New results for the Degree/Diameter Problem

Yawara ISHIDA

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Abstract

The Degree/Diameter Problem is one of the most famous problem in graph theory. We consider the problem for the case of diameter 2. A lower bound of the order of $(d, 2)$-graph is known as the Brown's construction. In this paper, we generalized the construction. Then we found two new records of degree 306 and 307 by our new construction. One is $(306, 2)$-graph with 88723 vertices, the other is $(307, 2)$-graph with 88724 vertices.

1 Introduction

A *graph* $G = (V, E)$ consists of a set V called *vertices* and a set $E \subset \{(v, w) \in V^2 | v \neq w\}$ called *edges*. If (v, w) is in E, it is said that v and w are *adjacent*, which is denoted by $v \sim w$. The *order* $|G|$ of the graph is the size of the set of vertices. The *neighbors* $N(v)$ of the vertex v is a set of vertices which are adjacent to v. The *degree* of the vertex $\delta(v)$ is a size of neighbors $|N(v)|$. The *degree* of the graph $\Delta(G)$ is the maximal degree of the vertex. The *distance* of the pair (v, w) of vertices is the shortest path length between v and w, which is denoted $d(v, w)$. The *diameter* $D(G)$ of the graph is the maximum distance of all pairs of vertices.

The *Degree/Diameter problem* is the problem of finding the largest possible number of vertices in a graph of given degree Δ and diameter D [3] [1]. The order of the graph of degree Δ ($\Delta > 2$) and diameter D is easily seen to be bounded by

$$1 + \Delta \sum_{k=1}^{D-1} (\Delta - 1)^k$$

which is called *Moore bound*. Moore bound is a general upper bound. On the other hand, already known lower bound of small degree and small diameter are availabe at `http://combinatoricswiki.org`. Especially for case of $D = 2$ and large degree there exists the general construction called the *Brown's construction* [3]. Given the finite field F_q where q is a power of prime, we can construct the graph $\mathrm{B}(F_q)$ whose vertices are lines in F_q^3 and two lines are adjacent if and only if they are orthogonal. We call it the Brown's graph. The order of $B(F_q)$ is $q^2 + q + 1$ and the degree of it is $q + 1$. The diameter of it is 2 because $B(F_q)$

includes many triangles. F_q^3 is isotropic so any lines are symmetric in F_q^3, the degree of the vertex of $\mathrm{B}(F_q)$ is $q+1$ or q. There exists $q+1$ vertices of degree q in $\mathrm{B}(F_q)$. If q is a power of 2, there exists $(q+1,2)$-graph with q^2+q+2 vertices by modifying $\mathrm{B}(F_q)$ [2]. In this paper, we generalize the Brown's construction, in which we replace a field with a commutative ring, and search new records of the Degree/Diameter Problem.

Let R be a commutative ring with unity. R^* denotes the set of invertible elements of R. R^3 is naturally seen as R-module. The addition and R-action are defined by coordinate-wise. The *inner product* $\cdot : R^3 \times R^3 \Rightarrow R$ is defined as follows

$$(v_1, v_2, v_3) \cdot (w_1, w_2, w_3) = v_1w_1 + v_2w_2 + v_3w_3$$

$\boldsymbol{v}$ and $\boldsymbol{w}$ are *orthogonal* if and only if the inner product vanishes, namely $\boldsymbol{v}\cdot\boldsymbol{w} = 0$. The *cross product* $\times : R^3 \times R^3 \Rightarrow R$ is defined as follows

$$(v_1, v_2, v_3) \cdot (w_1, w_2, w_3) = (v_2w_3 - v_3w_2, v_3w_1 - v_1w_3, v_1w_2 - v_2w_1)$$

A *domain* D is a commutative ring without zero divisors. A *Euclidean domain* is a domain E with a function called degree $d : E \setminus \{0\} \Rightarrow \mathbb{N}$ such that for all non-zero a, b in E there exists $q, r \in E, a = qb + r$ where $d(r) < d(b)$. Every *Euclidean domain* is a *unique factorization domain*, in which for all r in E, there exist prime elements u_i and natural numbers k_i such that $r = \Pi_i u_i^{k_i}$. The ring of integers $\mathbb{Z}$ is a example of the Euclidean domains whose degree function is an identity function.

