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Problem 1

a)

(i)

A queue takes the form A/B/c, where A: interarrival distribution, B: individual service time distribution, and c: number of servers. The UCC can be modeled as M/M/1 queue, where M stands for memoryless.

A: Arrivals to the UCC follows a Poission distribution, which is a memoryless stochastic process. B: the individual service times of the patients follows an exponential distribution, which is iid for each patient, and independent of arrivals. The exponential distribution is memoryless. c: There is only one bed in the UCC, that is, only one patient can be operated at a time.

(ii)

The number of patients either increase with one (birth) or decrease with one (death). All times until the next birth or death are independent and exponentially distributed. The UCC can therefore be regarded as an birth and death process, with birthrate equal to the rate of the poission distribution, λ, and deathrate equal to the parameter of the independent exponential distributions of the treatment times, μ.

(iii)

Littles Law states: $L = \lambda W$, where L: Average number of patients in the UCC, λ: interarrival rate, and W: average time spent in the UCC.

Since $\lambda < \mu$, we know that a long term limiting distribution π exists. Let π_n denote the long term amount of time we will be in state n. Note, n is both the state number and the number of patients in the UCC in that state. Then the average number of patients in the UCC can be expressed as $L = \sum_{n=0}^{\infty} n\pi_n$. An expression for π_n is given by:

$$\pi_n = \theta_n \pi_0$$

where:

$$\theta_n = \frac{\prod_{n=0}^{n-1} \lambda_n}{\prod_{n=1}^{n} \mu_n} = \left(\frac{\lambda}{\mu}\right)^n$$

since $\lambda_n = \lambda$ and $\mu_n = \mu$ in our case. Hence:

$$\pi_n = \left(\frac{\lambda}{\mu}\right)^n \pi_0 \tag{1}$$

$$L = \sum_{n=0}^{\infty} n \left(\frac{\lambda}{\mu}\right)^n \pi_0$$

since $\lambda_n = \lambda$ and $\mu_n = \mu$ in our case.

We need to find an expression for π_0. Using the fact that the limiting distrubutions sum up to one:

$$\sum_{n=0}^{\infty} \pi_n = 1$$

and combining this with 1, yields:

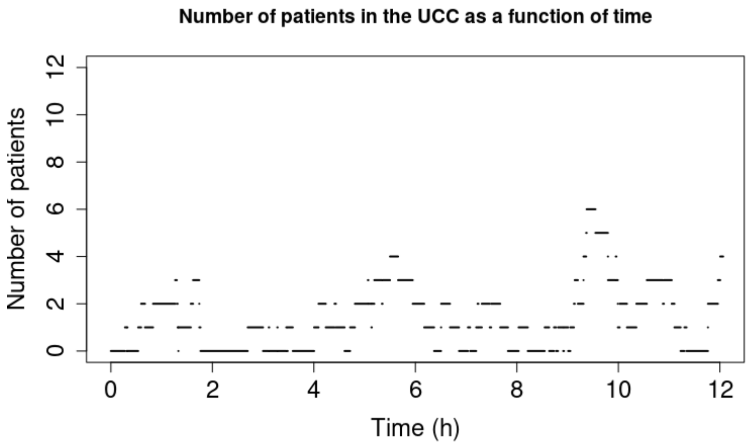

Figure 1: Number of patients in the UCC as a function of time, 12 first hours

$$\pi_0 = \frac{1}{\sum_{n=0}^{\infty} \left(\frac{\lambda}{\mu}\right)^n}$$

The denominator is a geometric sum, which converges since $\mu > \lambda$. Then:

$$\pi_0 = 1 - \frac{\lambda}{\mu}$$

which gives L:

$$L = \sum_{n=0}^{\infty} n \left(\frac{\lambda}{\mu}\right)^n \left(1 - \frac{\lambda}{\mu}\right) = \frac{\frac{\lambda}{\mu}}{1 - \frac{\lambda}{\mu}}$$

Using this to find the waiting time W:

$$W = \frac{L}{\lambda} = \frac{\frac{\lambda}{\mu}}{1 - \frac{\lambda}{\mu}} \frac{1}{\lambda} = \frac{1}{\mu - \lambda}$$

b)

(i), (ii), (iii)

See attached code file

(iv)

1 shows a realization of the number of patients in the UCC the first 12 hours. We can not see any clear convergence/divergence in this amount of time, so we can not conclude much from this plot.

