# TMA4180: Optimization 1 - Tensegrity Structures

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April 2024

## 1 Abstract

This paper uses an optimization-based model, specifically the BFGS method, for the form-finding of tensegrity structures. This implies indentifying the equilibrium positions of the structure's nodes that minimize the total energy, subject to various constraints on the system. There will be theoretical focus, and a section at the end about numerical results.

## 2 Introduction

Tensegrity structures are mechanical systems that maintain integrity through a continuous network of tensional forces. Their lightweight and adaptable nature, have led to their application in fields ranging from architectural design to aerospace engineering.

The mathematical representation for these structures is usually given as a directed graph, where vertices correspond to joints, and edges denote the connections between them.

This representation enables us to formulate the structure as an optimization problem.

## 3 Mathematical Model

### 3.1 Notations and Definitions

A tensegrity structure is defined as a directed graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$, with $\mathcal{V}$ representing the joints of the structure, and $\mathcal{E} \subset \mathcal{V} \times \mathcal{V}$ representing the connections (bars and cables) connecting these joints. Each edge $e_{ij}$ where, $i \neg j$, indicates a connection between joint $i$ and $j$ through a bar or cable.

The position of a joint $i$ is represented as $\mathbf{x}^{(i)} = (x_1^{(i)}, x_2^{(i)}, x_3^{(i)}) \in \mathbb{R}^3$. The goal is to find the vector $\mathbf{X} = (\mathbf{x}^{(1)}, \ldots, \mathbf{x}^{(N)}) \in \mathbb{R}^{3N}$ that minimizes the systems energy, which gives rise to an optimization problem.

### 3.2 Elastic and Gravitational Energy

The elastic energy of a bar connecting nodes $i$ and $j$ with a resting length $\ell_{ij}$ and new length $L(e_{ij}) = \|\mathbf{x}^{(i)} - \mathbf{x}^{(j)}\|$ is modeled as:

$$E_{\text{elast}}^{\text{bar}}(e_{ij}) = \frac{c}{2\ell_{ij}^2} \left(L(e_{ij}) - \ell_{ij}\right)^2, \tag{1}$$

where $c > 0$ is a constant related to the bar's material and cross-section, assuming uniform material properties. The rest lengths may vary. The gravitational potential energy of the bars in a field is given by:

$$E_{\text{grav}}^{\text{bar}}(e_{ij}) = \rho g \ell_{ij} \frac{x_3^{(i)} + x_3^{(j)}}{2}, \tag{2}$$

with $\rho$ representing the line density and $g$ the acceleration due to gravity.

For a cable, which is massless and only extends, the elastic energy when stretched beyond its resting length $\ell_{ij}$ is:

$$E_{\text{elast}}^{\text{cable}}(e_{ij}) = \begin{cases} \frac{k}{2\ell_{ij}^2}\left(\|\mathbf{x}^{(i)} - \mathbf{x}^{(j)}\| - \ell_{ij}\right)^2 & \text{if } \|\mathbf{x}^{(i)} - \mathbf{x}^{(j)}\| > \ell_{ij}, \\ 0 & \text{otherwise,} \end{cases} \tag{3}$$

where $k > 0$ depends on the cable's material.

### 3.3 External Loads

The external load energy at node $i$, loaded with a mass $m_i$, is calculated as:

$$E_{\text{ext}}(\mathbf{X}) = \sum_{i=1}^{N} m_i g x_3^{(i)}, \tag{4}$$

where each node is treated as a point mass.

### 3.4 Total Energy

The object function we aim to minimize is the total energy, calculated as:

$$E(\mathbf{X}) = \sum_{e_{ij} \in \mathcal{B}} \left(E_{\text{elast}}^{\text{bar}}(e_{ij}) + E_{\text{grav}}^{\text{bar}}(e_{ij})\right) + \sum_{e_{ij} \in \mathcal{C}} E_{\text{elast}}^{\text{cable}}(e_{ij}) + E_{\text{ext}}(\mathbf{X}). \tag{5}$$

where $\mathcal{B}$ and $\mathcal{C}$ represent the sets of bars and cables in the structure, respectively.

