Radiation 2: Line source

Consider the line source solution to (r7), which is logarithmic in $r = \sqrt{x^2+y^2}$. The only way this can make dimensional sense is if

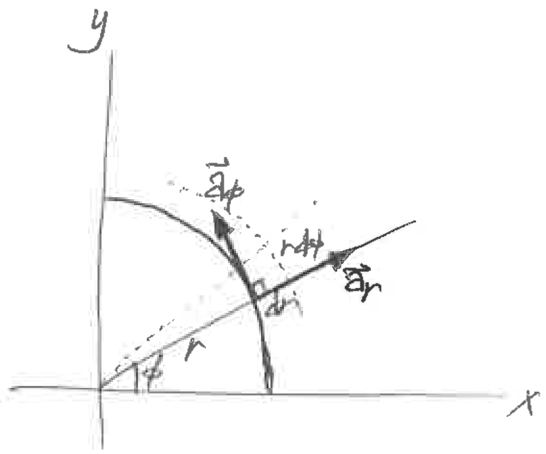
$$\boxed{T = T_0 \log \frac{r_0}{r}} \qquad \text{(r2.1)}$$

for some temperature T_0 and radius r_0. Note that the region $r > r_0$ is unphysical, since there $T < 0$.

1. Coordinates

We ~~have~~ work in polar coordinates (r, ϕ), ~~given by~~ for which:

$$x = r\cos\phi$$
$$y = r\sin\phi$$
$$\vec h_r = \cos\phi\, \vec a_x + \sin\phi\, \vec a_y$$
$$\vec h_\phi = -r\sin\phi\, \vec a_x + r\cos\phi\, \vec a_y$$
$$h_r = 1$$
$$h_\phi = r$$
$$\vec a_r = \cos\phi\, \vec a_x + \sin\phi\, \vec a_y$$
$$\vec a_\phi = -\sin\phi\, \vec a_x + \cos\phi\, \vec a_y$$



r is called the radius; ϕ is called the azimuthal angle.

2. Scaling

Put

$$\hat T = T/\tau \qquad \text{(r2.2)}$$
$$\hat r = r/\rho \qquad \text{(r2.3)}$$
$$\hat{\vec\nabla} = \rho\vec\nabla \qquad \text{(r2.4)}$$

and drop hats. Then (r8) and (r2.1) become

$$\frac{\tau}{\rho}\,\vec n\cdot\vec\nabla T = -c\tau^4 T^4$$
$$\tau T = T_0 \log\frac{r_0}{\rho r},$$

or

$$\vec n\cdot\vec\nabla T = -[c\rho\tau^3]\,T^4$$
$$T = \left[\frac{T_0}{\tau}\right]\log\frac{(r_0/\rho)}{r}.$$

We have 3 dimensionless groups but only 2 free scales τ and ρ, so one group cannot be made unity. My gut says make the logarithmic term as simple as possible, ~~see~~ so put

$$\boxed{\begin{aligned}\tau &= T_0 \\ \rho &= r_0\end{aligned}}$$ (r2.5) (r2.6)

and ~~let~~ define the dimensionless group

$$\boxed{A = \frac{1}{c r_0 T_0^3}}.$$ (r2.7)

Thus we have

$$\boxed{\begin{aligned}\vec{n}\cdot\vec{\nabla}T &= -\frac{T^4}{A} \\ T &= -\log r\end{aligned}}.$$ (r2.8) (r2.9)

~~where A is a dimensionless group~~

3. <u>Boundary tracing</u>

We have:

$$d\mu = dr$$ (r2.10)

$$d\upsilon = r\,d\phi$$ (r2.11)

$$P = \frac{\partial T}{\partial r} = -\frac{1}{r}$$ (r2.12)

$$Q = \frac{\partial T}{r\,\partial\phi} = 0$$ (r2.13)

$$F = -\frac{\log^4 r}{A}$$ (r2.14)

$$\Phi = P^2 + Q^2 - F^2 = \frac{1}{r^2} - \frac{\log^8 r}{A^2} = \frac{A^2 - r^2\log^8 r}{A^2 r^2}$$ (r2.15)

