Radiation 5: Cosine

radiation-5-cosine.pdf

Besides the one-dimensional solution to ~~the~~ Laplace's equation in Cartesian coordinates (radiation-1-plane.pdf) there ~~is~~ are also ~~the~~ harmonic functions which are a product of ~~exponential~~ trigonometric and hyperbolic functions. Since we would like to have $T = \text{const}$ along some $x = \text{const}$, to be taken later on for a Dirichlet boundary supplying the heat, consider the known solution

$$\boxed{T = T_0\left(1 - B\cos\frac{x}{L_0}\cosh\frac{y}{L_0}\right)} \tag{r5.1}$$

to (r7), for which $T = T_0$ along $x = \frac{\pi}{2}L_0$, where T_0 is a temperature, L_0 is a length scale, and B is some dimensionless constant.

1. Coordinates

$(u, v) = (x, y)$, $h_u = h_v = 1$ etc.

2. Scaling

Put

$$\hat{T} = T/\tau \tag{r5.2}$$

$$\hat{x} = x/\lambda \tag{r5.3}$$

$$\hat{y} = y/\lambda \tag{r5.4}$$

$$\hat{\vec{\nabla}} = \lambda\vec{\nabla} \tag{r5.5}$$

and drop hats. Then (r8) and (r1.1) become

$$\frac{\tau}{\lambda}\vec{n}\cdot\vec{\nabla}T = -c\tau^4 T^4$$

$$\tau T = T_0\left(1 - B\cos\frac{\lambda x}{L_0}\cosh\frac{\lambda y}{L_0}\right)$$

or

$$\vec{n}\cdot\vec{\nabla}T = -\left[c\lambda\tau^3\right]T^4$$

$$T = \left[\frac{T_0}{\tau}\right]\left(1 - B\cos\left(\left[\frac{\lambda}{L_0}\right]x\right)\cosh\left(\left[\frac{\lambda}{L_0}\right]y\right)\right).$$

Evidently it is sensible to choose the obvious scales

$$\boxed{\begin{aligned} \tau &= T_0 \\ \lambda &= L_0 \end{aligned}} \;. \qquad \begin{aligned} &(r5.6)\\ &(r5.7)\end{aligned}$$

As before, we put

$$\boxed{A = \frac{1}{c L_0 T_0^3}} \qquad (r5.8)$$

for a dimensionless group, leaving

$$\boxed{\begin{aligned} \vec{n}\cdot\vec{\nabla}T &= -\frac{T^4}{A} \\ T &= 1 - B\cos x \cosh y \end{aligned}} \;. \qquad \begin{aligned} &(r5.9)\\ &(r5.10)\end{aligned}$$

Note that in the context of complex function theory ~~[illegible]~~ (see (r4.14) and ~~so~~ surrounds), ~~so~~ the known solution (r5.10) arises from taking the real part of $1 - B\cos z$, and this is why I have chosen the name 'cosine' for this particular problem.

3. Boundary tracing

We have:

