Radiation 1 : Plane source

In Cartesian coordinates (x, y), the simplest possible solution to (r7) is

$$\boxed{T = h_0 x} \qquad (r1.1)$$

where h_0 is a temperature gradient. Note that the region $x<0$ is unphysical, since there $T<0$.

1. Coordinates

obvious stuff: $(u, v) = (x, y)$, $(\vec{a}_u, \vec{a}_v) = (\vec{a}_x, \vec{a}_y)$, $h_u = h_v = 1$ etc.

2. Scaling

Put

$$\hat{T} = T/\tau \qquad (r1.2)$$
$$\hat{x} = x/\lambda \qquad (r1.3)$$
$$\hat{y} = y/\lambda \qquad (r1.4)$$
$$\hat{\vec{\nabla}} = \lambda \vec{\nabla} \qquad (r1.5)$$

and drop hats. Then (r8) and (r1.1) become

$$\frac{\tau}{\lambda} n \cdot \vec{\nabla} T = -c\tau^4 T^4$$
$$\tau T = h_0 \lambda x$$

or

$$n \cdot \vec{\nabla} T = -[c\lambda\tau^3] T^4$$
$$T = \left[\frac{h_0 \lambda}{\tau}\right] x .$$

We have 2 dimensionless groups and 2 free scales, so both groups can be made unity; ~~put~~ putting $c\lambda\tau^3 = h_0\lambda/\tau = 1$ and ~~solve~~ solving for τ and λ, we obtain

$$\boxed{\begin{aligned} \tau &= \left(\frac{h_0}{c}\right)^{1/4} \\ \lambda &= \left(\frac{1}{ch_0^3}\right)^{1/4} \end{aligned}} \qquad \begin{aligned} &(r1.6) \\ &(r1.7) \end{aligned}$$

Hence

$$\boxed{\begin{aligned} n \cdot \vec{\nabla} T &= -T^4 \\ T &= x \end{aligned}} \qquad \begin{aligned} &(r1.8) \\ &(r1.9) \end{aligned}$$

3. Boundary tracing

We have:

$$d\mu = dx \tag{r1.10}$$

$$d\upsilon = dy \tag{r1.11}$$

$$P = \frac{1}{h_\mu}\frac{\partial T}{\partial \mu} = \frac{\partial T}{\partial x} = 1 \tag{r1.12}$$

$$Q = \frac{1}{h_\upsilon}\frac{\partial T}{\partial \upsilon} = \frac{\partial T}{\partial y} = 0 \tag{r1.13}$$

$$F = -x^4 \tag{r1.14}$$

$$\Phi = P^2 + Q^2 - F^2 = 1 - x^8 \tag{r1.15}$$

The viable domain (excluding the unphysical region $x<0$) is

$$0 \le x \le 1, \tag{r1.16}$$

and the tracing equation (46) becomes

$$\frac{dy}{dx} = \frac{0 \pm (-x^4)\sqrt{1-x^8}}{1-x^8}$$

$$\boxed{\frac{dy}{dx} = \mp \frac{x^4}{\sqrt{1-x^8}}} \tag{r1.17}$$

Thus

$$y = \mp \int \frac{x^4\,dx}{\sqrt{1-x^8}}$$

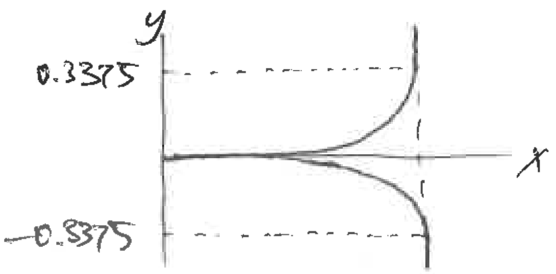
$$\boxed{y = \mp \frac{x^5}{5} \cdot {}_2F_1\left(\frac{1}{2}, \frac{5}{8}; \frac{13}{8}; x^8\right)} \tag{r1.18}$$

up to a constant. These attach smoothly onto the critical terminal curve

$$x = 1. \tag{r1.19}$$

Note that

$$y(1) = \mp \frac{\sqrt{\pi}\,\Gamma(13/8)}{5\,\Gamma(9/8)} = \mp 0.337488 \tag{r1.20}$$



A single spike constructed from (r1.18).

3.1 Near $x=0$

$$\frac{dy}{dx} = \mp \frac{x^4}{1+O(x^8)} = \mp x^4 + O(x^{12})$$

$$y = \mp \frac{x^5}{5} + O(x^{13})$$

3.2 Near $x=1$

Put $\xi = 1-x$.

$$\frac{dy}{d\xi} = -\frac{dy}{dx} = \pm \frac{(1-\xi)^4}{\sqrt{1-(1-\xi)^8}} = \pm \frac{1+O(\xi)}{\sqrt{8\xi + O(\xi^2)}} = \pm \frac{1+O(\xi)}{\sqrt{8\xi}\,(1+O(\xi))}$$

$$= \pm \frac{1}{\sqrt{8\xi}} + O(\sqrt{\xi}) = \pm \frac{1}{\sqrt{2}} \cdot \frac{1}{2\sqrt{\xi}} + O(\sqrt{\xi})$$

$$y = \pm \sqrt{\frac{\xi}{2}} + O(\xi^{3/2})$$

$$= \pm \sqrt{\frac{1-x}{2}} + O(1-x)^{3/2}$$

3.3 Convexity

Asserted boundaries may be pieced together from arbitrary translations of the spikes (r1.18), but unfortunately they are not convex, so we have no useful results here.