We assumed that all state variables are available for feedback. In practice, however, not all state variables are available for feedback. Then we need to estimate unavailable state variables.

Estimation of unmeasurable state variables is commonly called observation. A device (or a computer program) that estimates or observes the state variables is called a state observer.

If the state observer observes all state variables of the system, regardless of whether some state variables are available for direct measurement, it is called a full-order state observer.

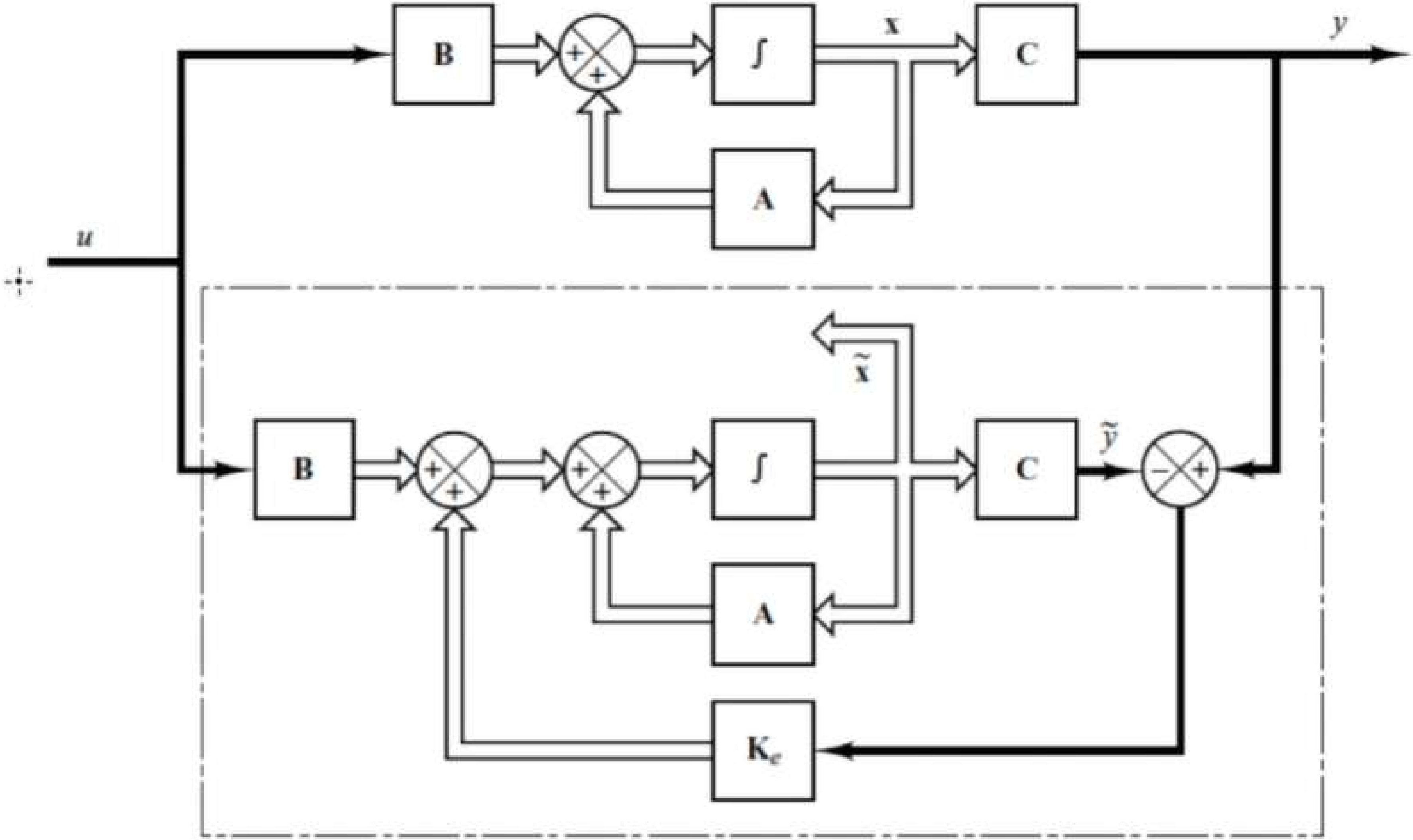
$u = -Kx$

Sometimes we will need observation of only the unmeasurable state variables, but not of those that are directly measurable as well.

