Condition for Complete State Controllability in the *s* Plane. The condition for complete state controllability can be stated in terms of transfer functions or transfer matrices.

It can be proved that a necessary and sufficient condition for complete state controllability is that no cancellation occur in the transfer function or transfer matrix. If cancellation occurs, the system cannot be controlled in the direction of the canceled mode.

The same conclusion can be obtained by writing this transfer function in the form of a state equation. A state-space representation is

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 2.5 & -1.5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u$$

Since

$$[\mathbf{B} \;\vdots\; \mathbf{AB}] = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$$

the rank of the matrix $[\mathbf{B} \;\vdots\; \mathbf{AB}]$ is 1. Therefore, we arrive at the same conclusion: The system is not completely state controllable.

Output Controllability. In the practical design of a control system, we may want to control the output rather than the state of the system. Complete state controllability is neither necessary nor sufficient for controlling the output of the system. For this reason, it is desirable to define separately complete output controllability.

Consider the system described by

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u} \tag{9–61}$$

$$\mathbf{y} = \mathbf{C}\mathbf{x} + \mathbf{D}\mathbf{u} \tag{9–62}$$

where $\mathbf{x}$ = state vector (n-vector)

$\mathbf{u}$ = control vector (r-vector)

$\mathbf{y}$ = output vector (m-vector)

$\mathbf{A}$ = $n \times n$ matrix

$\mathbf{B}$ = $n \times r$ matrix

$\mathbf{C}$ = $m \times n$ matrix

$\mathbf{D}$ = $m \times r$ matrix

The system described by Equations (9–61) and (9–62) is said to be completely output controllable if it is possible to construct an unconstrained control vector $\mathbf{u}(t)$ that will transfer any given initial output $\mathbf{y}(t_0)$ to any final output $\mathbf{y}(t_1)$ in a finite time interval $t_0 \le t \le t_1$.

It can be proved that the condition for complete output controllability is as follows: The system described by Equations (9–61) and (9–62) is completely output controllable if and only if the $m \times (n + 1)r$ matrix

$$\left[\mathbf{CB} \;\vdots\; \mathbf{CAB} \;\vdots\; \mathbf{CA}^2\mathbf{B} \;\vdots\; \cdots \;\vdots\; \mathbf{CA}^{n-1}\mathbf{B} \;\vdots\; \mathbf{D}\right]$$

is of rank m. (For a proof, see Problem **A–9–16**.) Note that the presence of the $\mathbf{Du}$ term in Equation (9–62) always helps to establish output controllability.

Uncontrollable System. An uncontrollable system has a subsystem that is physically disconnected from the input.

Stabilizability. For a partially controllable system, if the uncontrollable modes are stable and the unstable modes are controllable, the system is said to be stabilizable. For example, the system defined by

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u$$

is not state controllable. The stable mode that corresponds to the eigenvalue of -1 is not controllable. The unstable mode that corresponds to the eigenvalue of 1 is controllable. Such a system can be made stable by the use of a suitable feedback. Thus this system is stabilizable.

In this section we discuss the observability of linear systems. Consider the unforced system described by the following equations:

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} \tag{9–63}$$

$$\mathbf{y} = \mathbf{C}\mathbf{x} \tag{9–64}$$

where $\mathbf{x}$ = state vector (n-vector)
$\mathbf{y}$ = output vector (m-vector)
$\mathbf{A}$ = $n \times n$ matrix
$\mathbf{C}$ = $m \times n$ matrix

The system is said to be completely observable if every state $\mathbf{x}(t_0)$ can be determined from the observation of $\mathbf{y}(t)$ over a finite time interval, $t_0 \leq t \leq t_1$. The system is, therefore, completely observable if every transition of the state eventually affects every element of the output vector. The concept of observability is useful in solving the problem of reconstructing unmeasurable state variables from measurable variables in the minimum possible length of time. In this section we treat only linear, time-invariant systems. Therefore, without loss of generality, we can assume that $t_0 = 0$.

The concept of observability is very important because, in practice, the difficulty encountered with state feedback control is that some of the state variables are not accessible for direct measurement, with the result that it becomes necessary to estimate the unmeasurable state variables in order to construct the control signals. It will be shown in Section 10–5 that such estimates of state variables are possible if and only if the system is completely observable.

