Ex 2.2.

when $t = 1$, I is reduced by 20%
$\underset{8000}{\shortparallel}$

$I = I_0 e^{kt}$

$8000 = 10\,000\, e^{k \cdot 1}$

$e^k = \frac{8}{10}$

$k = \ln(0.8) < 0$

we have exponential decay

в минуте уйдет кол-во людей.

$\Rightarrow I = 10\,000\, e^{\ln(0.8)t}$

$I_0 = 10\,000$

$I = 1000$

$1000 = 10\,000\, e^{\ln(0.8)\cdot t}$

$\frac{1}{10} = e^{\ln(0.8)\cdot t}$

$t = \frac{\ln(0.1)}{\ln(0.8)} \approx 10.32.$

it talles about 10 and lil more years.

$\int_a^\infty f(x)\,dx$ type 1.

$\int_a^\infty f(x) = \lim_{t \to \infty} \int_a^t f(x)\,dt$ it exists ✓

if lim exists
I conv the lim value
if not we say that I div

$\Big\}$ it exists

Ex 5.

$\int_1^\infty \frac{dx}{x^p}$ p is cn.

* if $p = 1$, then $I = \int_1^\infty \frac{dx}{x^1}$

$f(x) = 1/x$ is cont. on $[1, +\infty[$ $\rightarrow$ integr.

it is integ. is an imp.int of type 1.

$\lim_{t \to \infty} \int_1^t \frac{dx}{x} = \lim_{t \to \infty} \ln|x| \Big|_1^t = \lim_{t \to \infty} (\ln|t| - 0) = \infty$

then the limit doesn't exist.

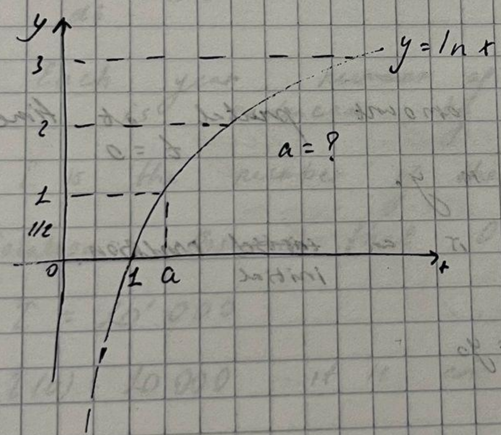
Thus, $I = \int_1^\infty \frac{dx}{x}$ is divergent if $x = 1$, then $\ln 1 = 0$

if $0<x<1$, then 1) $\int_1^x \frac{dt}{t}$ 2) $-\int_x^1 \frac{dt}{t}$

① $D_f =]0; +\infty[$

$R_f = \mathbb{R}$

$f(x) = \ln x$, continue at diff $\frac{d}{dx}(\ln x) = 1/x$



$\exists\, a \in \mathbb{R} \Rightarrow \ln x = 1$

$a = e$ (irrational)

by Interm th. e is a rear number

$\ln e = \int_1^e \frac{dt}{t}$

Ex.

$\frac{d}{dx}\ln|x| = ?$ $\qquad \frac{du}{dx} = \frac{x}{|x|}$

let $u = |x|$

$\frac{d}{dx}\ln|x| \Rightarrow \frac{d}{du}\ln \frac{du}{dx} \Rightarrow \frac{1}{u}\cdot\frac{x}{|x|} = \frac{1}{|x|}\cdot\frac{x}{|x|} \Rightarrow \frac{x}{x^2} = \frac{1}{x}$

$\ln 2 = ?$ hw

Chapter 5.

$f(x) = \ln x$ is diff on curve of $g(x) = e^x$.

☆ The natural logarithm is the function given by

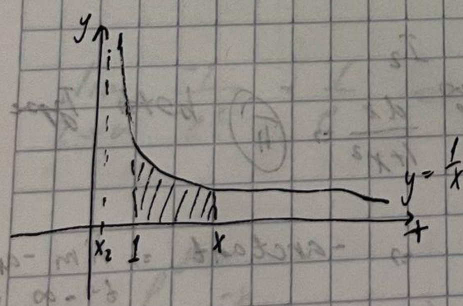
$$\ln x = \int_1^x \frac{dt}{t} \; ; \quad x > 0$$

If we consider $f(x) = \frac{1}{x}$ when $x > 0$

$$f(x) = \int_1^x \frac{1}{t}\, dt$$

In addition, $F'(x) = \frac{d}{dx}\left(\int_1^x \frac{dt}{t}\right) = f(x) = \frac{1}{x}$

$$\boxed{\frac{d}{dx}(\ln x) = \frac{1}{x}}$$



$\ln e^x = ?$

$\forall r \in \mathbb{R} \quad (g(x)) = e$

The change in y is proportional with the amount of y

$$\frac{dy}{dt} = ky \quad (k \text{ is const})$$

$\cancel{x = x(t)}$

$y = y(t) \Rightarrow$ the amount pruted at time $t = 0$

is ~~gbs~~ is y.

$y(0) = y_0$ is an ~~initial~~ initial condition.

$$\frac{dy}{dt} = ky \qquad y(0) = y_0$$

$$1/y \cdot \frac{dy}{dt} = k \qquad \ln|y| = k + c$$

$$\int \frac{1}{y} \cdot \frac{dy}{dt}\, dt = \int k\, dt \qquad |y| = e^{kt} \cdot e^{c}$$

$$y = Ae^{kt}, \quad A = \pm e^{c}$$

$$\int \frac{dy}{y} = \int k\, dt \qquad At \quad t = 0 \Rightarrow y(0) = y_0$$

$$y_0 = Ae^{0} \Rightarrow y_0 = A$$

$y = y_0 e^{kt}$

$\frac{dy}{dt} = ky \quad y(0) \overset{?}{=} y_0$ thus, $y = y_0 e^{kt}$

↑ $k > 0$, the change go exponential growth

$k < 0$, the change go expont. ↓ down

Ex 2.

$\frac{dI}{dt} = kI \qquad I = I(t)$

Each year number of infected people is reduced by 25 %.

I is the number of the infected people.

Solution: Assume that 0 is initial (time)

$I = 10'000$

$I(0) = 10\,000$ it is an initial condition

$\frac{dI}{dt} = kI \quad , \quad I(0) = 10'000$

Solut. $I = I_0 e^{kt} = 10'000 \cdot e^{kt}$ is the solution of the initial value problem.

$\int_a^b f(x)\,dx$ both $x=a$ & $x=b$ are singular points.

Choose a $c \in \,]a,b[$

$$\int_a^b f(x)\,dx = \underbrace{\int_a^c f(x)\,dx}_{\text{apply this type (2)}} + \underbrace{\int_c^b f(x)\,dx}_{\text{apply type (1)}}$$

3\. If f is discont. at $x = c \in \,]a,b[$

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$$

Ex.

$I = \int_2^5 \frac{dx}{\sqrt{x-2}}$ $\qquad f(x) = \frac{1}{\sqrt{x-2}}$ is discont. at $x=2$

cont. on

$f \in \,]2,5]$ $\qquad$ 2 — t — 5

$\forall t \in \,]2,5[\, : \int_t^5 f(x)\,dx \rightarrow$ Riemann int.

$\lim_{t\to 2^+} \int_t^5 \frac{dx}{\sqrt{x-2}}$ $\qquad dx = du$, $x-2 = u$, $x=t,\ u=t-2$, $x=5,\ u=3$ $\Rightarrow \int_t^3 \frac{dx}{\sqrt{u}}$ $\Rightarrow$

$$\int_t^3 \frac{dx}{\sqrt{x-2}} = \int_{t-2}^3 \frac{du}{\sqrt{u}} = \int_{t-2}^3 u^{-1/2}\,du = \left.\frac{u^{-\frac{1}{2}+1}}{-\frac{1}{2}+1}\right|_{t-2}^3$$

$$= 2\sqrt{u}\,\Big|_{t-2}^3 = 2\sqrt{3} - 2\sqrt{t-2} \quad (2)$$

(1) & (2) $\Rightarrow$ $\lim_{t\to 2^+} \int_t^5 \frac{dx}{\sqrt{x-2}} = \lim_{\substack{t\to 2^+\\ t>2\\ t-2>0}} (2\sqrt{3} - \underbrace{2\sqrt{t-2}}_{\to 0})$

$= \boxed{2\sqrt{3}}$

Remark: The definition is very similar to the definition of the limit of a function $f(x)$ as x tends to ∞.

Clearly; if $\lim_{x \to \infty} f(x) = L$, then $\lim_{n \to \infty} f(n) = L$

For $f(x) = \frac{1}{x}$ then $\lim_{x \to \infty} f(x) = 0$ then $\lim_{n \to \infty} f(n) = 0$

Converse is not true:

$\sin(n\pi) = 0$; $n \in \mathbb{Z}^+$

$\lim_{n \to \infty} f(n) = 0$

But for $f(x) = x \sin x$, $x \in \mathbb{R}$ does not exist for $\lim_{x \to \infty} f(x)$.

Example 3

1) $\lim_{n \to \infty} \frac{\ln n}{n}$

$f(x) = \frac{\ln x}{x}$

$\lim_{x \to \infty} \frac{\ln x}{x} \left(\frac{\infty}{\infty}\right) \lim_{x \to \infty} \frac{\frac{1}{x}}{1} = \lim_{x \to \infty} \frac{1}{x} = 0$

$\Rightarrow \lim_{n \to \infty} \frac{\ln n}{n} = 0$

2) $\lim_{n \to \infty} \left(\frac{n+1}{n-1}\right)^n$

$f(x) = \left(\frac{x+1}{x-1}\right)^x$

$\lim_{x \to \infty} \left(\frac{x+1}{x-1}\right)^x = 1^\infty$ undefined form

Let $y = \left(\frac{x+1}{x-1}\right)^x$

$\ln y = x \ln\left(\frac{x+1}{x-1}\right)$

$\lim_{x \to \infty} \ln y = \underbrace{\lim_{x \to \infty} x \ln\left(\frac{x+1}{x-1}\right)}_{0 \cdot \infty}$

$\lim_{x \to \infty} \ln y = \lim_{x \to \infty} \frac{\ln\left(\frac{x+1}{x-1}\right)}{\frac{1}{x}}$

$\left(\frac{0}{0}\right) = \frac{\frac{\left(\frac{x+1}{x-1}\right)'}{\frac{x+1}{x-1}}}{-\frac{1}{x^2}} = \frac{\frac{-2}{(x-1)^2} \cdot \frac{x-1}{x+1}}{\frac{-1}{x^2}}$

$x-1-x-1$

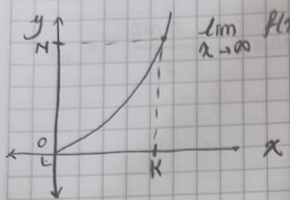
$1 \cdot (x-1) - 1(x+1)$

$\frac{\frac{-2}{(x-1)(x+1)}}{\frac{-1}{x^2}} = \lim_{x \to \infty} \frac{2x^2}{x^2-1}$

$\ln\left(\lim_{x \to \infty} y\right) = 2$ $\lim_{x \to \infty} y$ e^2

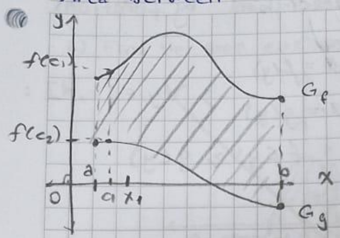
Definition 10.2 The sequence $(a_n)_{n \in \mathbb{Z}^+}$ diverges to infinity if:

$$\lim_{n \to \infty} a_n = +\infty \iff \forall M > 0 \quad \exists N \in \mathbb{Z}^+ \ni \forall n > N : a_n > M$$



$$\lim_{x \to \infty} f(x) = \infty$$

Area between curves



f and g are cont. with $f(x) \geq g(x)$ throughout $[a,b]$ then the area of the region between the curves

(through x-axis)

Choose $(n-1)$ points: $x_0 < x_1 < \dots < x_{k-1} < x_k < \dots < x_n$

$I = [a,b]$ is divided into n subintervals, let length of Δx_k of each subinterval $I_k = [x_{k-1}, x_k]$ equal to each other. Thus

$\Delta x_k = \frac{b-a}{n}$; $\forall k = 1, \dots, n$

$$\left.\begin{array}{l} h_{1st\,R} = f(c_1) - g(c_1) \\ \vdots \\ h_{kth\,R} = f(c_k) - g(c_k) \end{array}\right\} R_p = \overbrace{\sum_{k=1}^{n} [f(c_k) - g(c_k)] \cdot \Delta x_k}^{\text{is a Riemann sum for } f-g} \cong A$$

Since f and g is cont. on $[a,b]$ then $f-g$ is cont. on $[a,b]$ Thus $f-g$ is integrable so

$$\lim_{n \to \infty} \sum_{k=1}^{n} (f-g)(c_k) \cdot \Delta x_k \text{ exists equals to}$$

$$\int_a^b (f-g)(x)\,dx = A$$

// lnx homework

$f(x) = \ln x$ is diff. on curve of $g(x) = e^x$

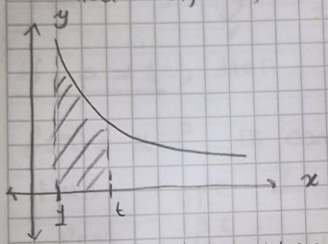
The natural logarithm is the function:

$$\ln x = \int_1^x \frac{1}{t}\, dt, \quad x > 0$$

If we consider $g(x) = \frac{1}{x}$ when $x > 0$

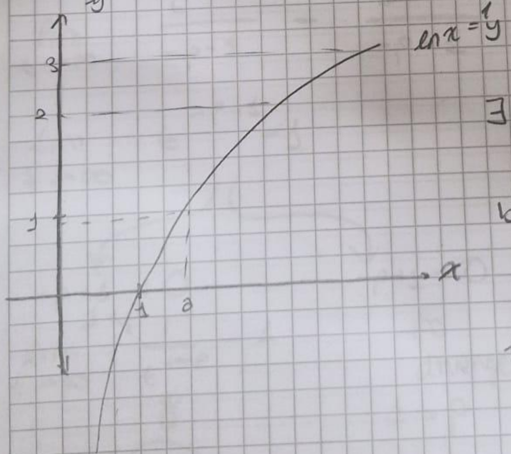
$$f(x) = \int_1^x \frac{1}{t}\, dt$$

In addition, $f'(x) = \frac{d}{dx}\left(\int_1^x \frac{dt}{t}\right) = g(x) = \frac{1}{x}$



if $x = t$ then $\ln t = 0$

if $0 < x < 1$ then $\ln x = \int_1^x \frac{dt}{t} = -\int_x^1 \frac{dt}{t}$



$\exists a \in \mathbb{R} \Rightarrow \ln x = 1$

(irrational)

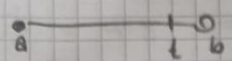
by intermediate value th.

e is a real number

$$\ln e = \int_1^e \frac{dt}{t}$$

Type II

f is cont on $[a, b[$



$$\lim_{t \to b^-} \int_a^t g(x)dx$$

Over $[a,t]$ f is cont.

if f is cont on $]a, b[$

$$\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$$

Average Value of a Continuous Function

f is integrable on $[a,b]$, then its average on $[a,b]$ is:

$$av(f) = \frac{1}{(b-a)} \cdot \int_a^b f(x)\,dx$$

Example 7: $h(x) = -|x|$ on $[-1,1]$

$$av(h(x)) = \frac{1}{2}\int_{-1}^{1} -|x|\,dx = -\frac{1}{2}\cdot\int_{-1}^{1} |x|\,dx =$$

$$-\frac{1}{2}\left[\int_{-1}^{0} |x|\,dx + \int_{0}^{1} |x|\,dx\right] = -\frac{1}{2}\left[\int_{-1}^{0} -x\,dx + \int_{0}^{1} x\,dx\right]$$

$$-\frac{1}{2}\left[\frac{-x^2}{2}\Big|_{1}^{0} + \frac{+x^2}{2}\Big|_{0}^{1}\right]$$

$h(x) = -|x|$ on $[1,3]$

$$av(h(x)) = \frac{1}{2}\int_{1}^{3} -|x|\,dx = -\frac{1}{2}\int_{1}^{3} x\,dx$$

$g(x) = |x| - 1$ on $[-1,1]$

$$av(g(x)) = \frac{1}{2}\int_{-1}^{1} |x|-1\,dx = \frac{1}{2}\left[\int_{-1}^{0} |x|-1\,dx + \int_{0}^{1} |x|-1\,dx\right]$$

Theorem 2: Sandwich Theorem

$a_n \leq b_n \leq c_n$ for $\forall n > N$ and for some $N \in \mathbb{N}$. If

$\lim_{n\to\infty} a_n = \lim_{n\to\infty} c_n = L$ then $\lim_{n\to\infty} b_n = L$

ex. 5:

$a_n = \frac{\sin n}{n}$ since $-1 \leq \sin x \leq 1$; $x \in \mathbb{R}$

$-1 \leq \sin n \leq 1$ $n \in \mathbb{Z}^+$

$-\frac{1}{n} \leq \frac{\sin n}{n} \leq \frac{1}{n}$

$\lim_{n\to\infty} -\frac{1}{n} \leq \lim_{n\to\infty} \frac{\sin n}{n} \leq \lim_{n\to\infty}$

* Thus $\frac{\sin n}{n}$ converges to 0.

b) $b_n = \left(\frac{1}{2}\right)^n$

Theorem 3: $\lim_{n\to\infty} |a_n| = 0$ if and only if $\lim_{n\to\infty} a_n = 0$

$\Updownarrow$

$\forall \varepsilon > 0 : \exists n_0 \in \mathbb{Z}^+ \ni \forall n \geq n_0 :$

$|a_{n_0} - 0| < \varepsilon$

Let $\varepsilon > 0$ and by hypothesis

$\exists N_0 \in \mathbb{Z}^+ \ni \forall n \geq N_0 : ||a_{n_0}| - 0| < \varepsilon$

$\lim_{n\to\infty} a_n = 0 \Rightarrow \lim_{n\to\infty} |a_n| = 0$

$\Updownarrow$

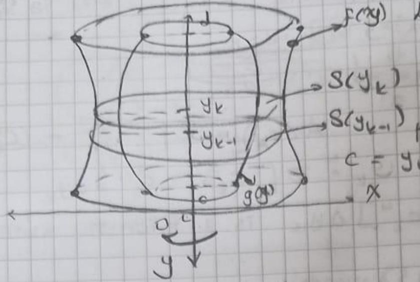
$\forall \varepsilon > 0 : \exists n_0 \in \mathbb{Z}^+ \ni \forall n \geq n_0 :$

$||a_n| - 0| < \varepsilon$

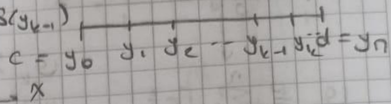
$|a_n| < \varepsilon$

Let $\varepsilon > 0$

$\exists N_0 \in \mathbb{Z}^+ \ni \forall n \geq N_0 : |a_n - 0| < \varepsilon$



Let a partition P of $[c,d]$



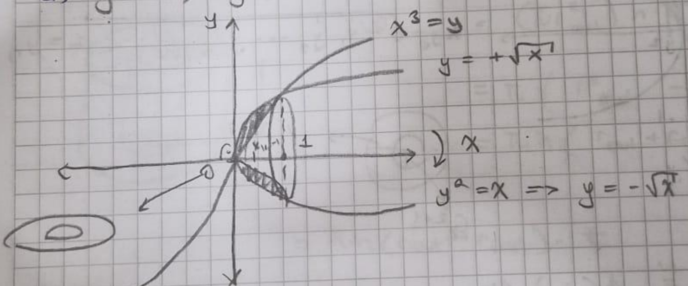
$$\vartheta_k \cong \pi\left[(f(y_{k-1}))^2 - (g(y_{k-1}))^2\right]\Delta y_k$$

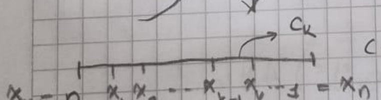
$$\vartheta \cong \sum_{k=1}^{n} \vartheta_k$$

$$\vartheta = \int_c^d \left[f^2(y) - g^2(y)\right]dy$$

Example 5:

a) $y^2 = x$, $y = x^3$, revolved around x-axis.