$$d\mu = dx \qquad (r5.11)$$

$$dv = dy \qquad (r5.12)$$

$$P = \frac{\partial T}{\partial x} = B \sin x \cosh y \qquad (r5.13)$$

$$Q = \frac{\partial T}{\partial y} = -B\cos x \sinh y \qquad (r5.14)$$

$$F = -\frac{(1 - B\cos x\cosh y)^4}{A} \qquad (r5.15)$$

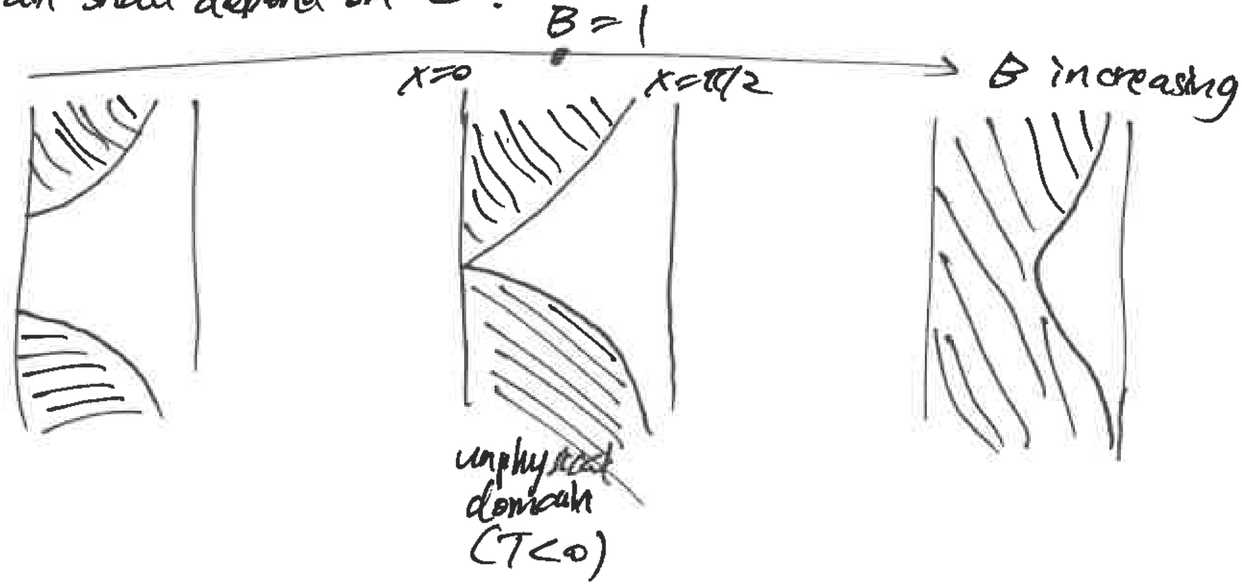
$$\begin{aligned} (\vec{\nabla}T)^2 = P^2 + Q^2 &= B^2(\sin^2 x \cosh^2 y + \cos^2 x \sinh^2 y) \\ &= B^2(\sin^2 x + \sinh^2 y) \end{aligned} \qquad (r5.16)$$

$$\begin{aligned} \Phi &= (\vec{\nabla}T)^2 - F^2 \\ &= B^2(\sin^2 x + \sinh^2 y) - \frac{(1 - B\cos x \cosh y)^8}{A^2} \end{aligned} \qquad (r5.17)$$

3.1 Physical domain

Since ultimately it is the straight boundary $x=\pi/2$ which shall supply heat via the Dirichlet condition $T=1$, we really only care about $x \le \pi/2$ (if $x>\pi/2$ then $T>1$). We also only care about $x \ge 0$, since ~~xxxx~~ T is even in x.

Now not all of the strip $0 \le x \le \pi/2$ is physical, since T vanishes along $\cos x \cosh y = 1/B$. Therefore the geometry of the physical domain shall depend on B:



The geometry of the viable domain is even more complicated, since there are two dimensionless parameters A & B which can be tweaked, and there is not much I can describe or do on paper at this point.

