1. Consider the triangle

$$a = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$
$$b = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$$
$$c = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

For each of the three vertices, determine the point that is twice as close to that vertex than to the other two (in the barycentric sense).

Solution: For a point to be "twice as close" to one than to the other two, the barycentric coordinates must be $\frac{1}{2}$, $\frac{1}{4}$, and $\frac{1}{4}$. We get:

$$p = \frac{1}{2}\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + \frac{1}{4}\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + \frac{1}{4}\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} \frac{3}{4} \\ \frac{1}{2} \\ 0 \end{pmatrix}$$
$$q = \frac{1}{4}\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + \frac{1}{2}\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + \frac{1}{4}\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} \frac{3}{4} \\ \frac{3}{4} \\ 0 \end{pmatrix}$$
$$r = \frac{1}{4}\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + \frac{1}{4}\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + \frac{1}{2}\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} \frac{1}{2} \\ \frac{3}{4} \\ 0 \end{pmatrix}$$

Where p is twice as close to a as to b and c, q is twice as close to b as to a and c, and r is twice as close to c as to a and b (in the barycentric sense).

2. Consider the triangle

$$a = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$
$$b = \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}$$
$$c = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

If a has the color $(132, 75, 0)$, b has the color $(12, 144, 234)$, and c has the color $(252, 99, 198)$, what color does the barycenter of the triangle have using linear interpolation?

The barycenter has, by definition, the barycentric coordinates $\frac{1}{3}, \frac{1}{3}, \frac{1}{3}$. To calculate the color, we weigh the vertex colors with these same coordinates:

$$\text{color}(p) = \frac{1}{3} \cdot (132, 75, 0) + \frac{1}{3} \cdot (12, 144, 234) + \frac{1}{3} \cdot (252, 99, 198) = (132, 106, 144)$$

3. Calculate the matrix to rotate a 3D vector by 135 degrees around the z axis, and **then** rotate it by 45 degrees around the x axis. Apply this matrix to the vector

$$\vec{v} = \begin{pmatrix} 0 \\ 0 \\ \sqrt{2} \end{pmatrix}$$

Note:

$$\sin(45) = \cos(45) = \sin(135) = \frac{1}{\sqrt{2}}, \cos(135) = -\frac{1}{\sqrt{2}}$$

The matrix to rotate a 3D vector by 135 degrees around the z axis is given by

$$R_z = \begin{pmatrix} -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 & 0 \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

The matrix to rotate a 3D vector by 45 degrees around the x axis is given by:

$$R_x = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

To rotate around the z-axis first, that matrix has to come **last** in the multiplication:

$$R = R_x \cdot R_z = \begin{pmatrix} -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 & 0 \\ \frac{1}{2} & -\frac{1}{2} & -\frac{1}{\sqrt{2}} & 0 \\ \frac{1}{2} & -\frac{1}{2} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

4. You are given a triangle with vertices

$$a = \begin{pmatrix} 4 \\ 3 \\ 1 \end{pmatrix}$$

$$b = \begin{pmatrix} 6 \\ 4 \\ -1 \end{pmatrix}$$

$$c = \begin{pmatrix} 3 \\ 3 \\ 3 \end{pmatrix}$$

and normal vector

$$\vec{n} = \begin{pmatrix} \frac{2}{3} \\ -\frac{2}{3} \\ \frac{1}{3} \end{pmatrix}$$

The camera is located at

$$cam = \begin{pmatrix} 7 \\ 1 \\ 3 \end{pmatrix}$$

and looking in direction

$$\vec{d} = \begin{pmatrix} -2 \\ -2 \\ -1 \end{pmatrix}$$

Does the triangle face the camera?

How far from the plane of the camera is each of the three vertices (hint: The direction vector points in the same direction as the normal of the camera plane)

To determine if the triangle "faces" the camera, we need to determine if the normal vector points in the opposite direction from the view direction of the camera. We do this using the dot product:

$$\vec{d} \cdot \vec{n} = \begin{pmatrix} -2 \\ -2 \\ -1 \end{pmatrix} \cdot \begin{pmatrix} \frac{2}{3} \\ -\frac{2}{3} \\ \frac{1}{3} \end{pmatrix} = -2 \cdot \frac{2}{3} + (-2) \cdot \left(-\frac{2}{3}\right) + (-1) \cdot \frac{1}{3} = -\frac{1}{3}$$