For example, since the output variables are observable and they are linearly related to the state variables, we need not observe all state variables, but observe only n-m state variables, where n is the dimension of the state vector and m is the dimension of the output vector.

An observer that estimates fewer than n state variables, where n is the dimension of the state vector, is called a reduced-order state observer.

If the order of the reduced-order state observer is the minimum possible, the observer is called a minimum-order state observer or minimum-order observer.



Full-order state observer

Full-Order State Observer. The order of the state observer that will be discussed here is the same as that of the plant. Assume that the plant is defined by Equations (10–55) and (10–56) and the observer model is defined by Equation (10–57).

To obtain the observer error equation, let us subtract Equation (10–57) from Equation (10–55):

$$\begin{aligned}\dot{\mathbf{x}} - \dot{\tilde{\mathbf{x}}} &= \mathbf{A}\mathbf{x} - \mathbf{A}\tilde{\mathbf{x}} - \mathbf{K}_e(\mathbf{C}\mathbf{x} - \mathbf{C}\tilde{\mathbf{x}}) \\ &= (\mathbf{A} - \mathbf{K}_e\mathbf{C})(\mathbf{x} - \tilde{\mathbf{x}}) \qquad (10\text{–}58)\end{aligned}$$

Define the difference between $\mathbf{x}$ and $\tilde{\mathbf{x}}$ as the error vector $\mathbf{e}$, or

$$\mathbf{e} = \mathbf{x} - \tilde{\mathbf{x}}$$

Then Equation (10–58) becomes

$\dot{X} = AX + Bu$

$$\dot{\mathbf{e}} = (\mathbf{A} - \mathbf{K}_e\mathbf{C})\mathbf{e} \qquad (10\text{–}59)$$

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$$= (\mathbf{A} - \mathbf{K}_e\mathbf{C})(\mathbf{x} - \tilde{\mathbf{x}}) \tag{10–58}$$

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Then Equation (10–58) becomes

$$\dot{\mathbf{e}} = (\mathbf{A} - \mathbf{K}_e\mathbf{C})\mathbf{e} \tag{10–59}$$

$\dot{X} = AX + \cancel{B}u$

$\dot{\tilde{X}} = A\tilde{X} + \cancel{B}u + (y - \tilde{y})Ke$

From Equation (10–59), we see that the dynamic behavior of the error vector is determined by the eigenvalues of matrix $\mathbf{A} - \mathbf{K}_e\mathbf{C}$. If matrix $\mathbf{A} - \mathbf{K}_e\mathbf{C}$ is a stable matrix, the error vector will converge to zero for any initial error vector $\mathbf{e}(0)$. That is, $\widetilde{\mathbf{x}}(t)$ will converge to $\mathbf{x}(t)$ regardless of the values of $\mathbf{x}(0)$ and $\widetilde{\mathbf{x}}(0)$. If the eigenvalues of matrix $\mathbf{A} - \mathbf{K}_e\mathbf{C}$ are chosen in such a way that the dynamic behavior of the error vector is asymptotically stable and is adequately fast, then any error vector will tend to zero (the origin) with an adequate speed.

If the plant is completely observable, then it can be proved that it is possible to choose matrix $\mathbf{K}_e$ such that $\mathbf{A} - \mathbf{K}_e\mathbf{C}$ has arbitrarily desired eigenvalues. That is, the observer gain matrix $\mathbf{K}_e$ can be determined to yield the desired matrix $\mathbf{A} - \mathbf{K}_e\mathbf{C}$. We shall discuss this matter in what follows.

