Surgeries and branched covers for 3d theories

量子场论及其拓展国际研讨会

SEUYC

shi cheng

SIMIS & Fudan

2025-11-27

Outline

0. Introduction & Motivation
1. Surgery construction
2. Branched cover construction

Ref:

[CCV] Cecotti, Córdova, Vafa, "Braids, Walls, and Mirrors". 1110.2115

[GGP] Gadde, Gukov, Putrov, "Fivebranes and 4-manifolds", 1306.4320

[SC] 2410.03852, 2310.07624,

Some work in progress with B. Haghighat, M. Romo

Some work in progress with Jiahua Tian

Introduction & Motivation

- Compactify M5-branes on 2-mfds & 3-mfds.

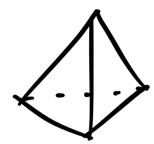
  2d / 4d correspondence $\rightsquigarrow$ 3d / 3d correspondence

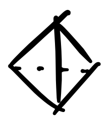
  $2+4=6$ $\qquad$ $3+3=6$

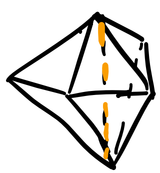
  class S, $T[\Sigma_{g,n}]$ $\qquad$ class H, $T[M_3]$

- We will only consider 3d $T[M_3]$. The 3d/3d correspondence is another thing.

- The first construction is DGG construction, [1108.4389]

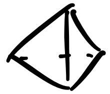
- Decompose hyperbolic 3-mfds into tetrahedrons 

  dictionary :  $\xleftrightarrow{1:1}$ Φ, chiral multiplet

  internal edg  $\longleftrightarrow$ $W = XYZ$ cubic superpotential

- A sign of sucess : geometrically realize SQED − XYZ duality.

Remark: DGG construction is not complete.

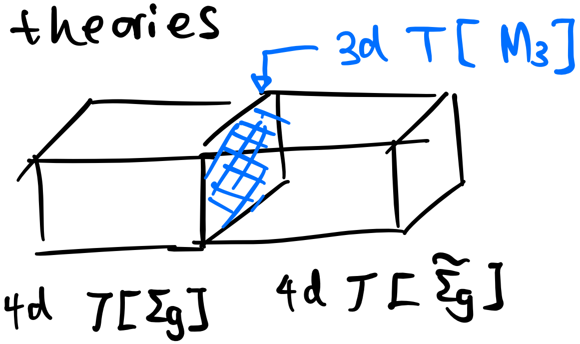
For instance: •  $\leftrightarrow \Phi$ is more like a conjecture

• Gauge group is not very obvious.

Almost at the same time, [CCV] construction emerges

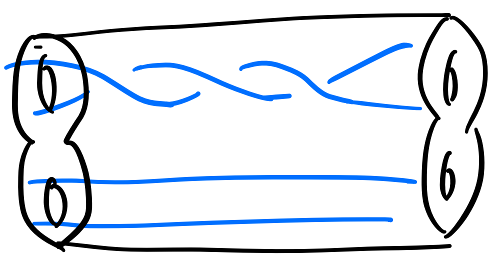
[CCV] construction

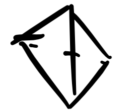
• 3d $T[M_3]$ is considered as the domain wall theory of class S theories



$M_3 \simeq \Sigma_g \times I$

• The time evolution of branched loci of 4d theories gives a braid



- [CCV] should be more generic than [DGG], because the correspondence between  $\leftrightarrow$ Φ is physically interpreted. The SQED-XYZ duality can also be interpreted.

- <u>However</u>, [CCV] construction (branched cover construction) is also <u>not</u> complete, because the 3-mfds have boundaries, and hence not compact.
  - The <u>braids</u> should be lifted to <u>knots</u> by capping boundaries.

branched loci

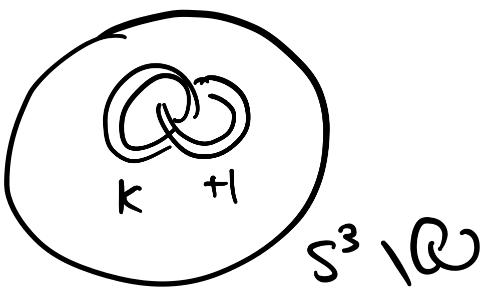
- <u>Not that</u>, these <u>braids & knots</u> are not those for knot complement $S^3 \backslash K$, The <u>knot complement</u> is used for <u>surgeries</u>. This reminds us another construction:

  [GGP] "Fivebranes & 4-mfds"

  ↳ but many parts of this paper are about 3-mfd.

[GGP] construction

- This construction is inspired by Witten's work 0307041
  "$SL(2,\mathbb{Z})$ action on 3d gauge theory"
- Abelian and pure 3d $\mathcal{N}=2$ gauge theories are geometrically realized by surgery properties of 3-mfds.
- $SL(2,\mathbb{Z})$ transformation is identical to Kirby moves.
- <u>Benefit</u>: [GGP] construction could address compact 3-mfd.
- However, [GGP] is also <u>not complete</u>, because matter fields are not considered and introduced.



surgery is an operation of drilling out and gluing back solid tori.

what we are doing ?

- We try to compare and complete surgery construction [G G P]
  and branched cover construction [C C V].
- We try to unify or identify [G G P] and [C C V].

what we can use ?

- Mathematical ideas from 3-mfds
- string dualities and brane webs.

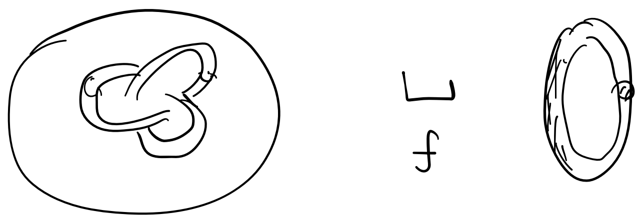
Surgery construction

try to complete [GGP]

- How to complete surgery construction?