(v)

The expected time is calculated by using Littles law, $W = \frac{L}{\lambda}$. To do so, we first calculate the average number of patients in the UCC, L. This is done by calculating the long term proportion of time spent in each state, and then using the fact that:

$$L = \sum_{n=0}^{N} n\pi_n$$

where N is the total number of states. π_n is calculated by taking the ratio between time spent in state n, and the total runtime of the simulation, which in our case is 12 hours.

Used 30 simulations to calculate a 95% confidence interval for the expected time a patient spends in the UCC. This interval came out to be [0.938463, 1.04684]. This matches up pretty good with the the theoretical calculated vaule:

$$W = \frac{L}{\lambda} = \frac{1}{\mu - \lambda} = \frac{1}{6-5}$$

.

This result gives us confidence in the simulation.

c)

(i)

$\{U(t) : t \geq 0\}$ is a queue and can be written as A/B/c. To be more precise, it is a M/M/1 queue since:

A: Arrivals of patients are poission, which is a memoryless stochastic process
B: The service time is iid exponentially distributed, and independent of arrivals
C: There is still one server. Only one operation can take place at a time.

(ii)

Arrival rate of all patients is still λ, but upon arrival, they are labeled as "urgent" with a propability p. Therefore, the arrival rate of "urgent patients" is given by λp. This means that "normal patients" now have the arrival rate $(1-p)\lambda$. If we sum up these rates, we get the total rate λ.

(iii)

Same derivation for L as in task a)(iii), but setting λ equal to λp, which is the arrival rate for "urgent patients". The death rate for "urgent patients" is the same as for "normal patients", as they get the same service. Therefore:

$$L_U = \frac{\frac{\lambda p}{\mu}}{1 - \frac{\lambda p}{\mu}}$$

d)

(i)

$\{N(t) : t \geq 0\}$ cannot be a M/M/1 queue because the probability of transitioning will be affected by which state one arrived from. For example, the transition $i+1 \to i$ can only happen if there are no urgent patients. However the transition $i-1 \to i$ can happen with arbitrary many urgent patients. The probability for a birth increases with the number of urgent patients. Thus, the rates and probabilities for transitioning depends on which state one came from.

(ii)

The long run number of patients can also be given by $L_X = L_U + L_N$, where:

$$L_U = \frac{\frac{p\lambda}{\mu}}{1 - \frac{p\lambda}{\mu}}$$

$$L_X = \frac{\frac{\lambda}{\mu}}{1 - \frac{\lambda}{\mu}}$$

and L_N is the number of long run "normal patients". This has to hold, as the total number of patients at any timepoint t is given by $X = L + U$. Since L_X and L_U are known, we can find the long run number of "normal patients":

$$L_N = L_X - L_U = \frac{\frac{\lambda}{\mu}}{1 - \frac{\lambda}{\mu}} - \frac{\frac{p\lambda}{\mu}}{1 - \frac{p\lambda}{\mu}} = \frac{\lambda\mu(1-p)}{(\mu - \lambda)(\mu - p\lambda)}$$

e)

(i)

$$W_U = L_U \frac{1}{\lambda_U} = \frac{\frac{p\lambda}{\mu}}{1 - \frac{p\lambda}{\mu}} \frac{1}{p\lambda} = \frac{\frac{p\lambda}{\mu}}{p\lambda - \frac{(p\lambda)^2}{\mu}} = \frac{\frac{1}{\mu}}{1 - \frac{p\lambda}{\mu}} = \frac{1}{\mu - p\lambda}$$

Which is what we wanted to show

(ii)

$$W_N = L_N \frac{1}{\lambda_N} = \frac{\lambda\mu(1-p)}{(\mu - \lambda)(\mu - p\lambda)} \frac{1}{(1-p)\lambda} = \frac{\mu}{(\mu - \lambda)(\mu - p\lambda)}$$

Which is what we wanted to show

f)

(i)

From 2, we see that the waiting time for "normal patients" (in red) increase as the number of "urgent patients" (in green) increase.