Dual Problem. The problem of designing a full-order observer becomes that of determining the observer gain matrix $\mathbf{K}_e$ such that the error dynamics defined by Equation (10–59) are asymptotically stable with sufficient speed of response. (The asymptotic stability and the speed of response of the error dynamics are determined by the eigenvalues of matrix $\mathbf{A} - \mathbf{K}_e\mathbf{C}$.) Hence, the design of the full-order observer becomes that of determining an appropriate $\mathbf{K}_e$ such that $\mathbf{A} - \mathbf{K}_e\mathbf{C}$ has desired eigenvalues. Thus, the problem here becomes the same as the pole-placement problem we discussed in Section 10–2. In fact, the two problems are mathematically the same. This property is called duality.

Consider the system defined by

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}u$$

$$y = \mathbf{C}\mathbf{x}$$

In designing the full-order state observer, we may solve the dual problem, that is, solve the pole-placement problem for the dual system

$$\dot{\mathbf{z}} = \mathbf{A}^*\mathbf{z} + \mathbf{C}^*v$$

$$n = \mathbf{B}^*\mathbf{z}$$

assuming the control signal v to be

$$v = -\mathbf{K}\mathbf{z}$$

x

If the dual system is completely state controllable, then the state feedback gain matrix $\mathbf{K}$ can be determined such that matrix $\mathbf{A}^* - \mathbf{C}^*\mathbf{K}$ will yield a set of the desired eigenvalues.

If $\mu_1, \mu_2, \dots, \mu_n$ are the desired eigenvalues of the state observer matrix, then by taking the same μ_i's as the desired eigenvalues of the state-feedback gain matrix of the dual system, we obtain

$$|s\mathbf{I} - (\mathbf{A}^* - \mathbf{C}^*\mathbf{K})| = (s - \mu_1)(s - \mu_2)\cdots(s - \mu_n)$$

Noting that the eigenvalues of $\mathbf{A}^* - \mathbf{C}^*\mathbf{K}$ and those of $\mathbf{A} - \mathbf{K}^*\mathbf{C}$ are the same, we have

$$|s\mathbf{I} - (\mathbf{A}^* - \mathbf{C}^*\mathbf{K})| = |s\mathbf{I} - (\mathbf{A} - \mathbf{K}^*\mathbf{C})|$$

Comparing the characteristic polynomial $|s\mathbf{I} - (\mathbf{A} - \mathbf{K}^*\mathbf{C})|$ and the characteristic polynomial $|s\mathbf{I} - (\mathbf{A} - \mathbf{K}_e\mathbf{C})|$ for the observer system [refer to Equation (10–57)], we find that $\mathbf{K}_e$ and $\mathbf{K}^*$ are related by

$$\mathbf{K}_e = \mathbf{K}^*$$

Thus, using the matrix $\mathbf{K}$ determined by the pole-placement approach in the dual system, the observer gain matrix $\mathbf{K}_e$ for the original system can be determined by using the relationship $\mathbf{K}_e = \mathbf{K}^*$. (See Problem **A–10–10** for the details.)

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$A^* = A$

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Necessary and Sufficient Condition for State Observation. As discussed, a necessary and sufficient condition for the determination of the observer gain matrix $\mathbf{K}_e$ for the desired eigenvalues of $\mathbf{A} - \mathbf{K}_e\mathbf{C}$ is that the dual of the original system

$$\dot{\mathbf{z}} = \mathbf{A}^*\mathbf{z} + \mathbf{C}^*v$$

be completely state controllable. The complete state controllability condition for this dual system is that the rank of

$$\boxed{\left[\mathbf{C}^* \,\vdots\, \mathbf{A}^*\mathbf{C}^* \,\vdots\, \cdots \,\vdots\, (\mathbf{A}^*)^{n-1}\mathbf{C}^*\right]}$$

be n. This is the condition for complete observability of the original system defined by Equations (10–55) and (10–56). This means that a necessary and sufficient condition for the observation of the state of the system defined by Equations (10–55) and (10–56) is that the system be completely observable.

