Integrable open spin chain in ABJM theory from giant graviton

Hui-Huang Chen
(Jiangxi Normal University)

Based on:
H.Ouyang,J.Wu&H.Chen[arXiv:1809.09941]
N.Bai,H.Ouyang,J.Wu &H.Chen[arXiv:1901.03949]
H.Chen [arXiv:1906.09886]

Shing-Tung Yau Center of Southeast University
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Motivation and Background

- The discovery of integrability in planar N=4 super Yang-Mills and ABJM has enabled a precise study of AdS/CFT.[Beisert et al. 10],[Basso et al. 15]
- Giant gravitons are multi-graviton states inherent to AdS/CFT which provides one of few well-established formulation of quantum gravity.[McGreevy et al. 00],[Balasubramanian et al. 01]
- In the AdS_5/CFT_4 case, strings attached to giant graviton is an integrable open system.[Berenstein et al. 05],[Hofman & Maldacena 07],[Bajnok et al. 12]

- Giant graviton in AdS_4/CFT_3 context has also been studied[Giovannoni et al. 11],[Cardona&Nastase 14], but the integrability aspects have not been clarified.
- Giant graviton are not single trace, but determinant like operators (Schur Polynomial). e.g.

$$
\begin{aligned}
\chi_{R_n}(Z) &= \frac{1}{n!}\sum_{\sigma\in S_n}\chi_{R_n}(\sigma)Z_{i_1}^{i_{\sigma(1)}}\cdots Z_{i_n}^{i_{\sigma(n)}} \\
&= c_0\mathrm{Tr}Z^n + c_1\mathrm{Tr}Z\mathrm{Tr}Z^{n-1}+\cdots+c_n(\mathrm{Tr}Z)^n.
\end{aligned}
$$

- Giant graviton are D-branes wrapping some cycles with some angular momentum.

ABJM theory and integrable closed chain

In ABJM theory, the scalar fields (Y^1, Y^2, Y^3, Y^4) transform in the fundamental representation of the $SU(4)$ R-symmetry group. the action of ABJM theory can be written as

$$
\begin{aligned}
S &= \int d^3x (L_{CS} + L_k - V_F - V_B), \\
L_{CS} &= \frac{k}{4\pi}\varepsilon^{\mu\nu\rho}\mathrm{tr}\Big(A_\mu\partial_\nu A_\rho + \frac{2\mathrm{i}}{3}A_\mu A_\nu A_\rho - \hat{A}_\mu\partial_\nu\hat{A}_\rho - \frac{2\mathrm{i}}{3}\hat{A}_\mu\hat{A}_\nu\hat{A}_\rho\Big), \\
L_k &= \mathrm{tr}(-D_\mu Y_I^\dagger D^\mu Y^I + \mathrm{i}\Psi^{\dagger I}\gamma^\mu D_\mu \Psi_I), \\
V_F &= \frac{2\pi i}{k}\mathrm{tr}\Big(Y_I^\dagger Y^I \Psi^{\dagger J}\Psi_J - 2Y_I^\dagger Y^J \Psi^{\dagger I}\Psi_J + \epsilon^{IJKL}Y_I^\dagger \Psi_J Y_K^\dagger \Psi_L \\
&\quad - Y^I Y_I^\dagger \Psi_J \Psi^{\dagger J} - 2Y^I Y_J^\dagger \Psi_I \Psi^{\dagger J} + \epsilon_{IJKL}Y^I \Psi^{\dagger J} Y^K \Psi^{\dagger L}\Big), \\
V_B &= -\frac{4\pi^2}{3k^2}\mathrm{tr}\Big(Y_I^\dagger Y^J Y_J^\dagger Y^K Y_K^\dagger Y^I + Y_I^\dagger Y^I Y_J^\dagger Y^J Y_K^\dagger Y^K \\
&\quad + 4Y_I^\dagger Y^J Y_K^\dagger Y^I Y_J^\dagger Y^K - 6Y_I^\dagger Y^I Y_J^\dagger Y^K Y_K^\dagger Y^J\Big).
\end{aligned}
$$

Covariant derivatives are defined as

$$D_\mu Y^I = \partial_\mu Y^I + iA_\mu Y^I - iY^I \hat{A}_\mu, \quad D_\mu Y_I^\dagger = \partial_\mu Y_I^\dagger + i\hat{A}_\mu Y_I^\dagger - iY_I^\dagger A_\mu$$
$$D_\mu \Psi_I = \partial_\mu \Psi_I + iA_\mu \Psi_I - i\Psi_I \hat{A}_\mu.$$

The 't Hooft coupling is defined as $\lambda = N/k$.
Consider the gauge invariant single-trace operator

$$\mathcal{O}_{[J]}^{[I]} = \mathrm{Tr}(Y^{I_1} Y_{J_1}^\dagger \cdots Y^{I_L} Y_{J_L}^\dagger).$$

which can be viewed as a closed spin chain. The two-loop renormalization of the single trace operator was considered in [Minahan& Zarembo 08][Bak & Rey 08]

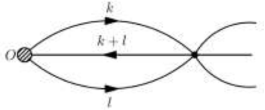

Figure: Two loop contribution of scalar sextet interaction.

$$\mathcal{H}_B/\lambda^2 = \frac{1}{2}\mathbb{I} - \mathbb{P}_{l,l+2} + \frac{1}{2}\mathbb{P}_{l,l+2}\mathbb{K}_{l,l+1} + \frac{1}{2}\mathbb{P}_{l,l+2}\mathbb{K}_{l+1,l+2} - \frac{1}{4}\mathbb{K}_{l,l+1} - \frac{1}{4}\mathbb{K}_{l+1,l+2},$$

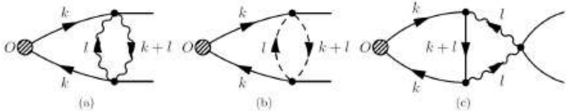

Figure: Two loop contribution of gauge and fermion exchange interaction.

$$\mathcal{H}_{gauge}/\lambda^2 = \frac{1}{4}\mathbb{I} - \frac{1}{2}\mathbb{K}_{l,l+1}, \qquad \mathcal{H}_F/\lambda^2 = \mathbb{K}_{l,l+1},$$

where the trace operator $\mathbb{K}$ and permutation operator $\mathbb{P}$ are defined as

$$(\mathbb{K}_{ij})^{I_i I_j}_{J_i J_j} = \delta^{I_i I_j}\delta_{J_i J_j}, \quad (\mathbb{P}_{ij})^{I_i I_j}_{J_i J_j} = \delta^{I_j}_{J_i}\delta^{I_j}_{J_i},$$

The closed spin chain Hamiltonian $\sum_{l=1}^{2L} \mathcal{H}_B + \mathcal{H}_F + \mathcal{H}_{gauge}$ is integrable.