2 Generalized Brown's Construction

Definition 1. *Let R be a commutative ring with unit. The vertex set V of the generalized Brown's graph* $\mathrm{B}(R)$ *is*

$$V = (R^3 \setminus \{\boldsymbol{v} | \exists r \in R, r \cdot \boldsymbol{v} = \boldsymbol{0}\}) / \sim$$

where $\boldsymbol{v} \sim \boldsymbol{w}$ if and only if there exists $k \in R^*$ such that $k \cdot \boldsymbol{v} = \boldsymbol{w}$. The two vertices $[\boldsymbol{v}]$ and $[\boldsymbol{w}]$ are adjacent if and only if $\boldsymbol{v} \cdot \boldsymbol{w} = 0$.

The adjacency of the definition above is well-defined because the orthogonality does not depends on the selection of representitives. We call the above construction which gives a graph from a ring the *generalized Brown's construction.* It is clear that the new construction coincides old one when the ring is a field.

Lemma 1. *Let E be a Euclidean domain and u be a prime element in E. If $E/(u^k)$ is a finite ring, then the degree of the vertex of* $\mathrm{B}(E/(u^k))$ *is Δ or $\Delta - 1$, where (u^k) is the principal ideal generated by u^k.*

Proof. It is clear that the degrees of vertices represented by $(1, 0, 0), (0, 1, 0), (0, 0, 1)$ are same. Let $\boldsymbol{v} = ([a], [b], [c])$ be a representitive of any vertex where $a, b, c \in$

E. If any element of $[a],[b],[c]$ is not invertible in $E/(u^k)$, there exist natural numbers $1 \leq l,m,n < k$ and some elements a',b',c' in E such that $a = u^l a', b = u^m b', c = u^n c'$. $\boldsymbol{v}$ is not a representitive of vertices, the equation $[u^{min(l,m,n)}]\cdot \boldsymbol{v} = ([0],[0],[0])$ holds. This is a contradiction. Therefore, at leatst one element of $[a],[b],[c]$ is invertible. If $[a]$ is invertible, there exists one-to-one correspondence $\overline{U} : N([(1,0,0)]) \to N([\boldsymbol{v}])$ such that for all $[\boldsymbol{w}] \in N([1,0,0])$, $\overline{U}([\boldsymbol{w}]) = [{}^t U^{-1}\boldsymbol{w}]$ where U is an invertible matrix defined as follows

$$U = \begin{pmatrix} [a] & 0 & 0 \\ [b] & 1 & 0 \\ [c] & 0 & 1 \end{pmatrix}$$

If $\boldsymbol{v}\cdot\boldsymbol{v} = 0$, then $\delta([\boldsymbol{v}]) = \delta([(1,0,0)] - 1$. If not so, $\delta([\boldsymbol{v}]) = \delta([(1,0,0)])$. In the same way, if $[b]/[c]$ is invertible, then $\delta([\boldsymbol{v}])$ is $\delta([(0,1,0)])/\delta([(0,0,1)])$ or $\delta([(1,0,0)]/\delta([(0,0,1)]) - 1$. Therefore, for all the vertex $[\boldsymbol{v}]$,

$$\delta([\boldsymbol{v}]) = \delta([(1,0,0)]) = \Delta$$

$\square$

Lemma 2. *Let E be a Euclidean domain and I be an ideal of E. The diameter of* $\mathrm{B}(E/I)$ *is 2.*

Proof. For any two distinct vertices represented by $\boldsymbol{v} = ([v_1],[v_2],[v_3])$ and $\boldsymbol{w} = ([w_1],[w_2],[w_3])$, consider the cross product $\boldsymbol{v}\times\boldsymbol{w}$. If $\boldsymbol{v}\times\boldsymbol{w} = \mathbf{0}$, then $[v_i]\cdot\boldsymbol{w} = [w_i]\cdot\boldsymbol{v}$ for $i = 1,2,3$. There exists $e \in E$ such that $I = (e)$ because any Euclidean domain is a principal ideal domain. If $\gcd(v_1,v_2,v_3,e)$ is not a unity, where gcd is a greatest common divisor, there exists $e' \neq 1$ in E such that $e = de'$. $\boldsymbol{v}$ is not a representitive because $[e']\cdot\boldsymbol{v} = \mathbf{0}$. This is a contradiction. Therefore d is a unity, namely v_1 and v_2, v_3, e are coprime. Then there exist $a,b,c,d \in E$ such that $av_1 + bv_2 + cv_3 + de = 1$ in E. Seeing this formula in E/I, we get $[a][v_1] + [b][v_2] + [c][v_3] = [1]$.

