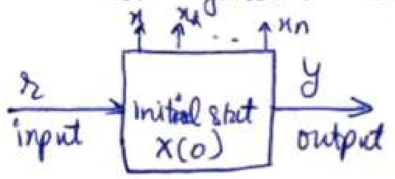
State Variable representation :

Consider a general SISO system.



$x_1 \rightarrow x_n$ — statevariables

We will denote the system state by n state variables $\{x_1(t), x_2(t) \dots x_n(t)\}$. , state vector.

$$X(t) \triangleq \begin{bmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_n(t) \end{bmatrix}$$

dimension of the state vector defines the order of the system. The set of eqns that describe the relationship among i/p o/p and state variables is called the state variable model. For nth order linear time invariant system, the state variable model takes the form.

$$\dot{x}_1(t) = \frac{dx_1(t)}{dt} = a_{11}x_1(t) + a_{12}x_2(t) - - - a_{1n}x_n(t) + b_1 r(t)$$

$$\dot{x}_2(t) = \frac{dx_2(t)}{dt} = a_{21}x_1(t) + a_{22}x_2(t) - - - a_{2n}x_n(t) + b_2 r(t)$$

$$\vdots$$

$$\dot{x}_n(t) = \frac{dx_n(t)}{dt} = a_{n1}x_1(t) + a_{n2}x_2(t) - - - a_{nn}x_n(t) + b_n r(t)$$

$$y(t) = c_1 x_1(t) + c_2 x_2(t) - - c_n x_n(t) + d r(t)$$

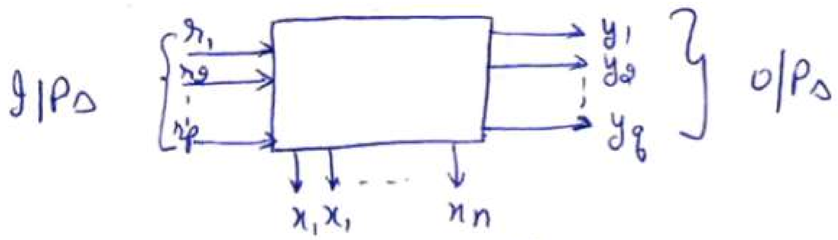
or $\dot{X}(t) = Ax(t) + br(t)$ state eqn

$y(t) = Cx(t) + d r(t)$ output eqn.

$x(t)$ - state $r(t)$ - i/p $y(t) = $ o/p $A \rightarrow n \times n$ $b \rightarrow n \times 1$, $C = 1 \times n$

$d =$ constant scalar.

MIMO system.



State variables

$$r(t) = \begin{bmatrix} r_1(t) \\ r_2(t) \\ \vdots \\ r_p(t) \end{bmatrix} \qquad y(t) = \begin{bmatrix} y_1(t) \\ y_2(t) \\ \vdots \\ y_q(t) \end{bmatrix}. \qquad x(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_n(t) \end{bmatrix}$$

$\dot{x}(t) = A\,x(t) + B\,r(t)$ state eqn

$y(t) = C\,x(t) + D\,r(t)$ output eqn.

$A \rightarrow n \times n$ $B \rightarrow n \times p$ $C \rightarrow q \times n$ $D \rightarrow q \times p$ constant matrices

OR $\dot{x}(t) = Ax(t) + bu(t) \quad x(t_0) \triangleq x^0$ state eqn.

$y(t) = Cx(t) + du(t)$ output eqn.