Determinant like operators in ABJM

We consider the two point function of following type operators

$$O_W = \epsilon_{a_1 \dots a_N} \epsilon^{b_1 \dots b_N} (A_1 B_1)^{a_1}_{b_1} \dots (A_1 B_1)^{a_{N-1}}_{b_{N-1}} W^{a_N}_{b_N},$$

with

$$W = Y^{I_1} Y^{\dagger}_{J_1} \cdots Y^{I_L} Y^{\dagger}_{J_L}.$$

- The dual descriptions of these operators are open strings attached to the giant graviton (D4-brane wrapping a $\mathbb{CP}^2$ inside $\mathbb{CP}^3$).
- When the W field is $A_1 B_1$, the operator is dual to D4-brane itself.

The operator O_W and its conjugate $\bar{O}_W$ can be rewritten as

$$O_W = \frac{1}{(N-1)!} \epsilon^{[I]_{N-1} c}_{[J]_{N-1} a} \epsilon^{[K]_{N-1} b}_{[L]_{N-1} c} (A_1)^{[J]_{N-1}}_{[I]_{N-1}} (B_1)^{[L]_{N-1}}_{[K]_{N-1}} W^a_b,$$
$$\bar{O}_W = \frac{1}{(N-1)!} \epsilon^{[M]_{N-1} f}_{[S]_{N-1} d} \epsilon^{[Q]_{N-1} e}_{[P]_{N-1} f} (A_1^\dagger)^{[P]_{N-1}}_{[Q]_{N-1}} (B_1^\dagger)^{[S]_{N-1}}_{[M]_{N-1}} \bar{W}^d_e.$$

Here we use the shorthand notations

$$[I]_{N-1} = I_1 ... I_{N-1}, \quad (A_1)^{[J]_{N-1}}_{[I]_{N-1}} = (A_1)^{J_1}_{I_1} ... (A_1)^{J_{N-1}}_{I_{N-1}},$$

make the following identification,

$$(A_1, A_2, B_1^\dagger, B_2^\dagger) = (Y^1, Y^2, Y^3, Y^4).$$

When $Y^{I_1} \neq A_1$ and $Y^\dagger_{J_L} \neq B_1$, we get

$$\langle O_W \bar{O}_W \rangle \sim (N-1)!^4 N^{2L+2}.$$

When $Y^{I_1} = A_1$ or $Y^\dagger_{J_L} = B_1$ the operator factorizes, and the contribution is suppressed by $1/N$.

Two loop open spin chain Hamiltonian

$$
\begin{aligned}
\langle O_W \bar{O}_{\tilde{W}} \rangle_{2-\text{loop}} \sim (N-2) \langle & \mathrm{tr}(W\bar{\tilde{W}})\mathrm{tr}(A_1 A_1^\dagger)\mathrm{tr}(B_1 B_1^\dagger) \\
& - \mathrm{tr}(\bar{\tilde{W}} W B_1^\dagger B_1)\mathrm{tr}(A_1 A_1^\dagger) - \mathrm{tr}(A_1 A_1^\dagger W \bar{\tilde{W}})\mathrm{tr}(B_1 B_1^\dagger) \\
& + \mathrm{tr}(W B_1^\dagger B_1 \bar{\tilde{W}} A_1 A_1^\dagger) \rangle_{2-\text{loop}} \\
& + \langle \mathrm{tr}(W\bar{\tilde{W}})\mathrm{tr}(A_1 B_1 B_1^\dagger A_1^\dagger) - \mathrm{tr}(W B_1^\dagger A_1^\dagger A_1 B_1 \bar{\tilde{W}}) \\
& - \mathrm{tr}(W\bar{\tilde{W}} A_1 B_1 B_1^\dagger A_1^\dagger) + \mathrm{tr}(W B_1^\dagger A_1^\dagger)\mathrm{tr}(A_1 B_1 \bar{\tilde{W}}) \rangle_{2-\text{loop}}.
\end{aligned}
$$

We first focus on the left boundary corresponding to the term

$$
\langle -\mathrm{tr}(A_1 A_1^\dagger W \bar{\tilde{W}})\mathrm{tr}(B_1 B_1^\dagger) \rangle_{2-\text{loop}} \to \langle -\mathrm{tr}(A_1 A_1^\dagger Y^{I_1} Y_{J_1}^\dagger Y^{M_1} Y_{N_1}^\dagger) \rangle_{2-\text{loop}}.
$$

Contributions from wave function renormalization (self-interactions) are proportional to $\delta^{I_1}_{N_1}$ and thus flavor blind. Because $Y^{I_1} \neq A_1$ and $Y^\dagger_{N_1} \neq A_1^\dagger$, contributions from gluon exchange and fermion exchange are also flavor blind.

We only need to consider contribution from sextet scalar potential V_B. Then we get

$$H'_{\text{left}} = \frac{\lambda^2}{2}\left(\frac{1}{2}\delta^{I_1}_{J_1}\delta^{M_1}_{N_1} + 2\delta^{M_1}_{1}\delta^{1}_{J_1}\delta^{I_1}_{N_1} - \delta^{I_1}_{N_1}\delta^{M_1}_{J_1} + C\delta^{I_1}_{N_1}\delta^{M_1}_{J_1}\right).$$

