# Generalized Macdonald polynomials at higher level

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**[Based on [JEB, Cassia, Stoyan 2508.19704]]**

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• This work follows the same philosophy. By studying **generalized Macdonald polynomials**, an object originally introduced in the gauge theory context, we find new results at the interface of mathematical physics, representation theory and algebraic combinatorics.

But more on this later...

## Summary

- I will present three of our main results:
  i. Pieri rules for generalized Macdonald polynomials
  ii. Factorization formula for the reproducing kernel
  iii. Extension of a relation between Macdonald polynomials and Whittaker vectors

# Outline

1. Toy model: quantum harmonic oscillator

2. Macdonald symmetric functions

3. Generalized Macdonald polynomials

4. Motivations and discussion

# 1. Toy model: quantum harmonic oscillator

## Quantum harmonic oscillator

- The (1d) quantum harmonic oscillator is one of the simplest quantum mechanical system. It is defined by the following operator acting on $L^2(\mathbb{R})$ called *Hamiltonian*,

$$H = -\frac{1}{2}\partial_x^2 + \frac{1}{2}x^2, \quad \partial_x = \frac{\partial}{\partial x}.$$

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- The stationary states are the eigenstates of this operator,

$$Hv_n(x) = E_n v_n(x) \quad \Rightarrow \quad v_n(x) = h_n(x)e^{-\frac{1}{2}x^2}, \quad E_n = n + \frac{1}{2},$$

where $h_n(x)$ are the Hermite polynomials, $n \in \mathbb{Z}^{\geq 0}$.

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- This system is a good toy model to introduce the notion of Fock space.

- For this purpose, we solve the system using the method of ladder operators,

$$a = \frac{1}{\sqrt{2}}(x + \partial_x), \quad a^\dagger = \frac{1}{\sqrt{2}}(x - \partial_x), \quad [a, a^\dagger] = 1,$$

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- The ground state is obtained from the vector $v_0$ such that $av_0 = 0$,

$$av_0(x) = \frac{1}{\sqrt{2}}(x + \partial_x)v_0(x) = 0 \quad \Rightarrow \quad v_0(x) = Ce^{-\frac{1}{2}x^2}, \quad Hv_0(x) = \frac{1}{2}v_0(x).$$

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- Excited states are obtained using the creation operators

$$v_n(x) = (a^\dagger)^n v_0(x) \quad \Rightarrow \quad v_n(x) = h_n(x)e^{-\frac{1}{2}x^2}, \quad Hv_n(x) = (n + 1/2)v_n(x).$$

## Fock space

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- In this simple example, the analogue of the Pieri rules are trivial. They encode the action of the algebra on $F$,

$$a^\dagger v_n = v_{n+1}, \quad z \times z^n = z^{n+1},$$
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This is a toy version of our first result!

# Reproducing kernel

- Introduce the inner product $(v_n, v_m) = n!\delta_{n,m}$ such that $a$ and $a^\dagger$ are adjoint of each other, and $H$ self-adjoint. Let

$$\Pi = \sum_{n\geq 0} \frac{v_n \otimes v_n}{(v_n, v_n)} \quad \Rightarrow \quad (v_n \otimes 1, \Pi) = v_n.$$

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- Under the isomorphism $F \otimes F \simeq \mathbb{C}[z] \otimes \mathbb{C}[w]$,

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This is a toy version of our second result!

## Whittaker vectors

• The quantization of classical solutions correspond to coherent states that are eigenstates of the annihilation operator,

$$aw_\alpha(x) = \alpha w_\alpha(x) \quad \Rightarrow \quad w_\alpha(x) = C_\alpha e^{-\frac{1}{2}x^2 + \alpha\sqrt{2}x}.$$

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- Under the isomorphism $F \simeq \mathbb{C}[z]$,

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- This notion can be extended to Lie algebras and Verma modules. In this context, the states $w_\alpha$ are called **Whittaker vectors**.

