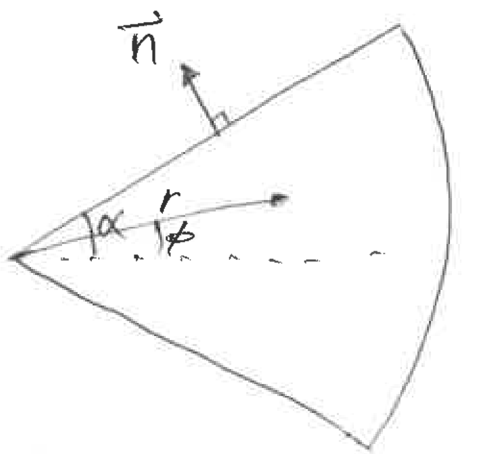
Capillary 1 : Small wedge numerics



Capillary boundary value problem after scaling:

$$\vec{\nabla} \cdot \frac{\vec{\nabla} T}{\sqrt{1+(\vec{\nabla} T)^2}} = T \qquad (c1.1)$$

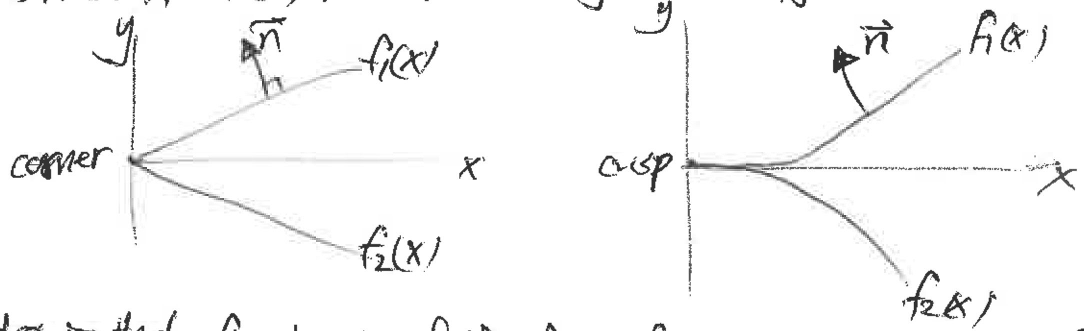
$$\vec{n} \cdot \frac{\vec{\nabla} T}{\sqrt{1+(\vec{\nabla} T)^2}} = \cos\gamma \qquad (c1.2)$$

For small wedges, i.e. $\alpha+\gamma<\pi/2$, corner height is infinite (see Concus & Finn (1970) On a class of capillary surfaces. J. Analyse Math., 23, 65–70), with leading order asymptote

$$T \sim \frac{\cos\phi - \sqrt{k^2-\sin^2\phi}}{kr} \qquad (c1.3)$$

1. Singularity removal idea, Aoki & De Sterck (2014)

See Aoki & De Sterck (2014). Numerical study of unbounded capillary surfaces. Pacific J. Math., 267(1). <https://doi.org/10.2140/pjm.2014.267.1>.



The idea is that, for domains of the form $f_2(x)<y<f_1(x)$, with $f_1(0^+)=f_2(0^+)=0$ and $f_1'(0^+)$ & $f_2'(0^+)$ finite, we have

$$T \sim \frac{O(1)}{f_1(x)-f_2(x)}. \qquad (c1.4)$$

Therefore, putting

$$T = \frac{V(x,y)}{f_1(x)-f_2(x)} \qquad (c1.5)$$

will result in bounded V, which is much nicer to compute than unbounded T. For general f_1, f_2, one needs an horrible non-orthogonal coordinate transformation. Since here we shall only consider simple, straight wedges, we stick with nice and orthogonal polar coordinates.

2. Change of coordinates

Following the idea of Aoki & De Sterck (2014), we put

$$T(r,\phi) = \frac{H(r,\phi)}{r} \tag{c1.6}$$

so that H will be bounded. For brevity, put

$$K = \frac{1}{\sqrt{1+(\vec{\nabla}T)^2}} \tag{c1.7}$$

Then (c1.1) & (c1.2) become

$$\vec{\nabla}\cdot[K\vec{\nabla}T] = T \tag{c1.8}$$

$$\vec{n}\cdot[K\vec{\nabla}T] = \cos\gamma \tag{c1.9}$$

2.1 Laplace-Young PDE

observe that

$$\vec{\nabla}T = \frac{\partial T}{\partial r}\vec{a}_r + \frac{\partial T}{r\partial\phi}\vec{a}_\phi \tag{c1.10}$$