Once we select the desired eigenvalues (or desired characteristic equation), the full-order state observer can be designed, provided the plant is completely observable. The desired eigenvalues of the characteristic equation should be chosen so that the state observer responds at least two to five times faster than the closed-loop system considered. As stated earlier, the equation for the full-order state observer is

$$\dot{\tilde{\mathbf{x}}} = (\mathbf{A} - \mathbf{K}_e\mathbf{C})\tilde{\mathbf{x}} + \mathbf{B}u + \mathbf{K}_e y \tag{10–60}$$

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$$\dot{\tilde{\mathbf{x}}} = (\mathbf{A} - \mathbf{K}_e\mathbf{C})\tilde{\mathbf{x}} + \mathbf{B}u + \mathbf{K}_e y \qquad (10\text{–}60)$$

$\dot{\tilde{X}} = AX + Bu + K_e(y - \tilde{y})$ $y =$

to be exactly the same as those of the physical plant. If there are discrepancies in **A**, **B**, and **C** in the observer and in the physical plant, the dynamics of the observer error are no longer governed by Equation (10–59). This means that the error may not approach zero as expected. Therefore, we need to choose $\mathbf{K}_e$ so that the observer is stable and the error remains acceptably small in the presence of small modeling errors.

Transformation Approach to Obtain State Observer Gain Matrix $\mathbf{K}_e$. By following the same approach as we used in deriving the equation for the state feedback gain matrix **K**, we can obtain the following equation:

$$\mathbf{K}_e = \mathbf{Q}\begin{bmatrix} \alpha_n - a_n \\ \alpha_{n-1} - a_{n-1} \\ \cdot \\ \cdot \\ \cdot \\ \alpha_1 - a_1 \end{bmatrix} = (\mathbf{W}\mathbf{N}^*)^{-1}\begin{bmatrix} \alpha_n - a_n \\ \alpha_{n-1} - a_{n-1} \\ \cdot \\ \cdot \\ \cdot \\ \alpha_1 - a_1 \end{bmatrix} \tag{10–61}$$

where $\mathbf{K}_e$ is an $n \times 1$ matrix,

$$\mathbf{Q} = (\mathbf{W}\mathbf{N}^*)^{-1}$$

and

$$\mathbf{N} = \left[\mathbf{C}^* \;\vdots\; \mathbf{A}^*\mathbf{C}^* \;\vdots\; \cdots \;\vdots\; (\mathbf{A}^*)^{n-1}\mathbf{C}^*\right]$$

It is noted that thus far we have assumed the matrices **A**, **B**, and **C** in the observer to be exactly the same as those of the physical plant. If there are discrepancies in **A**, **B**, and **C** in the observer and in the physical plant, the dynamics of the observer error are no longer governed by Equation (10–59). This means that the error may not approach zero as expected. Therefore, we need to choose $\mathbf{K}_e$ so that the observer is stable and the error remains acceptably small in the presence of small modeling errors.

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controllable

$T = \underline{MW}$

where $\mathbf{K}_e$ is an $n \times 1$ matrix,

$$\boxed{\mathbf{Q} = (\mathbf{W}\mathbf{N}^*)^{-1}} \rightarrow \text{observable}$$

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$K = [\alpha_i - a_i] T^{-1}$

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and

$$\mathbf{N} = \left[\mathbf{C}^* \mid \mathbf{A}^*\mathbf{C}^* \mid \cdots \mid (\mathbf{A}^*)^{n-1}\mathbf{C}^*\right]$$

$$\mathbf{W} = \begin{bmatrix} a_{n-1} & a_{n-2} & \cdots & a_1 & 1 \\ a_{n-2} & a_{n-3} & \cdots & 1 & 0 \\ \cdot & \cdot & & \cdot & \cdot \\ \cdot & \cdot & & \cdot & \cdot \\ \cdot & \cdot & & \cdot & \cdot \\ a_1 & 1 & \cdots & 0 & 0 \\ 1 & 0 & \cdots & 0 & 0 \end{bmatrix}$$