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- For instance, for $\mathfrak{sl}(2)$,

$$[e,f]=h, \quad [h,e]=2e, \quad [h,f]=-2f.$$

The Verma module is built upon $v_0$ with $fv_0=0$, $hv_0=\lambda v_0$, $V_\lambda=\mathrm{Span}_{n\in\mathbb{Z}\geq 0}\{e^n v_0\}$. The Whittaker vectors are

$$fw_\alpha=\alpha w_\alpha \quad \Rightarrow \quad w_\alpha=C_\alpha\sum_{n\geq 0}\frac{(-\alpha)^n}{n!}\frac{e^n v_0}{(\lambda+n-1)(\lambda+n-2)\cdots\lambda}$$

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In general, Whittaker vectors are not related to weight vectors,

but quantum toroidal $\mathfrak{gl}(1)$ is very special!!!

# 2. Macdonald symmetric functions

## Bosonic Fock space

- We consider infinitely many copies of the algebra of ladder operators $a_k$ and $a_k^\dagger = a_{-k}$,

$$[a_k, a_l] = k\delta_{k+l,0}, \quad k \in \mathbb{Z}^\times.$$

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- Upon ordering the creation operators, we can write a basis indexed by partitions $\lambda \in \mathcal{P}$,

$$\mathcal{F} = \operatorname*{Span}_{\lambda\in\mathcal{P}} \{a_{-\lambda} v_0\}, \quad a_{-\lambda} v_0 = a_{-\lambda_1} a_{-\lambda_2} \cdots a_{-\lambda_\ell} v_0.$$

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- Next we need to define the isomorphism with polynomials...

$\Rightarrow$ We need to consider an infinite number of variables!

- Symmetric functions can be defined as a limit $N \to \infty$ of symmetric polynomials $\Lambda_N[\boldsymbol{x}] = \mathbb{C}[x_1, x_2, \cdots, x_N]^{S_N}$. They form a ring $\Lambda[\boldsymbol{x}]$, and have a basis, indexed by partitions, obtained as a product of power sums,

$$\Lambda[\boldsymbol{x}] = \underset{\lambda \in \mathcal{P}}{\mathrm{Span}} \left\{ p_{\lambda_1}(\boldsymbol{x}) p_{\lambda_2}(\boldsymbol{x}) \cdots p_{\lambda_\ell}(\boldsymbol{x}) \right\}, \quad p_k(\boldsymbol{x}) = \sum_i x_i^k.$$

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- We have a ring isomorphism $\mathcal{F} \simeq \Lambda[\boldsymbol{x}]$,

$$v_0 \equiv 1, \quad a_{-\lambda} v_0 \equiv p_{\lambda_1}(\boldsymbol{x}) p_{\lambda_2}(\boldsymbol{x}) \cdots p_{\lambda_\ell}(\boldsymbol{x}), \quad a_{-k} \equiv p_k(x), \quad a_k \equiv k \frac{\partial}{\partial p_k(\boldsymbol{x})}, \quad (k > 0).$$

$\rightsquigarrow$ $p_k(\boldsymbol{x})$ plays the role of our variable $z$ in $F \simeq \mathbb{C}[z]$ previously.

# Macdonald polynomials

- Consider the Ruijsenaars Hamiltonian on $\Lambda_N[\boldsymbol{x}] = \mathbb{C}[x_1, \cdots, x_N]^{S_N}$,

$$H_N = (1 - t^{-1}) \sum_{i=1}^{N} \prod_{j \neq i} \frac{x_i - t^{-1} x_j}{x_i - x_j} \; q^{x_i \partial_{x_i}},$$

with the shift operator $q^{x_i \partial_{x_i}} f(x_1, \cdots, x_N) = f(x_1, \cdots, x_{i-1}, q x_i, x_{i+1}, \cdots, x_N)$.

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$\rightsquigarrow$ In the limit $N \to \infty$, it defines an operator $H$ on $\Lambda[\boldsymbol{x}]$ (Macdonald operator).

- Macdonald symmetric functions can be defined as the eigenvectors of $H$,

$$H P_\lambda(\boldsymbol{x}) = E_\lambda P_\lambda(\boldsymbol{x}), \quad \lambda \in \mathcal{P}.$$

Note: $\rightsquigarrow$ They form a basis of $\Lambda[\boldsymbol{x}]$.

$\rightsquigarrow$ By isomorphism, they can also be seen as vectors in $\mathcal{F}$.