### 3.5 Constraints

We will consider two constraints:

#### 3.5.1 Fixed points

The first constraint fixes certain nodes in space:

$$x^{(i)} = p^{(i)} \quad \text{for } i = 1, \dots, M, \tag{6}$$

where $p^{(i)}$ are the positions of fixed nodes. This simplifies the free optimization problem to a dimension of $3(M - N)$, with $1 < M < N$.

#### 3.5.2 Above ground

The second constraint ensures that the structure remains above ground:

$$x_3^{(i)} \geq f(x_1^{(i)}, x_2^{(i)}) \quad \text{for all } i = 1, \dots, N, \tag{7}$$

where $f : \mathbb{R}^2 \to \mathbb{R}$ represents the height profile of the ground.

## 4 Existence of Solutions

Considering the constraints in Subsection 3.5, we aim to determine wether solutions exists undere these conditions.

## 4.1 Fixed point

**Statement:** If the graph $\mathcal{G}$ is connected and the energy function 5 is continuous, then the optimization problem has a solution, provided $k$ and $c$ are non-zero.

Firstly, 5 is continous because it consists of sums of quadratic and linear terms, both of which are contionus functions over the node points.

Secondly, the domain is bounded and compact. By fixing M out of N points, the search space is significantly reduced. Given that $\mathcal{G}$ is a connected graph, all nodes are within a finite distance from any fixed node, addressing the physical limits of the length of bars and cables, thereby ensuring compactness.

Finally, The Extreme Value Theorem states that if a function $f$ is continuous on a compact set $S$ in $\mathbb{R}^n$, then $f$ attains its maximum and minimum on $S$. This applies in our case, and ensures that $E(X)$ attains a global minimum.

## 4.2 Above ground

**Statement:** Provided that the graph $\mathcal{G}$ is connected and the energy function 5 is continuous, the optimization problem admits a solution.

A crucial aspect is the coerciveness of the function $f$, which ensures that the $z$-coordinates of the nodes cannot decrease indefinitely. This is due to the ground constraints, as $x_3^{(i)}$ must be greater than or equal to a value that increases infinitely as $x_1^{(i)}$ or $x_2^{(i)}$ tend to infinity. This bounds the vertical positions of the nodes, providing a floor that rises indefinitely away from the origin. Mathematically, the domain defined by the constraints is:

$$S = \left\{ X \in \mathbb{R}^{3N} : x_3^{(i)} \geq f(x_1^{(i)}, x_2^{(i)}), \forall i \right\} \tag{8}$$

This domain is compact due to the coercive nature of $f$, which limits the node positions and prevents them from "escaping". With $E(\mathbf{X})$ being continous, the extreme value theorem applies:

$$\exists X^* \in S : E(X^*) \leq E(X), \quad \forall X \in S.$$

### 4.2.1 Flat ground

One might question whether the condition discussed above also applies for a flat ground profile, where $f(x_1, x_2) = 0 \forall (x_1, x_2) \in \mathbb{R}^2$.

The constraint simplifies to $x_3^{(i)} \geq 0$, ensuring all nodes remain above the flat ground. However, this lacks an upper bound, potentially allowing the total energy of the structure to decrease indefinitely as its spatial extent increases. Although physical limits would prevent this in reality, matematically proving the existence of a minimum in such a case becomes challenging without further constraints.

# 5 The Simple Case: Cable Net with fixed Nodes

In this model, we fix certain nodes and connect them solely with cables, leading to the following optimization problem:

$$\begin{aligned} &\min_X E(X) = \sum_{e_{ij} \in \mathcal{E}} E_{\text{elast}}^{\text{cable}}(e_{ij}) + E_{\text{ext}}(X) \\ &\text{s.t.} \quad x_{(i)} = p_{(i)}, \quad i = 1, \ldots, M. \end{aligned} \tag{9}$$

Before solving this we first need to establish some properties.

## 5.1 Differentiability

**Statement:** 9 is $C^1$

Analyzing the function term by term, we find:

$$\nabla E_{\text{ext}}(\mathbf{X}) = \nabla \sum_{i=1}^{N} m_i g x_3^{(i)} = (m_1 g, m_2 g, \ldots, m_N g). \tag{10}$$

which is clearly contionus and $C^\infty$.