  At least, we should find a way to add chiral multiplets.
- How to encode $SL(2,\mathbb{Z})$ & superpotentials of SQED-XYZ in the presence of chiral multiplets?

surgery (introduction)

- 

$S^3 \backslash K$ $\sqcup_f$ solid torus

- LW theorem: Any compact 3-mfd can be obtained by surgeries on a link.

  Surgeries on a knot can be turned into surgeries on a link by kirby moves.

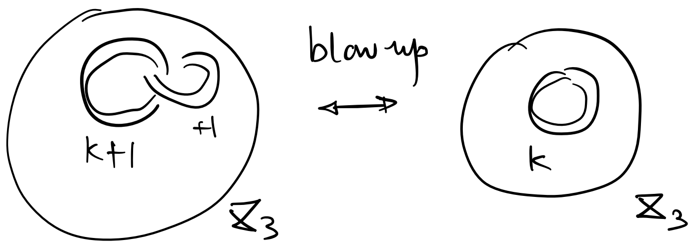
  So, we only need to consider links.

kirby moves (introduction)

- For a given 3-mfd, surgeries are not unique, there are infinite many equivalent surgeries related by kirby moves.

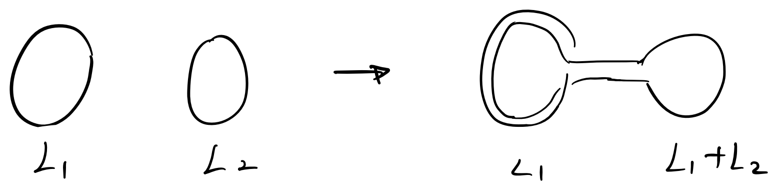
- blow-up/down

plumbed graphs:

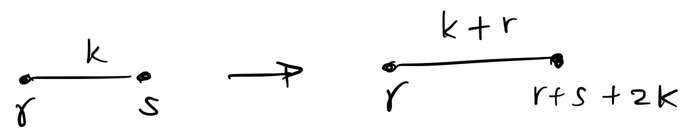


,

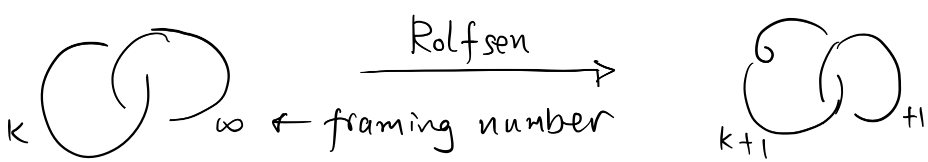
- handle slides: Linear recombinations of surgical circles



,



- An exception: identical surgery



Dictionary between surgery and gauge theory : [GGV]

surgery circle $\bigcirc_k$ $\longleftrightarrow$ $U(1)_K$ gauge group

$\uparrow$ Chern-Simons level

blow-up/down $\underset{k+1}{\bullet}\!\!-\!\!\underset{+1}{\circ} \to \underset{k}{\bullet}$ $\longleftrightarrow$ $U(1)_{k+1} \times U(1) \longrightarrow U(1)_k$

$\uparrow$ Integrate in/out gauge fields.

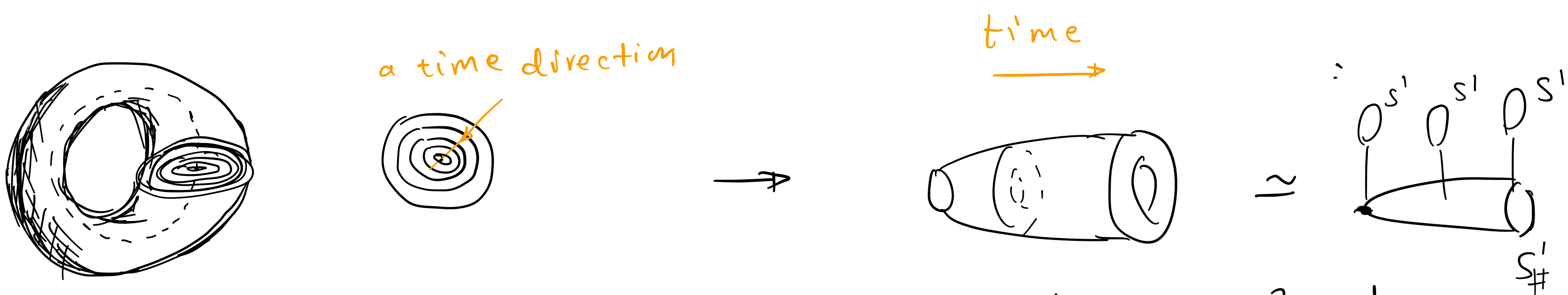
linking number $L_i \cdot L_j$ $\longleftrightarrow$ K_{ij}

$\uparrow$ mixed Chern-Simons levels

- How to understand this dictionary ?

  Answer : use M-theory / IIB duality, and foliation.

