything is classical.

Descendent dualities



- From the fusion, many descendent dualities can be derived. They all have geometric interpretations.

- The above is about how we complete [GGP] construction
- The below is about how to complete [ccv] construction

Branched cover construction

- At least we can cap (close) the boundaries of the braids.

braids $\xrightarrow{\text{Cap}}$ knots

Heegaard splitting

- For a given 3-mfd, there is always a Heegaard splitting

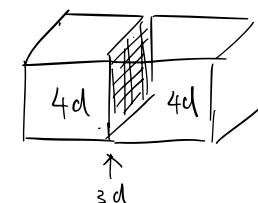


- Question: How to physically understand this splitting?

[CCV] construction (revisit)

- 3d T[Σ_3] is a domain wall theory.
- Consider T[$\Sigma_g \times I$]. Σ_g : Seiberg-Witten curve of 4d Class-S

$$\Sigma_3 = \Sigma_g \times I$$



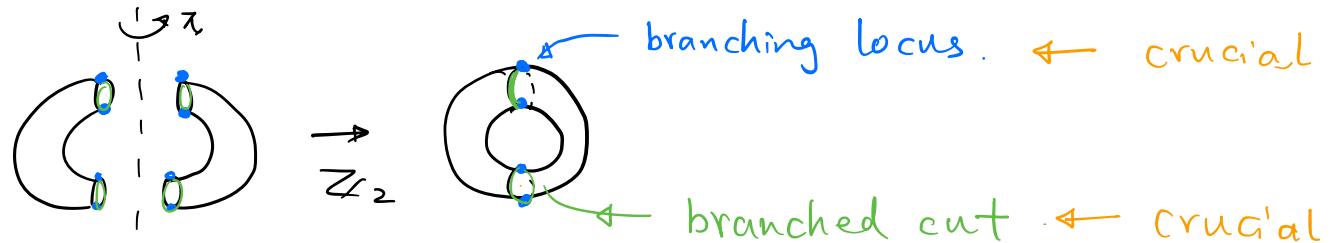
- Move one direction: $\underbrace{\mathbb{R}^3 \times \mathbb{R}}_{\text{4d}} \times \underbrace{\Sigma_g}_{\text{SW}} \simeq \underbrace{\mathbb{R}^3}_{\text{3d}} \times \underbrace{(\mathbb{R} \times \Sigma_g)}_{\text{3-mfd}}$

Double branched cover

- Recall that Σ_g is a double branch cover of a complex plane.

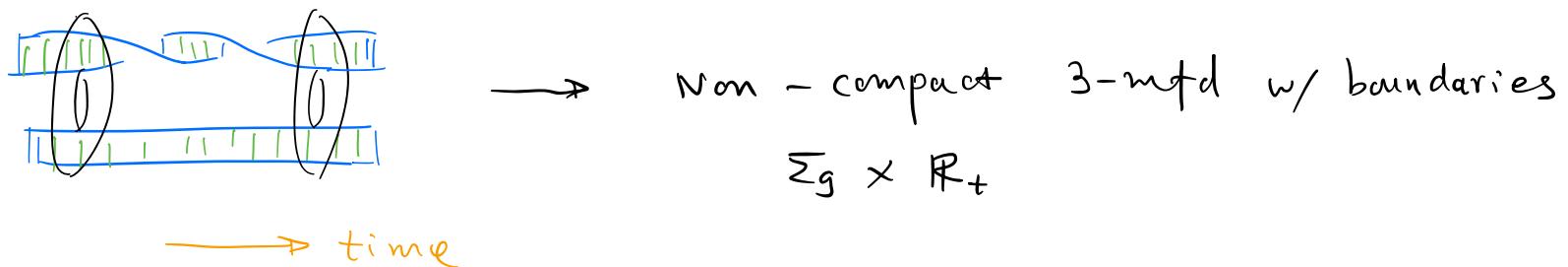
$$\Sigma_g \xrightarrow{2:1} \mathbb{C}$$

For example : A torus $T^2 = S^1 \times S^1$ is



Foliation (time evolution / flow)

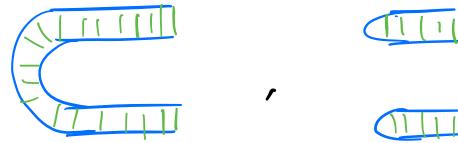
- We can twist (braid) the branched loci,



- Branched cut and branch loci bound a Seifert surface.
- To get compact 3-mfd, we should cap the braid. \rightsquigarrow knot
- Branched knots bound Seifert surfaces, $K = 2S$.

