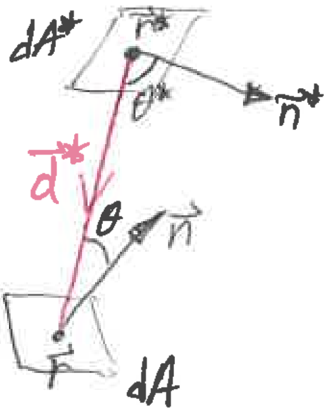
Radiation 6: Self-viewing radiation

radiation-6-self.pdf

In cases where the constructed domain is not convex, some outgoing radiation will not continue to infinity, but instead strike another portion of the domain boundary.

Consider a differential area dA at $\vec{r}$, with normal $\vec{n}$, which receives radiation from dA^* at $\vec{r}^*$, with normal $\vec{n}^*$. The displacement from $\vec{r}^*$ unto $\vec{r}$ is



$$\vec{d}^* = \vec{r} - \vec{r}^*. \tag{r6.1}$$

Let the normals make angles θ and θ^* with the connecting line of the displacement. It is well known that the view factor from dA^* to dA, that is, the proportion of radiation which leaves dA^* and strikes dA, is given by

$$\frac{\cos\theta^* \cos\theta \, dA^*}{\pi d^{*2}} . \tag{r6.2}$$

Thus, to account for self-viewing radiation, the boundary condition (r8) must be modified to

$$\boxed{\vec{n}\cdot\vec{\nabla}T = -cT^4 + \int \frac{cT^{*4}\cos\theta^*\cos\theta \, dA^*}{\pi d^{*2}}} , \tag{r6.3}$$

where the integral is taken over all elements dA^* which can see the element dA at the local position $\vec{r}$ under consideration. The ratio R between the new integral term and the existing term in (r6.3) is given by

$$R = \frac{1}{T^4}\int \frac{T^{*4}\cos\theta^*\cos\theta \, dA^*}{\pi d^{*2}} . \tag{r6.4}$$

Now observe that

$$\cos\theta^* = \frac{\vec{n}^*\cdot\vec{d}^*}{d^*} \tag{r6.5}$$

$$\cos\theta = \frac{\vec{n}\cdot(-\vec{d}^*)}{d^*} \tag{r6.6}$$

$$\cos\theta^*\cos\theta = \frac{(\vec{n}^*\cdot\vec{d}^*)(-\vec{n}\cdot\vec{d}^*)}{d^{*2}} . \tag{r6.7}$$

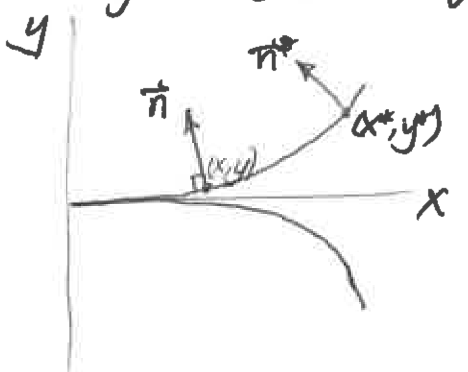
Therefore

$$\boxed{R = \frac{1}{T^4}\int \frac{T^{*4}(\vec{n}^*\cdot\vec{d}^*)(-\vec{n}\cdot\vec{d}^*)\,dA^*}{\pi d^{*4}}} \qquad (r6.8)$$

We now consider some specific cases:

1. <u>Plane source spike</u>

Consider the spike $y=y(x)$ from radiation-1-plane.pdf, which is everywhere concave. We have the following, where primes denote x-differentiation:



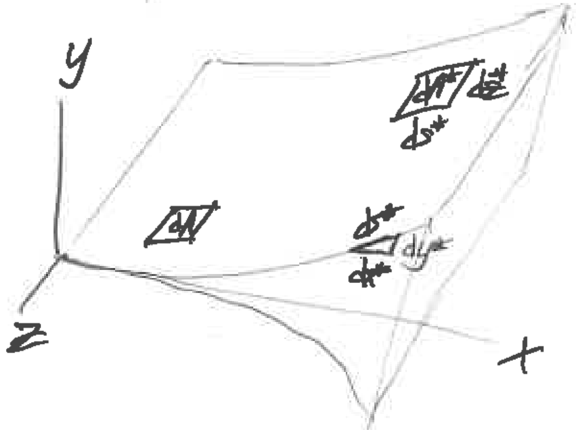
$$\vec{n} = \frac{-y'\vec{a}_x + \vec{a}_y}{\sqrt{1+y'^2}} \qquad (r6.9)$$

$$\vec{d}^* = (x-x^*)\vec{a}_x + (y-y^*)\vec{a}_y + (z-z^*)\vec{a}_z \qquad (r6.10)$$

