New results for the Degree/Diameter Problem

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July 25, 2015

Abstract

The Degree/Diameter Problem is one of the most famous problem in graph theory. We consider the problem for the case of diameter 2. A lower bound of the order of $(d, 2)$-graph is known as the Brown's construction. In this paper, we generalized his construction. Then we found two new records of degree 306 and 307 in the graphs generated by our new construction. One is $(306, 2)$-graph with 88732 vertices. The other is $(307, 2)$-graph with 88733 vertices.

1 Introduction

A *graph* $G = (V, E)$ consists of a set V called *vertices* and a set $E \subset \{(v, w) \in V^2 | v \neq w\}$ called *edges*. If (v, w) is in E, it is said that v and w are *adjacent*, which is denoted by $v \sim w$. The *order* $|G|$ of the graph is the size of the set of vertices. The *neighbors* $N(v)$ of the vertex v is a set of vertices which are adjacent to v. The *degree* of the vertex $\delta(v)$ is a size of neighbors $|N(v)|$. The *degree* of the graph $\Delta(G)$ is the maximal degree of the vertex. The *distance* for each pair (v, w) of vertices is the shortest path length between v and w. The *diameter* $D(G)$ of the graph is the maximum distance of all pairs of vertices.

The *Degree/Diameter problem* is the problem of finding the largest possible number of vertices in graphs of given degree Δ and diameter D [2]. The order of the graph of degree Δ $(\Delta > 2)$ and diameter D is easily seen to be bounded by

$$1 + \Delta \sum_{k=1}^{D-1} (\Delta - 1)^k$$

which is called *Moore bound*. Moore bound is a general upper bound. On the other hand, already known lower bound of small degree and small diameter are availabe at `http://combinatoricswiki.org`. Especially for case of $D = 2$ and large degree there exists the general construction called the *Brown's construction* [2]. Given the finite field F_q where q is a power of prime, we construct the graph $\mathrm{B}(F_q)$ whose vertices are lines in F_q^3 and two lines are adjacent if and only if they are orthogonal. We call it the Brown's graph. The order of $B(F_q)$ is $q^2 + q + 1$ and the degree of it is $q + 1$. The diameter of it is 2 because $B(F_q)$

includes many triangles. F_q is isotropic so any lines are symmetric in F_q, the degree of the vertex of $\mathrm{B}(F_q)$ is $q+1$ or q. There exists $q+1$ vertices of degree q in $\mathrm{B}(F_q)$. If q is a power of 2, there exists $(q+1,2)$-graph with q^2+q+2 vertices by modifying $\mathrm{B}(F_q)$ [1]. In this paper, we generalize the Brown's construction, in which we replace a field with a ring, and search new records of the Degree/Diameter Problem.

Let R be a ring with unity. R^* denotes the set of invertible elements of R. R^3 is naturally seen as R-module. The addition and R-action are defined by coordinate-wise. The *inner product* $\cdot : R^3 \times R^3 \Rightarrow R$ is defined as follows

$$(v_1, v_2, v_3) \cdot (w_1, w_2, w_3) = v_1w_1 + v_2w_2 + v_3w_3$$

$\boldsymbol{v}, \boldsymbol{w}$ are *orthogonal* if and only if the inner product vanishes, namely $\boldsymbol{v} \cdot \boldsymbol{w} = 0$. The *cross product* $\times : R^3 \times R^3 \Rightarrow R$ is defined as follows

$$(v_1, v_2, v_3) \cdot (w_1, w_2, w_3) = (v_2w_3 - v_3w_2, v_3w_1 - v_1w_3, v_1w_2 - v_2w_1)$$

A *domain* D is a ring without zero divisors. A *Euclidean domain* is a domain E with a function called degree $d : E \setminus \{0\} \Rightarrow \mathbb{N}$ such that for all non-zero $a, b \in E$ there exists $q, r \in E, a = qb + r$ where $d(r) < d(b)$. The ring of integers $\mathbb{Z}$ is a example of the Euclidean domains.

2 Generalized Brown's Construction

Definition 1. *Let R be a ring with unit. The vertex set V of the generalized Brown's graph $\mathrm{B}(R)$ is*

$$V = (R^3 \setminus \{\boldsymbol{v} | \exists r \in R, r \cdot \boldsymbol{v} = \boldsymbol{0}\})/\sim$$

where $\boldsymbol{v} \sim \boldsymbol{w}$ if and only if there exists $k \in R^*$ such that $k \cdot \boldsymbol{v} = \boldsymbol{w}$. The two vertices $[\boldsymbol{v}], [\boldsymbol{w}]$ are adjacent if and only if $\boldsymbol{v} \cdot \boldsymbol{w} = 0$.

The adjacency of the definition above is well-defined because the orthogonality does not depends on the selection of representitives. We call the above construction which gives a graph from a ring the *generalized Brown's construction.* It is clear that the new construction coincides old one when the ring is a field.