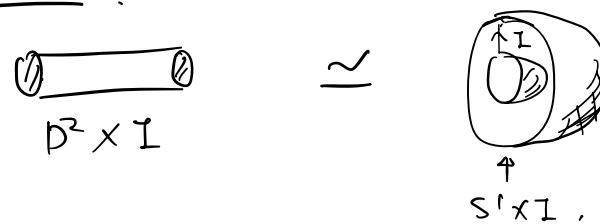
Cap the boundaries

- There are two choices :

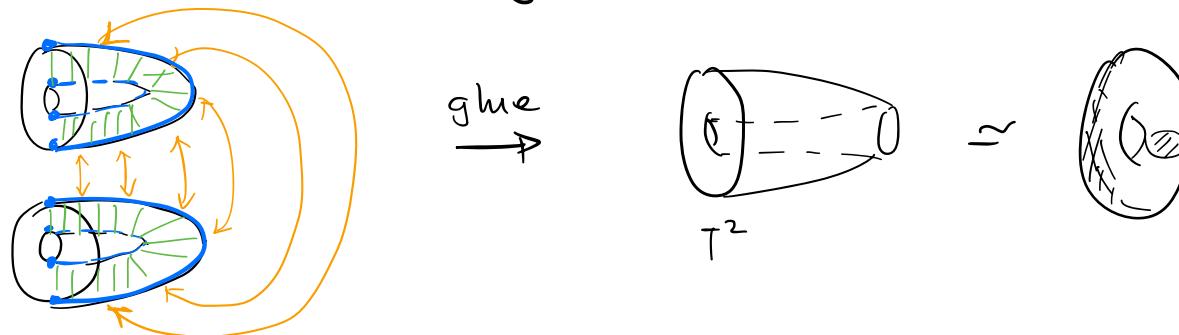


- To find which choice is right, we should rethink about double branched cover in terms of handles.

Handles :



- glue two handles along the boundary (a disc) \rightarrow a solid torus



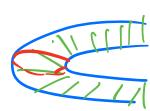
- In this gluing, we can keep track of the branched loci (cuts).

It turns out



branched cut \rightsquigarrow longitude

- Recall that the longitude is the matter circle, which is not shrinkable.



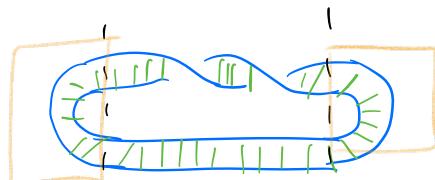
correspond
→



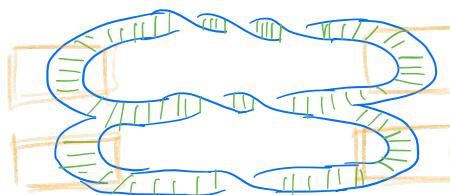
- This shows the branched cover is consistent with the surgery.
- so

Cap the braids

- example :

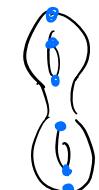


Lens space $L(2,1)$

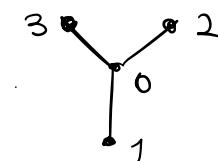


Seifert mfd

genus = 2

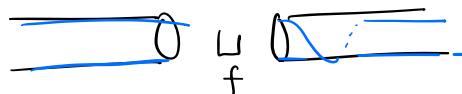


Σ_2



Dehn twist

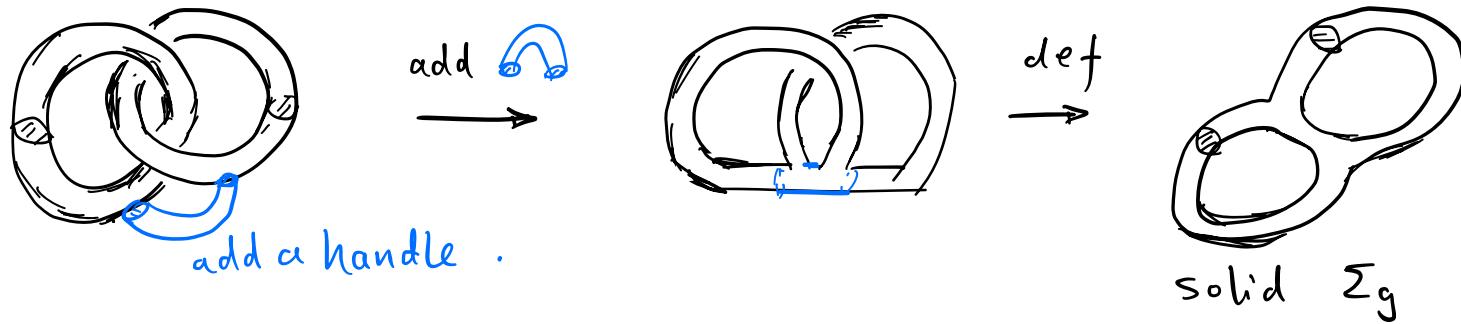
- is a Dehn twist.