Define

$$\boxed{\psi = r\log^4 r}$$ (r2.16)

Then

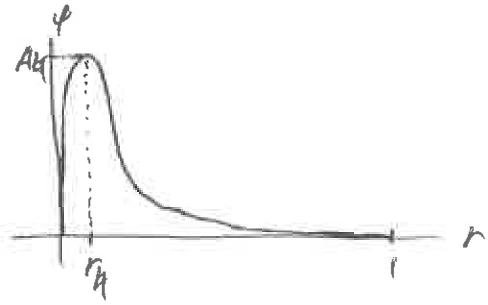
$$\Phi = \frac{A^2 - \psi^2}{A^2 r^2}.$$ (r2.17)

In scaled terms, the region $r>1$ is unphysical, so henceforth we only consider $0 \le r \le 1$.

3.1 Properties of $\psi(r)$

$$\psi(0) = 0$$
$$\psi(1) = 0$$
$$\psi' = \frac{d\psi}{dr} = r \cdot 4\log^3 r \cdot \frac{1}{r} + \log^4 r$$
$$= \underbrace{\log^3 r}_{\text{always neg.}} \; \underbrace{(4 + \log r)}_{\text{changes from neg. to pos. at } r = e^{-4}}$$

$$\psi'(0) = +\infty$$
$$\psi'(r_H) = 0$$
$$\psi'(1) = 0$$



where

$$\boxed{r_H = e^{-4}} \qquad (r2.18)$$

$$\psi'' = \frac{d^2\psi}{dr^2} = \frac{4\log^2 r}{r}(3 + \log r)$$
$$\psi''(r_H) = \frac{4(-4)^2}{e^{-4}}(3-4) = -64e^4 = -3494.28 < 0$$

We see that ψ has a single maximum on $0 \le r \le 1$ at

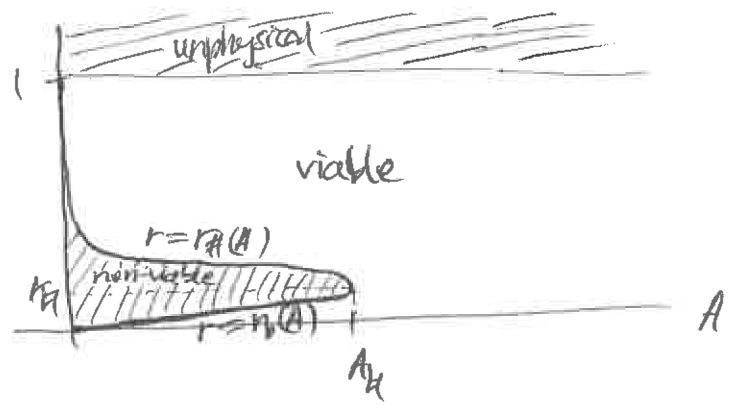
$$\boxed{r = r_H = e^{-4} = 0.01832} \qquad (r2.19)$$

where it takes the value

$$\boxed{\psi = A_H = \psi(r_H) = e^{-4}(-4)^4 = \left(\frac{4}{e}\right)^4 = 4.6888} \qquad (r2.20)$$

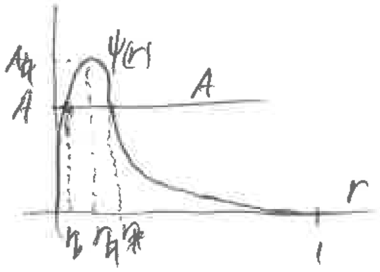
3.2 Viable domain

The geometry of the viable domain depends on the number of roots of the equation $\psi(r) = A$, which ~~also~~ specifies the terminal curves.



3.2.1 Hot regime

$$\boxed{A < A_\natural = 4.6888}$$



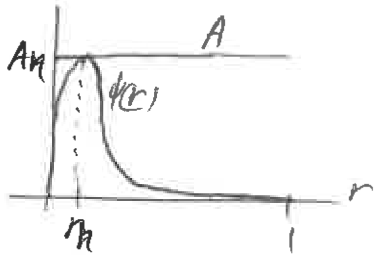
$\boxed{\psi(r) = A \text{ at } r_\flat, r_\sharp}$

where $0 < r_\flat < r_\natural < r_\sharp < 1$.

