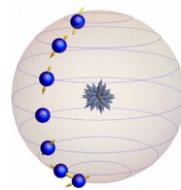


# Quantization by LLL Projection, Fuzzy Hemisphere, and Stuff



Mykola Dedushenko



Based on [2407.15948](#) and some old material  
(c.f. also [2407.15914](#) by Zhou-Zou)

QFT and Beyond, November 28, Southeast University

# Plan

- Formal Story
  - ◆ Geometric quantization + Landau quantization = Berezin-Toeplitz (BT) quantization
  - ◆ Bosonic particle on LLL → scalar BT
  - ◆ Superparticle on LLL → spinor BT
  - ◆ Fuzzy sphere and hemisphere
- Numerical Story
  - ◆ Model on fuzzy sphere
  - ◆ Boundary criticality in 3d Ising

# Formal Story

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Define the physical states as  $\psi \in \mathcal{H} \subset \Gamma(M, L)$ , such that  $\nabla_X \psi = 0$ ,  $X \in P$

(like saying that  $\psi$  only depends on positions)

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Let me give a physical derivation of this,  
leading to two versions of the BT quantization [MD'10]

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Landau quantization [MD'10]

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Symplectic volume form on M,  $\text{Pf } \omega = \frac{1}{n!} \omega \wedge \dots \wedge \omega = dx^1 \wedge \dots \wedge dx^{2n} \int d\psi^1 \dots d\psi^{2n} e^{\frac{1}{2} \omega_{ij} \psi^i \psi^j}$

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It has supersymmetry:  $\delta x^i = i\hbar \epsilon \psi^i, \quad \delta \psi^i = -\epsilon \dot{x}^i$  [Morozov-Niemi-Palo'91]

Every quantum-mechanical system has this hidden SUSY!

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SUSY extension is unique:

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$\mu \rightarrow 0$  projects on  $\ker Q$

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= supersymmetric ground states of the superparticle on M [MD'11]

= stationary zero-energy one-particle states of the complex spinor field on  $M \times \mathbb{R}$

(because  $D_{M \times \mathbb{R}} = D_M + \Gamma \partial_t$  and on stationary zero-energy states,  $\partial_t = 0$ )

# Fuzzy Sphere

Applying this to  $M=S^2$  with symplectic form:  $\omega=n \times Vol = n \times \frac{i dz \wedge d \bar{z}}{(1+|z|^2)^2}$   
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Geometric  
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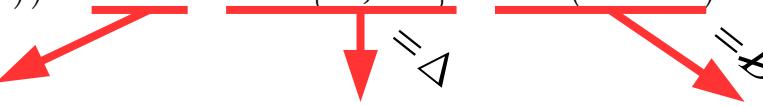
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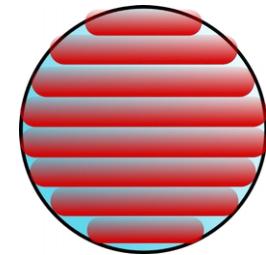
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Acting on  $\mathcal{H} = \mathbb{C}^{n+1} = \text{span} \langle z_0^n, z_0^{n-1} z_1, \dots, z_0 z_1^{n-1}, z_1^n \rangle$

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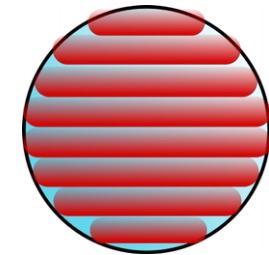
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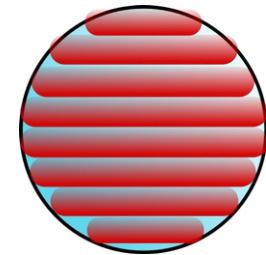


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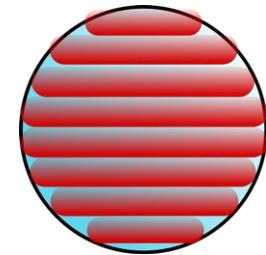
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Also, a fantastic toy model for studying the classical limit of quantum mechanics.  
(Topic for a separate discussion)

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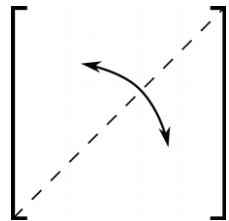
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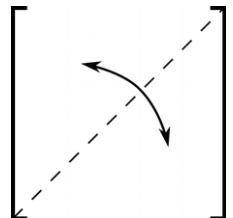
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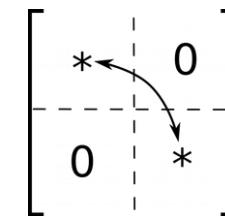
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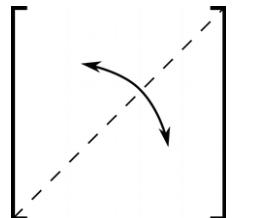
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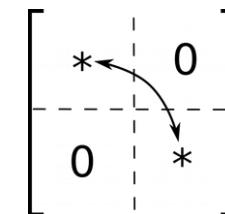
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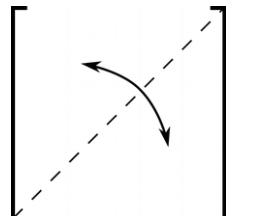
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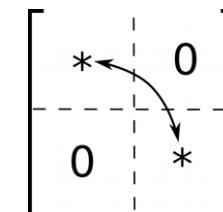
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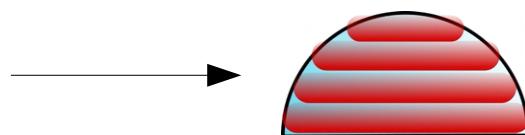
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Now  $[n/2]+1$  fuzzy orbitals.