(through x-axis)

$$A(c_k) = \pi\left[(\sqrt{c_k})^2 - (c_k^3)^2\right] = \pi\left[c_k - c_k^6\right]$$

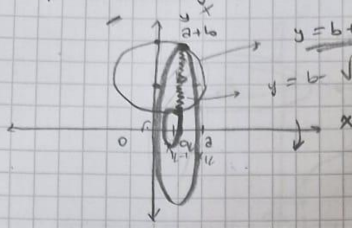
$$\vartheta_k \cong A(c_k)\,\Delta x_k = \pi\left[c_k - c_k^6\right]\Delta x_k$$

$\vartheta \cong \sum_{k=1}^{n} \pi\left[c_k - c_k^6\right]\Delta x_k$ is a Riemann sum

for $\pi(x - x^6)$ is cont. thus integrable.

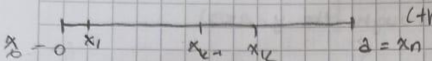
$$\vartheta = \pi\int_0^1 x - x^6\,dx$$

Example 6: $x^2 + (y-b)^2 = a^2$ around x-axis $(a<b)$

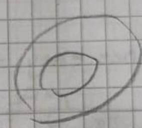


$y = b + \sqrt{a^2 - x^2}$ $\qquad$ $y - b = \sqrt{a^2 - x^2}$

$y = b - \sqrt{a^2 - x^2}$



(through x-axis)

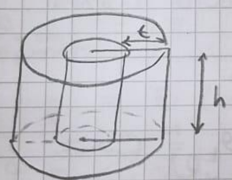


$$V_{out} = \pi \int_{-a}^{a} \left(b + \sqrt{a^2 - x^2}\right)^2 dx$$

$$V_{inn} = \pi \int_{-a}^{a} \left(b - \sqrt{a^2 - x^2}\right)^2 dx$$

$$V = V_{out} - V_{inn}$$

Volumes using Cylindrical Shells



$$V_{c.shell} = (\pi r_{out}^2) h - (\pi r_{inn}^2 h)$$

$$= \pi \left(r_{out}^2 - r_{inn}^2\right) h$$

$$= \pi h (r_{out} + r_{inn}) \underbrace{(r_{out} - r_{inn})}$$

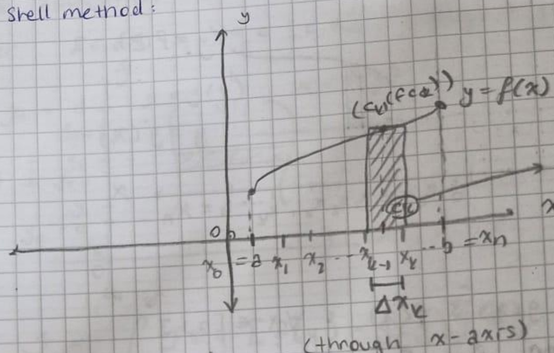
$t :=$ thickness of the shell

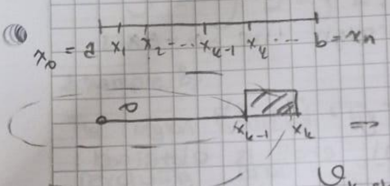
$$= \pi h (r_{out} + r_{inn}) \cdot t$$

$$= 2 \cdot \pi h \underbrace{\left(\frac{r_{out} + r_{inn}}{2}\right)}_{\bar{r} := \text{Avr radius}} t$$

$$= 2\pi h \cdot \bar{r} \cdot t$$

The shell method:





$\Rightarrow$ A cylinderical shell

$$V_{k\,shell} \cong C.h.t$$

$$\cong 2\pi \bar{r} h t \; ; \; \bar{r} = \frac{r_{out} + r_{in}}{2}$$

$$V_{k\,shell} \cong 2\pi \frac{x_k + x_{k-1}}{2} \cdot f(c_k) \cdot \Delta x_k$$

$$\cong \pi (x_k + x_{k-1}) \cdot f(c_k) \cdot \Delta x_k$$

$V \cong \sum_{k=1}^{n} V_{k\,shell}$ is a Riemann sum for

$h(x) = \pi 2x f(x)$ is conti. thus

$$V = \int_a^b 2\pi x f(x)\, dx$$

b) $\int_1^{\infty} \frac{dx}{\sqrt{x^2-0{,}1}}$ is an improper integral of type I.

Since $x \geq 1 \Rightarrow x^2 - 0{,}1 \leq x^2$

$\Rightarrow \sqrt{x^2-0{,}1} \leq |x| = x$

$$\frac{1}{\sqrt{x^2-0{,}1}} \geq \frac{1}{x}$$

$\forall x \geq 1$: $f(x) = \frac{1}{\sqrt{x^2-0{,}1}} \geq \frac{1}{x} = g(x)$

$\int_1^{\infty} \frac{dx}{x}$ is an improper integral of type I and since $p=1$ it is divergent.

$\Downarrow$

$\int_1^{\infty} \frac{dx}{\sqrt{x^2-0{,}1}}$ diverges

c $\Rightarrow$ Homework : hint compare to e^{-x}

Substitutions

Theorem 7: g' is cont. on $[a,b]$ and f is cont. on the range of $g(x) = u$, then $g'(x)\,dx = du$

$$\int_a^b \underbrace{f(g(x))}_{f(u)} \cdot \underbrace{g'(x)\,dx}_{du} = \int_{g(a)}^{g(b)} f(u)\,.\,du$$

in addition, if $x = a \Rightarrow u = g(a)$

$x = b \Rightarrow u = g(b)$

Example 7:

a) $\int_{-1}^{1} 3x^2 \sqrt{x^3+1}\,dx$ if we let $u = x^3+1$ then

$du = 3x^2\,dx$

in addition; if $x = 1 \Rightarrow u = 2$

$x = -1 \Rightarrow u = 0$

$$\int_0^2 \sqrt{u}\,du = \frac{u^{\frac{3}{2}}}{\frac{3}{2}}\Bigg|_0^2 = \frac{2}{3}\left[\sqrt{8} - 0\right] = \frac{4\sqrt{2}}{3}$$

Bounded Monotonic Sequences

If $\exists M \in \mathbb{R} \ni \forall n \in \mathbb{Z}^+ : a_n \leq M$ then bounded from above

If $\exists m \in \mathbb{R} \ni \forall n \in \mathbb{Z}^+ : m \leq a_n$

a) $(n)_{n \in \mathbb{Z}^+}$ $\quad \forall n \in \mathbb{Z}^+ : 0 \leq n$ thus $(a_n)_{n \in \mathbb{Z}^+}$ is bounded from below but not from above

$\left(\frac{n}{n+1}\right)_{n \in \mathbb{Z}^+}$ $\quad \forall n \in \mathbb{Z}^+ :$ $0 < \frac{n}{n+1}$ then bounded from below

$\forall n \in \mathbb{Z}^+$

$n < n+1$

$a_n = \frac{n}{n+1} < 1 \longrightarrow$ upper bound for a_n

Thus a_n is a bounded sequence

$\left(\frac{1}{2^n}\right)$ bounded from below

0 to $\frac{1}{2} \longrightarrow$ bounded

$((-1)^n)_{n \in \mathbb{Z}^+}$ $\quad 1$ or -1 anything smaller than -1 is lower bound

anything bigger than -1 is upper bound

~~g(x)~~ g) $\int_0^2 \lfloor x^2 \rfloor dx = I$

$0 \leq x < 1 \Rightarrow 0 \leq x^2 < 1 \Rightarrow \lfloor x^2 \rfloor = 0$

$1 \leq x < \sqrt{2} \Rightarrow 1 \leq x^2 < 2 \Rightarrow \lfloor x^2 \rfloor = 1$

$\sqrt{2} \leq x \leq \sqrt{3} \Rightarrow 2 \leq x^2 < 3 \Rightarrow \lfloor x^2 \rfloor = 2$

$\sqrt{3} < x \leq 2 \Rightarrow 3 \leq x^2 < 4 \Rightarrow \lfloor x^2 \rfloor = 3$

$$I = \int_0^1 \lfloor x^2 \rfloor dx + \int_1^{\sqrt{2}} \lfloor x^2 \rfloor dx + \int_{\sqrt{2}}^{\sqrt{3}} \lfloor x^2 \rfloor dx + \int_{\sqrt{3}}^{2} \lfloor x^2 \rfloor dx$$

$$= \int_0^1 0\,dx + \int_1^{\sqrt{2}} 1\,dx + \int_{\sqrt{2}}^{\sqrt{3}} 2\,dx + \int_{\sqrt{3}}^{2} 3\,dx$$

$$= 0 + x\Big|_1^{\sqrt{2}} + 2x\Big|_{\sqrt{2}}^{\sqrt{3}} + 3x\Big|_{\sqrt{3}}^{2}$$

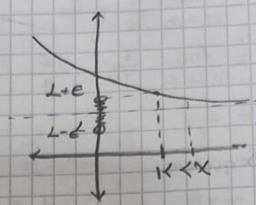
$$\sqrt{2} - 1 + 2\sqrt{3} - 2\sqrt{2} + 6 - 3\sqrt{3}$$

Convergence and Divergence

The sequence $(a_n)_{n \in \mathbb{Z}^+}$, $L \in \mathbb{R}$

$\lim_{n \to \infty} a_n = L$

$\lim_{n \to \infty} \frac{1}{2^n} = 0$



$f : \mathbb{R} \to \mathbb{R}$

$\quad x \to f(x)$

$\lim_{x \to \infty} f(x) = L$

$\forall \epsilon > 0 : \exists K \in \mathbb{R} \ni$

$\forall x > K : \ |f(x) - L| < \epsilon$

$\lim_{n \to \infty} a_n = L$

$\Updownarrow$

$\forall \epsilon > 0 : \exists n_0 \in \mathbb{R} \ni \forall n \geq n_0 :$

$|f(n) - L| < \epsilon$

$\quad \|$

$|a_n - L| < \epsilon$

Choosing from $\mathbb{Z}^+$ cuz domain

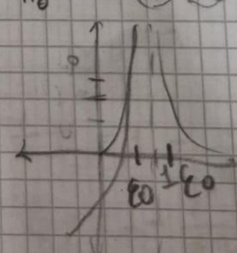
$\lim_{n \to \infty} a_n = L \iff \forall \epsilon > 0 : \exists n_0 \in \mathbb{Z}^+ \ni \forall n \geq n_0 :$

$|a_n - L| < \epsilon$

$(a_n)_{n \in \mathbb{Z}^+} \longrightarrow L$

$(a_n)_{n \in \mathbb{Z}^+} \not\longrightarrow L \iff \exists \epsilon_0 > 0$ s.t. $\forall N \in \mathbb{Z}^+ :$

$\exists n_0 > N$ and $|a_{n_0} - L| \geq \epsilon_0$



$\frac{1}{x-1}$

Ex. 7: Commonly occuring limits.

$$\lim_{n \to \infty} x^{1/n} = 1 \quad \text{whenever} \quad x > 0$$

Let $y = x^{1/n}$

$\ln y = \frac{1}{n} \ln x$

$\lim_{n \to \infty} \ln y = \lim_{n \to \infty} \frac{1}{n} \ln x = \ln x \lim_{n \to \infty} \frac{1}{n} = 0$

$\ln \lim_{n \to \infty} y = 0$

$\lim_{n \to \infty} y = e^0 = 1$

b) $\lim_{n \to \infty} x^n = 0 \quad$ whenever $\quad |x| < 1$

$f(t) = x^t$

$\lim_{t \to \infty} x^t = 0$

c) $\lim_{n \to \infty} \left(1 + \frac{x}{n}\right)^n = e^x \quad$ for any x

$f(t) = \left(1 + \frac{x}{t}\right)^t$ ----→ use log.

① Show that

$\lim_{n\to\infty} \frac{1}{n} = 0$

$\forall \varepsilon > 0 : \exists n_0 \in \mathbb{Z}^+ \ni \forall n \geq n_0 : |a_n - 0| < \varepsilon$

Let $\varepsilon > 0$ be chosen arbitrarily

$\exists n_0 \in \mathbb{Z}^+ \ni \frac{1}{n_0} < \varepsilon$

$\forall n \geq n_0 \implies \frac{1}{n} \leq \frac{1}{n_0} < \varepsilon$

$$\Downarrow$$

$$|a_n - 0| = |a_n| = \left|\frac{1}{n}\right| < \varepsilon$$

② $\lim_{n\to\infty} k = k$ (constant k)

$a_n = k$: $a_1 = k$, $a_2 = k$, ..., $a_{100} = k$, ...

③ $a_n = (-1)^n$ and $\lim_{n\to\infty} a_n \neq 1$. Can it be -1?

// Homework

$\exists \varepsilon_0 > 0 \ni \forall N \in \mathbb{Z}^+ : \exists n_0 \in \mathbb{Z}^+ \ni n_0 \geq N$ and

$|a_{n_0} - 1| > \varepsilon_0$

Let $N \in \mathbb{Z}^+$ be chosen arbitrarily. Thus:

$n_0 = 2N + 1 > N$ then $n_0 > N$ and $|a_{n_0} - 1| = |-1-1|$

$= 2 > \varepsilon_0 = \frac{1}{2}$

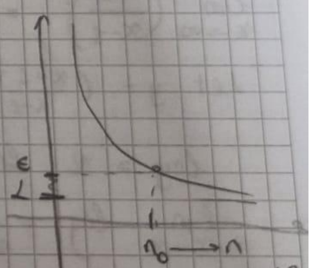
paz
pazt
sa

6.4 surface
~~7.1~~ log
~~10.1~~
Ser { 10.2
10.3

5.4
5.5
5.6
6.1
6.2
6.3
6.4

$\exists \varepsilon_0 > 0 : \forall N \in \mathbb{Z}^+ \ni$

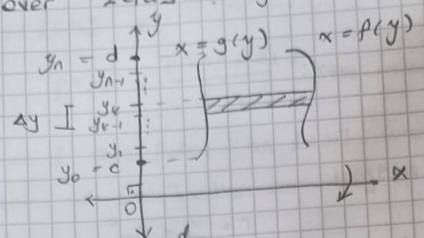
$\exists n_0 > N$ & $|a_n - L| \geq \varepsilon_0$



$|a_n - L| < \varepsilon$

Same application can be done for:

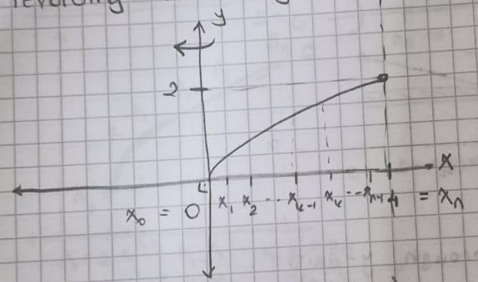
over $[c,d]$: $\forall y \in [c,d]$: $f(y) \geq g(y)$

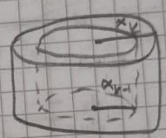


$$\vartheta = \int_c^d 2\pi y \left[f(y) - g(y) \right] dy$$

Example 8:

The region bounded by $y = \sqrt{x}$, x-axis, $x = 4$ by revolving around y-axis:



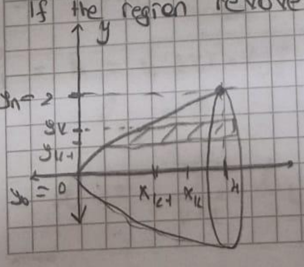


(through x-axis)

0 — 4

$$\vartheta = \int_0^4 2\pi x \sqrt{x}\, dx = \frac{128\pi}{5}$$

If the region revoveled around x-axis:



$x_0 = 0$ — x_{k-1} — x_k^* x_k — $4 = x_n$

$$\vartheta = \int_0^4 \pi (\sqrt{x})^2 dx \Big\} \ 1^{st} \text{ method}$$

(through y-axis)

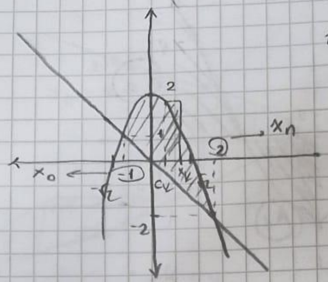
$y_0 = 0$ — $2 = y_n$

$$\vartheta = \int_0^2 2\pi y \left[4 - y^2 \right] dy$$

Others are homework

Example 7

a) Find the area enclosed by $y = 2 - x^2$ and $y = -x$

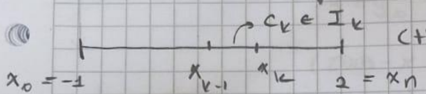


1^{st}: specify the interval of function:

$2 - x^2 = -x$

$x^2 - x - 2 = 0$

$x = 2 \quad x = -1$



(through the x-axis)

$(\forall k)\ \Delta x_k$ is equal to each other.

$h_{k^{th}\,R} = 2 - (c_k)^2 - (-c_k)$

$= 2 - c_k^2 + c_k$

$A \cong \sum_{k=1}^{n} (2 - c_k^2 + c_k)\Delta x_k$ is a Riemann sum for $h(x) = 2 - x^2 - x$ is cont. on the interval meaning is integrable.