The constant C can be determined using the fact that open spin chain vacuum with $W = (A_2B_2)^L$ has vanishing anomalous dimension in the large N limit. At the end, the Hamiltonian can be written as

$$H = \lambda^2\sum_{l=2}^{2L-3}\left(\mathbb{I} - \mathbb{P}_{l,l+2} + \frac{1}{2}\mathbb{P}_{l,l+2}\mathbb{K}_{l,l+1} + \frac{1}{2}\mathbb{P}_{l,l+2}\mathbb{K}_{l+1,l+2}\right)Q_1^AQ_{2L}^B$$
$$+ \lambda^2 Q_1^A\left(\mathbb{I} + \frac{1}{2}\mathbb{K}_{1,2} - \mathbb{P}_{1,3} + \frac{1}{2}\mathbb{P}_{1,3}\mathbb{K}_{1,2} + \frac{1}{2}\mathbb{P}_{1,3}\mathbb{K}_{2,3}\right)Q_1^AQ_{2L}^B$$
$$+ \lambda^2 Q_{2L}^B\left(\mathbb{I} + \frac{1}{2}\mathbb{K}_{2L-1,2L} - \mathbb{P}_{2L-2,2L} + \frac{1}{2}\mathbb{P}_{2L-2,2L}\mathbb{K}_{2L-2,2L-1} + \frac{1}{2}\mathbb{P}_{2L-2,2L}\mathbb{K}_{2L-1,2L}\right)$$
$$Q_1^AQ_{2L}^B + \lambda^2(\mathbb{I} - Q_2^{\bar{A}})Q_1^AQ_{2L}^B + \lambda^2(\mathbb{I} - Q_{2L-1}^{\bar{B}})Q_1^AQ_{2L}^B$$

where the Q operators are defined as

$$Q^\phi|\phi\rangle = 0, \quad Q^\phi|\psi\rangle = |\psi\rangle, \quad \text{for } \psi \neq \phi.$$

Reflection Matrices from Coordinate Bethe ansatz

The vacuum of this open chain is chosen to be

$$W = (A_2 B_2) \cdots (A_2 B_2).$$

The one-particle excitations include

bulk odd site	$(A_2 B_2) \cdots (A_1 B_2) \cdots (A_2 B_2)$	$\vert x\rangle_{A_1}, 2 \leq x \leq L$
	$(A_2 B_2) \cdots (B_1^\dagger B_2) \cdots (A_2 B_2)$	$\vert x\rangle_{B_1^\dagger}, 1 \leq x \leq L$
bulk even site	$(A_2 B_2) \cdots (A_2 B_1) \cdots (A_2 B_2)$	$\vert x\rangle_{B_1}, 1 \leq x \leq L-1$
	$(A_2 B_2) \cdots (A_2 A_1^\dagger) \cdots (A_2 B_2)$	$\vert x\rangle_{A_1^\dagger}, 1 \leq x \leq L$
left boundary	$(B_1^\dagger B_2) \cdots (A_2 B_2)$	$\vert 1\rangle_{B_1^\dagger}$
right boundary	$(A_2 B_2) \cdots (A_2 A_1^\dagger)$	$\vert L\rangle_{A_1^\dagger}.$

Let us begin with

$$|p\rangle_{B_1^\dagger} = \sum_{x=1}^{L} f_{B_1^\dagger}(x)|x\rangle_{B_1^\dagger},$$

where

$$f_{B_1^\dagger}(x) = F_{B_1^\dagger} e^{ipx} + \tilde{F}_{B_1^\dagger} e^{-ipx}.$$

On the states $|x\rangle_{B_1^\dagger}$, the Hamiltonian acts as follows

$$H|x\rangle_{B_1^\dagger} = \lambda^2(2|x\rangle_{B_1^\dagger} - |x+1\rangle_{B_1^\dagger} - |x-1\rangle_{B_1^\dagger}), \quad 2 \le x \le L-1$$

$$H|1\rangle_{B_1^\dagger} = \lambda^2(|1\rangle_{B_1^\dagger} - |2\rangle_{B_1^\dagger}),$$

$$H|L\rangle_{B_1^\dagger} = \lambda^2(2|L\rangle_{B_1^\dagger} - |L-1\rangle_{B_1^\dagger}).$$

So we get

$$H|p\rangle_{B_1^\dagger} = \lambda^2 \textstyle\sum_{x=2}^{L-2}(2f_{B_1^\dagger}(x) - f_{B_1^\dagger}(x-1) - f_{B_1^\dagger}(x+1))|x\rangle_{B_1^\dagger}$$
$$+\lambda^2(f_{B_1^\dagger}(1) - f_{B_1^\dagger}(2))|1\rangle_{B_1^\dagger} + \lambda^2(2f_{B_1^\dagger}(L) - f_{B_1^\dagger}(L-1))|L\rangle_{B_1^\dagger}.$$

Then equation

$$H|p\rangle_{B_1^\dagger} = E(p)|p\rangle_{B_1^\dagger},$$

leads to the following relations

$$E(p) = \lambda^2(2 - 2\cos p), f_{B_1^\dagger}(1) = f_{B_1^\dagger}(0), f_{B_1^\dagger}(L+1) = 0$$

Since the reflections of $B_1^\dagger$ excitation at both sides are diagonal, we define the left reflection coefficient to be

$$K_{L,\,B_1^\dagger} = F_{B_1^\dagger}/\tilde{F}_{B_1^\dagger},$$

and the right reflection coefficient to be

$$K_{R,\,B_1^\dagger} = e^{2ipL}F_{B_1^\dagger}/\tilde{F}_{B_1^\dagger}.$$

The results are

$$\begin{aligned} K_{L,\,B_1^\dagger} &= e^{-ip}, \\ K_{R,\,B_1^\dagger} &= -e^{-2ip}. \end{aligned}$$

The computation of reflection amplitude of other excitations are similar. Finally, with the order of the excitations as $A_1, B_1^\dagger, A_1^\dagger, B_1$, the left reflection matrix is

$$K_L = \begin{pmatrix} -e^{-2ip} & & & \\ & e^{-ip} & & \\ & & -1 & \\ & & & e^{-ip} \end{pmatrix},$$

and the right reflection matrix is

$$K_R = \begin{pmatrix} e^{-ip} & & & \\ & -e^{-2ip} & & \\ & & e^{-ip} & \\ & & & -1 \end{pmatrix}.$$