$x(t) = n \times 1$ state vector of nth order dynamic system

$u(t)$ = system input

$y(t)$ = defined output

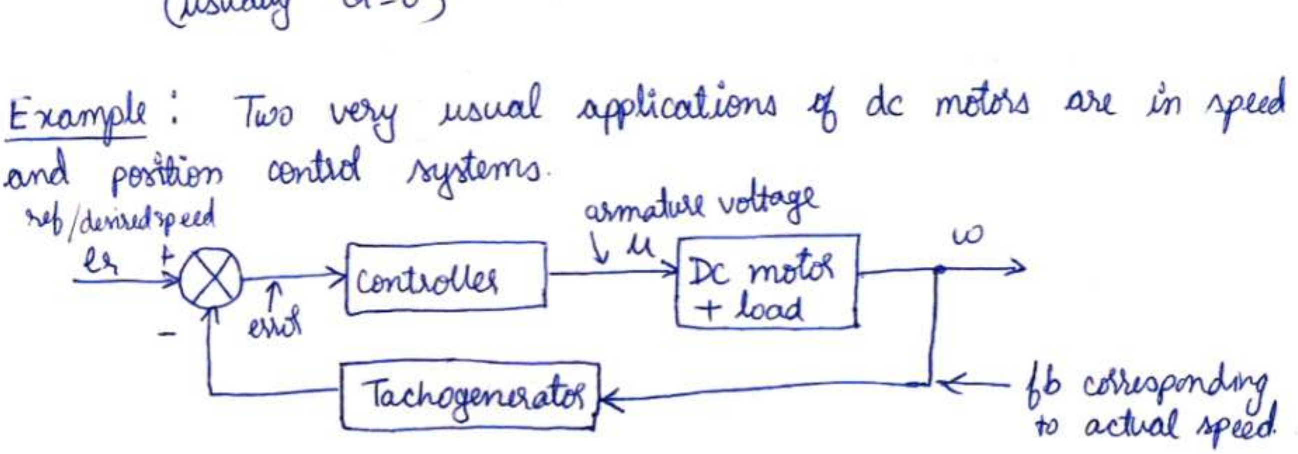
d = scalar represents direct coupling betⁿ I/P & O/P (usually $d=0$)

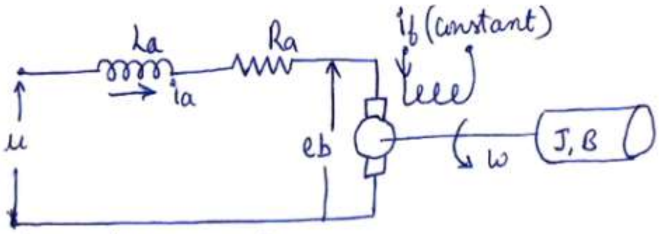
$A = n \times n$ matrix

$b = n \times 1$ column matrix

$C = 1 \times n$ row matrix

Example: Two very usual applications of dc motors are in speed and position control systems.





The voltage loop eqn is

$$u(t) = La \frac{di_a(t)}{dt} + Ra\, i_a(t) + e_b(t)$$

where La – inductance of armature wdg (H)
Ra – resistance of arm. wdg (Ω)
i_a – arm. current (A)
e_b – back emf (V)
u – applied arm. voltage (V)

Torque balance eqn

$$T_m(t) = J \frac{d\omega(t)}{dt} + B\omega(t)$$

T_m = Torque developed by motor (Nm)
J = eqv. MI of motor + load referred to motor shaft (Kgm^2)
B = eqv. viscous friction coeff of motor and load referred to motor shaft [Nm/(rad/sec)]
ω = angular velocity of motor shaft (rad/sec)

In servo applications, the dc motors are generally used in the linear range of mag. curve. $\therefore \phi \propto i_f$. For arm. controlled motor i_f is constant $\therefore T_m \propto i_a \phi$ can be expressed as $\underline{T_m(t)} = K_T \, i_a(t)$

K_T - motor torque constant (Nm/A)

The counter emf $e_b \propto \phi \omega$ is $\underline{e_b(t)} = K_b \, \omega(t)$

K_b = back emf constant [V/(rad/sec)]

$$\frac{di_a(t)}{dt} = -\frac{R_a}{L_a} i_a(t) - \frac{K_b}{L_a} \omega(t) + \frac{1}{L_a} u(t)$$

$$\frac{d\omega(t)}{dt} = \frac{K_T}{J} i_a(t) - \frac{B}{J} \omega(t)$$

$x_1(t) = \omega(t)$ $\quad x_2(t) = i_a(t)$ obvious choice of state variables

o/p variable is $y(t) = \omega(t)$

The plant model of the speed control system organized into the vector - matrix notation is given below