$A = \int_{-1}^{2} (2 - x^2 + x)\,dx$

Exp. 8: $I = \int_0^1 \frac{dx}{1-x}$

$f(x) = \frac{1}{1-x}$ is discont. at $x=1$ thus I is an imp. integral of type II

$0 \quad t \quad 1$ $\qquad \lim_{t\to1^-} \int_0^t \frac{dx}{1-x}$

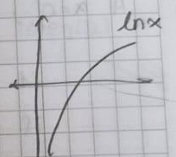
is a Riemann integral

Let $1-x=u$ then $-dx = du$

$x=0 \Rightarrow u=1$

$x=t \Rightarrow u=1-t$

$$\lim_{t\to1^-} \int_1^{1-t} \frac{-du}{u} = \lim_{t\to1^-} -\ln|u| \Big|_1^{1-t}$$



$$\lim_{t\to1^-} -\ln|1-t| + \ln 1 = \lim_{t\to1^-} -\ln|1-t|$$

Since $t<1$ then $t-1<0$ thus $1-t>0$

$$\lim_{t\to1^-} -\ln|1-t| = +\infty$$

Exp. 9: $I = \int_0^3 \frac{dx}{(x-1)^{\frac{2}{3}}}$ is discontinuous at $x=1 \in [0,3]$

Thus, I is an improper integral of type II.

$$\int_0^3 \frac{dx}{(x-1)^{\frac{2}{3}}} = \underbrace{\int_0^1 \frac{dx}{(x-1)^{\frac{2}{3}}}}_{I_1} + \underbrace{\int_1^3 \frac{dx}{(x-1)^{\frac{2}{3}}}}_{I_2}$$

For the convergent behaviour of $\rightarrow I_1 =$ $0 \quad t \quad 1$ is cont. over the interval

$$\lim_{t\to1^-} \int_0^t \frac{dx}{(x-1)^{\frac{2}{3}}} = \lim_{t\to1^-} \frac{(x-1)^{-\frac{2}{3}+1}}{\left(-\frac{2}{3}\right)+1} \Bigg|_0^t$$

$$(x-1)^{-\frac{2}{3}} dx = \lim_{t\to1^-} 3\sqrt[3]{x-1} \Big|_0^t = \lim_{t\to1^-} 3\sqrt[3]{t-1} - (-1) = 1$$

Arc Length

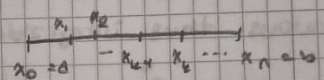
f is cont on $[a,b]$ and diff. on $]a,b[$, there exist $c \in [a,b]$ $\ni$

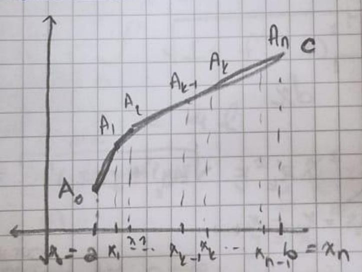
$$f'(c) = \frac{f(b)-f(a)}{b-a}$$; the mean value theorem

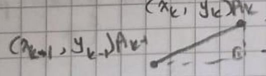
Let f be a func. on $[a,b]$ whose derivative f' is cont. on $[a,b]$, then contuinity of f' rules out the possibility of

↓ continue onwards

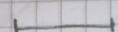
Length of a curve $y = f(x)$







$f(x_k)$ I $f(x_{k-1})$ $\Leftrightarrow$ $y_k - y_{k-1} = \Delta y_k$



x_{k-1} x_k $\Leftrightarrow$ $x_k - x_{k-1} = \Delta x_k$

By Pythagorous (?) theorem how to write this lmao!

$$l_k = h = \sqrt{(\Delta x)_k^2 + (\Delta y)_k^2}$$

$l_{c_k} = l_k$ thus;

$$l = \sum_{k=1}^{n} l_k = \sum_{k=1}^{n} \left[(\Delta x_k)^2 + (\Delta y_k)^2\right]^{\frac{1}{2}}$$

$|c_n| = \left| (-1)^n \frac{1}{n} \right| = \frac{1}{n}$

$\lim_{n \to \infty} |c_n| = 0 \Rightarrow \lim_{n \to \infty} a_n = 0$

<u>Theorem 4</u> Let $(a_n)_{n \in \mathbb{N}}$ be a sequence of real numbers, if $a_n \to L$ and if f is a function that is cont. at L and defined at all a_n then $f(a_n) \to f(a)$

a) $\sqrt{\frac{n+1}{n}} \longrightarrow 1$

$f : [0, +\infty[\longrightarrow \mathbb{R}$ is a cont. func.

$x \longrightarrow f(x) = \sqrt{x}$

f is cont. at $x = 1$

$\lim_{n \to \infty} \frac{n+1}{n} = 1 \Rightarrow \lim_{n \to \infty} f\left(\frac{n+1}{n}\right) = f(1)$ // Second one is homework

$\Updownarrow$

$\lim_{n \to \infty} \sqrt{\frac{n+1}{n}} = 1$

$2^{\frac{1}{n}} \longrightarrow 1$

$f : \underbrace{\mathbb{R}^*}_{\text{reels except } 0} \longrightarrow \mathbb{R} \; ; \; \mathbb{R}^* = \mathbb{R} - \{0\}$

$x \longrightarrow f(x) = 2^x$ is cont at $x = 1$

Let $b_n = \frac{1}{n} \Rightarrow \lim_{n \to \infty} b_n = 0$

$\Downarrow$

$\lim_{n \to \infty} f(b_n) = f(0)$

$\lim_{n \to \infty} 2^{\frac{1}{n}} = 1$

GIPTA

13.) $\int_0^3 \frac{dx}{x-1}$ ⟹ HOMEWORK

Convergence Tests for Improper Integrals of Type I

Theorem 1 : Direct Comparison Test

Let f and g be continuous on $[a, \infty[$ with $0 \leq f(x) \leq g(x)$

① $\int_a^\infty f(x)\,dx$ converges if $\int_a^\infty g(x)\,dx$ converges.

② $\int_a^\infty f(x)\,dx$ diverges if $\int_a^\infty g(x)\,dx$ diverges.

Exp. 13

a) $\int_1^\infty \frac{\cos^2 x}{x^2}\,dx$ is an improper integral of Type I.

Thus $I = \lim_{t \to \infty} \int_1^t \frac{\cos^2 x}{x^2}\,dx$

Let $f(x) = \frac{\cos^2 x}{x^2}$, $\forall x > 0$: $0 \leq \cos^2 x \leq 1$

$\Rightarrow 0 < \frac{\cos^2 x}{x^2} \leq \frac{1}{x^2}$

Thus $\forall x > 1$ if we let $g(x) = \frac{1}{x^2}$

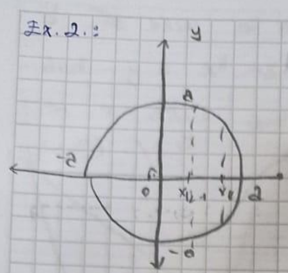
$0 \leq f(x) \leq g(x)$ is satisfied

$\int_1^\infty \frac{dx}{x^2}$ is convergent since $p = 2 > 1$

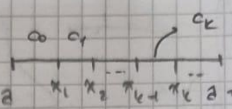
⇓ By comparison test

$\int_1^\infty \frac{\cos^2 x}{x^2}\,dx$ is convergent

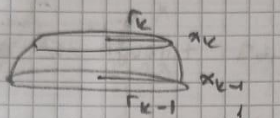
Ex. 2.:



$x^2+y^2=a^2$ is rotated about the x-axis to generate a sphere.



(through x-axis)



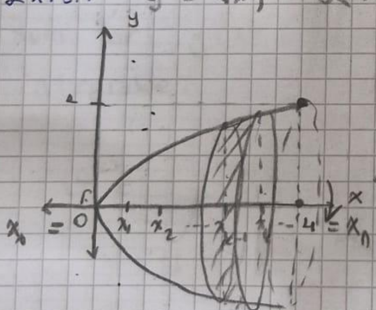
$$V_k \cong \pi\left(-c_k^2+a^2\right)^{\frac{1}{2}\,2}\Delta x_k$$

$$V \cong \sum_{k=1}^{n} \pi\left(a^2-c_k^2\right)^2 \Delta x_k$$

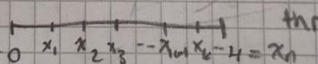
is a Riemann sum for $g(x)=\pi(f(x))^2$ is continuous thus integrable. Hence;

$$\Rightarrow V = \pi\int_{-a}^{a}[f(x)]^2\,dx = \pi\int_{-a}^{a}(a^2-x^2)\,dx$$

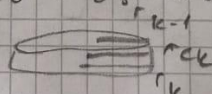
Ex. 3.: $y=\sqrt{x}$, $0\le x\le 4$, x-axis is revolved around x-axis to generate a solid.



Let be a partition of $[0,4]$



through x-axis



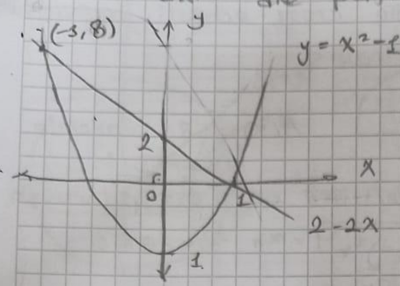
$$r_k = f(c_k)$$

$$V_k = f(c_k)^2\,\pi$$

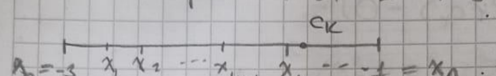
$$V \cong \sum_{k=1}^{n}\left(\sqrt{c_k}\right)^2 \cdot \pi \cdot \Delta x$$ is a Riemann sum for $g(x)=\pi x$ is continuous thus integrable.

$$V = \pi\int_{0}^{4} x\,dx$$

d The base of a solid is bounded by the curves $y = 2-2x$ and $y = x^2 - 1$ and the cross-sections perpendicular ⊥ to x-axis are perfect squares.



Take a partition of $[-3,1]$ through x-axis.



$$V_k \cong \left[\left(2 - 2c_k - c_k^2 + 1\right)\right]^2 \Delta x_k$$

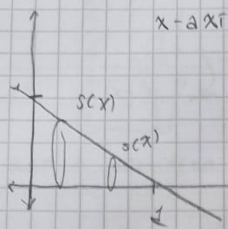
$$V \cong \sum_{k=1}^{n} V_k$$

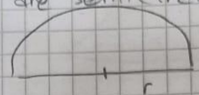
$$= \sum_{k=1}^{n} \left(2 - 2c_k - c_k^2 + 1\right)^2 \Delta x_k \text{ is a Riemann}$$

sum for $g(x) = (3 - 2x - x^2)^2$ is continuous thus integrable:

$$V = \int_{-3}^{1} (3 - 2x - x^2)^2 \, dx$$

1 c) The base of a solid is bounded by x and y axes, $x+y=1$, find the volume if the cross-sections perpendicular x-axis are semicircles





$A(x) :=$ a semicircle

$:= \frac{\pi r^2}{2}$

Let a partition of $[0,1]$ through x-axis

$x_0 = 0 \cdots x_{k-1} \; x_k \; 1 = x_n$

c_k

$r_k = (f(c_k) - 0)\frac{1}{2}$

$v_k \simeq \frac{\pi}{8}(1 - c_k)^2 \cdot \Delta x_k$

$v \cong \sum_{k=1}^{n} v_k$

$\cong \sum_{k=1}^{n} \frac{\pi}{8}(1 - c_k)^2 \cdot \Delta x_k$ is a Riemann sum for $g(x) = \frac{\pi}{8}(1-x)^2$ is cont.

$v := \lim_{\|P\| \to \infty} \sum_{k=1}^{n} \frac{\pi}{8}(1-c_k)^2 \Delta x_k$ then integ.

$\Updownarrow$

$n \to \infty$

$= \lim_{n\to\infty} \sum_{k=1}^{n} \frac{\pi}{8}(1-x)^2 \Delta x$

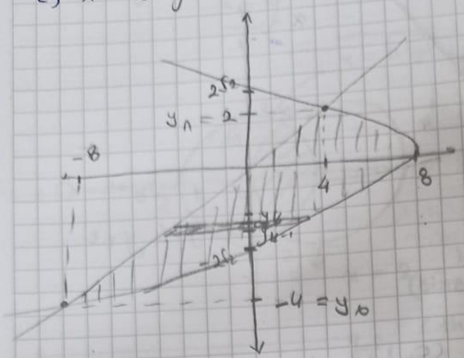
$:= \int_0^1 \frac{\pi}{8}(1-x)^2 \, dx$

// c and d are homework

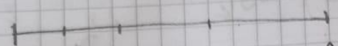
e) $x = 8 - y^2$, $x = 2y$

$y^2 + 2y - 8$

$y + 4 \quad y - 2$



(through y-axis)



$$\sum_{k=1}^{n} \left(8 - c_k^2 - 2c_k\right) \Delta y_k \cong A$$

is a Riemann sum for $8 - y^2 - 2y$

$$A = \int_{-4}^{2} 8 - y^2 - 2y \, dy$$

f) $y = x^3 - 6x^2 + 8x$, x-axis

$x(x-4)(x-2)$

	0	2	4	
$-$	$+$		$-$	$+$

$$\int_0^4 |f(x)| \, dx = \int_0^2 f(x) \, dx + \int_2^4 -f(x) \, dx$$

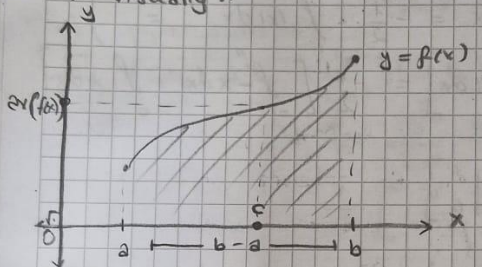
The Mean Value Theorem for Definite Integrals

If f is continuous on $[a,b]$, then at some point c in $[a,b]$

// The theorem claims that the av value of a cont. func. over $[a,b]$ is always taken on at least once by the function.

$$f(c) = \frac{1}{(b-a)} \int_a^b f(x)\,dx$$

// Visually:



Proof: By hypothesis, f is cont. on $[a,b]$, thus is integrable on $[a,b]$.

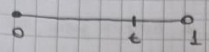
In addition; since f is cont. on $[a,b]$ it attains both max and min values on $[a,b]$ and thus by Th. 2. Prop. 6:

$$\min(f) \cdot (b-a) \leq \int_a^b f(x)\,dx \leq (b-a)\max(f)$$

$b>a$ $\Rightarrow$

$$\min(f) \leq \underbrace{\frac{1}{(b-a)} \int_a^b f(x)\,dx}_{av(f)} \leq \max(f)$$

$I_1 = \int_0^1 \frac{\arcsin x}{1-x} dx$ is an improper integral of type II.



$\int_0^t \frac{\arcsin x}{1-x} dx$ is a Riemann int. thus finite.

If $\lim_{t \to 1} \int_0^t \frac{\arcsin x}{1-x} dx$ exists.

Set the following limit:

$\lim_{x \to 1^-} (1-x)^p \cdot f(x) = \gamma$

$\lim_{x \to 1^-} (1-x)^p \cdot \left(\frac{\arcsin x}{1-x}\right) = \gamma$

$\forall p < 1$, $\gamma < 0$ thus we take

$p = 1$: $\lim_{x \to 1^-} \arcsin x = \arcsin 1 = \frac{\pi}{2}$

$\left.\begin{array}{l} p = 1 \geq 1 \\ \gamma > 0 \end{array}\right\} I_1$ diverges.

$I_2 = \int_1^2 \frac{\arcsin x}{1-x} dx$ is an improper integral of type II.

$\int_t^2 \frac{\arcsin x}{1-x} dx$ is a Riemann int. thus finite

If $\lim_{t \to 1^+} \int_t^2$

// d, e, f are homework + I_2

// Area equals integral for NONNEGATIVE functions.

$$\int_0^1 (x-1)\,dx$$

$x_0 = 0 \quad x_1 \quad x_2 \quad \cdots \quad 1 = x_n$

$$0 < x_1 < x_2 < \cdots < x_{n-1} < x_n$$

$$\Delta x_k = \frac{1}{n}$$

$\forall I_k = [x_{k-1}, x_k]$ c_k is chosen as the right hand end point of I_k

Thus $c_k = x_k = \frac{k}{n}$

An upper sum is;

$$\sum_{k=1}^{n} f(c_k)\,\Delta x_k = \sum_{k=1}^{n} f\left(\frac{k}{n}\right)\cdot\frac{1}{n}$$

$$= \frac{1}{n}\sum_{k=1}^{n} \frac{k}{n} - 1$$

$$= \frac{1}{n}\left[\frac{1}{n}\underbrace{\sum_{k=1}^{n} k}_{\frac{n(n+1)}{2}} - \sum_{k=1}^{n} 1\right]$$

$$= \frac{n(n+1)}{2n^2} - 1$$

Thus;

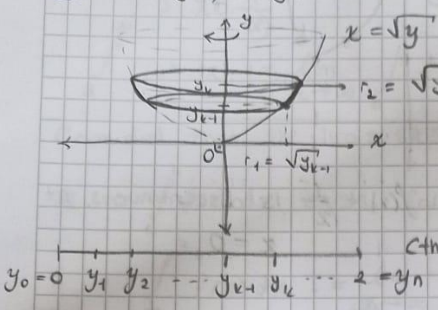
$$\int_0^1 (x-1)\,dx \leq \lim_{n\to\infty} \sum_{k=1}^{n} f(c_k)\,\Delta x_k \quad \left(= \frac{n(n+1)}{2n^2} - 1\right)$$

$$\Rightarrow \int_0^1 (x-1)\,dx \leq -\frac{1}{2}$$

// Example 6 is homewok

// exp 14 a,b homework

(14.c) $x = \sqrt{y}$, $0 \le y \le 2$ about y-axis $\rightarrow$ area of the Surface



$$l_k = \sqrt{(\Delta x_k)^2 + (\Delta y_k)^2} = \sqrt{\left(\frac{\Delta x_k}{\Delta y_k}\right)^2 + 1} \cdot \Delta y_k$$

By mean value theorem:

$$\exists y_k^* \in \,]y_{k-1}, y_k[\, \ni f'(y_k^*) = \frac{\Delta x_k}{\Delta y_k}$$

$$l_k = \sqrt{f'(y_k^*) + 1} \cdot \Delta y_k$$

$$r = \frac{\sqrt{y_k} + \sqrt{y_{k-1}}}{2}$$

$$S \cong \sum_{k=1}^{n} 2\pi \frac{\sqrt{y_k + y_{k-1}}}{2} \cdot \sqrt{1 + (f'(y_k^*))^2}\, \Delta y_k$$

$$S = \int_0^2 2\pi \sqrt{y} \cdot \sqrt{1 + f'(y)^2}\, dy$$

$$= \int_0^2 2\pi \sqrt{y} \sqrt{1 + \left(\frac{1}{2\sqrt{y}}\right)^2}\, dy$$

$$= 2\pi \int_0^2 \sqrt{y} \frac{\sqrt{1+4y}}{2\sqrt{y}}\, dy$$

$$= \pi \int_0^2 \sqrt{1+4y}\, dy \quad ; \quad 1 + 4y = u$$

$4\,dy = du$ $\Rightarrow \frac{\pi}{4} \frac{u^{1+1/2}}{1+1/2}$

$y = 0 \quad u = 1$

$y = 2 \quad u = 9$

$$= \pi \int_1^9 \sqrt{u} \frac{du}{4}$$

Theorem 8: Let f be cont. on the symmetric interval $[-a, a]$

a) f is even, then $\int_{-a}^{a} f(x)\,dx = 2\int_{0}^{a} f(x)\,dx$

b) f is odd, then $\int_{-a}^{a} f(x)\,dx = 0$

Proof: Assume f is even then:

$\forall x \in [-a, a] : f(-x) = f(x)$

$$\int_{-a}^{a} f(x)\,dx = \int_{-a}^{0} f(x)\,dx + \int_{0}^{a} f(x)\,dx$$

Let $a = -t$ $\quad x = -a \Rightarrow t = a$

$dx = -dt \Rightarrow \quad x = 0 \Rightarrow t = 0$

$$\int_{-a}^{0} f(x)\,dx = \int_{a}^{0} f(-t)\,.\,d(-t) = \int_{0}^{a} f(-t)\,.\,d(t) \overset{f \text{ is even}}{=}$$

$$\int_{0}^{a} f(x)\,dx = \int_{0}^{a} f(t)\,dt$$

Proof: Assume f is odd thus $\forall x \in [-a, a] : f(-x) = -f(x)$

$$\int_{-a}^{a} f(x)\,dx = \underbrace{\int_{-a}^{0} f(x)\,dx}_{I} + \int_{0}^{a} f(x)\,dx$$

$I = \int_{-a}^{0} f(x)\,dx$; let $x = -t \Rightarrow dx = d(-t)$

$x = -a \Rightarrow t = a$

$x = 0 \Rightarrow t = 0$

$$= \int_{a}^{0} f(-t)\,.\,d(-t)$$

$$\int_{0}^{a} f(-t)\,.\,d(t) \overset{f \text{ is odd}}{=} -\int_{0}^{a} f(t)\,dt$$

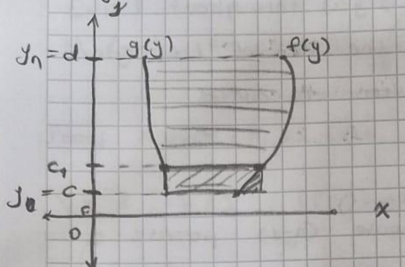
Integration with respect to y

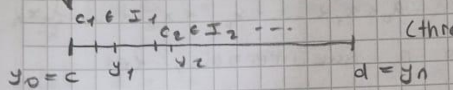
If a region's bounding curves are described by functions of y;

$x = f(y)$

$x = g(y)$ $\quad > \forall y \in [c,d]$:

$f(y) \geq g(y)$





(through y-axis)

Choose $(n-1)$ points such as;

$$y < y_1 < y_2 < \dots < y_{k-1} < y_k < \dots < y_{n-1} < y_n$$

$$\forall k \in 1, \dots, n : \Delta y_k = y_k - y_{k-1} = \frac{d-c}{n}$$

$$h_{1st\ R} = f(c_1) - g(c_1)$$

$$\vdots$$

$$h_{k\,th\ R} = f(c_k) - g(c_k)$$

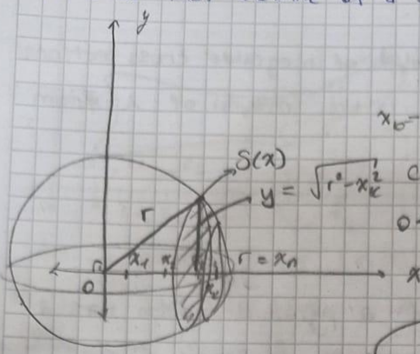
$$A \approx \sum_{k=1}^{n} f(c_k) - g(c_k)\, \Delta y_k$$

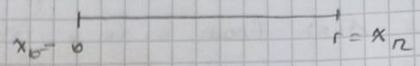
is a Riemann sum for $f-g$ which is cont. on $[c,d]$ thus integrable.

$$\lim_{n \to \infty} \sum_{k=1}^{n} f(c_k) - g(c_k)\, \Delta y_k \text{ exists.}$$

$$\int_c^d (f-g)(y)\, dy = A$$

b) Show that volume of a sphere is $\vartheta = \frac{4}{3}\pi r^3$





Choose $n-1$ points s.t.