Direct-Substitution Approach to Obtain State Observer Gain Matrix $\mathbf{K}_e$. Similar to the case of pole placement, if the system is of low order, then direct substitution of matrix $\mathbf{K}_e$ into the desired characteristic polynomial may be simpler. For example, if $\mathbf{x}$ is a 3-vector, then write the observer gain matrix $\mathbf{K}_e$ as

$$\mathbf{K}_e = \begin{bmatrix} k_{e1} \\ k_{e2} \\ k_{e3} \end{bmatrix}$$

Substitute this $\mathbf{K}_e$ matrix into the desired characteristic polynomial:

$$|s\mathbf{I} - (\mathbf{A} - \mathbf{K}_e\mathbf{C})| = (s - \mu_1)(s - \mu_2)(s - \mu_3)$$

By equating the coefficients of the like powers of s on both sides of this last equation, we can determine the values of k_{e1}, k_{e2}, and k_{e3}. This approach is convenient if $n = 1$, 2, or 3, where n is the dimension of the state vector $\mathbf{x}$. (Although this approach can be used when $n = 4, 5, 6, \ldots$, the computations involved may become very tedious.)

Another approach to the determination of the state observer gain matrix $\mathbf{K}_e$ is to use Ackermann's formula. This approach is presented in the following.

Ackermann's Formula. Consider the system defined by

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}u \tag{10–62}$$

$$y = \mathbf{C}\mathbf{x} \tag{10–63}$$

In Section 10–2 we derived Ackermann's formula for pole placement for the system defined by Equation (10–62). The result was given by Equation (10–18), rewritten thus:

$$\mathbf{K} = [0 \quad 0 \cdots 0 \quad 1][\mathbf{B} \,\vdots\, \mathbf{AB} \,\vdots\, \cdots \,\vdots\, \mathbf{A}^{n-1}\mathbf{B}]^{-1}\phi(\mathbf{A})$$

For the dual of the system defined by Equations (10–62) and (10–63),

$$\dot{\mathbf{z}} = \mathbf{A}^*\mathbf{z} + \mathbf{C}^*v$$

$$n = \mathbf{B}^*\mathbf{z}$$

the preceding Ackermann's formula for pole placement is modified to

$$\mathbf{K} = [0 \quad 0 \cdots 0 \quad 1][\mathbf{C}^* \,\vdots\, \mathbf{A}^*\mathbf{C}^* \,\vdots\, \cdots \,\vdots\, (\mathbf{A}^*)^{n-1}\mathbf{C}^*]^{-1}\phi(\mathbf{A}^*) \tag{10–64}$$

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where $\mathbf{K}_e$ is an $n \times 1$ matrix, and

$$\mathbf{N} = [\mathbf{C}^* \mid \mathbf{A}^*\mathbf{C}^* \mid \cdots \mid (\mathbf{A}^*)^{n-1}\mathbf{C}^*]$$

$$\mathbf{W} = \begin{bmatrix} a_{n-1} & a_{n-2} & \cdots & a_1 & 1 \\ a_{n-2} & a_{n-3} & \cdots & 1 & 0 \\ \cdot & \cdot & & \cdot & \cdot \\ \cdot & \cdot & & \cdot & \cdot \\ \cdot & \cdot & & \cdot & \cdot \\ a_1 & 1 & \cdots & 0 & 0 \\ 1 & 0 & \cdots & 0 & 0 \end{bmatrix}$$

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Substitute this $\mathbf{K}_e$ matrix into the desired characteristic polynomial:

By equating the coefficients of the like powers of s on both sides of this last equation, we can determine the values of k_{e1}, k_{e2}, and k_{e3}. This approach is convenient if $n = 1$, 2, or 3, where n is the dimension of the state vector $\mathbf{x}$. (Although this approach can be used when $n = 4, 5, 6, \ldots$, the computations involved may become very tedious.)

