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今有積二十三萬四千五百六十七步。問為方幾何?答曰:四百八十四步九百六十八分步之三百一十一。

Now we have area, two hundred and thirty-four thousand, five hundred and sixty-seven 步.
We ask: be this a square of edge how much?
Answer saith: four hundred and eighty-four 步 three hundred and eleven nine-hundred and sixty-eighths of a 步.

$$\sqrt{234567\,步^2} \simeq 484\frac{311}{968}\,步$$

Here we have an algorithm for computing the square root.

術曰:置積二十三萬四千五百六十七步為實。

Method saith: put down the area, two hundred and thirty-four thousand, five hundred and sixty-seven 步, as the dividend.

Dividend, 實; $d := 234567$

d	實	2 3 4 5 6 7

Next, borrow one rod, to be the lower divisor. Step it, jumping over one place, unto the hundreds, and stop. (even)

次借一算，為下法。步之，超一位，至百而止。

超：跳起，jumping over; or skipping
i.e. take double steps, so 100 is in fact 10000.

Number of digits of radicand:
$N := \lfloor \log_{10} d \rfloor + 1 = 6$

Number of digits of integer part of square root,
$n := \lceil \frac{N}{2} \rceil = 3$

Lower divisor, 下法: (take $n-1 = 2$ double steps)
$L := (10^{n-1})^2 = 10^4$

d	實	2	3	4	5	6	7
L	下法	1					

For the upper quotient, put down four hundred above the dividend.

上商置四百於實之上。

商：quotient (this is the integer part of the square root to be computed)

Largest a such that $a^2 L \leq d$,
or $10000\, a^2 \leq 234567$,
is $a := 4$; this is the hundreds digit of 上商.

		a					
	上商	4					
d	實	2	3	4	5	6	7
L	下法	1					

Versions A & C ~~are missing~~ have just 商 for 上商.

Subsidiarily, put down forty thousand below the dividend, and above the lower divisor: its name be the upright divisor.

副置四萬於實之下，下法之上，名為方法。

方 : (morally) upright
方法 is one of three named divisors,
- 方法 : upright divisor
- 廉法 : incorrupt divisor
- 隅法 : honest divisor　　隅，廉也

which I refer to henceforth as the straight divisors, ~~whose~~ straight as in 'straight copper', as opposed to 'bent copper'.

Upright divisor, 方法 (1st straight divisor):

$$p := aL = 4 \times 10000 = 40000$$

		a
	上商	4
d	實	2 3 4 5 6 7
p	方法	4
L	下法	1

Command the upper quotient's four hundred, to divide the dividend.

命上商四百除實。

除 : divide; or remove from

$$d := d - ap = 234567 - 4 \times 40000$$
$$= 74567$$

(Actually the text says $d := d - 400^2$, but the form above ~~generalise to~~ is what extends/generalises properly.)

	上商	4
d	實	7 4 5 6 7
p	方法	4
L	下法	1

The division finished, double the upright divisor.

除訖，倍方法，

$$p := 2p = 2 \times 40000 = 80000$$

		a				
	上商	4				
d	實	7	4	5	6	7
p	方法	8				
L	下法	1				

The upright divisor retreateth once; the lower divisor retreateth ~~twice~~ again.

再：~~twice; finally~~ again; or twice

方法一退，下法再退。

Versions A & C missing 方法 from 方法一退

$$p := p/10 = 80000/10 = 8000$$
$$L := L/100 = 10000/100 = 100$$

		a				
	上商	4				
d	實	7	4	5	6	7
p	方法		8			
L	下法			1		

Resume, putting for the upper quotient, eighty, to be next ~~from~~ to the former quotient digit.

復置上商八十，以次前商。

Largest b such that
$$b(p + bL) \le d$$
or $b(8000 + 100b) \le 74567$,
is $b := 8$; this is the ten's digit of 上商.

				a	b	
	上商			4	8	
d	實	7	4	5	6	7
p	方法		8			
L	下法			1		

副置八百於方法之下，下法之上，名為廉法。

Subsidiarily, put down eight hundred below the upright divisor, and above the lower divisor; its name be the incorrupt divisor.

Incorrupt divisor, 廉法 (2nd straight divisor):

$$q := bL = 8 \times 100 = 800$$

		a	b
	上商	4	8
d	實	74	567
p	方法	8	
q	廉法		8
L	下法		1

方、廉各命上商八十，以除實。

The upright and the incorrupt each command the upper quotient's eighty, to divide the dividend.

$$d := d - bp - bq = 74567 - 8 \times 8000 - 8 \times 800 = 4167$$

		a	b
	上商	4	8
d	實	4	167
p	方法	8	
q	廉法		8
L	下法		1

A, C missing 實.

除訖，倍廉法，上從方法。

The division finished, double the ~~upright~~ incorrupt divisor, which followeth the upright divisor alone.

$$q := 2q = 2 \times 800 = 1600$$

		a	b
	上商	4	8
d	實	4	167
p	方法	8	
q	廉法	16	
L	下法		1

A, C missing 除.

The upright divisor retreateth once;
the lower divisor retreateth again.