The non-viable domain forms a moat surrounding an inner viable island (containing the singularity $r=0$), and surrounded by an outer viable mainland. As A increases, the moat gets thinner. Both $r=r_\flat$ and $r=r_\sharp$ are critical terminal curves.

3.2.2 Transition

$$\boxed{A = A_\natural = 4.6888}$$



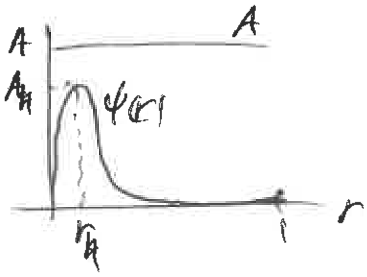
$\boxed{\psi(r) = A \text{ at } r_\natural}$ (tan.)

where $0 < r_\natural < 1$

The roots $r_\flat$ and $r_\sharp$ of the hot regime merge into $r_\natural$ and the moat disappears. The entire plane becomes viable, although $r=r_\natural$ is a critical terminal curve.

3.2.3 Cold regime

$$\boxed{A > A_\natural = 4.6888}$$



$\boxed{\psi(r) = A \text{ nowhere}}$

The entire plane is viable and there are no terminal curves.

3.3 Tracing ODE

The tracing ODE (47) becomes

$$\frac{dr}{r\,d\phi} = \frac{0 \mp F\sqrt{\Phi}}{-F^2} = \mp\frac{\sqrt{\Phi}}{-F} = \mp\frac{A}{\log^4 r}\sqrt{\frac{A^2-\psi^2}{A^2 r^2}}$$

$$\boxed{\frac{dr}{d\phi} = \mp\frac{\sqrt{A^2-\psi^2}}{\log^4 r} \cancel{= \mp\frac{r\sqrt{A^2-\psi^2}}{\psi}}} \qquad (r2.21)$$

Thus

$$\boxed{\phi = \mp\int \frac{\log^4 r\,dr}{\sqrt{A^2-\psi^2}}}. \qquad (r2.22)$$

~~Now we are seeking a closed curve $r=r(\phi)$ which surrounds the singularity $r=0$. Observe that $dr/d\phi<0$ in the upper branch of (r2.21)~~

Now we are seeking a closed curve $r=r(\phi)$, made by patching together the traced boundaries of (r2.21), which surrounds the singularity $r=0$. ~~Moving in the direction of increasing~~ Observe that $dr/d\phi$ is always negative for the upper branch of (r2.21) and positive for the lower branch, except along terminal curves $\psi(r)=A$ where it is zero as $r\to$ const ~~(r = const)~~. Travelling in the direction of increasing ϕ, a necessary condition for convexity ~~is that~~ when switching branches is that $dr/d\phi$ must not increase across the switch. It follows that switching from the upper to the lower branch must take place at a terminal point.

3.3.1 Cold regime $\boxed{A>A_H}$

Now there are no ~~critical~~ terminal curves, so the ~~lower~~ upper branch of (r2.21) has $dr/d\phi<0$ and the lower, $dr/d\phi>0$, everywhere. Thus a closed curve must consist of both the upper and the lower branch, and somewhere along it we must switch from upper to lower (in the direction of increasing ϕ). But there are no terminal points, so a concave corner will result.

3.3.2 Transition $\boxed{A=A_H}$

There is only one terminal curve $r=r_H$, a critical terminal curve, which is itself a traced boundary. Using a similar argument to the above, any other closed curve must have upper-to-lower switching ~~along~~ at a terminal point, but this is not possible since $r=r_H$ is in fact a limiting cycle (so no other traced boundary may join onto it in finite time); to see this consider a perturbation $r=r_H+\xi$. We have

$$\psi(r_H+\xi) = \psi(r_H)+\xi\psi'(r_H)+\frac{1}{2!}\xi^2\psi''(r_H)+O(\xi^3)$$
$$= A_H+\frac{1}{2}\xi^2\psi''(r_H)+O(\xi^3)$$
$$\psi^2(r_H+\xi) = A_H^2+A_H\psi''(r_H)\xi^2+O(\xi^3) \qquad (\text{recall } \psi''(r_H)<0)$$