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$$\underline{\mathbf{K} = [0 \quad 0 \cdots 0 \quad 1][\mathbf{B} \mid \mathbf{AB} \mid \cdots \mid \mathbf{A}^{n-1}\mathbf{B}]^{-1}\phi(\mathbf{A})}$$

For the dual of the system defined by Equations (10–62) and (10–63),

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$$\mathbf{K}^* = \left([0 \quad 0 \cdots 0 \quad 1][\mathbf{C}^* \mid \mathbf{A}^*\mathbf{C}^* \mid \cdots \mid (\mathbf{A}^*)^{n-1}\mathbf{C}^*]^{-1}\phi(\mathbf{A}^*) \right)^* \tag{10–64}$$

x

$= [\phi(A^*)]^* [[0\,0 \cdots 1][C^* \cdots A^*)^{n-1}$

As stated earlier, the state observer gain matrix $\mathbf{K}_e$ is given by $\mathbf{K}^*$, where $\mathbf{K}$ is given by Equation (10–64). Thus,

$$\mathbf{K}_e = \mathbf{K}^* = \phi(\mathbf{A}^*)^* \begin{bmatrix} \mathbf{C} \\ \mathbf{CA} \\ \cdot \\ \cdot \\ \cdot \\ \mathbf{CA}^{n-2} \\ \mathbf{CA}^{n-1} \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ 0 \\ \cdot \\ \cdot \\ \cdot \\ 0 \\ 1 \end{bmatrix} = \phi(\mathbf{A}) \begin{bmatrix} \mathbf{C} \\ \mathbf{CA} \\ \cdot \\ \cdot \\ \cdot \\ \mathbf{CA}^{n-2} \\ \mathbf{CA}^{n-1} \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ 0 \\ \cdot \\ \cdot \\ \cdot \\ 0 \\ 1 \end{bmatrix} \tag{10–65}$$

x

$$= \left[\phi(A^*)\right]^* \left[[0\ 0 \cdots 1]\left[C^* \cdots (A^*)^{n-1}C^*\right]\right]^*$$ 57

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$$\left[\phi(A^*)\right]^* = \phi(A)$$

$\phi(s)$ = desired ch. eqn

where $\phi(s)$ is the desired characteristic polynomial for the state observer, or

$$\phi(s) = (s - \mu_1)(s - \mu_2)\cdots(s - \mu_n)$$

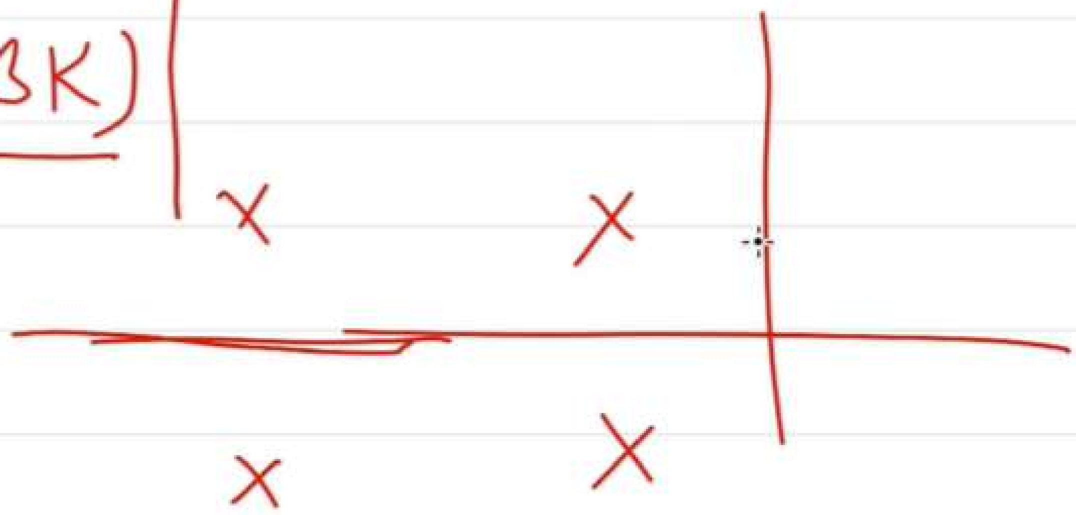
where $\mu_1, \mu_2, \ldots, \mu_n$ are the desired eigenvalues. Equation (10–65) is called Ackermann's formula for the determination of the observer gain matrix $\mathbf{K}_e$.