$0 < x_1 < x_2 < \dots < x_{k-1} < x_k < \dots < x_n = r$

$h = x_k - x_{k-1} = \Delta x_k$

$\rightarrow S(x_k) \longrightarrow A(x_k) =$

$x^2 + y^2 + z^2 = r^2 \Leftarrow \underbrace{f(x_k)}\,^2 \Delta x_k$

$= \pi\left(\sqrt{r^2 - x_k^2}\right)^2 \longleftarrow$ radius of $S(x_k)$

$= \pi (r^2 - x_k^2)$

$\frac{\vartheta}{2} \cong \sum_{k=1}^{n} A(x_k) \cdot \Delta x_k$ is a Riemann sum

for $A(x) = \pi(r^2 - x^2)$ it is conti. thus integrable

$$\frac{\vartheta}{2} = \int_0^r A(x)\,dx = \int_0^r \pi (r^2 - x^2)\,dx =$$

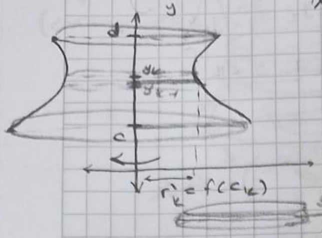
$$\pi\left(r^2 x - \frac{x^3}{3}\right)\Big|_0^r = \pi\left[\left(r^3 - \frac{r^3}{3}\right) - 0\right]$$

$$= \frac{2\pi r^3}{3} \quad \text{thus}$$

$$\vartheta = \frac{4\pi r^3}{3}$$

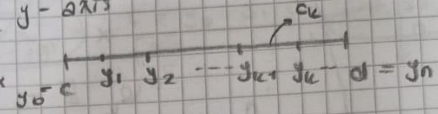
Volume by Disks for Rotations about the y-axis

Ex. 4.:



$x = f(y)$

Let be a partition of $[c, d]$ through y-axis



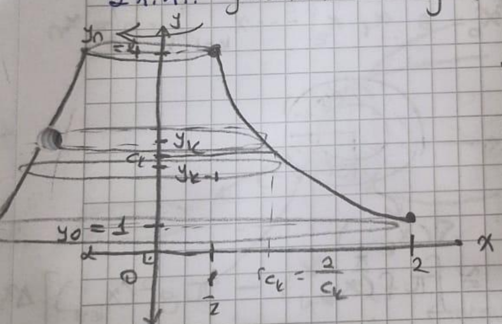
$$\vartheta_k = \pi f(c_k)^2 \Delta y_k$$

$\vartheta = \sum_{k=1}^{n} \pi f(c_k)^2 \cdot \Delta y_k$ is a Riemann sum for

$g(x) = \pi f(y)^2$ is cont. thus integrable.

$$\vartheta = \pi \int_c^d [f(y)]^2 dy$$

Ex. 4.: y-axis, $x = \frac{2}{y}$, $1 \leq y \leq 4$, y revolution axis



Let a partition P through y-axis

$$\vartheta_k \cong \pi r_k^2 \cdot \Delta y_k$$

$$\vartheta_k \cong \pi \left(\frac{2}{c_k}\right)^2 \Delta y_k$$

$$\vartheta \cong \sum_{k=1}^{n} \pi \left(\frac{2}{c_k}\right)^2 \Delta y_k$$

is a Riemann sum for $g(y) = \pi \left(\frac{2}{y}\right)^2$ is cont. on $[1, 4]$ thus integrable

$$\vartheta = \pi \int_1^4 \left(\frac{2}{y}\right)^2 dy$$

1. If $\int_a^t f(x)\,dx$ exists for all $t \geq a$, then we define

$$\int_a^{\infty} f(x)\,dx := \lim_{t \to \infty} \int_a^t f(x)\,dx$$

Provided that the limit exists as a finite number.

2. If $\int_t^b f(x)\,dx$ exists, for all $t \leq b$, then we define

$$\int_{-\infty}^{b} f(x)\,dx := \lim_{t \to \infty} \int_t^b f(x)\,dx$$

provided that the limit exists as a finite number.

3.

Example 1: Area under $y = e^{-x}$ over $[1, +\infty[$

$\forall t > 1$: $\int_1^t e^{-x}\,dx$ exists since $y = e^{-x}$ is continuous over $[1, t]$

$$\int_1^t e^{-x}\,dx = -e^{-x}\Big|_1^t = -e^{-t} + e^{-1} = \frac{-1}{e^t} + \frac{1}{e}$$

In addition ;

$$\lim_{t \to \infty} -\frac{1}{e^t} + \frac{1}{e} = \frac{1}{e}$$

$\Rightarrow \int_1^{\infty} e^{-x}\,dx$ converge to $\frac{1}{e}$

Infinite Sequences and Series

① $\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\dots+\frac{1}{2^n}+\dots$

we may assign a sequence to that infinite sum:

let $a_n = \frac{1}{2^n}$ for all $n \in \mathbb{N}$ then,

$s_1 = a_1 = \frac{1}{2}$

$s_2 = a_1 + a_2 = \frac{1}{2} + \frac{1}{4} = \frac{3}{4}$

$s_3 = a_1 + a_2 + a_3 = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} = \frac{7}{8}$

$s_4 = a_1 + a_2 + a_3 + a_4 = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} = \frac{15}{16}$

$(a_n)_{n \in \mathbb{Z}^+}$ this notation indicates a sequence

$\forall n \in \mathbb{Z}^+, \quad a_n = \frac{1}{2^n}$

Terms of the sequence is:

$a_1 + a_2 + \dots + a_n + a_{n-1} + \dots$

instead can be reprented as:

$\sum_{n=1}^{\infty} a_n$ is called series.

$$\left.\begin{array}{l} S_1 = a_1 \\ S_2 = a_1 + a_2 \dots \end{array}\right\} \text{sequence of partial sums}$$

$S_n = a_1 + a_2 + \dots + a_n = \sum_{k=1}^{n} a_k$

For $a_n = \frac{1}{2^n}$ the n^{th} or general term of the sequence S_n will be

$S_n = \frac{1}{2} + \frac{1}{2^2} + \dots + \frac{1}{2^n}$

$$S_n = \frac{1}{2} + \cancel{\frac{1}{2^2}} + \cancel{\frac{1}{2^3}} + \dots + \cancel{\frac{1}{2^{n-1}}} + \cancel{\frac{1}{2^n}}$$

$$\frac{1}{2} S_n = \cancel{\frac{1}{2^2}} + \frac{1}{2^3} + \cancel{\frac{1}{2^4}} + \dots + \cancel{\frac{1}{2^n}} + \frac{1}{2^{n+1}}$$

$$\Downarrow$$

$$S_n - \frac{1}{2} S_n = \frac{1}{2} - \frac{1}{2^{n-1}}$$

$$S_n = 1 - \frac{1}{2^n} \quad \text{Thus} \lim_{n \to +\infty} 1 - \frac{1}{2^n} = 1$$

$$0,\bar{3} = 0,3333\bar{3} = \frac{3}{10} + \frac{3}{100} + \frac{3}{1000} + \frac{3}{10000} + \dots = \frac{1}{3}$$

$x = 0,33333\bar{3}$

$10x = 3,3333\bar{3}$

$9x = 3, \ x = \frac{1}{3}$

By using the sequences:

$a_1 = \frac{3}{10}, \ a_2 = \frac{3}{100}, \ a_3 = \frac{3}{1000} \dots$

$a_n = \frac{3}{10^n}$

$S_1 = a_1$

$S_2 = a_1 + a_2$

$\vdots$

$$S_n = a_1 + \dots + a_n = \sum_{k=1}^{n} \frac{3}{10^k}$$

$$S_n = \frac{3}{10} + \frac{3}{10^2} + \dots + \frac{3}{10^n}$$

$$\frac{3}{10} \Big[\underbrace{1 + \frac{1}{10} + \frac{1}{10^2} + \dots + \frac{1}{10^{n-1}}}_{T} \Big]$$

$$T - \frac{1}{10} T = 1 - \frac{1}{10^n}$$

$$\frac{1}{10} T = \frac{1}{10} + \frac{1}{10^2} + \dots \frac{1}{10^n}$$

$$\frac{9}{10} T = 1 - \frac{1}{10^n}$$

For the convergent behaviour of I_2:

$$\lim_{t \to 1^+} \int_t^3 (x-1)^{-\frac{2}{3}}\, dx$$

I_2 is a Riemann sum.

$$\lim_{t \to 1^+} 3\sqrt[3]{x-1}\,\Big|_t^3 = \lim_{t \to 1^+} 3\left(\sqrt[3]{2} - \underbrace{\sqrt[3]{t-1}}_{\Downarrow\, 0}\right)$$

$$= 3\sqrt[3]{2} \text{ is convergent}$$

th about Type I.

Since $I = I_1 + I_2$

$$\Uparrow \quad I = 3\left(\sqrt[3]{2} + 1\right)$$

★ Exp. 10:

Show that $\displaystyle\int_0^1 \frac{dx}{x^p}$ converges if $p < 1$, diverges otherwise.

If $x = 0$, $\underbrace{f(x) = \frac{1}{x^p}}$ may be discontinuous.

if $p < 0$ then $f(x) = x^{-p}$ is continous thus integrable.

if $p > 0$ then $f(x) = x^p$ is discontinuous at $x = 0$. This should be considered as type II imp. integral

For the $p > 0$: we should investigate two situations separetaly, that is $p = 1$ and $p \neq 1$.

$\boxed{1)\ p = 1}$

$I = \displaystyle\int_0^1 \frac{dx}{x}$ is an imp. integral of type II. $x = 0$ is singular point

$$\lim_{t \to 0^+} \int_t^1 \frac{dx}{x} = \lim_{t \to 0^+} \ln|x|\,\Big|_t^1$$

is a Riemann sum

$$\lim_{t \to 0^+} \underbrace{\ln 1}_{\Downarrow\, 0} - \ln t = +\infty$$ thus asked integral is divergent.

Thus apply the limit test:

$$\lim_{x\to 1^-} (1-x)^p f(x) = \gamma$$

$$\lim_{x\to 1^-} (1-x)^p \frac{1}{\sqrt[3]{1-x^2}}$$

$$\lim_{x\to 1^-} (1-x)^p \frac{1}{\sqrt[3]{1-x}\sqrt[3]{1+x}}$$

if $p = \frac{1}{3}$ then $\lim_{x\to 1^-} (1-x)^{\frac{1}{3}} \cdot \frac{1}{\sqrt[3]{1-x}\sqrt[3]{1+x}}$ will be simplified as:

$$\lim_{x\to 1^-} \frac{1}{\sqrt[3]{1+x}} = \frac{1}{\sqrt[3]{2}} = \gamma$$ thus the asked improper integral converges.

② $$\int_0^2 \frac{\arcsin x}{1-x}\,dx = \int_0^1 \frac{\arcsin x}{1-x}\,dx + \int_1^2 \frac{\arcsin x}{1-x}\,dx$$

my soli

$$\lim_{x\to 1^-} (1-x)^p \cdot \frac{\arcsin x}{1-x}$$

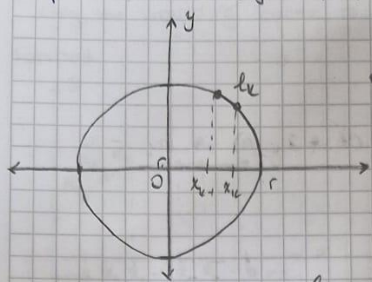
if $p = 1$ then $\lim_{x\to 1^-} \arcsin x = \pi$

$\int f(x) = \frac{\arcsin x}{1-x}\,dx$ has an infinite discont. at $x = 1$ thus is an improper integ. of Type II

$$\int_0^2 \frac{\arcsin x}{1-x}\,dx = \underbrace{\int_0^1 \frac{\arcsin x}{1-x}\,dx}_{I_1} + \underbrace{\int_1^2 \frac{\arcsin x}{1-x}\,dx}_{I_2}$$

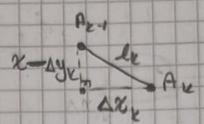
// c is homework

Exp. 11: Find the length of the circle with radius r



(through x-axis)

$0 = x_0$ $x_n = r$



$$l_k = \sqrt{(\Delta x_k)^2 + (-\Delta y_k)^2}$$

$l_c = \sum_{k=1}^{e} l_k$ on the interval $[x_{k-1}, x_k]$ by

M.V.T $\exists x_k^* \in]x_{k-1}, x_k[\ni f'(x_k^*) = \dfrac{f(x_k) - f(x_{k-1})}{x_k - x_{k-1}}$

$$\Delta y_k = f(x_k^*) \cdot \Delta x_k$$

$$l = \int_0^r \sqrt{1 + f'(x)^2}\, dx$$

$$l_{circle} = 4l$$

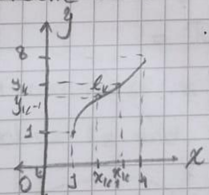
⑥ Arc length of $y^2 = x^3$ semi-cubical between $(1,1)$ and $(4,8)$:

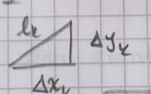
$\Downarrow$

$y = x^{3/2} \iff y' = \frac{3}{2} x^{\frac{1}{2}}$

Assume the curve: (through x-axis)

1 4





$$l_k = \sqrt{(\Delta y_k)^2 + (\Delta x_k)^2}$$

$$= \sqrt{1 + \left(\frac{\Delta y_k}{\Delta x_k}\right)^2} \cdot \Delta x_k$$

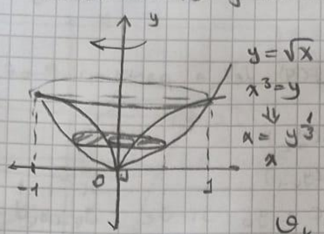
By mean value theorem on $[x_{k-1}, x_k]$ then $\rightarrow \Delta y_k$

$\exists x_k^* \in]x_{k-1}, x_k[\ni f'(x_k^*) = \dfrac{f(x_k) - f(x_{k-1})}{x_k - x_{k-1}}$

$$l_c \approx \sum_{k=1}^{4} \sqrt{1 + f'^2(x_k^*)}\, \Delta x_k$$

$$l_c = \int_1^4 \sqrt{1 + \frac{9}{4} x}\, dx$$

b) Revolved around y-axis:



Choose a $c_k \in [y_{k-1}, y_k]$

$$A(c_k) = \pi[\sqrt[3]{c_k} - (c_k^2)^2]$$

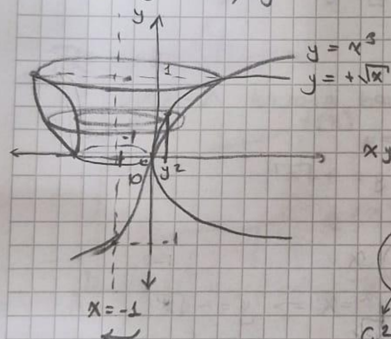
$$\vartheta_k \cong \pi[c_k^{\frac{2}{3}} - c_k^4]\Delta y_k$$

$\vartheta \cong \sum_{k=1}^{n} [c_k^{\frac{2}{3}} - c_k^4]\Delta y_k$ is a Riemann sum for

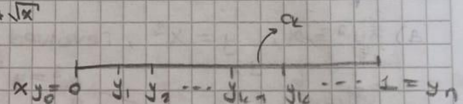
$h(y) = \pi(y^{\frac{2}{3}} - y^4)$ is cont, thus integrable

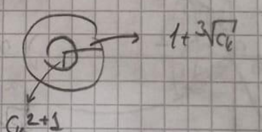
$$\vartheta = \int_0^1 \pi(y^{\frac{2}{3}} - y^4)\,dy$$

c) $y^2 = x$, $y = x^3$, revolved around $x = -1$



Let a partition of [0,1]





$$\vartheta_k \cong A(c_k)\Delta y_k$$

$$= \pi\left((1+\sqrt[3]{c_k})^2 - (1+c_k^2)^2\right)\Delta y_k$$

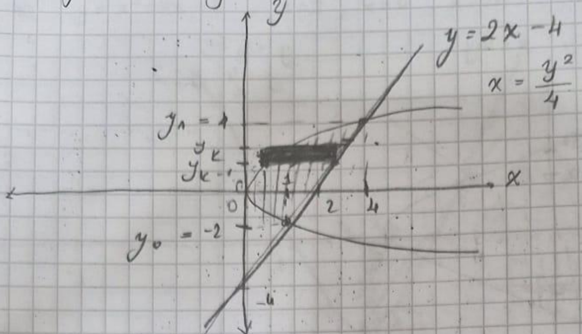
$\vartheta \cong \sum_{k=1}^{n} \vartheta_k$ is a Riemann sum for

$$h(y) = \pi[(1+\sqrt[3]{y})^2 - (1+y^2)^2]$$

$$\vartheta = \pi\int_0^1 [(1+\sqrt[3]{y})^2 - (1+y^2)^2]\,dy$$

// i to n are homeworks.

g) $y^2 = 4x$, $y = 2x - 4$



$y^2 - 2y + 8 = 0$

$y - 4 \quad y + 2$

Let P be a partition of

$[-2, 4]$ (y-axis)

$$A \approx \sum_{k=1}^{n} \left(\frac{c_k + 4}{2} - \frac{(c_k)^2}{4} \right) \Delta y_k$$

is a Riemann sum for $x = \dfrac{y+4}{2} - \dfrac{y^2}{4}$

cont. and integ.

$$A = \int_{-2}^{4} \left(\frac{y+4}{2} - \frac{y^2}{4} \right) dy$$

(h) $y = x^3 + x$ and $y = 3x^2 - x$

$x(x^2 - 3x + 2) = 0$

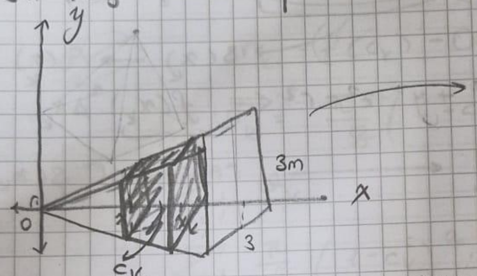
$$A = \int_0^2 |x^3 + x - 3x^2 + x| \, dx$$

$$A = \int_0^1 (x^3 + 2x - 3x^2) \, dx + \int_1^2 (-x^3 - 2x + 3x^2) \, dx$$

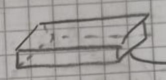
Definition The volume of a solid of integrable cross sectional area $A(x)$ from a to b is the integral of A from a to b.

$$V = \int_a^b A(x)\,dx$$

Example 1: find the volume of a pyramid which has a 3m height and a square base that is 3m on a side.