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} -B/J & K_T/J \\ -\frac{K_b}{L_a} & -R_a/L_a \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 1/L_a \end{bmatrix} u(t)$$

$$y(t) = x_1(t)$$

Let $R_a = 1\ \Omega$ $\quad L_a = 0.1\ H$ $\quad J = 0.1\ Kg\,m^2$

$B = 0.1\ \frac{Nm}{rad/s}$ $\quad K_b = K_T = 0.1$

then

$$\dot{x}(t) = A\,x(t) + b\,u(t)$$

$$y(t) = c\,x(t)$$

$$A = \begin{bmatrix} -1 & 1 \\ -1 & -10 \end{bmatrix} \quad b = \begin{bmatrix} 0 \\ 10 \end{bmatrix} \quad c = \begin{bmatrix} 1 & 0 \end{bmatrix}$$

Example: block diag is same. Now the controlled variable is angular shaft position $\theta(t)$, sensor — position sensor (fb)

$$\frac{d\theta(t)}{dt} = \omega(t)$$

$x_1(t) = \theta(t)$ $\quad x_2(t) = \omega(t)$ $\quad x_3(t) = i_a(t)$ $\quad y(t) = \theta(t)$

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \\ \dot{x}_3(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & -B/J & \frac{K_T}{J} \\ 0 & -K_b/L_a & -R_a/L_a \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1/L_a \end{bmatrix} u(t)$$

$$y(t) = x_1(t)$$

for the previous parameters the model is now.

$$\dot{x}(t) = A x(t) + b u(t)$$

$$y(t) = c\, x(t)$$

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & -1 & -10 \end{bmatrix} \quad b = \begin{bmatrix} 0 \\ 0 \\ 10 \end{bmatrix} \quad c = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}$$

The selected state variables are the physical quantities of the systems which can be measured. Sometimes it is necessary & adv to provide a f.b proportional to the state variables of the system, rather than o/p alone, for the purpose of stabilizing

stabilizing and improving the performance of a system. physical variables are chosen as state variables for help in implementation of design.

Transformation of state variables : State variable used in the original formulation of the dynamics of a system are not as convenient as another set of state variables. Instead of having to re-formulate the system dynamics, it is possible to transform the set. [A b c d] of the original formulation to a new set {$\bar{A}$ $\bar{b}$ $\bar{c}$ $\bar{d}$}. The change of variables is represented by a linear transformation

$$x = P\bar{x}$$

where $\bar{x}$ is a state vector in the new formulation and x is the state vector in the original formulation. Transformation matrix is assumed to be nonsingular $n \times n$ matrix so that

$$\bar{x} = P^{-1}x \qquad P\text{- constant matrix.}$$

The original dynamics are expressed by

$$\dot{x}(t) = Ax(t) + bu(t) \qquad x(t_0) \triangleq x^0$$

and the output by $\quad y(t) = cx(t) + du(t)$

Substitution of x into these eqns. $(x = P\bar{x})$

$$P\dot{\bar{x}}(t) = AP\bar{x}(t) + bu(t)$$

$$y(t) = cP\bar{x}(t) + du(t)$$

or

$$\dot{\bar{x}}(t) = \bar{A}\bar{x}(t) + \bar{b}u(t) \qquad \bar{x}(t_0) = P^{-1}x(t_0)$$

$$y(t) = \bar{c}\bar{x}(t) + \bar{d}u(t)$$

with $\bar{A} = P^{-1}AP \qquad \bar{b} = P^{-1}b$

$\bar{c} = cP \qquad \bar{d} = d$

both the systems have identical O/P responses for the same input. P is called an equivalence or similarity transformation. there are infinite eqv. systems ∵ P can be arbitrarily chosen. Some transformations are extensively used in the analysis and design. Five will be discussed.