3.2 Tracing ODE, coordinate parametrisation

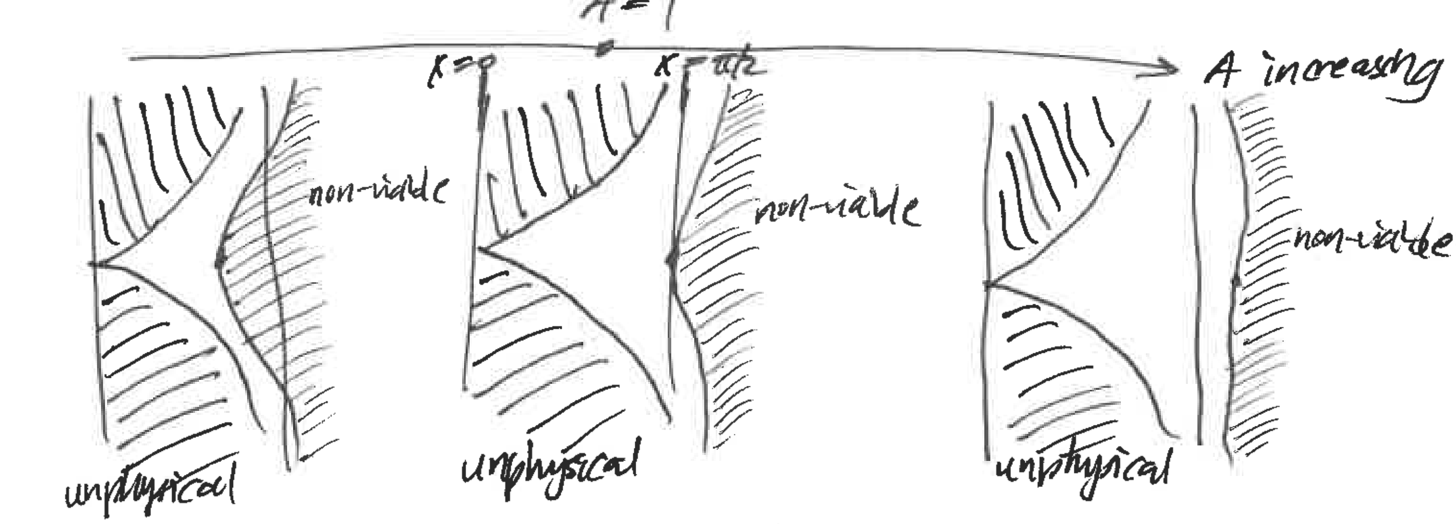
From ~~xxx~~ (47), we have ~~the xxxxxx~~

$$\boxed{\frac{dx}{dy} = \frac{PQ \mp F\sqrt{\Phi}}{Q^2 - F^2}}, \qquad (r5.18)$$

which is too horrible to write out in terms of x and y.

3.3 Simple case $B=1$

3.3.1 Viable domain & tracing



The terminal curve shifts toward the right with increasing A, passing through the straight line $x=\pi/2$ at $A=1$.
In all cases there is a single non-trivial critical terminal point $(x_0, 0)$, where x_0 ⊛ is the positive solution to

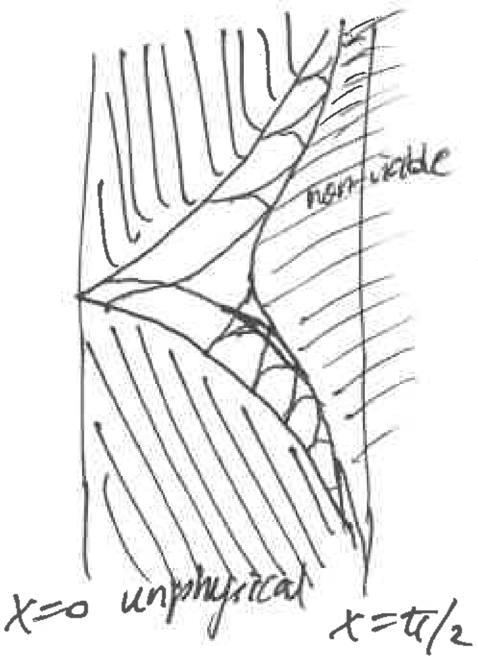
$$Z|_{y=0} = (1-\cos^2 x) - \frac{(1-\cos x)^8}{A^2} = 0, \qquad \text{(r519)}$$

a polynomial in $\cos x$. For all A the critical terminal point $(x_0, 0)$ is of hyperbolic type; the contour therethrough lies to the left, even on the viable side of the terminal curve.

Given how the contours meet the terminal curve away from $y=0$, that is, the viable side of each contour corresponds to the side where $|y|$ is greater, the lower branches will asymptote towards the terminal curve travelling in the direction of increasing y and T; they are repelled from the terminal curve into the edge $T=0$ of the physical domain for decreasing y; and vice-versa for the ~~lower~~ upper branch.