(From the previous sentence, the incorrupt divisor 廉法 is
to follow the upright divisor 方法 in its retreat)

A em has 一退方法 for 方法一退.

方法一退，下法再退。

$$p := p/10 = 8000/10 = 800$$
$$q := q/10 = 1600/10 = 160$$
$$L := L/100 = 100/100 = 1$$

			a	b	
	上商		4	8	
d	實	4	1	6	7
p	方法		8		
q	廉法		1	6	
L	下法				1

Resume, putting for the upper quotient four, to be next after the former.

復置上商四，以次前。

Largest c such that
$$c(p+q+cL) \le d$$
or $c(800+160+c) \le 4167$,
is $c := 4$, this is the ones digit of 上商.

			a	b	c
	上商		4	8	4
d	實	4	1	6	7
p	方法		8		
q	廉法		1	6	
L	下法				1

Subsidiarily, put down four below the upright divisor, and above the lower divisor; its name be called the honest divisor.

副置四於方法之下，下法之上，名曰隅法。

Honest divisor, 隅法 (3rd straight divisor):
$$r := cL = 4 \times 1 = 4$$

			a	b	c
	上商		4	8	4
d	實	4	1	6	7
p	方法		8		
q	廉法		1	6	
r	隅法				4
L	下法				1

方、廉、隅各命上商四，以除實。

The upright, the incorrupt and the honest each command the upper quotient's four, to divide the dividend.

$$d := d - cp - cq - cr$$
$$= 4167 - 4\times 800 - 4\times 160 - 4\times 4$$
$$= 311$$

		a	b	c
	上商	4	8	4
d	實	3	1	1
p	方法	8		
q	廉法	1	6	
r	隅法			4
L	下法			1

A missing 以。

除訖，倍隅法，從方法。

The division finished, double the honest divisor, which followeth the upright divisor.

$$r := 2r = 2\times 4 = 8$$

		a	b	c
	上商	4	8	4
d	實	3	1	1
p	方法	8		
q	廉法	1	6	
r	隅法			8
L	下法			1

A missing 倍隅法從方法

上商得四百八十四，下法得九百六十八，不盡三百一十一。

The upper quotient resulteth in four hundred and eighty-four, and the lower divisors result in nine hundred and sixty-eight, with remainder three hundred and eleven.

下法: lower divisor. Confusing, but here it means the total of the three straight divisors (upright, incorrupt and honest).

$$u := 100a + 10b + c = 484$$
$$L := p + q + r = 800 + 160 + 8 = 968$$
$$R := d = 311.$$

This be a square of edge four hundred and eighty-four 步 three hundred and eleven nine hundred and sixty-eighths of a 步.

是為方四百八十四步九百六十八分步之三百一十一。

$$\sqrt{234567\ 步^2} = \left(u + \frac{R}{L}\right) 步 = 484\frac{311}{968}\ 步.$$

In general, to compute the square root of an arbitrary positive integer x, the algorithm is as follows (I am using names a, b, c ~~rather than~~ rather than subscripts $a_1, a_2, a_3, \ldots$ for readability ~~to relate~~ and for ease of relating to the worked example above).

- Number of digits of radicand x
  $N := \lfloor \log_{10} x \rfloor + 1$
- Number of digits of integer part of $\sqrt{x}$
  $n := \lceil N/2 \rceil$
- Lower divisor (下法)
  $L := (10^{n-1})^2$
- Dividend (實)
  $d := x$
- Upper quotient (上商) digits
  abc-list := []
- Straight divisors, i.e. upright, incorrupt, honest divisors (方廉隅法)
  pqr-list := []

While True

- Determine largest integer α such that $\alpha[(p+q+\ldots)+\alpha L] \leq d$
- Append α to abc-list.
- Newest straight divisor:
  $p := \alpha L$
  Append p to pqr-list.
- ~~Update dividend~~ Divide the dividend (除法)
  $d := d - \alpha(p+q+r+\ldots)$
- Double the newest straight divisor
  $p := 2p$, where p is the last element of pqr-list.

- If not done, retreat; otherwise break loop.

```
If L > 1
    pqr-list := pqr-list / 10
           L := L / 100
Else
    Break
EndIf
EndWhile
```

- upper quotient

$U := 10^{n-1} \cdot a + 10^{n-2} \cdot b + 10^{n-3} \cdot c + \ldots$

$L := p + q + r + \ldots$

$R := d$

Return $U + R/L$.

For an implementation in Python, see /code/shuen-sqrt.py

Why does this work? Stare at the following identity(ies) for a long time:

$$(10a+b)^2 = 10^2 \cdot a^2 + 1 \cdot b(20a+b)$$

$$(100a+10b+c)^2 = 10^4 \cdot a^2 + 10^2 \cdot b(20a+b) + 1 \cdot c(200a+20b+c).$$

$$(1000a+100b+10c+d)^2 = 10^6 \cdot a^2 + 10^4 \cdot b(20a+b) + 10^2 \cdot c(200a+20b+c) + 1 \cdot d(2000a+200b+20c+d)$$

etc.

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