For the prev. system we have taken angular velocity $\omega(t)$ and armature current $i_a(t)$ as state variables:

$$x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} \omega \\ i_a \end{bmatrix}$$

We now define new state variables as

$$\bar{x}_1 = \omega \qquad \bar{x}_2 = -\omega + i_a$$

or

$$\bar{x} = \begin{bmatrix} \bar{x}_1 \\ \bar{x}_2 \end{bmatrix} = \begin{bmatrix} x_1 \\ -x_1 + x_2 \end{bmatrix} = \overbrace{\begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}}^{P^{-1}} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

We can now express velocity $x_1(t)$ and armature current $x_2(t)$ in terms of the variables $\bar{x}_1(t)$ and $\bar{x}_2(t)$

$$x = P\bar{x} \qquad P = \begin{bmatrix} 1 & 0 \\ +1 & 1 \end{bmatrix}$$

State variable model in terms of the transformed state vector $\bar{x}(t)$

$$\dot{\bar{x}}(t) = \bar{A}\,\bar{x}(t) + \bar{b}\,u(t)$$
$$y(t) = \bar{c}\,\bar{x}(t)$$

$$\bar{A} = P^{-1}AP = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} -1 & 1 \\ -1 & -10 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -11 & -11 \end{bmatrix}$$

$$\bar{b} = P^{-1}b = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} 0 \\ 10 \end{bmatrix} = \begin{bmatrix} 0 \\ 10 \end{bmatrix}$$

$$\bar{c} = cP = \begin{bmatrix} 1 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \end{bmatrix}$$

$\bar{x}_1(t_0) = x_1(t_0) \qquad \bar{x}_2(t_0) = -x_1(t_0) + x_2(t_0)$

$\bar{x}(t)$, $x(t)$ — qualify to be state vectors (characterise the system completely at time t)

$y(t)$ - uniquely determined from both models

Thus models are equivalent. P – equivalence / similarity transformation

Conversion of state variable models to transfer functions :

Consider the state variable model

$$\dot{x}(t) = Ax(t) + bu(t) \qquad x(t_0) \triangleq x^0$$

$$y(t) = cx(t) + du(t)$$

Taking the Laplace transform we obtain

Taking the Laplace transform

$$sX(s) - x^0 = AX(s) + bU(s)$$

$$Y(s) = cX(s) + dU(s)$$

where

$$X(s) \triangleq \mathcal{L}[x(t)] \quad U(s) = \mathcal{L}[u(t)] \quad Y(s) = \mathcal{L}[y(t)]$$

Manipulation of these equations gives.

$$(sI - A)X(s) = x^0 + bU(s)$$, I is $n \times n$ identity matrix

or

$$X(s) = (sI - A)^{-1}x^0 + \cancel{SA} (sI - A)^{-1} bU(s)$$

$$Y(s) = c(sI - A)^{-1}x^0 + [c(sI - A)^{-1} b + d] U(s)$$

If x^0 and $U(s)$ are known, $X(s)$ and $X(s)$ can be computed.

In the case of a zero initial state (i.e $x^0 = 0$) the I/P O/P behaviour of the system is determined entirely by the transfer function.

$$\frac{Y(s)}{U(s)} = G(s) = c\,(sI-A)^{-1}b + d.$$

inverse of matrix $(SI-A)$ is

$$(sI-A)^{-1} = (SI-A)^{+} / |sI-A|$$

T.F can be written as

$$G(s) = \frac{c\,(sI-A)^{+}b}{|SI-A|} + d$$

let

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} \qquad SI-A = \begin{bmatrix} s-a_{11} & -a_{12} & \cdots & -a_{1n} \\ -a_{21} & s-a_{22} & \cdots & -a_{2n} \\ \vdots & \vdots & & \vdots \\ -a_{n1} & -a_{n2} & \cdots & s-a_{nn} \end{bmatrix}$$

If we imagine calculating det $(sI-A)$, we see that one of the terms will be product of diagnol elements of $(SI-A)$

$$(s-a_{11})(s-a_{22}) \cdots (s-a_{nn}) = s^n + \alpha_1' s^{n-1} + \cdots + \alpha_n'$$

a polynomial of degree n with the leading coeff of unity. There will be other terms coming from the off diagonal elements of $(SI-A)$, but none will have a degree as high as n.

The $|sI-A|$ will be of the following form:

$$|sI-A| = \Delta(s) = s^n + \alpha_1 s^{n-1} + \cdots + \alpha_n$$

where α_i are constant scalars.

