Radiation 3 : Bipolar coordinates

Consider the solution to (r7) which consists of equal and opposite line sources at $(x,y) = (\pm a, 0)$. The only way this can make dimensional sense is if

$$\boxed{T = T_0 v} \qquad (r3.1)$$

for some temperature T_0, where v is the hyperbolic coordinate of the bipolar coordinate system. This is best seen by introducing the coordinates themselves:

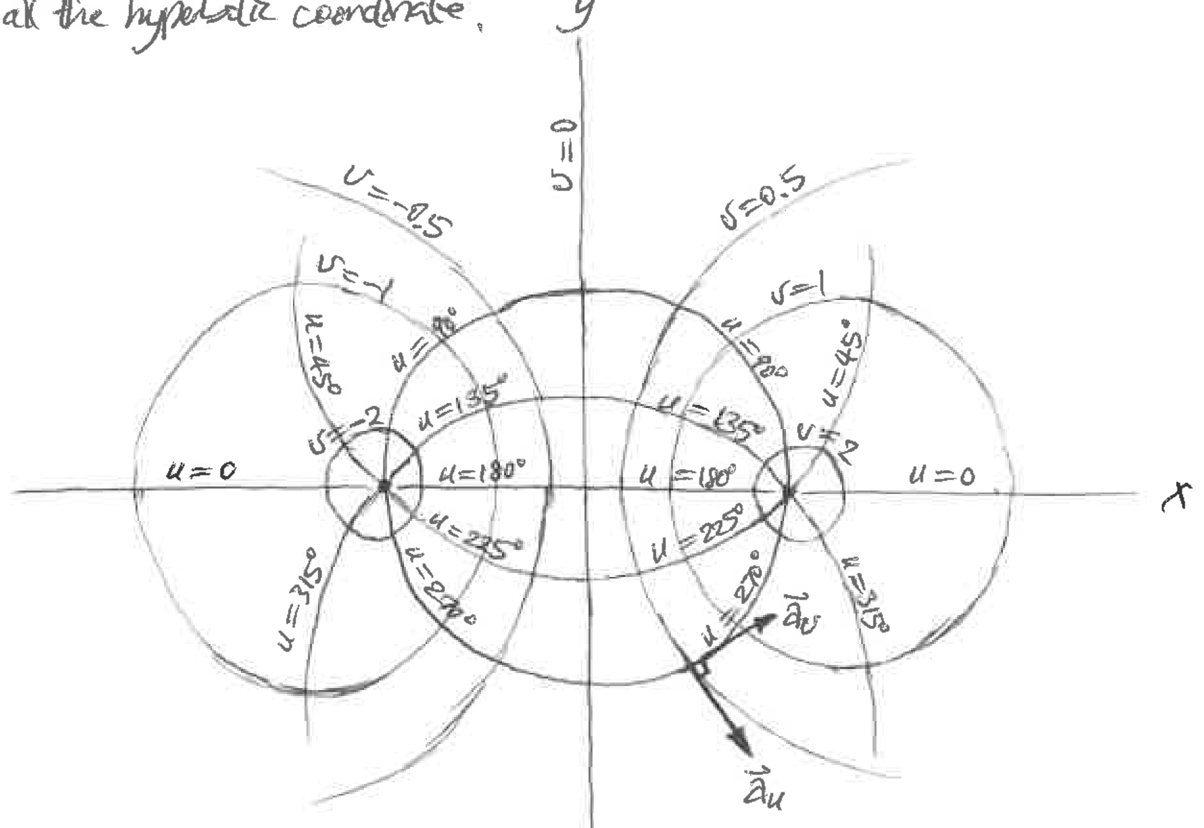
1. <u>Coordinates (unscaled)</u>

Bipolar coordinates (u, v) are given by

$$x = \frac{a \sinh v}{\cosh v - \cos u} \qquad (r3.2)$$

$$y = \frac{a \sin u}{\cosh v - \cos u}. \qquad (r3.3)$$

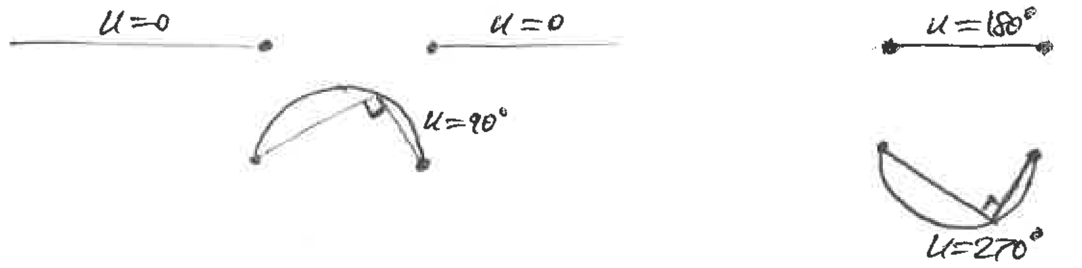
The length scale a is intrinsic to the coordinate system, in the location of the singularities $(x,y) = (\pm a, 0)$, which correspond to $v = \pm\infty$. u I shall call the angular coordinate, reckoned modulo 2π; v I shall call the hyperbolic coordinate.