(ii)

$p \approx 0$:

In this situation, there are for all practical reasons, only "normal patients" in the UCC. Then the waiting time is approximately equal to one (equal to one when $p = 0$), as the theoretical calculation suggest.

$p \approx 1$:

In this situation, there are for all practical reasons, only "urgent patients" in the UCC. Then the waiting time for these are one hour, as the theoretical calculation suggest. The waiting time for the "normal patients" tend to increases rapidly.

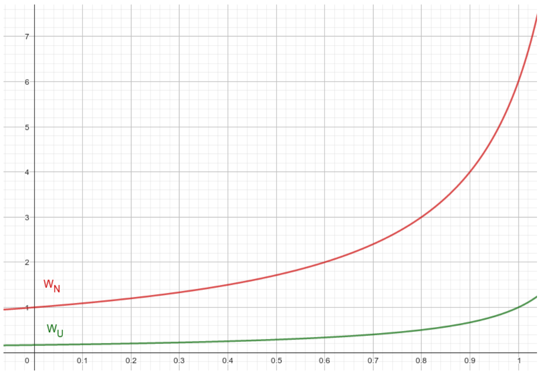

Figure 2: Number of patients in the UCC as a function of time, 12 first hours

(iii)

$p \approx 0$:

$$W_U \approx \frac{1}{\mu} = \frac{1}{6}$$

$$W_N \approx \frac{1}{(\mu - \lambda)\mu} = \frac{1}{\mu - \lambda} = \frac{1}{6-5} = 1$$

$p \approx 1$:

$$W_U \approx \frac{1}{\mu - \lambda} = \frac{1}{6-5} = 1$$

$$W_N = \frac{\mu}{(\mu - \lambda)^2} = \frac{6}{(6-5)^2} = 6$$

(iv)

$$W_N = \frac{\mu}{(\mu - \lambda)(\mu - p\lambda)} = 2 \Leftrightarrow p = \frac{2\mu - \frac{\mu}{(\mu - \lambda)}}{2\lambda} = \frac{\mu}{\lambda} - \frac{\mu}{2\lambda(\mu - \lambda)}$$

Let $\lambda = 5$ and $\mu = 6$:

$$p = \frac{6}{5} - \frac{6}{10(6-5)} = \frac{3}{5}$$

g)

(i), (ii)

See handed in code

(iii)

(iv)

While there are urgent patients in the UCC, the amount of normal patients in the UCC cannot decrease. Thus, these will increase until there are no more urgent patients in the UCC. This has

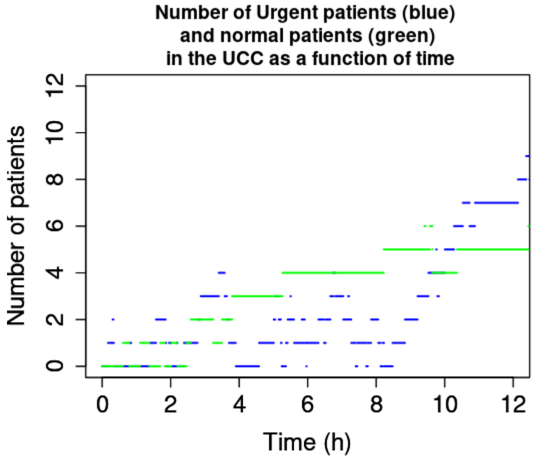

Figure 3: Number of urgent and normal patients in the UCC over 12 hours

been taken into account in the simulation. This is particularly evident in the time interval 2-4 hours in 3

The CI for the Urgent patients is given by [0.545, 0.589]. The theoretical calculated value is given by 0.5, which falls outside this interval. This can be a result of randomness, as other runs gave a CI where ths theoretical calculated value was in the CI.