# Numerical Story

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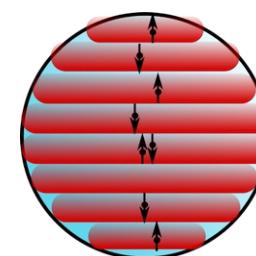
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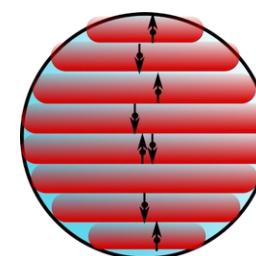
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→  $2N$  available slots. The full Hilbert space of this field on the FS is  $\mathbb{C}^{2^N}$

We will focus on the half-filled subspace:     $\mathbb{C}^{\binom{2N}{N}} \subset \mathbb{C}^{2^N}$

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Where  $V_1=1$ ,  $V_0$ ,  $h$  – spin-spin interaction strength and transverse magnetic field.

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Where  $V_1=1$ ,  $V_0$ ,  $h$  – spin-spin interaction strength and transverse magnetic field.

→ The model is defined on the half-filled Hilbert subspace  $\mathbb{C}^{\binom{2N}{N}} \subset \mathbb{C}^{2^{2N}}$

In the  $(V_0, h)$  plane, a line of quantum critical points described by 3D Ising CFT.

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- ◆ SO(3) rotations of the fuzzy sphere (the main advantage of the fuzzy sphere as a UV regulator)
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These symmetries are extremely useful for solving CFT

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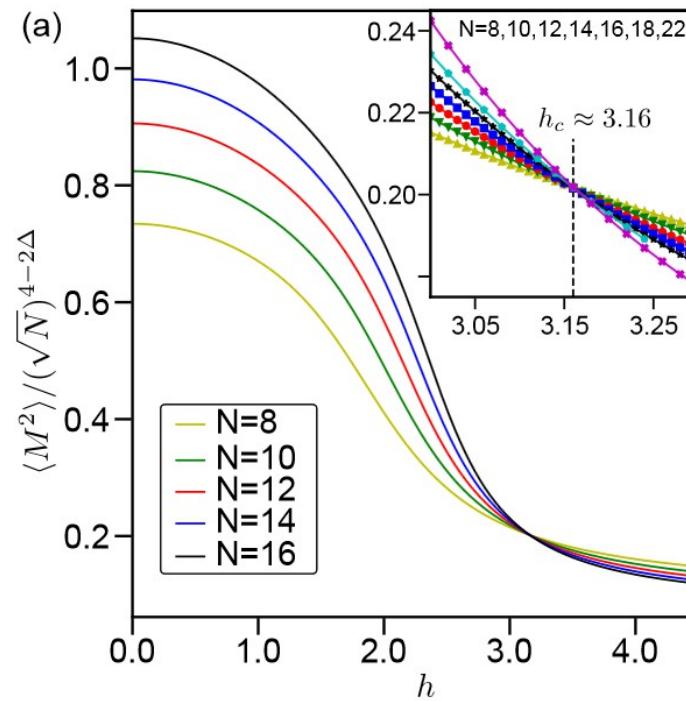
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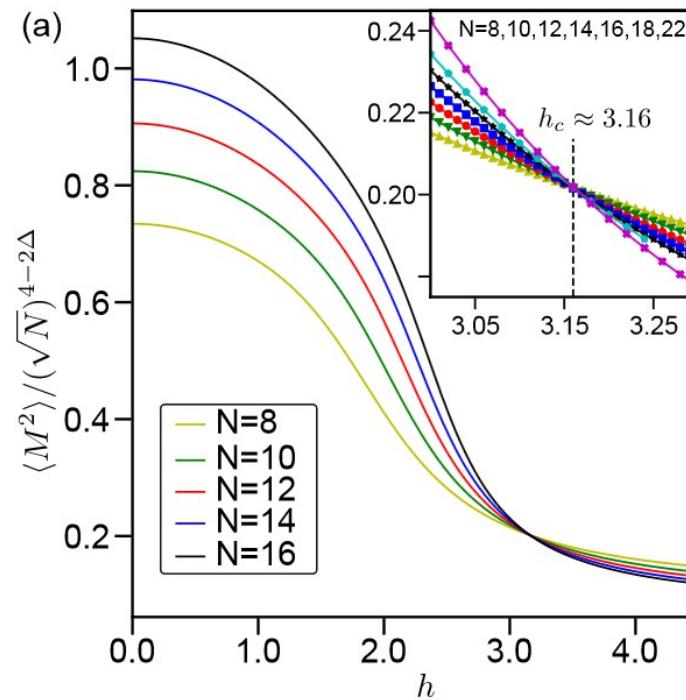
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We will do something similar on the hemisphere...

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- (Also shifts by irrelevant operators, but at least those are  $O(1/N^*)$ )

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- Symmetries on the hemisphere:
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What are the possible boundary conditions?