$S(x)$ is a square.



$\to A(c_k) = c_k^2$

$h = \Delta x_k$

(through x-axis)

$x_0 = 0 \quad x_1 \quad x_2 \; \cdots \; x_{k-1} \quad x_k \; \cdots \; 3 = x_n$

By similarity $\frac{c_k}{3} = \frac{a}{3} \Rightarrow c_k = a$

$$\sum_{k=1}^{n} V_k = \sum_{k=1}^{n} c_k^2 \, \Delta x_k \text{ is a Riemann sum}$$

for $A(x) = x^2$

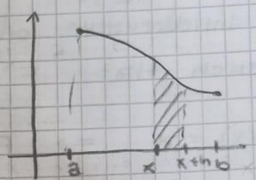
$$V = \int_0^3 x^2\,dx$$

Theorem 4:

f is cont. on $[a,b]$, then $F(x) = \int_a^x f(t)\,dt$ is cont. on $[a,b]$ and differentiable on $]a,b[$ and its derivative; i.e.

$$F'(x) = \frac{d}{dx}\left(\int_a^x f(t)\,dt\right) = f(x)$$

① Show $F'(x) = f(x) \longrightarrow$ diff. $\longrightarrow$ cont.



$$\Downarrow$$

$$F(x+h) - F(x) = \int_x^{x+h} f(t)\,dt$$

$$F'(x) = \lim_{h\to 0} \frac{F(x+h) - F(x)}{h}$$

$$= \lim_{h\to 0} \frac{\int_x^{x+h} f(t)\,dt}{h}$$

$$= \lim_{h\to 0} \frac{1}{h}\left[\int_a^{x+h} f(t)\,dt - \int_a^{x} f(t)\,dt\right]$$

$$= \lim_{h\to 0} \underbrace{\frac{1}{h}\int_x^{x+h} f(t)\,dt}_{av(f) \text{ on } [x,\,x+h]}$$

By mean value theorem $\exists\, c_x \in [x, x+h] \ni$

$f(c_x) = av(f)$

$$\lim_{h\to 0} f(c_x) = f(x)$$

$\int_1^\infty \frac{dx}{x^p}$; p is arbitrary number

if $p = 1$, then $I = \int_1^\infty \frac{dx}{x}$

$f(x) = \frac{1}{x}$ is cont. on $[1, +\infty[$ thus is an integer

thus type I.

$\lim_{t\to\infty} \int_1^t \frac{dx}{x} = \lim_{t\to\infty} \ln|x| \Big[_1^t = \lim_{t\to\infty} \ln|t| - 0$

$= \infty$

Thus $I = \int_1^\infty \frac{dx}{x}$ is divergent.

if $p \neq 1$, then $I = \int_1^\infty \frac{dx}{x^p} = \int_1^\infty x^{-p} dx$

$f(x) = x^{-p}$ is cont on $[1, +\infty[$ is an integ..

thus type I.

$\lim_{t\to\infty} \int_1^t x^{-p} dx \Rightarrow \lim_{t\to\infty} \frac{x^{-p+1}}{-p+1} \Big|_1^t \Rightarrow$

$\lim_{t\to\infty} \frac{t^{-p+1}}{-p+1} - \frac{1^{-p+1}}{-p+1} = \lim_{t\to\infty} \left(\frac{t^{-p+1} - 1}{-p+1} \right)$

$\frac{1}{-p+1} \lim_{t\to\infty} t^{-p+1} - 1$

$-p+1 < 0$

$1 < p$

$\lim_{t\to\infty} \frac{1}{t^{1-p}} - 1$

0

$\Rightarrow \frac{1}{1-p}$

$-p+1 > 0$

$+\infty$

divergent.

$1 > p$

$\int_1^\infty \frac{dx}{x^p}$

$p \leq 1$ divergent

$p > 1$ convergent

$\frac{1}{p-1}$

If $p \neq 1$

$I = \int_a^b \frac{dx}{(b-x)^p}$ is an improper integral of type II

$$\lim_{t \to b^-} \int_a^t \frac{dx}{(b-x)^p} = \lim_{t \to b^-} - \frac{(b-x)^{p+1}}{p+1} \Bigg|_a^t$$

$$\lim_{\substack{t \to b^- \\ t<b \\ b-t>0}} \left[\frac{-(b-t)^{1-p}}{1-p} + \frac{(b-a)^{1-p}}{1-p} \right]$$

$$= \begin{cases} \dfrac{(b-a)^{1-p}}{1-p} & ; \ -p+1 > 0 \iff p < 1 \\ +\infty & ; \ -p+1 < 0 \iff p > 1 \end{cases}$$

In conclusion;

$p < 0$, is a Riemann integral thus convergent.

$p = 1$, it diverges.

$0 < p < 1$, it is convergent.

$p > 1$, it diverges

◇ ◇ ◇

Definition 8.3.: Both Type I and Type II, it is called type III,

Ex. 12: $\int_0^\infty \frac{dx}{x^2} = I = \underbrace{\int_0^1 \frac{dx}{x^2}}_{} + \underbrace{\int_1^\infty \frac{dx}{x^2}}_{}$

Enough for exam if not "calculate" $\Longleftarrow$ $p = 2 > 1$ this convergent

$p = 2 > 1$ this it is convergent

$\Downarrow$

$$\lim_{t \to +\infty} \int_1^t x^{-2} \, dx$$

$$\lim_{t \to +\infty} \left[\frac{x^{-1}}{-1} \right]_1^t = 1$$

$$\frac{1}{t} + 1$$

f is cont on $[x_{k-1}, x_k]$ and diff. on $]x_{k-1}, x_k[$

Thus by Mean Value Theorem:

$$\exists x_k^* \in]x_{k-1}, x_k[\Rightarrow f'(x_k^*) = \frac{f(x_k) - f(x_{k-1})}{x_k - x_{k-1}}$$

$$f'(x_k^*) = \frac{\Delta y_k}{\Delta x_k} \iff f'(x_k^*)\,\Delta x_k = \Delta y_k$$

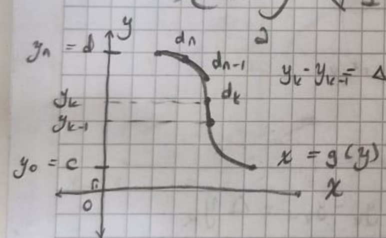
Thus:

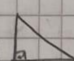
$$\ell \simeq \sum_{k=1}^{n} \sqrt{(\Delta x_k)^2 + (f'(x_k^*)\Delta x_k)^2}$$

$$= \sum_{k=1}^{n} \sqrt{1 + f'(x_k^*)}\ \Delta x_k$$

is a Riemann sum for $h(x) = \sqrt{1 + f'^2(x)}$ is continuous thus integrable;

$$\ell = \int_a^b \sqrt{1 + f'^2(x)}\ dx$$



$y_k - y_{k-1} = \Delta y_k$  $\Delta x_k = x_k - x_{k-1}$

$$\ell_k = \sqrt{(\Delta y_k)^2 + (\Delta x_k)^2} \simeq \ell_{c_k}$$

curve's k^{th} part

$$\ell \simeq \sum_{k=1}^{n} \sqrt{(\Delta y_k)^2 + (\Delta x_k)^2}$$

By mean value theorem; $\exists y_k^* \in]y_{k-1}, y_k[\Rightarrow f'(y_k^*) = \frac{f(y_k) - f(y_{k-1})}{y_k - y_{k-1}}$

Δx_k $\Uparrow$ $\Delta f(y_k)$ ... $\Vert$ Δy_k

$\Delta f(y_k)\ \Delta y_k = f'(y_k^*)$ thus:

$$\ell \simeq \sum_{k=1}^{n} \sqrt{1 + f'^2(y_k^*)}\ \Delta y_k$$

$$\ell = \int_c^d \sqrt{1 + f'(y)^2}\ dy$$

Proof:

Since f is cont; part I says that an antiderivative of f exists and it gives:

$$G(x) = \int_a^x f(t)\,dt \quad ; \text{ diff. cont. and } G'(x) = f(x)$$

In addition, if we consider another antiderivative of f and denote it by F:

$$F'(x) = G'(x)$$

then by M.V.T. there exists a constant C such that:

$$F(x) = G(x) + C \quad ; \quad \forall x \in]a,b[$$

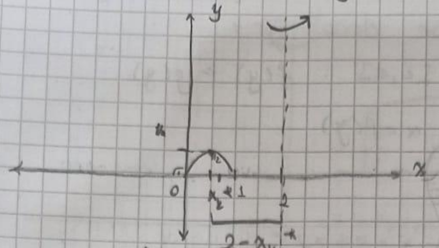
Thus:

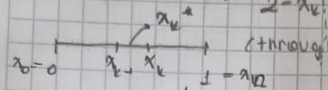
$$F(b) - F(a) = G(b) - G(a)$$

$$F(b) - F(a) = \int_a^b f(t)\,dt - \int_a^a f(t)\,dt$$

// exp. 10 is homework // 9.6 is homework with ~~for~~ other methods

Example 9: ⓐ $y = x - x^2$, $y = 0$, $x = 2$ is the revolution axis





(through x-axis)

$r = 2 - x_k^*$

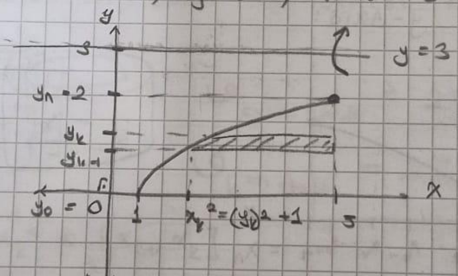
$$V = \int_0^1 \underbrace{2\pi\,[2 - x]}_{c} \cdot \underbrace{f(x)}_{h}\, \underbrace{dx}_{t}$$

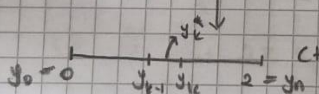
$t = dx$

$h = f(x)$

$$= \frac{\pi}{2}$$

ⓑ $y = \sqrt{x-1}$, $y = 0$, $x = 5$ about $y = 3$





(through y-axis)

$r = 3 - y_k^*$

$h = 5 - (y_k)^2 + 1$

$t = \Delta y_k$

$$V = \int_0^2 2\pi\,[3 - y] \cdot [4 - y^2]\, dy$$

Theorem 4: Limit Test

Let f has an infinite discontinuity at b, bounded on $[a,b[$

and $\lim_{x \to b^-} (b-x)^p f(x) = \gamma$

Since $\int_a^\infty \frac{dx}{x^p} = \begin{cases} \frac{a^{1-p}}{p-1} ; & p>1 \\ \infty ; & p \le 1 \end{cases}$

$\int_a^b \frac{dx}{(b-x)^p}$ converges $p<1$, diverges $p \ge 1$

1. Set the limit
2. Evaluate

1) if $0 \le \gamma < \infty$ and $p<1$, then $\int_a^b f(x)dx$ converges.

2) if $0 < \gamma \le \infty$ and $p \ge 1$, then $\int_a^b f(x)dx$ diverges.

Example 18:

$\int_0^1 \frac{dx}{\sqrt[3]{1-x^2}}$ $\qquad \lim_{x \to 1^-} (1-x)^p f(x) = \gamma$

$f(x) = \frac{1}{\sqrt[3]{1-x^2}}$ has an infinite discon. at $x=1$ thus is an improper integral of type II.

f is continuous over $[0,t]$ $\forall t \in [0,1[$ thus asked integral is a Riemann integral over $[0,t]$

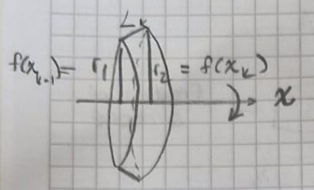
Hence we should evaluate;

$\lim_{t \to 1^-} \int_0^t \frac{dx}{\sqrt{1-x^2}}$ is not easy to evaluate.

Thus, the frustum of cone can be described as:

$$S_{frustum} = 2\pi \bar{r} L \; ; \; \bar{r} := \text{avg. radius}, \; L = \text{slant height}$$

Area of the k^{th} frustum of a cone:



$$L_k = \sqrt{(x_k - x_{k-1})^2 + (y_k - y_{k-1})^2}$$

$$= \sqrt{(\Delta x_k)^2 + (\Delta y_k)^2}$$

$$= \sqrt{1 + \left(\frac{\Delta y_k}{\Delta x_k}\right)^2} \cdot \Delta x_k$$

by Mean Value Theorem:

$\exists x_k^* \in]x_{k-1}, x_k[\;\ni\; f'(x_k^*) = \frac{\Delta y_k}{\Delta x_k}$; thus

$$L_k = \sqrt{1 + f'(x_k^*)^2} \cdot \Delta x_k$$

And $\bar{r} = \frac{f(x_{k-1}) + f(x_k)}{2}$

Thus, surface area of a frustum of a cone:

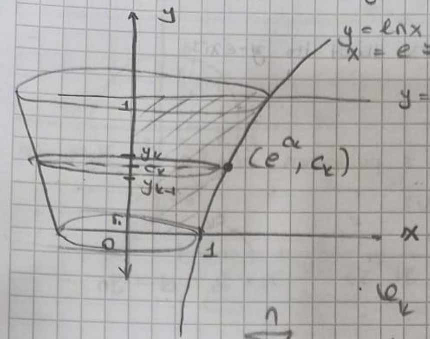
$$S_{FC}^{(k)} \cong 2\pi \frac{f(x_{k-1}) + f(x_k)}{2} \sqrt{1 + f'(x_k^*)^2} \cdot \Delta x_k$$

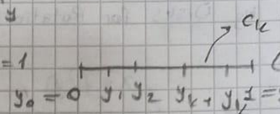
$$S \cong \sum_{k=1}^{n} S_{FC}^{(k)} = \sum_{k=1}^{n} 2\pi \frac{f(x_{k-1}) + f(x_k)}{2} \sqrt{1 + f'(x_k^*)^2} \, \Delta x_k$$

is a Riemann sum for $h(x) = \pi f(x) \sqrt{1 + f'(x)^2}$

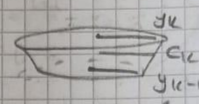
$$S = \int_a^b 2\pi f(x) \sqrt{1 + f'(x)^2} \, dx$$

b) $y = \ln x$, x-axis, y-axis, $y = 1$





(through y-axis)



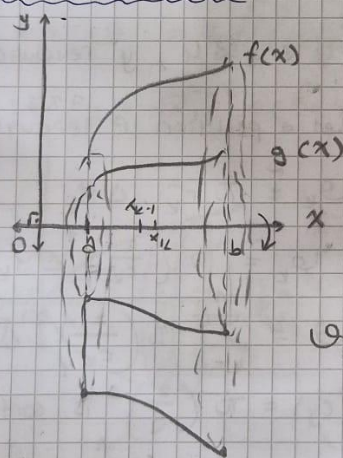
$$\vartheta_k \cong \pi \left(e^{c_k}\right)^2 \Delta y_k$$

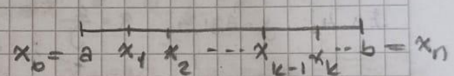
$\vartheta \cong \sum_1^n \pi \left(e^{c_k}\right)^2 \Delta y_k$ is a Riemann sum for $g(y) = \pi e^{2y}$ is cont. thus integrable

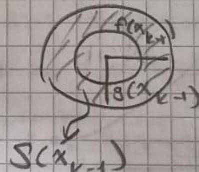
$$\vartheta = \int_0^1 \pi e^{2y}\, dy$$

The Washer Method

Let a partition P of [a,b]







$$\vartheta_k \cong \left[\pi f(x_{k-1}) - \pi g^2(x_{k-1})\right]\Delta x_k$$
$$\cong \pi\left[f^2(x_{k-1}) - g^2(x_{k-1})\right]\Delta x_k$$
$$\vartheta \cong \sum_{k=1}^n \vartheta_k = \sum_{k=1}^n \pi\left[f^2(x_{k-1}) - g^2(x_{k-1})\right]\Delta x_k$$

is a Riemann sum for $h(x) = \pi[f^2(x) - g^2(x)]$ is cont. thus integrable

$$\vartheta = \pi\int_a^b \left(f^2(x) - g^2(x)\right) dx$$

Example 5: A heavy rock blown straight up the ground by a dynamite blast. The velocity of the rock at any t during its motion is given as $v(t) = 49 - 9,8t$ m/s

solve acc to this

$$\int_{t_1}^{t_2} v(t)\,dt = x(t_2) - x(t_1)$$: the net change displacement

$$\int_{t_1}^{t_2} |v(t)|\,dt$$

The Relationship Between Integral and Diff.

f is cont. on $[a,b]$, then $F(x) = \int_a^x f(t)\,dt$ is cont. on $[a,b]$, diff. on $]a,b[$ and its derivative is $f(x)$

$$F(x) = \int_a^x f(t)\,dt$$

$$F'(x) = \frac{d}{dx}\int_a^x f(t)\,dt = f(x)$$

which means;

$$F(x) - F(a) = \int_a^x F'(t)\,dt$$

Determine whether $\left(\frac{\sin(n)}{\sqrt{n}}\right)_{n \in \mathbb{Z}^+}$ converge or diverge

$\forall n \in \mathbb{Z}^+ : \quad |\sin(n)| \leq 1 \iff -1 \leq \sin(n) \leq 1$

$\forall n \in \mathbb{Z}^+ : \quad \sqrt{n} > 0$

$-\frac{1}{\sqrt{n}} \leq \frac{\sin n}{\sqrt{n}} \leq \frac{1}{\sqrt{n}}$ By sandwich $\to 0$

Determine $(r^n)_{n \in \mathbb{Z}^+}$ $r \in \mathbb{R}$

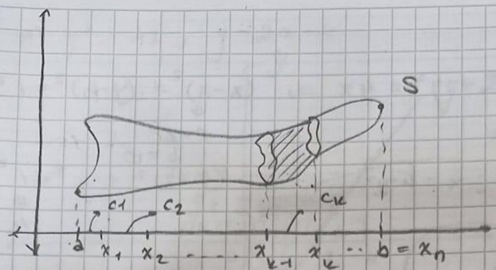
if $|r| < 1$ $\lim_{n \to \infty} r^n = 0$

if $|r| > 1$ $\lim_{n \to \infty} r^n = \infty$ diverges

if $|r| = 1$ $\forall n\ \mathbb{Z}^+ : r^n = 1 \Rightarrow \lim_{n \to \infty} r^n = 1$

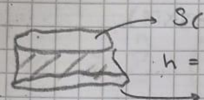
if $r = -1$ $r^n = (-1)^n$ it diverges

if $r < -1$ r^n diverges (no fixed value)



Let a partition P of $[a, b]$ by choosing $(n-1)$ points such as $a < x_1 < x_2 < \cdots < x_{k-1} < x_k < \cdots < x_{n-1} < x_n = b$

Slice the solid S with plane P_{x_k} through x_k



$$V_k = A(c_k)\,\Delta x_k$$

$$V \cong \sum_{k=1}^{n} V_k = \sum_{k=1}^{n} A(c_k)\,\Delta x_k$$

is a Riemann sum for $A(x)$ which describes the area of the k^{th} cross section.

$$\lim_{n \to \infty} \sum_{k=1}^{n} A(c_k) \cdot \Delta x_k$$, if exists then

$$V = \int_a^b A(x)\,dx$$

// exp 4 is homework, Exp 5 is VERY important

Example 2: $\int_1^\infty \frac{dx}{x}$

$\forall t > 1$: $\int_1^t \frac{dx}{x}$ exists and is a Riemann integral

$\int_1^t \frac{dx}{x} = \ln|x| \Big|_1^t = \ln|t| - \ln|1|$

$= \ln|t|$

$\lim_{t\to\infty} \int_1^t \frac{dx}{x} = \lim_{t\to\infty} \ln|t| = \infty$ thus the

limit does not exist, is divergent over the interval.