Each $u = \text{const}$ contour forms a circular arc delimited by the singularities $(x,y) = (\pm a, 0)$, of the form

$$x^2 + (y - a\cot u)^2 = a^2 \csc^2 u \tag{r3.4}$$

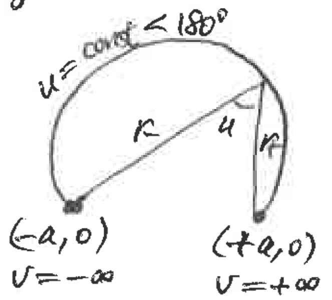
Geometrically, u is the angle subtended by the circular arc $u = \text{const}$. The two arcs corresponding to two angles which add to a full turn, make up a full circle:



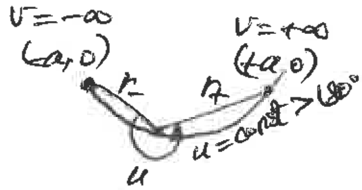
Each $v = \text{const}$ contour forms a circle of the form

$$(x - a\coth v)^2 + y^2 = a^2 \operatorname{csch}^2 v. \tag{r3.5}$$

Geometrically, v is the logarithm of the ratio of distances to each of the two singularities:



$$v = \log \frac{r_-}{r_+}$$



The $v = \text{const}$ contours are equipotentials for equal and opposite line charges at $(x,y) = (\pm a, 0)$. The inverse coordinate transformations are

$$u = \tan^{-1} \frac{2ay}{x^2+y^2-a^2} \tag{r3.6}$$

$$v = \tanh^{-1} \frac{2ax}{x^2+y^2+a^2}, \tag{r3.7}$$

where $\tan^{-1} \frac{Y}{X}$ means $\operatorname{atan2}(Y, X)$.

For brevity define

$$S = \sin u \sinh v \tag{r3.8}$$
$$C = \cos u \cosh v - 1 \tag{r3.9}$$
$$h = \frac{a}{\cosh v - \cos u}\,; \tag{r3.10}$$

it will come to pass that h is the scale factor (Lamé coefficient) for both u and v. After some algebra, we find that the local basis is

$$\vec{h}_u = \frac{h^2}{a}(-S\vec{a}_x + C\vec{a}_y)$$
$$\vec{h}_v = \frac{h^2}{a}(-C\vec{a}_x - S\vec{a}_y)\,.$$

Since

$$S^2 + C^2 = (\cosh v - \cos u)^2 = \left(\frac{a}{h}\right)^2\,, \tag{r3.11}$$

i.e. $\sqrt{S^2+C^2} = a/h$, it follows that both scale factors are

$$\boxed{h_u = h_v = \frac{h^2}{a}\cdot\frac{a}{h} = h}\,, \tag{r3.12}$$

and the ~~as~~ local orthonormal basis is

$$\boxed{\begin{aligned}\vec{a}_u &= \frac{h}{a}(-S\vec{a}_x + C\vec{a}_y) \quad (r3.13)\\ \vec{a}_v &= \frac{h}{a}(-C\vec{a}_x - S\vec{a}_y) \quad (r3.14)\end{aligned}}$$

The nice thing about the scale factor being the same coordinates is that the form of the Laplacian is the same as the Cartesian Laplacian, up to an overall scale factor:

$$\vec{\nabla} T = \frac{1}{h}\left[\frac{\partial T}{\partial u}\vec{a}_u + \frac{\partial T}{\partial v}\vec{a}_v\right] \tag{r3.15}$$

$$\vec{\nabla}\cdot\vec{F} = \frac{1}{h^2}\left[\frac{\partial}{\partial u}(hF_u) + \frac{\partial}{\partial v}(hF_v)\right] \tag{r3.16}$$

$$\vec{\nabla}^2 T = \frac{1}{h^2}\left[\frac{\partial^2 T}{\partial u^2} + \frac{\partial^2 T}{\partial v^2}\right] \tag{r3.17}$$

Thus (r3.1) is a solution to Laplace's equation (r7), as claimed. Note that the region $v<0$ (equivalent to $x<0$) is unphysical, since there $T<0$.

2. Scaling

Noting that the length scale has already been fixed by the length scale a which is ~~inherent~~ intrinsic to the bipolar coordinate system, we put

$$\hat{T} = T/\tau \tag{r3.18}$$
$$\hat{x} = x/a \tag{r3.19}$$
$$\hat{y} = y/a \tag{r3.20}$$
$$\hat{\vec{\nabla}} = a\vec{\nabla} \tag{r3.21}$$

and drop hats. Then (r8) and (r3.1) become

$$\frac{\tau}{a}\vec{n}\cdot\vec{\nabla}T = -c\tau^4 T^4$$
$$\tau T = T_0 v$$

or

$$\vec{n}\cdot\vec{\nabla}T = -[ca\tau^3]\,T^4$$
$$T = \left[\frac{T_0}{\tau}\right]v.$$

We have 2 dimensionless groups but only 1 free scale τ (remember a is tied to the bipolar coordinate system ~~to~~ by the placement of the two line sources), so one group cannot be made unity. Put

$$\boxed{\tau = T_0} \tag{r3.22}$$

and define

$$\boxed{A = \frac{1}{caT_0^3}}. \tag{r3.23}$$

Thus we have

$$\vec{n}\cdot\vec{\nabla}T = -\frac{T^4}{A} \tag{r3.24}$$
$$T = v \tag{r3.25}$$

3. Coordinates (scaled)

Given the length scale a, it is sensible to put

$$\hat{h} = h/a \qquad (r3.26)$$

and drop the hat for h too. Thus we have the following in scaled quantities:

$$\boxed{\begin{aligned}
x &= \frac{\sinh v}{\cosh v - \cos u} && (r3.27)\\
y &= \frac{\sin u}{\cosh v - \cos u} && (r3.28)\\
h &= \frac{1}{\cosh v - \cos u} && (r3.29)\\
\vec{a}_u &= h(-S\vec{a}_x + C\vec{a}_y) && (r3.30)\\
\vec{a}_v &= h(-C\vec{a}_x - S\vec{a}_y) && (r3.31)\\
u &= \tan^{-1}\frac{2y}{x^2+y^2-1} && (r3.32)\\
v &= \tanh^{-1}\frac{2x}{x^2+y^2+1} && (r3.33)
\end{aligned}}$$