# Boundary criticality in 3d Ising

- The standard description

- ◆ Classical Ising model on a 3D lattice with boundary:

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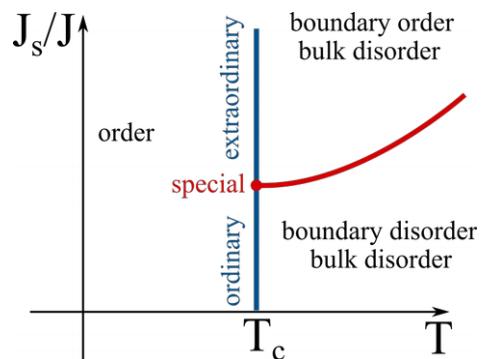
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Extraordinary = normal  $\oplus$  normal

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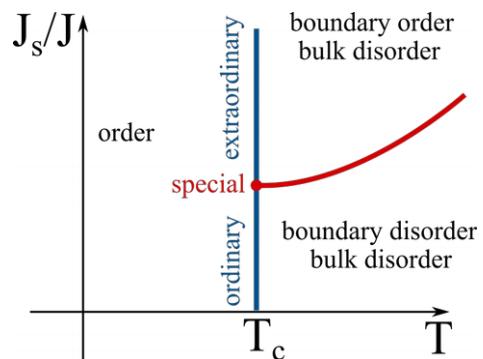
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[Binder'83, Cardy'96, Deng-Blote-Nightingale'05, Diehl'98, Liendo-Rastelli-van Rees'12, Gliozzi-Liendo-Meineri-Rago'15, Metlitski'20, Padayasi-Krishnan-Metlitski-Gruzberg-Meineri'21, Toldin-Metlitski, Trepanier'21]

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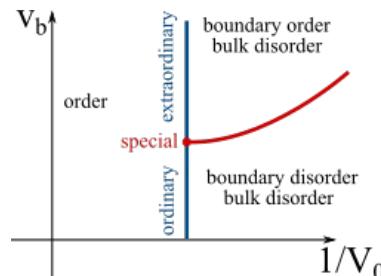
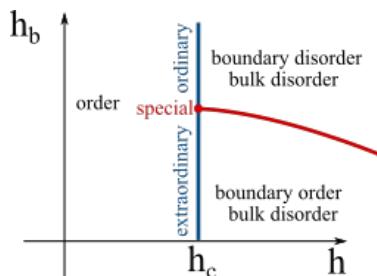
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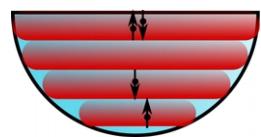
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(always half-filled)

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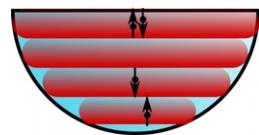


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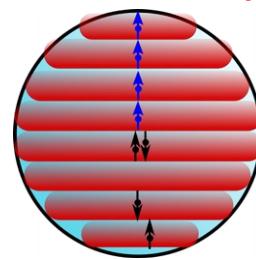
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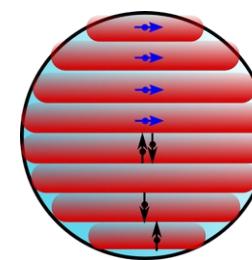


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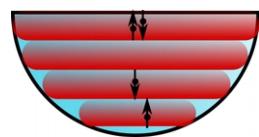


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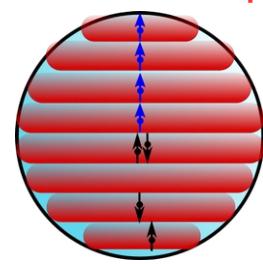
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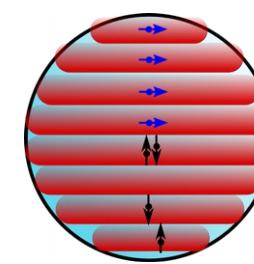
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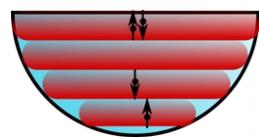
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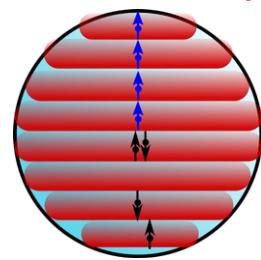
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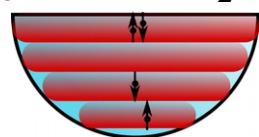
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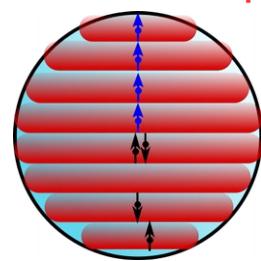
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What kind of boundary/equator interactions are useful?

- ◆ Boundary/equator spin-spin interaction  $v$ , e.g.  $V_0 \rightarrow V_0 + v$
- ◆ Boundary/equator transverse magnetic field  $h_b$

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Shifting  $V_0 \rightarrow V_0 + \nu$  only align the boundary.

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Shifting  $V_0 \rightarrow V_0 + \nu$  only along the boundary.

# Boundary interactions

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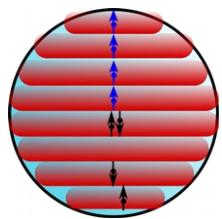
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What we always do: (1) shift the energy spectrum to start at 0;

(2) rescale the energy spectrum to ensure that  $D \propto T_{\perp\perp}$  is a dimension 3 scalar.  
displacement operator