For the cable's elastic energy gradient:

$$\nabla E_{\text{elast}}^{\text{cable}}(e_{ij}) = 0 \tag{11}$$

when the distance between nodes $i$ and $j$ does not exceed their resting length $\ell_{ij}$. Otherwise the gradient is calculated by:

$$\frac{\partial}{\partial x_s^{(i)}} E_{\text{elast}}^{\text{cable}}(e_{ij}) = \frac{k(x_s^{(i)} - x_s^{(j)})}{\ell_{ij}^2} \left(1 - \frac{\ell_{ij}}{\|x^{(i)} - x^{(j)}\|}\right) \tag{12}$$

which approaches zero as $\lim_{\|x^{(i)}-x^{(j)}\|} \to \ell_{ij}^+$. This shows that the function is $C^1$.

### 5.1.1 Is it $C^2$?

No, consider the second-order partial derivative of the cables elastic energy:

$$\frac{\partial}{\partial x_1^{(i)}} \frac{\partial}{\partial x_2^{(i)}} E_{\text{elast}}^{\text{cable}}(e_{ij}) = \frac{k(x_1^{(i)} - x_1^{(j)})(x_2^{(i)} - x_2^{(j)})}{\ell_{ij} \|x^{(i)} - x^{(j)}\|^3} \tag{13}$$

which does not approach 0 as $\lim_{\|x^{(i)}-x^{(j)}\|} \to \ell_{ij}^+$.

## 5.2 Convexity

**Statement:** 9 is convex, but not strictly convex

We consider the function term by term. Note, 4 is convex, due to its linearity. For equation 3, we simplify it as follows:

$$g(x^{(i)}, x^{(j)}) = \frac{k}{2\ell_{ij}^2} h(\|x^{(i)} - x^{(j)}\|) \tag{14}$$

where:

$$h(t) = \begin{cases} (t - \ell_{ij})^2 & t > \gamma, \\ 0 & \text{otherwise}, \end{cases} \tag{15}$$

and:

$$h'(t) = \begin{cases} 2(t - \ell_{ij}) & t > \gamma, \\ 0 & \text{otherwise}. \end{cases} \tag{16}$$

We now need to show that $g$ and $h$ are convex. $g$ is convex since it is a norm. $h$ is convex since $\forall t > \gamma, h'(t)$ is non-negative. As the sum of two convex functions is convex, 9 is convex. It is not strictly convex, as non of our "components" are strictly convex.

## 5.3 Necessary and Sufficient Optimality Conditions

Since we are dealing with a convex and differentiable function, the only condition for a solution $X^*$ is:

$$\nabla E(X^*) = 0 \tag{17}$$

This condition ensures that $X^*$ will be a global minimizer, thanks to the functions convexity.

# 6 Tensegrity-Domes

We now consider the full problem, 5, with the fixed point constraint of section 3.5. The resulting optimization problem reads:

$$\begin{aligned}\min_X \quad & E(X) = \sum_{e_{ij}\in B}\left(E^{\text{elast}}_{\text{bar}}(e_{ij}) + E^{\text{grav}}_{\text{bar}}(e_{ij})\right) + \sum_{e_{ij}\in C} E^{\text{elast}}_{\text{cable}}(e_{ij}) + E_{\text{ext}}(X) \\ \text{such that} \quad & x^{(i)} = p^{(i)}, \quad i = 1,\dots,M.\end{aligned} \tag{18}$$

## 6.1 Differentiability

We begin with examining the component $E^{\text{bar}}_{\text{elast}}$ of the total energy function:

$$\frac{\partial}{\partial x_s^{(i)}} E^{\text{bar}}_{\text{elast}}(e_{ij}) = \frac{c(x_s^{(i)} - x_s^{(j)})}{\ell_{ij}^2}\left(1 - \frac{\ell_{ij}}{\|x^{(i)} - x^{(j)}\|}\right) \tag{19}$$

This poses a potential issue as the norm expression in the denominator becomes zero when two nodes are directly connected by a bar, leading to non-differentiability of 5.

However, in practical applications, this lack of differentiability typically doesn't cause problems. The factor $(x_s^{(i)} - x_s^{(j)})$ ensures the norm remains finite as the points approach each other, preventing a numerical "divide by zero" error. Adding a small number (close to machine precision) could also circumvent this issue.

