A Generalization of Brown's Construction for the Degree/Diameter Problem

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Abstract

The degree/diameter problem is one of the most famous problem in graph theory. We consider the problem for the case of diameter 2. William G Brown gave a lower bound of the order of $(\Delta, 2)$-graph. In this paper, we give a generalization of his construction and improve the lower bounds for the case of $\Delta = 306$ and $\Delta = 307$. One is $(306, 2)$-graph with 88723 vertices, the other is $(307, 2)$-graph with 88724 vertices.

1 Introduction

A *graph* $G = (V, E)$ consists of a set V of *vertices* and a set $E \subset \{(v, w) \in V^2 | v \neq w\}$ of *edges*. If (v, w) is in E, it is said that v and w are *adjacent*, which is denoted by $v \sim w$. The *order* $|G|$ of the graph is the number of vertices. The *neighbors* $N(v)$ of the vertex v is a set of vertices which are adjacent to v. The *degree* of the vertex $\delta(v)$ is the number of neighbors $|N(v)|$. The *degree* of the graph $\Delta(G)$ is the maximum degree of vertices. The *distance* of the pair (v, w) of vertices is the shortest path length between v and w. The *diameter* $D(G)$ of the graph is the maximum distance of all pairs of vertices.

The *degree/diameter problem* is the problem of finding the largest possible number of vertices $n_{\Delta,D}$ in a graph of given degree Δ and diameter D [3] [1]. Let G be a graph with degree Δ ($\Delta > 2$) and diameter D, then we have

$$|G| \leq 1 + \Delta \sum_{k=1}^{D-1} (\Delta - 1)^k$$

The right hand side of the above equation is called *Moore bound*. Moore bound is a general upper bound. On the other hand, already known lower bound of small degree and small diameter are availabe at `http://combinatoricswiki.org`. Especially for case of $D = 2$ and large degree, there exists the general construction called the *Brown's construction* [3]. Given the finite field F_q where q is a power of prime, we can construct the graph $\mathrm{B}(F_q)$ whose vertices are lines in F_q^3 and two lines are adjacent if and only if they are orthogonal. We call it the

Brown's graph. The order of $B(F_q)$ is q^2+q+1 and the degree of it is $q+1$. The diameter of it is 2 because $B(F_q)$ includes many triangles. F_q^3 is isotropic so any lines are symmetric in F_q^3, the degree of the vertex of $\mathrm{B}(F_q)$ is $q+1$ or q. There exists $q+1$ vertices of degree q in $\mathrm{B}(F_q)$. If q is a power of 2, there exists $(q+1,2)$-graph with q^2+q+2 vertices by modifying $\mathrm{B}(F_q)$ [2]. In this paper, we generalize the Brown's construction, in which we replace a field with a commutative ring, and search new records of the degree/diameter problem.

Let R be a commutative ring with unity. R^* denotes the set of invertible elements of R. R^3 is naturally seen as R-module. The addition and R-action are defined by coordinate-wise. The *inner product* $\cdot : R^3 \times R^3 \Rightarrow R$ is defined as follows

$$(v_1,v_2,v_3)\cdot(w_1,w_2,w_3) = v_1w_1+v_2w_2+v_3w_3$$

$\boldsymbol{v}$ and $\boldsymbol{w}$ are *orthogonal* if and only if the inner product vanishes, namely $\boldsymbol{v}\cdot\boldsymbol{w} = 0$. The *cross product* $\times : R^3 \times R^3 \Rightarrow R$ is defined as follows

$$(v_1,v_2,v_3)\cdot(w_1,w_2,w_3) = (v_2w_3 - v_3w_2, v_3w_1 - v_1w_3, v_1w_2 - v_2w_1)$$

A *domain* D is a commutative ring without zero divisors. A *Euclidean domain* is a domain E with a function called degree $d : E \setminus \{0\} \Rightarrow \mathbb{N}$ such that for all non-zero a,b in E there exists $q,r \in E, a = qb+r$ where $d(r) < d(b)$. Every *Euclidean domain* is a *unique factorization domain*, in which for all r in E, there exist prime elements u_i and natural numbers k_i such that $r = \Pi_i u_i^{k_i}$. The ring of integers $\mathbb{Z}$ is a example of the Euclidean domains whose degree function is an identity function.