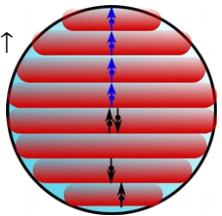
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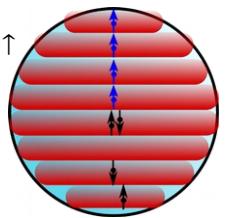
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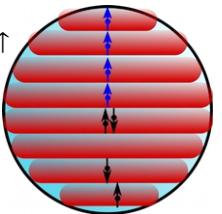
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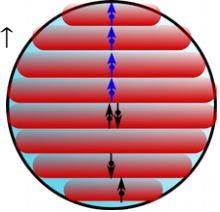
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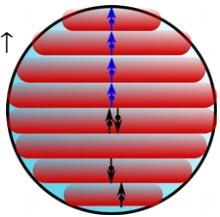
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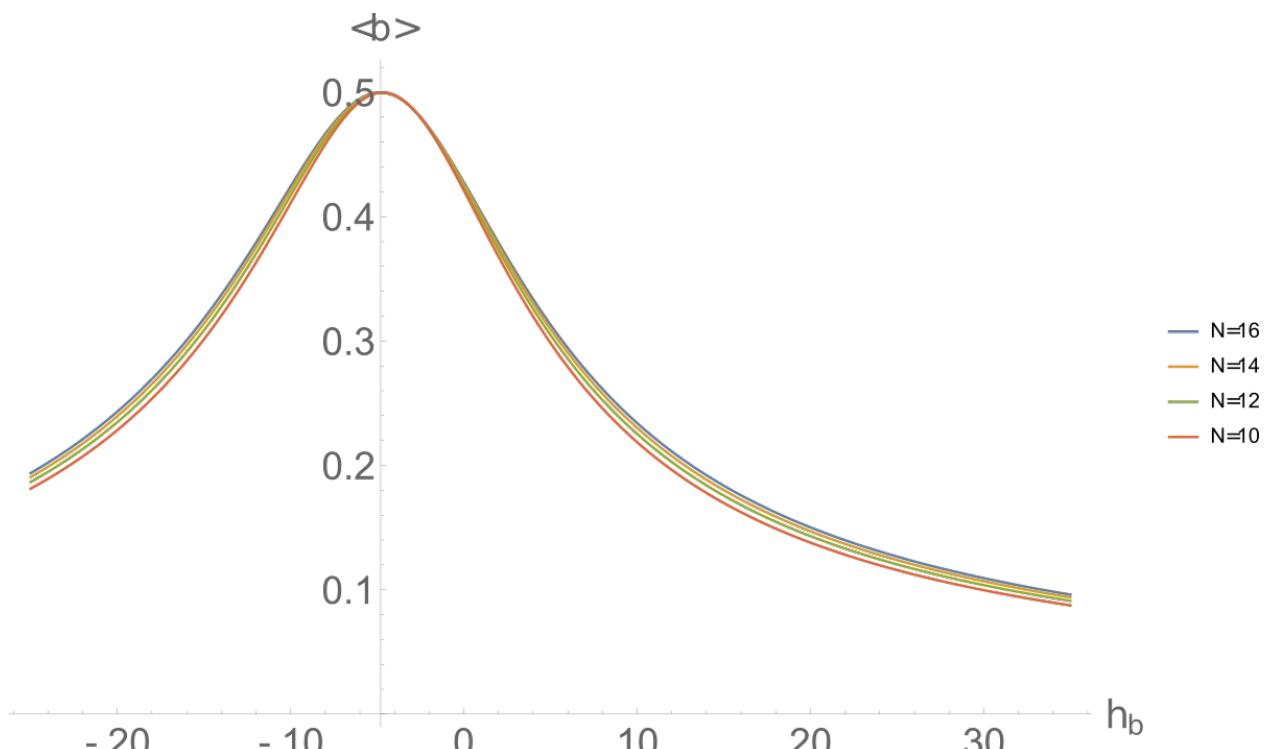
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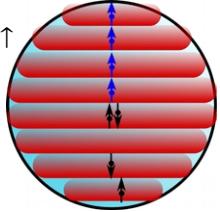
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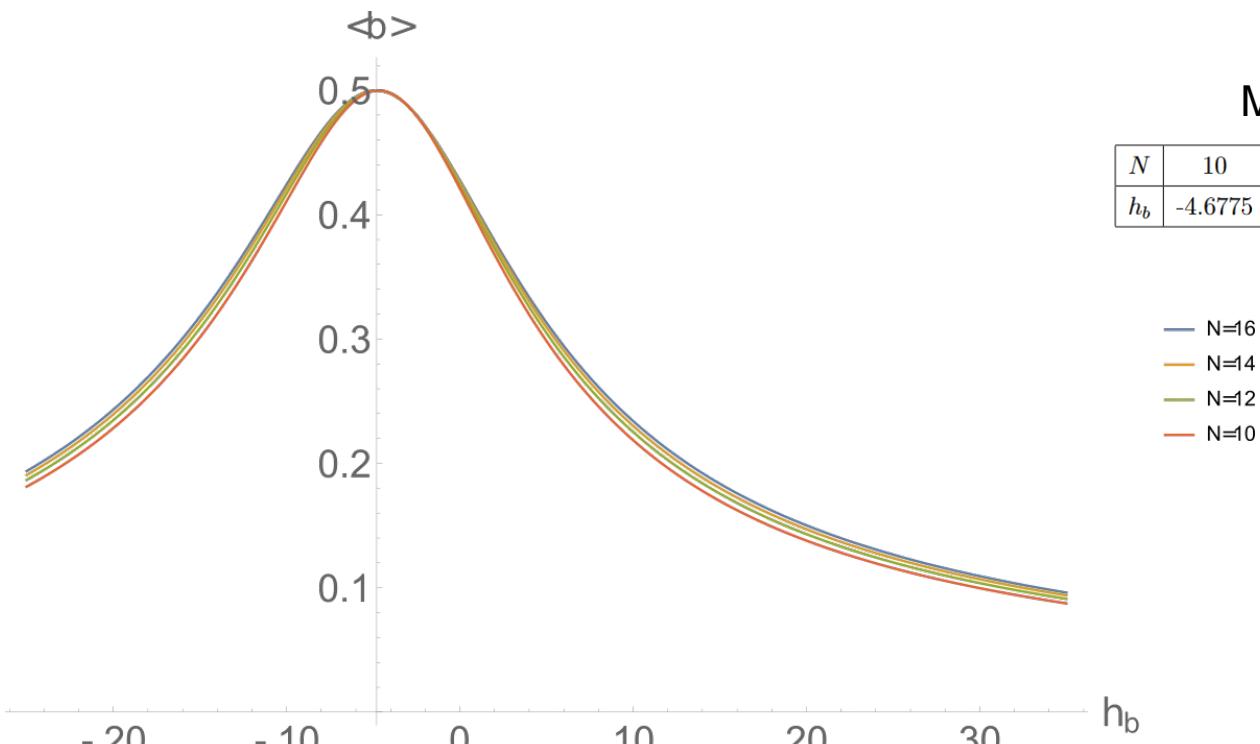
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N=22