Example 3:

$\int_{-\infty}^\infty \frac{dx}{1+x^2}$ since $f(x) = \frac{1}{x^2+1}$ is cont. on $\mathbb{R}$ it is a improper integral of Type 1.

$\int_{-\infty}^\infty \frac{dx}{1+x^2} = \underbrace{\int_{-\infty}^0 \frac{dx}{1+x^2}}_{I_1} + \underbrace{\int_0^\infty \frac{dx}{1+x^2}}_{I_2}$

I_1 and I_2 are both improper integrals of type 1.

$\forall t < 0$: $\int_t^0 \frac{dx}{1+x^2}$ is a Riemann Integral.

$\Downarrow$

$\arctan x \Big|_t^0 = -\arctan t$

$\lim_{t\to-\infty} -\arctan t = \frac{\pi}{2}$ is what I_1 converges to.

$I_2 = \int_0^\infty \frac{dx}{1+x^2}$

$\forall t > 0$: $\int_0^t \frac{dx}{1+x^2}$ is a Riemann integral.

$\Downarrow$

$\arctan x \Big|_0^t = \arctan t$

$\lim_{t\to\infty} \arctan t = \frac{\pi}{2}$ is what I_2 converges to.

Thus I converges to $I_1 + I_2 = \pi$

GIPTA

Example 2 : find dy/dx if :

(a) $y = \int_2^x (t^3+1)\,dt \Rightarrow \frac{dy}{dx} = \frac{d}{dx}\left(\int_2^x (t^3+1)\,dt\right)$

If $f(x) = x^3+1$ then $F(x) = \int_2^x (t^3+1)\,dt$ is an antiderivative for f (i.e. $F'(x) = f(x)$)

$= x^3+1$; cont. on $\mathbb{R}$

(b) $y = \int_0^x \cos t\,dt$; cos is cont. on $\mathbb{R}$

If $g(t) = \cos t$ then $G(t) = \int_0^x \cos t\,dt$ is an antiderivative for g ($G'(x) = g(x)$)

$\frac{dy}{dx} = \frac{d}{dx}\left(\int_0^x \cos t\right) = \frac{d}{dx} G(x) = G'(x) = g(x) = \cos x$

(c) $y = \int_0^{x^2} \cos t\,dt$; let $u = x^2$ and :

$y = \int_1^u \cos t\,dt$; $u = x^2$

$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} = \cos u . (2x) \overset{u=x^2}{=} \cos x^2 . (2x)$

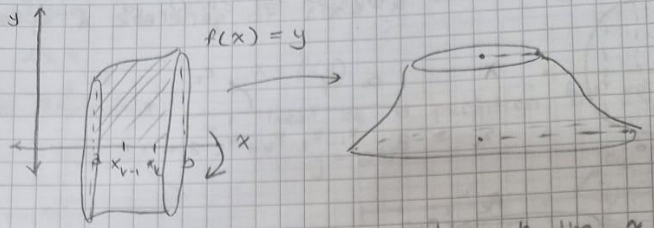
$\frac{d}{du}\left(\int_1^u \cos t\,dt\right) = \cos u$

(d) $y = \int_{1+3x^2}^4 \frac{dt}{2+t} dt = -\int_4^{1+3x^2} \frac{dt}{2+t} dt$; let $u = 1+3x^2$

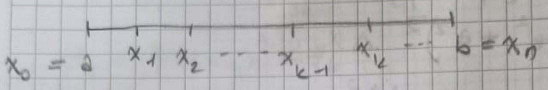
$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} = -\frac{1}{2+t}(6x) = -\frac{1}{3+3x^2} . (6x)$

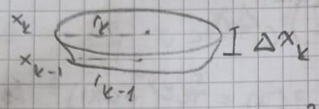
$\frac{d}{du}\left(\int_4^u \frac{dt}{2+t} dt\right) = -\frac{1}{2+t}$

Solids of Revolution : The Disk Method



Let a partition of $[a,b]$ through the x-axis :





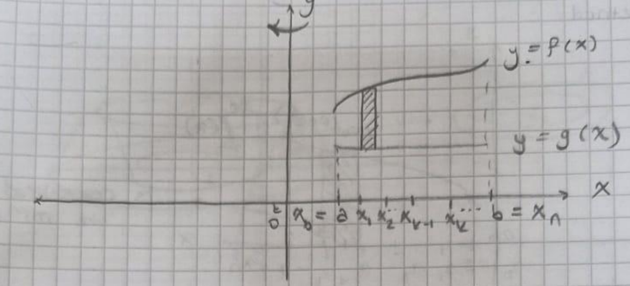
let $c_k \in [x_{k-1}, x_k]$

$$V_k \cong \pi \left[f(c_k)\right]^2 \Delta x_k$$

$$V \cong \sum_{k=1}^{n} \pi \left[f(c_k) \right]^2 \Delta x_k$$ is a Riemann sum for $g(x) = \pi f^2(x)$ is continuous thus integrable :

$$V = \int_a^b \pi f^2(x) \, dx$$

Revolving the region between two curves

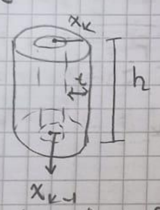


over $[a,b]$ $g(x) \leq f(x)$; $\forall x \in [a,b]$

Let a partition for $[a,b]$ through the x-axis

$x_0 = a, x_1, x_2 \cdots x_{k-1}, x_k, \cdots, x_n = b$

By revolution, such cylinderical shell is obtained:



$$V_{\text{cylin. shell}} = 2\pi \bar{r} h t$$

$$V_{k^{\text{th}} \text{ shell}} = 2\pi x_k^* (f(x_k^*) - g(x_k^*)) \Delta x_k$$

Let $x_k^* \in [x_{k-1}, x_k]$

$$V \cong \sum_{k=1}^{n} 2\pi x_k^* [f(x_k^*) - g(x_k^*)] \Delta x_k$$

is a Riemann sum for $h(x) = 2\pi x [f(x) - g(x)]$ and since $h(x)$ is continuous thus integrable:

$$V = \lim_{\|P\| \to 0} \sum_{k=1}^{n} 2\pi x_k^* [f(x_k^*) - g(x_k^*)] \Delta x_k$$

$$V = 2\pi \int_a^b x [f(x) - g(x)] dx$$

Example 6: Consider $f(x) = x^2 - 4$ on $I = [-2, 2]$

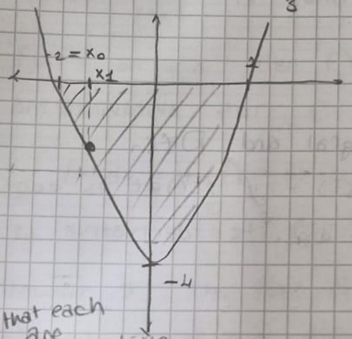
Compute integral over I between graph an x-axis.

$$\int_{-2}^{2} (x^2-4)\,dx \quad ; \quad x^2-4 = f(x)$$

$F(x) = \frac{x^3}{3} - 4x$ cont. over I

Since $F'(x) = f(x)$, $F(x)$ is an antiderivative for f. Thus:

$$\int_{-2}^{2} (x^2-4)\,dx = F(2) - F(-2)$$
$$= \frac{8}{3} - 8 - \left(-\frac{8}{3} - 8\right) = -\frac{32}{3}$$



through x-axis

$x_0 = -2$ —— $2 = x_n$

Choose $n-1$ points such that

$x_0 < x_1 < x_2 < \dots < x_{n-1} < x_n$

$\Delta x = \frac{2-(-2)}{n} = \frac{4}{n} = \Delta x_k$;

$\forall k = 1, \dots, n$

sure that each are negative

$A = |f(c_1)|\Delta x_1 + |f(c_2)|\Delta x_2 + \dots + |f(c_k)|\Delta x_k + \dots + |f(c_n)|\Delta x_n$

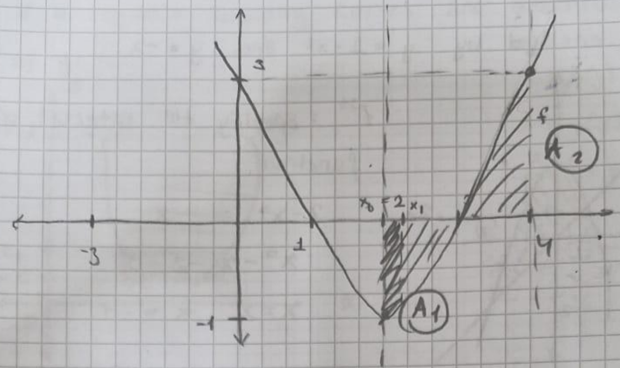
$= \sum_{k=1}^{n} f(c_k)\,\Delta x_k$ is a Riemann sum for $|f|$ over $[-2, 2]$ (*)

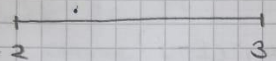
$$A = \int_{-2}^{2} |f(x)|\,dx$$

$$= \int_{-2}^{2} |x^2-4|\,dx = -\int_{-2}^{2} (x^2-4)\,dx = \frac{32}{3}$$

b) $y = x^2 - 4x + 3$, $x=2$, $x=4$, x-axis

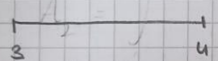
$16 - 16 + 3$





(through x-axis)

$$A_1 = \int_2^3 |f(x)|\,dx = -\int_2^3 f(x)\,dx = -\int_2^3 x^2 - 4x + 3\,dx$$



(through x-axis)

$$\int_3^4 f(x)\,dx = \int_3^4 x^2 - 4x + 3\,dx$$

d) $\lim\limits_{n\to\infty} \dfrac{x^n}{n!} = 0$ for any x.

Let $x=2$ then; $\lim\limits_{n\to\infty} \dfrac{2^n}{n!}$

$\forall n \in \mathbb{Z}^+ \quad 0 < \underbrace{\dfrac{2^n}{n!}}_{b_n} = \dfrac{\overbrace{2\cdot 2\cdot 2 \cdots 2}^{n}}{1\cdot 2\cdot 3 \cdots n}$

$= 2\cdot\dfrac{2}{2}\cdot\underbrace{\dfrac{2}{3}}_{\le 2/3}\cdot\underbrace{\dfrac{2}{4}}_{\le 2/3}\cdot\underbrace{\dfrac{2}{5}}_{\le 2/3}\cdots\underbrace{\dfrac{2}{n}}_{\le 2/3}$

$\le 2\cdot\left(\dfrac{2}{3}\right)^{n-2} = c_n$

$\lim\limits_{n\to\infty} c_n = \lim\limits_{n\to\infty} 2\cdot\left(\dfrac{2}{3}\right)^{n-2} = 0$

$\lim\limits_{n\to\infty} b_n = 0$

By sandwich theorem:

$\lim\limits_{n\to\infty} \dfrac{2^n}{n!} = 0$

$\lim\limits_{n\to\infty} \dfrac{x^n}{n!}$

$x \in \mathbb{R}^+ \quad 0 < \dfrac{\overbrace{x\cdot x\cdot x \cdots x}^{n}}{1\cdot 2\cdot 3 \cdots n} = x\ \dfrac{x}{2}\ \dfrac{x}{3} \cdots \dfrac{x}{k} \cdots \dfrac{x}{n}$

$\le \dfrac{x}{k} \quad \le \dfrac{x}{k}$

0

$x \le k \ni 1 \le k \le n$

$\dfrac{x}{n} \le \dfrac{x}{k} \quad \forall k \le n$

c)

d)

e) $\int_{0}^{\sqrt{\pi}} x \cos x^2 \, dx$ let $x^2 = u$ then $2x \cdot dx = du$

$\int_{0}^{\pi} \frac{\cos u}{2} \, du$ in addition $x = \sqrt{\pi} \Rightarrow u = \pi$

$x = 0 \Rightarrow u = 0$

$\frac{1}{2} \sin u \Big|_{0}^{\pi} = 0$

Thus $T = \frac{10}{9}\left[1 - \frac{1}{10^n}\right]$

$$\lim_{n\to\infty} S_n = \lim_{n\to\infty} \frac{3}{10} T$$

$$= \lim_{n\to\infty} \frac{3}{10} \cdot \frac{16}{9}\left(1 - \frac{1}{10^n}\right)$$

$$= \frac{1}{3}$$

Infinite Sequences

$f : A \to B \quad ; \quad A, B \subseteq \mathbb{R}$

$x \to f(x)$

Whenever $A = \mathbb{Z}^+$

$f : \mathbb{Z}^+ \to B$ or $f : \mathbb{N} \to \mathbb{R}$

$n \to f(n) = a_n$

is called a sequence.

★ $a_1, a_2, \dots, a_n, \dots$

★ $(a_n)_{n \in \mathbb{N} \text{ or } \mathbb{Z}^+}$

★ $(f(n))_{n \in \mathbb{Z}^+}$

Sequences can be described with general terms:

$a_n = \sqrt{n}$, $b_n = (-1)^{n+1}$

Ex. 7:

$$I = \int_0^{\pi} \sec^2 x \, dx$$

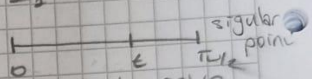
$\sec x = \dfrac{1}{\cos x}$ and it is discontinuous at $x = \frac{\pi}{2} \in [0, \pi]$

Thus I is an improper integral of type III.

Hence;

$$I = \int_0^{\pi} \sec^2 x \, dx = \underbrace{\int_0^{\pi/2} \sec^2 x \, dx}_{I_1} + \underbrace{\int_{\pi/2}^{\pi} \sec^2 x \, dx}_{I_2}$$

For convergent behaviour of I_1:



over $[0, t]$ $f(x) = \sec^2 x$ is con. thus integrable.

Thus, we shall investigate

$$\lim_{t \to \frac{\pi}{2}^-} \int_0^t \sec^2 x \, dx = \lim_{t \to \frac{\pi}{2}^-} \tan x \Big|_0^t = \lim_{t \to \pi/2^-} (\tan t - \tan 0)$$

$$= \lim_{t \to \pi/2^-} \tan t = +\infty$$

$\Downarrow$

Rest is homework.

Example 4: Evaluate $\int_0^1 \frac{dx}{x+1}$ and $\int_0^1 \frac{dx}{x^2+1}$

a) $f(x) = \frac{1}{x+1}$ is cont on $[0,1]$

$F(x) = \ln|x+1|$ then $F'(x) = f(x)$;

thus F is an antiderivative for f. Thus;

$$\int_0^1 f(x)\,dx = F(1) - F(0)$$
$$= \ln 2 - 0 = \ln 2$$

b) $f(x) = \frac{1}{1+x^2}$ is cont. on $[0,1]$

$F(x) = \arctan x$ then $F'(x) = f(x)$. Thus F is an antiderivative for f.

$$\int_0^1 f(x)\,dx = F(1) - F(0)$$
$$= \frac{\pi}{4} - 0 = \frac{\pi}{4}$$

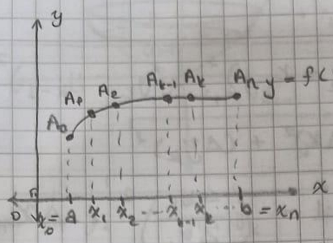
Theorem 5 : The Net Change Theorem

The net change in a diff. func. $F(x)$ over an $[a,b]$

: write rest

Area of Surface of Revolution

Only the bounding curve is revoved a surface is formed.

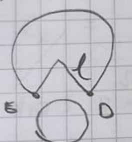


Frostum of a cone

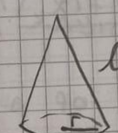
Consider a partition through x-axis over $[a, b]$:

$x_0 = a \quad x_1 \quad x_2 \; \cdots \; x_{k-1} \quad x_k \; \cdots \; b = x_n$

Given that any A_{k-1} and A_k is revoved, a frostum of a cone is formed:



Cone when unrolled



ℓ: slant height

r: base radius

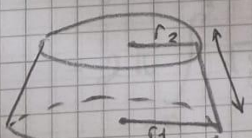
$C_{disk} = 2\pi\ell$

Area of the sector $= \dfrac{2\pi r}{2\pi \ell} \cdot \pi \ell^2 = \pi r \ell$

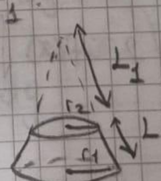
Thus, surface area of a cone:

$S_{cone} = \pi r \ell$

Frostum of a cone with slant height L, upper radius r_2 and lower radius r_1:



$\Rightarrow$



$S_{fc} = \pi r_1 (L_1 + L) - \pi r_2 L_1$

By using similarity: $\dfrac{r_2}{r_1} = \dfrac{L_1}{L_1 + L}$

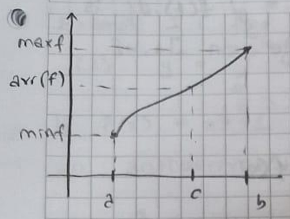
thus,

$r_2 L_1 + L r_2 = r_1 L_1$

$L_1 [r_2 - r_1] = -L r_2$

$S_{fc} = \pi L (r_1 + r_2) = 2\pi \underbrace{\left(\frac{r_1 + r_2}{2}\right)}_{\text{avg radius}} L$

Ex 1 here ↓ homework



Thus by Intermediate value theorem, f is cont.

