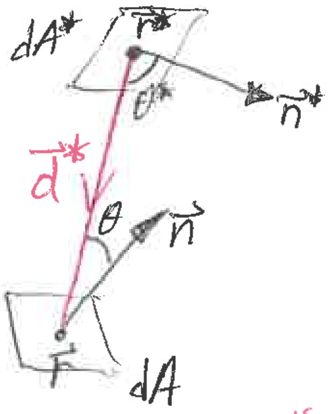
Radiation 6: Self-viewing radiation

In cases where the constructed domain is not convex, some outgoing radiation will not continue to infinity, but instead strike another portion of the domain boundary.

Consider a differential area dA at $\vec{r}$, with normal $\vec{n}$, which receives radiation from dA^* at $\vec{r}^*$, with normal $\vec{n}^*$. The displacement from $\vec{r}^*$ unto $\vec{r}$ is



$$\vec{d}^* = \vec{r} - \vec{r}^* . \tag{r6.1}$$

Let the normals make angles θ and θ^* with the connecting line of the displacement. It is well known that the view factor from dA^* to dA, that is, the proportion of radiation which leaves dA^* and strikes dA, is given by

Fraction of energy $dA^* \to dA$: $\boxed{\dfrac{\cos\theta^* \cos\theta \, dA}{\pi d^{*2}}}$.

Energy $dA^* \to dA$: $\boxed{\dfrac{cT^{*4} dA^* \cos\theta^* \cos\theta \, dA}{\pi d^{*2}}}$

Per dA: $\dfrac{cT^{*4} dA^* \cos\theta^* \cos\theta}{\pi d^{*2}}$ (r6.2)

Thus, to account for self-viewing radiation, the boundary condition (r8) must be modified to

$$\boxed{\vec{n} \cdot \vec{\nabla} T = -cT^4 + \int \frac{cT^{*4} \cos\theta^* \cos\theta \, dA^*}{\pi d^{*2}}} , \tag{r6.3}$$

where the integral is taken over all elements dA^* which can see the element dA at the local position $\vec{r}$ under consideration. The ratio R between the new integral term and the existing term in (r6.3) is given by

$$R = \frac{1}{T^4} \int \frac{T^{*4} \cos\theta^* \cos\theta \, dA^*}{\pi d^{*2}} . \tag{r6.4}$$

Now observe that

$$\cos\theta^* = \frac{\vec{n}^* \cdot \vec{d}^*}{d^*} \tag{r6.5}$$

$$\cos\theta = \frac{\vec{n} \cdot (-\vec{d}^*)}{d^*} \tag{r6.6}$$

$$\cos\theta^* \cos\theta = \frac{(\vec{n}^* \cdot \vec{d}^*)(-\vec{n} \cdot \vec{d}^*)}{d^{*2}} . \tag{r6.7}$$

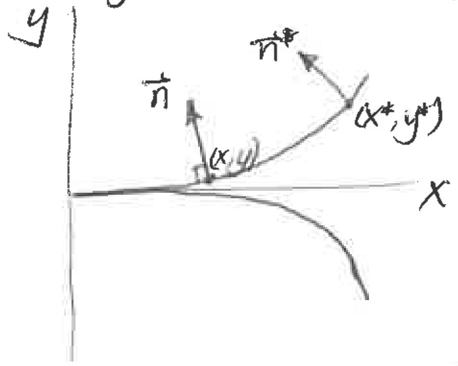
Therefore

$$\boxed{R = \frac{1}{T^4}\int \frac{T^{*4}(\vec{n}^*\cdot\vec{d}^*)(-\vec{n}\cdot\vec{d}^*)\,dA^*}{\pi d^{*4}}} . \qquad (r6.8)$$

We now consider some specific cases:

1. Plane source spike

Consider the spike $y = y(x)$ from radiation-1-plane.pdf, which is everywhere concave. We have the following, where primes denote x-differentiation:



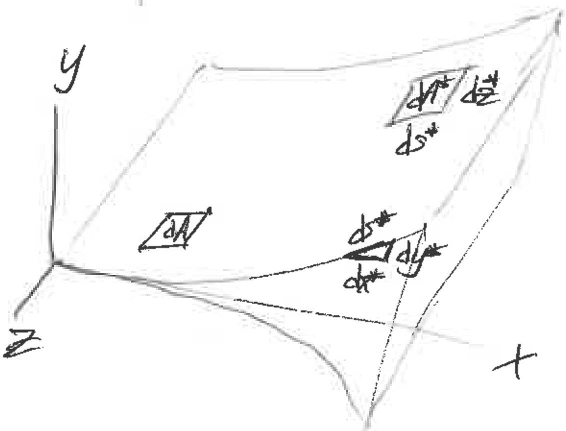
$$\vec{n} = \frac{-y'\vec{a}_x + \vec{a}_y}{\sqrt{1+y'^2}} \qquad (r6.9)$$

$$\vec{d}^* = (x-x^*)\vec{a}_x + (y-y^*)\vec{a}_y + (z-z^*)\vec{a}_z \qquad (r6.10)$$