The two loop bulk scalar scattering matrix is easily obtained from Beisert's $SU(2|2)$ invariant S-matrix and ABJM dressing phases

$$e^{ip} = \frac{u+\frac{i}{2}}{u-\frac{i}{2}}$$

$$
\left(\begin{array}{cccccccccccccccc}
\frac{u_1-u_2-i}{u_1-u_2+i} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \frac{u_1-u_2}{u_1-u_2+i} & 0 & 0 & -\frac{i}{u_1-u_2+i} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \frac{u_1-u_2}{u_1-u_2-i} & 0 & 0 & 0 & 0 & \frac{i}{u_1-u_2-i} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -\frac{i}{u_1-u_2+i} & 0 & 0 & \frac{u_1-u_2}{u_1-u_2+i} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & \frac{u_1-u_2-i}{u_1-u_2+i} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \frac{i}{u_1-u_2-i} & 0 & 0 & 0 & 0 & \frac{u_1-u_2}{u_1-u_2-i} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{u_1-u_2}{u_1-u_2-i} & 0 & 0 & 0 & 0 & \frac{i}{u_1-u_2-i} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{u_1-u_2-i}{u_1-u_2+i} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{u_1-u_2}{u_1-u_2+i} & 0 & 0 & -\frac{i}{u_1-u_2+i} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{i}{u_1-u_2-i} & 0 & 0 & 0 & 0 & \frac{u_1-u_2}{u_1-u_2-i} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{i}{u_1-u_2+i} & 0 & 0 & \frac{u_1-u_2}{u_1-u_2+i} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{u_1-u_2-i}{u_1-u_2+i}
\end{array}\right)
$$

Figure: ABJM two loop S-matrix $S(p_1, p_2)$ in the scalar sector.

Boundary Yang-Baxter equation

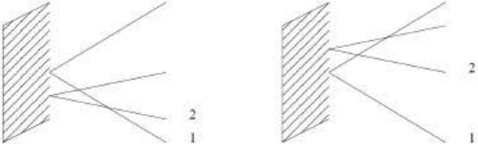

Figure: Pictorial representation of the boundary Yang-Baxter equation.

It can be straightforward to verify that reflection equations are satisfied

$$
\begin{aligned}
&K_{L2}(p_2)S_{12}(p_1,-p_2)K_{L1}(p_1)S_{21}(-p_2,-p_1)\\
&=S_{12}(p_1,p_2)K_{L1}(p_1)S_{21}(p_2,p_1)K_{L2}(p_2),\\
&K_{R2}(-p_2)S_{21}(p_2,-p_1)K_{R1}(-p_1)S_{12}(p_1,p_2)\\
&=S_{21}(-p_2,-p_1)K_{R2}(-p_1)S_{12}(p_1,-p_2)K_{R2}(-p_2).
\end{aligned}
$$

This gives strong evidence of the *Integrability* of the Hamiltonian.

Two loop integrability

With suitable choice of R-matrices and projected K-matrices(operator-valued), we can construct the following two transfer matrices for the alternating spin chain with dynamic boundaries,

$$\tau(u) = \mathrm{tr}_0 K^+_{01}(u) R_{02}(u) \cdots R_{0,2L-1}(u) K^-_{0,2L}(u) R_{0,2L-1}(u) \cdots R_{02}(u)$$
$$\bar{\tau}(u) = \mathrm{tr}_{\bar{0}} K^+_{\bar{0}1}(u) R_{\bar{0}2}(u) \cdots R_{\bar{0},2L-1}(u) K^-_{\bar{0},2L}(u) R_{\bar{0},2L-1}(u) \cdots R_{\bar{0}2}(u)$$

Then it can be shown that the transfer matrices obey the commutativity property

$$[\bar{\tau}(u), \bar{\tau}(v)] = [\tau(u), \bar{\tau}(v)] = [\tau(u), \tau(v)] = 0.$$

The Hamiltonian can be obtained from the transfer matrices by

$$H = \frac{d}{du} \log \tau(u) \Big|_{u=0} + \frac{d}{du} \log \bar{\tau}(u) \Big|_{u=0}.$$

This complete the proof of two loop *Integrability* of ABJM open spin chain from giant graviton.

Bethe equations and "gauge" transformation

The commutativity property of $\tau(u)$ and $\bar{\tau}(u)$ implies the existence of u-independent eigenstates $|\Lambda\rangle$ of both $\tau(u)$ and $\bar{\tau}(u)$,

$$\tau(u)\,|\Lambda\rangle = \Lambda(u)\,|\Lambda\rangle\,, \quad \bar{\tau}(u)\,|\Lambda\rangle = \bar{\Lambda}(u)\,|\Lambda\rangle\,.$$

We choose a reference state $|(A_2B_2)^L\rangle$ which is a eigenstate of both transfer matrices. According to the direct calculations of $L = 1, 2, 3$ cases, we conjecture that the eigenvalues of the reference state for a general L is given by

$$\begin{aligned}\Lambda_0(u) = \bar{\Lambda}_0(u) \quad &= \quad \frac{2}{d(u)}\Big[a(u)(u+1)^{2L}(u+2)^{2L} \\ &+ \quad b(u)u^{2L}(u+1)^{2L} - c(u)u^{2L}(u+2)^{2L}\Big],\end{aligned}$$

where

$$\begin{aligned} &a(u) = (2u+3)(u+1)^2, \quad b(u) = (2u+1)(u+1)^2, \\ &c(u) = 4(u+1)^3, \quad d(u) = (u+1)(2u+1)(2u+3). \end{aligned}$$

The eigenvalues of a generic state should have the “dressed” form

$$
\begin{aligned}
\Lambda(u|\{u_i\}) = \frac{2}{d(u)}\Bigg\{ & a(u)(u+1)^{2L}(u+2)^{2L}\frac{Q_4(iu-\frac{i}{2})}{Q_4(iu+\frac{i}{2})} + b(u)u^{2L}(u+1)^{2L}\frac{Q_{\bar{4}}(iu+\frac{5i}{2})}{Q_{\bar{4}}(iu+\frac{3i}{2})} \\
& - u^{2L}(u+2)^{2L}\left[c_1(u)\frac{Q_4(iu+\frac{3i}{2})Q_3(iu)}{Q_4(u+\frac{i}{2})Q_3(iu+i)} + c_2(u)\frac{Q_3(iu+2i)Q_{\bar{4}}(iu+\frac{i}{2})}{Q_3(iu+i)Q_{\bar{4}}(iu+\frac{3i}{2})}\right]\Bigg\}, \\
\bar{\Lambda}(u|\{u_i\}) = \frac{2}{d(u)}\Bigg\{ & a(u)(u+1)^{2L}(u+2)^{2L}\frac{Q_{\bar{4}}(iu-\frac{i}{2})}{Q_{\bar{4}}(iu+\frac{i}{2})} + b(u)u^{2L}(u+1)^{2L}\frac{Q_4(iu+\frac{5i}{2})}{Q_4(iu+\frac{3i}{2})} \\
& - u^{2L}(u+2)^{2L}\left[c_1(u)\frac{Q_{\bar{4}}(iu+\frac{3i}{2})Q_3(iu)}{Q_{\bar{4}}(u+\frac{i}{2})Q_3(iu+i)} + c_2(u)\frac{Q_3(iu+2i)Q_4(iu+\frac{i}{2})}{Q_3(iu+i)Q_4(iu+\frac{3i}{2})}\right]\Bigg\}.
\end{aligned}
$$