$$K\vec{\nabla}T = K\frac{\partial T}{\partial r}\vec{a}_r + \frac{K}{r}\frac{\partial T}{\partial\phi}\vec{a}_\phi \tag{c1.11}$$

$$\vec{\nabla}\cdot[K\vec{\nabla}T] = \frac{1}{r}\left[\frac{\partial}{\partial r}\left(rK\frac{\partial T}{\partial r}\right) + \frac{\partial}{\partial\phi}\left(\frac{K}{r}\frac{\partial T}{\partial\phi}\right)\right] \tag{c1.12}$$

$$= \frac{1}{r}\left[\frac{\partial}{\partial r}\left(rK\frac{\partial}{\partial r}\left\{\frac{H}{r}\right\}\right) + \frac{\partial}{\partial\phi}\left(\frac{K}{r}\frac{\partial}{\partial\phi}\left\{\frac{H}{r}\right\}\right)\right]$$

$$= \frac{1}{r}\left[\frac{\partial}{\partial r}\left(rK\left\{\frac{1}{r}\frac{\partial H}{\partial r} - \frac{H}{r^2}\right\}\right) + \frac{\partial}{\partial\phi}\left(\frac{K}{r^2}\frac{\partial H}{\partial\phi}\right)\right]$$

$$= \frac{1}{r}\left[\frac{\partial}{\partial r}\left(K\frac{\partial H}{\partial r} - \frac{K}{r}\cdot H\right) + \frac{\partial}{\partial\phi}\left(\frac{K}{r^2}\frac{\partial H}{\partial\phi}\right)\right]$$

$$= \frac{1}{r}\left[\frac{\partial}{\partial r}\left(K\frac{\partial H}{\partial r}\right) + \frac{\partial}{\partial\phi}\left(\frac{K}{r^2}\frac{\partial H}{\partial\phi}\right) - \frac{\partial}{\partial r}\left(\frac{K}{r}\cdot H\right)\right]$$

$$= \frac{1}{r}\left[\left(\frac{\partial}{\partial r}\ \frac{\partial}{\partial\phi}\right)\begin{pmatrix}K & 0\\ 0 & K/r^2\end{pmatrix}\begin{pmatrix}\partial H/\partial r\\ \partial H/\partial\phi\end{pmatrix} - \left(\frac{\partial}{\partial r}\ \frac{\partial}{\partial\phi}\right)\begin{pmatrix}K/r\\ 0\end{pmatrix}H\right]$$

$$= \frac{1}{r}\left(\frac{\partial}{\partial r}\ \frac{\partial}{\partial\phi}\right)\left[\begin{pmatrix}K & 0\\ 0 & K/r^2\end{pmatrix}\begin{pmatrix}\partial H/\partial r\\ \partial H/\partial\phi\end{pmatrix} - \begin{pmatrix}K/r\\ 0\end{pmatrix}H\right]$$

$$= \frac{1}{r}\mathbb{D}\cdot[\mathbb{K}\mathbb{D}H - \mathbb{V}H], \tag{c1.13}$$

where

$$\mathbb{D} = \begin{pmatrix}\partial/\partial r\\ \partial/\partial\phi\end{pmatrix}, \tag{c1.14}$$

