Boundary tracing for thermal radiation

1. Radiation

A surface ~~of area~~ of emissivity ε at temperature T has radiant emittance (or radiant exitance)

$$\boxed{j = \varepsilon \sigma T^4} \qquad (r1)$$

where σ is the Stefan-Boltzmann constant

$$\sigma = \frac{2\pi^5 k_B^4}{15 h^3 c^2} = 5.6704 \times 10^{-8}\ W\, m^{-2} K^{-4}.$$

Note that

$$[\varepsilon] = 1$$

$$[\sigma] = \frac{\text{Pow}}{\text{Area Temp}^4}$$

$$[T] = \text{Temp},$$

so that

$$[j] = \frac{\text{Pow}}{\text{Area}}.$$

2. Generic heat dissipation problem.

Consider a body ~~~~ Ω, with conductivity k and internal heat generation of power density q, ~~with~~ losing heat ~~to vacuum~~ by thermal radiation to vacuum along its surface $\partial\Omega$. At equilibrium, we have the steady-state heat equation

$$\vec{\nabla}\cdot(-k\vec{\nabla}T) = q, \qquad \Omega. \qquad (r2)$$

Now

$$[k\vec{\nabla}T] = [q]/[\vec{\nabla}] = \frac{\text{Pow}}{\text{Vol}}\cdot \text{Len} = \frac{\text{Pow}}{\text{Area}} = [j]$$

$$[\vec{n}] = 1;$$

thus, in light of (r1), the boundary condition along the surface is

$$\vec{n}\cdot(-k\vec{\nabla}T) = j = \varepsilon\sigma T^4, \qquad \partial\Omega. \qquad (r3)$$

Therefore, the generic radiation dissipation problem is

$$\vec{\nabla}\cdot(-k\vec{\nabla}T) = q\,, \qquad \Omega \tag{r4}$$
$$\vec{n}\cdot(-k\vec{\nabla}T) = \varepsilon\sigma T^4, \qquad \partial\Omega \tag{r5}$$

While the PDE is linear, the boundary condition is not, which makes the problem difficult. The idea is to take known solutions to (r4) and use boundary tracing to seek non-trivial boundary shapes $\partial\Omega$ along which (r5) holds.

3. <u>A simpler special case</u>

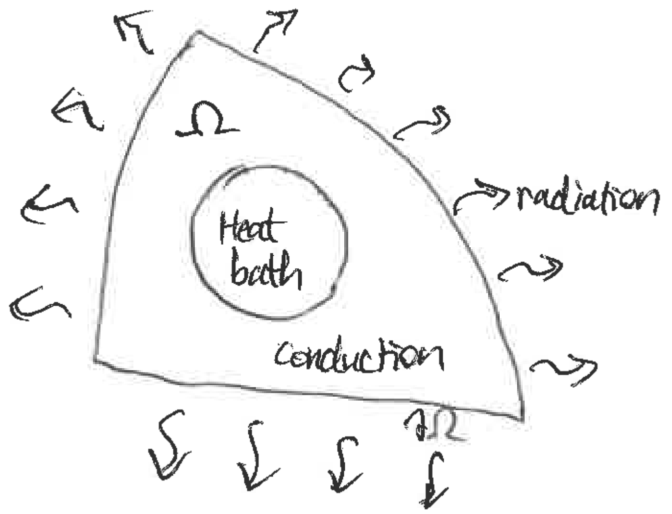
Assume that $k=\text{const}$ (i.e. an homogeneous & isotropic body) and that $q=0$ almost everywhere (i.e. no volumetric heat generation, but allow point/line/surface sources). Define

$$\boxed{c = \frac{\varepsilon\sigma}{k}}\,. \tag{r6}$$

Then we have the simpler boundary value problem

$$\boxed{\begin{aligned} \nabla^2 T &= 0\,, & \Omega \\ \vec{n}\cdot\vec{\nabla}T &= -cT^4, & \partial\Omega \end{aligned}}\,. \qquad \begin{aligned}(r7)\\(r8)\end{aligned}$$

While there is no volumetric source term on the right hand side of (r7), we allow for singularities (i.e. point/line/surface sources) and for ~~the~~ ~~generat~~ the supply of heat in the interior of Ω ~~along~~ via a Dirichlet condition ~~at~~ along an interior boundary (i.e. an internal heat bath).



Note that (r8) assumes there is no self-incident radiation: the body Ω <u>must be convex</u>.

While Laplace's equation (r7) is well studied, nonlinear flux conditions such as (r8) are not, so there should be a lot we can do. There are plenty of solutions to (r7), though we will focus on ones which have some sort of local maximum, which will eventually ~~[illegible]~~ correspond to a ~~[illegible]~~ body with hot interior, losing heat by radiation to vacuum along the boundary.

Things which pop to mind:

- 2D coordinate systems in which Laplace's equation (r7) is separable
  - Cartesian (plane source)
  - Polar (line source)
  - Elliptic (segment source)
  - Parabolic (probably won't work since it doesn't correspond to some type of singularity)
  - Bipolar (equal and opposite line sources)
- Perturbing the line source case with other line sources or images or singularities
- Regular arrays of line sources (e.g. equilateral triangle for 3-phase wire)
- Conformal mapping to get known solutions to (r7) in weird shapes
- ~~Extension~~ Generalisation to 3D: spherical polar coordinates (point source)