$\exists c \in [a,b] \ni f(c) = av(f)$

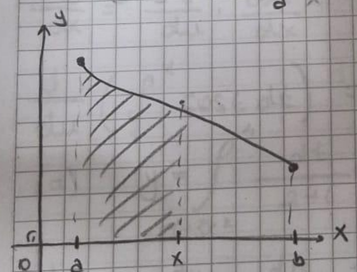
✶ ✶ ✶

An antiderivative of a function f is another function F such that $F'(x) = f(x)$

$f(x) = x \Rightarrow F(x) = \frac{x^2}{2} + C$

Assume that $f(t)$ is an integrable function over a finite interval I, then the integral from any fixed number $a \in I$ to another number $x \in I$ defines a new function F whose value at x is:

$$F(x) = \int_a^x f(t) \cdot d(t)$$



$$F(x) = \int_a^x f(t) \cdot dt$$

7) $x^2+y^2=4$, $x^2+y^2=4x \Rightarrow (x-a)^2+(y-b)^2=r^2$

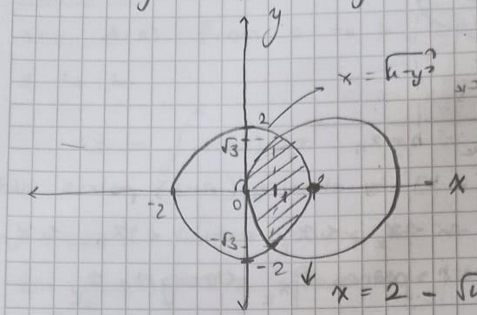
$a=2$

$b=0$

$(x-2)^2+y^2=r^2$

$\Downarrow$

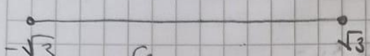
4



$x^2-4x=-y^2$

$\Downarrow$

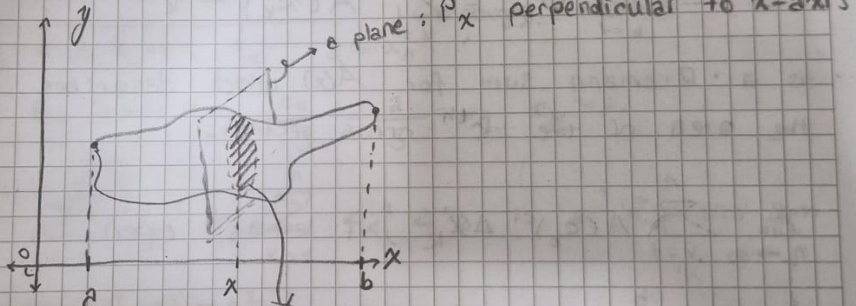
$x=2-\sqrt{4-y^2}$



(through y-axis)

$$\int_{-\sqrt{3}}^{\sqrt{3}} \sqrt{4-y^2}-\left(2-\sqrt{4-y^2}\right)dy = A$$

Volumes



$S(x) :=$ cross section

$A(x) :=$ the area of the cross section

a is homework $\{ h \to$ select a const c e-f-g homework $\}$

Example 3 :

(b) $\lim_{x\to 3} \frac{x}{x-3} \int_3^x \frac{\sin t}{t}\, dt \quad \left(\frac{0}{0}\right)$

$$\lim_{x\to 3} \frac{\frac{d}{dx}\left(x \int_3^x \frac{\sin t}{t}\, dt\right)}{\frac{d\,(x-3)}{dx}} = \frac{1 \cdot \int_3^x \frac{\sin t}{t}\, dt + x \cdot \left(\frac{\sin x}{x}\right)}{1}$$

$$= \sin 3$$

Theorem 4. Fundamental Theorem of Calc. Part 2

2nd Corollary of Mean Val. Theorem

If $f'(x) = g'(x)$ at each point $x \in]a,b[$ then there exists a constant c such that:

$$f(x) = g(x) + c \;; \quad \forall x \in]a,b[$$

$$* \quad F(a) \overset{F \text{ is cont.}}{=} \lim_{x\to a^+} F(x) \;; \quad F \text{ is cont on } [a,b]$$

$$= \lim_{x\to a^+} G(x) + c \;; \quad F'(x) = G'(x) \quad \forall x \in]a,b[$$

G is cont.

$$= G(a) + c$$

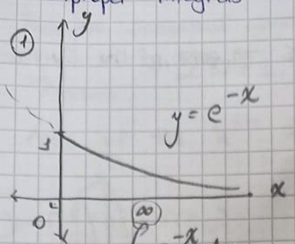
The fund. theorem:

f is cont on $[a,b]$ and F is any antiderivative of f on $[a,b]$ then

$$\int_a^b f(x)\,dx = F(b) - F(a)$$

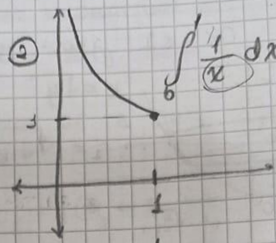
Improper Integrals

①



$$\int_0^{\infty} e^{-x}\,dx$$

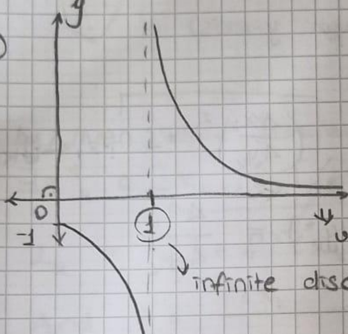
②



$$\int_0^1 \frac{1}{x}\,dx$$

$f(x) = \frac{1}{x}$ is discontinuous at $x = 0$

③



$$\int_0^{\infty} \frac{dx}{x-1}$$

upper bound is not finite

infinite discontiunity

#1 $\forall t > 0$ $y = e^{-x}$ is cont. over $[0,t]$ thus

$\int_0^t e^{-x}\,dx$ is a Riemann integral.

$$\lim_{t \to \infty} \int_0^t e^{-x}\,dx$$

#2 $\forall t \in \,]0,1[$: $\int_t^1 \frac{dx}{x}$ is a Riemann integral

$$\lim_{t \to 0^+} \int_t^1 \frac{dx}{x}$$

#3

2) $p \neq 1$

$$I = \int_0^1 \frac{dx}{x^p}$$

0 t 1

$$\lim_{t \to 0^+} \int_t^1 \frac{dx}{x^p} = \lim_{t \to 0^+} \frac{x^{-p+1}}{1-p}\Big|_t^1 = \lim_{t \to 0^+} \left[\frac{1}{1-p} - \frac{t^{-p+1}}{1-p}\right]$$

$$\frac{1}{1-p} \lim_{t \to 0^+} \left(1 - t^{1-p}\right)$$

$$= \begin{cases} \frac{1}{1-p} & ; \text{ if } 1-p > 0 \iff 1 > p \\ +\infty & ; \text{ if } 1-p < 0 \iff 1 < p \end{cases}$$

If p is negative, I is a Riemann integral

if $p = = 1$, it diverges.

if $p \neq 1$;

$0 < p < 1$ it converges

$p > 1$ it diverges.

Exp. 11: $\int_a^b \frac{dx}{(b-x)^p}$

if $p < 0$; $f(x) = \frac{1}{(b-x)^p} = (b-x)^{-p}$ is continuous

if $p > 0$; $f(x) = \frac{1}{(b-x)^p}$ has a singular point at $x = 0$

Thus I is an improper integral of Type III.

If $p = 1$; $\int_a^b \frac{dx}{b-x}$ is an improper integral of type II,

a t b

$\underbrace{\qquad}$ is a Riemann integral

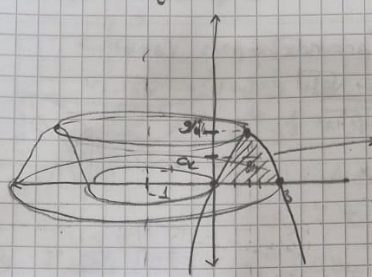
$$\lim_{t \to b^-} \int_a^t \frac{dx}{b-x} = \lim_{t \to b^-} -\ln|b-x|\Big|_a^t$$

$$\lim_{t \to b^-} -\ln|b-t| + \ln|b-a|$$

$$\lim_{t \to b^-} \ln\left|\frac{b-a}{b-t}\right| \quad \text{since } t < b \iff b-t < 0$$

$$= \infty$$

Ex.: x-axis, $y = 3x - x^2$ is revolved about $x = -1$



(through the y-axis)

$0 \qquad \frac{9}{4}$

Is not easy to evaluate the V by using this method.

Thus,

(through x-axis)

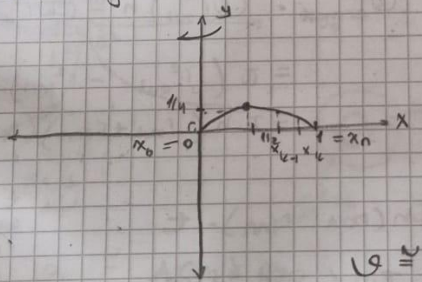
$x_0 = 0 \quad x_{k-1} \quad c_k \quad x_k \quad 3 = x_n$

$$\vartheta_k \cong C.h.t$$

$$= 2.\pi\bar{r}.\left(\frac{x_k + x_{k-1} + 2}{2}\right).f(c_k).\Delta x_k$$

$$V = \pi\int_0^3 (2x+2)\, f(x).\, dx$$

Ex. 7: $y = x - x^2$, revolved around x-axis



(through x-axis)

$0 \qquad 1$

$$\vartheta_k \cong C.h.t$$

$$\cong 2\pi\,\bar{r}.h.t$$

$$\cong 2\pi\,\frac{x_k + x_{k-1}}{2}.f(c_k).\Delta x_k$$

$$\vartheta \cong \sum_{k=1}^{n} \vartheta_k \quad \text{is a Riemann sum}$$

for π.

$$\vartheta = \int_0^1 \pi.2x.f(x).dx = 2\pi\int_0^1 f(x).x.dx$$

7 Domination

$f(x) \geq g(x)$ on $[a,b] \Rightarrow \int_a^b f(x)\,dx \geq \int_a^b g(x)\,dx$

$f(x) \geq 0$ on $[a,b] \Rightarrow \int_a^b f(x)\,dx \geq 0$

Example 2:

$\int_{-2}^{3} f(x)\,dx = 7$, $\int_{3}^{5} f(x)\,dx = 5$, $\int_{-2}^{3} h(x)\,dx = -3$ then:

$\int_5^3 f(x)\,dx = -5$

$\int_{-2}^{3} \left(2f(x) + 3h(x)\right) = 5$

// One more here

Example 3: Find an upper and lower bound for

$\int_1^3 \sqrt{3+\ln x}\,dx = f(x)$

By Prop 6 of Th. 2:

$\min f\,(b-a) \leq \int_a^b f(x)\,dx \leq \max f\,(b-a)$

$f'(x) = \frac{1}{2} \cdot \frac{1}{x} \cdot \frac{1}{\sqrt{3+\ln x}} > 0$; f is increasing

thus $x=3$ is max while $x=1$ is min.

$\min f = \sqrt{3}$, $\max f = \sqrt{3+\ln 3}$

$2\sqrt{3} \leq \int_1^3 \sqrt{3+\ln x}\,dx \leq 2\sqrt{3+\ln 3}$

// Can a func. be integ. on a point like 0 there?

* Remark is homework

$$\int_a^b \underset{\Downarrow}{f(x)}\,dx = J$$

integrand

$$\int_0^1 (1-x^2)\,dx = x - \frac{x^3}{3}\Big|_0^1 = 1 - \frac{1}{3} = 2/3$$

$$f(x) = \begin{cases} 1 ; & x \in \mathbb{Q} \\ 0 ; & x \in \complement_{\mathbb{R}} \mathbb{Q} \end{cases}$$

Over $[0,1]$ choose a $P = \{x_1, x_2, \dots, x_{n-1}\}$ with equal interval widths $\Delta x_k = \frac{1}{n}$, $\forall k = 1, \dots, n$

$$\left[0, \tfrac{1}{n}\right] \left[\tfrac{1}{n}, \tfrac{2}{n}\right] \cdots \left[\tfrac{k-1}{n}, \tfrac{k}{n}\right], \cdots \left[\tfrac{n-1}{n}, 1\right]$$

1.st Choose c_k as an rational number in each subinterval, then

$$\underset{\overset{\shortparallel}{U_p(f)}}{R_p} = \sum_{k=1}^{n} f(c_k)\cdot \Delta x_k = \sum_{k=1}^{n} 1 \cdot \frac{1}{n} = \frac{1}{n} \sum_{k=1}^{n} 1 = 1 \xrightarrow{n \to \infty} 1$$

2nd Choose c_k as an irrational number in each subinterval:

$$R_p = (f) = \sum_{k=1}^{n} f(c_k)\cdot \Delta x_k$$

$$= \sum_{k=1}^{n} 0 \cdot \frac{1}{n} = \sum_{k=1}^{n} 0 = 0 \xrightarrow{n \to \infty} 0$$

Thus f is not integrable over $[0,1]$

Theorem 1 : Integrability of Continuous Function

f is continuous over $[a,b]$, or if f has finitely many jump discontinuity, then f is integrable on $[a,b]$

// Not resp. for proof

Properties of Definite Integral

Theorem 2: ① when f and g are integrable over $[a,b]$, then:

$$\left\{ \int_a^b f(x)\,dx = -\int_b^a f(x)\,dx \right\}$$

$x_0 = a \quad x_1 \quad x_2 \quad \dots \quad x_n = b$

: $a < x_1 < x_2 < \dots < x_{n-1} < x_n$

$\Delta x_k = x_k - x_{k-1} > 0$

$\sum_{k=1}^{n} f(c_k)\,\Delta x_k$

$y = b \quad y_1 \quad y_2 \quad \dots \quad y_n = a$

: $b > y_1 > y_2 > \dots > y_{n-1} > a$

$\Delta y_k = y_k - y_{k-1} < 0$

$\sum_{k=1}^{n} f(c_k)\,\Delta y_k$

② $\int_a^a f(x)\,dx = 0$

$[a,a] = \{a\}$, $\Delta x_k = 0$

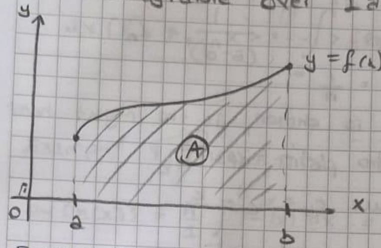
③ for any constant k:

$$\int_a^b k\,f(x)\,dx = k \cdot \int_a^b f(x)\,dx$$

Area under the graph of a nonnegative function:

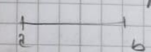
f is nonnegative ($\forall x \in [a,b] : f(x) \geq 0$)

f is integrable over $[a,b]$:



$$A = \int_a^b f(x)\,dx$$

$$a + k\frac{b-a}{n}$$

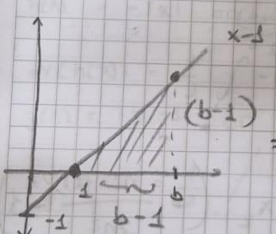


Example 4:

$\int_1^b (x-1)\,dx$, find the area under the curve $y = x-1$ over $[1,b]$

$$\int_1^b (x-1)\,dx = \frac{x^2}{2} - x \Big|_1^b = \frac{b^2}{2} - b - \frac{1}{2} + 1$$

$$= \frac{1}{2}(b^2 - 2b + 1)$$



$$\Rightarrow \frac{1}{2}(b-1)^2$$

$$\lim_{n\to\infty} \sum_{k=1}^{n} f(c_k)\Delta x_k$$

$$\overline{(b-1)^2}$$

$$2$$

$$\sum_{k=1}^{n} f(c_k) \cdot \Delta x_k$$

// WRITE AGAIN HER $x_0 = 1$ x_1 x_2 ... x_{n-1} $b = x_n$

$$\Delta x_k = \frac{b-1}{n}, \quad \forall k = 1, \dots, n$$

$n +$

$$c_1 = x_1 = x_0 + \Delta x_k = 1 + \frac{b-1}{n}$$

$$\sum_{k=1}^{n} f(c_k)\Delta x_k =$$

$$c_2 = x_1 + \Delta x_k = 1 + 2\left(\frac{b-1}{n}\right)$$

$$\vdots$$

$$\sum_{k=1}^{n} f\left(n + k\left(\frac{b-1}{n}\right)\right) \cdot \frac{b-1}{n}$$

$$c_k = x_{k-1} + \Delta x_k = 1 + k\left(\frac{b-1}{n}\right)$$

$$\frac{(b-1)^2}{n^2} \sum_{k=1}^{n} k = \frac{(b-1)^2}{n^2} \cdot \frac{n.(n-1)}{2}$$

Definite Integral

$$\lim_{\|P\| \to 0} S_P = J$$

$\forall \epsilon > 0 \; : \; \exists \delta > 0$ s.t. for each P satisfying $0 < \|P\| < \delta$ and each selection of numbers $c_k \in I_k = [x_{k-1}, x_k]$:

$$|S_P - J| < \epsilon$$

$$\sum_{k=1}^{n} f(x_k) \cdot \Delta x$$

Leibnitz Notation

The finite sums $\sum_{k=1}^{n} f(c_k) \Delta x_k$ are becoming an infinite sum of function values $f(x)$ multiplied by "infinitesi

$$\lim_{n \to \infty} \sum_{k=1}^{n} f(c_k) \cdot \Delta x_k = \sum_{k=1}^{\infty} f(c_k) \cdot \Delta x_k$$

$$\Updownarrow$$

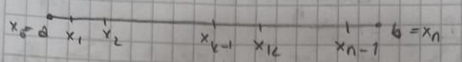
$$\|P\| \to 0$$

$$\forall k : \underset{\overset{\|}{dx}}{\Delta x_k} = x_k - x_{k-1} \longrightarrow 0$$

$$J = \int_a^b f(x) \cdot dx$$

Riemann Sum

f is a bounded func. on $[a,b]$



$x_0 < x_1 < x_2 < \dots < x_{k-1} < x_k < \dots < x_{n-1} < x_n$

$P = \{x_1, x_2, \dots, x_{n-1}\}$ is a partition of $[a,b]$

$[x_0, x_1]$, $[x_1, x_2]$ ... $[x_{n-1}, x_n]$ are subintervals

and width of each of them is described as:

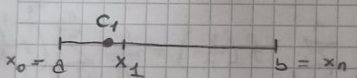
$\Delta x_1 = x_1 - x_0$, $\Delta x_2 = x_2 - x_1$, ..., $\Delta x_k = x_k - x_{k-1}$

$\Delta x_n = x_n - x_{n-1}$ thus

$\boxed{\Delta x_k = x_k - x_{k-1}}$

Then choose;

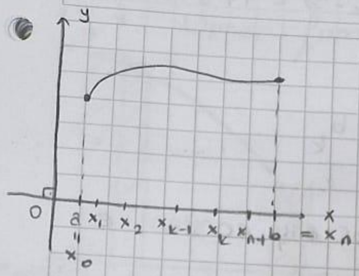
$c_1 \in [x_0, x_1]$, $c_2 \in [x_1, x_2]$, ..., $c_k \in [x_k, x_{k-1}]$



$\underbrace{\sum_{k=1}^{n} f(c_k) \cdot \Delta x_k}$ depends on partition P and the points c_k.

S_p is a Riemann Sum.

and : $LS \leq S_p \leq US$



Divide $[a,b]$ into subintervals n times.