$$\mathbb{K} = \begin{pmatrix}K & 0\\ 0 & K/r^2\end{pmatrix}, \tag{c1.15}$$

$$\mathbb{V} = \begin{pmatrix}K/r\\ 0\end{pmatrix}. \tag{c1.16}$$

Therefore (cl.8) becomes

$$\mathbb{D}\cdot[\mathbb{K}\mathbb{D}H - \mathbb{v}H] = H . \qquad (cl.17)$$

2.2 Constant contact angle boundary condition

Observe that

$$\vec{n} = n_r\vec{a}_r + n_\phi\vec{a}_\phi . \qquad (cl.18)$$

Dotting this with (cl.11), we have

$$\begin{aligned}
\vec{n}\cdot[K\vec{\nabla}T] &= n_r K\frac{\partial T}{\partial r} + n_\phi\frac{K}{r}\frac{\partial T}{\partial\phi} \qquad (cl.19)\\
&= n_r K\left\{\frac{1}{r}\frac{\partial H}{\partial r} - \frac{H}{r^2}\right\} + n_\phi\frac{K}{r}\left\{\frac{1}{r}\frac{\partial H}{\partial\phi}\right\}\\
&= \frac{n_r}{r}\left(K\frac{\partial H}{\partial r} - \frac{K}{r}H\right) + n_\phi\left(\frac{K}{r^2}\frac{\partial H}{\partial\phi}\right)\\
&= \frac{n_r}{r}\left(K\frac{\partial H}{\partial r}\right) + n_\phi\left(\frac{K}{r^2}\frac{\partial H}{\partial\phi}\right) - \frac{n_r}{r}\left(\frac{K}{r}H\right)\\
&= \begin{pmatrix}\frac{n_r}{r} & n_\phi\end{pmatrix}\begin{pmatrix}K & 0\\ 0 & K/r^2\end{pmatrix}\begin{pmatrix}\partial H/\partial r\\ \partial H/\partial\phi\end{pmatrix} - \begin{pmatrix}\frac{n_r}{r} & n_\phi\end{pmatrix}\begin{pmatrix}K/r\\ 0\end{pmatrix}H\\
&= \mathbb{n}\cdot[\mathbb{K}\mathbb{D}H] - \mathbb{n}\cdot[\mathbb{v}H]\\
&= \mathbb{n}\cdot[\mathbb{K}\mathbb{D}H - \mathbb{v}H]
\end{aligned}$$

where (cl.20)

$$\mathbb{n} = \begin{pmatrix}n_r/r\\ n_\phi\end{pmatrix} .$$

2.3 Boundary condition along $r=0$

Simply use the Dirichlet condition consistent with the asymptotic form (cl.3):

$$H|_{r=0} = \frac{\cos\phi - \sqrt{k^2 - \sin^2\phi}}{k} \qquad (cl.21)$$

2.4 Are $\mathbb{K}$ and $\mathbb{v}$ finite?

We see that $\mathbb{K}$ has the entry K/r^2, while $\mathbb{v}$ has K/r.

Observe that

$$\begin{aligned}
(\vec{\nabla}T)^2 &= \left(\frac{\partial T}{\partial r}\right)^2 + \left(\frac{\partial T}{r\partial\phi}\right)^2\\
&= \left\{\frac{1}{r}\frac{\partial H}{\partial r} - \frac{H}{r^2}\right\}^2 + \left\{\frac{1}{r^2}\frac{\partial H}{\partial\phi}\right\}^2\\
&= \frac{1}{r^4}\left[\left(r\frac{\partial H}{\partial r} - H\right)^2 + \left(\frac{\partial H}{\partial\phi}\right)^2\right]
\end{aligned} \qquad (cl.22)$$

Let[*]

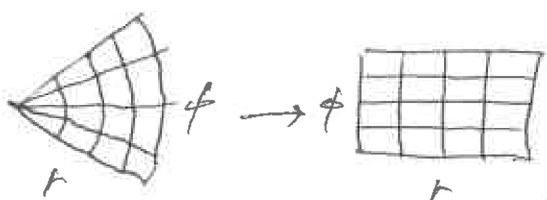
$$\boxed{\begin{aligned}\tilde{P} &= r\frac{\partial H}{\partial r} - H \\ \tilde{Q} &= \frac{\partial H}{\partial \phi}\ ,\end{aligned}}$$ (c1.23) (c1.24)

Then
$$(\vec{\nabla} T)^2 = \frac{\tilde{P}^2+\tilde{Q}^2}{r^4}\ ,$$
and hence
$$\begin{aligned} K &= \frac{1}{\sqrt{1+(\vec{\nabla}T)^2}} \\ &= \frac{1}{\sqrt{1+(\tilde{P}^2+\tilde{Q}^2)/r^4}} \\ &= \frac{r^2}{\sqrt{r^4+\tilde{P}^2+\tilde{Q}^2}} \qquad\qquad (c1.25) \\ &= r^2 C\ , \end{aligned}$$
where
$$\boxed{C = \frac{1}{\sqrt{r^4+\tilde{P}^2+\tilde{Q}^2}}\ ,} \qquad (c1.26)$$
which is finite, even at $r=0$.

2.5 Summary

with
$$\mathbb{D} = \begin{pmatrix}\partial/\partial r \\ \partial/\partial\phi\end{pmatrix}, \quad \mathbb{n} = \begin{pmatrix} n_r/r \\ n_\phi \end{pmatrix}, \qquad (c1.27)$$
$$\mathbb{K} = \begin{pmatrix} r^2C & 0 \\ 0 & C \end{pmatrix}, \qquad (c1.28)$$
$$\mathbb{V} = \begin{pmatrix} rC \\ 0 \end{pmatrix}, \qquad (c1.29)$$
the capillary boundary value problem becomes
$$\mathbb{D}\cdot[\mathbb{K}\mathbb{D}H - \mathbb{V}H] = H \qquad (c1.30)$$
$$\mathbb{n}\cdot[\mathbb{K}\mathbb{D}H - \mathbb{V}H] = \cos\gamma \qquad (c1.31)$$
$$H|_{r=0} = \frac{\cos\phi - \sqrt{k^2-\sin^2\phi}}{k}\ , \qquad (c1.32)$$
with (r,ϕ) to be interpreted as "formally rectangular" coordinates.

