4) $AQ = QB$

$$U_A \Sigma_A (V_A^* Q) = (Q U_B) \Sigma_B V_B^*$$

$$U \Sigma V = (Q U') \Sigma (V' Q^*)$$

$$\Sigma = \Sigma$$

Scratch Work

$$U \Sigma V = U \Sigma V$$

$$(U' U^*) \Sigma (V^* V')$$

$$(U' U^*) A (U^* V') = B$$

$$A = Q$$

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* True

$\Rightarrow$ let $A = QBQ^*$

$$U_A \Sigma_A V_A = Q U_B \Sigma_B V_B Q^*$$

$$\Sigma_A = (U_A^* Q U_B) \Sigma_B (V_B Q^* V_A^*)$$

And unitarily equivalent diagonal matrices must be equivalent up to a permutation, so

$$\Sigma_A = \Sigma_B$$

$\Leftarrow$ let $\Sigma_A = \Sigma_A$

Then

$$U_B \Sigma_A V_B = U_B \Sigma_B V_B = B$$

$$U_A U_B^* U_B \Sigma_A V_B V_B^* V_A = U_A U_B^* B V_B^* V_A$$

$$A = (U_A U_B^*) B (V_B^* V_A)$$

$$= Q B Q^*$$

3) If $A$ is real, then

$$(U \Sigma V^*)^* = (U \Sigma V^*)^T$$

$$V \Sigma U^* = \overline{V} \Sigma U^T$$

Which means

$$V = \overline{V}$$

and

$$U^* = U^T \Rightarrow \overline{U} = U$$

So both must be real