Pole placement $\quad u = -KX$

observer $\longrightarrow K_e$

$$\left|\left[sI - (\underline{A - K_e C})\right]\right|$$

$$\left|sI - (\underline{A - BK})\right|$$



Comments on Selecting the Best $\mathbf{K}_e$. Referring to Figure 10–11, notice that the feedback signal through the observer gain matrix $\mathbf{K}_e$ serves as a correction signal to the plant model to account for the unknowns in the plant. If significant unknowns are involved, the feedback signal through the matrix $\mathbf{K}_e$ should be relatively large. However, if the output signal is contaminated significantly by disturbances and measurement noises, then the output y is not reliable and the feedback signal through the matrix $\mathbf{K}_e$ should be relatively small. In determining the matrix $\mathbf{K}_e$, we should carefully examine the effects of disturbances and noises involved in the output y.

actual
ABC

ABC
model

Remember that the observer gain matrix $\mathbf{K}_e$ depends on the desired characteristic equation

$$(s - \mu_1)(s - \mu_2)\cdots(s - \mu_n) = 0$$

The choice of a set of $\mu_1, \mu_2, \ldots, \mu_n$ is, in many instances, not unique. As a general rule, however, the observer poles must be two to five times faster than the controller poles to make sure the observation error (estimation error) converges to zero quickly. This means that the observer estimation error decays two to five times faster than does the state vector $\mathbf{x}$. Such faster decay of the observer error compared with the desired dynamics makes the controller poles dominate the system response.

It is important to note that if sensor noise is considerable, we may choose the observer poles to be slower than two times the controller poles, so that the bandwidth of the system will become lower and smooth the noise. In this case the system response will be strongly influenced by the observer poles. If the observer poles are located to the right of the controller poles in the left-half s plane, the system response will be dominated by the observer poles rather than by the control poles.

In the design of the state observer, it is desirable to determine several observer gain matrices $\mathbf{K}_e$ based on several different desired characteristic equations. For each of the several different matrices $\mathbf{K}_e$, simulation tests must be run to evaluate the resulting system performance. Then we select the best $\mathbf{K}_e$ from the viewpoint of overall system performance. In many practical cases, the selection of the best matrix $\mathbf{K}_e$ boils down to a compromise between speedy response and sensitivity to disturbances and noises.

$$\begin{matrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{matrix}$$

Consider the system

observable

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}u$$

$$y = \mathbf{C}\mathbf{x}$$

where

$$\mathbf{A} = \begin{bmatrix} 0 & 20.6 \\ 1 & 0 \end{bmatrix}, \qquad \mathbf{B} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \qquad \mathbf{C} = [0 \quad 1]$$

We use the observed state feedback such that

$$u = -\mathbf{K}\tilde{\mathbf{x}}$$

Design a full-order state observer, assuming that the system configuration is identical to tha shown in Figure 10–11. Assume that the desired eigenvalues of the observer matrix are

$$\mu_1 = -10, \qquad \mu_2 = -10$$

The design of the state observer reduces to the determination of an appropriate observer gai matrix $\mathbf{K}_e$.

Let us examine the observability matrix. The rank of

$$[\mathbf{C}^* \;\vdots\; \mathbf{A}^*\mathbf{C}^*] = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

is 2. Hence, the system is completely observable and the determination of the desired observer gai matrix is possible. We shall solve this problem by three methods.

