New results for the Degree/Diameter Problem

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Abstract

The Degree/Diameter Problem is one of the most famous problem in graph theory. We consider the problem for the case of diameter 2. We extended the Brown's construction. We found a new graph by $(306, 2)$-graph with 88723 vertices and $(307, 2)$-graph with 88724 by our construction.

1 Introduction

A *graph* $G = (V, E)$ consists of a set V called *vertices* and a set $E \subset V^2$ called *edges*. If (v, w) is in E, it is said that v and w are *adjacent*, which is denoted by $v \sim w$. If the vertex v is adjacent to itself, the edge (v, v) is called *loop*. The graph G without any loops is *simple*. The *order* $|G|$ of the graph is the size of the set of vertices. The *neighbors* $N(v)$ of the vertex v is a set of vertices which are adjacent to v. The *degree* of the vertex $\delta(v)$ is a size of neighbors $|N(v)|$. The *degree* of the graph $\Delta(G)$ is the maximal degree of the vertex. The graph is *regular* if every vertex's degree are same. The *distance* for each pair (v, w) of vertices is the shortest path length between v and w. The *diameter* $D(G)$ of the graph is the maximum distance of all pairs of vertices. The *Degree/Diameter problem* is the problem of finding the graph with the maximum vertices for given degree Δ and diameter D. The order of a graph with degree Δ $(\Delta > 2)$ of diameter D is easily seen to be bounded by

$$1 + \Delta \sum_{k=1}^{D-1} (\Delta - 1)^k$$

which is called *Moore bound*.

The genaral constructions for small diameters are known. Especially for case of $D = 2$ there exists the general construction called the *Brown's construction*. Given the finite field F_q where q is a power of prime, we construct the graph $B(F_q)$ whose vertices are lines in F_q^3 and two lines are adjacent if and only if they are orthogonal. We call it the Brown's graph. The order of $B(F_q)$ is $q^2 + q + 1$ and the degree of it is $q + 1$. The diameter of it is 2 because $B(F_q)$ includes many triangles. Any lines are symmetric in F_q, so $B(F_q)$ is regular.

However it is not simple because of including some loops. Removing any loops from $B(F_q)$, we get the simple graph whose degree of vertices are $q+1$ or q.

Let R be a ring with unity. R^* denotes the set of invertible elements of R. R^3 is naturally seen as R-module. The addition and R-action are defined by coordinate-wise. The *inner product* $\cdot : R^3 \times R^3 \Rightarrow R$ is defined as follows

$$(v_1, v_2, v_3) \cdot (w_1, w_2, w_3) = v_1 w_1 + v_2 w_2 + v_3 w_3$$

$\boldsymbol{v}, \boldsymbol{w}$ are *orthogonal* if and only if the inner product vanishes, namely $\boldsymbol{v} \cdot \boldsymbol{w} = 0$. The *cross product* $\times : R^3 \times R^3 \Rightarrow R$ is defined as follows

$$(v_1, v_2, v_3) \cdot (w_1, w_2, w_3) = (v_2 w_3 - v_3 w_2, v_3 w_1 - v_1 w_3, v_1 w_2 - v_2 w_1)$$

A *domain* D is a ring without zero divisors. A *Euclidean domain* is a domain E with a function called degree $d : E \setminus \{0\} \Rightarrow \mathbb{N}$ such that for all non-zero $a, b \in E$ there exists $q, r \in E, a = qb + r$ where $d(r) < d(b)$. The ring of integers $\mathbb{Z}$ is a example of the Euclidean domains.

2 Extended Brown's Construction

Definition 1. *Let $(R, +, 0, *, 1)$ be a ring with unit. The vertex set V of the extended Brown's graph $\mathrm{EB}(R)$ is*

$$V = (R^3 \setminus \{\boldsymbol{v} | \exists r \in R, r \cdot \boldsymbol{v} = \boldsymbol{0}\}) / \sim$$

where $\boldsymbol{v} \sim \boldsymbol{w}$ if and only if $\exists k \in R^*, k \cdot \boldsymbol{v} = \boldsymbol{w}$. The two vertices $[\boldsymbol{v}], [\boldsymbol{w}]$ are adjacent if and only if $\boldsymbol{v} \cdot \boldsymbol{w} = 0$.

The adjacency of the definition above is well-defined because the orthogonality does not depends on the selection of representitives. We call the above construction whish gives a graph from a ring the *extended Brown's construction*. It is clear that the new construction coincides old one when the ring is a field.

