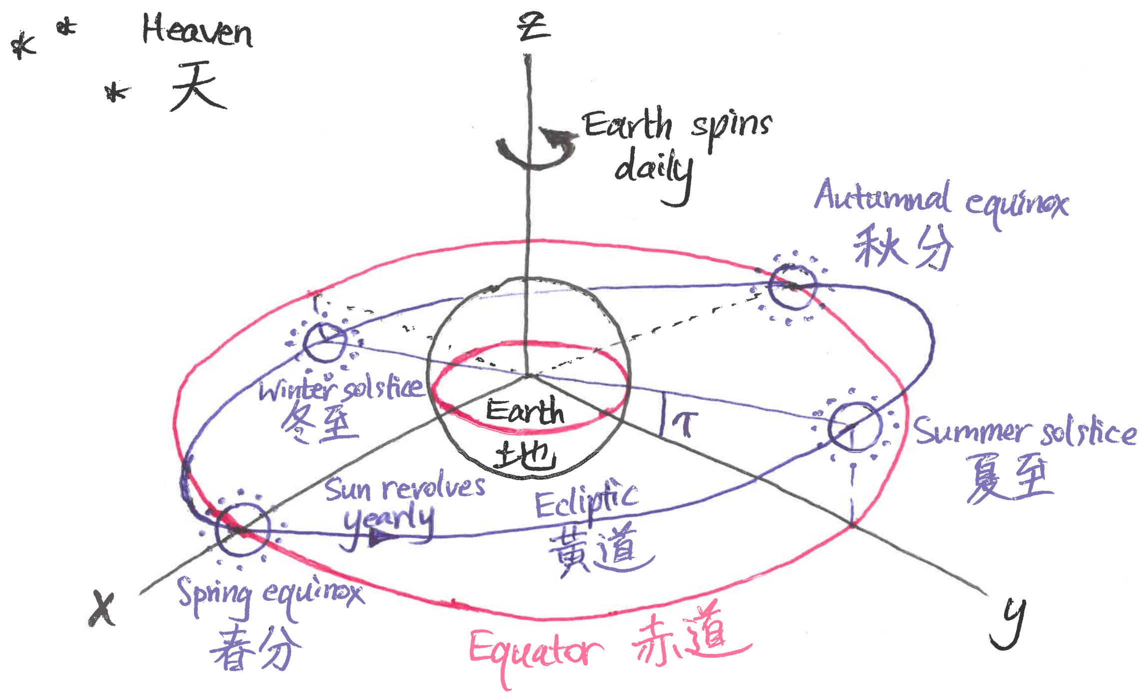
Conway

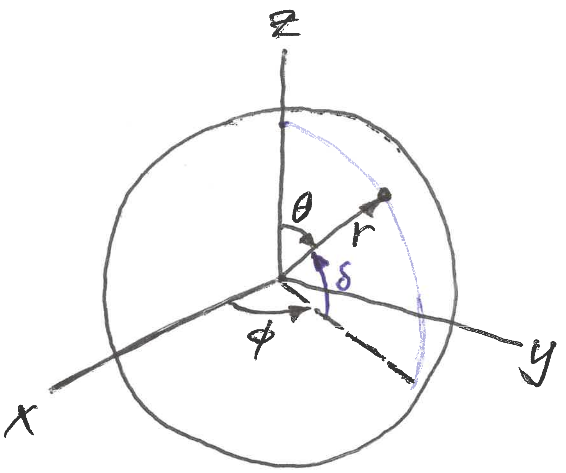
daytime.pdf

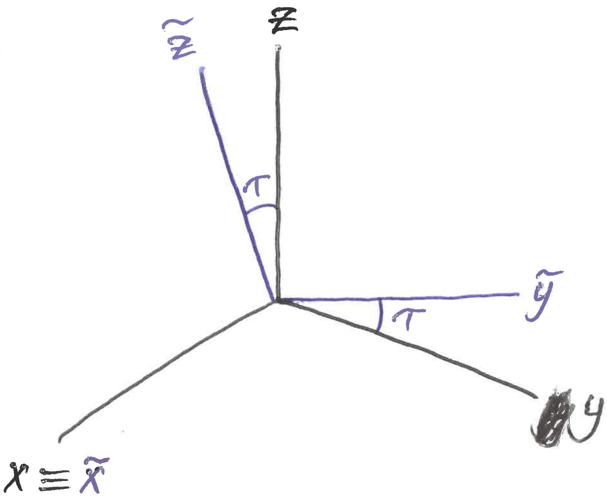
Manuscript for $\boxed{\text{Daytime: dependence on latitude and season}}$

Geometry

Equatorial coordinates: (x, y, z), (r, θ, ϕ)







Ecliptic coordinates: $(\tilde{x}, \tilde{y}, \tilde{z})$

Now

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix}_{sun} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\tau & -\sin\tau \\ 0 & \sin\tau & \cos\tau \end{pmatrix} \begin{pmatrix} \xi \\ \eta \\ \zeta \end{pmatrix}_{sun}$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\tau & -\sin\tau \\ 0 & \sin\tau & \cos\tau \end{pmatrix} R_{es} \begin{pmatrix} \cos\Upsilon \\ \sin\Upsilon \\ 0 \end{pmatrix}$$

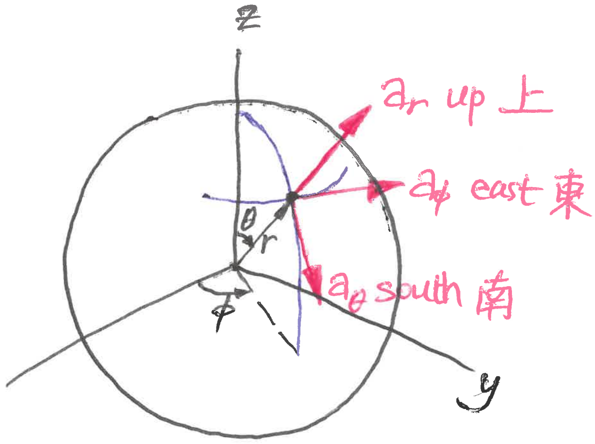
$$= R_{es} \begin{pmatrix} \cos\Upsilon \\ \cos\tau \sin\Upsilon \\ \sin\tau \sin\Upsilon \end{pmatrix}$$