Because the dot product is less than zero the vectors are pointing in opposite directions, and therefore the triangle is "facing" the camera. However, that is only really true if the triangle is also in front of the camera. To calculate this, we calculate the distance between the camera plane and the three vertices. What we need to do is calculate the projection of a vector pointing from the camera to a vertex onto the (unit) vector pointing into the view direction. This unit vector in camera view direction can be calculated by dividing the view direction vector by its length:

$$\vec{d_U} = \frac{\vec{d}}{|\vec{d}|} = \frac{\begin{pmatrix}-2\\-2\\-1\end{pmatrix}}{\sqrt{(-2)^2+(-2)^2+(-1)^2}} = \frac{\begin{pmatrix}-2\\-2\\-1\end{pmatrix}}{3} = \begin{pmatrix}-\frac{2}{3}\\-\frac{2}{3}\\-\frac{1}{3}\end{pmatrix}$$

Then we calculate the vector from the camera position to each vertex:

$$\vec{a}_d = a - cam = \begin{pmatrix}4\\3\\1\end{pmatrix} - \begin{pmatrix}7\\1\\3\end{pmatrix} = \begin{pmatrix}-3\\2\\-2\end{pmatrix}$$

$$\vec{b}_d = b - cam = \begin{pmatrix}6\\4\\-1\end{pmatrix} - \begin{pmatrix}7\\1\\3\end{pmatrix} = \begin{pmatrix}-1\\3\\-4\end{pmatrix}$$

$$\vec{c}_d = c - cam = \begin{pmatrix}3\\3\\3\end{pmatrix} - \begin{pmatrix}7\\1\\3\end{pmatrix} = \begin{pmatrix}-4\\2\\0\end{pmatrix}$$

The projection of each of these vectors onto the (unit) view direction vector will give us the distance of that vertex from the camera plane:

$$\vec{a}_d \cdot \vec{d_U} = \begin{pmatrix}-3\\2\\-2\end{pmatrix} \cdot \begin{pmatrix}-\frac{2}{3}\\-\frac{2}{3}\\-\frac{1}{3}\end{pmatrix} = (-3)\cdot\left(-\frac{2}{3}\right) + 2\cdot\left(-\frac{2}{3}\right) + (-2)\cdot\left(-\frac{1}{3}\right) = \frac{4}{3}$$

$$\vec{b}_d \cdot \vec{d_U} = \begin{pmatrix}-1\\3\\-4\end{pmatrix} \cdot \begin{pmatrix}-\frac{2}{3}\\-\frac{2}{3}\\-\frac{1}{3}\end{pmatrix} = (-1)\cdot\left(-\frac{2}{3}\right) + 3\cdot\left(-\frac{2}{3}\right) + (-4)\cdot\left(-\frac{1}{3}\right) = 0$$

$$\vec{c}_d \cdot \vec{d_U} = \begin{pmatrix}-4\\2\\0\end{pmatrix} \cdot \begin{pmatrix}-\frac{2}{3}\\-\frac{2}{3}\\-\frac{1}{3}\end{pmatrix} = (-4)\cdot\left(-\frac{2}{3}\right) + 2\cdot\left(-\frac{2}{3}\right) + 0\cdot\left(-\frac{1}{3}\right) = \frac{4}{3}$$

This means that a and c each are $\frac{4}{3}$ units away (in front of!) the camera plane, while b is **on** the camera plane.

5. The player is at

$$p = \begin{pmatrix}1\\2\\1\end{pmatrix}$$

and looking in direction

$$d = \begin{pmatrix}2\\1\\2\end{pmatrix}$$

If they can see anything in front of them, can they see the enemy at e?

$$e = \begin{pmatrix} 3 \\ -1 \\ -2 \end{pmatrix}$$

How large does their field of view have to be *at least* to be able to see the enemy?

To determine whether the enemy is in front of or behind the player, we use the dot product between the direction the player is looking into d and a vector from the player to the enemy $e - p$.

$$d \cdot (e - p) = \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix} \cdot \left(\begin{pmatrix} 3 \\ -1 \\ -2 \end{pmatrix} - \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} \right) = \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix} \cdot \begin{pmatrix} 2 \\ -3 \\ -3 \end{pmatrix} = -5$$

Because this dot product is negative, we know that the vectors are pointing in different directions, and therefore the enemy must be behind the player, and not in the field of view. To determine the field of view that would be required, we need the angle between these two vectors. We can get that angle (or, more accurately, its cosine) by dividing the value we just got by the product of the length of the two vectors.

$$\cos(\alpha) = \frac{-5}{\left| \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix} \right| \cdot \left| \begin{pmatrix} 2 \\ -3 \\ -3 \end{pmatrix} \right|} = \frac{-5}{3 \cdot \sqrt{22}} \approx -0.36$$

The angle itself is then about 110 degrees, which means the player would need a field of view of at least 220 degrees to be able to see the enemy.