Therefore,

$$\vec{n}^*\cdot\vec{d}^* = \frac{-(x-x^*)y'^* + (y-y^*)}{\sqrt{1+y'^{*2}}} \qquad (r6.11)$$

$$-\vec{n}\cdot\vec{d}^* = \frac{+(x-x^*)y' - (y-y^*)}{\sqrt{1+y'^2}}\,. \qquad (r6.12)$$



Also

$$dA^* = ds^*\,dz^* = \sqrt{1+y'^{*2}}\,dx^*\,dz^*\,. \qquad (r6.13)$$

If we are constructing a domain with traced boundaries taken from the interval from x_1 to x_2, we therefore have

$$R = \frac{1}{T^4}\int_{-\infty}^{\infty}\int_{x_1}^{x_2} \frac{T^{*4}[-(x-x^*)y'^* + (y-y^*)][(x-x^*)y' - (y-y^*)]\cancel{\sqrt{1+y'^{*2}}}\,dx^*\,dz^*}{\pi[(x-x^*)^2+(y-y^*)^2+(z-z^*)^2]^2\cancel{\sqrt{1+y'^{*2}}}\sqrt{1+y'^2}} \qquad (r6.14)$$

$$\boxed{R = \frac{1}{T^4}\int_{x_1}^{x_2} \frac{T^{*4}[-(x-x^*)y'^* + (y-y^*)][(x-x^*)y' - (y-y^*)]\,dx^*}{2[(x-x^*)^2+(y-y^*)^2]^{3/2}\sqrt{1+y'^2}}} \qquad (r6.15)$$

where we have used

$$\int_{-\infty}^{\infty}\frac{d\zeta}{\pi(\rho^2+\zeta^2)^2} = \frac{1}{2[\rho^2]^{3/2}}\,. \qquad (r6.16)$$

A crude upper bound for (r6-15) can be estimated by Taylor remainders. Observe that

$$y = y^* + (x-x^*)y'^* + \frac{(x-x^*)^2}{2!} y_c''^* \qquad (r6\text{-}17)$$

$$y^* = y + (x^*-x)y' + \frac{(x^*-x)^2}{2!} y_c'' \qquad (r6\text{-}18)$$

where $y_c''^* = y''|_{x=x_c^*}$, $y_c'' = y''|_{x=x_c}$, for some x_c^* and x_c between x^* and x. Therefore

$$-(x-x^*)y'^* + (y-y^*) = \frac{(x-x^*)^2}{2!} y_c''^* \qquad (r6\text{-}19)$$

$$(x-x^*)y' - (y-y^*) = \frac{(x-x^*)^2}{2!} y_c'' \,, \qquad (r6\text{-}20)$$

so we have

$$R = \frac{1}{T^4}\int_{x_1}^{x_2} \frac{T^{*4}(x-x^*)^2 y_c''^* (x-x^*)^2 y_c'' \, dx^*}{2\cdot 2!\, 2!\, [(x-x^*)^2+(y-y^*)^2]^{3/2}\sqrt{1+y'^2}}$$

$$= \frac{1}{8T^4\sqrt{1+y'^2}}\int_{x_1}^{x_2} \frac{T^{*4}(x-x^*)^4 y_c''^* y_c'' \, dx^*}{[(x-x^*)^2+(y-y^*)^2]^{3/2}} \,. \qquad (r6\text{-}21)$$