where $Q_l(u)$ is the Baxter polynomial

$$
Q_l(u) = \prod_{j=1}^{K_l}(u-u_{l,j})(u+u_{l,j}).
$$

The functions $c_1(u), c_2(u)$ must satisfy

$$c_1(u) + c_2(u) = c(u).$$

The crossing property of eigenvalues

$$\Lambda(-u-2|\{u_i\}) = \bar{\Lambda}(u|\{u_i\})$$

implies

$$c_1(-u-2) = -c_2(u), \quad c_2(-u-2) = -c_1(u).$$

These constraints cannot determine $c_1(u), c_2(u)$ uniquely. In fact, there are two solutions

$$c_1(u) = (2u+3)(u+1)^2, \quad c_2(u) = (u+1)^2(2u+1),$$

and

$$\tilde{c}_1(u) = u^2(2u+3), \quad \tilde{c}_2(u) = (u+2)^2(2u+1).$$

Thus two sets of Bethe equations can be derived

$$-1=\frac{u^{+}}{u^{-}}\frac{Q_3^{--}Q_4^{+}Q_{\bar{4}}^{+}}{Q_3^{++}Q_4^{-}Q_{\bar{4}}^{-}}\bigg|_{u_{3,k}},$$

$$1=\left(\frac{u-\frac{i}{2}}{u+\frac{i}{2}}\right)^{2L}\frac{Q_4^{++}Q_3^{-}}{Q_4^{--}Q_3^{+}}\bigg|_{u_{4,k}},$$

$$1=\left(\frac{u-\frac{i}{2}}{u+\frac{i}{2}}\right)^{2L}\frac{Q_{\bar{4}}^{++}Q_3^{-}}{Q_{\bar{4}}^{--}Q_3^{+}}\bigg|_{u_{\bar{4},k}}.$$

or

$$-1=\frac{u^{+}}{u^{-}}\frac{(u-i)^2}{(u+i)^2}\frac{Q_3^{--}Q_4^{+}Q_{\bar{4}}^{+}}{Q_3^{++}Q_4^{-}Q_{\bar{4}}^{-}}\bigg|_{u_{3,k}},$$

$$1=\left(\frac{u-\frac{i}{2}}{u+\frac{i}{2}}\right)^{2L+2}\frac{Q_4^{++}Q_3^{-}}{Q_4^{--}Q_3^{+}}\bigg|_{u_{4,k}},$$

$$1=\left(\frac{u-\frac{i}{2}}{u+\frac{i}{2}}\right)^{2L+2}\frac{Q_{\bar{4}}^{++}Q_3^{-}}{Q_{\bar{4}}^{--}Q_3^{+}}\bigg|_{u_{\bar{4},k}}.$$

the eigenvalues of the Hamiltonian are given by

$$E = \lambda^2 \sum_{j=1}^{K_4} \frac{1}{1/4 + u_{4,j}^2} + \lambda^2 \sum_{j=1}^{K_{\bar{4}}} \frac{1}{1/4 + u_{4,j}^2}.$$

In terms of the eigenvalues of double row transfer matrices, these above mentioned two solutions can be related by the transformation on the Baxter polynomial $Q_3(u)$

$$Q_3(u) \to u^2 Q_3(u).$$

The two sets of Bethe equations are also related by this “gauge” transformation.

The exact bulk S-matrices

Consider the full sector at the finite coupling. The ABJM giant graviton open spin chain has two types of particle which we named A-particle and B-particle and are charge conjugate to each other. The bulk dispersion relation is

$$\epsilon(p) = \frac{1}{2}\sqrt{1 + 16h^2(\lambda)\sin^2(\frac{p}{2})}$$

where the $h(\lambda)$ is the so called interpolation function with the weak coupling expansion

$$h(\lambda) = \lambda - \frac{\pi^2}{3}\lambda^3 + \mathcal{O}(\lambda^5),$$

The anomalous dimension of determinant like operator is related to the bulk energy of the spin chain as

$$\Delta = \sum_{j=1}^{K_A}\left(\epsilon(p_j^A) - \frac{1}{2}\right) + \sum_{j=1}^{K_B}\left(\epsilon(p_j^B) - \frac{1}{2}\right)$$

The scattering matrices of A-particle and B-particle are $\mathfrak{su}(2|2)$ invariant

$$\mathbb{S}^{AA}(p_1, p_2) = \mathbb{S}^{BB}(p_1, p_2) = S_0(p_1, p_2)S(p_1, p_2),$$
$$\mathbb{S}^{AB}(p_1, p_2) = \mathbb{S}^{BA}(p_1, p_2) = \tilde{S}_0(p_1, p_2)S(p_1, p_2).$$

Here $S(p_1, p_2)$ is matrix part of these $\mathfrak{su}(2|2)$ invariant matrices. At finite coupling, it's useful to introduce spectral parameter x

$$x^{\pm} + \frac{1}{x^{\pm}} = \frac{u \pm \frac{i}{2}}{h(\lambda)}, \quad x^{\pm} \equiv x(u \pm \frac{i}{2}).$$

The momentum p and energy ϵ of the fundamental magnon are

$$e^{ip} = \frac{x^+}{x^-}, \quad \epsilon = \frac{1}{2} + ih(\lambda)(\frac{1}{x^+} - \frac{1}{x^-}).$$

We can also write the momentum and energy in terms of Jacobi elliptic functions

$$x^{\pm} = \frac{1}{4h(\lambda)}(\frac{\mathrm{cn}z}{\mathrm{sn}z} \pm i)(1 + \mathrm{dn}z), \quad p(z) = 2\mathrm{am}z, \quad \epsilon(z) = \frac{1}{2}\mathrm{dn}(z).$$