4. Boundary tracing

Since $h_u = h_v = h$, we have:

$$d\mu = h\,du \qquad (r3.34)$$

$$d\nu = h\,dv \qquad (r3.35)$$

$$P = \frac{1}{h}\frac{\partial T}{\partial u} = 0 \qquad (r3.36)$$

$$Q = \frac{1}{h}\frac{\partial T}{\partial v} = \frac{1}{h} = \cosh v - \cos u \qquad (r3.37)$$

$$F = -\frac{v^4}{A} \qquad (r3.38)$$

$$\Phi = P^2+Q^2-F^2 = \frac{1}{h^2} - \frac{v^8}{A^2} = (\cosh v - \cos u)^2 - \frac{v^8}{A^2} \qquad (r3.39)$$

4.1 Viable domain

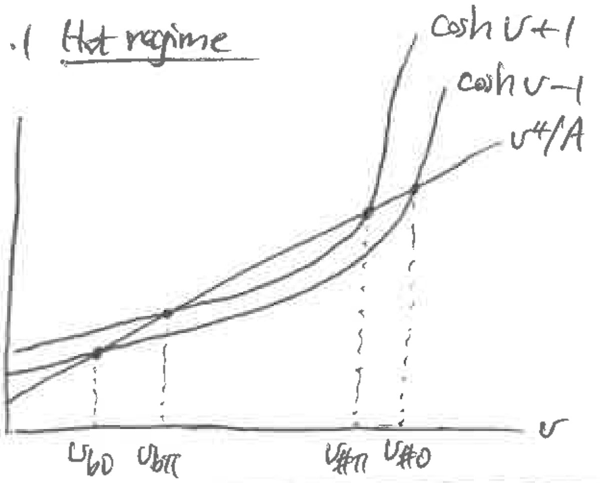
The viable domain $\Phi \geq 0$ can be written

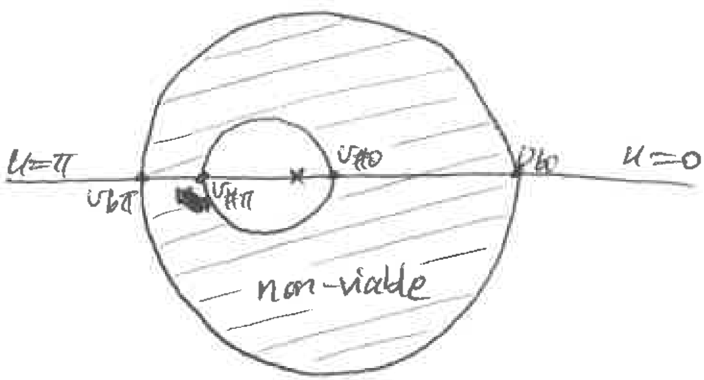
$$\cosh v - \cos u \geq \frac{v^4}{A}, \qquad \text{(r3.40)}$$

and its geometry depends on the value of A. It turns out that all critical terminal points lie on the x-axis, at $\cosh v - 1 = v^4/A$ for $u=0$ (but excluding the trivial solution $v=0$) and at $\cosh v + 1 = v^4/A$ for $u=\pi$.

4.1.1 Hot regime

$\boxed{A < A_{\sharp\pi} = 9.06433}$





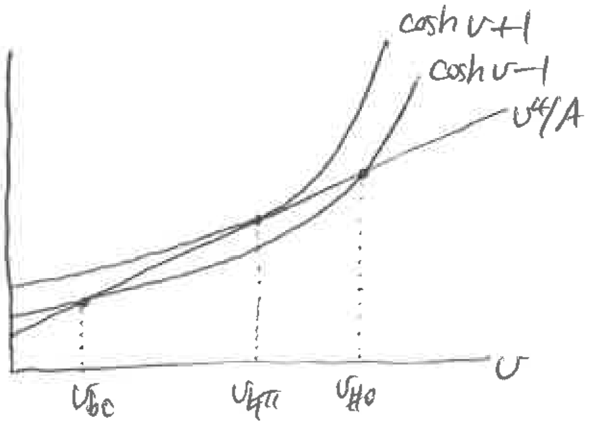
$$\boxed{\begin{array}{l} u=0:\ \cosh v - 1 = \dfrac{v^4}{A} \ \text{at } v_{\flat 0}, v_{\sharp 0} \\ u=\pi:\ \cosh v + 1 = \dfrac{v^4}{A} \ \text{at } v_{\flat\pi}, v_{\sharp\pi} \end{array}}$$

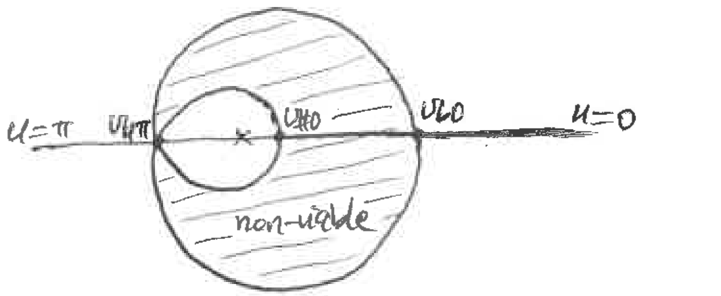
where $0 < v_{\flat 0} < v_{\flat\pi} < v_{\sharp\pi} < v_{\sharp 0}$.

The non-viable domain forms an avocado-like moat surrounding an inner viable island (containing the singularity $v=+\infty$) and surrounded by an outer viable mainland.