so (r2.21) ~~becomes~~ (inverted) becomes

$$\mp\frac{d\phi}{d\xi}=\mp\frac{d\phi}{dr}=\frac{\psi(r_H+\xi)}{(r_H+\xi)\sqrt{A_H^2-\psi^2(r_H+\xi)}}=\frac{A_H+O(\xi^2)}{(r_H+\xi)\sqrt{-A_H\psi''(r_H)\xi^2+O(\xi^3)}}$$
$$=\frac{\sqrt{A_H}}{r_H\sqrt{-\psi''(r_H)}}\frac{1}{\xi}+O(1)=\frac{(4/e)^2}{e^4\cdot 8e^2}\frac{1}{\xi}+O(1)=\frac{2}{\xi}+O(1)$$

Thus $d\xi/\xi \sim \mp d\phi/2$ or $\log\xi \sim \text{const} \mp \phi/2$, and so

$$\begin{aligned} r &= r_\natural + \xi \\ &\sim r_\natural + \text{const}\cdot e^{\mp\phi/2}, \end{aligned}$$

a limiting cycle as claimed. So the only possible closed curve is the (boring) critical terminal curve $r = r_\natural$.

Any non-trivial tracing must turn in (the non-viable moat lies without), which results in a nonconvex corner.

3.3.3 <u>Hot regime</u> $\boxed{A < A_\natural}$

There are two critical terminal curves, $r = r_\flat$ and $r = r_\sharp$. The inner $r = r_\flat$ instance is boring, since any upper branch ($dr/d\phi < 0$) curve will spiral in towards the singularity $r = 0$ unless one switches to the lower branch, but this will result in a non-convex corner, since one is no longer along the terminal curve.

Nontrivial closed curves are only possible coming off the $r = r_\sharp$ terminal curve. Consider $r = r_\sharp + \xi$ where ξ is positive. We have

$$\begin{aligned} \psi(r_\sharp+\xi) &= \psi(r_\sharp) + \xi\,\psi'(r_\sharp) + O(\xi^2) \\ &= A + \psi'(r_\sharp)\,\xi + O(\xi^2) \\ \psi^2(r_\sharp+\xi) &= A^2 + 2A\psi'(r_\sharp)\,\xi + O(\xi^2) \qquad (\text{note } \psi'(r_\sharp) < 0) \end{aligned}$$

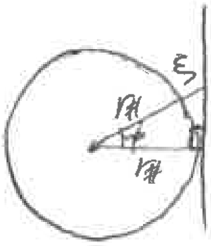
so (r2.21) (inverted) becomes

$$\begin{aligned} \mp\frac{d\phi}{d\xi} = \mp\frac{d\phi}{dr} &= \frac{\psi(r_\sharp+\xi)}{(r_\sharp+\xi)\sqrt{A^2-\psi^2(r_\sharp+\xi)}} = \frac{A+O(\xi)}{(r_\sharp+\xi)\sqrt{-2A\psi'(r_\sharp)\xi+O(\xi)}} \\ &= \frac{\sqrt{A}}{r_\sharp\sqrt{-2\psi'(r_\sharp)}}\frac{1}{\sqrt{\xi}} + O(\sqrt{\xi}) = \frac{1}{r_\sharp}\sqrt{\frac{2A}{-\psi'(r_\sharp)}}\,\frac{1}{2\sqrt{\xi}} + O(\sqrt{\xi}) \\ \mp\phi &= \frac{1}{r_\sharp}\sqrt{\frac{2A}{\psi'(r_\sharp)}}\,\sqrt{\xi} + O(\xi^{3/2}) = \frac{1}{r_\sharp}\sqrt{\frac{2r_\sharp\log^4 r_\sharp}{-\log^3 r_\sharp(4+\log r_\sharp)}}\,\sqrt{\xi} + O(\xi^{3/2}) \\ &= \sqrt{\frac{2(-\log r_\sharp)}{r_\sharp(4+\log r_\sharp)}}\,\sqrt{\xi} + O(\xi^{3/2}) \qquad \text{(r2.23a)} \end{aligned}$$

up to a constant, so

$$\xi \sim \frac{r_\sharp(4+\log r_\sharp)}{2(-\log r_\sharp)}\cdot\phi^2 \qquad \text{(r2.23)}$$