In discussing observability conditions, we consider the unforced system as given by Equations (9–63) and (9–64). The reason for this is as follows: If the system is described by

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u}$$

$$\mathbf{y} = \mathbf{C}\mathbf{x} + \mathbf{D}\mathbf{u}$$

then

$$\mathbf{x}(t) = e^{\mathbf{A}t}\mathbf{x}(0) + \int_0^t e^{\mathbf{A}(t-\tau)}\mathbf{B}\mathbf{u}(\tau)\,d\tau$$

and $\mathbf{y}(t)$ is

$$\mathbf{y}(t) = \mathbf{C}e^{\mathbf{A}t}\mathbf{x}(0) + \mathbf{C}\int_0^t e^{\mathbf{A}(t-\tau)}\mathbf{B}\mathbf{u}(\tau)\,d\tau + \mathbf{D}\mathbf{u}$$

Since the matrices $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$, and $\mathbf{D}$ are known and $\mathbf{u}(t)$ is also known, the last two terms on the right-hand side of this last equation are known quantities. Therefore, they may be subtracted from the observed value of $\mathbf{y}(t)$. Hence, for investigating a necessary and sufficient condition for complete observability, it suffices to consider the system described by Equations (9–63) and (9–64).

Complete Observability of Continuous-Time Systems. Consider the system described by Equations (9–63) and (9–64). The output vector $\mathbf{y}(t)$ is

$$\mathbf{y}(t) = \mathbf{C}e^{\mathbf{A}t}\mathbf{x}(0)$$

Referring to Equation (9–48) or (9–50), we have

$$e^{\mathbf{A}t} = \sum_{k=0}^{n-1} \alpha_k(t)\mathbf{A}^k$$

where n is the degree of the characteristic polynomial. [Note that Equations (9–48) and (9–50) with m replaced by n can be derived using the characteristic polynomial.]

Hence, we obtain

$$\mathbf{y}(t) = \sum_{k=0}^{n-1} \alpha_k(t)\mathbf{C}\mathbf{A}^k\mathbf{x}(0)$$

or

$$\mathbf{y}(t) = \alpha_0(t)\mathbf{C}\mathbf{x}(0) + \alpha_1(t)\mathbf{C}\mathbf{A}\mathbf{x}(0) + \cdots + \alpha_{n-1}(t)\mathbf{C}\mathbf{A}^{n-1}\mathbf{x}(0) \qquad (9\text{–}65)$$

If the system is completely observable, then, given the output $\mathbf{y}(t)$ over a time interval $0 \le t \le t_1$, $\mathbf{x}(0)$ is uniquely determined from Equation (9–65). It can be shown that this requires the rank of the $nm \times n$ matrix

$$\begin{bmatrix} \mathbf{C} \\ \hdashline \mathbf{CA} \\ \hdashline \cdot \\ \cdot \\ \cdot \\ \hdashline \mathbf{CA}^{n-1} \end{bmatrix}$$

to be n. (See Problem **A–9–19** for the derivation of this condition.)

From this analysis, we can state the condition for complete observability as follows: The system described by Equations (9–63) and (9–64) is completely observable if and only if the $n \times nm$ matrix

$$[\mathbf{C}^* \mid \mathbf{A}^*\mathbf{C}^* \mid \cdots \mid (\mathbf{A}^*)^{n-1}\mathbf{C}^*]$$

is of rank n or has n linearly independent column vectors. This matrix is called the *observability matrix*.

Consider the system described by

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ -2 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$$

$$y = [1 \quad 0] \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Is this system controllable and observable?

Since the rank of the matrix

$$[\mathbf{B} \;\vdots\; \mathbf{AB}] = \begin{bmatrix} 0 & 1 \\ 1 & -1 \end{bmatrix}$$

is 2, the system is completely state controllable.

For output controllability, let us find the rank of the matrix $[\mathbf{CB} \;\vdots\; \mathbf{CAB}]$. Since

$$[\mathbf{CB} \;\vdots\; \mathbf{CAB}] = [0 \quad 1]$$

the rank of this matrix is 1. Hence, the system is completely output controllable.