Foliation : A solid torus can be foliated by giving it a time direction

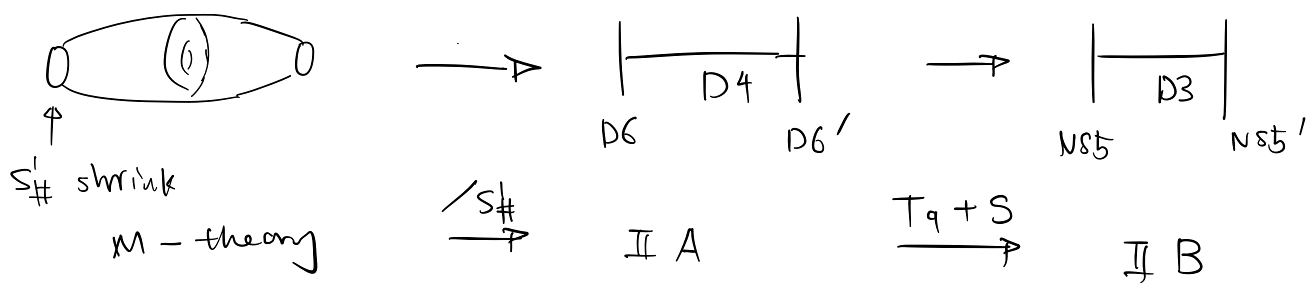


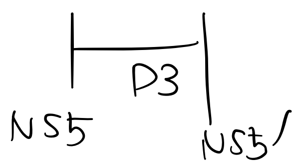
solid torus

Solid torus $D^2 \times S^1$

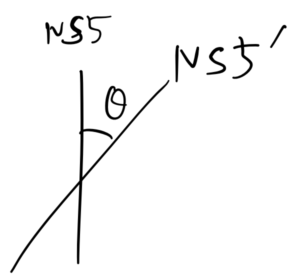
- At the tip of the cigar, a circle S^1 shrinks.
- If we put a solid torus in M-theory, shrinking a $S^1_\#$ implies a D0-brane which is EM dual to a D6-brane.

ex 1: Lens space $L(k,1)$, We put a M5-brane on it

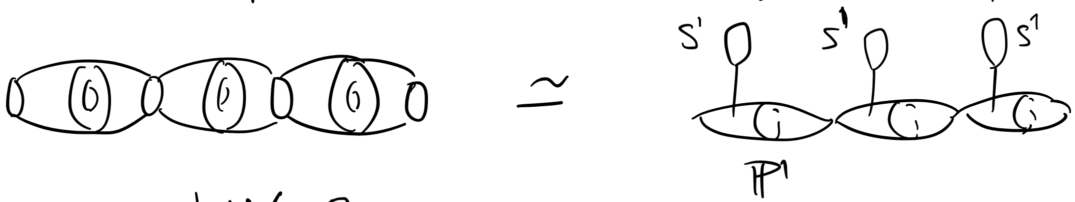


- 

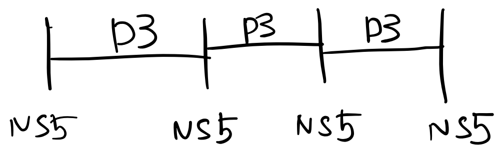
is the standard brane web for 3d $\mathcal{N}=2$, $U(1)_k$, $k = \tan\theta$, θ is the relative angle between NS5 and NS5'.



<u>ex 2</u> : ALE space : a chain of Lens space $S^2 \times S^1 = L(0,1)$



$\downarrow$ M/IIB



$\leftarrow$ a linear quiver theory $U(1) \times U(1) \times U(1)$
3d $\mathcal{N}=4$

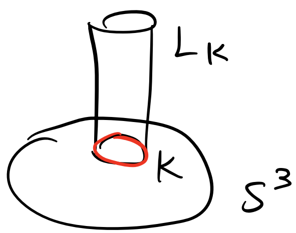
Now, we can see that some 3-mfds could realize abelian theories.

Then, how to realize <u>chiral multiplets</u>?

<u>Matter</u>

- Let us use Ooguri — Vafa construction, since It is hard to find other choices.

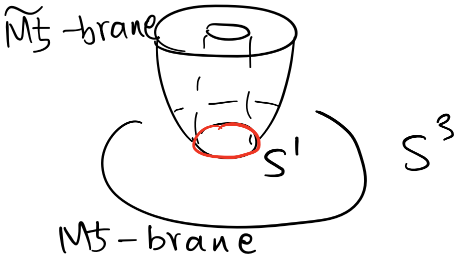
<u>OV-construction</u> : use for engineering open top. string & Wilson loops.



$L_K \cap S^3 = K$, $L_K \subset T^*S^3$ is a Lag. submfd.

- In our cases, $K =$  <u>unknot</u>.

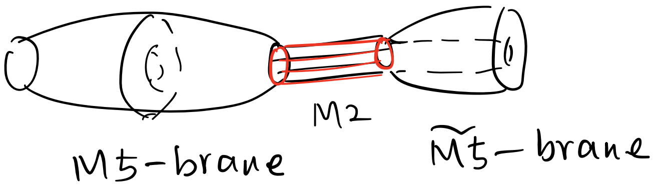
$L_K = D^2 \times S^1$ is a solid torus. So we can draw more precisely:



- A question emerges :

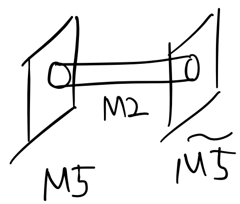
  Does this unknot link to the surgical circle for $S^3 = L(\pm 1, 1)$?

- To answer this question, let us foliate the three-sphere, then



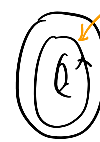
similarly

$\rightsquigarrow$

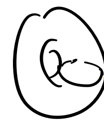


- M2-branes should stretch between these M5-branes, and give bifundamental hypermultiplets.

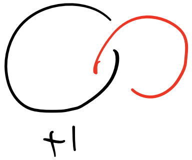
- the unknot $\longrightarrow$ S^1_q longitude of



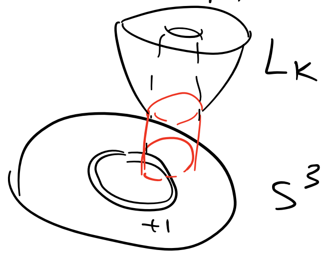
  surgery circle $\longrightarrow$ $S^1_\#$ meridian of



Hence, these two circle should form a Hopf link :

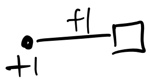


$\rightsquigarrow$



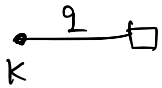
$\Big\}$ More precise picture for OV construction

- We can use plumbed graphs to represent



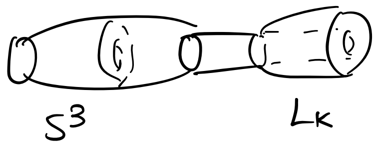
← this box denotes the matter circle  .