The CI for the Normal patients is given by [2.634, 3.624]. The theoretical calculated value is 3, which is in the CI.

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Problem 2

a)

To find the conditional mean and covariance matrix for the 51 grid point, the equation for conditional mean and covariance was used, i.e. for

$$X = (X_A, X_B)^T \sim N_n \left(\begin{pmatrix} \mu_A \\ \mu_B \end{pmatrix}, \begin{pmatrix} \Sigma_{AA} & \Sigma_{AB} \\ \Sigma_{BA} & \Sigma_{BB} \end{pmatrix} \right)$$

then the conditional mean and covariance are

$$\mu_C = \mu_A + \Sigma_{AB}\Sigma_{BB}^{-1}(x_B - \mu_B)$$
$$\Sigma_C = \Sigma_{AA} - \Sigma_{AB}\Sigma_{BB}^{-1}\Sigma_{BA},$$

where x_B are the values being conditioned on.
The conditional mean and covariance matrix is found by calculating an update to the original mean and covariance matrix.

Then a realization for $Y(\theta)$ for each θ using the rnorm function in R with its respective conditional mean and conditional marginal variance, see fig 4

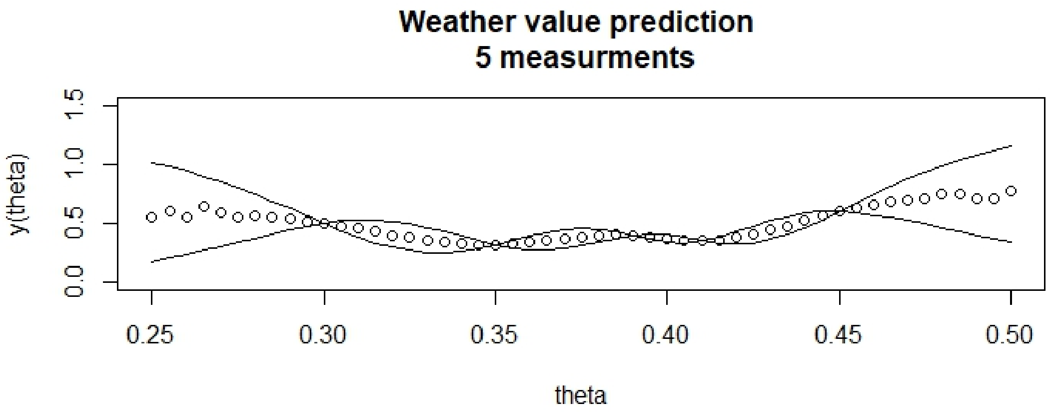

Figure 4: Weather value prediction with 90% prediction interval, conditioned on 5 measured data

b)

The probability that $Y(\theta) < 0.3$ for each θ was found by using the pnorm function in R with its respective conditional mean and conditional marginal variance. The plot of this can be seen in fig 5

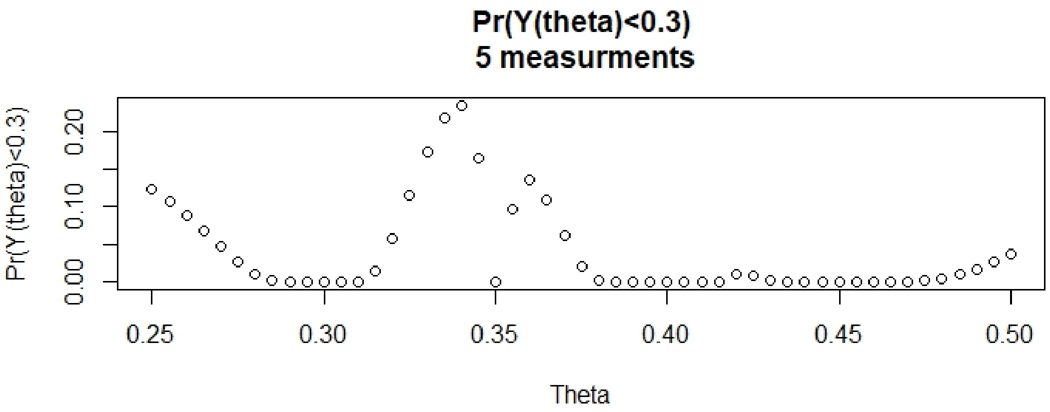

Figure 5: Plot of probability that $Y(\theta) < 0.3$ as a function of θ with 5 measured data

c)

