Determination of Matrix K Using Transformation Matrix T. Suppose that the system is defined by

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}u$$

and the control signal is given by

$$u = -\mathbf{K}\mathbf{x}$$

$|sI - (A - BK)| = (s - \mu_1) \cdots (s - \mu_n)$

The feedback gain matrix $\mathbf{K}$ that forces the eigenvalues of $\mathbf{A} - \mathbf{B}\mathbf{K}$ to be $\mu_1, \mu_2, \ldots, \mu_n$ (desired values) can be determined by the following steps (if μ_i is a complex eigenvalue, then its conjugate must also be an eigenvalue of $\mathbf{A} - \mathbf{B}\mathbf{K}$):

Step 1: Check the controllability condition for the system. If the system is completely state controllable, then use the following steps:

Step 2: From the characteristic polynomial for matrix $\mathbf{A}$, that is,

$$|s\mathbf{I} - \mathbf{A}| = s^n + a_1 s^{n-1} + \cdots + a_{n-1}s + a_n$$

determine the values of $a_1, a_2, \ldots, a_n$.

$\dot{X} = AX + Bu$

$u = -KX$

Step 3: Determine the transformation matrix $\mathbf{T}$ that transforms the system state equation into the controllable canonical form. (If the given system equation is already in the controllable canonical form, then $\mathbf{T} = \mathbf{I}$.) It is not necessary to write the state equation in the controllable canonical form. All we need here is to find the matrix $\mathbf{T}$. The transformation matrix $\mathbf{T}$ is given by Equation (10–4), or

$M = [B \ AB \ \cdots \ A^{n-1} B]$

$$\underline{\mathbf{T} = \mathbf{M}\mathbf{W}}$$

$W = [\quad]$

where $\mathbf{M}$ is given by Equation (10–5) and $\mathbf{W}$ is given by Equation (10–6).

Step 4: Using the desired eigenvalues (desired closed-loop poles), write the desired characteristic polynomial:

$$(s - \mu_1)(s - \mu_2) \cdots (s - \mu_n) = s^n + \alpha_1 s^{n-1} + \cdots + \alpha_{n-1} s + \alpha_n$$

a_i

and determine the values of $\alpha_1, \alpha_2, \ldots, \alpha_n$.

Step 5: The required state feedback gain matrix $\mathbf{K}$ can be determined from Equation (10–13), rewritten thus:

$$\mathbf{K} = [\alpha_n - a_n \mid \alpha_{n-1} - a_{n-1} \mid \cdots \mid \alpha_2 - a_2 \mid \alpha_1 - a_1]\mathbf{T}^{-1}$$

II

Determination of Matrix K Using Direct Substitution Method. If the system is of low order ($n \leq 3$), direct substitution of matrix $\mathbf{K}$ into the desired characteristic polynomial may be simpler. For example, if $n = 3$, then write the state feedback gain matrix $\mathbf{K}$ as

$$\mathbf{K} = \begin{bmatrix} k_1 & k_2 & k_3 \end{bmatrix}$$

Substitute this $\mathbf{K}$ matrix into the desired characteristic polynomial $|s\mathbf{I} - \mathbf{A} + \mathbf{BK}|$ and equate it to $(s - \mu_1)(s - \mu_2)(s - \mu_3)$, or

$$|s\mathbf{I} - \mathbf{A} + \mathbf{BK}| = (s - \mu_1)(s - \mu_2)(s - \mu_3)$$

Since both sides of this characteristic equation are polynomials in s, by equating the coefficients of the like powers of s on both sides, it is possible to determine the values of k1, k2 , and k3 . This approach is convenient if n=2 or 3. (For n=4, 5, 6, p , this approach may become very tedious.) Note that if the system is not completely controllable, matrix K cannot be determined. (No solution exists.)

Determination of Matrix K Using Ackermann's Formula

Consider the system

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}u$$

where we use the state feedback control $u = -\mathbf{K}\mathbf{x}$. We assume that the system is completely state controllable. We also assume that the desired closed-loop poles are at $s = \mu_1, s = \mu_2, \ldots, s = \mu_n$.