$\rightsquigarrow$ They play the role of the stationary states $v_n$ of the quantum harmonic oscillator.

$\rightsquigarrow$ Note that both $H$ and $P_\lambda(\boldsymbol{x})$ depend on the extra parameters $(q, t) \in \mathbb{C}^\times \times \mathbb{C}^\times$.

- Macdonald symm. functions are known to obey the Pieri rule (and its dual)

$$p_1(\boldsymbol{x})P_\lambda(\boldsymbol{x}) = \sum_{\square\in A(\lambda)} \psi_\lambda(\square)P_{\lambda+\square}(\boldsymbol{x})$$

$$\frac{\partial}{\partial p_1(\boldsymbol{x})}P_\lambda(\boldsymbol{x}) = \sum_{\square\in R(\lambda)} \psi_\lambda^*(\square)P_{\lambda-\square}(\boldsymbol{x}),$$

with known coefficients $\psi_\lambda(\square)$, $\psi_\lambda^*(\square)$. Here $A(\lambda)$ (resp. $R(\lambda)$) are the sets of boxes that can be added to (resp. removed from) $\lambda$.

$\Rightarrow$ This gives the action of the modes $a_{\pm 1}$ on $P_\lambda(\boldsymbol{x})$.

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This is the first result we want to generalize!

## Kernel

- Let $(\cdot,\cdot)$ be the Macdonald scalar product, which is such that $H$ is self-adjoint. We have $(P_\lambda(\boldsymbol{x}), P_\mu(\boldsymbol{y})) = b_\lambda^{-1}\delta_{\lambda,\mu}$, with a known combinatorial coefficient $b_\lambda$.

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- The reproducing kernel can be defined as

$$\Pi(\boldsymbol{x}|\boldsymbol{y}) = \sum_{\lambda\in\mathcal{P}} b_\lambda P_\lambda(\boldsymbol{x})P_\mu(\boldsymbol{y}) \quad \Rightarrow \quad (P_\lambda(\boldsymbol{x}), \Pi(\boldsymbol{x}|\boldsymbol{y})) = P_\lambda(\boldsymbol{y}).$$

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- **[Property]** The kernel has a factorized formula

$$\Pi(\boldsymbol{x}|\boldsymbol{y}) = \exp\left(\sum_{k>0} \frac{1}{k}\frac{1-t^k}{1-q^k} p_k(\boldsymbol{x}) p_k(\boldsymbol{y})\right).$$

This is the second result we want to generalize!

# Whittaker vectors

• Remark: Let $\lambda \in \mathcal{P}$, and introduce the infinite set of variables $\epsilon_\lambda = (t^{-1}q^{\lambda_1}, t^{-2}q^{\lambda_2}, \cdots)$ (with $\lambda_i = 0$ for $i > \ell(\lambda)$). We have

$$p_k(\epsilon_\lambda) = \sum_{i=1}^{\ell(\lambda)} t^{-i} q^{\lambda_i} - \frac{t^{-\ell(\lambda)}}{1-t}, \qquad (|t| > 1).$$

$\rightsquigarrow$ It implies that $P_\mu(\epsilon_\lambda)$ is well defined, since it has a finite decomposition on the power sum basis.

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$$p_k(\epsilon_\lambda) = \sum_{i=1}^{\ell(\lambda)} t^{-i}q^{\lambda_i} - \frac{t^{-\ell(\lambda)}}{1-t}, \qquad (|t|>1).$$

$\rightsquigarrow$ It implies that $P_\mu(\epsilon_\lambda)$ is well defined, since it has a finite decomposition on the power sum basis.

- Let's consider the specialisation

$$W_\lambda(\boldsymbol{x}) = \Pi(\boldsymbol{x}|\epsilon_\lambda) = \sum_{\mu\in\mathcal{P}} b_\mu P_\mu(\epsilon_\lambda)\ P_\mu(\boldsymbol{x}) \in \bar{\Lambda}[\boldsymbol{x}].$$

The factorized expression of the kernel implies that $W_\lambda(\boldsymbol{x})$ are eigenvectors of $a_k$ for $k>0$,

$$a_k W_\lambda(\boldsymbol{x}) = \frac{\partial}{\partial p_k(\boldsymbol{x})} W_\lambda(\boldsymbol{x}) = \frac{1-t^k}{1-q^k} p_k(\epsilon_\lambda)\ W_\lambda(\boldsymbol{x}).$$

$\rightsquigarrow$ They are Whittaker vectors of the algebra $\widehat{\mathfrak{gl}(1)}$ generated by $a_k$ with $k \in \mathbb{Z}^\times$.