Lemma 1. *Let E be a Euclidean domain and u be a prime element in E. If $E/(u^k)$ is a finite ring, then $\mathrm{EB}(E/(u^k))$ is a regular graph, where (u^k) is a principal ideal generated by u^k. Note that $\mathrm{EB}(E/(u^k))$ is not simple.*

Proof. It is clear that the degrees of vertices represented by $(1,0,0), (0,1,0), (0,0,1)$ are same. Let $\boldsymbol{v} = ([a], [b], [c])$ be a representitive of any vertex where $a, b, c \in E$. If any element of $[a], [b], [c]$ is not invertible in $E/(u^k)$, there exists natural numbers $l, m, n < k$ and some elements a', b', c' in E such that $a = u^l a', b = u^m b', c = u^n c'$. $\boldsymbol{v}$ is not a representitive of vertices because of $[u^{min(l,m,n)}] \cdot \boldsymbol{v} = ([0], [0], [0])$. Therefore, one element of $[a], [b], [c]$ is at leatst invertible. If [a] is invertible, $\square$

Lemma 2. *Let E be a Euclidean domain and I be ideal of E. The diameter of $\mathrm{EB}(E/I)$ is 2.*

Proof. For any two distinct vertices $[\boldsymbol{v}]$ and $[\boldsymbol{w}]$, consider the cross product $\boldsymbol{v} \times \boldsymbol{w}$. If $\boldsymbol{v} \times \boldsymbol{w} = \boldsymbol{0}$, $v_i \cdot \boldsymbol{w} = w_i \cdot \boldsymbol{v}$ for $i = 1, 2, 3$. For any vertex $[\boldsymbol{v}]$, the cordinate triple (v_1, v_2, v_3) of $\boldsymbol{v}$ are coprime, then there exists $a, b, c \in E/I$ such that $av_1 + bv_2 + cv_3 = 1$.

$$\boldsymbol{v} = 1 \cdot \boldsymbol{v} = (av_1 + bv_2 + cv_3)\boldsymbol{v} = (aw_1 + bw_2 + cw_3)\boldsymbol{w}$$

It is a contradiction then $\boldsymbol{v} \times \boldsymbol{w} \neq \boldsymbol{0}$. If $\boldsymbol{v} \times \boldsymbol{w}$ is a representitive of vertex, $[\boldsymbol{v} \times \boldsymbol{w}]$ is adjacent to $[\boldsymbol{v}]$ and $[\boldsymbol{w}]$. If $\boldsymbol{v} \times \boldsymbol{w}$ is not a representitive of vertex, there exist $k \in E/I$ and $\boldsymbol{u} \in (E/I)^3$ such that $\boldsymbol{v} \times \boldsymbol{w} = k \cdot \boldsymbol{u}$ and $\boldsymbol{u}$ is a representitive of vertex. $[\boldsymbol{u}]$ is adjacent to $[\boldsymbol{v}]$ and $[\boldsymbol{w}]$. □

Theorem. *The following equations hold.*

1. $|\mathrm{EB}(\mathbb{Z}_{p^k})| = p^{2k} + p^{2k-1} + p^{2k-2}$
2. $\Delta(\mathrm{EB}(\mathbb{Z}_{p^k})) = p^k + p^{k-1}$
3. $D(\mathbb{Z}_{p^k}) = 2$

Proof. It is straightforward to show the formula of the order of $\mathrm{EB}(\mathbb{Z}_{p^k})$.

$$\begin{aligned} |\mathrm{EB}(\mathbb{Z}_{p^k})| &= \frac{|\mathbb{Z}_{p^k}|^3 - |\{mp|0 \leq m < k\}|^3}{|\mathbb{Z}_{p^k}| - |\{mp|0 \leq m < k\}|} \\ &= \frac{(p^k)^3 - (p^{k-1})^3}{p^k - p^{k-1}} = p^{2k} + p^{2k-1} + p^{2k-2} \end{aligned}$$

It is straightforward to show the formula of the degree of the vertex represented by $(1, 0, 0)$

$$\begin{aligned} \delta([(1,0,0)]) &= \frac{|\mathbb{Z}_{p^k}|^2 - |\{mp|0 \leq m < k\}|^2}{|\mathbb{Z}_{p^k}| - |\{mp|0 \leq m < k\}|} \\ &= \frac{(p^k)^2 - (p^{k-1})^2}{p^k - p^{k-1}} = p^k + p^{k-1} \end{aligned}$$

In the same way,

$$\delta([(1,0,0)]) = \delta([(0,1,0)]) = \delta([(0,0,1)]) = p^k + p^{k-1}$$

holds. The degree of any other vertex is same to the degree of the vertex represented by $(1, 0, 0)$. For all a representitive $\boldsymbol{v} = (v_1, v_2, v_3)$ of the vertex, the triple (v_1, v_2, v_3) includes one invertible element at least. If v_1 is invertible element,

□

We search new graphs

Theorem.

3 Acknowledgement

Thank you!!!