The linking number is the charge of the chiral multiplet. In general, we have

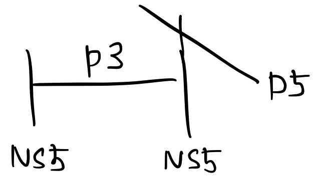


, $U(1)_K + \Phi_q$.

- Use M/IIB-duality again,



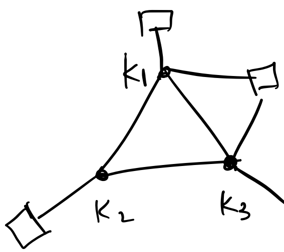
→



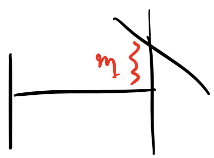
.

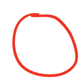
M5-brane on L_K $\overset{M/IIB}{\longleftrightarrow}$ D5

- OV construction & 3-mfd are much more generic than 3d brane web, since they realize abelian theories with mixed Chern-Simons levels and charged chiral multiplets. For example,



- <u>mass parameter</u> :



↔ 

mass parameter is the length of the matter circle.

Decoupling & geometric transition

- A problem left : 1 hyper = 1 chiral + 1 anti-chiral .

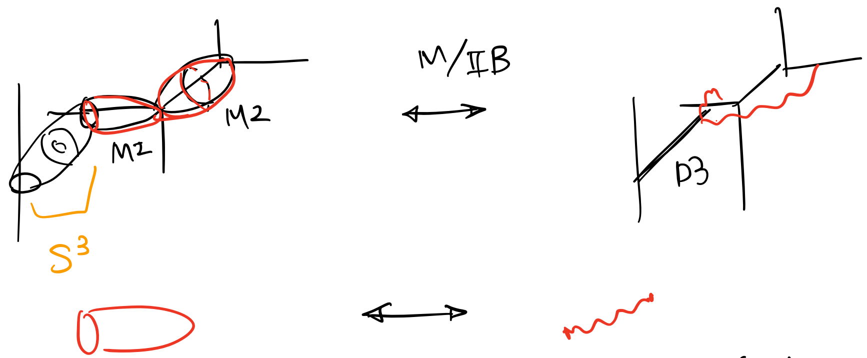
  How to decouple a chiral multiplet ?

- To begin with, think about the correspondence between

  toric diagram $\xleftrightarrow{M/IIB}$ 5-brane web

M2-brane $\longleftrightarrow$ (p,q)-string

- The geometric transition

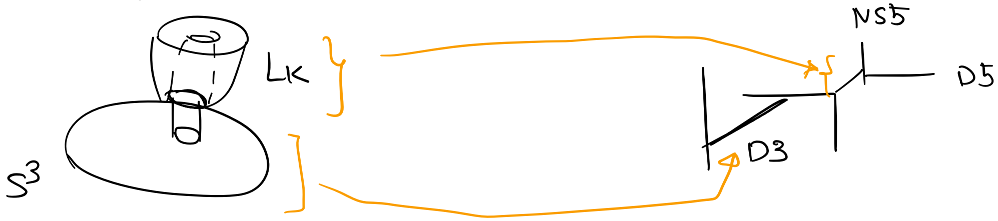


M2-brane $\longleftrightarrow$ string (chiral multiplet)

- In the toric geometry, the holonomy of ○ is equivalent to the length of strings, which is the mass parameter.

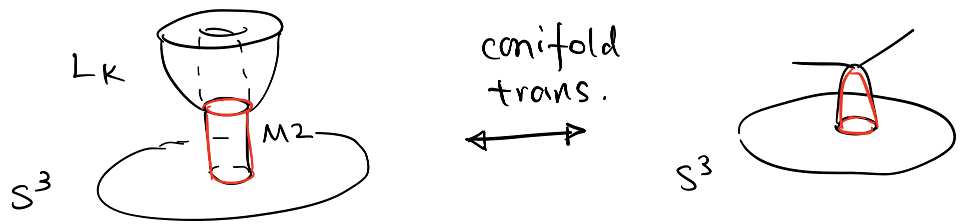
  Similarly, the length of surgery circle $\bigcirc_K$ is the FI parameter.

- There is a fiber-base switch :



- The conifold transition is applied on the L_K, rather than the base S^3.

  So, our 3-mfd & OV construction is consistent with toric diagrams, upto a geometric transition.



- The M2-brane becomes a disc after the transition.

  Finally, the problem becomes how the M2-brane ends on a M5-brane.

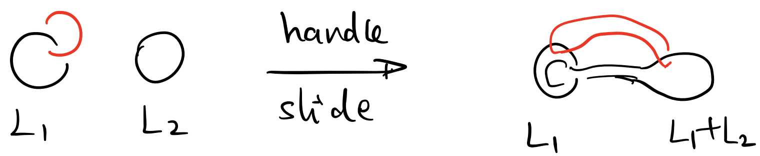
Kirby move with matter circles

- In [GGP] and previous slides, Kirby moves only apply on surgery circles.

  In the presence of matter circles O, Kirby moves still preserve.

  O can even extend Kirby moves.

example :



The matter circle could change its locations through Kirby moves.