$T = MW$ $A \to$ cont.

Method 1: We shall determine the observer gain matrix by use of Equation (10–61). The given system is already in the observable canonical form. Hence, the transformation matrix $\mathbf{Q} = (\mathbf{WN^*})^{-1}$ is $\mathbf{I}$. Since the characteristic equation of the given system is

$$|s\mathbf{I} - \mathbf{A}| = \begin{vmatrix} s & -20.6 \\ -1 & s \end{vmatrix} = s^2 - 20.6 = s^2 + a_1 s + a_2 = 0$$

$W \to a_i$

we have

$$a_1 = 0, \qquad a_2 = -20.6$$

The desired characteristic equation is

$$(s + 10)^2 = s^2 + 20s + 100 = s^2 + \alpha_1 s + \alpha_2 = 0$$

Hence,

$$\alpha_1 = 20, \qquad \alpha_2 = 100$$

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act

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−10

$$(s + 10)^2 = s^2 + 20s + 100 = s^2 + \alpha_1 s + \alpha_2 = 0$$

desired

Hence,

$$\alpha_1 = 20, \qquad \alpha_2 = 100$$

Then the observer gain matrix $\mathbf{K}_e$ can be obtained from Equation (10–61) as follows:

$$\mathbf{K}_e = (\mathbf{W}\mathbf{N}^*)^{-1}\begin{bmatrix} \alpha_2 - a_2 \\ \alpha_1 - a_1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 100 + 20.6 \\ 20 - 0 \end{bmatrix} = \begin{bmatrix} 120.6 \\ 20 \end{bmatrix}$$

Method 2: Referring to Equation (10–59):

$$\dot{\mathbf{e}} = (\mathbf{A} - \mathbf{K}_e\mathbf{C})\mathbf{e}$$

the characteristic equation for the observer becomes

$$|s\mathbf{I} - \mathbf{A} + \mathbf{K}_e\mathbf{C}| = 0$$

Define

$$\mathbf{K}_e = \begin{bmatrix} k_{e1} \\ k_{e2} \end{bmatrix}$$

Then the characteristic equation becomes

$$\left|\begin{bmatrix} s & 0 \\ 0 & s \end{bmatrix} - \begin{bmatrix} 0 & 20.6 \\ 1 & 0 \end{bmatrix} + \begin{bmatrix} k_{e1} \\ k_{e2} \end{bmatrix}[0 \quad 1]\right| = \begin{vmatrix} s & -20.6 + k_{e1} \\ -1 & s + k_{e2} \end{vmatrix}$$

$$= s^2 + k_{e2}s - 20.6 + k_{e1} = 0$$

Since the desired characteristic equation is

$$s^2 + 20s + 100 = 0$$

by comparing Equation (10–66) with this last equation, we obtain

$$k_{e1} = 120.6, \qquad k_{e2} = 20$$

or

$$\mathbf{K}_e = \begin{bmatrix} 120.6 \\ 20 \end{bmatrix}$$

Method 3: We shall use Ackermann's formula given by Equation (10–65):

$$\mathbf{K}_e = \phi(\mathbf{A})\begin{bmatrix} \mathbf{C} \\ \mathbf{CA} \end{bmatrix}^{-1}\begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

where

$$\phi(s) = (s - \mu_1)(s - \mu_2) = s^2 + 20s + 100$$

Thus,

$$\phi(\mathbf{A}) = \mathbf{A}^2 + 20\mathbf{A} + 100\mathbf{I}$$

and

$$\mathbf{K}_e = (\mathbf{A}^2 + 20\mathbf{A} + 100\mathbf{I})\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}^{-1}\begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} 120.6 & 412 \\ 20 & 120.6 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 120.6 \\ 20 \end{bmatrix}$$

As a matter of course, we get the same $\mathbf{K}_e$ regardless of the method employed.

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