- **[Theorem]** **[Garsia, Haiman, Tesler 2010]**

Whittaker vectors and Macdonald sym. functions are related by

$$W_\lambda(\boldsymbol{x}) = \mathrm{V}\,\frac{P_\lambda(\boldsymbol{x})}{P_\lambda(\epsilon_\emptyset)},$$

with

$$\mathrm{V} = \nabla e^{\sum_{k>0} \frac{(-)^k}{k(1-q^k)} p_k(\boldsymbol{x})}\, t^{-L_0}\, e^{\sum_{k>0} \frac{(-)^k}{1-t^k} \frac{\partial}{\partial p_k(\boldsymbol{x})}} \nabla,$$

$$L_0 P_\lambda(\boldsymbol{x}) = |\lambda| P_\lambda(\boldsymbol{x}), \quad \nabla P_\lambda(\boldsymbol{x}) = \prod_{(i,j)\in\lambda} t^{1-i} q^{j-1}\, P_\lambda(\boldsymbol{x}).$$

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This is the third result we want to generalize!

# 3. Generalized Macdonald polynomials

# Quantum toroidal $\mathfrak{gl}(1)$ algebra

- To introduce generalized Macdonald polynomials, we need to recall first the quantum toroidal $\mathfrak{gl}(1)$ action on $\mathcal{F} \simeq \Lambda[\boldsymbol{x}]$.

# Quantum toroidal $\mathfrak{gl}(1)$ algebra

- To introduce generalized Macdonald polynomials, we need to recall first the quantum toroidal $\mathfrak{gl}(1)$ action on $\mathcal{F} \simeq \Lambda[\boldsymbol{x}]$.

- The quantum toroidal $\mathfrak{gl}(1)$ algebra $\mathcal{E}$ depends on two quantum group parameters $(q,t)$. It is generated by the modes of four currents

$$x^{\pm}(z) = \sum_{k\in\mathbb{Z}} x_k^{\pm}, \quad \psi^{\pm}(z) = \psi_0^{\pm} e^{\sum_{k>0} z^{\mp k} h_{\pm k}},$$

and two central elements $c$ and $\psi_0^{\pm} = \gamma^{\mp\bar{c}}$ with $\gamma = t^{\frac{1}{2}} q^{-\frac{1}{2}}$.

## Quantum toroidal $\mathfrak{gl}(1)$ algebra

• To introduce generalized Macdonald polynomials, we need to recall first the quantum toroidal $\mathfrak{gl}(1)$ action on $\mathcal{F} \simeq \Lambda[\boldsymbol{x}]$.

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• Multiplicatively, the algebra has only six generators: $x_0^{\pm}$, $h_{\pm 1}$ and $(c,\bar{c})$.
(e.g. $x_k^{\pm} \propto [h_k, x_0^{\pm}]$)
$\Rightarrow$ It is sufficient to specify the action of these generators to define a representation!

- In the Fock representation $\rho_u^{(1,0)}$, the algebra $\mathcal{E}$ acts on $\Lambda[\boldsymbol{x}]$ as follows,

$$\rho_u^{(1,0)}(c,\bar{c}) = (1,0), \quad \rho_u^{(1,0)}(x_0^+) = uH, \quad \rho_u^{(1,0)}(x_0^-) = u^{-1}H^*, \quad \rho_u^{(1,0)}(h_{\pm 1}) \propto a_{\pm 1},$$

where $H^*$ is another operator diagonal on Macdonald polynomials, and $u \in \mathbb{C}^\times$ is the weight.