Extended Kirby moves (Rolfsen twist)

- Recall that

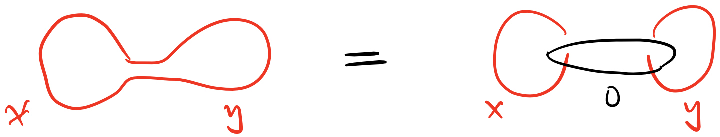
This extended Kirby move is just the 3d duality $1\Phi \overset{ST}{\longleftrightarrow} U(1)_{\pm 1} + 1\Phi$.
which correspondes to the $ST \in SL(2,\mathbb{Z})$.

- This extension is inspired by <u>a trick</u> : on the nbhd of a knot, there is alway an <u>identical surgery</u>.

Superpotential & Fusion

- Recall that one sucess of DGG construction is that it could interprete SQED - XYZ duality, and the superpotential.
- We did not directly find the object for superpotential, but find something more generic, which is the fusion or connected sum of matter circles.

Fusion of matter circles (M2-branes)



The fusion formula: $$\frac{(xy,q)_n}{(q,q)_n} = \sum_{k=0}^{n} \frac{(x,q)_{n-k}}{(q,q)_{n-k}} \cdot \frac{(y,q)_k}{(q,q)_k} \cdot x^k$$

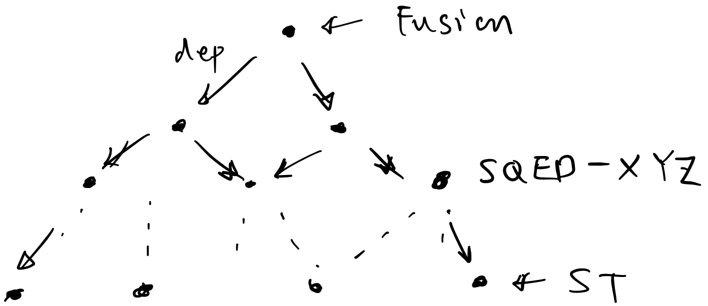
- By decoupling mass parameters x and y in different patterns, one can get the vortex partition functions for SQED$-XYZ$

  Fusion $\rightsquigarrow$ SQED$-XYZ$.

  So, the cubic superpotential $W = XYZ$ is hiding in the fusion.

- Note that partition functions of SQED$-XYZ$ is the pentagon realation, which is derived through quantum relation of wilson loops, but here everything is classical.

Descendent dualities



- From the fusion, many descendent dualities can be derived. They all have geometric interpretations.

- The above is about how we complete [GGP] construction
- The below is about how to complete [CCV] construction

Branched cover construction

- At least we can cap (close) the boundaries of the braids.

$$\text{braids} \xrightarrow{\text{cap}} \text{knots}$$

Heegaard splitting

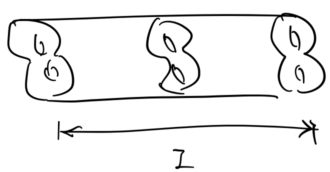
- For a given 3-mfd, there is always a Heegaard splitting

- Question: How to physically understand this splitting?

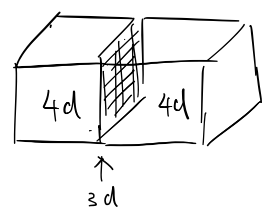
[CCV] construction (revisit)

- 3d $T[Z_3]$ is a domain wall theory.
- Consider $T[\Sigma_g \times I]$. Σ_g : Seiberg–Witten curve of 4d Class-S

$Z_3 = \Sigma_g \times I$



$\xrightarrow[\text{M5-brane}]{\text{wrap a}}$



- Move one direction: $\underbrace{\mathbb{R}^3 \times \mathbb{R}}_{4d} \times \underbrace{\Sigma_g}_{SW} \simeq \underbrace{\mathbb{R}^3}_{3d} \times \underbrace{(\mathbb{R} \times \Sigma_g)}_{3\text{-mfd}}$

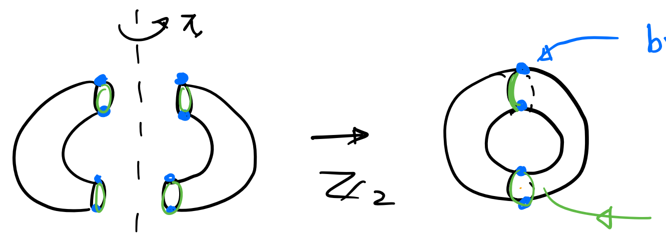
$2+4 = 3+3$

Double branched cover

- Recall that Σ_g is a double branch cover of a complex plane.

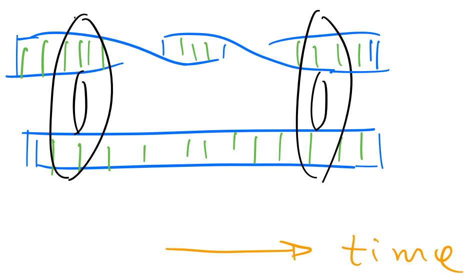
$$\Sigma_g \xrightarrow{2:1} \mathbb{C}$$

For example: A torus $T^2 = S^1 \times S^1$ is



Foliation (time evolution / flow)

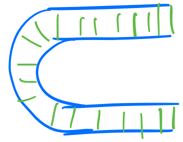
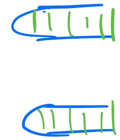
- We can twist (braid) the branched loci,



→ Non-compact 3-mfd w/ boundaries
$\Sigma_g \times \mathbb{R}_+$

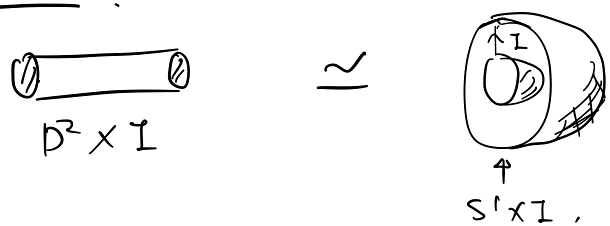
- Branched cut and branch loci bound a Seifert surface.
- To get compact 3-mfd, we should cap the braid. ⇝ knot
- Branched knots bound Seifert surfaces, $K = \partial S$.