Claim $(n-1)$ points between a and b $[x_0, x_1]$, $[x_1, x_2]$ ... $[x_{n-1}, x_n]$

Choose a point in each sub interval.

$\Delta x_1 = x_1 - x_0$: length of 1^{st} interval

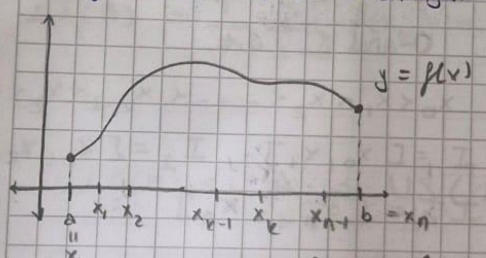
$\vdots$

$\Delta x_n = x_n - x_{n-1}$: length of n^{th} interval

$\Delta x_k = \Delta x$; $\forall k = 1, \dots, n$ thus subinterval lengths are equal.

$$f(c_1).\Delta x + f(c_2).\Delta x + \dots + f(c_n).\Delta x = \sum_{k=1}^{n} f(c_k).\Delta x$$

Average Value of a Nonnegative Cont. Function



$$Avg\,(f(x)) \approx \frac{f(x_0) + f(x_1) + f(x_2) + \dots + f(x_n)}{n} = \frac{\sum_{k=1}^{n} f(x_k)}{n}$$

And since $\Delta x_1 = \Delta x_2 = \dots = \Delta x_n = \dfrac{b-a}{n}$

$$Avg\,(f(x)) \approx \frac{\sum_{k=1}^{n} f(x_k)}{n.\Delta x} . \Delta x = \frac{\sum_{k=1}^{n} f(x_k)}{(b-a)} . \Delta x$$

$$\approx \frac{\sum_{k=1}^{n} f(x_k)}{(b-a)} . \Delta x$$

if all subintervals are equal:

$\frac{b-a}{n} = \Delta x_k$; $\forall k = 1, 2, \dots, n$

$$S_p = \sum_{k=1}^{n} f(x_k) \frac{b-a}{n} = \frac{b-a}{n} \sum_{k=1}^{n} f(x_k)$$

$\|P\| = \max\limits_{1 \le k \le n} \Delta x_k$ } norm of partition P

$[0,1]$ $P_1 = \left\{ \frac{1}{8}, \frac{1}{4}, \frac{1}{2}, \frac{3}{4} \right\}$

$\|P\| = \max \{ \Delta x_1, \Delta x_2, \Delta x_3, \Delta x_4, \Delta x_5 \}$

$\max \left\{ \frac{1}{8}, \frac{1}{8}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4} \right\} = \frac{1}{4}$

$P_2 = \left\{ \frac{1}{2}, \frac{3}{4} \right\}$

$\|P_2\| = \max \left\{ \frac{1}{2}, \frac{1}{4}, \frac{1}{4} \right\} = \frac{1}{2}$

$P_3 = \left\{ \frac{1}{8}, \frac{1}{2} \right\}$

$\|P_3\| = \max \left\{ \frac{1}{8}, \frac{3}{8}, \frac{1}{2} \right\} = \frac{1}{2}$

$\|P\| \longrightarrow 0$ $\Rightarrow$ $n \longrightarrow \infty$

$\not\Leftarrow$

$\Leftarrow$ if the subinterval widths are equal.

④ $$\int_a^b (f(x) \pm g(x))\,dx = \int_a^b f(x)\,dx \pm \int_a^b g(x)\,dx$$

⑤ $$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx, \quad c \in]a,b[$$

⑥ Max – min equality f assumes its maximum and minimum on $[a,b]$

$$\int_a^b f(x)\,dx :$$

$c_0 \in \quad c_1 \in \quad c_2 \in \quad \dots$

$x_0 = a \quad x_1 \quad x_2 \quad \dots \quad x_n = b$

$$\Delta x_k = x_k - x_{k-1} = \Delta x$$

$$\int_a^b f(x)\,dx = \lim_{\|P\| \to 0} \sum_{k=1}^{n} f(c_k)\,\Delta x_k$$

$$= \lim_{n \to \infty} \sum_{k=1}^{n} f(c_k)\,\frac{b-a}{n}$$

$$\min f \cdot (b-a) = \min f \sum_{k=1}^{n} \Delta x_k$$

$$= \sum_{k=1}^{n} \min f \cdot \Delta x_k$$

Over each $[x_{k-1}, x_k]$:

$$\min f \leq f(c_k) \leq \sum_{k=1}^{n} f(c_k)\,\Delta x_k \leq \sum_{k=1}^{n} (\max f)\,\Delta x_k$$

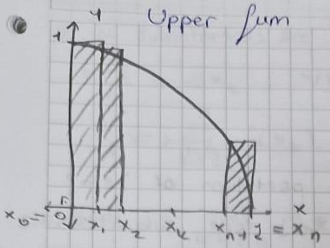
$$\lim_{n \to \infty} (\min f)(b-a) = \min f\,(b-a)$$

$$\lim_{n \to \infty} (\max f)(b-a) = \max f\,(b-a)$$

Squeeze Th. $\Longrightarrow$ $$(\min f)(b-a) \leq \lim_{n \to \infty} \sum_{k=1}^{n} f(c_k)\,\Delta x_k \leq (\max f)(b-a)$$

$$(\min f)(b-a) \leq \int_a^b f(x)\,dx \leq (\max f)(b-a)$$

Upper fum



$$\Delta x = \frac{1}{n}$$

$$A \cong \Delta x \left[f(x_0) + f(x_1) + \cdots + f(x_n) \right]$$

$$\cong \frac{1}{n} \cdot \sum_{k=0}^{n-1} \left(f(x_k) \right);$$

this approximation is always bound to be bigger than its exact value.

$$\sum_{k=1}^{n-1} k^2$$

$$US_n(A) = \frac{1}{n} \sum_{k=0}^{n-1} f(x_k) = \frac{1}{n} \cdot \sum_{k=0}^{n-1} f\left(\frac{k}{n}\right)$$

$$= \frac{1}{n} \sum_{k=0}^{n-1} \left(1 - \left(\frac{k}{n}\right)^2 \right) = \frac{1}{n} \left[\sum_{k=0}^{n-1} 1 - \frac{1}{n^2} \sum_{k=0}^{n-1} k^2 \right]$$

$$= \frac{1}{n} \left[n - \frac{1}{n^2} \frac{(n-1)n(2n-1)}{6} \right]$$

$$LS_n(A) = \frac{1}{n} \sum_{k=1}^{n} f(x_k) = \frac{1}{n} \sum_{k=1}^{n} f\left(\frac{k}{n}\right) = \frac{1}{n} \sum_{k=1}^{n} \left(1 - \left(\frac{k}{n}\right)^2\right)$$

$$= \frac{1}{n} \left(n - \frac{1}{n^2} \frac{n(n+1)(2n+1)}{6} \right.$$

$$LS(A) \leq A \leq US(A)$$

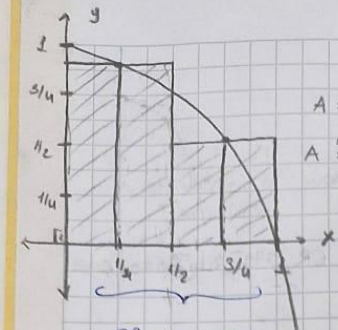
$$\lim_{n \to \infty} LS_n(A) \leq A \leq \lim_{n \to \infty} US_n(A)$$

$$\lim_{n \to \infty} \left(1 - \frac{(n+1)(2n+1)}{6n^2} \right) \qquad \lim_{n \to \infty} \left(1 - \frac{(2n-1)(n-1)}{6n^2} \right)$$

$$\Downarrow$$

$$1 - \frac{1}{3} = \frac{2}{3} \qquad 1 - \frac{1}{3} = \frac{2}{3}$$

Thus $A = \frac{2}{3}$.



Midpoint Rule

$$A \cong f\left(\tfrac{1}{4}\right)\cdot\tfrac{1}{2} + f\left(\tfrac{3}{4}\right)\cdot\tfrac{1}{2}$$

$$A \cong \tfrac{15}{16}\cdot\tfrac{1}{2} + \tfrac{7}{16}\cdot\tfrac{1}{2} = \tfrac{11}{16}$$

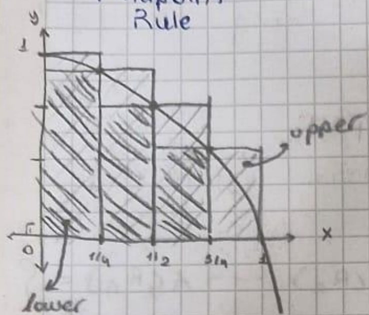
$x_0 = 0 \quad 1/4 \quad 1/2 \quad 3/4 \quad 1 = x_4$ (through x-axis)

$h_{R_1} = f(0)$ $\qquad \Delta x_1 = x_1 - x_0 = \frac{1}{4}$

$h_{R_2} = f\left(\frac{1}{4}\right)$ $\qquad \Delta x_2 = x_2 - x_1 = \frac{1}{4}$

$h_{R_3} = f\left(\frac{1}{2}\right)$ $\qquad \Delta x_3 = \Delta x_4 = \frac{1}{4}$

$h_{R_4} = f\left(\frac{3}{4}\right)$



Upper Sum:

$$A \cong f(0)\cdot\tfrac{1}{4} + f\left(\tfrac{1}{4}\right)\cdot\tfrac{1}{4} + f\left(\tfrac{1}{2}\right)\cdot\tfrac{1}{4} + f\left(\tfrac{3}{4}\right)\cdot\tfrac{1}{4}$$

$$\cong \tfrac{25}{32} = 0.78125$$

Lower Sum:

$h_{R_1} = f\left(\frac{1}{4}\right)$, $h_{R_2} = f\left(\frac{1}{2}\right)$, $h_{R_3} = f\left(\frac{3}{4}\right)$, $h_{R_4} = f(1)$

$$A \cong f\left(\tfrac{1}{4}\right)\cdot\tfrac{1}{4} + f\left(\tfrac{1}{2}\right)\cdot\tfrac{1}{4} + f\left(\tfrac{3}{4}\right)\cdot\tfrac{1}{4} + f(1)\cdot\tfrac{1}{4}$$

$$\cong \tfrac{17}{32} = 0.53125$$

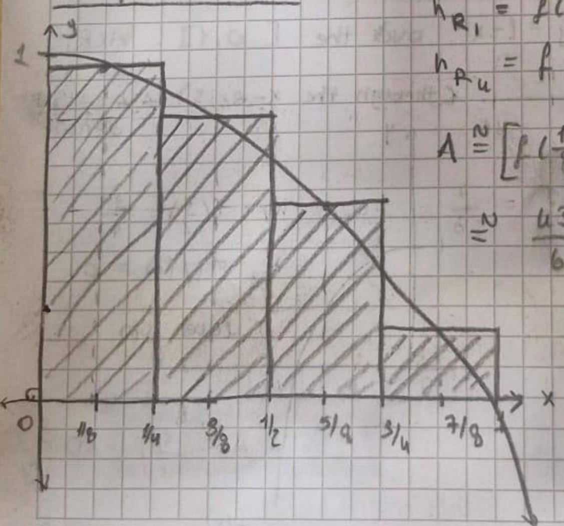
Midpoint Rule:

$h_{R_1} = f\left(\frac{1}{8}\right)$, $h_{R_2} = f\left(\frac{3}{8}\right)$, $h_{R_3} = f\left(\frac{5}{8}\right)$

$h_{R_4} = f\left(\frac{7}{8}\right)$

$$A \cong \left[f\left(\tfrac{1}{8}\right) + f\left(\tfrac{3}{8}\right) + f\left(\tfrac{5}{8}\right) + f\left(\tfrac{7}{8}\right)\right]\tfrac{1}{4}$$

$$\cong \tfrac{43}{64} = 0.67185$$



$1 - x^2 = f(x)$

Area and Estimating with Finite Sums

Motivations

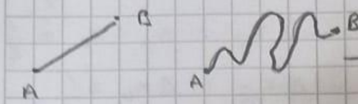
①

a non-regular shape → How to find the S?

②

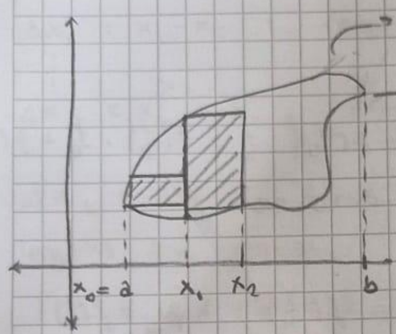
How to find its volume?

OR its surface area?



→ How to find ③ its length?

Idea: Divide into smaller calculatable parts



→ $f_2(x)$

→ $f_1(x)$

$A(R_1) + A(R_2) \cdots A(R_n) = S$

$\Downarrow$

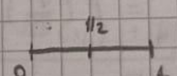
$[f_2(x_0) - f_1(x_0)]\,\Delta x_1$

$$S = \sum_{k=1}^{n} A(R_k) = \sum_{k=1}^{n} [f_2(x_k) - f_1(x_k)]\,\Delta x_k$$

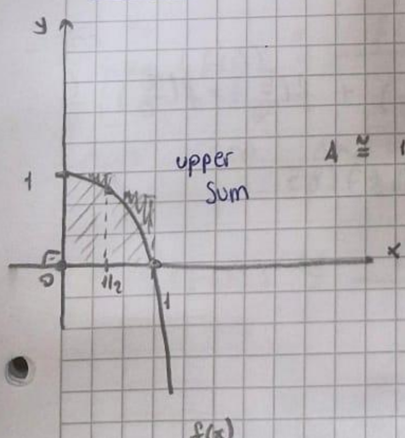
if $n \to \infty$, we could say $\cong A(R)$

the error (ϵ) must get as small as possible.

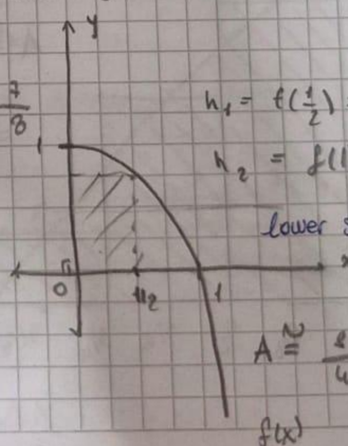
Examples Consider R below $y = 1 - x^2$ over the $[0,1]$. $A(R)$?



(through the x-axis) ** always denote



$A \cong 1 \cdot \frac{1}{2} + \frac{3}{4} \cdot \frac{1}{2} = \frac{7}{8}$



$h_1 = f(\frac{1}{2}) = \frac{3}{4}$

$h_2 = f(1) = 0$

lower sum

$A \cong \frac{3}{4} \cdot \frac{1}{2} + 0 = \frac{3}{8}$

Sigma Notation and Limits of Finite Sums

$\sum_{k=1}^{6} k = 1+2+\dots+6$ $\quad$ $\sum_{k=2}^{5} (2k-3) = 1+3+5+7$

$\sum_{k=1}^{4} (2k+1) = 1+3+\dots+9$

// $\sum_{k=1}^{n} (a_k \mp b_k) = \sum_{k=1}^{n} a_k \mp \sum_{k=1}^{n} b_k$

// $\sum_{k=1}^{n} c a_k = c \cdot \sum_{k=1}^{n} a_k$; for any constant c

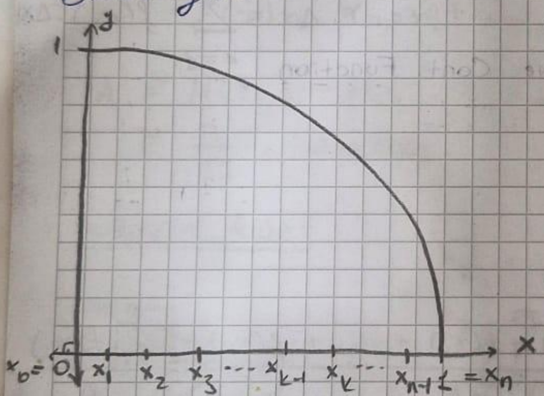
// $\sum_{k=1}^{n} c = nc \iff c \cdot \sum_{k=1}^{n} 1 = c\,(\underbrace{1+1+1\dots+1}_{n})$

// $\sum_{k=1}^{n} = \frac{n(n+1)}{2}$ $\quad$ // $\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$

proofs can be found on Internet

// $\sum_{k=1}^{n} k^3 = \left[\frac{n(n+1)}{2}\right]^2$

Lower Sum



$x_0 = 0$ —— $1 = x_n$

Choose $(n-1)$ points between 0 and 1 s.t.

$x_0 < x_1 < x_2 < \dots < x_{k-1} < x_k < \dots < x_n$

$I_1 = [x_0, x_1]$, $I_2 = [x_1, x_2] \dots$, $I_n = [x_{n-1}, x_n]$

$\Delta x_1 = x_2 - x_1$, $\Delta x_2 = x_2 - x_1$

$\Delta x_k = x_k - x_{k-1} \dots \Delta x_n = x_n - x_{n-1}$

If we assume that:

$\Delta x_0 = \Delta x_1 = \Delta x_2 = \dots = \Delta x_k = \dots = \Delta x_n = \Delta x$

then $\Delta x = \frac{1}{n}$.

$A \cong f(x_1)\Delta x + f(x_2)\Delta x \dots + f(x_n)\Delta x$

$\cong \frac{1}{n} \sum_{K=1}^{n} f(x_k)$; lower sum always gives an app. that is smaller than exact value

✱ if $p \neq 1$, then $I = \int_1^\infty \frac{dx}{x^p} = \int_1^\infty x^{-p}\,dx$

$f(x) = x^{-p}$ is cont on $[1, +\infty)$ & integ.

I is an improp. integ. of type 1

$\lim_{t\to\infty} \int_1^t x^{-p}\,dx \Rightarrow \lim_{t\to\infty} \frac{x^{-p+1}}{-p+1}\Big|_1^t \Rightarrow$

$\Rightarrow \lim_{t\to\infty}\left(\frac{t^{1-p}}{1-p} - \frac{1}{1-p}\right) \Rightarrow \lim_{t\to\infty}\left(\frac{t^{1-p}-1}{1-p}\right) \Rightarrow$

$\Rightarrow \frac{1}{1-p}\lim_{t\to\infty}(t^{1-p}-1) \Rightarrow \begin{cases} +\infty, & 1-p>0 \quad 1>p \\ 1/_{1-p}, & 1-p<0 \quad 1<p \end{cases}$

if $p \leq 1$, I is divergent

$p > 1$, I ~~congress~~ the $1/_{p-1}$ convergent.

$a>0$, conv. the beh of $\int_a^\infty dx/x^p$

Unbounded

Improper Integrals by Type 2.

$$\left[\int_1^\infty \frac{dx}{x^p} \begin{array}{lll} \text{is} & \text{divergent} & p \leq 1 \\ \text{is} & \text{convergent} & p > 1 \end{array}\right]$$

Integrals with unbounded integrands on the Domain of integration are improper integrals of Type II (2)

① f is cont on $[a, b[$, dis cont on b

$\int_a^b f(x)\,dx$ ⇒ Type 2.

a t b $\lim_{t\to b^-} \int_a^t f(x)\,dx$ $\left(\lim_{t\to a^+} \int_t^b f(x)\,dx\right)$ a b $x=a$ discont.

Over $[a, t]$ f is ∞b

$\int_a^b f(x)\,dx$ congress to the limit value

Otherwise, it is divergent.