## 6.2 Necessary and Sufficient Optimality Conditions

Generally, the conditions to be met are:

$$\nabla E(X^*) = 0 \tag{20}$$

$$x^T H_f(X^*) x \geq 0 \forall x \tag{21}$$

In typical $C^2$ optimization problems, these are the necessary conditions. However, our function is not $C^2$, so we lack second-order conditions. Additionally, since $E$ is not convex, the necessary condition $\nabla E(X^*) = 0$ is not sufficient. Nonetheless, this condition must still be met. Practically, this lack of sufficiency doesn't pose major issues, as local minimizers still correspond to stable physical states of the system.

## 6.3 Convexity

**Statement:** 5 is not convex

To determine convexity, the function $f : \mathbb{R} \to \mathbb{R}$ must satisfy the condition $\forall x, y \in \mathbb{R}$, and $\forall \lambda \in [0, 1]$:

$$f(\lambda x + (1-\lambda)y) \leq \lambda f(x) + (1-\lambda) f(y) \tag{22}$$

For our specific case, using $\lambda = \frac{1}{2}$ and $y = -x$, we calculate:

$$E(\lambda x + (1-\lambda)y) = E(\frac{1}{2}x - \frac{1}{2}x) = E(0) = \sum_{e_{ij}\in\mathcal{B}} \frac{k}{2\ell_{ij}^2}(\|0\| - \ell_{ij})^2 = \sum_{e_{ij}\in\mathcal{B}} \frac{k}{2} \tag{23}$$

and:

$$\lambda E(x) + (1-\lambda)E(y) = \frac{1}{2}E(x) + \frac{1}{2}E(-x) = \frac{1}{2}\sum_{e_{ij}\in\mathcal{B}} \frac{k}{2\ell_{ij}^2}(\|x^{(i)} - x^{(j)}\| - \ell_{ij})^2 - \frac{1}{2}\sum_{e_{ij}\in\mathcal{B}} \frac{k}{2\ell_{ij}^2}(\|x^{(i)} - x^{(j)}\| - \ell_{ij})^2 = 0 \tag{24}$$

We see that condition 22 does not hold as $\sum_{e_{ij}\in\mathcal{B}} \frac{k}{2} \nleq 0$

# 7 Free-Standing Structures

We now consider the full problem, 5, with the ground constraint of section 3.5. The resulting optimisation problem reads:

$$\begin{aligned} \min_X \quad & E(X) = \sum_{e_{ij}\in B} \left(E_{\text{bar}}^{\text{elast}}(e_{ij}) + E_{\text{bar}}^{\text{grav}}(e_{ij})\right) \\ & \quad + \sum_{e_{ij}\in C} E_{\text{cable}}^{\text{elast}}(e_{ij}) + E_{\text{ext}}(X) \\ \text{s.t.} \quad & x_3^{(i)} \geq f\left(x_1^{(i)}, x_2^{(i)}\right), \quad i = 1, \dots, N. \end{aligned} \tag{25}$$

## 7.1 Optimality Conditions

In this constrained optimization problem, we start by defining the Lagrangian:

$$\mathcal{L}(X, \lambda) = E(X) - \sum_{i\in\mathcal{I}} \lambda_i (f(x_1^{(i)}, x_2^{(i)}) - x_3^{(i)}) \tag{26}$$

Under the Karush-Kuhn-Tucker (KKT) conditions, specifically stationarity, the gradient of the Lagrangian must be zero at the solution:

$$\nabla_x \mathcal{L}(X^*, \lambda^*) = 0 \tag{27}$$

The KKT feasibility condition is:

$$x_3^{(i)} \geq f(x_1^{(i)}, x_2^{(i)}), \quad \forall i = 1, \dots, N. \tag{28}$$

where

$$\lambda_i \geq 0, \quad \forall i = 1, \dots, N. \tag{29}$$

The KKT conditions are necessary for a local minumum if the problem satisfies certain regularity conditions. There are several types of regularity conditions, but we consider:

- **Linear Independence Constraint Qualification (LICQ):** Requires that the gradients of the active inequality constraints and all equality constraints at the solution are linearly independent. This condition ensures that the Lagrange multipliers associated with each constraint are unique.

# 8 Numerical Optimization

## 8.1 Methods

Based on our analysis of different setups, BFGS is a resonable choice to solve the optimization problems at hand.

