Mathematical Foundations of Computing

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One-dimensional space - 1D

- What is an equation in 1D?
- What is a linear equation in 1D?
- Is it hard to solve a linear equation in 1D?
- Geometrical interpretation
- What is a system of equations?

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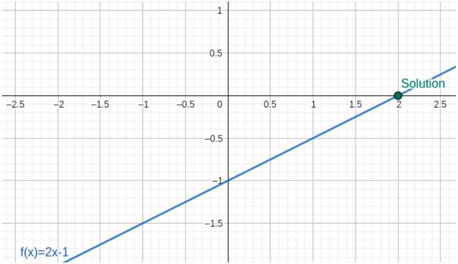
- Is it hard to solve a linear equation in 1D?
  Closed form solution $x = -\frac{b}{a}$

- Geometrical interpretation



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- What about 2D?

$$\begin{cases} 2x + 5y = 8, \\ 2x - y = 1. \end{cases}$$

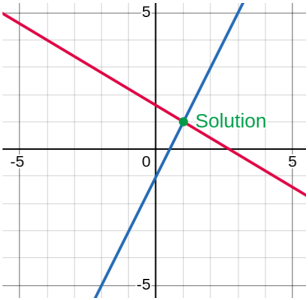
Algebraic interpretation:

- We have two unknowns: x and y.
- We need to find their values such that both equations are satisfied.
- Here we can "easily" guess, that $x^* = 1, y^* = 1$ is a solution.

$$\begin{cases} 3x + 5y = 8, \\ 2x - y = 1. \end{cases}$$

Geometric interpretation:

- Each equation corresponds to a straight line.
- A solution (x^*, y^*) is the intersection point of these lines.



Geometric Picture:

- If the lines intersect at unique point, there is a unique solution.
- If the lines are parallel (no intersection), there is no solution.
- If the lines overlap (coincide), there are infinitely many solutions.

Solving by substitution or elimination in 2D:

- **Substitution:** Solve one equation for y in terms of x, and substitute.
- **Elimination:** Combine equations to remove one variable at a time.

Generic 2D SLE:

$$\begin{cases} a_{11}x_1 + a_{12}x_2 = b_1, \\ a_{21}x_1 + a_{22}x_2 = b_2. \end{cases}$$

Gaussian Elimination

1. Choose a pivot equation (e.g., the first).
2. Multiply that equation by a factor $k_{12} = -\frac{a_{21}}{a_{11}}$. If you add it to the second equation, x_1 variable will be eliminated.
3. Solve for x_2, then back-substitute to find x_1.

$$
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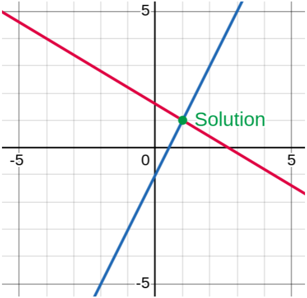
$$



Why we call it 2D?

- **1D**: $x \in \mathbb{R}$
- **2D**: $x \in \mathbb{R}^2$; $x = (x_1, x_2)$
- **3D**: $x \in \mathbb{R}^3$; $x = (x_1, x_2, x_3)$
- **nD**: $x \in \mathbb{R}^n$; $x = (x_1, \ldots, x_n)$

System of linear equations in nD:

$$\text{Find } x \in \mathbb{R}^n \text{ such, that } Ax + b = 0; A \in \mathbb{R}^n x \mathbb{R}^n, b \in \mathbb{R}^n.$$

$$A = \left(a_{ij}\right)_{n \times n} = \begin{pmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22} & \ldots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \ldots & a_{nn} \end{pmatrix}, \; x = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}, \; b = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix}$$

- What is Ax?

Please, multiply

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 1 \\ -1 & 0 & 1 \end{pmatrix},\ b = \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix}$$

Generic n-dimensional linear system:

$$Ax = b,$$

or

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1, \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2, \\ \quad \vdots \\ a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n = b_n. \end{cases}$$

Key ideas:

- Same elimination approach: use row operations to systematically eliminate variables.
- The process is repeated from the first row to the last, "sweeping" down to get a triangular form, then back-substitute.
- What if we encounter 0 on the leading position?

Test System of Linear Equations (SLE):

$$\begin{cases} x_1 + x_2 + x_3 = 6 \\ 3x_1 + 2x_2 + x_3 = 10 \\ 2x_1 - x_2 + 4x_3 = 12 \end{cases}$$

Good versus Evil

- Are all SLEs with an unique solution are equally good?
- Let's come with idea of a “bad” solvable system!
- Easiest intuition is based on geometry.