Let $u = u_x a_x + u_y a_y + u_z a_z$ be the unit vector (direction) from the observer towards the sun. The observer lies at radius R_e from the origin; the sun at radius R_{es}. Since $R_e \ll R_{es}$, we simply have

$$u_x = \cos\Upsilon$$
$$u_y = \cos\tau \sin\Upsilon$$
$$u_z = \sin\tau \sin\Upsilon .$$

Transform to the local spherical basis, i.e. put

$$u = u_r a_r + \cancel{}\, u_\theta a_\theta + u_\phi a_\phi .$$



Then

$$\begin{pmatrix} u_r \\ u_\theta \\ u_\phi \end{pmatrix} = \begin{pmatrix} \sin\theta\cos\phi & \sin\theta\sin\phi & \cos\theta \\ \cos\theta\cos\phi & \cos\theta\sin\phi & -\sin\theta \\ -\sin\phi & \cos\phi & 0 \end{pmatrix} \begin{pmatrix} u_x \\ u_y \\ u_z \end{pmatrix}$$

$$= \begin{pmatrix} \cos\delta\cos D & \cos\delta\sin D & \sin\delta \\ \sin\delta\cos D & \sin\delta\sin D & -\cos\delta \\ -\sin D & \cos D & 0 \end{pmatrix} \begin{pmatrix} \cos\Upsilon \\ \cos\tau\sin\Upsilon \\ \sin\tau\sin\Upsilon \end{pmatrix}$$

since $\theta_{obs} = \frac{\pi}{2} - \delta$ and $\phi_{obs} = D$.