Use of the state feedback control

$$u = -\mathbf{K}\mathbf{x}$$

modifies the system equation to

$$\dot{\mathbf{x}} = (\mathbf{A} - \mathbf{B}\mathbf{K})\mathbf{x} \tag{10–14}$$

Let us define

$$\widetilde{\mathbf{A}} = \mathbf{A} - \mathbf{B}\mathbf{K}$$

The desired characteristic equation is

$$\begin{aligned} |s\mathbf{I} - \mathbf{A} + \mathbf{B}\mathbf{K}| = |s\mathbf{I} - \widetilde{\mathbf{A}}| &= (s - \mu_1)(s - \mu_2)\cdots(s - \mu_n) \\ &= s^n + \alpha_1 s^{n-1} + \cdots + \alpha_{n-1}s + \alpha_n = 0 \end{aligned}$$

Since the Cayley–Hamilton theorem states that $\widetilde{\mathbf{A}}$ satisfies its own characteristic equation, we have

$$\phi(\widetilde{\mathbf{A}}) = \widetilde{\mathbf{A}}^n + \alpha_1 \widetilde{\mathbf{A}}^{n-1} + \cdots + \alpha_{n-1}\widetilde{\mathbf{A}} + \alpha_n \mathbf{I} = \mathbf{0} \tag{10–15}$$

We shall utilize Equation (10–15) to derive Ackermann's formula. To simplify the derivation, we consider the case where $n = 3$. (For any other positive integer n, the following derivation can be easily extended.)

Consider the following identities:

$$\begin{aligned}
\mathbf{I} &= \mathbf{I} \\
\widetilde{\mathbf{A}} &= \mathbf{A} - \mathbf{BK} \\
\widetilde{\mathbf{A}}^2 &= (\mathbf{A} - \mathbf{BK})^2 = \mathbf{A}^2 - \mathbf{ABK} - \mathbf{BK}\widetilde{\mathbf{A}} \\
\widetilde{\mathbf{A}}^3 &= (\mathbf{A} - \mathbf{BK})^3 = \mathbf{A}^3 - \mathbf{A}^2\mathbf{BK} - \mathbf{ABK}\widetilde{\mathbf{A}} - \mathbf{BK}\widetilde{\mathbf{A}}^2
\end{aligned}$$

Since the Cayley–Hamilton theorem states that $\widetilde{\mathbf{A}}$ satisfies its own characteristic equation, we have

$$\phi(\widetilde{\mathbf{A}}) = \widetilde{\mathbf{A}}^n + \alpha_1 \widetilde{\mathbf{A}}^{n-1} + \cdots + \alpha_{n-1} \widetilde{\mathbf{A}} + \alpha_n \mathbf{I} = \mathbf{0} \tag{10–15}$$

We shall utilize Equation (10–15) to derive Ackermann's formula. To simplify the derivation, we consider the case where $n = 3$. (For any other positive integer n, the following derivation can be easily extended.)

Consider the following identities:

$$(A - BK)(A - BK)$$

$$= A^2 - ABK - BKA + (BK)^2$$

$$\mathbf{I} = \mathbf{I}$$

$$\widetilde{\mathbf{A}} = \mathbf{A} - \mathbf{BK}$$

$$\widetilde{\mathbf{A}}^2 = (\mathbf{A} - \mathbf{BK})^2 = \mathbf{A}^2 - \mathbf{ABK} - \mathbf{BK}\widetilde{\mathbf{A}}$$

$$\widetilde{\mathbf{A}}^3 = (\mathbf{A} - \mathbf{BK})^3 = \mathbf{A}^3 - \mathbf{A}^2\mathbf{BK} - \mathbf{ABK}\widetilde{\mathbf{A}} - \mathbf{BK}\widetilde{\mathbf{A}}^2$$

$$= A^2 - ABK - BK(A - BK)$$

Multiplying the preceding equations in order by $\alpha_3, \alpha_2, \alpha_1$, and α_0 (where $\alpha_0 = 1$), respectively, and adding the results, we obtain

$$\begin{aligned}
&\alpha_3 \mathbf{I} + \alpha_2 \widetilde{\mathbf{A}} + \alpha_1 \widetilde{\mathbf{A}}^2 + \widetilde{\mathbf{A}}^3 \\
&= \alpha_3 \mathbf{I} + \alpha_2(\mathbf{A} - \mathbf{BK}) + \alpha_1(\mathbf{A}^2 - \mathbf{ABK} - \mathbf{BK}\widetilde{\mathbf{A}}) + \mathbf{A}^3 - \mathbf{A}^2\mathbf{BK} \\
&\quad - \mathbf{ABK}\widetilde{\mathbf{A}} - \mathbf{BK}\widetilde{\mathbf{A}}^2 \\
&= \alpha_3 \mathbf{I} + \alpha_2 \mathbf{A} + \alpha_1 \mathbf{A}^2 + \mathbf{A}^3 - \alpha_2 \mathbf{BK} - \alpha_1 \mathbf{ABK} - \alpha_1 \mathbf{BK}\widetilde{\mathbf{A}} - \mathbf{A}^2\mathbf{BK} \\
&\quad - \mathbf{ABK}\widetilde{\mathbf{A}} - \mathbf{BK}\widetilde{\mathbf{A}}^2
\end{aligned} \tag{10–16}$$