Lemma 1. *Let E be a Euclidean domain and u be a prime element in E. If $E/(u^k)$ is a finite ring, then $\mathrm{B}(E/(u^k))$ is a regular graph, where (u^k) is a principal ideal generated by u^k. Note that $\mathrm{B}(E/(u^k))$ is not simple.*

Proof. It is clear that the degrees of vertices represented by $(1,0,0), (0,1,0), (0,0,1)$ are same. Let $\boldsymbol{v} = ([a], [b], [c])$ be a representitive of any vertex where $a, b, c \in E$. If any element of $[a], [b], [c]$ is not invertible in $E/(u^k)$, there exists natural numbers $l, m, n < k$ and some elements a', b', c' in E such that $a = u^l a', b = u^m b', c = u^n c'$. $\boldsymbol{v}$ is not a representitive of vertices because of $[u^{min(l,m,n)}] \cdot \boldsymbol{v} = ([0], [0], [0])$. Therefore, one element of $[a], [b], [c]$ is at leatst invertible. If $[a]$ is invertible, there exists one-to-one correspondence $\overline{U} : N([(1,0,0)]) \to N([\boldsymbol{v}])$ such that for all $[\boldsymbol{w}] \in N([1,0,0])$, $\overline{U}([\boldsymbol{w}]) = [{}^t U^{-1} \boldsymbol{w}]$ where U is an invertible matrix defined as follows

$$U = \begin{pmatrix} [a] & 0 & 0 \\ [b] & 1 & 0 \\ [c] & 0 & 1 \end{pmatrix}$$

Therefore, $\delta([\boldsymbol{v}]) = \delta([(1,0,0)])$. In the same way, if $[b]/[c]$ is invertible, then $\delta([\boldsymbol{v}]) = \delta([(0,1,0)])/\delta([(0,0,1)])$.

□

Lemma 2. *Let E be a Euclidean domain and I be an ideal of E. The diameter of* $\mathrm{B}(E/I)$ *is 2.*

Proof. For any two distinct vertices represented by $\boldsymbol{v} = ([v_1],[v_2],[v_3])$ and $\boldsymbol{w} = ([w_1],[w_2],[w_3])$, consider the cross product $\boldsymbol{v} \times \boldsymbol{w}$. If $\boldsymbol{v} \times \boldsymbol{w} = \boldsymbol{0}$, then $[v_i] \cdot \boldsymbol{w} = [w_i] \cdot \boldsymbol{v}$ for $i = 1,2,3$. There exists $e \in E$ such that $I = (e)$ because any Euclidean domain is a princpal ideal domain. Let d be the greatest common divisor of v_1 and v_2, v_3, e. If d is not a unity, there exists $e' \neq 1$ in E such that $e = de'$. $\boldsymbol{v}$ is not a representitive because $[e'] \cdot \boldsymbol{v} = \boldsymbol{0}$. Therefore d is a unity, namely v_1 and v_2, v_3, e are coprime. Then there exist $a,b,c,d \in E$ such that $av_1 + bv_2 + cv_3 + de = 1$ in E. Seeing this formula in E/I, we get $[a][v_1] + [b][v_2] + [c][v_3] = [1]$.

$$\boldsymbol{v} = [1] \cdot \boldsymbol{v} = ([a][v_1] + [b][v_2] + [c][v_3])\boldsymbol{v} = ([a][w_1] + [b][w_2] + [c][w_3])\boldsymbol{w}$$

means $[\boldsymbol{v}] = [\boldsymbol{w}]$, which is a contradiction then $\boldsymbol{v} \times \boldsymbol{w} \neq \boldsymbol{0}$. If $\boldsymbol{v} \times \boldsymbol{w}$ is a representitive of vertex, $[\boldsymbol{v} \times \boldsymbol{w}]$ is adjacent to $[\boldsymbol{v}]$ and $[\boldsymbol{w}]$. If $\boldsymbol{v} \times \boldsymbol{w} = ([k_1],[k_2],[k_3])$ is not a representitive of vertex, there exist $[d]$ in E/I, where d is a greatest common divisor of k_1, k_2 and k_3, and $\boldsymbol{u}$ in $(E/I)^3$ such that $\boldsymbol{v} \times \boldsymbol{w} = [d] \cdot \boldsymbol{u}$ and $\boldsymbol{u}$ is a representitive of vertex. $[\boldsymbol{u}]$ is adjacent to $[\boldsymbol{v}]$ and $[\boldsymbol{w}]$. □

Theorem. *The following equations hold.*

1. $|\mathrm{B}(\mathbb{Z}_{p^k})| = p^{2k} + p^{2k-1} + p^{2k-2}$
2. $\Delta(\mathrm{B}(\mathbb{Z}_{p^k})) = p^k + p^{k-1}$
3. $D(\mathbb{Z}_{p^k}) = 2$

Proof. It is straightforward to show the formula of the order of $\mathrm{B}(\mathbb{Z}_{p^k})$.

$$\begin{aligned} |\mathrm{B}(\mathbb{Z}_{p^k})| &= \frac{|\mathbb{Z}_{p^k}|^3 - |\{mp | 0 \leq m < k\}|^3}{|\mathbb{Z}_{p^k}| - |\{mp | 0 \leq m < k\}|} \\ &= \frac{(p^k)^3 - (p^{k-1})^3}{p^k - p^{k-1}} = p^{2k} + p^{2k-1} + p^{2k-2} \end{aligned}$$

It is only enough to show that the degree of the vertex represented by $(1,0,0)$ satisfy the formula of the degree of $\mathrm{B}(\mathbb{Z}_{p^k})$. It is clear that $\mathbb{Z}_p^k$ satisfies the assumption of Lemma 1, then $\mathrm{B}(\mathbb{Z}_p^k)$ is a regular graph.

$$\begin{aligned}\Delta(\mathrm{B}(\mathbb{Z}_p^k)) &= \delta([(1,0,0)]) = \frac{|\mathbb{Z}_{p^k}|^2 - |\{mp|0 \leq m < k\}|^2}{|\mathbb{Z}_{p^k}| - |\{mp|0 \leq m < k\}|} \\ &= \frac{(p^k)^2 - (p^{k-1})^2}{p^k - p^{k-1}} = p^k + p^{k-1}\end{aligned}$$

Applying Lemma 2, we get $D(\mathrm{B}(\mathbb{Z}_p^k)) = 2$

□

We search new records among graphs by generalized Brown's construction.

3 Acknowledgement

Thank you, Ryosuke Mizuno, Nobuhito Tamaki, Sakie Suzuki !!!

References

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