Since $1 > r_\sharp > r_\natural = e^{-4}$, both $(4+\log r_\sharp)$ and $(-\log r_\sharp)$ are positive. The traced boundaries on the outer viable mainland attach smoothly to the critical terminal curve $r = r_\sharp$. Whether or not they are convex at the point of attachment depends on the size of the coefficient in (r2.23), which depends on $r_\sharp$, which depends on A.



Now the equation of a line tangent to the circle $r=r_\#$, up to a constant in ϕ,

$$r_\# = (r_\# + \xi)\cos\phi$$

i.e.

$$\begin{aligned}\xi &= r_\#(\sec\phi - 1)\\ &= r_\#\left(1+\frac{\phi^2}{2}+O(\phi^4)-1\right)\\ &= \frac{r_\#}{2}\cdot\phi^2 + O(\phi^4). \qquad (r2.24)\end{aligned}$$

Thus the traced boundaries (r2.23) will be convex at the point of tangency to $r=r_\#$ if the coefficient of ϕ^2 in (r2.23) does not exceed that in (r2.24), i.e. if

$$\begin{aligned}\frac{r_\#(4+\log r_\#)}{2(-\log r_\#)} &\le \frac{r_\#}{2}\\ 4+\log r_\# &\le -\log r_\#\\ r_\# &\le e^{-2} \qquad (r2.25)\end{aligned}$$

Remembering that $r_\# > r_4 = e^{-4}$, the total range over which we can construct convex ~~boundaries~~ (at least at the point of tangency at $r=r_\#$) boundaries is

$$e^{-4} < r_\# \le e^{-2}, \qquad (r2.26)$$

and since $A = \psi(r_\#)$ is decreasing in $r_\#$ over this range, this corresponds to

$$\psi(e^{-2}) \le A < \psi(e^{-4})$$

$$e^{-2}(-2)^4 \le A < e^{-4}(-4)^4$$

$$\boxed{2.1654 = \left(\frac{4}{e}\right)^2 \le A < \left(\frac{4}{e}\right)^4 = 4.6888} \qquad (r2.27)$$

If A is outside this range, the traced boundaries are assuredly concave. But (r2.27) only guarantees convexity at the point of tangency at $r=r_\#$; ~~if r is sufficiently close to the moving toe~~ travelling towards $r=1$, eventual concavity is inevitable, since there, $\phi \sim \mp(1-r)^5/(5A)$ up to a constant.

To determine where the traced boundaries become concave, we seek points of inflection.

With primes denoting r differentiation, observe that

$$\vec{a}_r' = -\sin\phi \cdot \phi' \vec{a}_x + \cos\phi \cdot \phi' \vec{a}_y = \phi' \vec{a}_\phi$$
$$\vec{a}_\phi' = -\cos\phi \cdot \phi' \vec{a}_x - \sin\phi \cdot \phi' \vec{a}_y = -\phi' \vec{a}_r .$$

Now from (23),

$$d\vec{r} = dr\, \vec{a}_r + r\, d\phi\, \vec{a}_\phi ,$$

and dividing this by dr gives the velocity vector for a curve parametrised in terms of r,

$$\vec{r}' = \frac{d\vec{r}}{dr} = \vec{a}_r + r\frac{d\phi}{dr}\vec{a}_\phi$$
$$= \vec{a}_r + r\phi'\vec{a}_\phi .$$

Another r derivative is taken for the acceleration,

$$\vec{r}'' = \frac{d^2\vec{r}}{dr^2} = \vec{a}_r' + r\phi'\vec{a}_\phi' + r\phi''\vec{a}_\phi + 1\phi'\vec{a}_\phi$$
$$= \phi'\vec{a}_\phi - r\phi'^2\vec{a}_r + r\phi''\vec{a}_\phi + \phi'\vec{a}_\phi$$
$$= -r\phi'^2\vec{a}_r + (r\phi'' + 2\phi')\vec{a}_\phi ,$$

and the curvature of the curve has the same sign as

$$\vec{a}_z \cdot \vec{r}' \times \vec{r}'' = 1(r\phi'' + 2\phi') + (r\phi'^2)(r\phi')$$
$$= r\phi'' + 2\phi' + r^2\phi'^3 . \qquad (r2.28)$$