Therefore,

$$\vec{n}^*\cdot\vec{d}^* = \frac{-(x-x^*)y'^* + (y-y^*)}{\sqrt{1+y'^{*2}}} \qquad (r6.11)$$

$$-\vec{n}\cdot\vec{d}^* = \frac{+(x-x^*)y' - (y-y^*)}{\sqrt{1+y'^2}} . \qquad (r6.12)$$



Also

$$dA^* = ds^*\,dz^* = \sqrt{1+y'^{*2}}\,dx^*\,dz^* . \qquad (r6.13)$$

If we are constructing a domain with traced boundaries taken from the interval from x_1 to x_2, we therefore have

$$R = \frac{1}{T^4}\int_{-\infty}^{\infty}\int_{x_1}^{x_2} \frac{T^{*4}\left[-(x-x^*)y'^* + (y-y^*)\right]\left[(x-x^*)y' - (y-y^*)\right]\sqrt{1+y'^{*2}}\,dx^*\,dz^*}{\pi\left[(x-x^*)^2 + (y-y^*)^2 + (z-z^*)^2\right]^2\sqrt{1+y'^{*2}}\sqrt{1+y'^2}} \qquad (r6.14)$$

$$\boxed{R = \frac{1}{T^4}\int_{x_1}^{x_2} \frac{T^{*4}\left[-(x-x^*)y'^* + (y-y^*)\right]\left[(x-x^*)y' - (y-y^*)\right]dx^*}{2\left[(x-x^*)^2 + (y-y^*)^2\right]^{3/2}\sqrt{1+y'^2}}} \qquad (r6.15)$$

where we have used

$$\int_{-\infty}^{\infty}\frac{d\zeta}{\pi(\rho^2+\zeta^2)^2} = \frac{1}{2[\rho^2]^{3/2}} . \qquad (r6.16)$$

A crude upper bound for (r6-15) can be estimated by Taylor remainders. Observe that

$$y = y^* + (x-x^*)y'^* + \frac{(x-x^*)^2}{2!} y_c''^* \qquad (r6\text{-}17)$$

$$y^* = y + (x^*-x)y' + \frac{(x^*-x)^2}{2!} y_c'' \qquad (r6\text{-}18)$$

where $y_c''^* = y''|_{x=x_c^*}$, $y_c'' = y''|_{x=x_c}$, for some x_c^* and x_c between x^* and x. Therefore

$$-(x-x^*)y'^* + (y-y^*) = \frac{(x-x^*)^2}{2!} y_c''^* \qquad (r6\text{-}19)$$

$$(x-x^*)y' - (y-y^*) = \frac{(x-x^*)^2}{2!} y_c'' , \qquad (r6\text{-}20)$$

so we have

$$R = \frac{1}{T^4}\int_{x_1}^{x_2} \frac{T^{*4}(x-x^*)^2 y_c''^* (x-x^*)^2 y_c'' \, dx^*}{2\cdot 2!\, 2!\, [(x-x^*)^2+(y-y^*)^2]^{3/2}\sqrt{1+y'^2}}$$

$$= \frac{1}{8T^4\sqrt{1+y'^2}}\int_{x_1}^{x_2} \frac{T^{*4}(x-x^*)^4 y_c''^* y_c'' \, dx^*}{[(x-x^*)^2+(y-y^*)^2]^{3/2}} . \qquad (r6\text{-}21)$$