To test the observability condition, examine the rank of $[\mathbf{C}^* \;\vdots\; \mathbf{A}^*\mathbf{C}^*]$. Since

$$[\mathbf{C}^* \;\vdots\; \mathbf{A}^*\mathbf{C}^*] = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

the rank of $[\mathbf{C}^* \;\vdots\; \mathbf{A}^*\mathbf{C}^*]$ is 2. Hence, the system is completely observable.

Conditions for Complete Observability in the s Plane. The conditions for complete observability can also be stated in terms of transfer functions or transfer matrices. The necessary and sufficient conditions for complete observability is that no cancellation occur in the transfer function or transfer matrix. If cancellation occurs, the canceled mode cannot be observed in the output.

Show that the following system is not completely observable:

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}u$$
$$y = \mathbf{C}\mathbf{x}$$

where

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}, \quad \mathbf{A} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -6 & -11 & -6 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \quad \mathbf{C} = \begin{bmatrix} 4 & 5 & 1 \end{bmatrix}$$

Note that the control function u does not affect the complete observability of the system. To examine complete observability, we may simply set $u = 0$. For this system, we have

$$[\mathbf{C}^* \mid \mathbf{A}^*\mathbf{C}^* \mid (\mathbf{A}^*)^2\mathbf{C}^*] = \begin{bmatrix} 4 & -6 & 6 \\ 5 & -7 & 5 \\ 1 & -1 & -1 \end{bmatrix}$$

Note that

$$\begin{vmatrix} 4 & -6 & 6 \\ 5 & -7 & 5 \\ 1 & -1 & -1 \end{vmatrix} = 0$$

Hence, the rank of the matrix $[\mathbf{C}^* \;\vdots\; \mathbf{A}^*\mathbf{C}^* \;\vdots\; (\mathbf{A}^*)^2\mathbf{C}^*]$ is less than 3. Therefore, the system is not completely observable.

In fact, in this system, cancellation occurs in the transfer function of the system. The transfer function between $X_1(s)$ and $U(s)$ is

$$\frac{X_1(s)}{U(s)} = \frac{1}{(s+1)(s+2)(s+3)}$$

and the transfer function between $Y(s)$ and $X_1(s)$ is

$$\frac{Y(s)}{X_1(s)} = (s+1)(s+4)$$

Therefore, the transfer function between the output $Y(s)$ and the input $U(s)$ is

$$\frac{Y(s)}{U(s)} = \frac{(s+1)(s+4)}{(s+1)(s+2)(s+3)}$$

Clearly, the two factors $(s+1)$ cancel each other. This means that there are nonzero initial states $\mathbf{x}(0)$, which cannot be determined from the measurement of $y(t)$.

Comments. The transfer function has no cancellation if and only if the system is completely state controllable and completely observable. This means that the canceled transfer function does not carry along all the information characterizing the dynamic system.

Alternative Form of the Condition for Complete Observability. Consider the system described by Equations (9–63) and (9–64), rewritten

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} \tag{9–66}$$

$$\mathbf{y} = \mathbf{C}\mathbf{x} \tag{9–67}$$

Suppose that the transformation matrix $\mathbf{P}$ transforms $\mathbf{A}$ into a diagonal matrix, or

$$\mathbf{P}^{-1}\mathbf{A}\mathbf{P} = \mathbf{D}$$

where $\mathbf{D}$ is a diagonal matrix. Let us define

$$\mathbf{x} = \mathbf{P}\mathbf{z}$$

Then Equations (9–66) and (9–67) can be written

$$\dot{\mathbf{z}} = \mathbf{P}^{-1}\mathbf{A}\mathbf{P}\mathbf{z} = \mathbf{D}\mathbf{z}$$

$$\mathbf{y} = \mathbf{C}\mathbf{P}\mathbf{z}$$

Hence,

$$\mathbf{y}(t) = \mathbf{C}\mathbf{P}e^{\mathbf{D}t}\mathbf{z}(0)$$

or

$$\mathbf{y}(t) = \mathbf{CP}\begin{bmatrix} e^{\lambda_1 t} & & & & 0 \\ & e^{\lambda_2 t} & & & \\ & & \cdot & & \\ & & & \cdot & \\ & & & & \cdot \\ 0 & & & & e^{\lambda_n t} \end{bmatrix}\mathbf{z}(0) = \mathbf{CP}\begin{bmatrix} e^{\lambda_1 t} z_1(0) \\ e^{\lambda_2 t} z_2(0) \\ \cdot \\ \cdot \\ \cdot \\ e^{\lambda_n t} z_n(0) \end{bmatrix}$$