4.1.2 Warm-to-hot transition

$\boxed{A = A_{\sharp\pi} = 9.06433}$





$$\boxed{\begin{array}{l} u=0:\ \cosh v - 1 = \dfrac{v^4}{A} \ \text{at } v_{\flat 0}, v_{\sharp 0} \\ u=\pi:\ \cosh v + 1 = \dfrac{v^4}{A} \ \text{at } v_{\sharp\pi} \ \text{(tan.)} \end{array}}$$

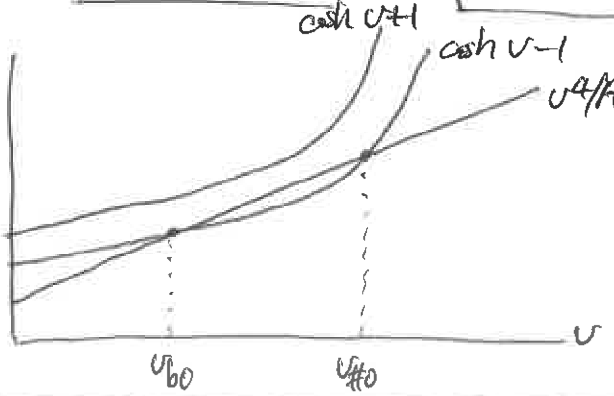
where $0 < v_{\flat 0} < v_{\sharp\pi} < v_{\sharp 0}$.

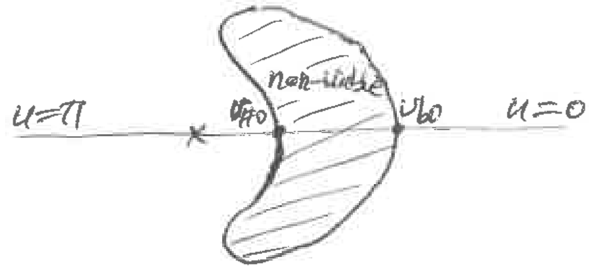
The roots $v_{\flat\pi}$ and $v_{\sharp\pi}$ merge into $v_{\sharp\pi}$ when $A = A_{\sharp\pi}$, for which $\cosh v + 1$ is tangential to v^4/A at $v = v_{\sharp\pi}$.

The moat is pincered along $u=\pi$, and the inner viable island touches the outer viable mainland at $(u,v) = (\pi, v_{\sharp\pi})$.

4.1.3 Warm regime

$$\boxed{9.06433 = A_{\sharp\pi} < A < A_{\natural 0} = 9.76206}$$





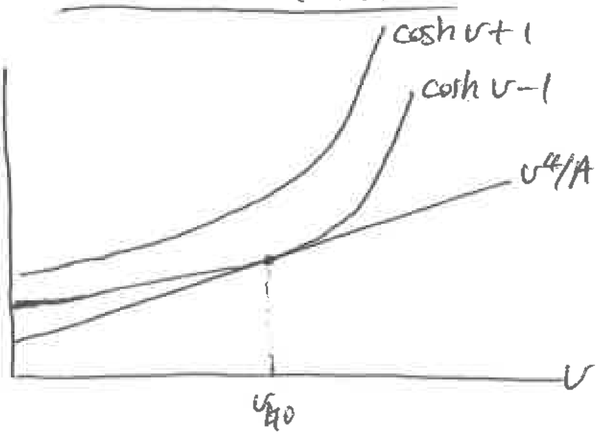
$$\boxed{\begin{array}{l} u=0:\ \cosh v-1 = \frac{v^4}{A} \text{ at } v_{\flat 0}, v_{\sharp 0} \\ u=\pi:\ \cosh v+1 = \frac{v^4}{A} \text{ nowhere} \end{array}}$$

where $0 < v_{\flat 0} < v_{\sharp 0}$

The inner viable island has joined with the outer viable mainland, and the non-viable domain is now a crescent-shaped lake.

4.1.4 Cold-to-warm transition

$$\boxed{A = A_{\natural 0} = 9.76206}$$



$$\boxed{\begin{array}{l} u=0:\ \cosh v-1 = \frac{v^4}{A} \text{ at } v_{\natural 0} \text{ (tan.)} \\ u=\pi:\ \cosh v+1 = \frac{v^4}{A} \text{ nowhere} \end{array}}$$

where $0 < v_{\natural 0}$.

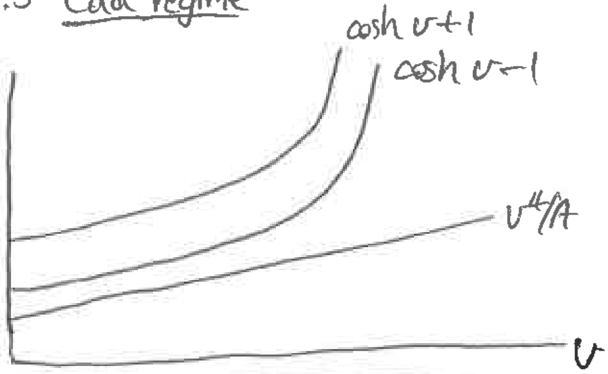
The roots $v_{\flat 0}$ and $v_{\sharp 0}$ merge into $v_{\natural 0}$ when $A = A_{\natural 0}$, for which $\cosh v - 1$ is tangential to v^4/A at $v = v_{\natural 0}$.

The non-viable lake has completely dried up at the critical terminal point $(u, v) = (0, v_{\natural 0})$.

4.1.5 Cold regime

$$\boxed{A > A_{\natural 0} = 9.76206}$$

The entire plane is viable, and there are no critical terminal points.



$$\boxed{\cosh v \pm 1 = \frac{v^4}{A} \text{ nowhere}}$$

4.1.6 Critical terminal points

Of the up to 4 critical terminal points, those along $u=\pi$ are of elliptic type (i.e. useless) with no traced boundaries passing through, while those along $u=0$ are of hyperbolic type (i.e. good) with 2 traced boundaries passing through.