$L_z$	Dimensions
0	0, 3, 5.088, 5.858, 7.032, 7.819, 8.239, 8.498, 9.075, 9.328, 9.496, 9.860, 10.075, 10.139, 10.358, 10.379, <b>10.433</b> , 10.514, 10.605, 10.664, 10.684, 10.964, 11.202, 11.400, 11.631, <b>11.810</b> , 11.927, 12.081, <b>12.097</b> , 12.146
1	4, 6.225, 6.788, 8.014, 8.783, 9.000, 9.085, 9.445, 9.721, 9.986, 10.168, 10.204, 10.302, 10.361, 10.451, 10.509, 10.862, 10.983, 11.312, 11.533, 11.549, 11.622, 11.855, 11.974, 12.023, 12.159, 12.221, 12.237, 12.402, 12.535
2	5, 7.148, 7.589, 7.816, 8.498, 9.265, 9.501, 9.682, 9.888, 9.908, 10.010, 10.036, 10.126, 10.182, 10.385, 10.662, 10.719, 11.101, 11.227, 11.334, 11.656, 11.760, 11.904, 12.057, 12.077, 12.161, 12.186, 12.203, 12.255, 12.440
3	6, 7.781, 8.400, 8.729, 8.815, 9.275, 9.517, 9.653, 9.744, 9.799, 9.985, 10.526, 10.681, 10.824, 10.987, 11.196, 11.238, 11.287, 11.527, 11.580, 11.681, 11.692, 11.811, 11.887, 12.188, 12.298, 12.428, 12.545, 12.605, 12.736

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$L_z = 0 : \Delta = 3, 5.088, 7.032, 9.075,$  correspond to  $D, \partial\bar{\partial}D, (\partial\bar{\partial})^2D, (\partial\bar{\partial})^3D$

$L_z = 1 : \Delta = 4, 6.225, 8.014$  must be  $\partial D, \partial(\partial\bar{\partial})D, \partial(\partial\bar{\partial})^2D$

$L_z = 2 : \Delta = 5, 7.148$  must be  $\partial^2D, \partial^2(\partial\bar{\partial})D$

$L_z = 3 : \Delta = 6, 7.781$  must be  $\partial^3D, \partial^3(\partial\bar{\partial})D.$

$L_z = 0 : \Delta = 5.858, 7.819, 9.860$  correspond to  $\mathcal{O}, \partial\bar{\partial}\mathcal{O}, (\partial\bar{\partial})^2\mathcal{O}$

Next scalar primary.

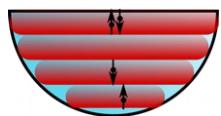
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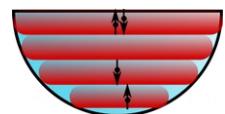
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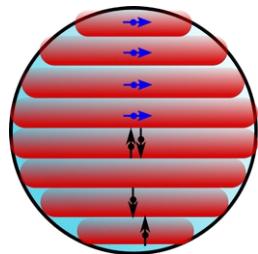
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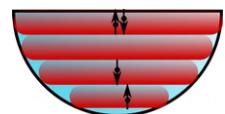
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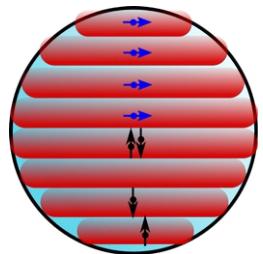
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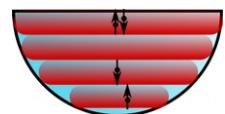


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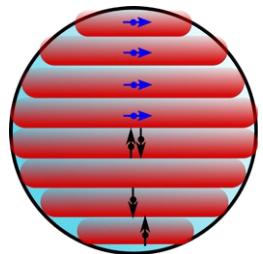
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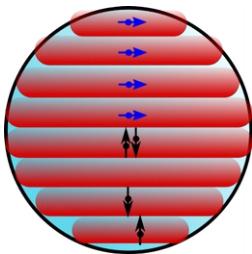
And one of the following spin-spin couplings:  $v(c_{-1/2,\uparrow}^+ c_{-1/2,\uparrow}^- - c_{-1/2,\downarrow}^+ c_{-1/2,\downarrow}^-)^2$

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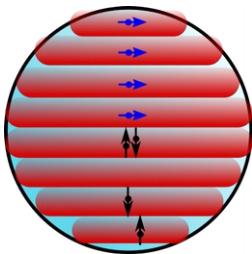