Referring to Equation (10–15), we have

$$\alpha_3 \mathbf{I} + \alpha_2 \widetilde{\mathbf{A}} + \alpha_1 \widetilde{\mathbf{A}}^2 + \widetilde{\mathbf{A}}^3 = \phi(\widetilde{\mathbf{A}}) = \mathbf{0}$$

Also, we have

$$\alpha_3 \mathbf{I} + \alpha_2 \mathbf{A} + \alpha_1 \mathbf{A}^2 + \mathbf{A}^3 = \phi(\mathbf{A}) \neq \mathbf{0}$$

Substituting the last two equations into Equation (10–16), we have

$$\phi(\widetilde{\mathbf{A}}) = \phi(\mathbf{A}) - \alpha_2 \mathbf{BK} - \alpha_1 \mathbf{BK}\widetilde{\mathbf{A}} - \mathbf{BK}\widetilde{\mathbf{A}}^2 - \alpha_1 \mathbf{ABK} - \mathbf{ABK}\widetilde{\mathbf{A}} - \mathbf{A}^2\mathbf{BK}$$

Since $\phi(\widetilde{\mathbf{A}}) = \mathbf{0}$, we obtain

$$\begin{aligned}
\phi(\mathbf{A}) &= \mathbf{B}(\alpha_2 \mathbf{K} + \alpha_1 \mathbf{K}\widetilde{\mathbf{A}} + \mathbf{K}\widetilde{\mathbf{A}}^2) + \mathbf{AB}(\alpha_1 \mathbf{K} + \mathbf{K}\widetilde{\mathbf{A}}) + \mathbf{A}^2\mathbf{BK} \\
&= \left[\mathbf{B} \;\vdots\; \mathbf{AB} \;\vdots\; \mathbf{A}^2\mathbf{B}\right] \begin{bmatrix} \alpha_2 \mathbf{K} + \alpha_1 \mathbf{K}\widetilde{\mathbf{A}} + \mathbf{K}\widetilde{\mathbf{A}}^2 \\ \alpha_1 \mathbf{K} + \mathbf{K}\widetilde{\mathbf{A}} \\ \mathbf{K} \end{bmatrix}
\end{aligned} \tag{10–17}$$

Since the system is completely state controllable, the inverse of the controllability matrix

$$\left[\mathbf{B} \;\vdots\; \mathbf{AB} \;\vdots\; \mathbf{A}^2\mathbf{B}\right]$$

exists. Premultiplying both sides of Equation (10–17) by the inverse of the controllability matrix, we obtain

$$\left[\mathbf{B} \;\vdots\; \mathbf{AB} \;\vdots\; \mathbf{A}^2\mathbf{B}\right]^{-1} \phi(\mathbf{A}) = \begin{bmatrix} \alpha_2 \mathbf{K} + \alpha_1 \mathbf{K}\widetilde{\mathbf{A}} + \mathbf{K}\widetilde{\mathbf{A}}^2 \\ \alpha_1 \mathbf{K} + \mathbf{K}\widetilde{\mathbf{A}} \\ \mathbf{K} \end{bmatrix}$$

Premultiplying both sides of this last equation by $[0 \quad 0 \quad 1]$, we obtain

$$[0 \quad 0 \quad 1][\mathbf{B} \mid \mathbf{AB} \mid \mathbf{A}^2\mathbf{B}]^{-1}\phi(\mathbf{A}) = [0 \quad 0 \quad 1]\begin{bmatrix} \alpha_2\mathbf{K} + \alpha_1\mathbf{K}\widetilde{\mathbf{A}} + \mathbf{K}\widetilde{\mathbf{A}}^2 \\ \alpha_1\mathbf{K} + \mathbf{K}\widetilde{\mathbf{A}} \\ \mathbf{K} \end{bmatrix} = \mathbf{K}$$

which can be rewritten as

$$\mathbf{K} = [0 \quad 0 \quad 1][\mathbf{B} \mid \mathbf{AB} \mid \mathbf{A}^2\mathbf{B}]^{-1}\phi(\mathbf{A})$$

This last equation gives the required state feedback gain matrix **K**.