~~① Note that $P = \tilde{P}/r^2$ & $K =$~~



3. Boundary tracing

3.1 Derivatives

We have

$$P = \frac{\partial T}{\partial r} = \frac{1}{r}\frac{\partial H}{\partial r} - \frac{H}{r^2} = \frac{1}{r^2}\left(r\frac{\partial H}{\partial r} - H\right) = \frac{\tilde{P}}{r^2} \qquad (c1.33)$$

$$Q = \frac{\partial T}{r\partial\phi} = \frac{1}{r^2}\frac{\partial H}{\partial \phi} = \frac{\tilde{Q}}{r^2}, \qquad (c1.34)$$

with $\tilde{P}, \tilde{Q}$ defined as in (c1.23) and (c1.24).

3.2 Flux function

We have

$$F = \cos\gamma\sqrt{1+(\vec{\nabla}T)^2} \qquad (c1.35)$$

$$= \cos\gamma\sqrt{1+(\tilde{P}^2+\tilde{Q}^2)/r^4}$$

$$= \frac{1}{r^2}\cos\gamma\sqrt{r^4+\tilde{P}^2+\tilde{Q}^2}$$

$$= \frac{\tilde{F}}{r^2}, \qquad (c1.36)$$

where

$$\boxed{\tilde{F} = \cos\gamma\sqrt{r^4+\tilde{P}^2+\tilde{Q}^2}} \qquad (c1.37)$$

3.3 Viability function

We have

$$\Phi = \sin^2\gamma(\vec{\nabla}T)^2 - \cos^2\gamma \qquad (c1.38)$$

$$= \sin^2\gamma\cdot\frac{\tilde{P}^2+\tilde{Q}^2}{r^4} - \cos^2\gamma$$

$$= \frac{1}{r^4}\left[\sin^2\gamma\,(\tilde{P}^2+\tilde{Q}^2) - r^4\cos^2\gamma\right]$$

$$= \frac{1}{r^4}\tilde{\Phi}, \qquad (c1.39)$$

where

$$\boxed{\tilde{\Phi} = \sin^2\gamma\,(\tilde{P}^2+\tilde{Q}^2) - r^4\cos^2\gamma}, \qquad (c1.40)$$

and hence

$$\sqrt{\Phi} = \frac{1}{r^2}\sqrt{\tilde{\Phi}} \qquad (c1.41)$$

3.4 Tracing system of ODEs

And therefore, (48) becomes

$$\boxed{\begin{aligned} \frac{dr}{ds} &= \frac{-(\tilde{Q}/r^2)(\tilde{F}/r^2) \pm (\tilde{P}/r^2)(\sqrt{\tilde{\Phi}}/r^2)}{(\tilde{P}^2+\tilde{Q}^2)/r^4} = \frac{-\tilde{Q}\tilde{F} \pm \tilde{P}\sqrt{\tilde{\Phi}}}{\tilde{P}^2+\tilde{Q}^2} && \text{(c1.42)} \\ \frac{r\,d\phi}{ds} &= \frac{+(\tilde{P}/r^2)(\tilde{F}/r^2) \pm (\tilde{Q}/r^2)(\sqrt{\tilde{\Phi}}/r^2)}{(\tilde{P}^2+\tilde{Q}^2)/r^4} = \frac{+\tilde{P}\tilde{F} \pm \tilde{Q}\sqrt{\tilde{\Phi}}}{\tilde{P}^2+\tilde{Q}^2}. && \text{(c1.43)} \end{aligned}}$$

Basically nothing has changed, except that all quantities on the right hand side now carry tildes.

3.5 Critical terminal point

This is $(r,\phi) = (r_0, 0)$, r_0 being the solution to

$$-\left.\frac{\partial I}{\partial r}\right|_{\phi=0} = \cot\gamma_0$$

or

$$\boxed{-\left.\frac{\tilde{P}}{r^2}\right|_{\phi=0} = \cot\gamma_0}, \qquad \text{(c1.44)}$$

where γ_0 is the tracing contact angle.