6. Given a triangle with vertices

$$a = \begin{pmatrix} 3 \\ 1 \\ 2 \end{pmatrix}$$
$$b = \begin{pmatrix} 4 \\ 1 \\ 3 \end{pmatrix}$$
$$c = \begin{pmatrix} -1 \\ -2 \\ 4 \end{pmatrix}$$

Calculate the Barycenter of this triangle. The vertices are assigned the colors

$$\text{color}(a) = (42, 126, 222)$$
$$\text{color}(b) = (126, 204, 54)$$
$$\text{color}(c) = (252, 252, 252)$$

What is the color of the Barycenter?

Which colors do the middle points of the three edges have?

First, the barycenter has, by definition, the barycentric coordinates $(\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$, and therefore is located at:

$$p = \frac{1}{3} \cdot \begin{pmatrix} 3 \\ 1 \\ 2 \end{pmatrix} + \frac{1}{3} \cdot \begin{pmatrix} 4 \\ 1 \\ 3 \end{pmatrix} + \frac{1}{3} \cdot \begin{pmatrix} -1 \\ -2 \\ 4 \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \\ 3 \end{pmatrix}$$

The *color* of the barycenter is calculated as the same mixture of the colors of the vertices as the point itself:

$$\begin{aligned} \mathrm{color}(p) =& \frac{1}{3} \cdot \mathrm{color}(a) + \frac{1}{3} \cdot \mathrm{color}(b) + \frac{1}{3} \cdot \mathrm{color}(c) = \\ & \frac{1}{3} \cdot (42, 126, 222) + \frac{1}{3} \cdot (126, 204, 54) + \frac{1}{3} \cdot (252, 252, 252) = (140, 194, 176) \end{aligned}$$

Finally, the middle point of the edges each have two barycentric coordinates with values $\frac{1}{2}$ and 0 for the third one, and the same holds for the mixtures of the colors. Let ab be the point halfway between a and b, bc the point halfway between b and c, and ac the point halfway between a and c. Then we have for their colors:

$$\begin{aligned} \mathrm{color}(ab) &= \frac{1}{2} \cdot \mathrm{color}(a) + \frac{1}{2} \cdot \mathrm{color}(b) = \frac{1}{2} \cdot (42, 126, 222) + \frac{1}{2} \cdot (126, 204, 54) = (84, 165, 138) \\ \mathrm{color}(bc) &= \frac{1}{2} \cdot \mathrm{color}(b) + \frac{1}{2} \cdot \mathrm{color}(c) = \frac{1}{2} \cdot (126, 204, 54) + \frac{1}{2} \cdot (252, 252, 252) = (189, 228, 153) \\ \mathrm{color}(ac) &= \frac{1}{2} \cdot \mathrm{color}(a) + \frac{1}{2} \cdot \mathrm{color}(c) = \frac{1}{2} \cdot (42, 126, 222) + \frac{1}{2} \cdot (252, 252, 252) = (147, 189, 237) \end{aligned}$$

7. Consider the triangle

$$a = \begin{pmatrix} 2 \\ 2 \\ 2 \end{pmatrix}$$

$$b = \begin{pmatrix} 3 \\ 3 \\ 4 \end{pmatrix}$$

$$c = \begin{pmatrix} 3 \\ 1 \\ 3 \end{pmatrix}$$

with the normal vector

$$\vec{n} = \begin{pmatrix} \frac{-1}{\sqrt{6}} \\ \frac{-1}{\sqrt{6}} \\ \frac{2}{\sqrt{6}} \end{pmatrix}$$

and a point

$$p = \begin{pmatrix} 3\sqrt{6} \\ 0 \\ 2\sqrt{6} \end{pmatrix}$$

Does the point p lie in front of or behind the triangle?