For an arbitrary positive integer n, we have

$$\mathbf{K} = [0 \quad 0 \quad \cdots \quad 0 \quad 1][\mathbf{B} \mid \mathbf{AB} \mid \cdots \mid \mathbf{A}^{n-1}\mathbf{B}]^{-1}\phi(\mathbf{A}) \tag{10–18}$$

Equation (10–18) is known as Ackermann's formula for the determination of the state feedback gain matrix K

Regulator systems (where the reference input is constant, including zero) and control systems (where the reference input is time varying).

The first step in the pole-placement design approach is to choose the locations of the desired closed-loop poles. The most frequently used approach is to choose such poles based on experience in the root-locus design, placing a dominant pair of closed-loop poles and choosing other poles so that they are far to the left of the dominant closed-loop poles.

Note that if we place the dominant closed-loop poles far from the jw axis, so that the system response becomes very fast, the signals in the system become very large, with the result that the system may become nonlinear. This should be avoided.

Note that requiring a high-speed response implies requiring large amounts of control energy. Also, in general, increasing the speed of response requires a larger, heavier actuator, which will cost more.

Consider the regulator system shown in Figure 10–2. The plant is given by

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}u$$

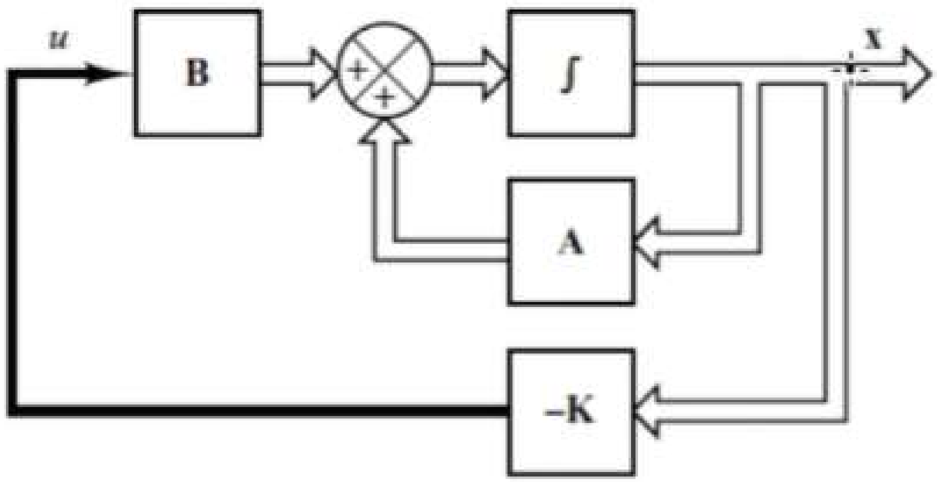
where

$$\mathbf{A} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -5 & -6 \end{bmatrix}, \qquad \mathbf{B} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

The system uses the state feedback control $\mathbf{u} = -\mathbf{K}\mathbf{x}$. Let us choose the desired closed-loop poles at

$$s = -2 + j4, \qquad s = -2 - j4, \qquad s = -10$$

(We make such a choice because we know from experience that such a set of closed-loop poles will result in a reasonable or acceptable transient response.) Determine the state feedback gain matrix **K**.



First, we need to check the controllability matrix of the system. Since the controllability matrix **M** is given by

$$\mathbf{M} = \left[\mathbf{B} \; \vdots \; \mathbf{AB} \; \vdots \; \mathbf{A}^2\mathbf{B}\right] = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & -6 \\ 1 & -6 & 31 \end{bmatrix}$$

we find that $|\mathbf{M}| = -1$, and therefore, rank $\mathbf{M} = 3$. Thus, the system is completely state controllable and arbitrary pole placement is possible.

Method 1: The first method is to use Equation (10–13). The characteristic equation for the system is

$\alpha_i \quad a_i$

$$|s\mathbf{I} - \mathbf{A}| = \begin{vmatrix} s & -1 & 0 \\ 0 & s & -1 \\ 1 & 5 & s+6 \end{vmatrix}$$
$$= s^3 + 6s^2 + 5s + 1$$
$$= s^3 + a_1 s^2 + a_2 s + a_3 = 0$$

Hence,

$-2 \pm j4$

$$a_1 = 6, \qquad a_2 = 5, \qquad a_3 = 1$$

The desired characteristic equation is

$$(s + 2 - j4)(s + 2 + j4)(s + 10) = s^3 + 14s^2 + 60s + 200$$
$$= s^3 + \alpha_1 s^2 + \alpha_2 s + \alpha_3 = 0$$

Hence,

$$\alpha_1 = 14, \qquad \alpha_2 = 60, \qquad \alpha_3 = 200$$

Referring to Equation (10–13), we have

$$\boxed{\mathbf{K} = [\alpha_3 - a_3 \;\vdots\; \alpha_2 - a_2 \;\vdots\; \alpha_1 - a_1]\mathbf{T}^{-1}}$$

where $\mathbf{T} = \mathbf{I}$ for this problem because the given state equation is in the controllable canonical form.