4.2 Values of A and v at the transitions

4.2.1 Warm-to-hot transition $\boxed{A = A_{4\pi} = 9.06433}$

$\cosh v + 1$ is tangential to v^4/A at $(A, v) = (A_{4\pi}, v_{4\pi})$, so ~~Thus~~ $(A_{4\pi}, v_{4\pi})$ is the solution to the system

$$\cosh v + 1 = \frac{v^4}{A} \qquad (r3.41)$$

$$\sinh v = \frac{4v^3}{A} \qquad (r3.42)$$

given by

$$\boxed{\begin{aligned} \cosh v + 1 &= \frac{v \sinh v}{4} \quad \text{near } v = 4.13 \qquad (r3.43) \\ A &= \frac{4v^3}{\sinh v} \qquad (r3.44) \end{aligned}},$$

which solves numerically to give

$$\boxed{\begin{aligned} v = v_{4\pi} &= 4.13068 \qquad (r3.45) \\ A = A_{4\pi} &= 9.06433 \qquad (r3.46) \end{aligned}}.$$

4.2.2 Cold-to-warm transition $\boxed{A = A_{40} = 9.76206}$

$\cosh v - 1$ is tangential to v^4/A at $(A, v) = (A_{40}, v_{40})$, so (A_{40}, v_{40}) is the potential solution to the system

$$\cosh v - 1 = \frac{v^4}{A} \qquad (r3.47)$$

$$\sinh v = \frac{4v^3}{A} \qquad (r3.48)$$

given by

$$\boxed{\begin{aligned} \cosh v - 1 &= \frac{v \sinh v}{4} \quad \text{near } v = 3.83 \\ A &= \frac{4v^3}{\sinh v} \end{aligned}} ,$$

(r3.49)
(r3.50)

which solves numerically to give

$$\boxed{\begin{aligned} v = v_{H0} &= 3.83002 \\ A = A_{H0} &= 9.76206 \end{aligned}}$$

(r3.51)
(r3.52)

4.3 Tracing ODE

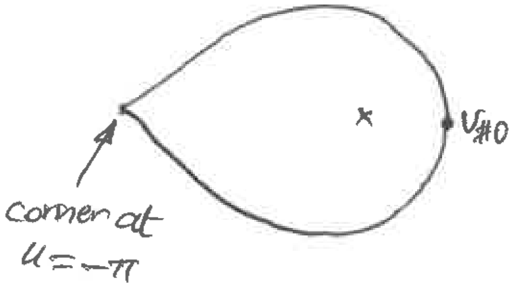
The tracing ODE (46) becomes

$$\frac{h\,dv}{h\,du} = \frac{0 \pm F\sqrt{\Phi}}{-F^2} = \frac{\pm\sqrt{\Phi}}{-F} = \pm \frac{A}{v^4}\sqrt{(\cosh v - \cos u)^2 - \frac{v^8}{A^2}}$$

$$\boxed{\frac{dv}{du} = \pm \frac{A}{v^4}\sqrt{(\cosh v - \cos u)^2 - \frac{v^8}{A^2}}} \qquad \text{(r3.53)}$$

Similar ~~the same~~ arguments for branch switching as in the line source case apply here (see r2-5): since we seek a closed, convex boundary surrounding the singularity $v = +\infty$, at ~~least~~ one switching must occur at a hyperbolic critical terminal point. Recall ~~these are~~ (see 4.1.6) these are located along $u=0$, at $v = v_{H0}$, v_{b0} for $A < A_{H0}$, and merge into $v = v_{H0}$ for $A = A_{H0}$, and are non-existent for $A > A_{H0}$.

we find that only the boundary through $(u, v) = (0, v_{H0})$ (for $A < A_{H0}$) and $(u, v) = (0, v_{H0})$ (for $A = A_{H0}$) appears to work; all others either fail to join up again, or form a concave corner. In particular, we obtain what appear to be a convex domain by taking (r3.53) through $(0, v_{H0})$, and using the upper branch for $-\pi < u < 0$ and the lower branch when $0 < u < \pi$:



This we shall call the "candidate boundary", and it exists and is unique whenever $0 < A \leq A_{H0}$.

4.4 Convexity

I ~~have~~ suspect that the candidate boundary is not convex near $A = A_{40}$, and to confirm this, we must check the sign of the traced boundary curvature. Let primes denote u differentiation, and for brevity, define

$$D = \sinh v - v' \sin u \tag{r3.54}$$
$$E = v' \sinh v + \sin u . \tag{r3.55}$$

Following some algebra, we get

$$(hS)' = h^2 CD \tag{r3.56}$$
$$(hC)' = -h^2 SD , \tag{r3.57}$$

and differentiating (r3.30) & (r3.31) with respect to u, we have

$$\vec{a}_u{}' = -h^2 CD \vec{a}_x - h^2 SD \vec{a}_y = hD \vec{a}_v \tag{r3.58}$$
$$\vec{a}_v{}' = h^2 SD \vec{a}_x - h^2 CD \vec{a}_y = -hD \vec{a}_u . \tag{r3.59}$$

From (23),

$$d\vec{r} = h\, du\, \vec{a}_u + h\, dv\, \vec{a}_v , \tag{r3.60}$$

which, divided by du, gives

$$\vec{r}\,' = \frac{d\vec{r}}{du} = h \vec{a}_u + h v' \vec{a}_v , \tag{r3.61}$$

which implies

$$\vec{r}\,'' = \frac{d^2\vec{r}}{du^2} = (h \vec{a}_u + h v' \vec{a}_v)'$$
$$= -h^2 (Dv' + E) \vec{a}_u + h (v'' - hEv' + hD) \vec{a}_v \tag{r3.62}$$

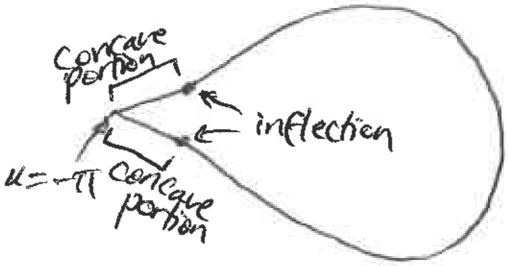
after some more algebra. Thus the curvature has the same sign as

$$\vec{a}_z \cdot \vec{r}\,' \times \vec{r}\,'' = h^2 (v'' - hEv' + hD) + h^3 v' (Dv' + E)$$
$$= h^3 \left[\frac{v''}{h} - Ev' + D + Dv'^2 + Ev' \right]$$
$$= h^3 \left[D(1 + v'^2) + \frac{v''}{h} \right] , \tag{r3.63}$$

which may be evaluated as desired along any traced boundary (r3.53).