To determine which side of the triangle the point is one we use the dot product: If the normal vector of the triangle and a vector from any of the three vertices of the triangle to the point have the same direction, the point is in front of the triangle.

$$\vec{n} \cdot (p - a) = \begin{pmatrix} \frac{-1}{\sqrt{6}} \\ \frac{-1}{\sqrt{6}} \\ \frac{2}{\sqrt{6}} \end{pmatrix} \cdot \left(\begin{pmatrix} 3\sqrt{6} \\ 0 \\ 2\sqrt{6} \end{pmatrix} - \begin{pmatrix} 2 \\ 2 \\ 2 \end{pmatrix} \right) = \begin{pmatrix} \frac{-1}{\sqrt{6}} \\ \frac{-1}{\sqrt{6}} \\ \frac{2}{\sqrt{6}} \end{pmatrix} \cdot \begin{pmatrix} 3\sqrt{6} - 2 \\ -2 \\ 2\sqrt{6} - 2 \end{pmatrix} =$$
$$\frac{-1}{\sqrt{6}} \cdot (3\sqrt{6} - 2) + \frac{-1}{\sqrt{6}} \cdot (-2) + \frac{2}{\sqrt{6}} \cdot (2\sqrt{6} - 2) = -3 + 4 + \frac{2 + 2 - 4}{\sqrt{6}} = 1$$

Because this value is greater than 0 this means that the point is in front of the triangle.

8. Consider the triangle

$$a = \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix}$$
$$b = \begin{pmatrix} 4 \\ 3 \\ 4 \end{pmatrix}$$
$$c = \begin{pmatrix} 3.5 \\ 2.5 \\ 4.5 \end{pmatrix}$$

The texture coordinates are

$$t(a) = \left(\frac{3}{5}, \frac{3}{8} \right)$$
$$t(b) = \left(\frac{3}{10}, \frac{3}{16} \right)$$
$$t(c) = \left(\frac{3}{7}, \frac{3}{10} \right)$$

The point p is the closest point to c on the line from a to b. What texture coordinates does p have?

First, we need to determine the point p (or rather, its barycentric coordinates). We can calculate the distance between a and p as the projection of the vector from a to c onto the (unit) vector in direction from a to b. Let us call the vector from a to c $\vec{v}$, and the (unit) vector from a into the direction of b $\vec{d}$. Then we have:

$$\vec{v} = c - a = \begin{pmatrix} 3.5 \\ 2.5 \\ 4.5 \end{pmatrix} - \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix} = \begin{pmatrix} 1.5 \\ 1.5 \\ 1.5 \end{pmatrix} = \begin{pmatrix} \frac{3}{2} \\ \frac{3}{2} \\ \frac{3}{2} \end{pmatrix}$$

$$\vec{d} = \frac{b-a}{|b-a|} = \frac{\begin{pmatrix} 4 \\ 3 \\ 4 \end{pmatrix} - \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix}}{\left|\begin{pmatrix} 4 \\ 3 \\ 4 \end{pmatrix} - \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix}\right|} = \frac{\begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix}}{\left|\begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix}\right|} = \frac{\begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix}}{3} = \begin{pmatrix} \frac{2}{3} \\ \frac{2}{3} \\ \frac{1}{3} \end{pmatrix}$$

The distance of p from a is then the projection of $\vec{v}$ onto $\vec{d}$, using the dot product:

$$\vec{v} \cdot \vec{d} = \begin{pmatrix} \frac{3}{2} \\ \frac{3}{2} \\ \frac{3}{2} \end{pmatrix} \cdot \begin{pmatrix} \frac{2}{3} \\ \frac{2}{3} \\ \frac{1}{3} \end{pmatrix} = \frac{3}{2} \cdot \frac{2}{3} + \frac{3}{2} \cdot \frac{2}{3} + \frac{3}{2} \cdot \frac{1}{3} = 1 + 1 + \frac{1}{2} = 2.5$$

Note that the **total** distance from a to b, $|b-a|$ was calculated above as 3, so the distance from p to b is 0.5. With these values we can now calculate the barycentric coordinates. Basically, we need to "scale" these values to be between 0 and 1 (and sum to 1). The contribution of c to p is 0, the contribution of b is $\frac{2.5}{3} = \frac{5}{6}$, and the contribution of a is $\frac{0.5}{3} = \frac{1}{6}$ (note that the point that is closer to p also contributes more, i.e. the distances from p to each vertex have to be reversed to get the weight). To calculate the texture coordinates of p we can then use these weights directly:

$$t(p) = \frac{1}{6} \cdot t(a) + \frac{5}{6} \cdot t(b) + 0 \cdot t(c) = \frac{1}{6} \cdot \left(\frac{3}{5}, \frac{3}{8}\right) + \frac{5}{6} \cdot \left(\frac{3}{10}, \frac{3}{16}\right) = \left(\frac{7}{20}, \frac{6}{32}\right)$$