⊛ In Section 3.4.2, x_0 is called $x_\#$. The trivial point at $x=0$ corresponds to x_b.

The only way to avoid a concave spike is to switch boundaries at the hyperbolic critical terminal point $(x_0, 0)$, and we obtain a smooth traced boundary which is practically indistinguishable from the terminal curve; indeed evaluating the residual of the radiation condition (r5.9) along the terminal curve, the relative error is no more than 1% whenever $A \leq 1$.



And since we want $x_0 \leq \pi/2$, it is only $A \leq 1$ which we are interested in. However, should A be too small, the smooth traced boundary which is almost indistinguishable from the terminal curve, henceforth called the candidate boundary, shall inflect at some $x < \pi/2$, which would be bad.

3.3.2 Curvature

The good thing about Cartesian coordinates is that inflection points of a traced boundary $x = x(y)$ are simply given by sign changes in the second derivative x'' (primes denoting y differentiation). Now from (r5.18),

$$x'' = \frac{d}{dy}\left(\frac{PQ \mp F\sqrt{\Xi}}{Q^2 - F^2}\right) \qquad \text{(r5.20)}$$

a most awful expression. Now we seek the A for which inflection occurs along $x = \pi/2$, therefore we evaluate thereat to obtain the less awful

$$x''\Big|_{x=\frac{\pi}{2}} = \frac{A^2 \cosh y}{\sqrt{A^2\cosh^2 y - 1}}\left[2\sinh y - \cosh^2 y\left(A^2 \sinh y + 4\sqrt{A^2\cosh^2 y - 1}\right)\right]. \qquad \text{(r5.21)}$$

Only the bracketed $[\cdot]$ factor of (r5.21), ~~a polynomial~~ will change sign; so (r5.21) will only vanish if

$$\left[2S - (1+S^2)\left(A^2 S + 4\sqrt{A^2(1+S^2)-1}\right)\right] = 0\,, \qquad \text{(r5.22)}$$

where for brevity

$$S = \sinh y. \qquad \text{(r5.23)}$$

(r5.22) is effectively a polynomial in S, and computer algebra shows it only has one solution $S = S(A)$ on $0 < A \leq 1$, thus the critical value of y for a given A will be

$$y = y(A) = \sinh^{-1} S(A)\,. \qquad \text{(r5.24)}$$

~~Evaluating this along a traced boundary will give~~

$$\cancel{x = x(y) = x(y(A))},$$

Since we seek inflection along $x = \pi/2$, we seek $A = A_i$ which solves

$$x(y(A)) = \pi/2 \qquad \text{(r5.25)}$$

along the candidate boundaries $x = x(y)$. The bisection algorithm yields

$$A_i = 0.79718\,; \qquad \text{(r5.26)}$$

thus the candidate boundary only corresponds to a convex domain for

$$A_i \leq A < 1\,. \qquad \text{(r5.27)}$$

A simpler way to determine an approximate value of A_i is to observe that the candidate boundary is approximated well by the terminal curve $\Phi = 0$, which crosses $x = \pi/2$ at

$$\Phi\Big|_{x=\frac{\pi}{2}} = 1 + \sinh^2 y - \frac{1}{A^2}$$

$$y = \cosh^{-1}(1/A)$$

$$= \operatorname{sech}^{-1} A\,. \qquad \text{(r5.28)}$$

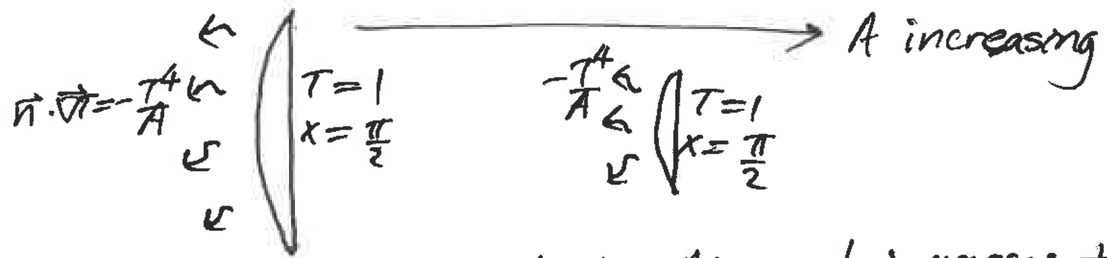
Thus the curvature of the contour $\Phi=0$ at $(x,y)=\left(\frac{\pi}{2}, \operatorname{sech}^{-1}A\right)$ is given by