We find for instance that the (upper branch of) the candidate boundary for $A = A_{40}$ has a point of inflection at $u = u_i = -3.04701$, so that the tip $-\pi < u < u_i$ is in fact concave, and not acceptable as a radiation boundary.

Thus there is in fact a small interval $A_i < A \leq A_{40}$ over which the candidate boundary is concave near unto the corner $u = -\pi$.



To determine A_i, we evaluate the curvature (r3.63) at the corner $u = -\pi$:

$$\vec{e}_z \cdot \vec{r}' \times \vec{r}''\Big|_{u=-\pi} = \frac{2A^2}{v^4}\left(-2 + v \tanh \frac{v}{2}\right) \qquad \text{(r3.64)}$$

This changes sign at $v = v_i$, the solution to the transcendental equation

$$\frac{v}{2} \tanh \frac{v}{2} = 1, \qquad \text{(r3.65)}$$

numerically,

$$v = v_i = 2.39936\,. \qquad \text{(r3.66)}$$

It follows that the candiate boundary $v = V(u)$ shall be concave near $u = -\pi$ if

$$V(-\pi) < v_i\,. \qquad \text{(r3.67)}$$

Thus the critical value A_i is given by the A for which

$$V(-\pi) = v_i\,; \qquad \text{(r3.68)}$$

using the bisection algorithm, we obtain

$$A_i = 9.76036\,, \qquad \text{(r3.69)}$$

which is within 0.002 of $A_{40} = 9.76206$ (see (r3.52)).

5. Summary

Restore dropped hats. Boundary tracing says that if

$$T = T_0 v = T_0 \tanh^{-1} \frac{2ax}{x^2+y^2+a^2} \qquad \text{(r3.70)}$$

is the known temperature profile, a candidate boundary (see the bottom of page r3-9) may be constructed for

$$0 < A \leq A_{H0} = 9.76206, \qquad \text{(r3.71)}$$

where

$$A = \frac{1}{caT_0^3} \qquad \text{(r3.72)}$$

is dimensionless. This boundary is given by

$$\frac{dv}{du} = \pm \frac{A}{v^4}\sqrt{(\cosh v - \cos u)^2 - \frac{v^8}{A^2}} \qquad \text{(r3.73)}$$

$$v(0) = v_{H0} \qquad \text{(r3.74)}$$

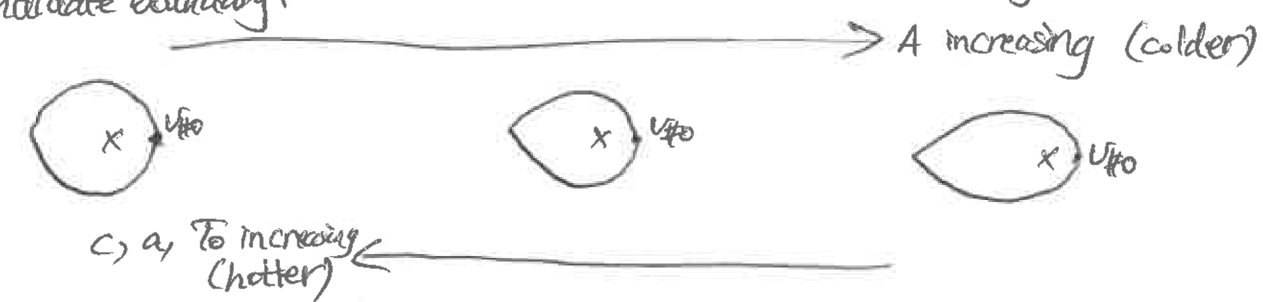
where the upper branch is taken on $-\pi < u < 0$, and the lower one on $0 < u < \pi$. Here, $v_{H0} = v_{H0}(A)$ is the largest non-trivial solution to the transcendental equation

$$\cosh v - 1 = \frac{v^4}{A}. \qquad \text{(r3.75)}$$

Note that (r3.76a)

$$v_{H0}(A) \geq v_{H0} = 3.83002 \qquad \text{~~(r3.76b)~~}$$

given (r3.71), with equality if and only if $A = A_{H0}$. The effect of increasing A (moving to colder regimes) is to increase the amount of asymmetry in the candidate boundary:



This makes sense; for large A (small a), the line source at $x = -a$ is significant, whereas for small A (large a), the line source at $x = -a$ is negligible. In the limiting case $A = 0$ ($a = \infty$), the candidate boundary reduces to the inner terminal curve $r = r_b$ in the hot regime of the ~~pole single line source case~~ single-line-source case, see Sections 3.21 and 3.33.

While ~~the~~ a unique candidate boundary exists for all A in the range (r3.71), i.e.

$$0 < A \le A_{40} = 9.76206,$$

a careful analysis shows that the tip thereof near the corner $u = -\pi$ will in fact be concave if

$$V(-\pi) < V_i \,, \tag{r3.76}$$

where

$$V = V_i = 2.39936 \tag{r3.77}$$

is the root of the transcendental equation

$$\frac{V}{2} \tanh \frac{V}{2} = 1 \,. \tag{r3.78}$$

High precision numerical integration shows that (r3.76) occurs for

$$A_i < A \le A_{40} \tag{r3.79}$$

where

$$A_i = 9.76036 \tag{r3.80}$$

(this is within 0.002 of A_{40}!); thus the candidate boundary is only convex for

$$0 < A \le A_i = 9.76036 \,. \tag{r3.81}$$

Still, the concavity of the tip over the small interval (r3.79) is very minute; for all practical purposes the concave portions of the tip are indistinguishable from straight lines.