From (r2.21),

$$\boxed{\phi' = \frac{d\phi}{dr} = \mp\frac{\log^4 r}{\sqrt{A^2 - r^2\log^8 r}} = \mp\frac{L^4}{\sqrt{A^2 - r^2L^8}}} \qquad (r2.29)$$

where, for brevity,

$$L = \log r . \qquad (r2.30)$$

Thus

$$\phi'' = \mp\left[\frac{4L^3/r}{\sqrt{A^2 - r^2L^8}} - \frac{L^4(-r^2 8L^7/r - 2rL^8)}{2(A^2 - r^2L^8)^{3/2}}\right]$$
$$= \mp\left[\frac{4L^3}{r\sqrt{A^2 - r^2L^8}} + \frac{L^4(8rL^7 + 2rL^8)}{2(A^2 - r^2L^8)^{3/2}}\right]$$
$$= \mp\frac{L^3}{r(A^2 - r^2L^8)}\left[4(A^2 - \cancel{r^2L^8}) + (\cancel{4r^2L^8} + r^2L^9)\right]$$
$$= \mp\frac{L^3}{r(A^2 - r^2L^8)^{3/2}}\left[4A^2 + r^2L^9\right] \qquad (r2.31)$$

so the curvature is

$$
\begin{aligned}
& r\phi'' + 2\phi' + r^2\phi'^3 \\
&= \mp \frac{L^3[4A^2 + r^2L^9]}{(A^2 - r^2L^8)^{3/2}} \mp \frac{2L^4}{\sqrt{A^2 - r^2L^8}} \mp \frac{r^2L^{12}}{(A^2 - r^2L^8)^{3/2}} \\
&= \mp \frac{L^3}{(A^2 - r^2L^8)^{3/2}}\left[4A^2 + \cancel{r^2L^9} + 2L(A^2 - \cancel{r^2L^8}) + \cancel{r^2L^9}\right] \\
&= \mp \frac{2A^2L^3}{(A^2 - r^2L^8)^{3/2}}\left[2 + L\right] \qquad (r2.32)
\end{aligned}
$$

which crosses zero at $2+L = 2+\log r = 0$ only. Thus inflection occurs at

$$\boxed{r = r_i = e^{-2}}$$

so the traced boundaries on the outer viable island are convex if and only if $r \le r_i = e^{-2}$. This is consistent with the earlier result (r2.25).

In practice the actual traced boundaries are determined by numerically integrating (r2.29). It is also possible to continue the series expansion (r2.23a) to more terms. Using computer algebra,

$$
\begin{aligned}
\phi \sim & \sqrt{\frac{2(-L_\#)}{r_\#(4+L_\#)}}\sqrt{\xi} + \frac{L_\#^2 - 4L_\# - 36}{6\sqrt{2(-L_\#)}\,[r_\#(4+L_\#)]^{3/2}}\cdot\xi^{3/2} \\
& + \frac{9L_\#^4 - 136L_\#^3 - 1624L_\#^2 - 3744L_\# + 144}{240\sqrt{2}\,[-L_\#]^{3/2}\,[r_\#(4+L_\#)]^{5/2}}\cdot\xi^{5/2} \\
& + \dots \qquad (r2.33)
\end{aligned}
$$

where

$$L_\# = \log r_\# . \qquad (r2.34)$$

The series (r2.33) may be continued indefinitely, but almost certainly ~~the radius of convergence~~ it is asymptotic, and it is unclear if the convergence is good enough to be useful.

