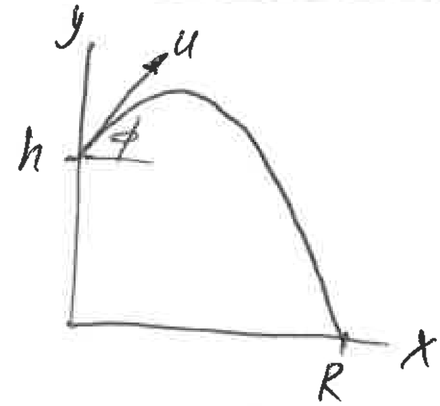
Conway platform.pdf

Manuscript for $\boxed{\text{Projectile motion: optimal launch angle from a platform}}$



Motion

$$x(t) = ut\cos\phi$$
$$y(t) = ut\sin\phi - \tfrac{1}{2}gt^2 + h$$

Flight time

$$y(t) = 0$$
$$t = \frac{1}{g}\left(u\sin\phi + \sqrt{u^2\sin^2\phi + 2gh}\right)$$
$$= \frac{u}{g}\left(\sin\phi + \sqrt{\sin^2\phi + C}\right)$$

where

$$C = \frac{2gh}{u^2} \quad \text{(initial potential-to-kinetic energy ratio)}$$

Range

$$R = x(t) = \frac{u^2\cos\phi}{g}\left(\sin\phi + \sqrt{\sin^2\phi + C}\right)$$

Optimal angle

$$\frac{\partial R}{\partial\phi} = \frac{u^2}{g}\Big[\cos\phi\cdot\cos\phi - \sin\phi\cdot\sin\phi + \cos\phi\cdot\frac{2\sin\phi\cos\phi}{2\sqrt{\sin^2\phi + C}} - \sin\phi\sqrt{\sin^2\phi + C}\Big]$$

$$= \frac{u^2}{g}\left[\cos^2\phi\left(1 + \frac{\sin\phi}{\sqrt{\sin^2\phi + C}}\right) - \sin\phi\left(\sin\phi + \sqrt{\sin^2\phi + C}\right)\right]$$

Let $\beta = \sin\phi$.

$$\frac{\partial R}{\partial \phi} = \frac{u^2}{g}\left[(1-\beta^2)\left(1+\frac{\beta^2}{\sqrt{\beta^2+C}}\right) - \beta\left(\beta+\sqrt{\beta^2+C}\right)\right]$$

$$= \frac{u^2}{g}\left(\beta+\sqrt{\beta^2+C}\right)\left[\frac{1-\beta^2}{\sqrt{\beta^2+C}} - \beta\right]$$

$$= \frac{2h}{C}\left(\beta+\sqrt{\beta^2+C}\right)\left[\frac{1-\beta^2}{\sqrt{\beta^2+C}} - \beta\right]$$

$$= 0$$

Case 1

If $C = \infty$, then

$$\frac{\partial R}{\partial \phi} = \frac{2h}{C}(\sqrt{C})[-\beta] = -2\beta \cdot \frac{h}{\sqrt{C}}$$

This vanishes if h is finite. But $C = 2gh/u^2$ is infinite, so either $g=\infty$ or $u=0$. Both have minimal $R=0$.

Case 2

If $C=0$ and $\beta<0$, then

$$\frac{\partial R}{\partial \phi} = \frac{2h}{C}\left(\beta+(-\beta)\left(1+\frac{C}{2\beta^2}\right)\right)\left[\frac{1-\beta^2}{-\beta}+\frac{\beta^2}{-\beta}\right]$$

$$= \frac{2h}{C}\left(-\frac{C}{2\beta}\right)\left[-\frac{1}{\beta}\right]$$

$$= \frac{h}{\beta^2}$$

This vanishes only if $h=0$. Now $\phi = \sin^{-1}\beta < 0$, so this is launching downwards from ground level. Again $R=0$.

Case 3

If $\left[\frac{1-\beta^2}{\sqrt{\beta^2+C}}-\beta\right]=0$, then $\frac{\partial R}{\partial \phi}=0$ unconditionally.

Thus

$$(1-\beta^2)^2=\beta^2(\beta^2+C)$$

$$1-2\beta^2+\cancel{\beta^4}=\cancel{\beta^4}+C\beta^2$$

$$1=(C+2)\beta^2$$

$$\beta=\frac{1}{\sqrt{C+2}}$$

$$\begin{aligned}
R &= \frac{u^2}{g}\sqrt{1-\beta^2}\,\left(\beta+\sqrt{\beta^2+C}\right)\\
&= \frac{u^2}{g}\sqrt{1-\frac{1}{C+2}}\left(\frac{\cancel{\beta^2}}{\beta}+\frac{1-\cancel{\beta^2}}{\beta}\right)\\
&= \frac{u^2}{g}\sqrt{\frac{C+1}{\cancel{C+2}}}\cdot\cancel{\frac{1}{\beta}}\\
&= \frac{u^2}{g}\sqrt{C+1}.
\end{aligned}$$

Hence optimal angle is

$$\phi=\sin^{-1}\frac{1}{\sqrt{2gh/u^2+2}}$$

achieving maximum range

$$R=\frac{u^2}{g}\sqrt{2gh/u^2+1}.$$

In particular;

- For $h=0$, $\phi = \sin^{-1}(1/\sqrt{2}) = 45°$ and $R = u^2/g$ as expected.
  For ~~small~~ h small compared to u^2/g,

$$\phi = \sin^{-1}\frac{1}{\sqrt{2gh/u^2+2}}$$

$$= \sin^{-1}\left(\frac{1}{\sqrt{2}}\frac{1}{\sqrt{1+gh/u^2}}\right)$$

$$\sim \sin^{-1}\left(\frac{1}{\sqrt{2}} - \frac{1}{2\sqrt{2}}\frac{gh}{u^2}\right)$$

$$\sim \sin^{-1}\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{1-(1/\sqrt{2})^2}} \cdot \frac{1}{2\sqrt{2}}\frac{gh}{u^2}$$

$$= 45° - \frac{1}{\sqrt{1/2}} \cdot \frac{1}{2\sqrt{2}}\frac{gh}{u^2}$$

$$= 45° - \frac{gh}{2u^2} \cdot \frac{180°}{\pi}$$

$$R = \frac{u^2}{g}\sqrt{2gh/u^2+1}$$

$$\sim \frac{u^2}{g}\left(1+\frac{gh}{u^2}\right)$$

$$= \frac{u^2}{g} + h .$$

- For $h=\infty$, $\phi = \sin^{-1}(1/\sqrt{\infty}) = 0$ and $R = u^2/g \cdot \sqrt{\infty} = \infty$.
  Asymptotically, i.e. for ~~h~~ h large compared to u^2/g,

$$\phi \sim \frac{1}{\sqrt{2gh/u^2}} = \frac{u}{\sqrt{2gh}} \cdot \frac{180°}{\pi}$$

$$R \sim \frac{u^2}{g}\sqrt{2gh/u^2} = u\sqrt{\frac{2h}{g}} .$$

Note optimal ϕ depends only on the dimensionless group

$$C = \frac{2gh}{u^2} .$$