Cap the boundaries

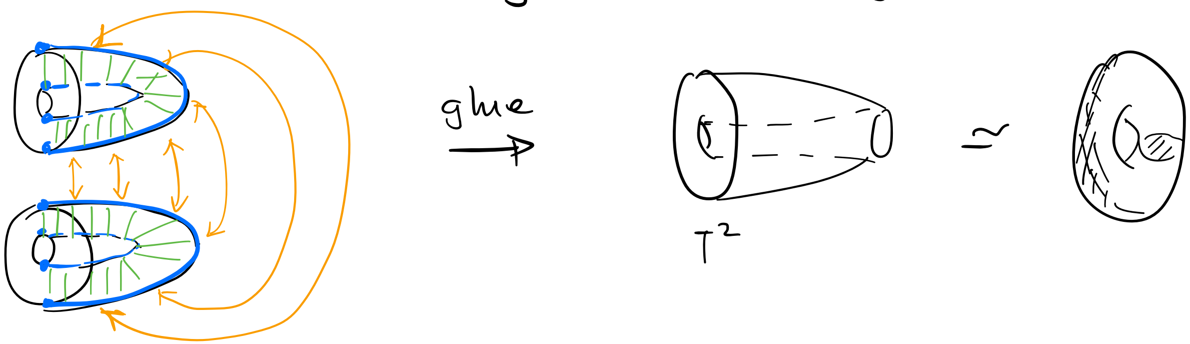
- There are two choices :  , 

- To find which choice is right, we should rethink about double branched cover in terms of handles.

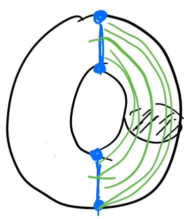
Handles :



- glue two handles along the boundary (a disc) $\to$ a solid torus

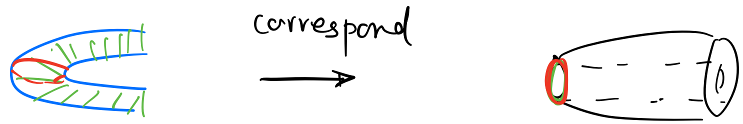


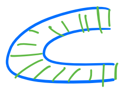
- In this gluing, we can keep track of the branched loci (cuts).

It turns out : 

branched cut $\rightsquigarrow$ longitude

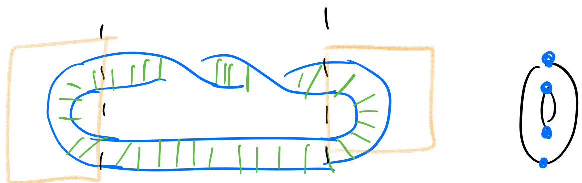
- Recall that the longitude is the matter circle, which is not shrinkable.



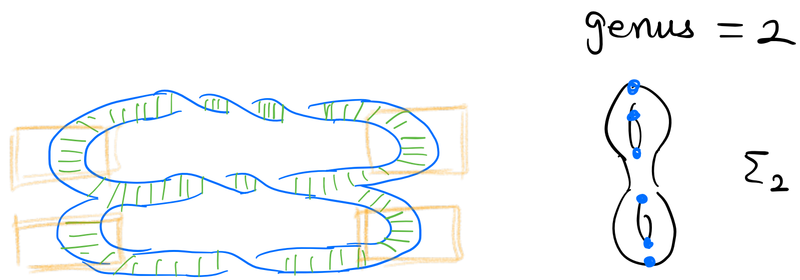
- This shows the branched cover is consistent with the surgery.
- so  is the right choice.

Cap the braids

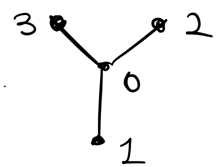
- example :



Lens space $L(2,1)$

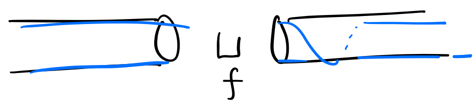


Seifert mfd.



Dehn twist

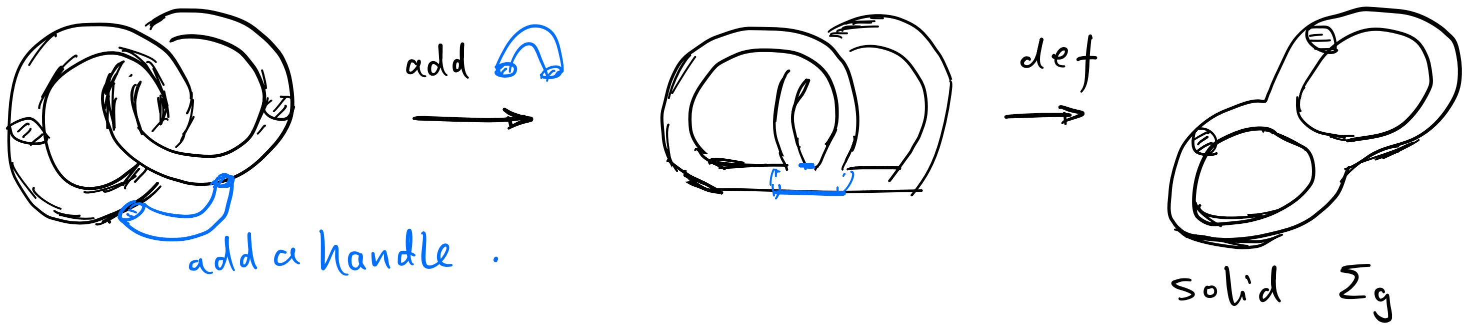
-  is a Dehn twist,



Recall that surgery is also Dehn twist. This implies a dictionary between branched cover and surgery.