### 8.1.1 BFGS

The BFGS method is highly effective for convex optimization and adaptable for non-convex problems due to its reliance on gradients. We have adapted the BFGS algorithm following the approach outlined in Algorithm 2, from Marco Sutti's Notes on BFGS Marco Sutti's Notes on BFGS. This approach uses first-order information to iteratively refine the inverse Hessian approximation. This adaptation includes a line search that meets the Wolfe conditions.

Meeting the Wolfe conditions offers multiple advantages: the Armijo condition ensures each step sufficiently reduces the function value, avoiding slow progress, while the curvature condition ensures a significant increase in the slope of the objective function, preventing overly large steps. This enhances convergence.

The line search method we have implemented employs specific constants, and is fully defined by these:

| **Constant** | **Value** |
|---|---|
| $\rho$: Controls extrapolation | 3 |
| $c_1$: Armijo condition | 0.1 |
| $c_2$: Curvature condition | 1 |

Table 1: Constants defining the line search process

These constants have been determined through trial and error, seeking the best results.

## 8.2 Cable Net with Fixed Nodes

We consider a model with 4 free nodes and 4 fixed nodes:

$$\text{Fixed nodes:} p^{(1)} = (5, 5, 0), p^{(2)} = (-5, 5, 0), p^{(3)} = (-5, -5, 0), p^{(4)} = (5, -5, 0)$$

$$\text{Cables:} \ell_{04} = \ell_{15} = \ell_{26} = \ell_{37} = \ell_{45} = \ell_{56} = \ell_{67} = \ell_{74} = 3$$

$$\text{Constants:} k = 3, mg = \frac{1}{6}$$

This setup is the only convex scenario in our study, allowing us to consistently find the global solution, irrespective of the initial starting point.

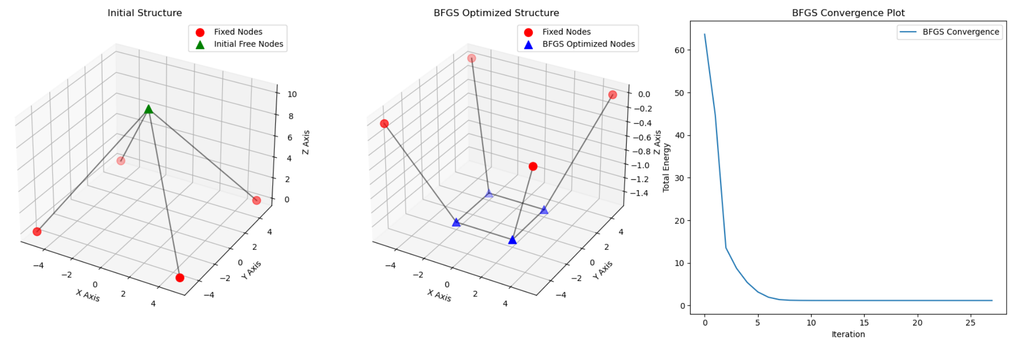


Figure 1: aaa

Table 2: Results for Cable Net with Fixed Nodes

| Node | Initial Guess | Solution (BFGS) | Exact Solution |
|---|---|---|---|
| $x^{(5)}$ | (0, 0, 10) | (2, 2, -1.5) | (2, 2, -1.5) |
| $x^{(6)}$ | (0, 0, 10) | (-2, 2, -1.5) | (-2, 2, -1.5) |
| $x^{(7)}$ | (0, 0, 10) | (-2, -2, -1.5) | (-2, -2, -1.5) |
| $x^{(8)}$ | (0, 0, 10) | (2, -2, -1.5) | (2, -2, -1.5) |
| | | BFGS | |
| Iterations | | 27 | |
| Runtime | | 0.15 seconds | |

The function's convexity ensures continuous and reliable convergence to the desired configuration from any starting point, confirming the presence of a globally attainable solution.