The system is completely observable if none of the columns of the $m \times n$ matrix **CP** consists of all zero elements. This is because, if the ith column of **CP** consists of all zero elements, then the state variable $z_i(0)$ will not appear in the output equation and therefore cannot be determined from observation of $\mathbf{y}(t)$. Thus, $\mathbf{x}(0)$, which is related to $\mathbf{z}(0)$ by the nonsingular matrix **P**, cannot be determined. (Remember that this test applies only if the matrix $\mathbf{P}^{-1}\mathbf{AP}$ is in diagonal form.)

If the matrix **A** cannot be transformed into a diagonal matrix, then by use of a suitable transformation matrix **S**, we can transform **A** into a Jordan canonical form, or

$$\mathbf{S}^{-1}\mathbf{AS} = \mathbf{J}$$

where **J** is in the Jordan canonical form.

Principle of Duality. We shall now discuss the relationship between controllability and observability. We shall introduce the principle of duality, due to Kalman, to clarify apparent analogies between controllability and observability.

Consider the system S_1 described by

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u}$$

$$\mathbf{y} = \mathbf{C}\mathbf{x}$$

where $\mathbf{x}$ = state vector (n-vector)

$\mathbf{u}$ = control vector (r-vector)

$\mathbf{y}$ = output vector (m-vector)

$\mathbf{A}$ = $n \times n$ matrix

$\mathbf{B}$ = $n \times r$ matrix

$\mathbf{C}$ = $m \times n$ matrix

and the dual system S_2 defined by

$$\dot{\mathbf{z}} = \mathbf{A}^*\mathbf{z} + \mathbf{C}^*\mathbf{v}$$
$$\mathbf{n} = \mathbf{B}^*\mathbf{z}$$

where $\mathbf{z}$ = state vector (n-vector)
$\mathbf{v}$ = control vector (m-vector)
$\mathbf{n}$ = output vector (r-vector)
$\mathbf{A}^*$ = conjugate transpose of $\mathbf{A}$
$\mathbf{B}^*$ = conjugate transpose of $\mathbf{B}$
$\mathbf{C}^*$ = conjugate transpose of $\mathbf{C}$

The principle of duality states that the system S_1 is completely state controllable (observable) if and only if system S_2 is completely observable (state controllable).

To verify this principle, let us write down the necessary and sufficient conditions for complete state controllability and complete observability for systems S_1 and S_2.

For system S_1:

1. A necessary and sufficient condition for complete state controllability is that the rank of the $n \times nr$ matrix

$$[\mathbf{B} \mid \mathbf{AB} \mid \cdots \mid \mathbf{A}^{n-1}\mathbf{B}]$$

be n.

2. A necessary and sufficient condition for complete observability is that the rank of the $n \times nm$ matrix

$$[\mathbf{C}^* \mid \mathbf{A}^*\mathbf{C}^* \mid \cdots \mid (\mathbf{A}^*)^{n-1}\mathbf{C}^*]$$

be n.

For system S_2:

1. A necessary and sufficient condition for complete state controllability is that the rank of the $n \times nm$ matrix

$$[\mathbf{C}^* \mid \mathbf{A}^*\mathbf{C}^* \mid \cdots \mid (\mathbf{A}^*)^{n-1}\mathbf{C}^*]$$

be n.

2. A necessary and sufficient condition for complete observability is that the rank of the $n \times nr$ matrix

$$[\mathbf{B} \mid \mathbf{AB} \mid \cdots \mid \mathbf{A}^{n-1}\mathbf{B}]$$

be n.