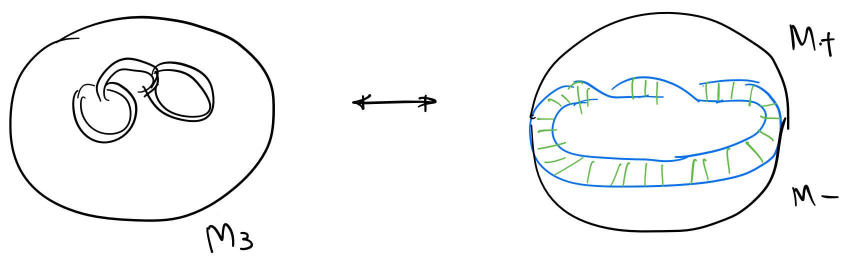
A key technique in the proof of LW theorem: add handles.

- Let us start from a surgical knot, and then add a handle

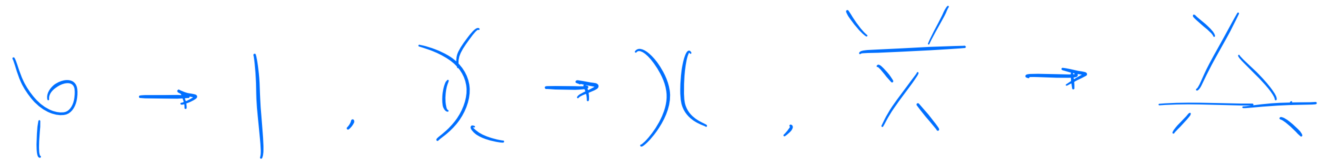


- So, if we have a surgery, we can always turn it into solid Σ_g.

$$\boxed{\text{surgery} \xleftrightarrow{\text{handle}} \text{Heegaard splitting}}$$

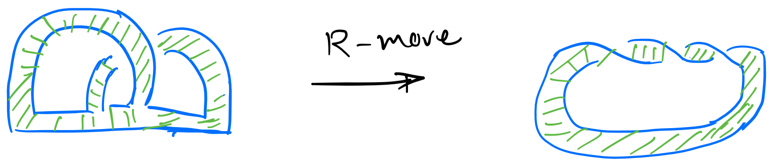


R-move of the knot

- R-move 

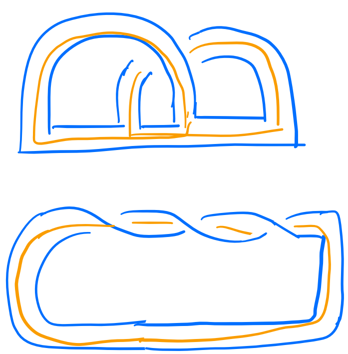
Question : Could we apply R-moves on branched knots. Answer : Yes.

ex : surgery $\overset{\text{handle}}{\rightsquigarrow}$ Seifert surface



Does this change something ? Yes, it changes Seifert matrices.

Seifert matrix

- 

$S_{ij} = \alpha_i \cdot \alpha_j$ is the linking matrix of the orange circle α_i.

$\nparallel$

$\tilde{S}_{ij} = [3]$

- Obviously, these two Seifert matrices are different.
- Note that, there are many different definitions for Seifert matrices.

- Here we only use this definition, such that branched covers could be translated into surgeries.

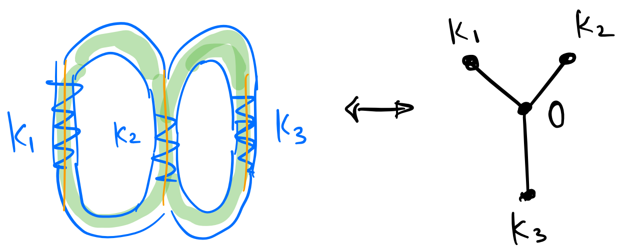
Dictionary

- For a given 3-mfd, there are two equivalent descriptions:

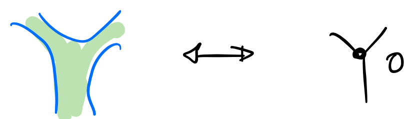
$$\begin{tikzcd} \text{Seifert surface} \arrow[r, leftrightarrow, "1:1"] \arrow[d, leftrightarrow, "R-\text{move}"] & \text{surgery} \arrow[d, leftrightarrow, "\text{Kirby move}"] \\ \widetilde{\text{Seifert surface}} \arrow[r, leftrightarrow, "1:1"] & \widetilde{\text{surgery}} \end{tikzcd}$$

- R-moves of branched knots are translated as Kirby moves of surgery links.

example:

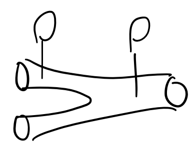


Seifert mfd.



connected sum.

pant:



Remark: • Because of this dictionary, Seifert matrix is interpreted as mixed Chern–Simons levels:

$$S_{ij} = K_{ij}$$

• The orange circles on the Seifert surface correspondе to the gauge group $U(1)_K$.

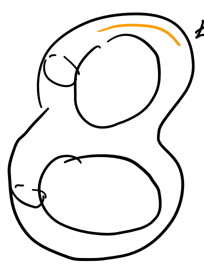
• The twist number $\rightsquigarrow$ Chern–Simons levels.

Unify [GGP] and [CCV]

• Because of this dictionary, [GGP] and [CCV] are unified.