4. Summary

Restore dropped hats. Boundary tracing says that if

$$T = T_0 \log \frac{r_0}{r} \qquad \text{(r2.35)}$$

is the known temperature profile, $\overset{\text{non-trivial}}{\text{convex}}$ boundaries may be constructed for

$$2.1654 = \left(\frac{4}{e}\right)^2 \leq A < \left(\frac{4}{e}\right)^4 = 4.6888\,, \qquad \text{(r2.36)}$$

where

$$A = \frac{1}{c r_0^2 T_0^3} \qquad \text{(r2.37)}$$

is dimensionless. These boundaries are given by

$$\frac{d\theta}{d\hat{r}} = \mp \frac{\log^4 \hat{r}}{\sqrt{A^2 - \hat{r}^2 \log^8 \hat{r}}} = \mp \frac{\psi}{\hat{r}\sqrt{A^2 - \psi^2}} \qquad \text{(r2.38)}$$

on $\hat{r}_\# \leq \hat{r} \leq \hat{r}_i = e^{-2}$ (if $\hat{r} > \hat{r}_i$ the boundaries become concave), and they attach smoothly to the terminal curve $\hat{r} = \hat{r}_\#$. Here, $\hat{r}_\#$ is the solution to

$$\psi(\hat{r}) = \hat{r} \log^4 \hat{r} = A\,, \qquad \text{(r2.39)}$$

on $e^{-2} = \hat{r}_i < \hat{r} < 1$, which is given by

$$\hat{r}_\# = \exp\left[4 W\left(-\frac{A^{1/4}}{4}\right)\right]$$

where $W(z)$ is the Lambert W function or product log, the principal solution to $z = We^W$. The restriction on $\hat{r}_\#$ which corresponds to (r2.36) is

$$e^{-2} \geq \hat{r}_\# > e^{-4}. \qquad \text{(r2.40)}$$

Note that

$$\hat{r}_\# = \frac{r_\#}{r_0} < 1$$

is physically the ratio between the terminal radius $r_\#$ and the radius r_0 at which the known temperature profile (r2.35) vanishes.

4.1 Physical temperature range

In practice one is probably only interested in objects of a given size. Fix $r_\#$, which is effectively the radius of any convex domain which results from boundary tracing. Since A is restricted to the interval (r2.36), equivalent to the restriction (r2.40) in $\hat{r}_\#$, the range of temperatures which can be accounted for is also restricted.

Observe that

$$r_0 = \frac{r_\#}{\hat{r}_\#} \tag{r2.41}$$

and from (r2.37),

$$T_0 = \left(\frac{1}{c r_0 A}\right)^{1/3} = \left(\frac{1}{c(r_\#/\hat{r}_\#)\,\hat{r}_\# \log^4 \hat{r}_\#}\right)^{1/3} = \left(\frac{1}{c r_\#}\,\frac{1}{\log^4 \hat{r}_\#}\right)^{1/3} \tag{r2.42}$$

so the temperature $T_\#$ along $r = r_\#$ is given by

$$\begin{aligned} T_\# &= T_0 \log \frac{r_0}{r_\#} = \left(\frac{1}{c r_\#}\,\frac{1}{\log^4 \hat{r}_\#}\right)^{1/3} \log \frac{1}{\hat{r}_\#} = \left(\frac{1}{c r_\#}\,\frac{1}{[-\log \hat{r}_\#]^4}\right)^{1/3} [-\log \hat{r}_\#] \\ &= \left(\frac{1}{c r_\#}\,\frac{1}{[-\log \hat{r}_\#]}\right)^{1/3} = \left(\frac{1}{c r_\# \log(1/\hat{r}_\#)}\right)^{1/3} \end{aligned} \tag{r2.43}$$

Now (r2.40) implies that

$$\begin{aligned} e^2 &\le 1/\hat{r}_\# < e^4 \\ 2 &\le \log(1/\hat{r}_\#) < 4 \\ \frac{1}{2} &\geq \frac{1}{\log(1/\hat{r}_\#)} > \frac{1}{4} \end{aligned} \tag{r2.44a}$$

thus the ~~possible~~ range of possible temperatures is

$$\left(\frac{1}{4 c r_\#}\right)^{1/3} < T_\# \le \left(\frac{1}{2 c r_\#}\right)^{1/3}, \tag{r2.44}$$

or

$$\left(\frac{k}{4 \varepsilon \sigma r_\#}\right)^{1/3} < T_\# \le \left(\frac{k}{2 \varepsilon \sigma r_\#}\right)^{1/3}. \tag{r2.45}$$