Crudely, we have the upper bound

$$\boxed{R \le \frac{{y''_{max}}^2}{8T^4\sqrt{1+y'^2}}\int_{x_1}^{x_2} \frac{T^{*4}(x-x^*)^4 \, dx^*}{[(x-x^*)^2+(y-y^*)^2]^{3/2}}} \qquad (r6\text{-}22)$$

where $y''_{max} = \max_{x_1\le x\le x_2} y''$. The bound (r6-22) has a derivative-free integrand, but still requires taking an integral. For an ultra-crude bound which requireth not taking an integral, note that

$$y-y^* = (x-x^*)\, y_c'^* \qquad (r6\text{-}23)$$

$$(y-y^*)^2 \ge (x-x^*)^2 {y'_{min}}^2$$

$$(x-x^*)^2 + (y-y^*)^2 \ge (x-x^*)^2 (1+{y'_{min}}^2)$$

therefore

$$R \le \frac{{y''_{max}}^2}{8T_{min}^4\sqrt{1+{y'_{min}}^2}}\int_{x_1}^{x_2} \frac{T_{max}^4 (x-x^*)^4 \, dx^*}{[(x-x^*)^2(1+{y'_{min}}^2)]^{3/2}}$$

$$= \frac{{y''_{max}}^2 \, T_{max}^4}{8T_{min}^4\sqrt{1+{y'_{min}}^2}} \frac{1}{(1+{y'_{min}}^2)^{3/2}}\int_{x_1}^{x_2} |x-x^*| \, dx^* . \qquad (r6\text{-}24)$$

Since we are only considering $x_1 \le x \le x_2$, and integrating over $x_1 \le x^* \le x_2$, we have

$$|x - x^*| \le (x_2 - x_1). \qquad (r6.25)$$

Therefore

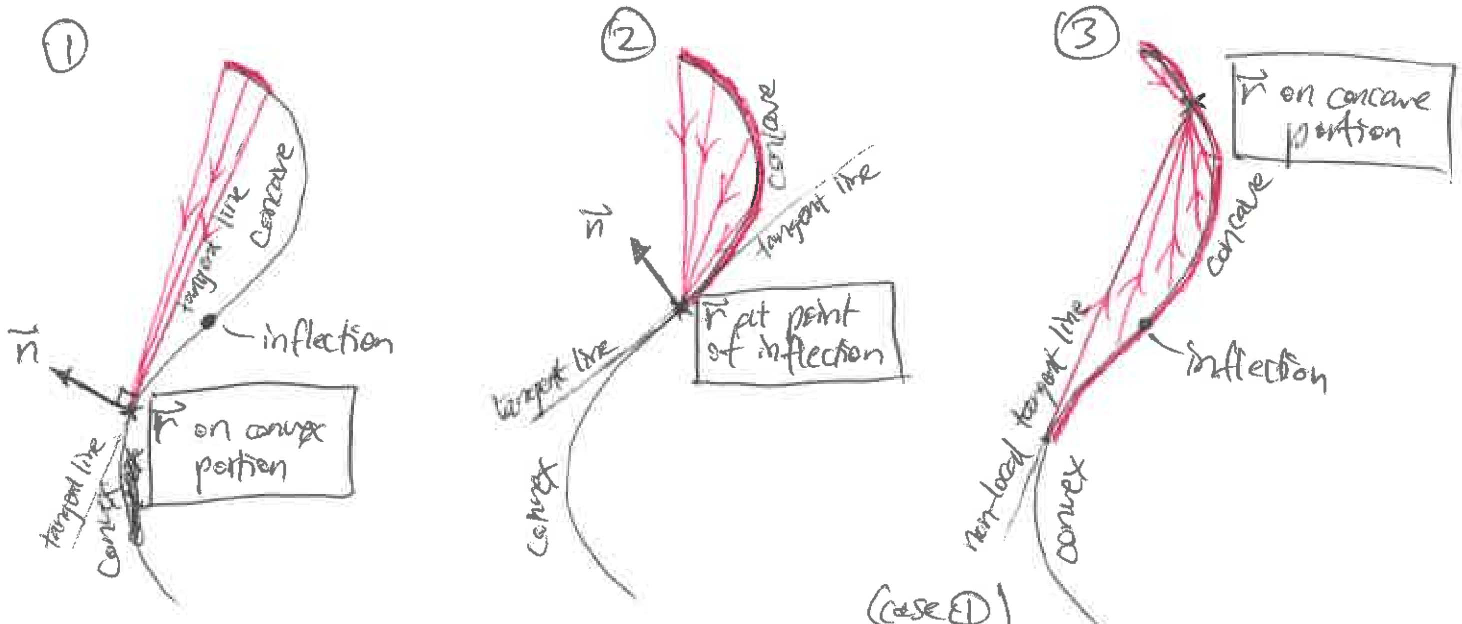
$$R \le \frac{y''^{2}_{max}\, T^4_{max}}{8\, T^4_{min}\sqrt{1+y'^{2}_{min}}} \, \frac{1}{(1+y'^{2}_{min})^{3/2}} \cdot (x_2 - x_1)^2$$

$$\boxed{R \le \frac{(x_2-x_1)^2 (T_{max}/T_{min})^4\, y''^{2}_{max}}{8\,(1+y'^{2}_{min})^2}}. \qquad (r6.26)$$