Crudely, we have the upper bound

$$\boxed{R \le \frac{y_{max}''^{\,2}}{8T^4\sqrt{1+y'^2}}\int_{x_1}^{x_2} \frac{T^{*4}(x-x^*)^4 \, dx^*}{[(x-x^*)^2+(y-y^*)^2]^{3/2}}} \qquad (r6\text{-}22)$$

where $y_{max}'' = \max_{x_1\le x\le x_2} y''$. The bound (r6-22) has a derivative-free integrand, but still requires taking an integral. For an ultra-crude bound which requireth not taking an integral, note that

$$y-y^* = (x-x^*)\, y_c'^* \qquad (r6\text{-}23)$$

$$(y-y^*)^2 \ge (x-x^*)^2\, y_{min}'^{\,2}$$

$$(x-x^*)^2 + (y-y^*)^2 \ge (x-x^*)^2\,[1+y_{min}'^{\,2}]$$

therefore

$$R \le \frac{y_{max}''^{\,2}}{8T_{min}^4\sqrt{1+y_{min}'^{\,2}}}\int_{x_1}^{x_2} \frac{T_{max}^4 (x-x^*)^4\, dx^*}{[(x-x^*)^2(1+y_{min}'^{\,2})]^{3/2}}$$

$$= \frac{y_{max}''^{\,2}\, T_{max}^4}{8T_{min}^4\sqrt{1+y_{min}'^{\,2}}}\,\frac{1}{(1+y_{min}'^{\,2})^{3/2}}\int_{x_1}^{x_2} |x-x^*|\, dx^* \,. \qquad (r6\text{-}24)$$

Since we are only considering $x_1 \le x \le x_2$, and integrating over $x_1 \le x^* \le x_2$, we have

$$|x - x^*| \le (x_2 - x_1) . \tag{r6.25}$$

Therefore

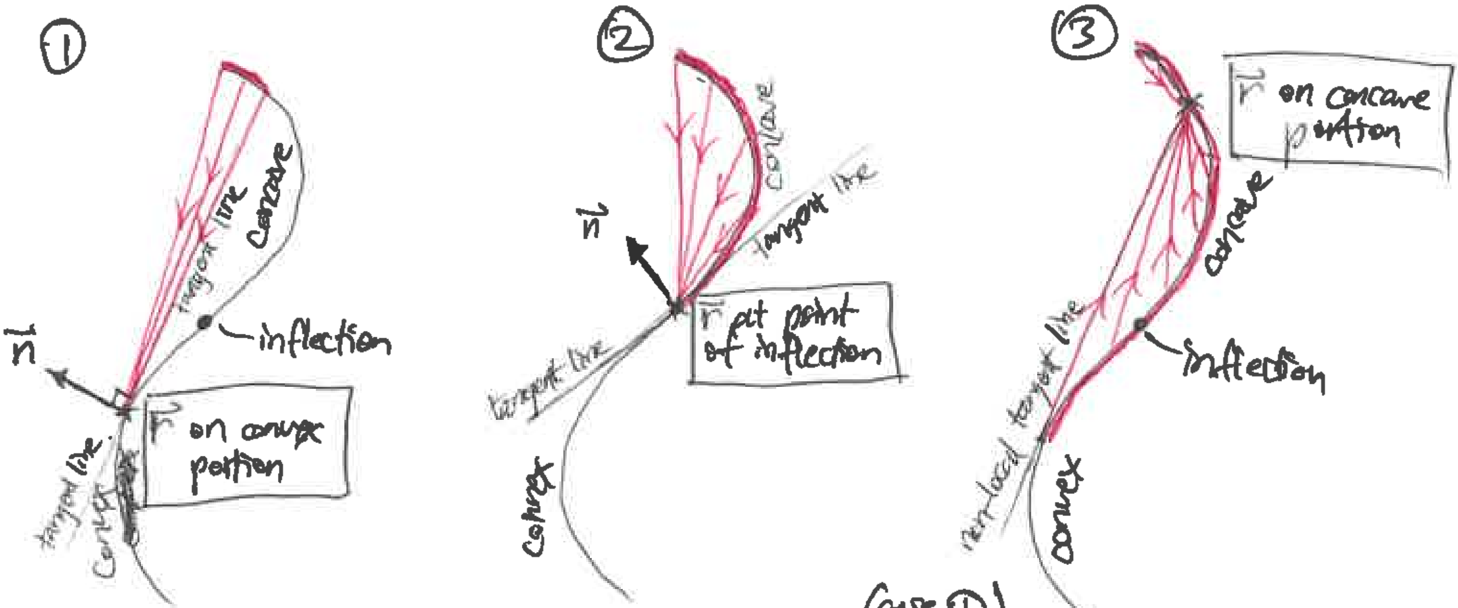
$$R \le \frac{y''^2_{max} T_{max}^4}{8 T_{min}^4 \sqrt{1+y'^2_{min}}} \frac{1}{(1+y'^2_{min})^{3/2}} \cdot (x_2 - x_1)^2$$

$$\boxed{R \le \frac{(x_2-x_1)^2 (T_{max}/T_{min})^4 \, y''^2_{max}}{8\,(1+y'^2_{min})^2}} . \tag{r6.26}$$