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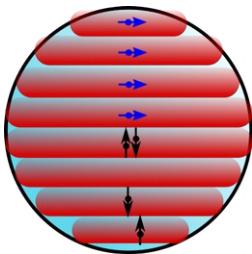
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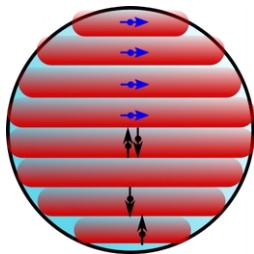
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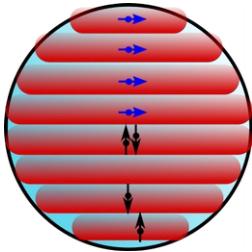
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0	+1	0, 3, 4.89, 5.31, 6.28, 6.66, <span style="border: 1px solid black; padding: 2px;">6.84</span> , 7.17, 7.42, 7.51
	-1	1.26, 3.26, 4.81, 5.07, 6.10, 6.86, 7.10, 7.35, 7.53, 7.59
1	+1	4, 5.57, 5.77, 6.27, 7.00, 7.09, 7.49, 7.76, 7.93, 8.05
	-1	2.26, 4.26, 5.64, 6.05, 6.67, 7.29, 7.46, 7.47, 7.49, 7.58
2	+1	5, 5.36, 6.76, 6.93, 7.14, 7.46, 7.88, 8.20, 8.51, 8.60
	-1	3.26, 4.91, 6.07, 6.67, 6.88, 6.95, 7.29, 7.33, 7.34, 7.41
3	+1	6, 6.12, 7.25, 7.49, 7.80, 8.00, 8.15, 8.28, 8.61, 8.75
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	-1	1.26, 3.26, 4.81, 5.07, 6.10, 6.86, 7.10, 7.35, 7.53, 7.59
1	+1	4, 5.57, 5.77, 6.27, 7.00, 7.09, 7.49, 7.76, 7.93, 8.05
	-1	2.26, 4.26, 5.64, 6.05, 6.67, 7.29, 7.46, 7.47, 7.49, 7.58
2	+1	5, 5.36, 6.76, 6.93, 7.14, 7.46, 7.88, 8.20, 8.51, 8.60
	-1	3.26, 4.91, 6.07, 6.67, 6.88, 6.95, 7.29, 7.33, 7.34, 7.41
3	+1	6, 6.12, 7.25, 7.49, 7.80, 8.00, 8.15, 8.28, 8.61, 8.75
	-1	4.26, 5.50, 6.63, 7.15, 7.21, 7.33, 7.34, 7.40, 7.50, 7.55

One relevant boundary primary:

$$\hat{\sigma}, \quad \Delta_{\hat{\sigma}} \approx 1.26$$

Descendants:

$$L_z = 0, \quad \Delta = 3.26, 4.81 : \quad \partial\bar{\partial}\hat{\sigma}, \quad (\partial\bar{\partial})^2\hat{\sigma},$$

$$L_z = 1, \quad \Delta = 2.26, 4.26 : \quad \partial\hat{\sigma}, \quad \partial(\partial\bar{\partial})\hat{\sigma},$$

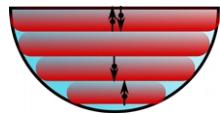
$$L_z = 2, \quad \Delta = 3.26, 4.91 : \quad \partial^2\hat{\sigma}, \quad \partial^2(\partial\bar{\partial})\hat{\sigma},$$

$$L_z = 3, \quad \Delta = 4.26, 5.50 : \quad \partial^3\hat{\sigma}, \quad \partial^3(\partial\bar{\partial})\hat{\sigma}$$

# Ordinary $\mathbb{Z}_2$ -preserving

Other possibilities:

- 



$$-h_b(c_{-1/2,\uparrow}^+ c_{-1/2,\downarrow} + c_{-1/2,\downarrow}^+ c_{-1/2,\uparrow}) \quad \text{With:}$$

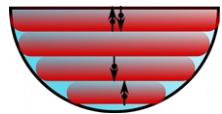
$$\nu(c_{-1/2,\uparrow}^+ c_{-1/2,\uparrow} - c_{-1/2,\downarrow}^+ c_{-1/2,\downarrow})^2 \quad \text{or} \quad \nu \int_{S^1} n_\uparrow(\varphi) n_\downarrow(\varphi) d\varphi$$

Breaks P, yet, surprisingly, seems to give better precision for the spectrum.

# Ordinary $\mathbb{Z}_2$ -preserving

Other possibilities:

- 

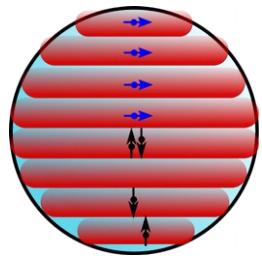


$$-h_b(c_{-1/2,\uparrow}^+ c_{-1/2,\downarrow} + c_{-1/2,\downarrow}^+ c_{-1/2,\uparrow}) \quad \text{With:}$$

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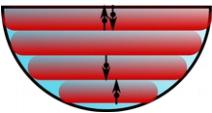


$$\text{With } -h_b(c_{-1/2,\uparrow}^+ c_{-1/2,\downarrow} + c_{-1/2,\downarrow}^+ c_{-1/2,\uparrow}) \quad \text{and} \quad \nu \int_{S^1} n^z(\varphi) n^z(\varphi) d\varphi$$

Both preserves P and seems to give better precision.  
However, computationally heavy. I could only implement up to N=20:

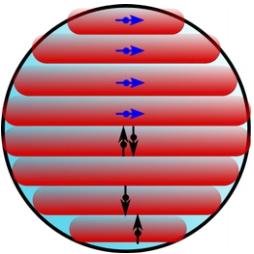
# Ordinary $\mathbb{Z}_2$ -preserving

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Breaks P, yet, surprisingly, seems to give better precision for the spectrum.
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With  $-h_b(c_{-1/2,\uparrow}^+ c_{-1/2,\downarrow} + c_{-1/2,\downarrow}^+ c_{-1/2,\uparrow})$  and  $\nu \int_{S^1} n^z(\varphi) n^z(\varphi) d\varphi$