$$\vec{\nabla}\cdot\frac{\vec{\nabla}\Phi}{\|\vec{\nabla}\Phi\|}\bigg|_{x=\frac{\pi}{2},\,y=\operatorname{sech}^{-1}A} = \frac{A(A^6-A^4+44A^2-28)}{(16+A^2-A^4)^{3/2}}, \quad (r5.29)$$

which crosses zero at

$$A_i(\text{approx}) = 0.79982\,. \quad (r5.30)$$

The convex domains for $A_i \le A < 1$ resemble thin lenses:



The aspect ratio is 25.6 at $A=A_i$, and increases to infinity as A increases to 1.

3.4 General case B arbitrary

3.4.1 Terminology

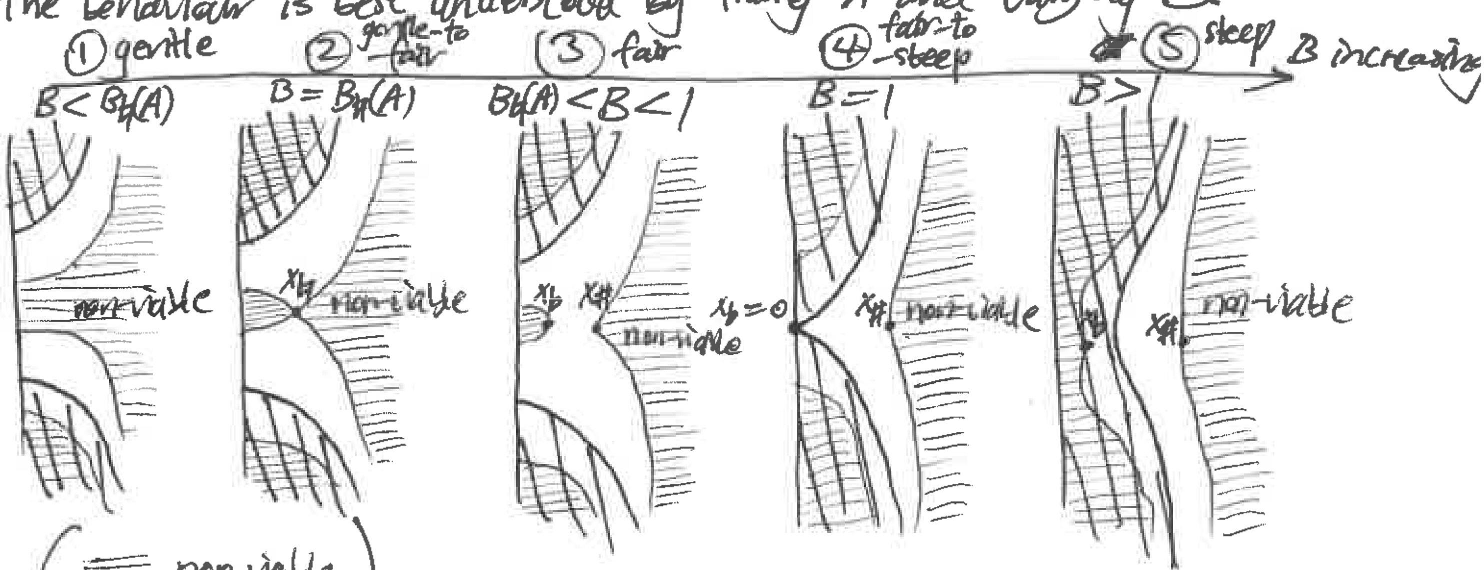
hot $\qquad A=\frac{1}{c_h T_0^3} \qquad$ cold $\longrightarrow$ A increasing