Compare with class-S theory

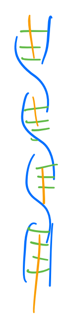
4d



the tube correspondes to a gauge group $U(1)$

Σ_{SW}

3d

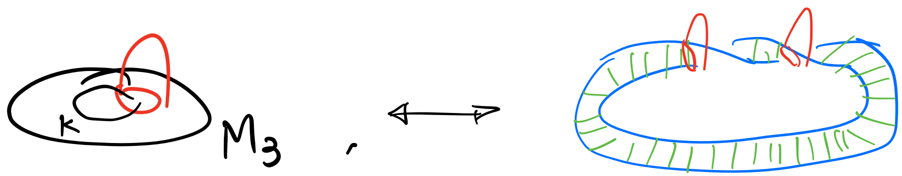


← each twisted belt correspondes to a 3d gauge group $U(1)_K$

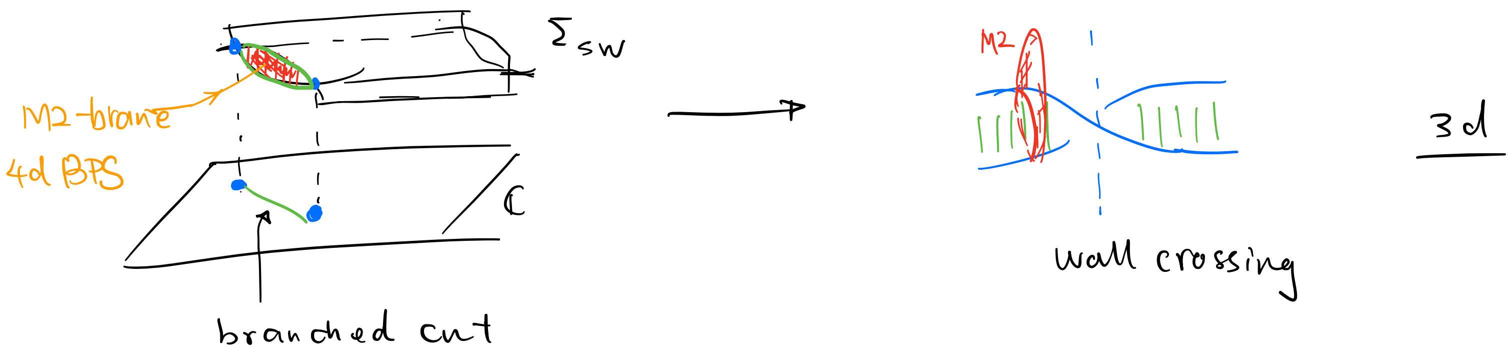
Matter

- In the last section, matter comes from M2-branes ending on a M5-brane.

  We can translate this picture to branched covers.



- In class-S theory, 4d BPS has a nice picture,



- The boundary of M2-brane is the branched cut.

  M2-branes locate on the nbhd of the Seifert surfaces.

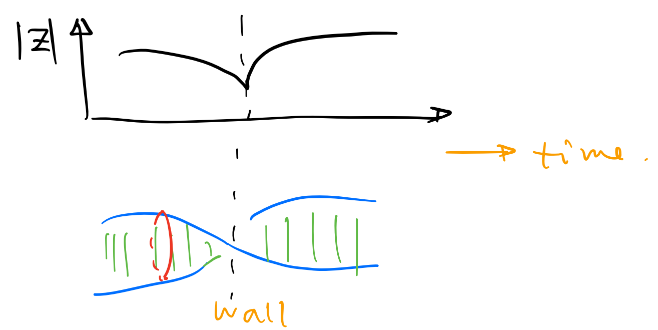
- Remark: all interesting things are determined by the Seifert surfaces given by branched knots and cuts.

3d BPS & wall crossing

- In [CCV], 3d BPS comes from 4d BPS trapped on the wall (cross).
- The problem is that mass parameter (central charge) in 4d is complex, while in 3d mass parameter is real.

On the wall (cross), the imaginary part $\mathrm{Im}(Z^{4d})$ should vanishes.

Such that the BPS has minimal energy:



$$|Z| = \sqrt{m_r^2 + m_{Im}^2}$$

- $$m_{cpx}^{4d} = \int_{M_2} \omega + iB = \int_{\bigcirc} \lambda + i\tilde{A} \xrightarrow{\text{wall}} \int_{\bigcirc} \lambda$$

kähler B field

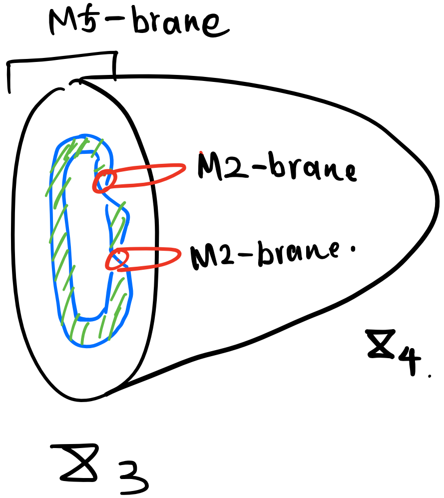
Since 3-mfd is Lagrangian submanifold, $F + B|_{\mathcal{S}_3} = 0$.

$F = dA$ is from the gauge field on 3-mfds.

The cross $\asymp$ is a reflection, which rules out F and hence B.

4 - mfd

- 3 - mfd is the boundary of a 4 - mfd : $X_3 = \partial X_4$



We wish this could has connections to Witten's work.

Summary:

Section 1:

- We introduce chiral multiplet to [GGP] construction
- We find surgery interpretations for many 3d dualities

Section 2:

- We find a way to cap the boundaries of braids.
- We find the dictionary and unify [GGP] and [CCV].
- More details on 3d BPS are coming, such as punctures.
- Results can be obtained from only the topology of 3-mfds.

Outlook:

- short term: 3d rank 0, R-twist
- long term: knot homology

Thank you!