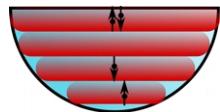
Both preserves P and seems to give better precision.  
However, computationally heavy. I could only implement up to N=20:

$\Delta$	0	1.25	3	3.27	4.60	4.95	5.10	5.44	6.04	6.12
$\mathbb{Z}_2$	+	-	+	-	-	+	-	+	-	+
$P$	+	+	+	+	+	+	+	+	+	+

# Ordinary $\mathbb{Z}_2$ -preserving

Other possibilities:

- 

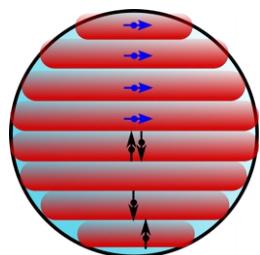


$$-h_b(c_{-1/2,\uparrow}^+ c_{-1/2,\downarrow}^- + c_{-1/2,\downarrow}^+ c_{-1/2,\uparrow}^-) \quad \text{With:}$$

$$\nu(c_{-1/2,\uparrow}^+ c_{-1/2,\uparrow}^- - c_{-1/2,\downarrow}^+ c_{-1/2,\downarrow}^-)^2 \quad \text{or} \quad \nu \int_{S^1} n_\uparrow(\varphi) n_\downarrow(\varphi) d\varphi$$

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$P$	+	+	+	+	+	+	+	+	+	+

Need to clean up this mess...

# Special $\mathbb{Z}_2$ -preserving

**Partial results only**

# Special $\mathbb{Z}_2$ -preserving

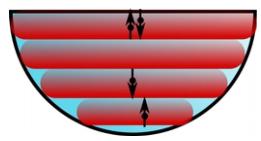
## Partial results only

- Strategy: vary the boundary couplings  $(\nu, h_b)$ , identify the special phase transition.

# Special $\mathbb{Z}_2$ -preserving

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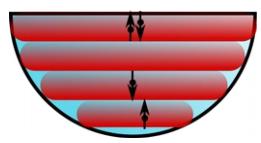
$V_0 \rightarrow V_0 + \nu$  along the boundary

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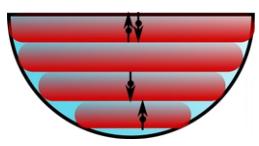
Look at the behavior of the boundary order parameter:

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Look at the behavior of the boundary order parameter:

$$\langle b^2 \rangle = \langle 0 | b^2 | 0 \rangle$$

At the *special* BCFT line, we expect:  $\langle b^2 \rangle \propto N^{1-\Delta_{\hat{\sigma}}}$ , where  $\Delta_{\hat{\sigma}} \approx 0.42$

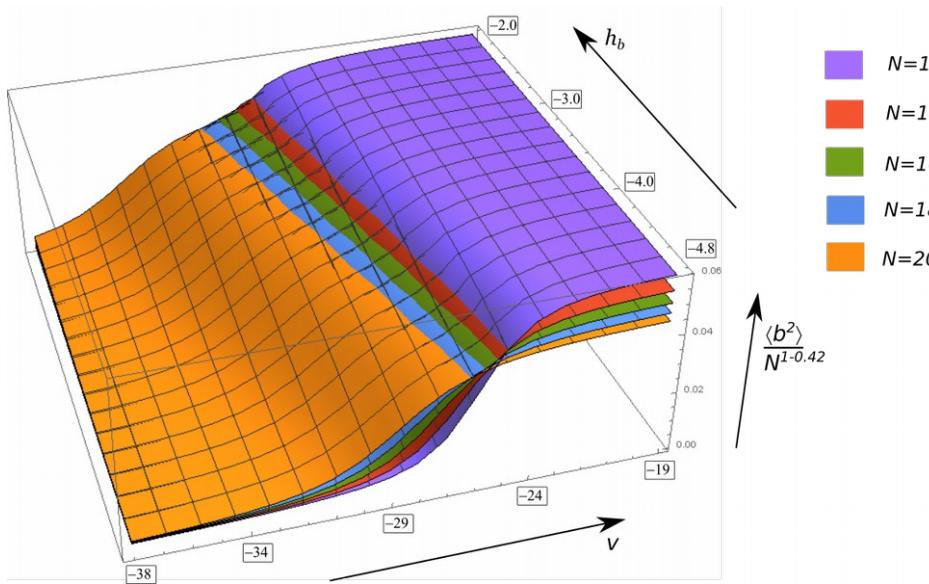
[Liendo-Rastelli-van Rees'12]

$$\langle b^2 \rangle / N^{1-\Delta_{\hat{\sigma}}}$$

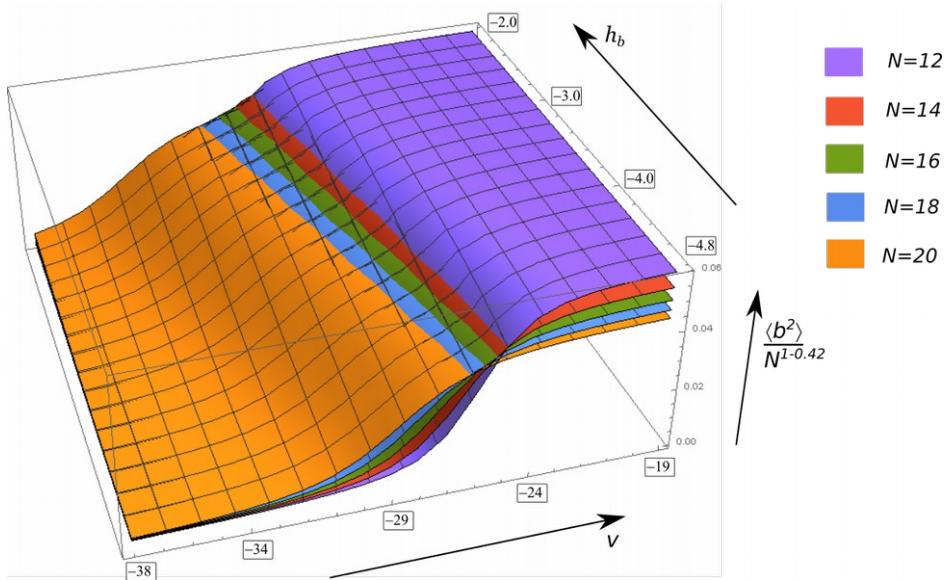
must be N-independent, like in

[Zhu-Han-Huffman-Hofmann-He'22]