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- Representations of higher level can be obtained using the coproduct of the algebra ,

$$\Delta(x^+(z)) = x^+(z) \otimes 1 + \psi^-(\gamma^{c_1/2}z) \otimes x^+(\gamma^{c_1}z),$$

$$\Delta(x^-(z)) = x^-(\gamma^{c_2}z) \otimes \psi^+(\gamma^{c_2/2}z) + 1 \otimes x^-(z),$$

$$\Delta(\psi^\pm(z)) = \psi^\pm(\gamma^{\pm c_2/2}z) \otimes \psi^\pm(\gamma^{\mp c_1/2}z),$$

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and $\Delta(c) = c_1 + c_2$ (with $c_1 = c \otimes 1$, $c_2 = 1 \otimes c$).

- For instance, a representation of levels $(2,0)$ acting on $\Lambda[\boldsymbol{x},\boldsymbol{y}] = \Lambda[\boldsymbol{x}] \otimes \Lambda[\boldsymbol{y}]$ is defined as

$$\rho_{u_1,u_2}^{(2,0)} = \left(\rho_{u_1}^{(1,0)} \otimes \rho_{u_2}^{(1,0)}\right) \circ \Delta.$$

$\Rightarrow$ It produces a new Hamiltonian as $\rho_{u_1,u_2}^{(2,0)}(x_0^+) = u_2 H_2(Q)$, with $Q = u_1/u_2$.

- Generalized Macdonald polynomials at level two are defined as the eigenvectors of $H_2(Q)$,

$$H_2(Q)P_{\lambda,\mu}(\boldsymbol{x},\boldsymbol{y}|Q) = (QE_\lambda + E_\mu)P_{\lambda,\mu}(\boldsymbol{x},\boldsymbol{y}|Q).$$

$\rightsquigarrow$ The action of $H_2(Q)$ on $P_\lambda(\boldsymbol{x})P_\mu(\boldsymbol{y})$ is triangular, and so it is diagonalized by a triangular transformation,

$$P_{\lambda,\mu}(\boldsymbol{x},\boldsymbol{y}|Q) = \sum_{(\rho,\sigma)\prec(\lambda,\mu)} G_{\lambda,\mu}^{\rho,\sigma}(Q)P_\rho(\boldsymbol{x})P_\sigma(\boldsymbol{y}).$$

$\rightsquigarrow$ It implies that $\{P_{\lambda,\mu}(\boldsymbol{x},\boldsymbol{y}|Q)\}_{\lambda,\mu\in\mathcal{P}}$ form a basis of $\Lambda[\boldsymbol{x},\boldsymbol{y}]$.

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Examples:

$$P_{\square,\emptyset}(\boldsymbol{x},\boldsymbol{y}|Q)=P_\square(\boldsymbol{x}),\quad P_{\emptyset,\square}(\boldsymbol{x},\boldsymbol{y}|Q)=P_\square(\boldsymbol{y})+\frac{1-tq^{-1}}{1-Q}P_\square(\boldsymbol{x}),$$

$$P_{1,1}(\boldsymbol{x},\boldsymbol{y}|Q)=P_1(\boldsymbol{x})P_1(\boldsymbol{y})+\frac{(1-q)(1-tq^{-1})(1+t)}{(1-t^{-1}Q)(1-qt)}P_{1^2}(\boldsymbol{x})+\frac{1-tq^{-1}}{1-qQ}P_2(\boldsymbol{x}).$$

## Main results

Disclaimer: For simplicity, I will restrict myself to the level two, but similar results have been established for any integer level.

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1. Proof of the Pieri formulas:

$$(p_1(\boldsymbol{x})+p_1(\boldsymbol{y}))P_{\lambda,\mu}(\boldsymbol{x},\boldsymbol{y}|Q)=\sum_{\square\in A(\lambda)}\Psi_\mu(Q\chi_\square)\psi_\lambda(\square)P_{\lambda+\square,\mu}(\boldsymbol{x},\boldsymbol{y}|Q)+\sum_{\square\in A(\mu)}\psi_\mu(\square)P_{\lambda,\mu+\square}(\boldsymbol{x},\boldsymbol{y}|Q),$$

$$\left(\frac{\partial}{\partial p_1(\boldsymbol{x})}+q_3\frac{\partial}{\partial p_1(\boldsymbol{y})}\right)P_{\lambda,\mu}(\boldsymbol{x},\boldsymbol{y}|Q)=\sum_{\square\in R(\lambda)}\psi^*_\lambda(\square)\ P_{\lambda-\square,\mu}(\boldsymbol{x},\boldsymbol{y}|Q)+\sum_{\square\in R(\mu)}\Psi_\lambda(\chi_\square/Q)\psi^*_\mu(\square)\ P_{\lambda,\mu-\square}(\boldsymbol{x},\boldsymbol{y}|Q),$$

where $\Psi_\lambda(z)$ is a known rational function ($\ell$-weight of $\mathcal{E}$'s evaluation representation).