5.1 Physical temperature range

In practice one is probably only interested in objects of a given size. Observe that the candidate boundary is approximately the level curve $v = v_{\#0}$ (a circle) enlarged on the left side by a wedge (due to the asymmetry brought about by the line charge at $x = -a$ ($v = -\infty$)). From (r3.5), the circle $v = v_{\#0}$ has radius

$$r_{\#0} = a \operatorname{csch} v_{\#0} = \frac{a}{\sinh v_{\#0}} \qquad \text{(r3.82)}$$

so

$$a = r_{\#0} \sinh v_{\#0} \,. \qquad \text{(r3.83)}$$

From (r3.72), we have

$$T_0 = \left(\frac{1}{caA}\right)^{1/3} = \left(\frac{1}{cr_{\#0}} \frac{1}{A \sinh v_{\#0}}\right)^{1/3} , \qquad \text{(r3.84)}$$

so the temperature $T_{\#0}$ along $r = r_{\#0}$ is given by

$$T_{\#0} = T_0 v_{\#0} = \left(\frac{1}{cr_{\#0}} \frac{v_{\#0}^3}{A \sinh v_{\#0}}\right)^{1/3}$$

$$= \left(\frac{1}{cr_{\#0}} \frac{\cosh v_{\#0} - 1}{v_{\#0} \sinh v_{\#0}}\right)^{1/3} \qquad \text{(r3.85)}$$

where in the last step we have used (r3.75). Define

$$\boxed{\omega = \frac{\cosh v_{\#0} - 1}{v_{\#0} \sinh v_{\#0}}} \qquad \text{(r3.86)}$$

Then

$$T_{\#0} = \left(\frac{1}{cr_{\#0}} \cdot \omega(v_{\#0})\right)^{1/3} . \qquad \text{(r3.87)}$$

Note that

$$\omega' = \frac{d\omega}{dv_{\#0}} = \frac{v_{\#0} - \sinh v_{\#0}}{v_{\#0}^2 (1 + \cosh v_{\#0})} < 0 \qquad \text{(r3.88)}$$

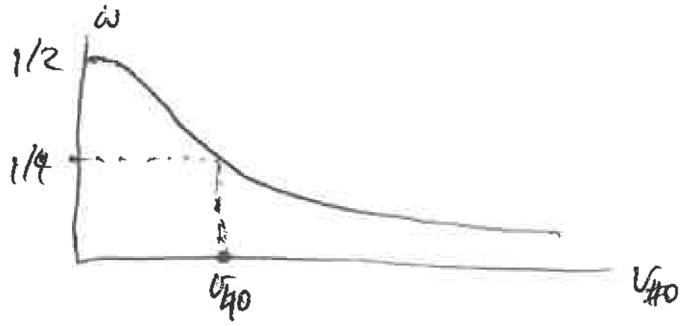
$$\omega(0) = \frac{1}{2}$$

$$\omega(\infty) = 0$$

$$\omega'(0) = 0$$

$$\omega \sim \frac{1}{2} - \frac{v_{\#0}^2}{24}, \qquad v_{\#0} \text{ small} \tag{r3.89a}$$

$$\omega \sim \frac{1}{v_{\#0}} - \frac{2e^{-v_{\#0}}}{v_{\#0}}, \qquad v_{\#0} \text{ large}. \tag{r3.89b}$$



We see that ω is a decreasing function of $v_{\#0}$, and since we have (r3.76a), we have

$$0 < \omega \le \omega(v_{40}) = \frac{1}{4}. \tag{r3.89}$$

Thus the range of possible temperatures is

$$0 < T_{\#0} \le \left(\frac{1}{4cr_{\#0}}\right)^{1/3} \tag{r3.90}$$

or

$$0 < T_{\#0} \le \left(\frac{k}{4\varepsilon\sigma r_{\#0}}\right)^{1/3}. \tag{r3.91}$$

(The upper bound here is very similar to the lower bound of (r2.45).)

<u>5.1.1 Example</u>

Using the same quantities as in Example 4.1.1, except with $r_{\#0}$ replacing $r_{\#}$, (r3.91) becomes

$$0 < T_{\#0} \le 626\text{ K}.$$

Now if we choose a sensible temperature, say $T_{\#0} = 323\text{K}$ (50°C), what $v_{\#0}$ and A does this correspond to? From (r3.87), we have

$$\frac{323\text{ K}}{626\text{ K}} = \left[\frac{\omega(v_{\#0})}{1/4}\right]^{1/3}$$

$$\omega(v_{\#0}) = \left(\frac{323}{626}\right)^3 \cdot \frac{1}{4}$$

$$= 0.0343$$

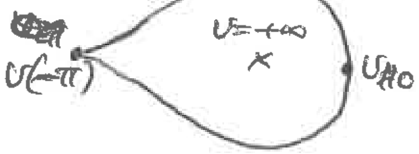
Using (r3.89b), $v_{\#0} \sim 1/\omega = 1/0.0343 = 29.1$, and using (r3.78), we have

$$A = \frac{v_{\#0}^{4}}{\cosh v_{\#0} - 1} = 3.2 \times 10^{-7} .$$

This is so close to the limiting case $A = 0$ (for which asymmetry caused by the line source at $x = -a$ ($v = -\infty$) is negligible) that the candidate boundary is practically indistinguishable from a circle; this is rather disappointing.