# Special $\mathbb{Z}_2$ -preserving



# Special $\mathbb{Z}_2$ -preserving



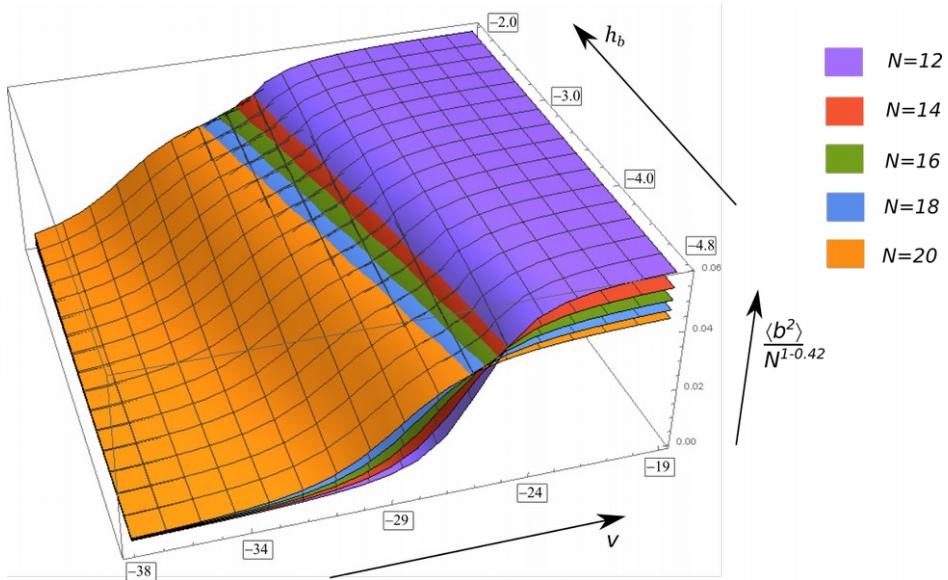
$$(h_b, v) = (-1.9, -24)$$

$N=20$

$\Delta$	0	0.42	2.20	2.44	3	3.46	4.28	4.30	4.48	4.90	4.97
$\mathbb{Z}_2$	+	-	+	-	+	-	-	+	-	-	+

$$\hat{\sigma} \quad : \hat{\sigma}^2 : \quad \Delta \hat{\sigma} \quad D$$

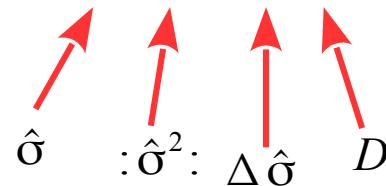
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$\mathbb{Z}_2$	+	-	+	-	+	-	-	+	-	-	+



Tried improving precision by adding :

$$v_1 \int_{S^1} n_\uparrow(\varphi) n_\downarrow(\varphi) d\varphi$$

$$(h_b, v, v_1) = (-2.6, -24, -3.563)$$

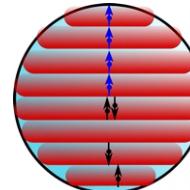
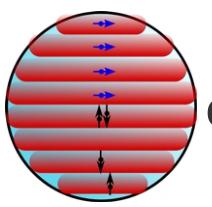
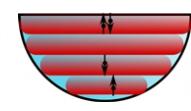
$\Delta$	0	0.42	1.93	2.42	3	3.40	4.16	4.37	4.61	4.92	4.99
$\mathbb{Z}_2$	+	-	+	-	+	-	+	-	-	-	+

$$(h_b, v, v_1) = (-2.6, -26, -2.771)$$

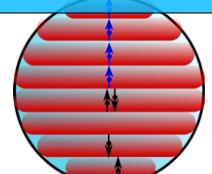
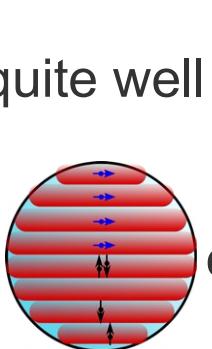
$\Delta$	0	0.42	1.57	2.42	3	3.03	3.78	4.32	4.45	4.64	4.72
$\mathbb{Z}_2$	+	-	+	-	+	-	+	-	-	+	-

Need a closer look at the boundary couplings...

# Take-home messages

- First part:
  - ◆ Quantization by the LLL projection is a powerful and *rigorous* alternative to geometric quantization or brane quantization.
  - ◆ Two versions: particle on  $M$  and superparticle on  $M$ .
  - ◆ Simplest example: fuzzy sphere.
- Second part:
  - ◆ Ising model on fuzzy hemisphere (or half-frozen sphere)  $\rightarrow$  BCFT
  - ◆ Extraordinary/normal b.c. is probed quite well by:
  - ◆ Ordinary b.c. is probed quite well by: or  (undecided)
  - ◆ Features of special b.c. are seen, but more analysis is needed.

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or  

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