Recall that surgery is also Dehn twist. This implies a dictionary between branched cover and surgery.

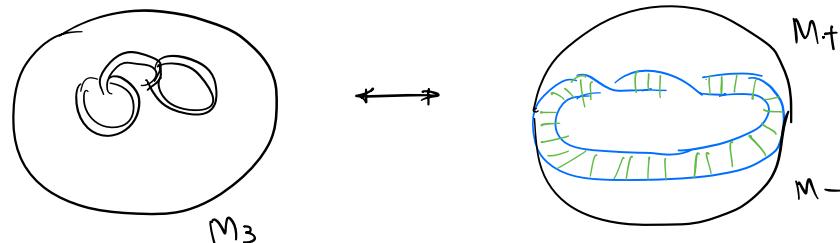
A key technique in the proof of Lw theorem: add handles

- Let us start from a surgical knot, and then add a handle



- So, if we have a surgery, we can always turn it into solid Σ_g .

$$\text{surgery} \xleftrightarrow{\text{handle}} \text{Heegaard splitting}$$



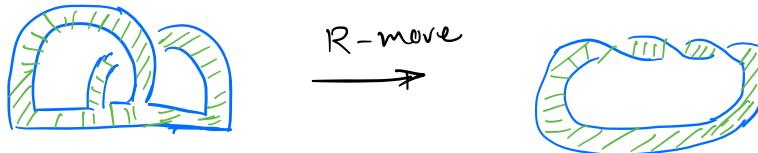
R-move of the knot

- R-move



Question : Could we apply R-moves on branched knots. Answer: Yes.

ex : surgery $\xrightarrow{\text{handle}}$ Seifert surface



Does this change something? Yes, it changes Seifert matrices.

Seifert matrix

-



$S_{ij} = d_i \cdot d_j$ is the linking matrix of the orange circle d_i .

$$\tilde{S}_{ij} = [3]$$



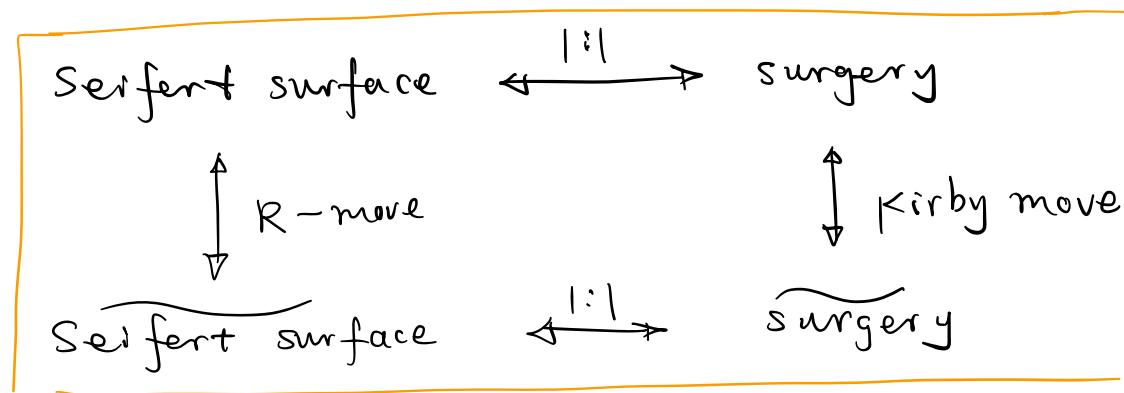
- Obviously, these two Seifert matrices are different.

- Note that, there are many different definitions for Seifert matrices.

- Here we only use this definition, such that branched covers could be translated into surgeries.

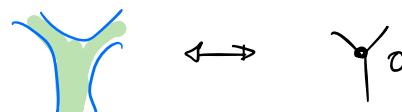
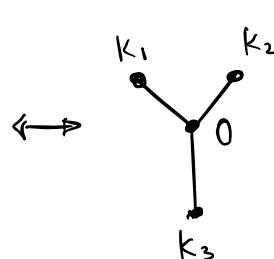
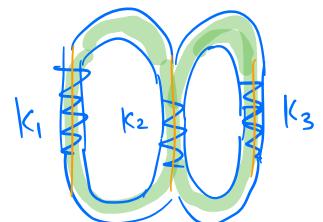
Dictionary

- For a given 3-mfd, there are two equivalent descriptions:



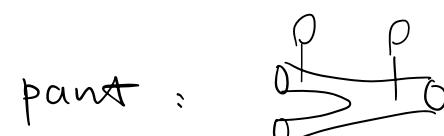
- R-moves of branched knots are translated as Kirby moves of surgery links.

example :



connected sum.

Seifert mfd .



pant :

Remark : • Because of this dictionary , Seifert matrix is interpreted as mixed Chern-Simons levels :

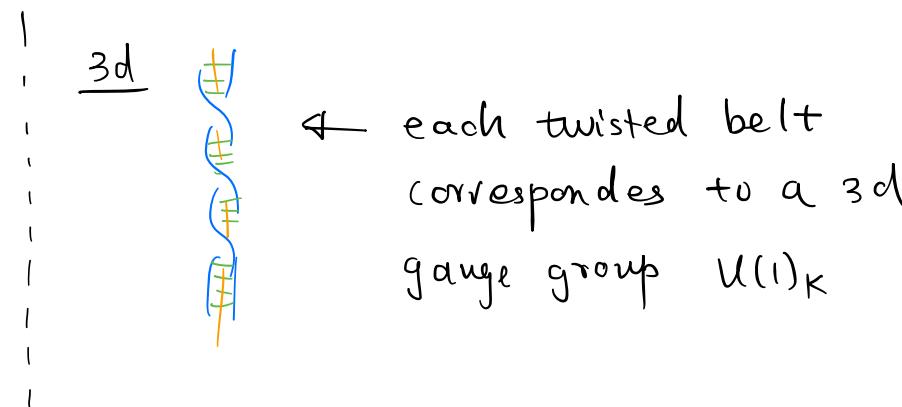
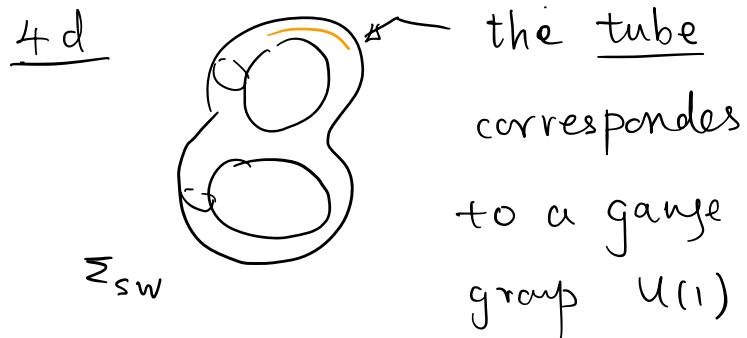
$$S_{ij} = K_{ij}$$

- The orange circles on the Seifert surface correspond to the gauge group $U(1)_k$.
- The twist number \rightsquigarrow Chern-Simons levels.

Unify [GGP] and [CCV]

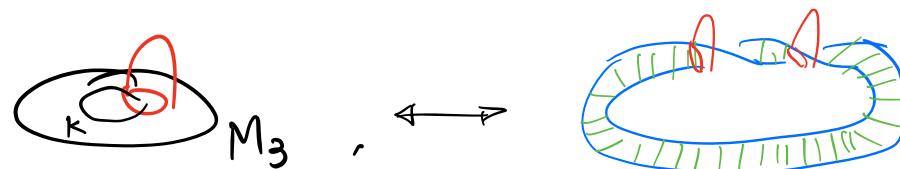
- Because of this dictionary , [GGP] and [CCV] are unified.

Compare with class-S theory

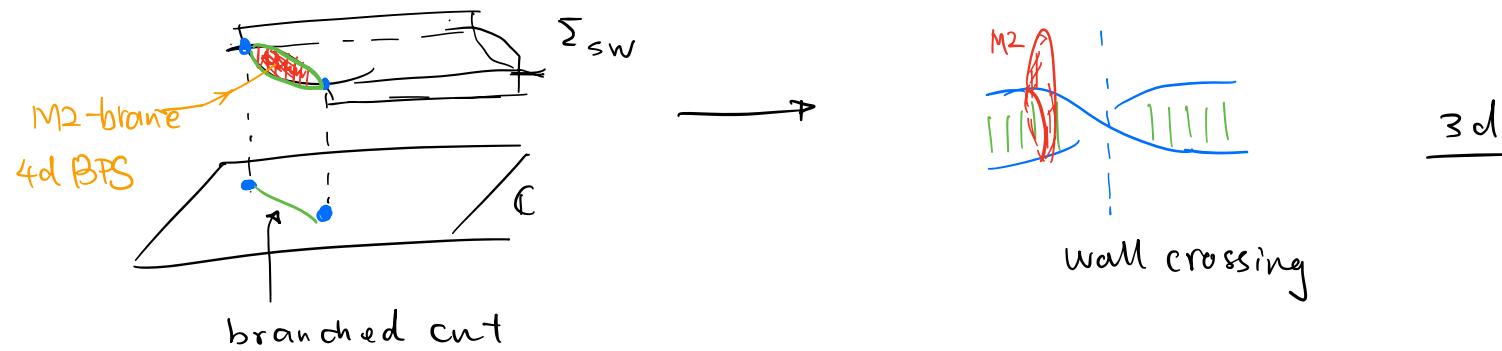


Matter

- In the last section, matter comes from M2-branes ending on a M5-brane.
We can translate this picture to branched covers.



- In class-S theory, 4d BPS has a nice picture,



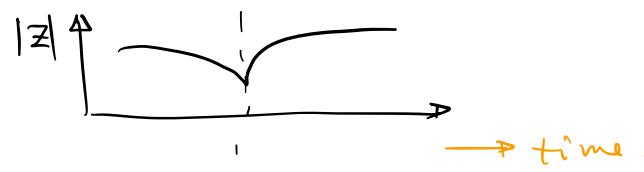
- The boundary of M2-brane is the branched cut.
M2-branes locate on the nbhd of the Seifert surfaces.
- Remark : all interesting things are determined by the Seifert surfaces given by branched knots and cuts.

3d BPS & wall crossing

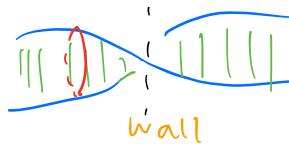
- In [CCV], 3d BPS comes from 4d BPS trapped on the wall (cross).
- The problem is that mass parameter (central charge) in 4d is complex, while in 3d mass parameter is real.

On the wall (cross), the imaginary part $\text{Im}(Z^{4d})$ should vanishes.

Such that the BPS has minimal energy:



$$|Z| = \sqrt{m_r^2 + m_{im}^2}$$



$$m_{CPX}^{4d} = \int w + iB = \int x + i\tilde{A} \xrightarrow{\text{wall}} \int \lambda$$

$\uparrow M_2$ \uparrow
 Kähler B field

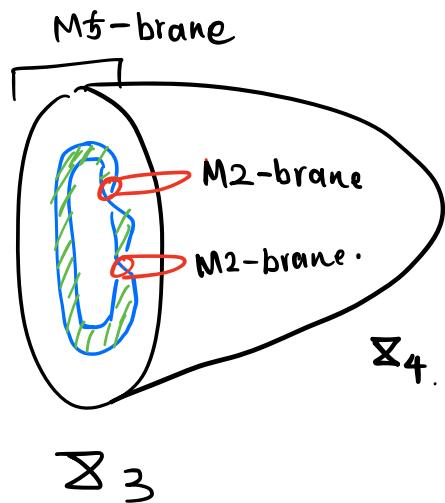
Since 3-mfd is Lagrangian submanifold, $F + B \Big|_{S_3} = 0$.

$F = dA$ is from the gauge field on 3-mfds.

The cross \times is a reflection, which rules out F and hence B .

4-mfd

- 3-mfd is the boundary of a 4-mfd : $\Sigma_3 = \partial \Sigma_4$



We wish this could has connections to Witten's work.

Summary :

Section 1 :

- We introduce chiral multiplet to [GGP] construction
- We find surgery interpretations for many 3d dualities

Section 2 :

- We find a way to cap the boundaries of braids.
- We find the dictionary and unify [GGP] and [CCV].
- More details on 3d BPS are coming, such as punctures.
- Results can be obtained from only the topology of 3-mfd's.

Outlook :

- short term : 3d rank 0, R-twist
- long term : knot homology

Thank you !