$\Rightarrow$ This gives the action $\rho^{(2,0)}(h_{\pm1})$ on the generalized Macdonald basis.

$\rightsquigarrow$ Proof: Corrolary of **[Fukuda, Okhubo, Shiraishi 2019]**.

## 2. Factorization of the reproducing kernel

Let

$$\Pi(\boldsymbol{x},\boldsymbol{y}|\boldsymbol{a},\boldsymbol{b}|Q)=\sum_{\lambda,\mu\in\mathcal{P}} b_\lambda b_\mu P_{\lambda,\mu}(\boldsymbol{x},\boldsymbol{y}|Q)P_{\mu,\lambda}(\boldsymbol{a},\boldsymbol{b}|Q^{-1}).$$

Then, we show that

$$\Pi(\boldsymbol{x},\boldsymbol{y}|\boldsymbol{a},\boldsymbol{b}|Q)=\Pi(\boldsymbol{x}|\boldsymbol{b})\Pi(\boldsymbol{y}|\boldsymbol{a})e^{\sum_{k>0}\frac{1}{k}\frac{1-t^k}{1-q^k}(1-t^kq^{-k})p_k(\boldsymbol{x})p_k(\boldsymbol{a})}.$$

⇝ Conjectured in **[Zenkevich 2014]**.

⇝ Proof: Use the interplay between the coproduct structure and an anti-automorphism realizing the adjoint action.

## 3. Generalization of the GHT formula

We define the Whittaker states by its algebraic characterization under $\mathcal{E}$ ($k > 0$, $l \geq 0$),

$$\rho^{(2,0)}_{u_1,u_2}(h_k) W_{\lambda,\mu}(\boldsymbol{x},\boldsymbol{y}|\boldsymbol{v}) = \left( (-\gamma v_1)^k p_k(\epsilon_\lambda) + (-\gamma v_2)^k p_k(\epsilon_\mu) \right) W_{\lambda,\mu}(\boldsymbol{x},\boldsymbol{y}|\boldsymbol{v}),$$

$$\begin{aligned}\rho^{(2,0)}_{u_1,u_2}(x_l^+) W_{\lambda,\mu}(\boldsymbol{x},\boldsymbol{y}|\boldsymbol{v}) = &\sum_{\square\in A(\lambda)} (v_1\chi_\square)^l \Psi_\mu(Q\chi_\square)\psi_\lambda(\square) W_{\lambda+\square,\mu}(\boldsymbol{x},\boldsymbol{y}|\boldsymbol{v}) \\ &+ \sum_{\square\in A(\mu)} (v_2\chi_\square)^l \psi_\mu(\square) W_{\lambda,\mu+\square}(\boldsymbol{x},\boldsymbol{y}|\boldsymbol{v})\end{aligned}$$

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$\Rightarrow$ Obtain explicit expression using vertex operator construction **[Awata, Feigin, Shiraishi 2012]**.

- Using these expressions, we propose a generalization of the GHT formula,

$$W_{\lambda,\mu}(\boldsymbol{x},\boldsymbol{y}|\boldsymbol{v}) = \mathrm{V} P_{\lambda,\mu}(\boldsymbol{x},\boldsymbol{y}|Q),$$

with $Q = v_1/v_2$. The operator $\mathrm{V}$ is constructed explicitly in terms of vertex operators.

Proof? In progress....

$\rightsquigarrow$ Shown that $\mathrm{V}$ possess the required algebraic properties.

$\rightsquigarrow$ Then, induction on $(\lambda,\mu)$. We are missing $(\lambda,\mu)=(\emptyset,\emptyset)$ but computer checks.