We assume $\mathbb{S}^{AA}$ and $\mathbb{S}^{AB}$ satisfy the unitary conditions, which imply

$$S_0(z_1,z_2)S_0(z_2,z_2)=1, \qquad \tilde{S}_0(z_1,z_2)\tilde{S}_0(z_2,z_1)=1.$$

The identification of the B-particles as charge conjugates of the A-particles suggests the following crossing relations

$$\begin{aligned} C_1^{-1}\mathbb{S}_{12}^{AAt_1}(z_1,z_2)C_1\mathbb{S}_{12}^{AB}(z_1+\omega_2,z_2) &= I_{12}, \\ \mathbb{S}_{12}^{AAt_2}(z_1,z_2)C_2\mathbb{S}_{12}^{AB}(z_1,z_2-\omega_2)C_2^{-1} &= I_{12}. \end{aligned}$$

Then using the relation , the scalar factor should satisfy

$$S_0(z_1,z_2)\tilde{S}_0(z_1+\omega_2,z_2)=S_0(z_1,z_2)\tilde{S}_0(z_1,z_2-\omega_2)=f(x_1,x_2).$$

where C is the charge conjugation matrix

$$C=\begin{pmatrix} \sigma_2 & 0 \\ 0 & i\sigma_2 \end{pmatrix}.$$

Here σ_2 is the Pauli matrix and $f(x_1,x_2)$ is a scalar function defined by

$$f(x_1,x_2)=\frac{(x_1^+-x_2^-)(1-\frac{1}{x_1^-x_2^-})}{(x_1^+-x_2^+)(1-\frac{1}{x_1^-x_2^+})}.$$

These constraints can be solved

$$S_0(z_1,z_2)=\frac{x_1^+-x_2^-}{x_1^--x_2^+}\frac{1-\frac{1}{x_1^+x_2^-}}{1-\frac{1}{x_1^-x_2^+}}\sqrt{\frac{x_1^-}{x_1^+}}\sqrt{\frac{x_2^+}{x_2^-}}\sigma(z_1,z_2),$$

$$\tilde{S}_0(z_1,z_2)=\sqrt{\frac{x_1^-}{x_1^+}}\sqrt{\frac{x_2^+}{x_2^-}}\sigma(z_1,z_2),$$

where σ is the BES dressing phase.

The exact boundary scattering matrices

Quite similar to SYM case. Symmetry preserved by the boundary of our integrable open system is $\mathfrak{su}(1|2)$, which can fix the right boundary reflection matrices up to some scalar factors

$$\mathbb{R}^{A-}(p) = R_0^{A-}(p)R^-(p) = R_0^{A-}(p)\mathrm{diag}(e^{-i\frac{p}{2}}, -e^{i\frac{p}{2}}, 1, 1),$$
$$\mathbb{R}^{B-}(p) = R_0^{A-}(p)R^-(p) = R_0^{B-}(p)\mathrm{diag}(e^{-i\frac{p}{2}}, -e^{i\frac{p}{2}}, 1, 1).$$

The left boundary reflection matrices are related to the right ones as

$$\mathbb{R}^{A+}(p) = \mathbb{R}^{A-}(-p), \quad \mathbb{R}^{B+}(p) = \mathbb{R}^{B-}(-p).$$

There are two types of particles in the bulk, both are transformed under fundamental representation of $\mathfrak{su}(2|2)$. Therefore the bulk Zamolodchikov-Faddeev (ZF) algebra is described by two kinds of creating operators

$$\mathbb{A}_i^\dagger(p), \mathbb{B}_i^\dagger(p), \qquad i = 1, \cdots, 4$$

which satisfy

$$
\begin{aligned}
\mathbb{A}_i^\dagger(p_1)\mathbb{A}_j^\dagger(p_2) &= \mathbb{S}^{AA}(p_1,p_2)_{ij}^{i'j'}\mathbb{A}_{j'}^\dagger(p_2)\mathbb{A}_{i'}^\dagger(p_1), \\
\mathbb{B}_i^\dagger(p_1)\mathbb{B}_j^\dagger(p_2) &= \mathbb{S}^{BB}(p_1,p_2)_{ij}^{i'j'}\mathbb{B}_{j'}^\dagger(p_2)\mathbb{B}_{i'}^\dagger(p_1), \\
\mathbb{A}_i^\dagger(p_1)\mathbb{B}_j^\dagger(p_2) &= \mathbb{S}^{AB}(p_1,p_2)_{ij}^{i'j'}\mathbb{B}_{j'}^\dagger(p_2)\mathbb{A}_{i'}^\dagger(p_1), \\
\mathbb{B}_i^\dagger(p_1)\mathbb{A}_j^\dagger(p_2) &= \mathbb{S}^{BA}(p_1,p_2)_{ij}^{i'j'}\mathbb{A}_{j'}^\dagger(p_2)\mathbb{B}_{i'}^\dagger(p_1).
\end{aligned}
$$

To treat the boundary, we introduce the boundary creating operator $\mathcal{B}_R$ satisfying the boundary ZF algebra

$$
\begin{aligned}
\mathbb{A}_i^\dagger(p)\mathcal{B}_R &= \mathbb{R}^{A-}(p)_i^{i'}\mathbb{A}_{i'}^\dagger(-p)\mathcal{B}_R, \\
\mathbb{B}_i^\dagger(p)\mathcal{B}_R &= \mathbb{R}^{B-}(p)_i^{i'}\mathbb{B}_{i'}^\dagger(-p)\mathcal{B}_R.
\end{aligned}
$$

which imply

$$
R_0^{A-}(p)R_0^A(-p) = 1, \quad R_0^{B-}(p)R_0^B(-p) = 1.
$$

To derive the boundary crossing relation, we introduce the singlet operator

$$\mathbb{I}(p) = C^{ij}\mathbb{A}_i^\dagger(p)\mathbb{B}_j^\dagger(\bar{p}) + C^{ij}\mathbb{B}_i^\dagger(p)\mathbb{A}_j^\dagger(\bar{p}),$$

where $\bar{p}$ is the crossed momentum, defined by

$$x^\pm(\bar{p}) = \frac{1}{x^\pm(p)}.$$

Scattering the singlet operator off the right boundary, we must have

$$\mathbb{I}(p)\mathcal{B}_R = \mathbb{I}(-\bar{p})\mathcal{B}_R.$$

Using the bulk and boundary ZF algebra, we obtain

$$C^{ij}\mathbb{R}^{B-}(\bar{p})_j^{j'}(\bar{p})\mathbb{S}^{AB}(p,-\bar{p})_{ij'}^{i'j''}\mathbb{R}^{A-}(p)_{i'}^{i''} = C^{j''i''},$$
$$C^{ij}\mathbb{R}^{A-}(\bar{p})_j^{j'}(\bar{p})\mathbb{S}^{BA}(p,-\bar{p})_{ij'}^{i'j''}\mathbb{R}^{B-}(p)_{i'}^{i''} = C^{j''i''}.$$