$$\boldsymbol{v} = [1]\cdot\boldsymbol{v} = ([a][v_1] + [b][v_2] + [c][v_3])\boldsymbol{v} = ([a][w_1] + [b][w_2] + [c][w_3])\boldsymbol{w}$$

means $[\boldsymbol{v}] = [\boldsymbol{w}]$, which is a contradiction to that two vertices are distinct, then $\boldsymbol{v}\times\boldsymbol{w} \neq \mathbf{0}$. If $\boldsymbol{v}\times\boldsymbol{w}$ is a representitive of vertex, $[\boldsymbol{v}\times\boldsymbol{w}]$ is adjacent to $[\boldsymbol{v}]$ and $[\boldsymbol{w}]$. If $\boldsymbol{v}\times\boldsymbol{w} = ([k_1],[k_2],[k_3])$ is not a representitive of vertex, $\boldsymbol{v}\times\boldsymbol{w} = [\gcd(k_1,k_2,k_3)]\cdot\boldsymbol{u}$ and $\boldsymbol{u}$ is a representitive of vertex. $[\boldsymbol{u}]$ is adjacent to $[\boldsymbol{v}]$ and $[\boldsymbol{w}]$. $\square$

Theorem. *The following equations hold.*

1. $|\mathrm{B}(\mathbb{Z}_{p^k})| = p^{2k} + p^{2k-1} + p^{2k-2}$
2. $\Delta(\mathrm{B}(\mathbb{Z}_{p^k})) = p^k + p^{k-1}$
3. $D(\mathbb{Z}_{p^k}) = 2$

Proof. It is straightforward to show the formula of the order of $\mathrm{B}(\mathbb{Z}_{p^k})$.

$$\begin{aligned}
|\mathrm{B}(\mathbb{Z}_{p^k})| &= \frac{|\mathbb{Z}_{p^k}|^3 - |\{mp|0 \leq m < k\}|^3}{|\mathbb{Z}_{p^k}| - |\{mp|0 \leq m < k\}|} \\
&= \frac{(p^k)^3 - (p^{k-1})^3}{p^k - p^{k-1}} = p^{2k} + p^{2k-1} + p^{2k-2}
\end{aligned}$$

Using Lemma 1, it is only enough to show that the degree of the vertex represented by $(1,0,0)$ satisfy the formula of the degree of $\mathrm{B}(\mathbb{Z}_{p^k})$.

$$\begin{aligned}
\Delta(\mathrm{B}(\mathbb{Z}_p^k)) &= \delta([(1,0,0)]) = \frac{|\mathbb{Z}_{p^k}|^2 - |\{mp|0 \leq m < k\}|^2}{|\mathbb{Z}_{p^k}| - |\{mp|0 \leq m < k\}|} \\
&= \frac{(p^k)^2 - (p^{k-1})^2}{p^k - p^{k-1}} = p^k + p^{k-1}
\end{aligned}$$

Using Lemma 2, we get $D(\mathrm{B}(\mathbb{Z}_p^k)) = 2$

□

We search new records among graphs by generalized Brown's construction. Using the above theorem, $\mathrm{B}(\mathbb{Z}_{17^2})$ has degree 306 and diameter 2 and 88723 vertices. It is a new record of $(306, 2)$ of the Degree-Diameter Problem because it cannot be obtained from ordinary Brown's construction. The power of prime less than $305 = 306 - 1$ is 293^1 and the graph $\mathrm{B}(\mathbb{Z}_{293})$ obtained from ordinary Brown's construction of 293^1 has $294 = 293+1$ degree and $86143 = 293^2+293+1$ vertices. The old record of $306 = 294 + 12$ is $86156 = 86143 + 12$ obtained from $\mathrm{B}(\mathbb{Z}_{293})$ by duplicating vertices. In the same way, the graph obtained from $\mathrm{B}(\mathbb{Z}_{17^2})$ by duplicating any one vertex, whose order is 88724, is a new record of $(307, 2)$ because the power of prime less than $306 = 307 - 1$ is 293^1.

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