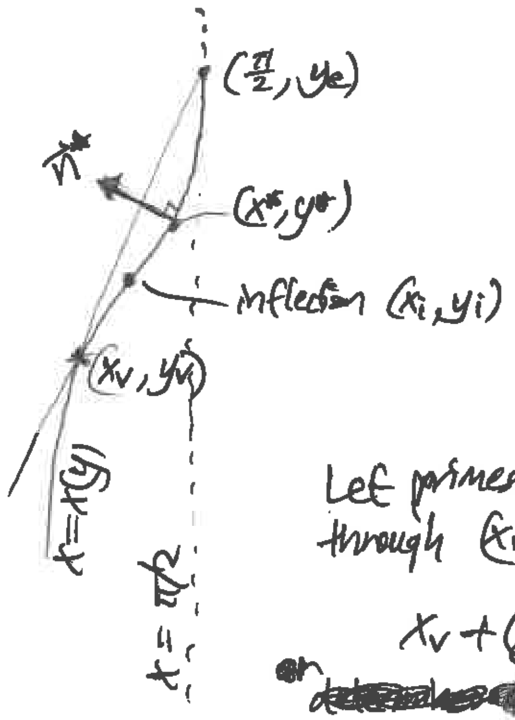
2. Non-convex lens

One supervisor said that it would be good to evaluate R for the non-convex lens-like domain for the cosine simple ($B=1$) case, see Page r5-5, Sec. 3.3.2 in radiation-5-cosine.pdf.

But that situation is actually quite complicated; there is self-viewing even into the convex portions; this is best seen with an exaggerated sketch:



We see that if $\vec{r}$ is on the convex portion of the lens (case ①), the self-viewing ~~[illegible]~~ ratio R is probably less than for cases ② & ③, but between ② & ③ it is unclear which has greater R. In particular, for case ③, the local point $\vec{r}$ receives radiation from both the entire concave portion, but also some of the convex portion too.



The simplest way forward is to do a bounding on $y_v \le y^* \le y_e$, where y_e is the end of the lens,

$$x(y)\big|_{y=y_e} = \pi/2 , \qquad (r6.27)$$

and (x_v, y_v) is the lowest part (in the top half of the lens) which is visible to the end.

Let primes denote y-differentiation. The tangent line through (x_v, y_v) meets the endpoint $(\pi/2, y_e)$, therefore

$$x_v + (y_e - y_v)\, x'_v = \pi/2 \qquad (r6.28)$$

or

$$x(y_v) + (y_e - y_v)\, x'(y_v) = \pi/2 \qquad (r6.29)$$

determines y_v. Now for $x = X(y)$, we have instead of (r6.9),

$$\hat{n} = \frac{-\hat{e}_x + x'\hat{e}_y}{\sqrt{x'^2+1}} \qquad (r6.30)$$

so we apply $\{y' \to 1,\ 1 \to x'\}$ unto (r6.15), and get

$$R = \frac{1}{T^*}\int_{y_1}^{y_2} \frac{T^{*4}\left[-(x-x^*) + (y-y^*)x'^*\right]\left[(x-x^*) - (y-y^*)x'\right] dy^*}{2\left[(x-x^*)^2 + (y-y^*)^2\right]^{3/2}\sqrt{x'^2+1}} . \qquad (r6.31)$$

and so

$$R \le \frac{1}{T^4}\int_{y_v}^{y_e} \frac{T^{*4}(y-y^*)^2\,|x''_c|\,(y-y^*)^2\,|x'_c|\, dy^*}{8\left[(y-y^*)^2\right]^{3/2}\left(x'^2+1\right)^{3/2}\sqrt{x'^2+1}} \quad |x'|^2_{min}$$

$$\le \left(\frac{T_{max}}{T_{min}}\right)^4 \frac{|x''|^2_{max}}{8\left(|x'|^2_{min}+1\right)^2}\int_{y_v}^{y_e} |y-y^*|\, dy^*$$

$$\boxed{R \le \frac{(y_e-y_v)^2 (T_{max}/T_{min})^4 |x''|^2_{max}}{8\left(1+|x'|^2_{min}\right)^2}} , \qquad (r6.32)$$

very crudely.