TMA4212 Numerical Solution of Differential Equations by Difference Methods Project 2

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March 2023

The purpose of this report is to develope and implement finite element schemes for elliptic problems in 1D. This is done by studying the 1D stationary convection diffusion problem. The model we use is a boundary problem for a Poisson like equation:

$$-\partial_x(\alpha(x)\partial_x u) + \partial_x(b(x)u) + c(x)u = f(x)$$

where u is the concentration (of the substance), $\alpha(x) > 0$ is the diffusion coefficient, $b(x)$ the fluid velocity, $c(x) \geq 0$ is the decay rate of the substance, and $f(x)$ is the source term. There is no time dependence, since we look at a stationary time limit.

Assume $\alpha(x) \geq \alpha_0 > 0$, $\|\alpha\|_{L^\infty} + \|b\|_{L^\infty} + \|c\|_{L^\infty} + \|f\|_{L^2} \leq K$ and the Dirichlet boundary conditions $u(0) = 0 = u(1)$

We will now use integration by parts to show that the solution u of this problem satisfies

$$a(u, v) = F(v) \;\; \forall v \in H_0^1(0, 1)$$

Using integration by parts gives

$$\int_0^1 (-\partial_x(\alpha\partial_x u) + \partial_x(bu) + cu)v dx = (-\alpha u_x + bu)v\Big|_0^1 - \int_0^1 (-\alpha u_x + bu)v_x dx + \int_0^1 cuvdx.$$

For $v \in H_0^1(0, 1)$, we get

$$a(u, v) = \int_0^1 \alpha u_x v_x - buv_x + cuv \; dx \int_0^1 fvdx = F(v).$$

Further, we want to show that $a(u, v)$ is bilinear and continuous function on $H^1 \times H^1$, and that $F(v)$ is a linear continous functional on H^1. We first show bilearity. By linearity of integrals and derivatives we obtain

$$a(\beta_1 u_1 + \beta_2 u_2, v) = \int_0^1 \alpha(\beta_1(u_1)_x + \beta_2(u_2)_x)v_x - b(\beta_1 u_1 + \beta_2 u_2)v_x + ((\beta_1 u_1 + \beta_2 u_2)v)_x,$$

and similar argumant for linearity in second argument. Similarly, it can be shown that F is linear,

$$F(\beta_1 v_1 + \beta_2 v_2) = \int_0^1 f(\beta_1 v_1 + \beta_2 v_2)dx = \beta_1 \int_0^1 f v_1 + \beta_2 \int_0^1 f v_2 = \beta_1 F(v_1) + \beta_2 F(v_2).$$

Further, we want to explore if F is bounded and continous.

$$\|F\| = \frac{\|F(v)\|}{\|v\|_{H^1}} \leq \frac{\|f\|_2 \|v\|_2}{\|v\|_{H^1}} \leq \|f\|_2 \leq K$$

This is a result of $\|v\|_{H^2} \geq \|v\|_{L^2}, \|u'\|_{L^2}$. Thus, F is continous, since it is bounded. Further, we want to show that $a(u,v)$ is continous on $H^1 \times H^1$

$$\begin{aligned} a(u,v) = \int_0^1 \alpha u_x v_x - uv_x + cuvdx \ &\leq \left| \int_0^1 \alpha u_x v_x - buv_x + cuvdx \right| \\ &\leq |\alpha| \int_0^1 |u_x||v_x| \ dx + |b| \int_0^1 |u||v_x| \ dx + |c| \int_0^1 |u||v| \ dx \\ &\leq |\alpha| \|u_x\|_{L^2} \|v_x\|_{L^2} + |b| \|u\|_{L^2} \|v_x\|_{L^2} + |c| \|u\|_{L^2} \|v\|_L^2 \leq K \|u\|_{H^1} \|v\|_{H^1}, \end{aligned}$$

where the fact that $\|v\|_{H^2} \geq \|v\|_{L^2}, \|u'\|_{L^2}$ is used again. Thus, $a(u,v)$ is continuous on $H^1 \times H^1$

Further, let α, b, c be nonzero constants. We want to show that $a(u,v)$ satisfies the following Gårding inequality:

$$a(u,v) \geq (\alpha - \frac{\epsilon}{2}|b|) \int_0^1 u_x^2 \ dx + (c - \frac{1}{2\epsilon}|b|) \int_0^1 u^2 \ dx \quad \forall \, \epsilon > 0$$

It is appropiate for us that $a(u,v)$ satisfies the Gårding inequality, because it provides a way to establish the well-existance and uniqeness, as well as stability, for the solution.

$$\begin{aligned} a(u,v) &= \int_0^1 \alpha u_x^2 dx - \int_0^1 buu_x dx + \int_0^1 cu^2 dx \geq \int_0^1 \alpha u_x^2 dx - \frac{1}{2} \int_0^1 \frac{1}{\epsilon}(bu)^2 + \epsilon u_x^2 dx + \int_0^1 cu^2 dx \\ &\overset{\epsilon \to |b|\epsilon}{=} (\alpha - \frac{\epsilon}{2}|b|) \int_0^1 u_x^2 dx + (c - \frac{1}{2\epsilon}|b|) \int_0^1 u^2 dx, \end{aligned}$$

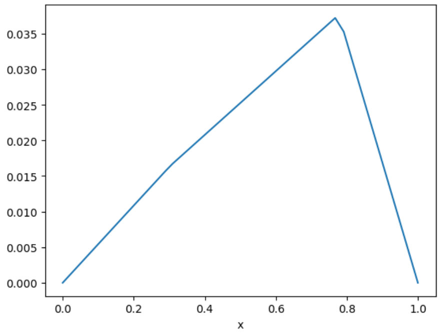
which is what we wanted to show. Further, we will use this inequality to prove that $a(u,v)$ coercive assuming $c > \frac{|b|^2}{2\alpha}$. Let $\epsilon = \frac{\alpha}{|b|}$, then:

$$\alpha - \frac{\epsilon}{2}|b| = \alpha - \frac{\alpha}{2} = \frac{\alpha}{2} > 0, \qquad c - \frac{1}{2\epsilon}|b| = c - \frac{|b|^2}{2} > 0.$$

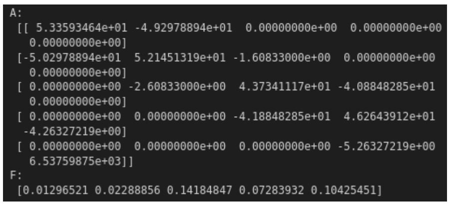
Thus, letting $M = \min\{\frac{\alpha}{2}, c - \frac{|b|^2}{2\alpha}\}$ we find that

$$a(u,u) \geq M \|u\|_{H^1}^2.$$

Since, in this case, $a(u,v)$ is a contours bilinear from on $H^1 \times H^1$ and $F(v)$ is a continuous linear functional on H^1, by the Lax-Milgram theorem there exists a unique solution $u \in H_0^1$ such that $a(u,v) = F(v) \ \forall v \in H_0^1$. This yields:



(a) Numerical solution for test function x^2, using 6 nodes, and $a = b = c = 1$



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A:
 [[ 5.33593464e+01 -4.92978894e+01  0.00000000e+00  0.00000000e+00
   0.00000000e+00]
 [-5.02978894e+01  5.21451319e+01 -1.60833000e+00  0.00000000e+00
   0.00000000e+00]
 [ 0.00000000e+00 -2.60833000e+00  4.37341117e+01 -4.08848285e+01
   0.00000000e+00]
 [ 0.00000000e+00  0.00000000e+00 -4.18848285e+01  4.62643912e+01
  -4.26327219e+00]
 [ 0.00000000e+00  0.00000000e+00  0.00000000e+00 -5.26327219e+00
   6.53759875e+03]]
F:
 [0.01296521 0.02288856 0.14184847 0.07283932 0.10425451]
```

(b) The stiffness matrix (A) and load vector (F) used in (a).

$$\phi_i = \begin{cases} \frac{x - x_{i-1}}{h_{i-1}}, x \in (x_{i-1}, x_i) \\ \frac{x_{i+1} - x}{h_i} x \in (x_i, x_{i+1}) \end{cases}$$

We now want to solve the problem $a(u, v) = F(v)$ with a P_1 FEM on a general grid on (0,1). We want to find $u \in V_h$ such that $a(u_h, v_h) = F(v_h) \forall v_h \; V_h$, where $V_h = X_h^1(0,1) \cap H_0^1(0,1)$ (this is the space of f continuous functions with zero boundary values, that are piecewise linear on the "triangulation" given by the grid). We let $\alpha \geq 0$, $b, c \geq 0$ be nonzero constants.

We write the problem as: $A\vec{U} = \vec{F}$, where

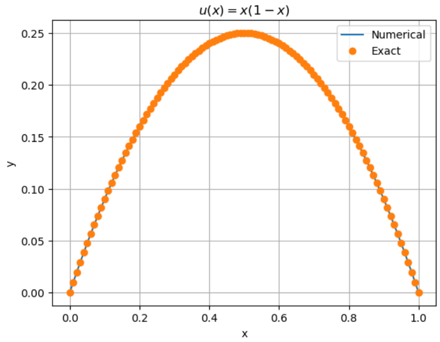
$$A = (a(\phi_j, \phi_i))_{i,j=1}^n, \quad \vec{F} = (F(\phi_i))_{i=1}^n$$

$$\phi_i = \begin{cases} \frac{x - x_{i-1}}{h_{i-1}} & x \in (x_{i-1}, x_i), \quad h_i = x_i - x_{i-1} \\ \frac{x_{i+1} - x}{h_i} & x \in (x_i, x_{i+1}), \quad h_{i+1} = x_{i+1} - x_i \end{cases}$$

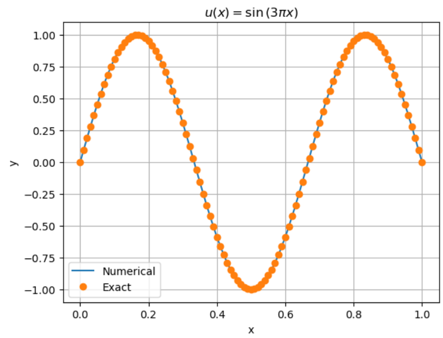
Then, the elements of A are of the form

$$A_{ij} = \begin{cases} \alpha \left(\frac{1}{h_i} + \frac{1}{h_{i+1}} \right) + \frac{c}{3}(h_i + h_{i+1}) & j = i \\ -\frac{\alpha}{h_{i+1}} + \frac{b}{2} + \frac{c}{6} h_{i+1} & j = i+1 \\ -\frac{\alpha}{h_i} - \frac{b}{2} + \frac{c}{6} h_i & j = i-1 \end{cases}$$

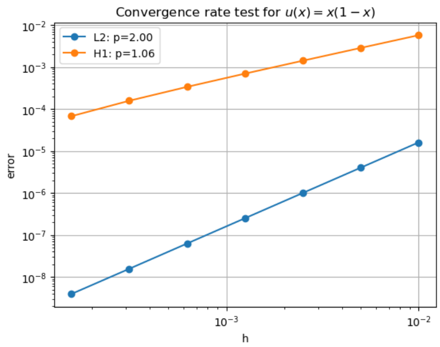
We are testing the method and numerically finding the convergence rate, in the L^2 and H^1 norm, for the test solutions $x(1-x)$ and $\sin(3\pi x)$.



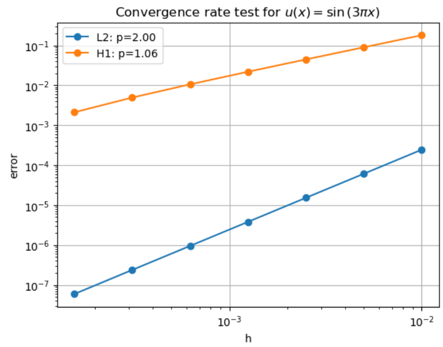
(a) Exact- and Numerical solution, 100 nodes, $a = b = c = 1$



(b) Exact- and Numerical solution, 100 nodes, $a = b = c = 1$



(c) Convergence rates



(d) Convergence rates

Figure 2: Numerical and exact solutions for $x(1-x)$ and $\sin(3\pi x)$, and corresponding error and convergence plots. In both (a) and (b) M = 100 is used.

The convergence rate in L^2 is 2 and the convergence rate in H^1 is 1, for both functions. The reason the L^2 norm has faster convergence rate than H^1 is because the H^1 norm requires additional information about the gradient to the exact solution. The faster convergence in L^2 can be seen as a result of weaker requirements.

We now want to show Galerkin orthogonality and Cea's lemma holds for our method, as this will lead to a H^1 error bound. Let $u \in H_0^1$ be the solution to $a(u, v) = F(v)$ $\forall v \in H^1$, and likewise let $u_h \in X_h^1$ be the solution to the finite-dimensional case $a(u_h, v_h) = F(v_h)$ $\forall v_h \in X_h^1$, then we get Galerkin orthogonality since

$$a(u - u_h, v_h) = a(u, v_h) - a(u_h, v_h) = F(v_h) - F(v_h) = 0 \quad \forall v_h \in X_h^1.$$

Using this we Cea's lemma.

$$\begin{aligned}
\|u - u_h\|_{H_1}^2 &\leq \frac{1}{M} a(u - u_h, u - u_h) = \frac{1}{M}(a(u - u_h, u - v_h) + a(u - u_h, v_h - u_h)) \\
&= \frac{1}{M} a(u - u_h, u - v_h) \leq \frac{K}{M} \|u - u_h\|_{H^1} \|u - v_h\|_{H^1} \\
\Rightarrow \|u - u_h\| &\leq \frac{K}{M} \|u - v_h\| \qquad \forall v_h \in X_h^1.
\end{aligned}$$

Using Cea's lemma we have that $\|u - u_h\|_{H_1}^2 \leq \frac{K}{M} \|u - I_h u\|$, where $I_h u$ is the piecewise linear interpolation of the solution u. If $u \in H^2(0, 1)$, then the interpolation error can be estimated by

$$\|(u - I_h u)'\|_{L_2} \leq h \|u''\|_{L^2}$$
$$\|u - I_h u\|_{L_2} \leq h^2 \|u''\|_{L^2}$$

Using this with Cea's lemma, we have that

$$\|u - u_h\| \leq \frac{K}{M}\sqrt{(h^2 + h^4) \|u''\|_{L^2}^2} \leq \sqrt{2}h \|u''\|_{L^2},$$

where $h^4 \leq h^2$ for $0 < h \leq 1$ is used in the last inequality. Hence the method is first order in H^1, which is consistent with the numerical results.

We now want to test our method on irregular functions. We start by showing that

$$w_1(x) = \begin{cases} 2x, x \in (0, \frac{1}{2}) \\ 2(1-x), x \in (\frac{1}{2}, 1) \end{cases} \qquad \text{and} \qquad w_2(x) = x - |x|^{\frac{2}{3}}$$

belongs to $H^1(0,1)$ and not to $H^2(0,1)$.

We start by showing this for w_1:

$$\int_0^1 |w_1|^2 dx = \frac{1}{6} < \infty \Rightarrow w_1 \in L^2(0,1)$$

Using the definition of a weak derivative we have that

$$\int_0^1 w_1' v \ dx = -\int_0^1 w_1 v' \ dx = \int_0^{\frac{1}{2}} 2v \ dx + \int_{\frac{1}{2}}^1 -2v \ dx \Rightarrow w_1'(x) = \begin{cases} 2 & x \in (0, \frac{1}{2}) \\ -2 & x \in (\frac{1}{2}, 1) \\ \alpha & x = \frac{1}{2} \end{cases}$$

where $\alpha \in \mathbb{R}$, furthermore we have that

$$\int_0^1 |w_1'|^2 dx = 4 < \infty \Rightarrow w_1' \in L^2(0,1).$$

Want to now see if w_1 has a weak second derivative, we find that

$$\int_0^1 w_1'' v dx = -\int_0^1 w_1' v' dx = -4v(\frac{1}{2}) \Rightarrow w_1'' = -4\delta(x - \frac{1}{2}),$$

but the delta-function is not a function. Thus w_1 does not have a weak second derivative, which means that $w_1 \notin H^2(0,1)$. For $w_2(x) = x - x^{-2/3}$ we have

$$\int_0^1 w_2' v \ dx = -\int_0^1 w_2 v' \ dx = \int_0^1 v dx - \frac{2}{3}\int_0^1 x^{-1/3} v dx$$

Thus we have that $w_2' = 1 - \frac{2}{3}x^{-1/3}$

In order to check if w_2 belongs in $H^1(0,1)$ we need to check that $w_2' \in L^2(0,1)$. We start by

$$\int_0^1 \mid w_2' \mid^2 \, dx = \int_0^1 \mid 1 - \frac{2}{3}x^{-\frac{1}{3}} \mid dx = -2((\frac{2}{3})^3 - (\frac{2}{3})^2) < \infty$$

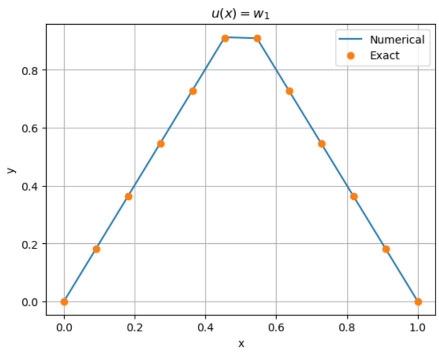
Because the integral is lower than infinity, we know that $w_2 \in L^2(0,1)$.
We now want to find w_2''
We have

$$\int_0^1 w_2'' v dx = -\int_0^1 w_2' v' dx = \frac{2}{3}[x^{-\frac{1}{3}}v]_0^1 + \frac{2}{9}\int_0^1 x^{-\frac{4}{3}} v dx$$

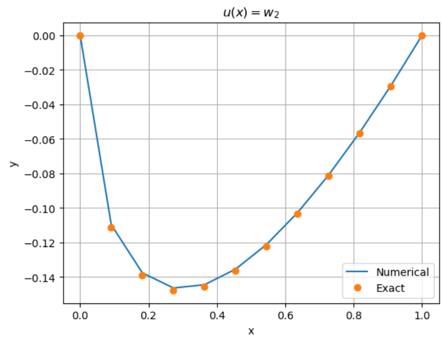
We see that $x^{-\frac{1}{3}}$ is not defined for $x = 0$ and thus we can argue that the weak derivative of w_2' does not exist, and therefore $w_2 \notin H^2(0,1)$. We now try to use the functions w_1, w_2 as test solutions, to avoid the second derivative in the F we use integration by parts on the first term to obtain