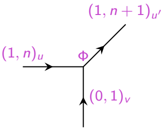
# Connections with refined topological strings

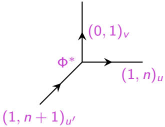
- One of the key ingredient is the vertex operator for quantum toroidal $\mathfrak{gl}(1)$,

**[Awata, Feigin, Shiraishi 2011]**

$$\Phi^{(1,n)}[u,v]\left(\rho_v^{(0,1)}\otimes\rho_u^{(1,n)}\ \Delta(e)\right)=\rho_{u'}^{(1,n+1)}(e)\,\Phi^{(1,n)}[u,v],$$

$$\Phi^{(1,n)*}[u,v]\,\rho_{u'}^{(1,n+1)}(e)=\left(\rho_v^{(0,1)}\otimes\rho_u^{(1,n)}\ \Delta'(e)\right)\Phi^{(1,n)*}[u,v].$$



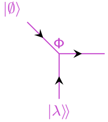




$\rightsquigarrow$ Matrix elements of these vertex operators reproduce the refined topological vertex.

**[Iqbal, Kozcaz, Vafa 2009] [Awata, Kanno 2009]**

- In this language, the rank one Whittaker vector reads

$$W_\lambda(\boldsymbol{x}) = \Phi_\lambda^{(1,-1)} \, |\emptyset\rangle =$$



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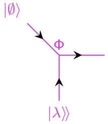
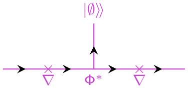



- The operator $V$ has the form

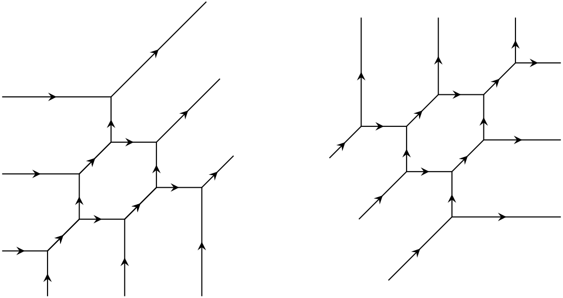
$$V = \nabla \Phi_\emptyset^* \nabla =$$



- To generalize the formula to level rank, we introduce higher vertex operators

$$\Phi^{(r,\boldsymbol{n})}\left(\rho_{\boldsymbol{v}}^{(0,r)}\otimes\rho_{\boldsymbol{u}}^{(r,\boldsymbol{n})}\right)\circ\Delta(e)=\rho_{\boldsymbol{u}'}^{(r,\boldsymbol{n}')}(e)\Phi^{(r,\boldsymbol{n})},$$

$$\Phi^{(r,\boldsymbol{n})*}\rho_{\boldsymbol{u}'}^{(r,\boldsymbol{n}')}(e)=\left(\rho_{\boldsymbol{v}}^{(0,r)}\otimes\rho_{\boldsymbol{u}}^{(r,\boldsymbol{n})}\right)\circ\Delta'(e)\Phi^{(r,\boldsymbol{n})*},$$



$\rightsquigarrow$ These operators are associated to the trinion diagrams $T_r$ of topological strings.

**[Coman, Pomoni, Teschner 2019]**

# 4. Motivations and discussion

# Motivations

- Generalized Macdonald polynomials of level $r$ were introduced in the context of the 5d AGT correspondence. This correspondence relates partition functions of 5d $\mathcal{N} = 1$ gauge theories on $\mathbb{C}_q \times \mathbb{C}_{t^{-1}} \times S^1$ to conformal blocks of q-deformed W-algebras (lift of 2d CFT).

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But we have also a different motivation in mind!!!

# Integrable hierarchies

- Integrable hierarchies are infinite systems of compatible non-linear differential equations.

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- Schur symmetric functions play an essential role in the theory of integrable hierarchies. They provide polynomial solutions for the KP (Kadomtsev–Petviashvili) hierarchy. But, more importantly, they also serve as coordinates on the space of solutions (Sato's infinite Grassmaniann), the so-called **Plucker coordinates**. This special role is due to the following property

$$\Psi(\boldsymbol{x}, \boldsymbol{y}) s_\lambda(\boldsymbol{x}) s_\lambda(\boldsymbol{y}) = 0,$$

where the operator $\Psi(\boldsymbol{x}, \boldsymbol{y})$ is the Casimir of the $\widehat{\mathfrak{gl}(\infty)}$ algebra.