## 8.3 Tensegrity-Domes

We modify the model from previous task by adding bars, resulting in a non-convex objective function. Here are the model parameters:

$$\text{Fixed nodes:}p^{(1)} = (1, 1, 0), p^{(2)} = (-1, 1, 0), p^{(3)} = (-1, -1, 0), p^{(4)} = (1, -1, 0)$$
$$\text{Cables:}\ell_{18} = \ell_{25} = \ell_{36} = \ell_{47} = 8, \ell_{56} = \ell_{67} = \ell_{78} = \ell_{58} = 1$$
$$\text{Bars:}\ell_{15} = \ell_{26} = \ell_{37} = \ell_{48} = 10$$
$$\text{Constants:}c = 1, k = 0.1, g\rho = 0, mg = 0$$

Table 3: Results for Tensegrity-Domes

| Node | Initial Guess | Solution (BFGS) | Exact Solution |
|---|---|---|---|
| $x^{(5)}$ | (0, 0, 3) | (-0.710, -0.0000963, 9.54) | (-0.70970, 0, 9.54287) |
| $x^{(6)}$ | (0, 0, 3) | (-0.0000242, -0.710, 9.54) | (0, -0.70970, 9.54287) |
| $x^{(7)}$ | (0, 0, 3) | (0.710, -0.000024, 9.54) | (0.70970, 0, 9.54287) |
| $x^{(8)}$ | (0, 0, 3) | (-0.0000964, 0.710, 9.54) | (0, 0.70970, 9.54287) |
| Iterations | | 66 | |
| Runtime | | 0.62 seconds | |

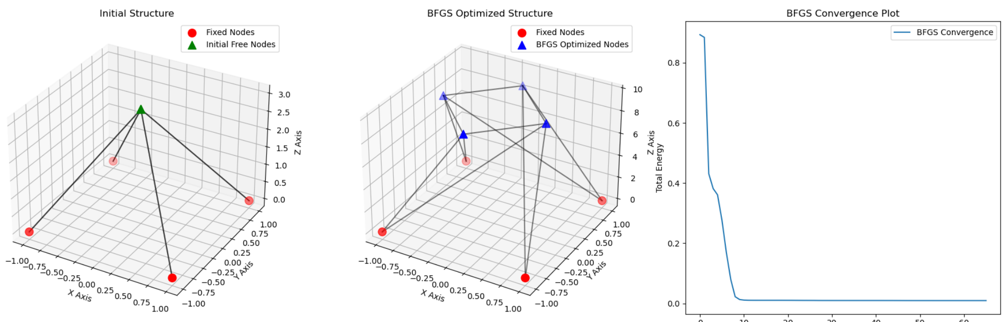


Figure 2: aaa

The BFGS algorithm approaches the analytical solution closely but does not match it exactly due to the problem's non-convexity, which can lead to local minima traps depending on the starting point. An alternative solution was also explored, illustrating the challenge with different starting points:

Table 4: Results for Tensegrity-Domes with alternative initialization

| Node | Initial Guess | Solution (BFGS) | Exact Solution |
|---|---|---|---|
| $x^{(5)}$ | (1, 0, 3) | (-0.391, -0.251, 9.576) | (-0.70970, 0, 9.54287) |
| $x^{(6)}$ | (-1, 0, 4) | (0.347, -0.917, 9.439) | (0, -0.70970, 9.54287) |
| $x^{(7)}$ | (0, 0, 5) | (0.833, -0.037, 9.470) | (0.70970, 0, 9.54287) |
| $x^{(8)}$ | (0.5, 0, 6) | (0.119, 0.612, 9.592) | (0, 0.70970, 9.54287) |
| Iterations | | 101 | |
| Runtime | | 0.89 seconds | |

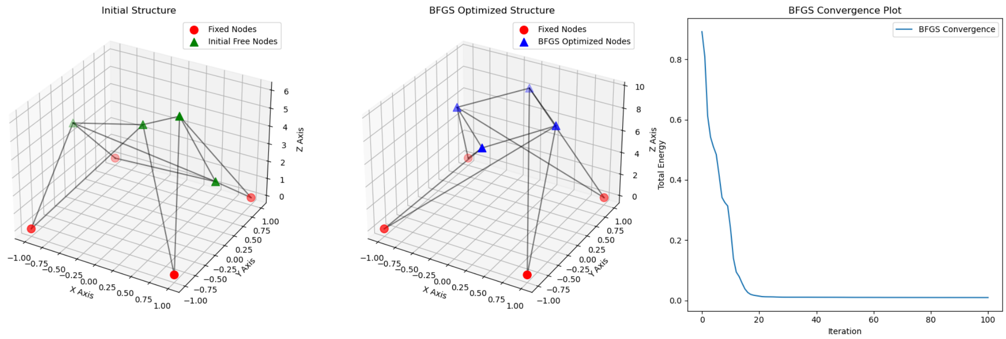


Figure 3: aaa

The local minima are indicated by fluctuations in the energy function, reflecting potential traps during optimization.