2 Generalized Brown's Construction

Definition 1. *Let R be a commutative ring with unit. The vertex set V of the generalized Brown's graph $\mathrm{B}(R)$ is*

$$V = (R^3 \setminus \{\boldsymbol{v} | \exists r \in R, r\cdot\boldsymbol{v} = \boldsymbol{0}\})/\sim$$

where $\boldsymbol{v} \sim \boldsymbol{w}$ if and only if there exists $k \in R^*$ such that $k\cdot\boldsymbol{v} = \boldsymbol{w}$. The two vertices $[\boldsymbol{v}]$ and $[\boldsymbol{w}]$ are adjacent if and only if $\boldsymbol{v}\cdot\boldsymbol{w} = 0$.

The adjacency of the definition above is well-defined because the orthogonality does not depends on the selection of representitives. We call the above construction which gives a graph from a ring the *generalized Brown's construction*. It is clear that the new construction coincides old one when the ring is a field.

Lemma 1. *Let E be a Euclidean domain and u be a prime element in E. If $E/(u^k)$ is a finite ring, then the degree of the vertex of $\mathrm{B}(E/(u^k))$ is Δ or $\Delta - 1$, where (u^k) is the principal ideal generated by u^k.*

Proof. It is clear that the degrees of vertices represented by $(1,0,0),(0,1,0),(0,0,1)$ are same. Let $\boldsymbol{v} = ([a],[b],[c])$ be a representitive of any vertex where $a,b,c \in$

E. If any element of $[a], [b], [c]$ is not invertible in $E/(u^k)$, there exist natural numbers $1 \le l, m, n < k$ and some elements a', b', c' in E such that $a = u^l a', b = u^m b', c = u^n c'$. $\boldsymbol{v}$ is not a representitive of vertices, the equation $[u^{min(l,m,n)}] \cdot \boldsymbol{v} = ([0], [0], [0])$ holds. This is a contradiction. Therefore, at leatst one element of $[a], [b], [c]$ is invertible. If $[a]$ is invertible, there exists one-to-one correspondence $\overline{U} : N([(1, 0, 0)]) \to N([\boldsymbol{v}])$ such that for all $[\boldsymbol{w}] \in N([1, 0, 0])$, $\overline{U}([\boldsymbol{w}]) = [{}^t U^{-1} \boldsymbol{w}]$ where U is an invertible matrix defined as follows

$$U = \begin{pmatrix} [a] & 0 & 0 \\ [b] & 1 & 0 \\ [c] & 0 & 1 \end{pmatrix}$$

If $\boldsymbol{v} \cdot \boldsymbol{v} = 0$, then $\delta([\boldsymbol{v}]) = \delta([(1, 0, 0)] - 1$. If not so, $\delta([\boldsymbol{v}]) = \delta([(1, 0, 0)])$. In the same way, if $[b]/[c]$ is invertible, then $\delta([\boldsymbol{v}])$ is $\delta([(0, 1, 0)])/\delta([(0, 0, 1)])$ or $\delta([(1, 0, 0)]/\delta([(0, 0, 1)]) - 1$. Therefore, for all the vertex $[\boldsymbol{v}]$,

$$\delta([\boldsymbol{v}]) = \delta([(1, 0, 0)]) = \Delta$$

□

Lemma 2. *Let E be a Euclidean domain and I be an ideal of E. The diameter of* $\mathrm{B}(E/I)$ *is 2.*

Proof. For any two distinct vertices represented by $\boldsymbol{v} = ([v_1], [v_2], [v_3])$ and $\boldsymbol{w} = ([w_1], [w_2], [w_3])$, consider the cross product $\boldsymbol{v} \times \boldsymbol{w}$. If $\boldsymbol{v} \times \boldsymbol{w} = \mathbf{0}$, then $[v_i] \cdot \boldsymbol{w} = [w_i] \cdot \boldsymbol{v}$ for $i = 1, 2, 3$. There exists $e \in E$ such that $I = (e)$ because any Euclidean domain is a principal ideal domain. If $\gcd(v_1, v_2, v_3, e)$ is not a unity, where gcd is a greatest common divisor, there exists $e' \neq 1$ in E such that $e = de'$. $\boldsymbol{v}$ is not a representitive because $[e'] \cdot \boldsymbol{v} = \mathbf{0}$. This is a contradiction. Therefore d is a unity, namely v_1 and v_2, v_3, e are coprime. Then there exist $a, b, c, d \in E$ such that $av_1 + bv_2 + cv_3 + de = 1$ in E. Seeing this formula in E/I, we get $[a][v_1] + [b][v_2] + [c][v_3] = [1]$.