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• In **[JEB, Garbali 2023]**, a $(q,t)$-deformation of integrable hierarchies was introduced. It is argued that the Casimir $\Psi$ should be replaced by screening charges of the q-Virasoro algebra.

What plays the role of Schur polynomials in this context?

- Schur polynomials are known to be $(q,t)$-deformed into Macdonald polynomials. However, the non-trivial coproduct structure makes it unlikely that the simple product $P_\lambda(\boldsymbol{x})P_\mu(\boldsymbol{y})$ will do. So, we probably need to consider a **double of Sato's Grassmaniann**.

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- What comes to mind for its coordinates is $P_{\lambda,\mu}(\boldsymbol{x},\boldsymbol{y}|Q)$. Experimentally, we find e.g.

$$\Psi_n^+(\boldsymbol{x},\boldsymbol{y})P_{\lambda,\mu}(\boldsymbol{x},\boldsymbol{y}|Q) = \Psi_{\lambda,\mu}(Q)P_{\lambda',\mu'}(\boldsymbol{x},\boldsymbol{y}|Q'),$$

which is very encouraging! However, the formula imposes $Q=q^nt$, $Q'=q^nt^{-1}$. **For these degenerate values, many generalized Macdonald polynomials become singular!!!**

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• In fact, a similar problem occurs when considering the fusion product of $U_q(\widehat{\mathfrak{g}})$ modules **[Hernandez 2003]**. It can be analysed in the case $\mathfrak{g} = \mathfrak{sl}(2)$, which provides a good toy model. We observe that a triangular change of basis can be used to remove the singularities, but at the cost of Jordan blocks in the action of the Cartan **[JEB, V. Sopin in progress...]**

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$\rightsquigarrow$ **Applying this procedure to GMP $P_{\lambda,\mu}(\boldsymbol{x}, \boldsymbol{y}|Q)$ should provide us with the good basis of $\Lambda[\boldsymbol{x}, \boldsymbol{y}]$ we are looking for!**

# Discussion

• Similar results have been obtained recently for **wreath Macdonald polynomials**. This is a different generalization associated to higher rank instead of higher level (i.e. level $(1,0)$ representation of quantum toroidal $\mathfrak{sl}(r)$).

**[J. Wen 2019] [Orr, Shimonozo 2021] [Orr, Shimonozo, Wen 2022]**.

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• There is another important generalization which is expected to play a role in the **Bethe/gauge correspondence**. It is the surface-defect deformation of the quantum toroidal $\mathfrak{sl}(n)$ algebra **[JEB, Jeong 2019]**. It is similar to the case of wreath Macdonald polynomials, but involves a different coloring of the partitions. The algebra acting on the Fock space is also different... $\Rightarrow$ In this context, almost nothing is understood!!!

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# Thank you !!!

# Quantum Algebras Meet

# Gauge Theory and String Theory

**January 12 - 16, 2026**

**Auditorium in 18th Floor @ SIMIS**

**Key Topics:**

- **Quantum Algebras & BPS Counting, BPS/CFT Correspondence**
- **Integrable Structures in Non-Perturbative Gauge Theories**
- **Vertex Algebras in Gauge and String Theory**
- **Brane Constructions & Representation Theory**

**Confirmed Speakers:**

**Tomoyuki Arakawa**
**Hidetoshi Awata**
**Jiakang Bao**
**Ryo Fujita**
**Nathan Haouzi**
**Chiung Hwang**
**Saebyeok Jeong**
**Norton Lee**
**Alexandre Minets**
**Takahiro Nishinaka**
**Go Noshita**
**Andrei Okounkov**
**Jun'ichi Shiraishi**
**Eric Vasserot**
**Jun'ya Yagi**
**Masahito Yamazaki**
**Yegor Zenkevich**
**Keyou Zeng**
**(and more ...)**

**Organizers:**

Arkadij Bojko, Jean-Emile Bourgine, Mykola Dedushenko, Zhengping Gui, Nafiz Ishtiaque, Tomoki Nosaka, Valerii Sopin, Yehao Zhou, Hao Zou

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