In terms of the scalar factors $R_0^{A-}(p), R_0^{B-}(p)$, the above boundary crossing relations imply

$$R_0^{A-}(p)R_0^{B-}(\bar{p}) = \frac{1}{\sigma(p, -\bar{p})}.$$

We define

$$f_b(p) = \frac{x^- + \frac{1}{x^-}}{x^+ + \frac{1}{x^+}}.$$

A solution is given by the ansatz

$$R_0^{A-}(p) = R_0^{B-}(p) = R_0^-(p),$$

where

$$R_0^{-2}(p) = F(p)\sigma(p, -p)\frac{1}{\sqrt{f_b(p)}}$$

and $F(p)$ is a CDD-type factor satisfies

$$F(p)F(\bar{p}) = 1, \quad F(p)F(-p) = 1.$$

which can fixed (see below)

$$F(p) = -e^{-\frac{ip}{2}}.$$

The shortest operator described the one particle excitation has the form $|p\rangle_{B_1^\dagger}$=$f_{B_1^\dagger}(1)\,|1\rangle_{B_1^\dagger} + f_{B_1^\dagger}(2)\,|2\rangle_{B_1^\dagger}$ with $L = 2$.The anomalous dimension of this operator is related to bulk energy $\epsilon(p)$ of the magnon as

$$\Delta = \epsilon(p) - \frac{1}{2} = \frac{1}{2}\sqrt{1 + 16h^2(\lambda)\sin^2(\frac{p}{2})} - \frac{1}{2} = 4\lambda^2\sin^2(\frac{p}{2}) + \mathcal{O}(\lambda^3).$$

The eigenvalue equation is

$$H\,|p\rangle_{B_1^\dagger} = \Delta\,|p\rangle_{B_1^\dagger} = \begin{pmatrix} \lambda^2 & -\lambda^2 \\ -\lambda^2 & 2\lambda^2 \end{pmatrix}\begin{pmatrix} f_{B_1^\dagger}(1) \\ f_{B_1^\dagger}(2) \end{pmatrix}.$$

we can get the first eigenvalue when $f_{B_1^\dagger}(1)/f_{B_1^\dagger}(2) = \frac{1-\sqrt{5}}{2}$ i.e. when $p = 3\pi/5$

$$\Delta_+ = \frac{3+\sqrt{5}}{2}\lambda^2 = 4\lambda^2\sin^2(\frac{3\pi}{10}),$$

and the second eigenvalue with $f_{B_1^\dagger}(1)/f_{B_1^\dagger}(2)=\frac{1+\sqrt{5}}{2}$, which means $p=\frac{\pi}{5}$

$$\Delta_-=\frac{3-\sqrt{5}}{2}\lambda^2=4\lambda^2\sin^2(\frac{\pi}{10}).$$

In a similar way, we find the possible momentum values of a single A_1 excitation to be $p=\frac{\pi}{5},\frac{3\pi}{5}$ with $L=3$.
For $L=4$, we obtain $p=\frac{\pi}{7},\frac{3\pi}{7},\frac{5\pi}{7}$.

For a single particle excitation, the boundary Bethe-Yang equation reads

$$e^{-2ipL}\mathbb{R}^+(-p)\mathbb{R}^-(p) = e^{-2ipL}R_0^2(p)\mathrm{diag}(e^{-ip}, e^{ip}, 1, 1) = 1.$$

At leading order, this reduce to

$$e^{-2ipL}\mathrm{diag}(e^{-ip}, e^{ip}, 1, 1) = -1.$$

For a single $B_1^\dagger$ excitation, we obtain the quantized momentum

$$p_n = \frac{n\pi}{2L+1}, \qquad n = 1, 3, \cdots, 2L-1.$$

Similarly, for a single A_1 excitation, the quantized momentum is

$$p_n = \frac{n\pi}{2L-1}, \qquad n = 1, 3, \cdots, 2L-3.$$

In order to obtain the right boundary Bethe-Yang equations from the double row transfer matrices, we should define the fundamental double row transfer matrices as following

$$\mathbb{D}(p,\{p_i^A,p_i^B\})$$
$$=\mathrm{Tr}_a\left(\prod_{i=K^I}^{K_A^I+1}\mathbb{S}_{ai}^{AB}(p,p_i^B)\prod_{i=K_A^I}^{1}\mathbb{S}_{ai}^{AA}(p,p_i^A)\mathbb{R}_a^-(p)\prod_{i=1}^{K_A^I}\mathbb{S}_{ia}^{AA}(p_i^A,-p)\prod_{i=K_A^I+1}^{K^I}\mathbb{S}_{ia}^{AB}(p_{\tilde{i}}^B,-p)\breve{\mathbb{R}}_a^+(-p)\right)$$
$$\tilde{\mathbb{D}}(p,\{p_i^A,p_i^B\})$$
$$=\mathrm{Tr}_a\left(\prod_{i=K^I}^{K_A^I+1}\mathbb{S}_{ai}^{BB}(p,p_i^B)\prod_{i=K_A^I}^{1}\mathbb{S}_{ai}^{BA}(p,p_i^A)\mathbb{R}_a^-(p)\prod_{i=1}^{K_A^I}\mathbb{S}_{ia}^{BA}(p_i^A,-p)\prod_{i=K_A^I+1}^{K^I}\mathbb{S}_{ia}^{BB}(p_{\tilde{i}}^B,-p)\breve{\mathbb{R}}_a^+(-p)\right)$$

where

$$\mathbb{R}^-(p)=R_0(p)R^-(p),\quad \breve{\mathbb{R}}^+(-p)=\breve{R}_0^+(-p)\breve{R}^+(-p),$$
$$K^I=K_A^I+K_B^I,\qquad \tilde{i}=i-K_A^I.$$

and

$$R^-(p) = \mathrm{diag}(e^{-\frac{ip}{2}}, -e^{\frac{ip}{2}}, 1, 1).$$

$\breve{\mathbb{R}}^+(p)$ is defined through

$$\mathbb{R}_a^-(p) = \mathrm{Tr}_{a'}(\mathbb{P}_{aa'}\mathbb{S}_{aa'}^{AA}(p,-p)\breve{\mathbb{R}}_{a'}^+(-p))$$

such that the boundary Bethe-Yang equations can be obtained from the double row transfer matrices as

$$e^{-2ip_j^A L}\mathbb{D}(p_j^A, \{p_i^A, p_i^B\}) = -1, \quad e^{-2ip_j^B L}\tilde{\mathbb{D}}(p_j^B, \{p_i^A, p_i^B\}) = -1.$$

Using the explicit form of the S-matrix, one can solve the equation as

$$\begin{aligned}
\breve{R}_0^+(-p) &= \frac{e^{-ip}R_0^-(p)}{S_0(p,-p)\rho(p)}, \\
\breve{R}^+(-p) &= (-1)^F R^-(-p) = \mathrm{diag}(e^{\frac{ip}{2}}, -e^{-\frac{ip}{2}}, -1, -1),
\end{aligned}$$

where

$$\rho = \frac{(1+(x^-)^2)(x^+ + x^-)}{2x^+(1+x^+x^-)}.$$

Collecting the scalar factors, we have

$$\mathbb{D}(p,\{p_i^A,p_i^B\}) = d(p)D(p,\{p_i^A,p_i^B\}),$$
$$\tilde{\mathbb{D}}(p,\{p_i^A,p_i^B\}) = \tilde{d}(p)D(p,\{p_i^A,p_i^B\}),$$

The eigenvalue $\Lambda(p)$ of $D(p,\{p_i^A,p_i^B\})$ is known [Bajnok,Nepomechie,Palla&Suzuki 12]. The main or physical Bethe equations for the massive roots are given by

$$e^{-2ip_j^A L}d(p_j^A)\Lambda(p_i^A) = -1, \qquad e^{-2ip_j^B L}\tilde{d}(p_j^B)\Lambda(p_j^B) = -1,$$

Bethe equations for the auxiliary roots are obtained from the analytic condition of $\Lambda(p)$.

Reducing to the scalar sector at two loop order

In the weak coupling limit $h(\lambda) \to 0$,the asymptotic Bethe ansatz equations reduced to

$$
\begin{aligned}
1 &= \frac{Q_2^-}{Q_2^+}\bigg|_{u_{1,k}}, \\
-1 &= \frac{u^-}{u^+}\frac{Q_1^- Q_3^- Q_2^{++}}{Q_1^+ Q_3^+ Q_2^{--}}\bigg|_{u_{2,k}}, \\
1 &= \frac{Q_2^- Q_4^+ Q_{\bar{4}}^+}{Q_2^+ Q_4^- Q_{\bar{4}}^-}\bigg|_{u_{3,k}}, \\
1 &= \left(\frac{u-\frac{i}{2}}{u+\frac{i}{2}}\right)^{2L'} \frac{Q_4^{++} Q_3^-}{Q_4^{--} Q_3^+}\bigg|_{u_{4,k}}, \\
1 &= \left(\frac{u-\frac{i}{2}}{u+\frac{i}{2}}\right)^{2L'} \frac{Q_{\bar{4}}^{++} Q_3^-}{Q_{\bar{4}}^{--} Q_3^+}\bigg|_{u_{\bar{4},k}},
\end{aligned}
$$

where $L' = L + \frac{K_1 - K_3}{2} + \frac{K_4 + K_{\bar{4}} - 1}{2}$.

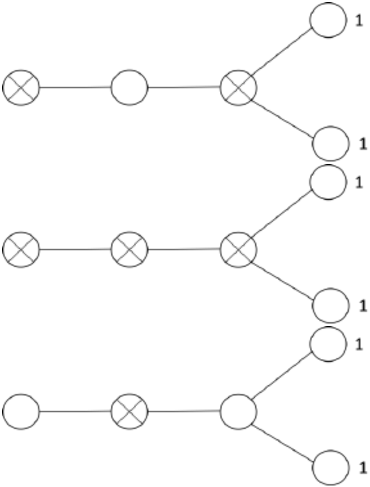

Figure: The chain of Fermionic duality.

After a chain of fermionic duality, the following Bethe equations can be derived

$$-1 = \frac{u^-}{u^+} \frac{Q_{\tilde{1}}^{++} Q_{\tilde{2}}^{-}}{Q_{\tilde{1}}^{--} Q_{\tilde{2}}^{+}}\bigg|_{\tilde{u}_{1,k}},$$

$$1 = \frac{Q_{\tilde{1}}^{-} Q_{3}^{+}}{Q_{\tilde{1}}^{+} Q_{3}^{-}}\bigg|_{\tilde{u}_{2,k}},$$

$$-1 = \frac{u^+}{u^-} \frac{Q_{\tilde{2}}^{+} Q_{3}^{--} Q_{4}^{+} Q_{\bar{4}}^{+}}{Q_{\tilde{2}}^{-} Q_{3}^{++} Q_{4}^{-} Q_{\bar{4}}^{-}}\bigg|_{u_{3,k}},$$

$$1 = \left(\frac{u-\frac{i}{2}}{u+\frac{i}{2}}\right)^{2\tilde{L}'} \frac{Q_{4}^{++} Q_{3}^{-}}{Q_{4}^{--} Q_{3}^{+}}\bigg|_{u_{4,k}},$$

$$1 = \left(\frac{u-\frac{i}{2}}{u+\frac{i}{2}}\right)^{2\tilde{L}'} \frac{Q_{\bar{4}}^{++} Q_{3}^{-}}{Q_{\bar{4}}^{--} Q_{3}^{+}}\bigg|_{u_{\bar{4},k}}.$$

Removing the the first and second type of Bethe roots $\tilde{u}_{1,k}, \tilde{u}_{2,k}$ and identify $\tilde{L}' = L$, we obtain the Same equations as before.

Thank you for your attention!