## 8.4 Free-Standing Structures

For the free standing structures, we will be using a structure similar to those of the Tensegrity domes, so that we can verify our result. We will make one addition to the model, which is:

$$\text{Add cables: } l_{12} = l_{23} = l_{34} = l_{14} = 2$$
$$g\rho = 10^{-10}$$
$$\text{Height profile: } f(x_1, x_2) = \frac{(x_1^2 + x_2^2)}{20}$$

The height constraint is numerically enforced by adding a penalty to the total energy, with a penalty scale of 0.1. This design prompts the algorithm to minimize the energy and the penalty simultaneously, effectively meeting the height profile constraint.

Table 5: Results Free-Standing Structures

| Node | Initial Guess | Solution (BFGS) | Exact Solution |
|---|---|---|---|
| $x^{(1)}$ | (0.5, 0.5, 0) | (0.989, 1.020, 1.261) | (1,1,0) |
| $x^{(2)}$ | (-0.5, 0.5, 0) | (-1.011, 0.988, 1.263) | (-1,1,0) |
| $x^{(3)}$ | (-0.5, -0.5, 0) | (-0.980, -1.011, 1.267) | (-1,-1,0) |
| $x^{(4)}$ | (0.5, -0.5, 0) | (1.020, -0.980, 1.265) | (1, -1, 0) |
| $x^{(5)}$ | (-0.4, 0.1, 10) | (-0.449, 0.174, 9.176) | (-0.70970, 0, 9.54287) |
| $x^{(6)}$ | (0.4, 0.1, 10) | (0.008, -0.448, 9.165) | (0, -0.70970, 9.54287) |
| $x^{(7)}$ | (0.4, 0.1, 10) | (0.448, 0.008, 9.165) | (0.70970, 0, 9.54287) |
| $x^{(8)}$ | (0.1, 0.4, 10) | (0.174, 0.449, 9.176) | (0, 0.70970, 9.54287) |
| Iterations | | 32 | |
| Runtime | | 0.50 seconds | |

These results, mimicking a tensegrity dome, validate our algorithm for free-standing structures.

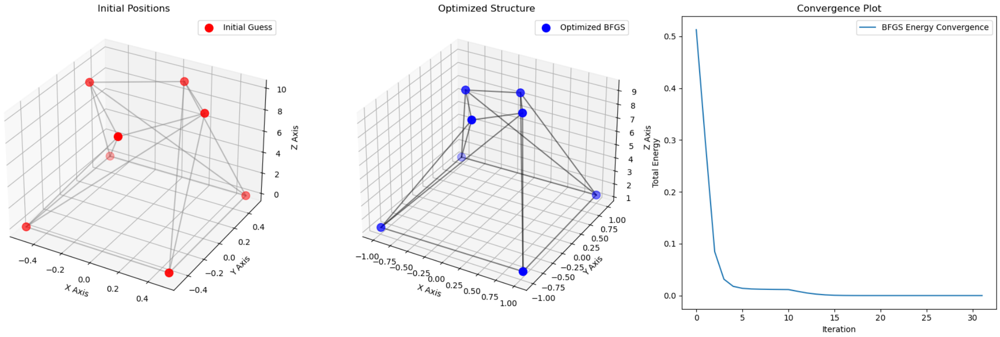


Figure 4: aaa

# 9 Conclusion

In this paper, we explored tensegrity structures under various conditions, demonstrating intriguing outcomes through basic assumptions like the pursuit of minimal energy states, and attempted to find these states using the BFGS method. Additionally, we engage in a theoretical examination of the feasibility of numerical solutions, focusing on aspects such as convexity, differentiability, and the requisite conditions for solutions.

# References

[1] Marco Sutti, *Notes on line search, trust regions, and BFGS method*, 2023, Available at: `https://www.marcosutti.net/files/Notes_BFGS.pdf`.