If we want greater asymmetry, we will need temperatures $T_{\#0}$ closer to the upper bound in (r3.91).

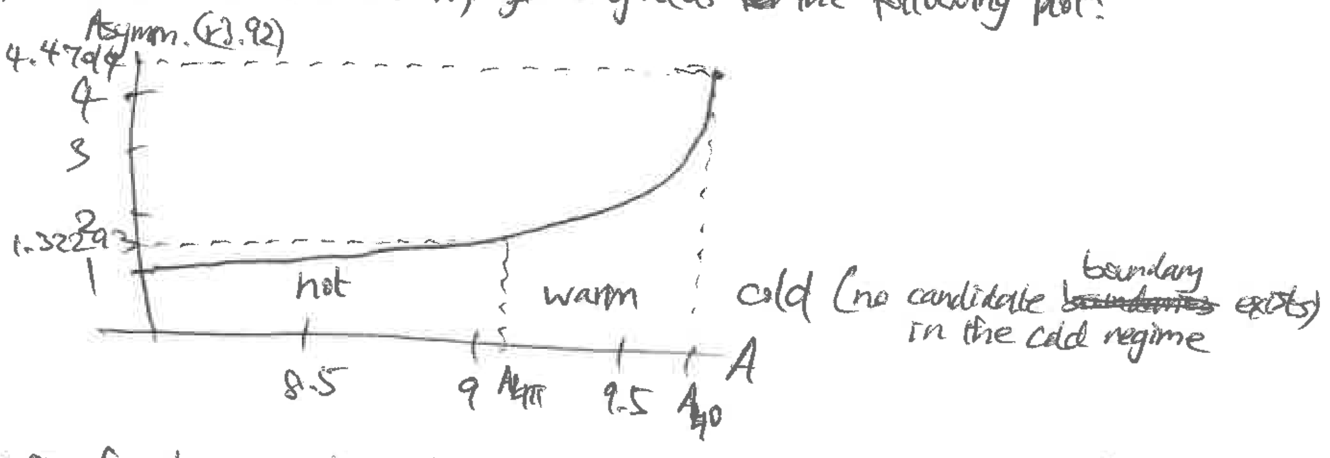
5.2 Physical asymmetry



Given any of the candidate boundaries, a measure of its asymmetry (or deviation from a perfect circle) can be obtained by evaluating the ratio of distances to the singularity at $x = +a$ ($v = +\infty$) from each of the hyperbolic critical terminal point $(u, v) = (0, v_{\#0})$ at the right end and the corner $(u, v) = (-\pi, v(-\pi))$ on the left end. This ratio, which we shall call the asymmetry, is given by

$$\frac{1 - \hat{x}(-\pi)}{\hat{x}_{\#0} - 1} = \frac{1 - \dfrac{\sinh v(-\pi)}{\cosh v(-\pi) + 1}}{\dfrac{\sinh v_{\#0}}{\cosh v_{\#0} - 1} - 1} = \frac{e^{v_{\#0}} - 1}{e^{v(-\pi)} + 1}, \qquad \text{(r3.92)}$$

which, evaluated at various A, yields the following plot:

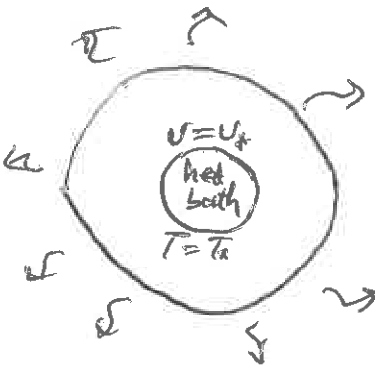


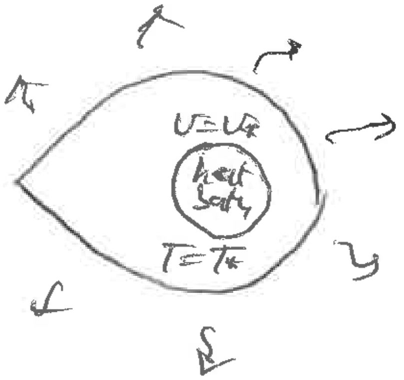
Significant asymmetry only appears at A greater than $A_{\#\pi}$ (the cold-to-warm transition), i.e. A in the warm regime.

5.3 A hypothetical situation

Given a radius $r_{\#0}$, we can construct a candidate boundary with a spike which admits the solution (r3.70) to the boundary value problem (r7) & (r8) in the temperature range (r3.91). The heat may be generated by any circular heat bath $v = v_*$ at temperature T_* satisfying

$$T_* = T_0 v_* .$$





Note that the candidate boundary is minutely concave for A in the interval (r3.79).

5.4 Physical power per length range

Equivalently, the heat bath $T = T_*$ condition may be replaced by a prescribed power per length

$$\begin{aligned}
P &= \int_0^{2\pi} \vec{n} \cdot \left[-k\nabla T\right] h \, du \\
&= \int_0^{2\pi} (-\vec{a}_v) \cdot \frac{-kT_0}{h} \vec{a}_v \, h \, du \\
&= 2\pi k T_0 \\
&= 2\pi k \left(\frac{1}{c r_{\#0}} \cdot \frac{\omega(v_{\#0})}{v_{\#0}^3} \right)^{1/3} .
\end{aligned} \qquad (r3.93)$$

It can be shown that $\omega(v_{\#0})/v_{\#0}^3$ is decreasing, so, given (r3.76a), the range of power per length possible is

$$0 < P \leq 2\pi k \left(\frac{1}{4 c r_{\#0}} \right)^{1/3} \frac{1}{v_{\#0}} \qquad (r3.94)$$

$$0 < P \leq \frac{2\pi k^{4/3}}{(4 \epsilon \sigma r_{\#0})^{1/3} \, v_{\#0}} . \qquad (r3.95)$$