Note: argument to arcsine will exceed unity at some point if

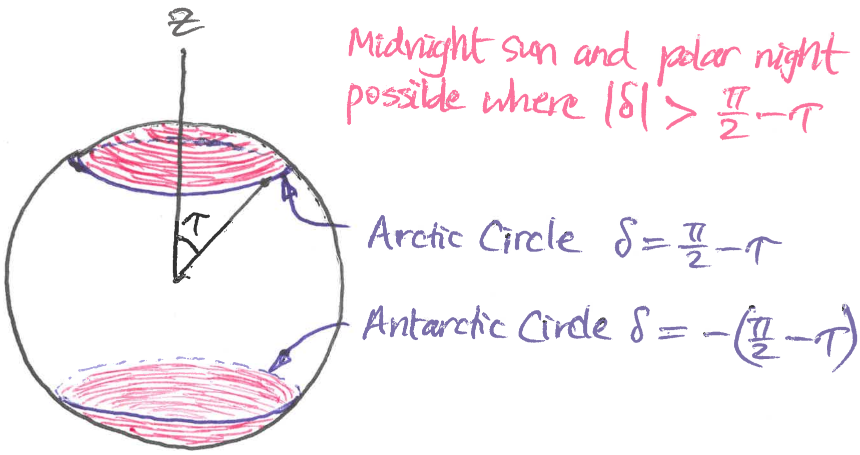
$$|\tan\tau \tan\delta| > 1$$

$$\tan|\delta| > \cot\tau$$

$$= \tan\left(\frac{\pi}{2} - \tau\right)$$

$$|\delta| > \frac{\pi}{2} - \tau \; .$$

In this case there is no real solution to the sunrise/sunset equation, corresponding to midnight sun and polar night, north of the Arctic Circle and south of the Antarctic Circle.



However, the real part of T still gives the correct daytime, since

$$\mathrm{Re}(\sin^{-1} w) = \begin{cases} +\pi/2, & w > 1 \\ -\pi/2, & w < -1 \end{cases}$$

so we get

$$\mathrm{Re}(T) = 24 \text{ hr} \quad \text{for midnight sun},$$
$$\mathrm{Re}(T) = 0 \text{ hr} \quad \text{for polar night}.$$

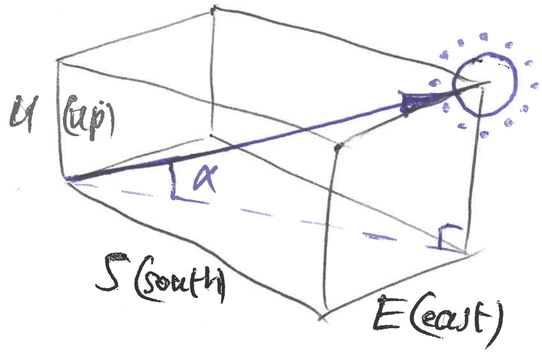
Assorted quantities

Let

$$U = \cos\delta\cos(D-\varphi) + \tan\tau\sin\delta\sin\varphi$$
$$S = \sin\delta\cos(D-\varphi) - \tan\tau\cos\delta\sin\varphi$$
$$E = -\sin(D-\varphi).$$

Sun's elevation angle (or altitude)

$$\alpha = \tan^{-1}\frac{U}{\sqrt{S^2+E^2}}$$



Sun's bearing (clockwise from north)

$$\beta = \tan^{-1}\frac{E}{-S} \quad \text{(up to quadrant identification)}$$

Shadow length of a vertical pole of ~~length~~ height h

$$\ell/h = \frac{\sqrt{S^2+E^2}}{U}$$

$$\ell = \frac{\sqrt{S^2+E^2}}{U}\cdot h.$$

Noon is at $D=\varphi$

Sunrise/sunset are at

$$D = D_{\pm} = \varphi \pm \left\{\frac{\pi}{2} + \sin^{-1}(\tan\tau\tan\delta\sin\varphi)\right\}.$$