This is known as the characteristic polynomial of the matrix. It's roots are called the characteristic roots or eigenvalues of matrix A. The roots determine essential features of the unforced dynamic behaviour of the system.

The adjoint of an $n \times n$ matrix is itself an $n \times n$ matrix whose elements are the cofactors of the original matrix. Each cofactor is obtained by computing the determinant of the matrix that remains when a row and a column of the original matrix are deleted. It thus follows that each element in $(sI - A)^{+}$ is a polynomial in s of max degree $(n-1)$. Adjoint of $(sI-A)$ can $\therefore$ be expressed as

$$(sI-A)^{+} = Q_1 s^{n-1} + Q_2 s^{n-2} + \cdots + Q_{n-1} s + Q_n$$

Q_i — constant $n \times n$ matrices. We can express transfer function $G(s)$ in the following form:

$$G(s) = \frac{c\left[Q_1 s^{n-1} + Q_2 s^{n-2} + \cdots + Q_{n-1} s + Q_n\right] b}{s^n + \alpha_1 s^{n-1} + \cdots + \alpha_{n-1} s + \alpha_n} + d$$

$G(s)$ — rational function of s. When $d=0$, the degree of numerator polynomial of $G(s)$ is strictly less than degree of denom. polynomial and $\therefore$ the resulting T.F. is a strictly proper transfer function. When $d \neq 0$, the degree of num. polynomial of $G(s)$ = degree of denom polynomial giving a proper transfer function.

Further $d = \lim_{s \to \infty} [G(s)]$

The char polynomial of matrix A is same as the denom polynomial of the corresponding TF G(s). If there are no cancellations between the numerator and denom, the eigen values of matrix A are same as the poles of G(s).

$$\dot{x}_1(t) = \frac{dx_1(t)}{dt} = a_{11}x_1(t) + a_{12}x_2(t) \cdots a_{1n}x_n(t) + b_{11}r_1(t) + b_{12}r_2(t) \cdots + b_{1p}r_p(t)$$

$\vdots$

$$\dot{x}_2(t) = \frac{dx_2(t)}{dt} =$$

$$= a_{n1}x_1(t) + a_{n2}x_2(t) \cdots a_{nn}x_n(t) + b_{n1}r_1(t) + b_{n2}r_2(t) + \cdots b_{np}r_p(t)$$

$$y_1(t) = c_{11}x_1(t) + c_{12}x_2(t) \cdots c_{1n}x_n(t) + d_{11}r_1(t) + d_{12}r_2(t) \cdots + d_{1p}r_p(t)$$

$\vdots$

$$y_q(t) = c_{q1}x_1(t) + c_{q2}x_2(t) + \cdots c_{qn}x_n(t) + d_{q1}r_1(t) + d_{q2}r_2(t) + \cdots d_{qp}r_p(t)$$

$$\begin{bmatrix} \dot{x}_1(t) \\ \vdots \\ \dot{x}_n(t) \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ & & & \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} \begin{bmatrix} x_1(t) \\ \vdots \\ x_n(t) \end{bmatrix} + \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1p} \\ \vdots & & & \\ b_{n1} & b_{n2} & \cdots & b_{np} \end{bmatrix} \begin{bmatrix} r_1(t) \\ r_2(t) \\ \vdots \\ r_p(t) \end{bmatrix}$$

$$\begin{bmatrix} y_1(t) \\ \vdots \\ y_q(t) \end{bmatrix} = \begin{bmatrix} c_{11} & c_{12} & \cdots & c_{1n} \\ \vdots & & & \\ c_{q1} & c_{q2} & \cdots & c_{qn} \end{bmatrix} \begin{bmatrix} x_1(t) \\ \vdots \\ x_n(t) \end{bmatrix} + \begin{bmatrix} d_{11} & d_{12} & \cdots & d_{1p} \\ \vdots & & & \\ d_{q1} & d_{q2} & \cdots & d_{qp} \end{bmatrix} \begin{bmatrix} r_1(t) \\ \vdots \\ r_p(t) \end{bmatrix}$$