4.1.1 Example

Consider PVC wire coating:

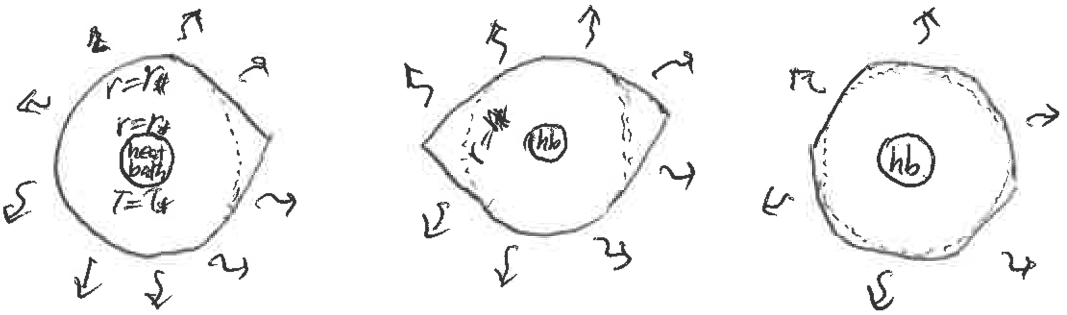
$$\varepsilon = 0.9$$
$$k = 0.15\,\mathrm{W\,m^{-1}\,K^{-1}}$$
$$r_\# = 3\,\mathrm{mm}$$

(r2.45) becomes $626\,K < T_\# \leq 788\,K$: which is too bad, since PVC distorts at 330 K and melts at around 400 K, So we need to be looking at much poorer conductors or much thicker cylinders, otherwise the temperatures will be too high.

4.2 A hypothetical situation

Given an inradius $r_\#$, we can construct a domain with spikes coming off $r = r_\#$, and it will ~~correspond to~~ admit the solution (r2.35) to the boundary value problem (r7) & (r8'). The heat may be generated by any circular heat bath at radius r_* and temperature T_* satisfying

$$T_* = T_0 \log \frac{r_0}{r_*} .$$



~~Equivalently, the~~ The heat bath $T = T_*$ condition may be replaced with an equivalent prescribed power per length p:

4.3 Physical power per length range

Given the known solution (r.2.35), we have (vector) power density

$$-k\vec{\nabla} T = -k\left(-T_0 \cdot \frac{1}{r}\,\vec{a}_r\right)$$
$$= \frac{kT_0}{r}\,\vec{a}_r .$$

Thus the power per length dissipated ~~through the surface by radiation~~ by the heat bath (and thus through the surface via radiation) is

$$\mathcal{P} = \int_0^{2\pi} \vec{n} \cdot \left[-k\vec{\nabla}T\right] r\,d\phi$$

where $\vec{n}$ is the normal to the bath $r = r_\#$. Thus

$$\mathcal{P} = \int_0^{2\pi} \vec{a}_r \cdot \frac{kT_0}{r}\vec{a}_r\, r\,d\phi = 2\pi k T_0$$

$$= 2\pi k \left(\frac{1}{c r_\#} \frac{1}{\log^4 r_\#}\right)^{1/3} . \qquad \text{(r2.46)}$$

Using (r2.44a), we have

$$2\pi k \left(\frac{1}{c r_\#} \frac{1}{4^4}\right)^{1/3} < \mathcal{P} \leq 2\pi k \left(\frac{1}{c r_\#} \frac{1}{2^4}\right)^{1/3}$$

$$\frac{2\pi k^{4/3}}{(256\,\varepsilon\sigma r_\#)^{1/3}} < \mathcal{P} \leq \frac{2\pi k^{4/3}}{(16\,\varepsilon\sigma r_\#)^{1/3}} . \qquad \text{(r2.47)}$$

4.3.1 Example

Using the values from Example 4.1.1, (r2.47) becomes

$$147\ \mathrm{W\,m^{-1}} < \mathcal{P} \leq 371\ \mathrm{W\,m^{-1}} .$$