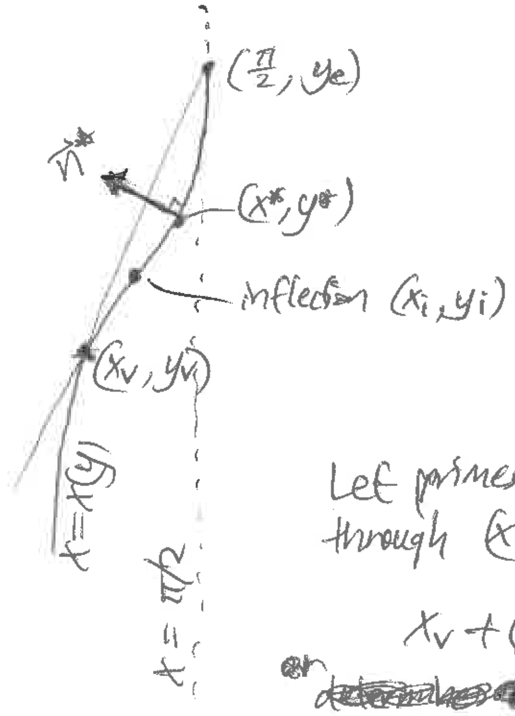
2. Non-convex lens

One supervisor saith that it would be good to evaluate R for the non-convex lens-like domains for the cosine simple ($B=1$) case, see Page r5-5, Sec. 3.3.2 in radiation-5-cosine.pdf.

But that situation is actually quite complicated; there is self-viewing even into the convex portions; this is best seen with an exaggerated sketch:



We see that if $\vec{r}$ is on the convex portion of the lens (case ①), the self-viewing ~~[illegible]~~ ratio R is probably less than for cases ② & ③, but between ② & ③ it is unclear which has greater R. In particular for case ③, the local point $\vec{r}$ receives radiation from both the entire concave portion, but also some of the convex portion too.



The simplest way forward is to do a bounding on $y_v \le y^* \le y_e$, where y_e is the end of the lens,

$$x(y)\big|_{y=y_e} = \pi/2 , \qquad \text{(r6.27)}$$

and (x_v, y_v) is the lowest part (in the top half of the lens) which is visible to the end.

Let primes denote y-differentiation. The tangent line through (x_v, y_v) meets the endpoint $(\pi/2, y_e)$, therefore

$$x_v + (y_e - y_v)\, x'_v = \pi/2 \qquad \text{(r6.28)}$$

or

$$x(y_v) + (y_e - y_v)\, x'(y_v) = \pi/2 \qquad \text{(r6.29)}$$

determines y_v. Now for $x = x(y)$, we have instead of (r6.9),

$$\vec{n} = \frac{-\vec{e}_x + x'\vec{e}_y}{\sqrt{x'^2+1}} \qquad \text{(r6.30)}$$

so we apply $\{y' \to 1,\ 1 \to x'\}$ unto (r6.15), and get

$$R = \frac{1}{T^4}\int_{y_1}^{y_2} \frac{T^{*4}\left[-(x-x^*) + (y-y^*)x'^*\right]\left[(x-x^*) - (y-y^*)x'\right] dy^*}{2\left[(x-x^*)^2 + (y-y^*)^2\right]^{3/2}\sqrt{x'^2+1}} . \qquad \text{(r6.31)}$$

and so

$$R \le \frac{1}{T^4}\int_{y_v}^{y_e} \frac{T^{*4}(y-y^*)^2\,|x''_c|\,(y-y^*)^2\,|x''_c|\, dy^*}{8\left[(y-y^*)^2\right]^{3/2}\left(|x'|^2_{min}+1\right)^{3/2}\sqrt{x'^2+1}}$$

$$\le \left(\frac{T_{max}}{T_{min}}\right)^4 \frac{|x''|^2_{max}}{8\left(|x'|^2_{min}+1\right)^2}\int_{y_v}^{y_e} |y-y^*|\, dy^*$$

$$\boxed{R \le \frac{(y_e-y_v)^2\,(T_{max}/T_{min})^4\,|x''|^2_{max}}{8\left(1+|x'|^2_{min}\right)^2}} , \qquad \text{(r6.32)}$$

very orderly.