Then we have

$$\mathbf{K} = [200 - 1 \,\vdots\, 60 - 5 \,\vdots\, 14 - 6]$$
$$= [199 \quad 55 \quad 8]$$

Method 2: By defining the desired state feedback gain matrix **K** as

$$\mathbf{K} = [k_1 \quad k_2 \quad k_3]$$

and equating $|s\mathbf{I} - \mathbf{A} + \mathbf{BK}|$ with the desired characteristic equation, we obtain

$$|s\mathbf{I} - \mathbf{A} + \mathbf{BK}| = \left| \begin{bmatrix} s & 0 & 0 \\ 0 & s & 0 \\ 0 & 0 & s \end{bmatrix} - \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -5 & -6 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} [k_1 \quad k_2 \quad k_3] \right|$$

$$= \begin{vmatrix} s & -1 & 0 \\ 0 & s & -1 \\ 1 + k_1 & 5 + k_2 & s + 6 + k_3 \end{vmatrix}$$

$$= s^3 + (6 + k_3)s^2 + (5 + k_2)s + 1 + k_1$$

$$= s^3 + 14s^2 + 60s + 200$$

Thus,

$$6 + k_3 = 14, \qquad 5 + k_2 = 60, \qquad 1 + k_1 = 200$$

from which we obtain

$$k_1 = 199, \qquad k_2 = 55, \qquad k_3 = 8$$

or

$$\mathbf{K} = [199 \quad 55 \quad 8]$$

Method 3: The third method is to use Ackermann's formula. Referring to Equation (10–18), we have

$$\mathbf{K} = [0 \quad 0 \quad 1][\mathbf{B} \;\vdots\; \mathbf{AB} \;\vdots\; \mathbf{A}^2\mathbf{B}]^{-1}\phi(\mathbf{A})$$

Since

$$\phi(s) = (s-\mu_1)\cdots(s-\mu_j)$$

$$s \rightarrow A$$

$$\begin{aligned}
\phi(\mathbf{A}) &= \mathbf{A}^3 + 14\mathbf{A}^2 + 60\mathbf{A} + 200\mathbf{I} \\
&= \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -5 & -6 \end{bmatrix}^3 + 14\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -5 & -6 \end{bmatrix}^2 \\
&\quad + 60\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -5 & -6 \end{bmatrix} + 200\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \\
&= \begin{bmatrix} 199 & 55 & 8 \\ -8 & 159 & 7 \\ -7 & -43 & 117 \end{bmatrix}
\end{aligned}$$

and

$$\left[\mathbf{B} \;\vdots\; \mathbf{AB} \;\vdots\; \mathbf{A}^2\mathbf{B}\right] = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & -6 \\ 1 & -6 & 31 \end{bmatrix}$$

we obtain

$$\mathbf{K} = [0 \quad 0 \quad 1]\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & -6 \\ 1 & -6 & 31 \end{bmatrix}^{-1}\begin{bmatrix} 199 & 55 & 8 \\ -8 & 159 & 7 \\ -7 & -43 & 117 \end{bmatrix}$$

$$= [0 \quad 0 \quad 1]\begin{bmatrix} 5 & 6 & 1 \\ 6 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}\begin{bmatrix} 199 & 55 & 8 \\ -8 & 159 & 7 \\ -7 & -43 & 117 \end{bmatrix}$$

$$= [199 \quad 55 \quad 8]$$

Submit

As a matter of course, the feedback gain matrix **K** obtained by the three methods are the same. With this state feedback, the closed-loop poles are placed at $s = -2 \pm j4$ and $s = -10$, as desired.

It is noted that if the order n of the system were 4 or higher, methods 1 and 3 are recommended, since all matrix computations can be carried out by a computer. If method 2 is used, hand computations become necessary because a computer may not handle the characteristic equation with unknown parameters $k_1, k_2, \ldots, k_n$.