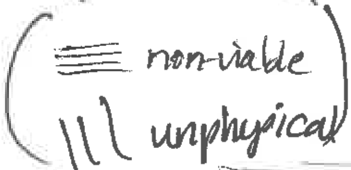
gentle $\qquad B=\frac{\partial T}{\partial x}\Big|_{x=\frac{\pi}{2},\,y=0} \qquad$ steep $\longrightarrow$ B increasing

Small A is called 'hot' because it corresponds to large T_0, and small B is called 'gentle' (as in the opposite of 'steep') because it corresponds to small $\partial T/\partial x|_{x=\pi/2,\,y=0}$.

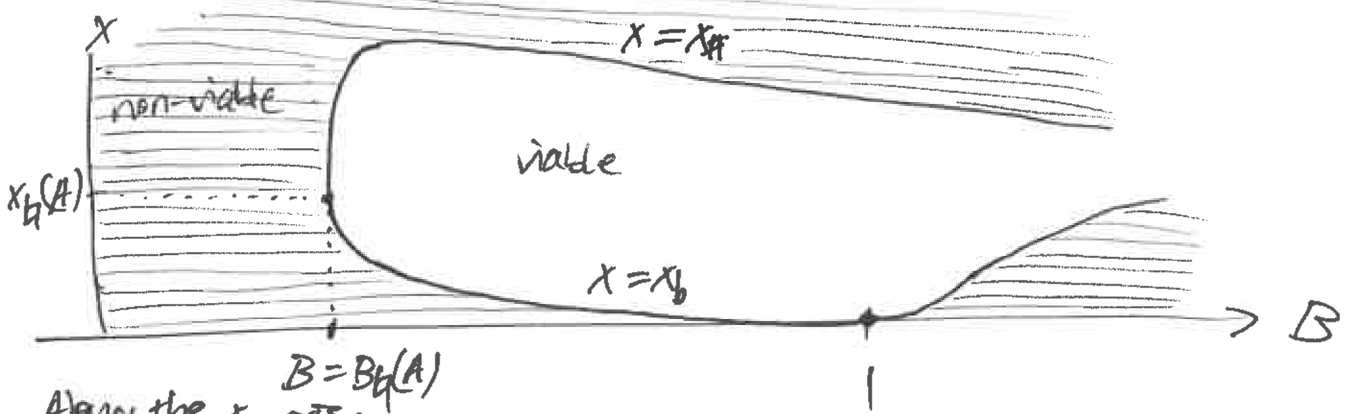
3.4.2 Viable domain & critical terminal points

The behaviour is best understood by fixing A and varying B.





Critical terminal points along x-axis, A fixed.



Along the x-axis:

① $\boxed{B < B_{\natural}(A)}$ (gentle) There are no critical terminal points.

② $\boxed{B = B_{\natural}(A)}$ (gentle-to-fair) There is one critical terminal point, $x = x_{\natural}(A)$, hyperbolic.

③ $\boxed{B_{\natural}(A) < B < 1}$ (fair) That in ② splits into two, $x = x_b$ & $x = x_\#$, both hyperbolic.

④ $\boxed{B = 1}$ (fair-to-~~steep~~ steep) $x = x_b$ becomes zero, and meets with edge $T = 0$ of the unphysical domain. (See 3.3 simple case).

⑤ $\boxed{B > 1}$ (steep) ~~[illegible]~~ While $x = x_b$ becomes positive again, it lies in the unphysical region $T < 0$.

Names:

① gentle
② gentle-to-fair
③ fair
④ fair-to-steep, or simple
⑤ steep

3.4.3 Tracing

Generally speaking the situation is fairly complicated when both A & B are allowed to vary, and I cannot give an exhaustive treatment here.

① gentle has no critical terminal points, and the entire x-axis is non-viable.

④ fair-to-steep, $B=1$, we have already considered; thin lenses result.