The same process from 2a) and 2b) is used here, but with an extra measurement. See figure 6 for a realization of $Y(\theta)$, and see figure 7 for the probability that $Y(\theta) < 0.3$ as a function of θ.

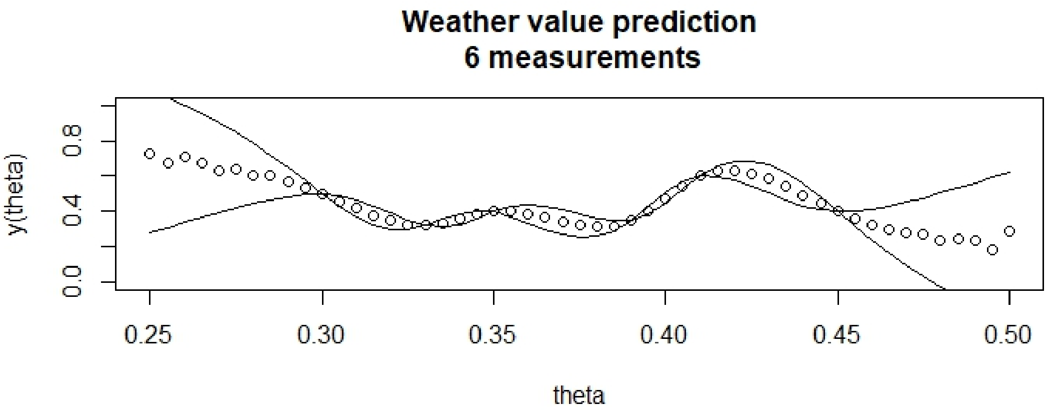

Figure 6: Weather value prediction with 90% prediction interval, conditioned on 6 measured data

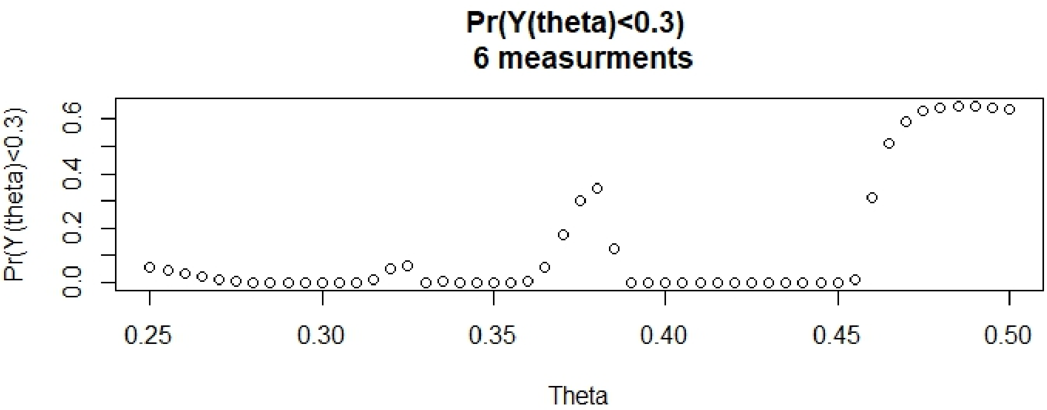

Figure 7: Plot of probability that $Y(\theta) < 0.3$ as a function of θ with 6 measured data

To have the best chance to achieve that $y(\theta) < 0.3$ scientist should do a simulation with $\theta = 0.485$. The reason being that from figure 7 $\theta = 0.485$ has the highest chance to give $Y(\theta) < 0.3$ with the measured data.