$$\boldsymbol{v} = [1] \cdot \boldsymbol{v} = ([a][v_1] + [b][v_2] + [c][v_3])\boldsymbol{v} = ([a][w_1] + [b][w_2] + [c][w_3])\boldsymbol{w}$$

means $[\boldsymbol{v}] = [\boldsymbol{w}]$, which is a contradiction to that two vertices are distinct, then $\boldsymbol{v} \times \boldsymbol{w} \neq \mathbf{0}$. If $\boldsymbol{v} \times \boldsymbol{w}$ is a representitive of vertex, $[\boldsymbol{v} \times \boldsymbol{w}]$ is adjacent to $[\boldsymbol{v}]$ and $[\boldsymbol{w}]$. If $\boldsymbol{v} \times \boldsymbol{w} = ([k_1], [k_2], [k_3])$ is not a representitive of vertex, $\boldsymbol{v} \times \boldsymbol{w} = [\gcd(k_1, k_2, k_3)] \cdot \boldsymbol{u}$ and $\boldsymbol{u}$ is a representitive of vertex. $[\boldsymbol{u}]$ is adjacent to $[\boldsymbol{v}]$ and $[\boldsymbol{w}]$. □

Theorem. *The following equations hold.*

1. $|\mathrm{B}(\mathbb{Z}_{p^k})| = p^{2k} + p^{2k-1} + p^{2k-2}$
2. $\Delta(\mathrm{B}(\mathbb{Z}_{p^k})) = p^k + p^{k-1}$
3. $D(\mathbb{Z}_{p^k}) = 2$

Proof. It is straightforward to show the formula of the order of $\mathrm{B}(\mathbb{Z}_{p^k})$.

$$\begin{aligned} |\mathrm{B}(\mathbb{Z}_{p^k})| &= \frac{|\mathbb{Z}_{p^k}|^3 - |\{mp|0 \leq m < k\}|^3}{|\mathbb{Z}_{p^k}| - |\{mp|0 \leq m < k\}|} \\ &= \frac{(p^k)^3 - (p^{k-1})^3}{p^k - p^{k-1}} = p^{2k} + p^{2k-1} + p^{2k-2} \end{aligned}$$

Using Lemma 1, it is only enough to show that the degree of the vertex represented by $(1, 0, 0)$ satisfy the formula of the degree of $\mathrm{B}(\mathbb{Z}_{p^k})$.

$$\begin{aligned} \Delta(\mathrm{B}(\mathbb{Z}_p^k)) &= \delta([(1,0,0)]) = \frac{|\mathbb{Z}_{p^k}|^2 - |\{mp|0 \leq m < k\}|^2}{|\mathbb{Z}_{p^k}| - |\{mp|0 \leq m < k\}|} \\ &= \frac{(p^k)^2 - (p^{k-1})^2}{p^k - p^{k-1}} = p^k + p^{k-1} \end{aligned}$$

Using Lemma 2, we get $D(\mathrm{B}(\mathbb{Z}_p^k)) = 2$

□

We search new records among graphs by generalized Brown's construction. Using the above theorem, $\mathrm{B}(\mathbb{Z}_{17^2})$ has degree 306 and diameter 2 and 88723 vertices. It is a new record of $(306, 2)$ of the degree/diameter problem because it cannot be obtained from ordinary Brown's construction. The power of prime less than $305 = 306 - 1$ is 293^1 and the graph $\mathrm{B}(\mathbb{Z}_{293})$ obtained from ordinary Brown's construction of 293^1 has $294 = 293+1$ degree and $86143 = 293^2+293+1$ vertices. The old record of $306 = 294 + 12$ is $86156 = 86143 + 12$ obtained from $\mathrm{B}(\mathbb{Z}_{293})$ by duplicating vertices. In the same way, the graph obtained from $\mathrm{B}(\mathbb{Z}_{17^2})$ by duplicating any one vertex, whose order is 88724, is a new record of $(307, 2)$ because the power of prime less than $306 = 307 - 1$ is 293^1.

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