$$F(v) = \int_0^1 \alpha w_1' v' + b w_1' v c w_1 v \, dx,$$

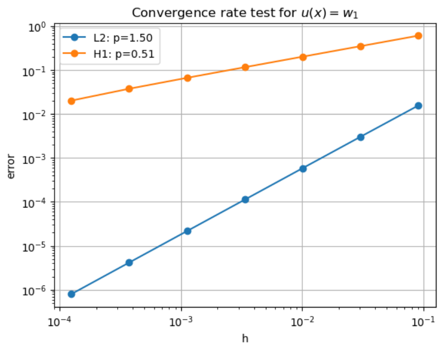
and similarly for when solving w_2. Here we could either use a general quadrature from scipy or solve the integrals analytically, where in this case the latter was chosen.



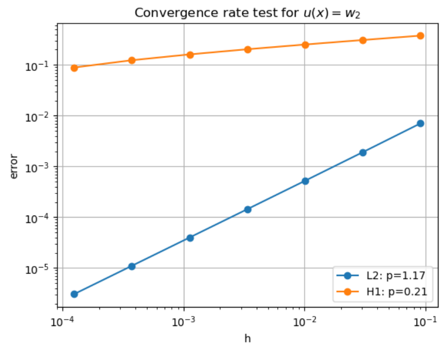
(a) Exact- and Numerical solution, using 11 nodes, $a = b = c = 1$



(b) Exact- and Numerical solution, using 11 nodes, $a = b = c = 1$



(c) Error and Convergence rates

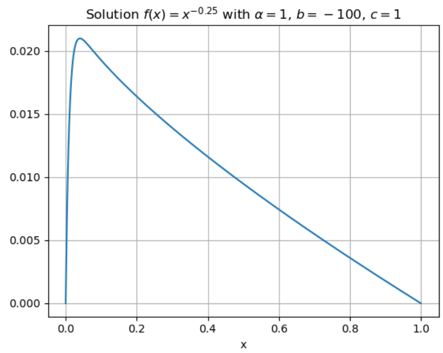


(d) Error and Convergence rates

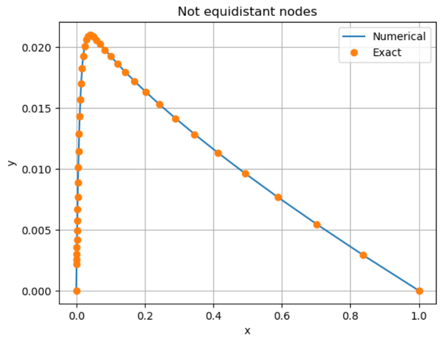
Figure 3: Numerical and exact solutions for w_1 and w_2, and corresponding error and convergence plots. In both (a) and (b) M = 40 is used.

We observe that w_1 have convergence rate 1.5 in L^2 and 0.51 in H^1, and w_2 have convergence rate 1.17 in L^2 and 0.21 in H^1. As discussed earlier, it is to expect that L^2 have a greater convergence rate than H^1. The convergence rate is much lower than that observed in REF, because w_1 and w_2 are not in H_2.

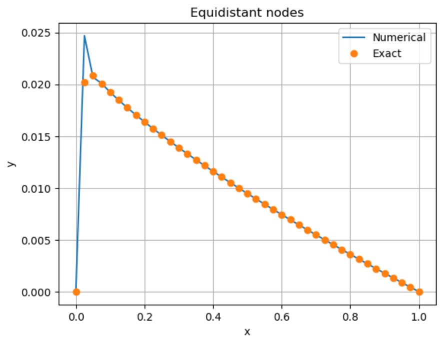
Up until this point, we have used equidistant nodes for our numerical methods. Now we want to study how not equidistant nodes can be used to decrease the error. We will test this by solving a differential equation with $f(x) = x^{-\frac{1}{4}}$



(a) Approximate "exact" solution using 1000 equidistant points, $a = c = 1, b = -100$



(b) Exact and numerical solution - 40 not equidistant nodes



(c) Exact and numerical solution - 40 equidistant nodes

The error in L^2 norm for equidistant nodes was $6.448 \cdot 10^{-4}$ and for H^1 norm it was $9.893 \cdot 10^{-2}$. The error in L^2 norm for not equidistant nodes was $1.175 \cdot 10^{-5}$ and for H^1 norm it was $3.375 \cdot 10^{-4}$. We observe that the error is lower in the case where we put more points around where the function changes dramatically. In our case, the point where the function changes the most happens near $x = 0$.