⑤ steep results in lenses which are even thinner than those for $B=1$, if at all. (The entire x-axis is non-convex.)

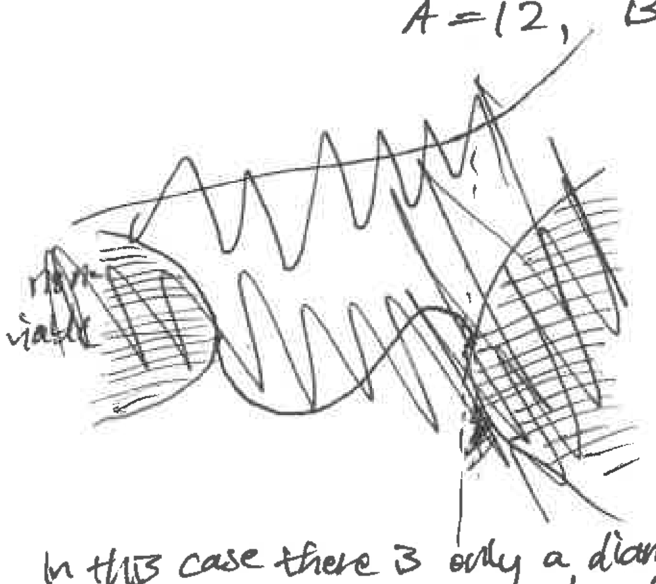
Thus we restrict attention to ② & ③, $B_H(A) \le B < 1$.

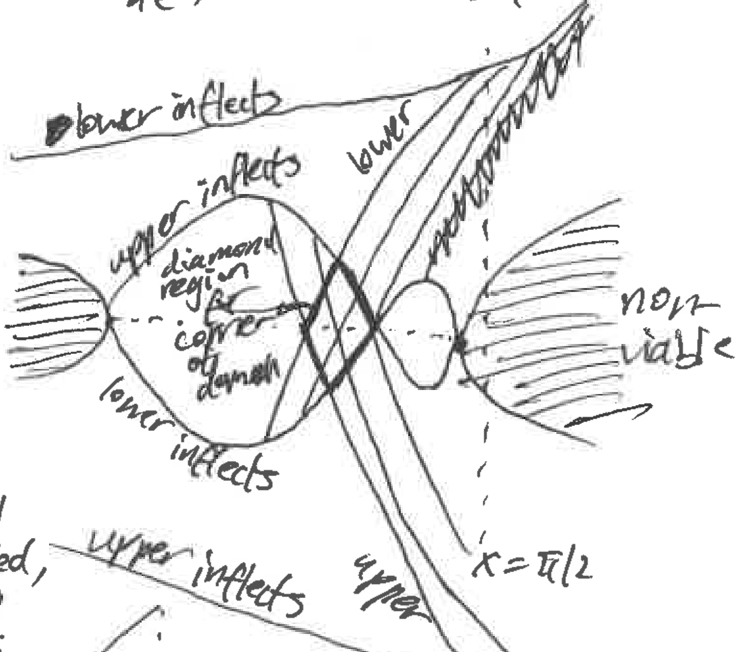
Broadly speaking, we will have domains with a Dirichlet boundary $T=1$ along $x=\pi/2$, and ~~the~~ radiation boundaries formed by traced boundaries with one ~~some portion~~ of them passing through the x-axis, more specifically through the viable segment $x_b \le x \le x_H$, $y=0$. Moreover, ~~or more~~ it must pass through ~~~~ the ~~subsegment~~ subset of this which ~~~~ corresponds to convexity, and also ~~~~ the traced boundaries must make it ~~~~ to $x=\pi/2$ without inflecting.

Therefore, only for very well-chosen pairs (A, B) can convex boundaries be formed. Note that the lens-like boundary usually makes it, but one requires $x_H < \pi/2$.

The situation ~~~~ is best illustrated by a graphical example: ~~~~

$$A=12, \quad B = 1.05\, B_H(A) = 0.0832989$$





In this case there is only a diamond region where a corner can be located, for constructing a domain thus: