Conway resistance.pdf

Manuscript for $\boxed{\text{Projectile motion: optimal launch angle for weak quadratic drag}}$

Equations of motion

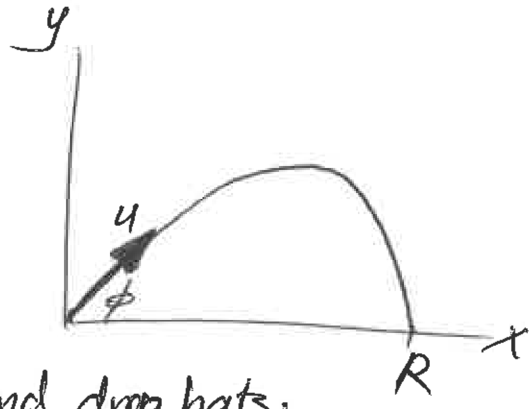
$$m\ddot{x} = \qquad -b\dot{x}\sqrt{\dot{x}^2+\dot{y}^2}, \quad \dot{x}(0) = u\cos\phi, \quad x(0)=0$$
$$m\ddot{y} = -mg - b\dot{y}\sqrt{\dot{x}^2+\dot{y}^2}, \quad \dot{y}(0) = u\sin\phi, \quad y(0)=0$$

Scaling

Use dragless $(b=0)$ scales:

Length $L = u^2/g$

Time $T = u/g$



Put $\hat{x} = \frac{x}{L}$, $\hat{y} = \frac{y}{L}$, $\hat{t} = \frac{t}{T}$ and drop hats:

$$\frac{mL}{T^2}\ddot{x} = \qquad -b\frac{L^2}{T^2}\dot{x}\sqrt{\dot{x}^2+\dot{y}^2}, \quad \frac{L}{T}\dot{x}(0) = u\cos\phi, \quad x(0)=0$$
$$\frac{mL}{T^2}\ddot{y} = -mg - b\frac{L^2}{T^2}\dot{y}\sqrt{\dot{x}^2+\dot{y}^2}, \quad \frac{L}{T}\dot{y}(0) = u\sin\phi, \quad y(0)=0$$

Now

$$\frac{mg}{mL/T^2} = \frac{gT^2}{L} = \frac{g\,u^2/g^2}{u^2/g} = 1,$$

$$\frac{bL^2/T^2}{mL/T^2} = \frac{bL}{m} = \frac{bu^2}{mg},$$

$$\frac{u}{L/T} = \frac{u}{(u^2/g)/(u/g)} = 1$$

Hence

$$\ddot{x} = -B\dot{x}\sqrt{\dot{x}^2+\dot{y}^2}, \quad \dot{x}(0) = \cos\phi, \quad x(0) = 0$$
$$\ddot{y} = -1 - B\dot{y}\sqrt{\dot{x}^2+\dot{y}^2}, \quad \dot{y}(0) = \sin\phi, \quad y(0) = 0$$

where

$$B = \frac{bu^2}{mg} \quad \text{(initial drag-to-weight ratio)}.$$

Let c be terminal speed. Then

$$1 = \frac{bc^2}{mg}.$$

Divide to get

$$B = \frac{u^2}{c^2}.$$

Since optimal ϕ is dimensionless, it depends only on B, and only on

$$\sqrt{B} = \frac{u}{c} \quad \text{(initial-to-terminal speed ratio)}.$$

Perturbed trajectory

Weak drag means $B \ll 1$, i.e. $u^2 \ll c^2$. Put

$$\dot{x} = \dot{x}_0 + B\dot{x}_1 + B^2\dot{x}_2$$
$$\dot{y} = \dot{y}_0 + B\dot{y}_1 + B^2\dot{y}_2$$

~~To linear order~~

Let

$$\boxed{\begin{aligned} v_0 &= \sqrt{\dot{x}_0^2+\dot{y}_0^2} \\ w_1 &= \dot{x}_0\dot{x}_1 + \dot{y}_0\dot{y}_1 \end{aligned}}.$$

To linear order,

$$\sqrt{\dot{x}^2+\dot{y}^2} = \sqrt{(\dot{x}_0+B\dot{x}_1)^2+(\dot{y}_0+B\dot{y}_1)^2}$$
$$= \sqrt{(\dot{x}_0^2+\dot{y}_0^2)+B(2\dot{x}_0\dot{x}_1+2\dot{y}_0\dot{y}_1)}$$
$$= \sqrt{v_0^2+B\cdot 2w_1}$$
$$= v_0\sqrt{1+B\cdot\frac{2w_1}{v_0^2}}$$
$$= v_0\left(1+B\cdot\frac{w_1}{v_0^2}\right)$$
$$= \boxed{v_0+B\cdot\frac{w_1}{v_0}}$$

Thus, to quadratic order, the equations of motion are

$$\ddot{x}_0+B\ddot{x}_1+B^2\ddot{x}_2 = -B(\dot{x}_0+B\dot{x}_1)\left[v_0+B\cdot\frac{w_1}{v_0}\right]$$
$$= -B[\dot{x}_0v_0]-B^2\left[\dot{x}_1v_0+\frac{\dot{x}_0w_1}{v_0}\right]$$
$$\ddot{y}_0+B\ddot{y}_1+B^2\ddot{y}_2 = -1-B(\dot{y}_0+B\dot{y}_1)\left[v_0+B\cdot\frac{w_1}{v_0}\right]$$
$$= -1-B[\dot{y}_0v_0]-B^2\left[\dot{y}_1v_0+\frac{\dot{y}_0w_1}{v_0}\right]$$

Hence

$$\ddot{x}_0 = 0,\quad \dot{x}_0(0)=\cos\phi,\quad x_0(0)=0$$
$$\ddot{y}_0 = -1,\quad \dot{y}_0(0)=\sin\phi,\quad y_0(0)=0\quad,$$

$$\ddot{x}_1 = -\dot{x}_0v_0,\quad \dot{x}_1(0)=0,\quad x_1(0)=0$$
$$\ddot{y}_1 = -\dot{y}_0v_0,\quad \dot{y}_1(0)=0,\quad y_1(0)=0\quad,$$

$$\ddot{x}_2 = -\left[\dot{x}_1v_0+\frac{\dot{x}_0w_1}{v_0}\right],\quad \dot{x}_2(0)=0,\quad x_2(0)=0,$$
$$\ddot{y}_2 = -\left[\dot{y}_1v_0+\frac{\dot{y}_0w_1}{v_0}\right],\quad \dot{y}_2(0)=0,\quad y_2(0)=0$$

etc.

For brevity let

$$\boxed{\begin{aligned}\alpha &= \cos\phi \\ \beta &= \sin\phi\end{aligned}}$$

$\boxed{\text{Order 1}}$

We have

$$\ddot{x}_0 = 0, \quad \dot{x}_0(0) = \alpha, \quad x_0(0) = 0$$

Thus

$$\boxed{\begin{aligned}\ddot{x}_0 &= 0 \\ \dot{x}_0 &= \alpha \\ x_0 &= \alpha t\end{aligned}}$$

~~For brevity let~~

Also

$$\ddot{y}_0 = -1, \quad \dot{y}_0(0) = \beta, \quad y_0(0) = 0$$

so

$$\boxed{\ddot{y}_0 = -1}$$

$$\dot{y}_0 = \beta - t$$

$$y_0 = \beta t - \frac{t^2}{2} .$$

Later it shall be convenient to work in terms of

$$\boxed{\tau = t - \beta}$$

so that

$$\boxed{\dot{y}_0 = -\tau}$$

In general, when the initial condition to an ODE is homogeneous, i.e.

$$\dot{F} = f, \quad F(0) = 0,$$

we have

$$F = \int_0^t f \, dt.$$

Moving to $\tau = t - \beta$, this becomes

$$\begin{aligned} F &= \int_{t=0}^{t=t} f \, d(t-\beta) \\ &= \int_{t-\beta=-\beta}^{t-\beta=t-\beta} f \, d(t-\beta) \\ &= \int_{\tau=-\beta}^{\tau=\tau} f \, d\tau \\ &= \int_{-\beta}^{\tau} f \, d\tau. \end{aligned}$$

Thus

$$\begin{aligned} y_0 &= \int_{-\beta}^{\tau} \dot{y}_0 \, d\tau \\ &= \int_{-\beta}^{\tau} -\tau \, d\tau \end{aligned}$$

$$\boxed{y_0 = -\frac{\tau^2}{2} + \frac{\beta^2}{2}}.$$

$\boxed{\text{Order } \beta}$

Note that

$$\begin{aligned} v_0 &= \sqrt{\dot{x}_0^2 + \dot{y}_0^2} \\ &= \sqrt{\alpha^2 + \tau^2}, \end{aligned}$$

$$\tau(0) = \tau|_{t=0} = -\beta,$$

$$v_0(0) = \sqrt{\alpha^2 + \beta^2} = 1.$$

i.e. $v_0|_{\tau=-\beta} = 1$.

Henceforth <u>all initial conditions are homogeneous</u>.

We have

$$\ddot{x}_1 = -\dot{x}_0 v_0$$

$$\boxed{\ddot{x}_1 = -\alpha v_0}$$

① $$\int v_0 \, d\tau = \int \sqrt{\alpha^2+\tau^2}\, d\tau$$

$$= \frac{\tau}{2}\sqrt{\alpha^2+\tau^2} + \frac{\alpha^2}{2}\log\left(\tau+\sqrt{\alpha^2+\tau^2}\right) + \text{const}$$

$$= \frac{\tau v_0}{2} + \frac{\alpha^2}{2}\log(\tau+v_0) + \text{const}$$

② $$\int_{-\beta}^{\tau} v_0 \, d\tau = \frac{\tau v_0}{2} + \frac{\beta}{2} + \frac{\alpha^2}{2}\log\frac{\tau+v_0}{-\beta+1} \qquad \text{by ①}$$

$$= \frac{\beta}{2} + \frac{\tau v_0}{2} + \frac{\alpha^2}{2}\log r$$

where

$$\boxed{r = \frac{\tau+v_0}{-\beta+1}}.$$

Note that

$$r(0) = r|_{t=0} = r|_{\tau=-\beta} = 1.$$

Thus

$$\dot{x}_1 = -\alpha\int_{-\beta}^{\tau} v_0 \, d\tau$$

$$\boxed{\dot{x}_1 = -\frac{\alpha\beta}{2} - \frac{\alpha}{2}\tau v_0 - \frac{\alpha^3}{2}\log r} \qquad \text{by ②}.$$

③ $\int \tau v_0 \, d\tau = \int \tau \sqrt{\alpha^2+\tau^2} \, d\tau$

$$= \frac{1}{2} \int \sqrt{\alpha^2+\tau^2} \, d(\alpha^2+\tau^2)$$

$$= \frac{1}{3} (\alpha^2+\tau^2)^{3/2} + \text{const}$$

$$= \frac{v_0^3}{3} + \text{const}$$

④ $\int_{-\beta}^{\tau} \tau v_0 \, d\tau = \frac{v_0^3}{3} - \frac{1}{3}$ by ③

⑤ $\frac{dv_0}{d\tau} = \frac{d}{d\tau} \sqrt{\alpha^2+\tau^2} = \frac{\tau}{\sqrt{\alpha^2+\tau^2}} = \frac{\tau}{v_0}$

⑥ $\frac{dr}{d\tau} = \frac{1 + dv_0/d\tau}{-\beta+1}$

$$= \frac{1+\tau/v_0}{-\beta+1}$$ by ⑤

$$= \frac{1}{v_0} \cdot \frac{v_0+\tau}{-\beta+1}$$

$$= \frac{r}{v_0}$$

⑦ $d \log r = \frac{dr}{r} = \frac{d\tau}{v_0}$ by ⑥

⑧ $\int \frac{d\tau}{v_0} = \log r + \text{const}$ by ⑦

⑨ $\int_{-\beta}^{\tau} \frac{d\tau}{v_0} = \log r$ by ⑧

(10)

$$\int \log r \, d\tau = \tau \log r - \int \tau \, d \log r$$

$$= \tau \log r - \int \frac{\tau \, d\tau}{v_0} \qquad \text{by (6)}$$

$$= \tau \log r - \int dv_0 \qquad \text{by (5)}$$

$$= -v_0 + \tau \log r + \text{const}$$

(11)

$$\int_{-\beta}^{\tau} \log r \, d\tau = 1 - v_0 + \tau \log r \qquad \text{by (10)}$$

(12)

$$\int_{-\beta}^{\tau} \frac{\tau}{v_0} \, d\tau = \int_{-\beta}^{\tau} dv_0 \qquad \text{by (5)}$$

$$= v_0 - 1 .$$

$$x_1 = -\frac{\alpha\beta}{2}\int_{-\beta}^{\tau} d\tau - \frac{\alpha}{2}\int_{-\beta}^{\tau} \tau v_0 \, d\tau - \frac{\alpha^3}{2}\int_{-\beta}^{\tau} \log r \, d\tau$$

$$= -\frac{\alpha\beta}{2}(\tau+\beta) - \frac{\alpha}{2}\left(\frac{v_0^3}{3} - \frac{1}{3}\right) - \frac{\alpha^3}{2}\left(1 - v_0 + \tau \log r\right) \qquad \text{by (4), (11)}$$

$$= \left[-\frac{\alpha\beta^2}{2} + \frac{\alpha}{6} - \frac{\alpha^3}{2}\right] - \left[\frac{\alpha\beta}{2}\right]\tau + \left[\frac{\alpha^3}{2}\right] v_0 - \left[\frac{\alpha}{6}\right] v_0^3 - \left[\frac{\alpha^3}{2}\right]\tau \log r$$

(13)

$$\left[-\frac{\alpha\beta^2}{2} + \frac{\alpha}{6} - \frac{\alpha^3}{2}\right] = \left[-\frac{\alpha(1-\alpha^2)}{2} + \frac{\alpha}{6} - \frac{\alpha^3}{2}\right]$$

$$= -\frac{\alpha}{3}$$

$$\boxed{x_1 = \left[-\frac{\alpha}{3}\right] - \left[\frac{\alpha\beta}{2}\right]\tau + \left[\frac{\alpha^3}{2}\right] v_0 - \left[\frac{\alpha}{6}\right] v_0^3 - \left[\frac{\alpha^3}{2}\right]\tau \log r} \qquad \text{by (13)}$$

$$\ddot{y}_1 = -\dot{y}_0 v_0$$
$$= -(-\tau) v_0$$
$$\boxed{\ddot{y}_1 = \tau v_0}$$
$$\dot{y}_1 = \int_{-\beta}^{\tau} \tau v_0 \, d\tau$$
$$\boxed{\dot{y}_1 = \frac{v_0^3}{3} - \frac{1}{3}}$$ by ⑭

⑭ $$\int v_0^3 \, d\tau = \int (\alpha^2+\tau^2)^{3/2} \, d\tau$$
$$= \left(\frac{5\alpha^2\tau}{8} + \frac{\tau^3}{4}\right)\sqrt{\alpha^2+\tau^2} + \frac{3\alpha^4}{8}\log\left(\tau+\sqrt{\alpha^2+\tau^2}\right) + \text{const}$$
$$= \left(\frac{5\alpha^2}{8} + \frac{\tau^2}{4}\right)\tau v_0 + \frac{3\alpha^4}{8}\log r + \text{const}$$
$$= \left(\frac{5\alpha^2}{8} + \frac{v_0^2-\alpha^2}{4}\right)\tau v_0 + \frac{3\alpha^4}{8}\log r + \text{const}$$
$$= \left[\frac{3\alpha^2}{8}\right]\tau v_0 + \left[\frac{1}{4}\right]\tau v_0^3 + \left[\frac{3\alpha^4}{8}\right]\log r + \text{const}$$

⑮ $$\int_{-\beta}^{\tau} v_0^3 \, d\tau = \left[\frac{3\alpha^2}{8}\right]\tau v_0 + \left[\frac{3\alpha^2}{8}\beta\right] + \left[\frac{1}{4}\right]\tau v_0^3 + \left[\frac{1}{4}\beta\right] + \left[\frac{3\alpha^4}{8}\right]\log r$$ by ⑭

$$y_1 = -\frac{1}{3}\int_{-\beta}^{\tau} d\tau + \frac{1}{3}\int_{-\beta}^{\tau} v_0^3 \, d\tau$$
$$= -\frac{1}{3}(\tau+\beta) + \left[\frac{\alpha^2}{8}\right]\tau v_0 + \left[\frac{\alpha^2}{8}\beta\right] + \left[\frac{1}{12}\right]\tau v_0^3 + \left[\frac{1}{12}\beta\right] + \left[\frac{\alpha^4}{8}\right]\log r$$ by ⑮
$$= \left[-\frac{\beta}{3} + \frac{\alpha^2}{8}\beta + \frac{1}{12}\beta\right] - \left[\frac{1}{3}\right]\tau + \left[\frac{\alpha^2}{8}\right]\tau v_0 + \left[\frac{1}{12}\right]\tau v_0^3 + \left[\frac{\alpha^4}{8}\right]\log r$$
$$\boxed{y_1 = \left[-\frac{1}{4} + \frac{\alpha^2}{8}\right]\beta - \left[\frac{1}{3}\right]\tau + \left[\frac{\alpha^2}{8}\right]\tau v_0 + \left[\frac{1}{12}\right]\tau v_0^3 + \left[\frac{\alpha^4}{8}\right]\log r}$$

$\boxed{\text{Order } \beta^2}$

$$\begin{aligned}
w_1 &= \dot{x}_0 \dot{x}_1 + \dot{y}_0 \dot{y}_1 \\
&= \alpha\left(-\frac{\alpha\beta}{2} - \frac{\alpha}{2}\tau v_0 - \frac{\alpha^3}{2}\log r\right) - \tau\left(\frac{v_0^3}{3} - \frac{1}{3}\right) \\
&= -\left[\frac{\alpha^2\beta}{2}\right] + \left[\frac{1}{3}\right]\tau - \left[\frac{\alpha^2}{2}\right]\tau v_0 - \left[\frac{1}{3}\right]\tau v_0^3 - \left[\frac{\alpha^4}{2}\right]\log r
\end{aligned}$$

$$\begin{aligned}
\ddot{x}_2 &= -\dot{x}_1 v_0 - \frac{\dot{x}_0 w_1}{v_0} \\
&= -v_0\left(-\frac{\alpha\beta}{2} - \frac{\alpha}{2}\tau v_0 - \frac{\alpha^3}{2}\log r\right) \\
&\quad - \frac{\alpha}{v_0}\left(-\left[\frac{\alpha^2\beta}{2}\right] + \left[\frac{1}{3}\right]\tau - \left[\frac{\alpha^2}{2}\right]\tau v_0 - \left[\frac{1}{3}\right]\tau v_0^3 - \left[\frac{\alpha^4}{2}\right]\log r\right) \\
&= \left[\frac{\alpha\beta}{2}\right]v_0 + \left[\frac{\alpha}{2}\right]\tau v_0^2 + \left[\frac{\alpha^3}{2}\right]v_0 \log r \\
&\quad + \left[\frac{\alpha^3\beta}{2}\right]\frac{1}{v_0} - \left[\frac{\alpha}{3}\right]\frac{\tau}{v_0} + \left[\frac{\alpha^3}{2}\right]\tau + \left[\frac{\alpha}{3}\right]\tau v_0^2 + \left[\frac{\alpha^5}{2}\right]\frac{\log r}{v_0} \\
&= \left[\frac{\alpha^3}{2}\right]\tau + \left[\frac{5\alpha}{6}\right]\tau v_0^2 + \left[\frac{\alpha^3\beta}{2}\right]\frac{1}{v_0} - \left[\frac{\alpha}{3}\right]\frac{\tau}{v_0} + \left[\frac{\alpha\beta}{2}\right]v_0 \\
&\quad + \left[\frac{\alpha^5}{2}\right]\frac{\log r}{v_0} + \left[\frac{\alpha^3}{2}\right]v_0 \log r
\end{aligned}$$

(16) $\frac{\alpha^3}{2} + \frac{5\alpha}{6}v_0^2 = \frac{\alpha^3}{2} + \frac{5\alpha}{6}(\alpha^2+\tau^2) = \left[\frac{4\alpha^3}{3}\right] + \left[\frac{5\alpha}{6}\right]\tau^2$

$$\boxed{\begin{aligned}
\ddot{x}_2 = \left[\frac{4\alpha^3}{3}\right]\tau + \left[\frac{5\alpha}{6}\right]\tau^3 + \left[\frac{\alpha^3\beta}{2}\right]\frac{1}{v_0} - \left[\frac{\alpha}{3}\right]\frac{\tau}{v_0} + \left[\frac{\alpha\beta}{2}\right]v_0 \\
+ \left[\frac{\alpha^5}{2}\right]\frac{\log r}{v_0} + \left[\frac{\alpha^3}{2}\right]v_0 \log r
\end{aligned}}$$ by (16)

(17) $\int_\beta^\tau \frac{\log r}{v_0}\,d\tau = \int_\beta^\tau \log r \; d\log r$ by (7)

$$= \frac{1}{2}\log^2 r$$

⑱ $\int_{-\beta}^{\tau} v_0 \log r \, d\tau$

$= \int_{-\beta}^{\tau} \log r \, d\left[\frac{\tau v_0}{2} + \frac{\alpha^2}{2}\log r\right]$ by ①

$= \frac{1}{2}\int_{-\beta}^{\tau} \log r \, d(\tau v_0) + \frac{\alpha^2}{2}\int_{-\beta}^{\tau} \log r \, d\log r$

$= \frac{1}{2}\left[\tau v_0 \log r - \int_{-\beta}^{\tau} \tau v_0 \, d\log r\right] + \frac{\alpha^2}{2}\cdot\frac{\log^2 r}{2}$

$= \frac{1}{2}\left[\tau v_0 \log r - \int_{-\beta}^{\tau} \tau \, d\tau\right] + \frac{\alpha^2}{2}\frac{\log^2 r}{2}$ by ⑦

$= \frac{1}{2}\left[\tau v_0 \log r - \frac{\tau^2}{2} + \frac{\beta^2}{2}\right] + \frac{\alpha^2}{4}\log^2 r$

$= \left[\frac{\beta^2}{4}\right] - \left[\frac{1}{4}\right]\tau^2 + \left[\frac{1}{2}\right]\tau v_0 \log r + \left[\frac{\alpha^2}{4}\right]\log^2 r$

$\dot{x}_2 = \left[\frac{4\alpha^3}{3}\right]\int_{-\beta}^{\tau} \tau\, d\tau + \left[\frac{5\alpha}{6}\right]\int_{-\beta}^{\tau} \tau^3 d\tau + \left[\frac{\alpha^3\beta}{2}\right]\int_{-\beta}^{\tau}\frac{d\tau}{v_0} - \left[\frac{\alpha}{3}\right]\int_{-\beta}^{\tau}\frac{\tau\, d\tau}{v_0}$

$\quad + \left[\frac{\alpha\beta}{2}\right]\int_{-\beta}^{\tau} v_0 \, d\tau + \left[\frac{\alpha^5}{2}\right]\int_{-\beta}^{\tau}\frac{\log r}{v_0}d\tau + \left[\frac{\alpha^3}{2}\right]\int_{-\beta}^{\tau} v_0 \log r \, d\tau$

$= \left[\frac{4\alpha^3}{3}\right]\left(\frac{\tau^2}{2} - \frac{\beta^2}{2}\right) + \left[\frac{5\alpha}{6}\right]\left(\frac{\tau^4}{4} - \frac{\beta^4}{4}\right) + \left[\frac{\alpha^3\beta}{2}\right]\log r - \left[\frac{\alpha}{3}\right](v_0 - 1)$ by ⑨, ⑫

$\quad + \left[\frac{\alpha\beta}{2}\right]\left(\frac{\beta}{2} + \frac{\tau v_0}{2} + \frac{\alpha^2}{2}\log r\right)$ by ②

$\quad + \left[\frac{\alpha^5}{2}\right]\left(\frac{1}{2}\log^2 r\right) + \left[\frac{\alpha^3}{2}\right]\left(\left[\frac{\beta^2}{4}\right] - \left[\frac{1}{4}\right]\tau^2 + \left[\frac{1}{2}\right]\tau v_0 \log r + \left[\frac{\alpha^2}{4}\right]\log^2 r\right)$ by ⑰, ⑱

$= \left[-\frac{2\alpha^3\beta^2}{3} - \frac{5\alpha\beta^4}{24} + \frac{\alpha}{3} + \frac{\alpha\beta^2}{4} + \frac{\alpha^3\beta^2}{8}\right] + \left[\frac{2\alpha^3}{3} - \frac{\alpha^3}{8}\right]\tau^2 + \left[\frac{5\alpha}{24}\right]\tau^4$

$\quad - \left[\frac{\alpha}{3}\right]v_0 + \left[\frac{\alpha\beta}{4}\right]\tau v_0 + \left[\frac{\alpha^3\beta}{2} + \frac{\alpha^3\beta}{4}\right]\log r$

$\quad + \left[\frac{\alpha^3}{4}\right]\tau v_0 \log r + \left[\frac{\alpha^5}{4} + \frac{\alpha^5}{8}\right]\log^2 r$

$= \left[\frac{\alpha}{3} + \frac{\alpha\beta^2}{4} - \frac{5\alpha\beta^4}{24} - \frac{13\alpha^3\beta^2}{24}\right] + \left[\frac{13\alpha^3}{24}\right]\tau^2 + \left[\frac{5\alpha}{24}\right]\tau^4$

$\quad - \left[\frac{\alpha}{3}\right]v_0 + \left[\frac{\alpha\beta}{4}\right]\tau v_0 + \left[\frac{3\alpha^3\beta}{4}\right]\log r + \left[\frac{\alpha^3}{4}\right]\tau v_0 \log r + \left[\frac{3\alpha^5}{8}\right]\log^2 r$

(19) $\left[\frac{\alpha}{3}+\frac{\alpha\beta^2}{4}-\frac{5\alpha\beta^4}{24}-\frac{13\alpha^3\beta^2}{24}\right]$

$$=\left[\frac{\alpha}{3}+\frac{\alpha(1-\alpha^2)}{4}-\frac{5\alpha(1-2\alpha^2+\alpha^4)}{24}-\frac{13\alpha^3(1-\alpha^2)}{24}\right]$$

$$=\left[\left(\frac{1}{3}+\frac{1}{4}-\frac{5}{24}\right)\alpha+\left(-\frac{1}{4}+\frac{10}{24}-\frac{13}{24}\right)\alpha^3+\left(-\frac{5}{24}+\frac{13}{24}\right)\alpha^5\right]$$

$$=\left[\frac{3\alpha}{8}-\frac{3\alpha^3}{8}+\frac{\alpha^5}{3}\right]$$

$$\boxed{\dot{x}_2=\left[\frac{3\alpha}{8}-\frac{3\alpha^3}{8}+\frac{\alpha^5}{3}\right]+\left[\frac{13\alpha^3}{24}\right]\tau^2+\left[\frac{5\alpha}{24}\right]\tau^4-\left[\frac{\alpha}{3}\right]v_0+\left[\frac{\alpha\beta}{4}\right]\tau v_0+\left[\frac{3\alpha^3\beta}{4}\right]\log r+\left(\frac{\alpha^3}{4}\right)\tau v_0\log r+\left[\frac{3\alpha^5}{8}\right]\log^2 r} \quad \text{by (19)}$$

(20) $\int_{-\beta}^{\tau}\tau v_0\log r\,d\tau$

$$=\int_{-\beta}^{\tau}\log r\,d\left[\frac{v_0^3}{3}\right] \qquad \text{by (3)}$$

$$=\frac{1}{3}v_0^3\log r-\frac{1}{3}\int_{-\beta}^{\tau}v_0^3\,d\log r$$

$$=\frac{1}{3}v_0^3\log r-\frac{1}{3}\int_{-\beta}^{\tau}v_0^3\cdot\frac{d\tau}{v_0} \qquad \text{by (7)}$$

$$=\frac{1}{3}v_0^3\log r-\frac{1}{3}\int_{-\beta}^{\tau}v_0^2\,d\tau$$

$$=\frac{1}{3}v_0^3\log r-\frac{1}{3}\int_{-\beta}^{\tau}(\alpha^2+\tau^2)\,d\tau$$

$$=\frac{1}{3}v_0^3\log r-\frac{1}{3}\left[\alpha^2(\tau+\beta)+\frac{\tau^3}{3}+\frac{\beta^3}{3}\right]$$

$$=\left[-\frac{\alpha^2\beta}{3}-\frac{\beta^3}{9}\right]-\left[\frac{\alpha^2}{3}\right]\tau-\left[\frac{1}{9}\right]\tau^3+\left[\frac{1}{3}\right]v_0^3\log r$$

(21) $\int\log^2 r\,d\tau=\tau\log^2 r-\int\tau\,d\log^2 r$

$$=\tau\log^2 r-\int\tau\cdot 2\log r\,d\log r$$

$$=\tau\log^2 r-2\int\tau\log r\cdot\frac{d\tau}{v_0} \qquad \text{by (7)}$$

$$=\tau\log^2 r-2\int\log r\cdot\frac{\tau\,d\tau}{v_0}$$

$$=\tau\log^2 r-2\int\log r\cdot dv_0 \qquad \text{by (12)}$$

(21)$^{+}$

$$\int \log^2 r \, d\tau = \tau \log^2 r - 2\int \log r \, dv_0$$
$$= \tau \log^2 r - 2v_0 \log r + 2\int v_0 \, d\log r$$
$$= \tau \log^2 r - 2v_0 \log r + 2\int d\tau \qquad \text{by (7)}$$
$$= 2\tau - 2v_0 \log r + \tau \log^2 r + \text{const}$$

(22) $$\int_{-\beta}^{\tau} \log^2 r \, d\tau = 2(\tau+\beta) - 2v_0 \log r + \tau \log^2 r \qquad \text{by (21)}$$

$$\tau_2 = \left[\frac{3\alpha}{8} - \frac{3\alpha^3}{8} + \frac{\alpha^5}{3}\right]\int_{-\beta}^{\tau} d\tau + \left[\frac{13\alpha^3}{24}\right]\int_{-\beta}^{\tau} \tau^2 d\tau + \left[\frac{5\alpha}{24}\right]\int_{-\beta}^{\tau} \tau^4 d\tau$$
$$- \left[\frac{\alpha}{3}\right]\int_{-\beta}^{\tau} v_0 d\tau + \left[\frac{\alpha\beta}{4}\right]\int_{-\beta}^{\tau} \tau v_0 d\tau + \left[\frac{3\alpha^3\beta}{4}\right]\int_{-\beta}^{\tau} \log r \, d\tau$$
$$+ \left[\frac{\alpha^3}{4}\right]\int_{-\beta}^{\tau} \tau v_0 \log r \, d\tau + \left[\frac{3\alpha^5}{8}\right]\int_{-\beta}^{\tau} \log^2 r \, d\tau$$
$$= \left[\frac{3\alpha}{8} - \frac{3\alpha^3}{8} + \frac{\alpha^5}{3}\right](\tau+\beta) + \left[\frac{13\alpha^3}{24}\right]\left(\frac{\tau^3}{3} + \frac{\beta^3}{3}\right) + \left[\frac{5\alpha}{24}\right]\left(\frac{\tau^5}{5} + \frac{\beta^5}{5}\right)$$
$$- \left[\frac{\alpha}{3}\right]\left(\frac{\beta}{2} + \frac{\tau v_0}{2} + \frac{\alpha^2}{2}\log r\right) + \left[\frac{\alpha\beta}{4}\right]\left(\frac{v_0^3}{3} - \frac{1}{3}\right) + \left[\frac{3\alpha^3\beta}{4}\right]\left(1 - v_0 + \tau \log r\right) \qquad \text{by (2), (4), (11)}$$
$$+ \left[\frac{\alpha^3}{4}\right]\left(\left[-\frac{\alpha^2\beta}{3} - \frac{\beta^3}{9}\right] - \left[\frac{\alpha^2}{3}\right]\tau - \left[\frac{1}{9}\right]\tau^3 + \left[\frac{1}{3}\right]v_0^3 \log r\right) \qquad \text{by (20)}$$
$$+ \left[\frac{3\alpha^5}{8}\right]\left(2(\tau+\beta) - 2v_0 \log r + \tau \log^2 r\right) \qquad \text{by (22)}$$
$$= \left[\frac{3\alpha}{8} - \frac{3\alpha^3}{8} + \frac{\alpha^5}{3} + \frac{13\alpha^3\beta^2}{72} + \frac{\alpha\beta^4}{24} - \frac{\alpha}{6} - \frac{\alpha}{12} + \frac{3\alpha^3}{4} - \frac{\alpha^5}{12} - \frac{\alpha^3\beta^2}{36} + \frac{3\alpha^5}{4}\right]\beta$$
$$+ \left[\frac{3\alpha}{8} - \frac{3\alpha^3}{8} + \frac{\alpha^5}{3} - \frac{\alpha^5}{12} + \frac{3\alpha^5}{4}\right]\tau + \left[\frac{13\alpha^3}{72} - \frac{\alpha^3}{36}\right]\tau^3 + \left[\frac{\alpha}{24}\right]\tau^5$$
$$- \left[\frac{3\alpha^3\beta}{4}\right]v_0 - \left[\frac{\alpha}{6}\right]\tau v_0 + \left[\frac{\alpha\beta}{12}\right]v_0^3 - \left[\frac{\alpha^3}{6}\right]\log r + \left[\frac{3\alpha^3\beta}{4}\right]\tau \log r$$
$$- \left[\frac{3\alpha^5}{4}\right]v_0 \log r + \left[\frac{\alpha^3}{12}\right]v_0^3 \log r + \left[\frac{3\alpha^5}{8}\right]\tau \log^2 r$$

$$㉓\ \left[\frac{3\alpha}{8}-\frac{3\alpha^3}{8}+\frac{\alpha^5}{3}+\frac{13\alpha^3\beta^2}{72}+\frac{\alpha\beta^4}{24}-\frac{\alpha}{6}-\frac{\alpha}{12}+\frac{3\alpha^3}{4}-\frac{\alpha^5}{12}-\frac{\alpha^3\beta^2}{36}+\frac{3\alpha^5}{4}\right]$$

$$=\left[\frac{3\alpha}{8}-\frac{3\alpha^3}{8}+\frac{\alpha^5}{3}+\frac{13\alpha^3(1-\alpha^2)}{72}+\frac{\alpha(1-2\alpha^2+\alpha^4)}{24}-\frac{\alpha}{6}-\frac{\alpha}{12}+\frac{3\alpha^3}{4}-\frac{\alpha^5}{12}\right.$$
$$\left.-\frac{\alpha^3(1-\alpha^2)}{36}+\frac{3\alpha^5}{4}\right]$$

$$=\left[\left(\frac{3}{8}+\frac{1}{24}-\frac{1}{6}-\frac{1}{12}\right)\alpha+\left(-\frac{3}{8}+\frac{13}{72}-\frac{2}{24}+\frac{3}{4}-\frac{1}{36}\right)\alpha^3\right.$$
$$\left.+\left(\frac{1}{3}-\frac{13}{72}+\frac{1}{24}-\frac{1}{12}+\frac{1}{36}+\frac{3}{4}\right)\alpha^5\right]$$

$$=\left[\frac{\alpha}{6}+\frac{4\alpha^3}{9}+\frac{8\alpha^5}{9}\right]$$

$$㉔\ \left[\frac{3\alpha}{8}-\frac{3\alpha^3}{8}+\frac{\alpha^5}{3}-\frac{\alpha^5}{12}+\frac{3\alpha^5}{4}\right]=\left[\frac{3\alpha}{8}-\frac{3\alpha^3}{8}+\alpha^5\right]$$

$$㉕\ \left[\frac{13\alpha^3}{72}-\frac{\alpha^3}{36}\right]=\left[\frac{11\alpha^3}{72}\right]$$

$$\boxed{\begin{aligned}x_2=&\left[\frac{\alpha}{6}+\frac{4\alpha^3}{9}+\frac{8\alpha^5}{9}\right]\beta+\left[\frac{3\alpha}{8}-\frac{3\alpha^3}{8}+\alpha^5\right]\tau+\left[\frac{11\alpha^3}{72}\right]\tau^3+\left[\frac{\alpha}{24}\right]\tau^5\\&-\left[\frac{3\alpha^3\beta}{4}\right]v_0-\left[\frac{\alpha}{6}\right]\tau v_0+\left[\frac{\alpha\beta}{12}\right]v_0^3-\left[\frac{\alpha^3}{6}\right]\log r+\left[\frac{3\alpha^3\beta}{4}\right]\tau\log r\\&-\left[\frac{3\alpha^5}{4}\right]v_0\log r+\left[\frac{\alpha^3}{12}\right]v_0^3\log r+\left[\frac{3\alpha^5}{8}\right]\tau\log^2 r\end{aligned}}\quad \text{by ㉓, ㉔, ㉕}$$

$$\ddot{y}_2=-\dot{y}_1v_0-\frac{\dot{y}_0w_1}{v_0}$$
$$=-v_0\left(\frac{v_0^3}{3}-\frac{1}{3}\right)+\frac{\tau}{v_0}\left(-\left[\frac{\alpha^2\beta}{2}\right]+\left[\frac{1}{3}\right]\tau-\left[\frac{\alpha^2}{2}\right]\tau v_0-\left[\frac{1}{3}\right]\tau v_0^3-\left[\frac{\alpha^4}{2}\right]\log r\right)$$
$$=-\frac{v_0^4}{3}+\frac{v_0}{3}-\left[\frac{\alpha^2\beta}{2}\right]\frac{\tau}{v_0}+\left[\frac{1}{3}\right]\frac{\tau^2}{v_0}-\left[\frac{\alpha^2}{2}\right]\tau^2-\left[\frac{1}{3}\right]\tau^2v_0^2-\left[\frac{\alpha^4}{2}\right]\frac{\tau\log r}{v_0}$$
$$\cancel{=-\frac{1}{3}(\alpha^2+\tau^2)^2+\frac{v_0}{3}-\left[\frac{\alpha^2\beta}{2}\right]\frac{\tau}{v_0}+\left[\frac{1}{3}\right]\frac{\tau^2}{v_0}-\left[\frac{\alpha^2}{2}\right]\tau^2-\left[\frac{1}{3}\right]\tau^2v_0^2-\left[\frac{\alpha^4}{2}\right]\frac{\tau\log r}{v_0}}$$
$$=-\frac{1}{3}(\alpha^2+\tau^2)^2+\frac{v_0}{3}-\left[\frac{\alpha^2\beta}{2}\right]\frac{\tau}{v_0}+\left[\frac{1}{3}\right]\frac{\tau^2}{v_0}-\left[\frac{\alpha^2}{2}\right]\tau^2-\left[\frac{1}{3}\right]\tau^2(\alpha^2+\tau^2)$$
$$-\left[\frac{\alpha^4}{2}\right]\frac{\tau\log r}{v_0}$$

$$\ddot{y}_2 = -\left[\frac{1}{3}\right]\alpha^4 + \left[-\frac{2\alpha^2}{3} - \frac{\alpha^2}{2} - \frac{\alpha^2}{3}\right]\tau^2 + \left[-\frac{1}{3} - \frac{1}{3}\right]\tau^4 + \left[\frac{1}{3}\right] v_0$$
$$- \left[\frac{\alpha^2\beta}{2}\right]\frac{\tau}{v_0} + \left[\frac{1}{3}\right]\frac{\tau^2}{v_0} - \left[\frac{\alpha^4}{2}\right]\frac{\tau \log r}{v_0}$$

$$\cancel{= -\left[\frac{1}{3}\right]\alpha^4 - \left[\frac{3\alpha^2}{2}\right]\tau^2}$$

$$\boxed{\ddot{y}_2 = -\left[\frac{\alpha^4}{3}\right] - \left[\frac{3\alpha^2}{2}\right]\tau^2 - \left[\frac{2}{3}\right]\tau^4 + \left[\frac{1}{3}\right] v_0 - \left[\frac{\alpha^2\beta}{2}\right]\frac{\tau}{v_0} + \left[\frac{1}{3}\right]\frac{\tau^2}{v_0} - \left[\frac{\alpha^4}{2}\right]\frac{\tau\log r}{v_0}}$$

(26) $\int_{-\beta}^{\tau} \frac{\tau^2}{v_0} d\tau = \int_{-\beta}^{\tau} \tau \cdot \frac{\tau d\tau}{v_0}$

$= \int_{-\beta}^{\tau} \tau \cdot dv_0$ by (12)

$= \tau v_0 + \beta - \int_{-\beta}^{\tau} v_0 \, d\tau$

$= \tau v_0 + \beta - \left(\frac{\beta}{2} + \frac{\tau v_0}{2} + \frac{\alpha^2}{2}\log r\right)$ by (2)

$= \left[\frac{\beta}{2}\right] + \left[\frac{1}{2}\right]\tau v_0 - \left[\frac{\alpha^2}{2}\right]\log r$

(27)

$\int_{-\beta}^{\tau} \frac{\tau \log r}{v_0} d\tau = \int_{-\beta}^{\tau} \log r \cdot \frac{\tau d\tau}{v_0}$

$= \int_{-\beta}^{\tau} \log r \cdot dv_0$ by (12)

$= v_0 \log r - \int_{-\beta}^{\tau} v_0 \, d\log r$

$= v_0 \log r - \int_{-\beta}^{\tau} d\tau$ by (7)

$= v_0 \log r - (\tau + \beta)$

$= -\beta - \tau + v_0 \log r$

$$\dot{y}_2 = -\left[\frac{\alpha^4}{3}\right]\int_{-\beta}^{\tau} d\tau - \left[\frac{3\alpha^2}{2}\right]\int_{-\beta}^{\tau}\tau^2 d\tau - \left[\frac{2}{3}\right]\int_{-\beta}^{\tau}\tau^4 d\tau + \left[\frac{1}{3}\right]\int_{-\beta}^{\tau} v_0 \, d\tau$$
$$- \left[\frac{\alpha^2\beta}{2}\right]\int_{-\beta}^{\tau}\frac{\tau}{v_0} d\tau + \left[\frac{1}{3}\right]\int_{-\beta}^{\tau}\frac{\tau^2}{v_0} d\tau - \left[\frac{\alpha^4}{2}\right]\int_{-\beta}^{\tau}\frac{\tau \log r}{v_0} d\tau$$

$$\dot{y}_2 = -\left[\frac{\alpha^4}{3}\right](\tau+\beta) - \left[\frac{3\alpha^2}{2}\right]\left(\frac{\tau^3}{3}+\frac{\beta^3}{3}\right) - \left[\frac{2}{3}\right]\left(\frac{\tau^5}{5}+\frac{\beta^5}{5}\right)$$

$$+\left[\frac{1}{3}\right]\left(\frac{\beta}{2}+\frac{\tau v_0}{2}+\cancel{\frac{\alpha^2}{2}\log r}\right) - \left[\frac{\alpha^2\beta}{2}\right](v_0-1) \qquad \text{by (2), (12)}$$

$$+\left[\frac{1}{3}\right]\left(\left[\frac{\beta}{2}\right]+\left[\frac{1}{2}\right]\tau v_0 - \cancel{\left[\frac{\alpha^2}{2}\right]\log r}\right) - \left[\frac{\alpha^4}{2}\right](-\beta-\tau+v_0\log r) \qquad \text{by (26), (27)}$$

$$= \left[-\frac{\alpha^4}{3}-\frac{\alpha^2\beta^2}{2}-\frac{2\beta^4}{15}+\frac{1}{6}+\frac{\alpha^2}{2}+\frac{1}{6}+\frac{\alpha^4}{2}\right]\beta + \left[-\frac{\alpha^4}{3}+\frac{\alpha^4}{2}\right]\tau$$

$$-\left[\frac{\alpha^2}{2}\right]\tau^3 - \left[\frac{2}{15}\right]\tau^5 - \left[\frac{\alpha^2\beta}{2}\right]v_0 + \left[\frac{1}{6}+\frac{1}{6}\right]\tau v_0 - \left[\frac{\alpha^4}{2}\right]v_0\log r$$

(28) $$\left[-\frac{\alpha^4}{3}-\frac{\alpha^2\beta^2}{2}-\frac{2\beta^4}{15}+\frac{1}{6}+\frac{\alpha^2}{2}+\frac{1}{6}+\frac{\alpha^4}{2}\right]$$

$$= \left[-\frac{\alpha^4}{3}-\frac{\alpha^2(1-\alpha^2)}{2}-\frac{2(1-2\alpha^2+\alpha^4)}{15}+\frac{1}{6}+\frac{\alpha^2}{2}+\frac{1}{6}+\frac{\alpha^4}{2}\right]$$

$$\cancel{= \left[\left(-\frac{1}{6}+\frac{1}{6}\right)+\left(-\frac{1}{2}\right.\right.}$$

$$= \left[\left(-\frac{2}{15}+\frac{1}{6}+\frac{1}{6}\right)+\left(-\frac{1}{2}+\frac{4}{15}+\frac{1}{2}\right)\alpha^2+\left(-\frac{1}{3}+\frac{1}{2}-\frac{2}{15}+\frac{1}{2}\right)\alpha^4\right]$$

$$= \left[\frac{1}{5}+\frac{4\alpha^2}{15}+\frac{8\alpha^4}{15}\right]$$

$$\boxed{\begin{aligned}\dot{y}_2 = {} & \left[\frac{1}{5}+\frac{4\alpha^2}{15}+\frac{8\alpha^4}{15}\right]\beta + \left[\frac{\alpha^4}{6}\right]\tau - \left[\frac{\alpha^2}{2}\right]\tau^3 - \left[\frac{2}{15}\right]\tau^5 \\ & -\left[\frac{\alpha^2\beta}{2}\right]v_0 + \left[\frac{1}{3}\right]\tau v_0 - \left[\frac{\alpha^4}{2}\right]v_0\log r\end{aligned}} \qquad \text{by (28)}$$

$$y_2 = \left[\frac{1}{5}+\frac{4\alpha^2}{15}+\frac{8\alpha^4}{15}\right]\beta\int_{-\beta}^{\tau}d\tau + \left[\frac{\alpha^4}{6}\right]\int_{-\beta}^{\tau}\tau\,d\tau - \left[\frac{\alpha^2}{2}\right]\int_{-\beta}^{\tau}\tau^3\,d\tau$$

$$-\left[\frac{2}{15}\right]\int_{-\beta}^{\tau}\tau^5\,d\tau - \left[\frac{\alpha^2\beta}{2}\right]\int_{-\beta}^{\tau}v_0\,d\tau + \left[\frac{1}{3}\right]\int_{-\beta}^{\tau}\tau v_0\,d\tau - \left[\frac{\alpha^4}{2}\right]\int_{-\beta}^{\tau}v_0\log r\,d\tau$$

$$= \left[\frac{1}{5}+\frac{4\alpha^2}{15}+\frac{8\alpha^4}{15}\right]\beta(\tau+\beta) + \left[\frac{\alpha^4}{6}\right]\left(\frac{\tau^2}{2}-\frac{\beta^2}{2}\right) - \left[\frac{\alpha^2}{2}\right]\left(\frac{\tau^4}{4}-\frac{\beta^4}{4}\right)$$

$$-\left[\frac{2}{15}\right]\left(\frac{\tau^6}{6}-\frac{\beta^6}{6}\right) - \left[\frac{\alpha^2\beta}{2}\right]\left(\frac{\beta}{2}+\frac{\tau v_0}{2}+\frac{\alpha^2}{2}\log r\right) \qquad \text{by (2)}$$

$$+\left[\frac{1}{3}\right]\left(\frac{v_0^3}{3}-\frac{1}{3}\right) - \left[\frac{\alpha^4}{2}\right]\left(\left[\frac{\beta^2}{4}\right]-\left[\frac{1}{4}\right]\tau^2+\left[\frac{1}{2}\right]\tau v_0\log r+\left[\frac{\alpha^2}{4}\right]\log^2 r\right) \qquad \text{by (4), (18)}$$

$$y_2 = \left[\left(\frac{1}{5}+\frac{4\alpha^2}{15}+\frac{8\alpha^4}{15}\right)\beta^2 - \frac{\alpha^4\beta^2}{12} + \frac{\alpha^2\beta^4}{8} + \frac{\beta^6}{45} - \frac{\alpha^2\beta^2}{4} - \frac{1}{9} - \frac{\alpha^4\beta^2}{8}\right]$$
$$+ \left[\left(\frac{1}{5}+\frac{4\alpha^2}{15}+\frac{8\alpha^4}{15}\right)\beta\right]\tau + \left[\frac{\alpha^4}{12}+\frac{\alpha^4}{8}\right]\tau^2 - \left[\frac{\alpha^2}{8}\right]\tau^4 - \left[\frac{1}{45}\right]\tau^6$$
$$- \left[\frac{\alpha^2\beta}{4}\right]\tau v_0 + \left[\frac{1}{9}\right]v_0^3 - \left[\frac{\alpha^4\beta}{4}\right]\log r - \left[\frac{\alpha^4}{4}\right]\tau v_0 \log r - \left[\frac{\alpha^6}{8}\right]\log^2 r$$

(29)
$$\left[\left(\frac{1}{5}+\frac{4\alpha^2}{15}+\frac{8\alpha^4}{15}\right)\beta^2 - \frac{\alpha^4\beta^2}{12} + \frac{\alpha^2\beta^4}{8} + \frac{\beta^6}{45} - \frac{\alpha^2\beta^2}{4} - \frac{1}{9} - \frac{\alpha^4\beta^2}{8}\right]$$
$$= \left[\left(\frac{1}{5}+\frac{4\alpha^2}{15}+\frac{8\alpha^4}{15}\right)(1-\alpha^2) - \frac{\alpha^4(1-\alpha^2)}{12} + \frac{\alpha^2(1-2\alpha^2+\alpha^4)}{8} + \frac{1-3\alpha^2+3\alpha^4-\alpha^6}{45} - \frac{\alpha^2(1-\alpha^2)}{4} - \frac{1}{9} - \frac{\alpha^4(1-\alpha^2)}{8}\right]$$
$$= \left[\left(\frac{1}{5}+\frac{1}{45}-\frac{1}{9}\right) + \left(\frac{4}{15}-\frac{1}{5}+\frac{1}{8}-\frac{3}{45}-\frac{1}{4}\right)\alpha^2 + \left(\frac{8}{15}-\frac{4}{15}-\frac{1}{12}-\frac{2}{8}+\frac{3}{45}+\frac{1}{4}-\frac{1}{8}\right)\alpha^4 + \left(-\frac{8}{15}+\frac{1}{12}+\frac{1}{8}-\frac{1}{45}+\frac{1}{8}\right)\alpha^6\right]$$
$$= \left[\frac{1}{9} - \frac{\alpha^2}{8} + \frac{\alpha^4}{8} - \frac{2\alpha^6}{9}\right]$$

$$\boxed{\begin{aligned} y_2 = {} & \left[\frac{1}{9} - \frac{\alpha^2}{8} + \frac{\alpha^4}{8} - \frac{2\alpha^6}{9}\right] + \left[\left(\frac{1}{5}+\frac{4\alpha^2}{15}+\frac{8\alpha^4}{15}\right)\beta\right]\tau + \left[\frac{5\alpha^4}{24}\right]\tau^2 \\ & - \left[\frac{\alpha^2}{8}\right]\tau^4 - \left[\frac{1}{45}\right]\tau^6 - \left[\frac{\alpha^2\beta}{4}\right]\tau v_0 + \left[\frac{1}{9}\right]v_0^3 - \left[\frac{\alpha^4\beta}{4}\right]\log r \\ & - \left[\frac{\alpha^4}{4}\right]\tau v_0 \log r - \left[\frac{\alpha^6}{8}\right]\log^2 r \end{aligned}}$$ by (29)

If you go to order B^3, you will find that the following terms have no closed-form expression:

$$\int \frac{\tau \log r}{v_0^2}\,d\tau \quad , \quad \int \frac{\log^2 r}{v_0^3}\,d\tau$$

or, equivalently,

$$\int \frac{\tau \log\left(\tau+\sqrt{\alpha^2+\tau^2}\right)}{\alpha^2+\tau^2}\,d\tau \quad , \quad \int \frac{\log^2\left(\tau+\sqrt{\alpha^2+\tau^2}\right)}{(\alpha^2+\tau^2)^{3/2}}\,d\tau .$$

Thus we stop at order B^2.

Recap

General

$$\alpha = \cos\phi$$
$$\beta = \sin\phi$$

$$\tau = t - \beta$$

Order 1

$$\ddot{x}_0 = 0$$
$$\dot{x}_0 = \alpha$$
$$x_0 = \alpha t$$

$$\ddot{y}_0 = -1$$
$$\dot{y}_0 = -\tau$$
$$y_0 = -\frac{\tau^2}{2} + \frac{\beta^2}{2}$$

$$v_0 = \sqrt{\alpha^2 + \tau^2}$$

$$r = \frac{\tau + v_0}{-\beta + 1}$$

Order B

$$\ddot{x}_1 = -\alpha v_0$$
$$\dot{x}_1 = -\left[\frac{\alpha\beta}{2}\right] - \left[\frac{\alpha}{2}\right]\tau v_0 - \left[\frac{\alpha^3}{2}\right]\log r$$
$$x_1 = -\left[\frac{\alpha}{3}\right] - \left[\frac{\alpha\beta}{2}\right]\tau + \left[\frac{\alpha^3}{2}\right]v_0 - \left[\frac{\alpha}{6}\right]v_0^3 - \left[\frac{\alpha^3}{2}\right]\tau\log r$$

$$\ddot{y}_1 = \tau v_0$$

$$\dot{y}_1 = \frac{v_0^3}{3} - \frac{1}{3}$$

$$y_1 = \left[-\frac{1}{4} + \frac{\alpha^2}{8}\right]\beta - \left[\frac{1}{3}\right]\tau + \left[\frac{\alpha^2}{8}\right]\tau v_0 + \left[\frac{1}{12}\right]\tau v_0^3 + \left[\frac{\alpha^4}{8}\right]\log r$$

Order β^2

$$\ddot{x}_2 = \left[\frac{4\alpha^3}{3}\right]\tau + \left[\frac{5\alpha}{6}\right]\tau^3 + \left[\frac{\alpha^3\beta}{2}\right]\frac{1}{v_0} - \left[\frac{\alpha}{3}\right]\frac{\tau}{v_0} + \left[\frac{\alpha\beta}{2}\right]v_0 + \left[\frac{\alpha^5}{2}\right]\frac{\log r}{v_0} + \left[\frac{\alpha^3}{2}\right]v_0 \log r$$

$$\dot{x}_2 = \left[\frac{3\alpha}{8} - \frac{3\alpha^3}{8} + \frac{\alpha^5}{3}\right] + \left[\frac{13\alpha^3}{24}\right]\tau^2 + \left[\frac{5\alpha}{24}\right]\tau^4 - \left[\frac{\alpha}{3}\right]v_0 + \left[\frac{\alpha\beta}{4}\right]\tau v_0 + \left[\frac{3\alpha^3\beta}{4}\right]\log r + \left[\frac{\alpha^3}{4}\right]\tau v_0 \log r + \left[\frac{3\alpha^5}{8}\right]\log^2 r$$

$$x_2 = \left[\frac{\alpha}{6} + \frac{4\alpha^3}{9} + \frac{8\alpha^5}{9}\right]\beta + \left[\frac{3\alpha}{8} - \frac{3\alpha^3}{8} + \alpha^5\right]\tau + \left[\frac{11\alpha^3}{72}\right]\tau^3 + \left[\frac{\alpha}{24}\right]\tau^5 - \left[\frac{3\alpha^3\beta}{4}\right]v_0 - \left[\frac{\alpha}{6}\right]\tau v_0 + \left[\frac{\alpha\beta}{12}\right]v_0^3 - \left[\frac{\alpha^3}{6}\right]\log r + \left[\frac{3\alpha^3\beta}{4}\right]\tau\log r - \left[\frac{3\alpha^5}{4}\right]v_0\log r + \left[\frac{\alpha^3}{12}\right]v_0^3\log r + \left[\frac{3\alpha^5}{8}\right]\tau\log^2 r$$

$$\ddot{y}_2 = -\left[\frac{\alpha^4}{3}\right] - \left[\frac{3\alpha^2}{2}\right]\tau^2 - \left[\frac{2}{3}\right]\tau^4 + \left[\frac{1}{3}\right]v_0 - \left[\frac{\alpha^2\beta}{2}\right]\frac{\tau}{v_0} + \left[\frac{1}{3}\right]\frac{\tau^2}{v_0} - \left[\frac{\alpha^4}{2}\right]\frac{\tau\log r}{v_0}$$

$$\dot{y}_2 = \left[\frac{1}{5} + \frac{4\alpha^2}{15} + \frac{8\alpha^4}{15}\right]\beta + \left[\frac{\alpha^4}{6}\right]\tau - \left[\frac{\alpha^2}{2}\right]\tau^3 - \left[\frac{2}{15}\right]\tau^5 - \left[\frac{\alpha^2\beta}{2}\right]v_0 + \left[\frac{1}{3}\right]\tau v_0 - \left[\frac{\alpha^4}{2}\right]v_0\log r$$

$$y_2 = \left[\frac{1}{9} - \frac{\alpha^2}{8} + \frac{\alpha^4}{8} - \frac{2\alpha^6}{9}\right] + \left[\frac{1}{5} + \frac{4\alpha^2}{15} + \frac{8\alpha^4}{15}\right]\beta\tau + \left[\frac{5\alpha^4}{24}\right]\tau^2 - \left[\frac{\alpha^2}{8}\right]\tau^4 - \left[\frac{1}{45}\right]\tau^6 - \left[\frac{\alpha^2\beta}{4}\right]\tau v_0 + \left[\frac{1}{9}\right]v_0^3 - \left[\frac{\alpha^4\beta}{4}\right]\log r - \left[\frac{\alpha^4}{4}\right]\tau v_0\log r - \left(\frac{\alpha^6}{8}\right)\log^2 r$$

Flight time

$$0 = y(t)$$
$$= \left(y_0 + B y_1 + B^2 y_2\right)\Big|_{t = t_0 + B t_1 + B^2 t_2}$$
$$= y_0(t_0) + (B t_1 + B^2 t_2)\,\dot{y}_0(t_0) + \frac{1}{2!}(B t_1)^2\,\ddot{y}_0(t_0)$$
$$+ B y_1(t_0) + (B t_1)\, B \dot{y}_1(t_0)$$
$$+ B^2 y_2(t_0)$$
$$= y_0(t_0) + B\left[t_1 \dot{y}_0(t_0) + y_1(t_0)\right]$$
$$+ B^2\left[t_2 \dot{y}_0(t_0) + \frac{1}{2} t_1^2 \ddot{y}_0(t_0) + t_1 \dot{y}_1(t_0) + y_2(t_0)\right]$$

The unperturbed flight time is given by

$$y_0(t_0) = \beta t_0 - \frac{t_0^2}{2} = t_0\left(\beta - \frac{t_0}{2}\right) = 0$$

$$\boxed{t_0 = 2\beta}$$

corresponding to

$$\tau_0 = \tau\big|_{t = t_0} = t_0 - \beta = 2\beta - \beta$$

$$\boxed{\tau_0 = \beta}$$

The subsequent coefficients follow directly:

$$t_1 = y_1(t_0) / -\dot{y}_0(t_0)$$

$$t_2 = \left[\frac{1}{2} t_1^2 \ddot{y}_0(t_0) + t_1 \dot{y}_1(t_0) + y_2(t_0)\right] / -\dot{y}_0(t_0).$$

We now do the actual computations, which requires evaluating the relevant expressions at $t = t_0$, i.e. $\tau = \tau_0 = \beta$.

At $t = t_0$, i.e. $\tau = \tau_0 = \beta$, note that

$$u_0(t_0) = \sqrt{\alpha^2+\beta^2} = 1$$

$$r(t_0) = \frac{\beta+1}{-\beta+1} = \frac{1+\beta}{1-\beta}.$$

For brevity put

$$\boxed{\rho = r(t_0) = \frac{1+\beta}{1-\beta}}.$$

We have

$$\boxed{\ddot{y}_0(t_0) = -1}$$

$$\boxed{\dot{y}_0(t_0) = -\beta}$$

$$\dot{y}_1(t_0) = \frac{1}{3} - \frac{1}{3}$$

$$\boxed{\dot{y}_1(t_0) = 0}$$

$$y_1(t_0) = \left[-\frac{1}{4}+\frac{\alpha^2}{8}\right]\beta - \left[\frac{1}{3}\right]\beta + \left[\frac{\alpha^2}{8}\right]\beta + \left[\frac{1}{12}\right]\beta + \left[\frac{\alpha^4}{8}\right]\log\rho$$

$$= \left[\left(-\frac{1}{4}-\frac{1}{3}+\frac{1}{12}\right)+\left(\frac{1}{8}+\frac{1}{8}\right)\alpha^2\right]\beta + \left[\frac{\alpha^4}{8}\right]\log\rho$$

$$\boxed{y_1(t_0) = \left[-\frac{1}{2}+\frac{\alpha^2}{4}\right]\beta + \left[\frac{\alpha^4}{8}\right]\log\rho}$$

$$y_2(t_0) = \left[\frac{1}{9}-\frac{\alpha^2}{8}+\frac{\alpha^4}{8}-\frac{2\alpha^6}{9}\right] + \left[\frac{1}{5}+\frac{4\alpha^2}{15}+\frac{8\alpha^4}{15}\right]\beta^2 + \left[\frac{5\alpha^4}{24}\right]\beta^2$$
$$-\left[\frac{\alpha^2}{8}\right]\beta^4 - \left[\frac{1}{45}\right]\beta^6 - \left[\frac{\alpha^2\beta}{4}\right]\beta + \left[\frac{1}{9}\right] - \left[\frac{\alpha^4\beta}{4}\right]\log\rho$$
$$-\left[\frac{\alpha^4}{4}\right]\beta\log\rho - \left[\frac{\alpha^6}{8}\right]\log^2\rho$$

$$\cancel{= \left[\frac{1}{9}-\frac{\alpha^2}{8}+\frac{\alpha^4}{8}-\frac{2\alpha^6}{9}\right]}$$

$$y_2(t_0) = \left[\left(\frac{1}{9}+\frac{1}{9}\right) - \frac{\alpha^2}{8} + \frac{\alpha^4}{8} - \frac{2\alpha^6}{9}\right] + \left[\frac{1}{5} + \left(\frac{4}{15}-\frac{1}{4}\right)\alpha^2 + \left(\frac{8}{15}+\frac{5}{24}\right)\alpha^4\right]\beta^2$$
$$- \left[\frac{\alpha^2}{8}\right]\beta^4 - \left[\frac{1}{45}\right]\beta^6 - \left[\frac{1}{4}+\frac{1}{4}\right]\alpha^4\beta\log\rho - \left[\frac{\alpha^6}{8}\right]\log^2\rho$$
$$= \left[\frac{2}{9} - \frac{\alpha^2}{8} + \frac{\alpha^4}{8} - \frac{2\alpha^6}{9}\right] + \left[\frac{1}{5} + \frac{\alpha^2}{60} + \frac{89\alpha^4}{120}\right]\beta^2$$
$$- \left[\frac{\alpha^2}{8}\right]\beta^4 - \left[\frac{1}{45}\right]\beta^6 - \left[\frac{\alpha^4\beta}{2}\right]\log\rho - \left[\frac{\alpha^6}{8}\right]\log^2\rho$$
$$= \left[\frac{2}{9} - \frac{\alpha^2}{8} + \frac{\alpha^4}{8} - \frac{2\alpha^6}{9}\right] + \left[\frac{1}{5} + \frac{\alpha^2}{60} + \frac{89\alpha^4}{120}\right](1-\alpha^2)$$
$$- \left[\frac{\alpha^2}{8}\right](1-2\alpha^2+\alpha^4) - \left[\frac{1}{45}\right](1-3\alpha^2+3\alpha^4-\alpha^6) - \left[\frac{\alpha^4\beta}{2}\right]\log\rho - \left[\frac{\alpha^6}{8}\right]\log^2\rho$$
$$= \left[\left(\frac{2}{9}+\frac{1}{5}-\frac{1}{45}\right) + \left(-\frac{1}{8}+\frac{1}{60}-\frac{1}{5}-\frac{1}{8}+\frac{3}{45}\right)\alpha^2\right.$$
$$\left. + \left(\frac{1}{8}+\frac{89}{120}-\frac{1}{60}+\frac{2}{8}-\frac{3}{45}\right)\alpha^4 + \left(-\frac{2}{9}-\frac{89}{120}-\frac{1}{8}+\frac{1}{45}\right)\alpha^6\right]$$
$$- \left[\frac{\alpha^4\beta}{2}\right]\log\rho - \left[\frac{\alpha^6}{8}\right]\log^2\rho$$

$$\boxed{y_2(t_0) = \left[\frac{2}{5} - \frac{11\alpha^2}{30} + \frac{31\alpha^4}{30} - \frac{16\alpha^6}{15}\right] - \left[\frac{\alpha^4\beta}{2}\right]\log\rho - \left(\frac{\alpha^6}{8}\right)\log^2\rho}$$

Now

$$-\dot{y}_0(t_0) = -(-\beta) = \beta\ ,$$

hence

$$\boxed{t_1 = \left[-\frac{1}{2} + \frac{\alpha^2}{4}\right] + \left[\frac{\alpha^4}{8\beta}\right]\log\rho}\ .$$

~~Also~~

$$\frac{1}{2}t_1^2\ddot{y}_0(t_0)$$
$$= \frac{1}{2}\left(\left[-\frac{1}{2}+\frac{\alpha^2}{4}\right]^2 + 2\left[-\frac{1}{2}+\frac{\alpha^2}{4}\right]\left[\frac{\alpha^4}{8\beta}\right]\log\rho + \left[\frac{\alpha^4}{8\beta}\right]^2\log^2\rho\right)(-1)$$
$$= -\frac{1}{2}\left(\left[\frac{1}{4} - \frac{\alpha^2}{4} + \frac{\alpha^4}{16}\right] + \left[-1+\frac{\alpha^2}{2}\right]\left[\frac{\alpha^4}{8\beta}\right]\log\rho + \left(\frac{\alpha^8}{64\beta^2}\right)\log^2\rho\right)$$
$$= \left[-\frac{1}{8} + \frac{\alpha^2}{8} - \frac{\alpha^4}{32}\right] + \left[1-\frac{\alpha^2}{2}\right]\left[\frac{\alpha^4}{16\beta}\right]\log\rho - \left[\frac{\alpha^8}{128\beta^2}\right]\log^2\rho$$

$$\left[\tfrac{1}{2}t_1^2\ddot{y}_0(t_0)+t_1\dot{y}_1(t_0)+y_2(t_0)\right]$$

$$= \left[-\frac{1}{8}+\frac{\alpha^2}{8}-\frac{\alpha^4}{32}\right]+\left[1-\frac{\alpha^2}{2}\right]\left[\frac{\alpha^4}{16\beta}\right]\log\rho-\left[\frac{\alpha^8}{128\beta^2}\right]\log^2\rho$$

$$+\,0+\left[\frac{2}{5}-\frac{11\alpha^2}{30}+\frac{31\alpha^4}{30}-\frac{16\alpha^6}{15}\right]-\left[\frac{\alpha^4\beta}{2}\right]\log\rho-\left[\frac{\alpha^6}{8}\right]\log^2\rho$$

$$=\left[\left(-\frac{1}{8}+\frac{2}{5}\right)+\left(\frac{1}{8}-\frac{11}{30}\right)\alpha^2+\left(-\frac{1}{32}+\frac{31}{30}\right)\alpha^4-\frac{16\alpha^6}{15}\right]$$

$$\left[\left(1-\frac{\alpha^2}{2}\right)\left(\frac{\alpha^4}{16}\right)-\frac{\alpha^4(1-\alpha^2)}{2}\right]\frac{1}{\beta}\log\rho-\left[\frac{\alpha^8}{128}+\frac{\alpha^6(1-\alpha^2)}{8}\right]\frac{1}{\beta^2}\log^2\rho$$

$$=\left[\frac{11}{40}-\frac{29\alpha^2}{120}+\frac{481\alpha^4}{480}-\frac{16\alpha^6}{15}\right]$$

$$+\left[\left(\frac{1}{16}-\frac{1}{2}\right)\alpha^4+\left(-\frac{1}{32}+\frac{1}{2}\right)\alpha^6\right]\frac{1}{\beta}\log\rho-\left[\frac{\alpha^6}{8}+\left(\frac{1}{128}-\frac{1}{8}\right)\alpha^8\right]\frac{1}{\beta^2}\log^2\rho$$

$$=\left[\frac{11}{40}-\frac{29\alpha^2}{120}+\frac{481\alpha^4}{480}-\frac{16\alpha^6}{15}\right]+\left[-\frac{7\alpha^4}{16}+\frac{15\alpha^6}{32}\right]\frac{\log\rho}{\beta}+\left[-\frac{\alpha^6}{8}+\frac{15\alpha^8}{128}\right]\frac{\log^2\rho}{\beta^2}$$

Hence

$$\boxed{t_2=\left[\frac{11}{40}-\frac{29\alpha^2}{120}+\frac{481\alpha^4}{480}-\frac{16\alpha^6}{15}\right]\frac{1}{\beta}+\left[-\frac{7\alpha^4}{16}+\frac{15\alpha^6}{32}\right]\frac{\log\rho}{\beta^2}+\left[-\frac{\alpha^6}{8}+\frac{15\alpha^8}{128}\right]\frac{\log^2\rho}{\beta^3}}$$

Range

$$x(t) = (x_0 + Bx_1 + B^2 x_2)\Big|_{t = t_0 + Bt_1 + B^2 t_2}$$

$$= x_0(t_0) + (Bt_1 + B^2 t_2)\dot{x}_0(t_0) + \cancel{\tfrac{1}{2!}(Bt_1)^2 \ddot{x}_0(t_0)}$$
$$+ Bx_1(t_0) + (Bt_1)B\dot{x}_1(t_0)$$
$$+ B^2 x_2(t_0)$$

$$= x_0(t_0) + B[t_1 \dot{x}_0(t_0) + x_1(t_0)]$$
$$+ B^2[t_2 \dot{x}_0(t_0) + t_1 \dot{x}_1(t_0) + x_2(t_0)]$$

$$= R_0 + B \cdot R_1 + B^2 R_2 ;$$

this is the range R to ~~quad~~ quadratic order.

We have

$$\boxed{\dot{x}_0(t_0) = \alpha}$$

$$x_0(t_0) = \alpha t_0 = \alpha \cdot 2\beta$$

$$\boxed{x_0(t_0) = 2\alpha\beta}$$

$$\dot{x}_1(t_0) = -\left[\frac{\alpha\beta}{2}\right] - \left[\frac{\alpha}{2}\right]\beta - \left[\frac{\alpha^3}{2}\right]\log\rho$$

$$\boxed{\dot{x}_1(t_0) = -[\alpha\beta] - \left[\frac{\alpha^3}{2}\right]\log\rho}$$

$$x_1(t_0) = -\left[\frac{\alpha}{3}\right] - \left[\frac{\alpha\beta}{2}\right]\beta + \left[\frac{\alpha^3}{2}\right] - \left[\frac{\alpha}{6}\right] - \left[\frac{\alpha^3}{2}\right]\beta\log\rho$$
$$= \left[\left(-\frac{1}{3} - \frac{1}{6}\right)\alpha + \left(-\frac{1}{2}\alpha(1-\alpha^2) + \frac{\alpha^3}{2}\right)\right] - \left[\frac{\alpha^3}{2}\right]\beta\log\rho$$
$$= \left[-\frac{\alpha}{2} - \frac{\alpha}{2} + \frac{\alpha^3}{2} + \frac{\alpha^3}{2}\right] - \left[\frac{\alpha^3}{2}\right]\beta\log\rho$$

$$\boxed{x_1(t_0) = [-\alpha + \alpha^3] - \left[\frac{\alpha^3}{2}\right]\beta\log\rho}$$

$$x_2(t_0) = \left[\frac{\alpha}{6} + \frac{4\alpha^3}{9} + \frac{8\alpha^5}{9}\right]\beta + \left[\frac{3\alpha}{8} - \frac{3\alpha^3}{8} + \alpha^5\right]\beta + \left[\frac{11\alpha^3}{72}\right]\beta^3 + \left[\frac{\alpha}{24}\right]\beta^5$$
$$- \left[\frac{3\alpha^3\beta}{4}\right] - \left[\frac{\alpha}{6}\right]\beta + \left[\frac{\alpha\beta}{12}\right] - \left[\frac{\alpha^3}{6}\right]\log\rho + \left[\frac{3\alpha^3\beta}{4}\right]\beta\log\rho$$
$$- \left[\frac{3\alpha^5}{4}\right]\log\rho + \left[\frac{\alpha^3}{12}\right]\log\rho + \left[\frac{3\alpha^5}{8}\right]\beta\log^2\rho$$

$$= \left[\frac{\alpha}{6} + \frac{4\alpha^3}{9} + \frac{8\alpha^5}{9} + \frac{3\alpha}{8} - \frac{3\alpha^3}{8} + \alpha^5 + \frac{11\alpha^3}{72}\cdot(1-\alpha^2) + \frac{\alpha}{24}\cdot(1-2\alpha^2+\alpha^4)\right.$$
$$\left. - \frac{3\alpha^3}{4} - \frac{\alpha}{6} + \frac{\alpha}{12}\right]\beta + \left[-\frac{\alpha^3}{6} + \frac{3\alpha^3}{4}(1-\alpha^2) - \frac{3\alpha^5}{4} + \frac{\alpha^3}{12}\right]\log\rho$$
$$+ \left[\frac{3\alpha^5}{8}\right]\beta\log^2\rho$$

$$= \left[\left(\frac{1}{6} + \frac{3}{8} + \frac{1}{24} - \frac{1}{6} + \frac{1}{12}\right)\alpha + \left(\frac{4}{9} - \frac{3}{8} + \frac{11}{72} - \frac{2}{24} - \frac{3}{4}\right)\alpha^3\right.$$
$$\left. + \left(\frac{8}{9} + 1 - \frac{11}{72} + \frac{1}{24}\right)\alpha^5\right]\beta$$
$$+ \left[\left(-\frac{1}{6} + \frac{3}{4} + \frac{1}{12}\right)\alpha^3 + \left(-\frac{3}{4} - \frac{3}{4}\right)\alpha^5\right]\log\rho + \left[\frac{3\alpha^5}{8}\right]\beta\log^2\rho$$

$$\boxed{x_2(t_0) = \left[\frac{\alpha}{2} - \frac{11\alpha^3}{18} + \frac{16\alpha^5}{9}\right]\beta + \left[\frac{2\alpha^3}{3} - \frac{3\alpha^5}{2}\right]\log\rho + \left[\frac{3\alpha^5}{8}\right]\beta\log^2\rho}$$

$$R_0 = x_0(t_0)$$

$$\boxed{R_0 = 2\alpha\beta}$$

$$\cancel{R_1 = t_1\dot{x}_0(t_0) + x_1(t_0)}$$

$$\cancel{\boxed{R_1 = \left[-\frac{\alpha}{2} + \frac{\alpha^3}{4}\right] + \left[\frac{\alpha^5}{8\beta}\right]\log\rho}}$$

$$R_1 = t_1\dot{x}_0(t_0) + x_1(t_0)$$
$$= \left[-\frac{\alpha}{2} + \frac{\alpha^3}{4}\right] + \left[\frac{\alpha^5}{8\beta}\right]\log\rho + \left[-\alpha + \alpha^3\right] - \left[\frac{\alpha^3}{2}\right]\beta\log\rho$$
$$= \left[\left(-\frac{1}{2} - 1\right)\alpha + \left(\frac{1}{4} + 1\right)\alpha^3\right] + \left[\frac{\alpha^5}{8} - \frac{\alpha^3}{2}(1-\alpha^2)\right]\frac{\log\rho}{\beta}$$

$$\boxed{R_1 = \left[-\frac{3\alpha}{2} + \frac{5\alpha^3}{4}\right] + \left[-\frac{\alpha^3}{2} + \frac{5\alpha^5}{8}\right]\frac{\log\rho}{\beta}}$$

$t_1 \dot{x}_1(t_0)$

$$= \left(\left[-\frac{1}{2}+\frac{\alpha^2}{4}\right]+\left[\frac{\alpha^4}{8\beta}\right]\log\rho\right)\left(-[\alpha\beta]-\left[\frac{\alpha^3}{2}\right]\log\rho\right)$$

$$= \left[\frac{\alpha}{2}-\frac{\alpha^3}{4}\right]\beta+\left[\frac{\alpha^3}{4}-\frac{\alpha^5}{8}-\frac{\alpha^5}{8}\right]\log\rho-\left[\frac{\alpha^7}{16\beta}\right]\log^2\rho$$

$$= \left[\frac{\alpha}{2}-\frac{\alpha^3}{4}\right]\beta+\left[\frac{\alpha^3}{4}-\frac{\alpha^5}{4}\right]\log\rho-\left[\frac{\alpha^7}{16\beta}\right]\log^2\rho$$

$$R_2 = t_2\dot{x}_0(t_0)+t_1\dot{x}_1(t_0)+x_2(t_0)$$

$$= \left[\frac{11\alpha}{40}-\frac{29\alpha^3}{120}+\frac{481\alpha^5}{480}-\frac{16\alpha^7}{15}\right]\frac{1}{\beta}+\left[-\frac{7\alpha^5}{16}+\frac{15\alpha^7}{32}\right]\frac{\log\rho}{\beta^2}$$

$$+\left[-\frac{\alpha^7}{8}+\frac{15\alpha^9}{128}\right]\frac{\log^2\rho}{\beta^3}$$

$$+\left[\frac{\alpha}{2}-\frac{\alpha^3}{4}\right]\beta+\left[\frac{\alpha^3}{4}-\frac{\alpha^5}{4}\right]\log\rho-\left[\frac{\alpha^7}{16\beta}\right]\log^2\rho$$

$$+\left[\frac{\alpha}{2}-\frac{11\alpha^3}{18}+\frac{16\alpha^5}{9}\right]\beta+\left[\frac{2\alpha^3}{3}-\frac{3\alpha^5}{2}\right]\log\rho+\left[\frac{3\alpha^5}{8}\right]\beta\log^2\rho$$

$$= \left[\frac{11\alpha}{40}-\frac{29\alpha^3}{120}+\frac{481\alpha^5}{480}-\frac{16\alpha^7}{15}+\left(\frac{\alpha}{2}-\frac{\alpha^3}{4}\right)(1-\alpha^2)+\left(\frac{\alpha}{2}-\frac{11\alpha^3}{18}+\frac{16\alpha^5}{9}\right)(1-\alpha^2)\right]\frac{1}{\beta}$$

$$+\left[-\frac{7\alpha^5}{16}+\frac{15\alpha^7}{32}+\left(\frac{\alpha^3}{4}-\frac{\alpha^5}{4}\right)(1-\alpha^2)+\left(\frac{2\alpha^3}{3}-\frac{3\alpha^5}{2}\right)(1-\alpha^2)\right]\frac{\log\rho}{\beta^2}$$

$$+\left[-\frac{\alpha^7}{8}+\frac{15\alpha^9}{128}-\frac{\alpha^7}{16}(1-\alpha^2)+\frac{3\alpha^5}{8}(1-2\alpha^2+\alpha^4)\right]\frac{\log^2\rho}{\beta^3}$$

$$= \left[\left(\frac{11}{40}+\frac{1}{2}+\frac{1}{2}\right)\alpha+\left(-\frac{29}{120}-\frac{1}{2}-\frac{1}{4}-\frac{1}{2}-\frac{11}{18}\right)\alpha^3+\left(\frac{481}{480}+\frac{1}{4}+\frac{11}{18}+\frac{16}{9}\right)\alpha^5\right.$$

$$\left.+\left(-\frac{16}{15}-\frac{16}{9}\right)\alpha^7\right]\frac{1}{\beta}$$

$$+\left[\left(\frac{1}{4}+\frac{2}{3}\right)\alpha^3+\left(-\frac{7}{16}-\frac{1}{4}-\frac{1}{4}-\frac{2}{3}-\frac{3}{2}\right)\alpha^5+\left(\frac{15}{32}+\frac{1}{4}+\frac{3}{2}\right)\alpha^7\right]\frac{\log\rho}{\beta^2}$$

$$+\left[\frac{3\alpha^5}{8}+\left(-\frac{1}{8}-\frac{1}{16}-\frac{3}{4}\right)\alpha^7+\left(\frac{15}{128}+\frac{1}{16}+\frac{3}{8}\right)\alpha^9\right]\frac{\log^2\rho}{\beta^3}$$

$$\boxed{\begin{aligned} R_2 &= \left[\frac{51\alpha}{40}-\frac{757\alpha^3}{360}+\frac{5243\alpha^5}{1440}-\frac{128\alpha^7}{45}\right]\frac{1}{\beta} \\ &+\left[\frac{11\alpha^3}{12}-\frac{149\alpha^5}{48}+\frac{71\alpha^7}{32}\right]\frac{\log\rho}{\beta^2}+\left[\frac{3\alpha^5}{8}-\frac{15\alpha^7}{16}+\frac{71\alpha^9}{128}\right]\frac{\log^2\rho}{\beta^3}\end{aligned}}$$

Optimal launch angle

$$
\begin{aligned}
0 &= R'(\phi) \\
&= \left(R_0' + BR_1' + B^2R_2'\right)\Big|_{\phi=\phi_0+B\phi_1+B^2\phi_2} \\
&= R_0'(\phi_0) + (B\phi_1 + B^2\phi_2)\,R_0''(\phi_0) + \frac{1}{2!}(B\phi_1)^2 R_0'''(\phi_0) \\
&\quad + BR_1'(\phi_0) + (B\phi_1)BR_1''(\phi_0) \\
&\quad + B^2R_2'(\phi_0) \\
&= R_0'(\phi_0) + B\left[\phi_1 R_0''(\phi_0) + R_1'(\phi_0)\right] \\
&\quad + B^2\left[\phi_2 R_0''(\phi_0) + \tfrac{1}{2}\phi_1^2 R_0'''(\phi_0) + \phi_1 R_1''(\phi_0) + R_2'(\phi_0)\right]
\end{aligned}
$$

The unperturbed optimal launch angle is given by

$$
\begin{aligned}
R_0'(\phi_0) &= \frac{d}{d\phi}\left[2\alpha\beta\right]\Big|_{\phi=\phi_0} \\
&= \frac{d}{d\phi}\left[2\cos\phi\sin\phi\right]\Big|_{\phi=\phi_0} \\
&= \frac{d}{d\phi}\sin(2\phi)\Big|_{\phi=\phi_0} \\
&= 2\cos(2\phi_0) \\
&= 0
\end{aligned}
$$

yielding the familiar

$$\boxed{\phi_0 = \frac{\pi}{4}},$$

in the absence of air resistance.

The subsequent coefficients follow directly:

$$\phi_1 = R_1'(\phi_0) \,/\, -R_0''(\phi_0)$$

$$\phi_2 = \left[\tfrac{1}{2}\phi_1^2 R_0'''(\phi_0) + \phi_1 R_1''(\phi_0) + R_2'(\phi_0)\right] \,/\, -R_0''(\phi_0)$$

(Lots of differentiation ahead)

We have

$$R_0 = 2\alpha\beta = \sin(2\phi)$$
$$R_0' = 2\cos(2\phi)$$
$$R_0'' = -4\sin(2\phi)$$
$$R_0''' = -8\cos(2\phi)$$

hence

$$R_0''(\phi_0) = -4\sin\frac{\pi}{2}$$
$$\boxed{R_0''(\phi_0) = -4}$$
$$R_0'''(\phi_0) = -8\cos\frac{\pi}{2}$$
$$\boxed{R_0'''(\phi_0) = 0}$$

Note that $\alpha = \cos\phi$, $\beta = \sin\phi$, so

$$\boxed{\begin{aligned} \alpha' &= -\beta \\ \beta' &= \alpha \end{aligned}}\ ;$$

also

$$\log\rho = \log(1+\beta) - \log(1-\beta)$$

so

$$(\log\rho)' = \frac{\beta'}{1+\beta} + \frac{\beta'}{1-\beta}$$
$$= \alpha\left(\frac{1}{1+\beta} + \frac{1}{1-\beta}\right)$$
$$= \alpha \cdot \frac{2}{1-\beta^2}$$
$$= \alpha \cdot \frac{2}{\alpha^2}$$

$$\boxed{(\log\rho)' = \frac{2}{\alpha}}$$

$$R_1 = \left[-\frac{3\alpha}{2} + \frac{5\alpha^3}{4}\right] + \left[-\frac{\alpha^3}{2} + \frac{5\alpha^5}{8}\right]\frac{\log\rho}{\beta}$$

$$R_1' = \left[-\frac{3}{2} + \frac{15\alpha^2}{4}\right]\alpha' + \left[-\frac{\alpha^3}{2} + \frac{5\alpha^5}{8}\right]\left\{\frac{(\log\rho)'}{\beta} - \frac{\beta'\log\rho}{\beta^2}\right\} + \left[-\frac{3\alpha^2}{2} + \frac{25\alpha^4}{8}\right]\alpha'\cdot\frac{\log\rho}{\beta}$$

$$= \left[\frac{3}{2} - \frac{15\alpha^2}{4}\right]\beta + \left[-\frac{\alpha^3}{2} + \frac{5\alpha^5}{8}\right]\left\{\frac{2}{\alpha\beta} - \frac{\alpha\log\rho}{\beta^2}\right\} + \left[\frac{3\alpha^2}{2} - \frac{25\alpha^4}{8}\right]\log\rho$$

$$= \left[\frac{3}{2} - \frac{15\alpha^2}{4}\right]\beta + \left[-\alpha^2 + \frac{5\alpha^4}{4}\right]\frac{1}{\beta} + \left[\frac{\alpha^4}{2} - \frac{5\alpha^6}{8}\right]\frac{\log\rho}{\beta^2} + \left[\frac{3\alpha^2}{2} - \frac{25\alpha^4}{8}\right]\log\rho$$

$$= \left[\left(\frac{3}{2} - \frac{15\alpha^2}{4}\right)(1-\alpha^2) - \alpha^2 + \frac{5\alpha^4}{4}\right]\frac{1}{\beta} + \left[\frac{\alpha^4}{2} - \frac{5\alpha^6}{8} + \left(\frac{3\alpha^2}{2} - \frac{25\alpha^4}{8}\right)(1-\alpha^2)\right]\frac{\log\rho}{\beta^2}$$

$$= \left[\frac{3}{2} + \left(-\frac{3}{2} - \frac{15}{4} - 1\right)\alpha^2 + \left(\frac{15}{4} + \frac{5}{4}\right)\alpha^4\right]\frac{1}{\beta} + \left[\frac{3\alpha^2}{2} + \left(\frac{1}{2} - \frac{3}{2} - \frac{25}{8}\right)\alpha^4 + \left(-\frac{5}{8} + \frac{25}{8}\right)\alpha^6\right]\frac{\log\rho}{\beta^2}$$

$$= \left[\frac{3}{2} - \frac{25\alpha^2}{4} + 5\alpha^4\right]\frac{1}{\beta} + \left[\frac{3\alpha^2}{2} - \frac{33\alpha^4}{8} + \frac{5\alpha^6}{2}\right]\frac{\log\rho}{\beta^2}.$$

$$R_1'' = \left[\frac{3}{2} - \frac{25\alpha^2}{4} + 5\alpha^4\right]\frac{-\beta'}{\beta^2} + \left[-\frac{25\alpha}{2} + 20\alpha^3\right]\frac{\alpha'}{\beta} + \left[\frac{3\alpha^2}{2} - \frac{33\alpha^4}{8} + \frac{5\alpha^6}{2}\right]\left\{\frac{(\log\rho)'}{\beta^2} - \frac{2\beta'\log\rho}{\beta^3}\right\} + \left[3\alpha - \frac{33\alpha^3}{2} + 15\alpha^5\right]\alpha'\cdot\frac{\log\rho}{\beta^2}$$

$$= \left[-\frac{3\alpha}{2} + \frac{25\alpha^3}{4} - 5\alpha^5\right]\frac{1}{\beta^2} + \left[\frac{25\alpha}{2} - 20\alpha^3\right] + \left[\frac{3\alpha^2}{2} - \frac{33\alpha^4}{8} + \frac{5\alpha^6}{2}\right]\left\{\frac{2}{\alpha\beta^2} - \frac{2\alpha\log\rho}{\beta^3}\right\} + \left[-3\alpha + \frac{33\alpha^3}{2} - 15\alpha^5\right]\frac{\log\rho}{\beta}$$

$$R_1'' = \left[-\frac{3\alpha}{2} + \frac{25\alpha^3}{4} - 5\alpha^5 + \left(\frac{25\alpha}{2} - 20\alpha^3\right)(1-\alpha^2) + 3\alpha - \frac{33\alpha^3}{4} + 5\alpha^5\right]\frac{1}{\beta^2}$$
$$+ \left[-3\alpha^3 + \frac{33\alpha^5}{4} - 5\alpha^7 + \left(-3\alpha + \frac{33\alpha^3}{2} - 15\alpha^5\right)(1-\alpha^2)\right]\frac{\log\rho}{\beta^3}$$
$$= \left[\left(-\frac{3}{2} + \frac{25}{2} + 3\right)\alpha + \left(\frac{25}{4} - \frac{25}{2} - 20 - \frac{33}{4}\right)\alpha^3 + (-5+20+5)\alpha^5\right]\frac{1}{\beta^2}$$
$$+ \left[-3\alpha + \left(-3+3+\frac{33}{2}\right)\alpha^3 + \left(\frac{33}{4} - \frac{33}{2} - 15\right)\alpha^5 + (-5+15)\alpha^7\right]\frac{\log\rho}{\beta^3}$$
$$= \left[14\alpha - \frac{69\alpha^3}{2} + 20\alpha^5\right]\frac{1}{\beta^2} + \left[-3\alpha + \frac{33\alpha^3}{2} - \frac{93\alpha^5}{4} + 10\alpha^7\right]\frac{\log\rho}{\beta^3}$$

$$R_2 = \left[\frac{51\alpha}{40} - \frac{757\alpha^3}{360} + \frac{5243\alpha^5}{1440} - \frac{128\alpha^7}{45}\right]\frac{1}{\beta}$$
$$+ \left[\frac{11\alpha^3}{12} - \frac{149\alpha^5}{48} + \frac{71\alpha^7}{32}\right]\frac{\log\rho}{\beta^2} + \left[\frac{3\alpha^5}{8} - \frac{15\alpha^7}{16} + \frac{71\alpha^9}{128}\right]\frac{\log^2\rho}{\beta^3}$$

$$R_2' = \left[\frac{51\alpha}{40} - \frac{757\alpha^3}{360} + \frac{5243\alpha^5}{1440} - \frac{128\alpha^7}{45}\right]\frac{-\beta'}{\beta^2}$$
$$+ \left[\frac{51}{40} - \frac{757\alpha^2}{120} + \frac{5243\alpha^4}{288} - \frac{896\alpha^6}{45}\right]\frac{\alpha'}{\beta}$$
$$+ \left[\frac{11\alpha^3}{12} - \frac{149\alpha^5}{48} + \frac{71\alpha^7}{32}\right]\left\{\frac{(\log\rho)'}{\beta^2} - \frac{2\beta'\log\rho}{\beta^3}\right\}$$
$$+ \left[\frac{11\alpha^2}{4} - \frac{745\alpha^4}{48} + \frac{497\alpha^6}{32}\right]\frac{\alpha'\log\rho}{\beta^2}$$
$$+ \left[\frac{3\alpha^5}{8} - \frac{15\alpha^7}{16} + \frac{71\alpha^9}{128}\right]\left\{\frac{2\log\rho(\log\rho)'}{\beta^3} - \frac{3\beta'\log^2\rho}{\beta^4}\right\}$$
$$+ \left[\frac{15\alpha^4}{8} - \frac{105\alpha^6}{16} + \frac{639\alpha^8}{128}\right]\frac{\alpha'\log^2\rho}{\beta^3}$$

$$
\begin{aligned}
R_2' = {} & \left[-\frac{51\alpha^2}{40}+\frac{757\alpha^4}{360}-\frac{5243\alpha^6}{1440}+\frac{128\alpha^8}{45}\right]\frac{1}{\beta^2} \\
& +\left[-\frac{51}{40}+\frac{757\alpha^2}{120}-\frac{5243\alpha^4}{288}+\frac{896\alpha^6}{45}\right] \\
& +\left[\frac{11\alpha^3}{12}-\frac{149\alpha^5}{48}+\frac{71\alpha^7}{32}\right]\left\{\frac{2}{\alpha\beta^2}-\frac{2\alpha\log\rho}{\beta^3}\right\} \\
& +\left[-\frac{11\alpha^2}{4}+\frac{745\alpha^4}{48}-\frac{497\alpha^6}{32}\right]\frac{\log\rho}{\beta} \\
& +\left[\frac{3\alpha^5}{8}-\frac{15\alpha^7}{16}+\frac{71\alpha^9}{128}\right]\left\{\frac{4\log\rho}{\alpha\beta^3}-\frac{3\alpha\log^2\rho}{\beta^4}\right\} \\
& +\left[-\frac{15\alpha^4}{8}+\frac{105\alpha^6}{16}-\frac{639\alpha^8}{128}\right]\frac{\log^2\rho}{\beta^2} \\
= {} & \left[-\frac{51\alpha^2}{40}+\frac{757\alpha^4}{360}-\frac{5243\alpha^6}{1440}+\frac{128\alpha^8}{45}+\left(-\frac{51}{40}+\frac{757\alpha^2}{120}-\frac{5243\alpha^4}{288}+\frac{896\alpha^6}{45}\right)(1-\alpha^2)\right. \\
& \quad \left.+\frac{11\alpha^2}{6}-\frac{149\alpha^4}{24}+\frac{71\alpha^6}{16}\right]\frac{1}{\beta^2} \\
& +\left[-\frac{11\alpha^4}{6}+\frac{149\alpha^6}{24}-\frac{71\alpha^8}{16}+\left(-\frac{11\alpha^2}{4}+\frac{745\alpha^4}{48}-\frac{497\alpha^6}{32}\right)(1-\alpha^2)\right. \\
& \quad \left.+\frac{3\alpha^4}{2}-\frac{15\alpha^6}{4}+\frac{71\alpha^8}{32}\right]\frac{\log\rho}{\beta^3} \\
& +\left[-\frac{9\alpha^6}{8}+\frac{45\alpha^8}{16}-\frac{213\alpha^{10}}{128}+\left(-\frac{15\alpha^4}{8}+\frac{105\alpha^6}{16}-\frac{639\alpha^8}{128}\right)(1-\alpha^2)\right]\frac{\log^2\rho}{\beta^4} \\
= {} & \left[-\frac{51}{40}+\left(-\frac{51}{40}+\frac{51}{40}+\frac{757}{120}+\frac{11}{6}\right)\alpha^2+\left(\frac{757}{360}-\frac{757}{120}-\frac{5243}{288}-\frac{149}{24}\right)\alpha^4\right. \\
& \quad \left.+\left(-\frac{5243}{1440}+\frac{5243}{288}+\frac{896}{45}+\frac{71}{16}\right)\alpha^6+\left(\frac{128}{45}-\frac{896}{45}\right)\alpha^8\right]\frac{1}{\beta^2} \\
& +\left[-\frac{11\alpha^2}{4}+\left(-\frac{11}{6}+\frac{11}{4}+\frac{745}{48}+\frac{3}{2}\right)\alpha^4+\left(\frac{149}{24}-\frac{745}{48}-\frac{497}{32}-\frac{15}{4}\right)\alpha^6\right. \\
& \quad \left.+\left(-\frac{71}{16}+\frac{497}{32}+\frac{71}{32}\right)\alpha^8\right]\frac{\log\rho}{\beta^3} \\
& +\left[-\frac{15\alpha^4}{8}+\left(-\frac{9}{8}+\frac{15}{8}+\frac{105}{16}\right)\alpha^6+\left(\frac{45}{16}-\frac{105}{16}-\frac{639}{128}\right)\alpha^8+\left(-\frac{213}{128}+\frac{639}{128}\right)\alpha^{10}\right]\frac{\log^2\rho}{\beta^4}
\end{aligned}
$$

$$R_2' = \left[-\frac{51}{40} + \frac{977\alpha^2}{120} - \frac{4579\alpha^4}{160} + \frac{3113\alpha^6}{80} - \frac{256\alpha^8}{15}\right]\frac{1}{\beta^2}$$

$$+\left[-\frac{11\alpha^2}{4} + \frac{287\alpha^4}{16} - \frac{915\alpha^6}{32} + \frac{213\alpha^8}{16}\right]\frac{\log\beta}{\beta^3}$$

$$+\left[-\frac{15\alpha^4}{8} + \frac{117\alpha^6}{16} - \frac{1119\alpha^8}{128} + \frac{213\alpha^{10}}{64}\right]\frac{\log^2\beta}{\beta^4}$$

Observe that

$$\alpha|_{\phi=\phi_0} = \beta|_{\phi=\phi_0} = \frac{1}{\sqrt{2}}.$$

For brevity, let

$$\begin{aligned} p &= \log \rho|_{\phi=\phi_0} \\ &= \log \frac{1+1/\sqrt{2}}{1-1/\sqrt{2}} \\ &= 2\tanh^{-1}\frac{1}{\sqrt{2}} \\ &= 2\coth^{-1}\sqrt{2}. \end{aligned}$$

Then

$$\begin{aligned} R_1'(\phi_0) &= \left[\frac{3}{2} - \frac{25}{4}\cdot\frac{1}{2} + 5\cdot\frac{1}{2^2}\right]\sqrt{2} \\ &\quad + \left[\frac{3}{2}\cdot\frac{1}{2} - \frac{33}{8}\cdot\frac{1}{2^2} + \frac{5}{2}\cdot\frac{1}{2^3}\right]2p \end{aligned}$$

$$\boxed{R_1'(\phi_0) = -\frac{3}{8}\sqrt{2} + \frac{p}{16}}$$

$$\begin{aligned} R_1''(\phi_0) &= \left[14 - \frac{69}{2}\cdot\frac{1}{2} + 20\cdot\frac{1}{2^2}\right]\frac{1}{\sqrt{2}}\cdot 2 \\ &\quad + \left[-3 + \frac{33}{2}\cdot\frac{1}{2} - \frac{93}{4}\cdot\frac{1}{2^2} + 10\cdot\frac{1}{2^3}\right]\frac{1}{\cancel{\sqrt{2}}}\cdot p\cdot 2\cancel{\sqrt{2}} \end{aligned}$$

$$\boxed{R_1''(\phi_0) = \frac{7}{4}\sqrt{2} + \frac{11p}{8}}$$

$$R_2'(\phi_0) = \left[-\frac{51}{40} + \frac{977}{120}\cdot\frac{1}{2} - \frac{4579}{160}\cdot\frac{1}{2^2} + \frac{3113}{80}\cdot\frac{1}{2^3} - \frac{256}{15}\cdot\frac{1}{2^4}\right]2$$

$$+\left[-\frac{11}{4}\cdot\frac{1}{2} + \frac{287}{16}\cdot\frac{1}{2^2} - \frac{915}{32}\cdot\frac{1}{2^3} + \frac{213}{16}\cdot\frac{1}{2^4}\right]2\sqrt{2}\cdot p$$

$$+\left[-\frac{15}{8}\cdot\frac{1}{2^2} + \frac{117}{16}\cdot\frac{1}{2^3} - \frac{1119}{128}\cdot\frac{1}{2^4} + \frac{213}{64}\cdot\frac{1}{2^5}\right]2^2\cdot p^2$$

$$\boxed{R_2'(\phi_0) = -\frac{539}{480} + \frac{47}{64}p\sqrt{2} + \frac{3}{256}p^2}$$

Hence

$$\phi_1 = R_1'(\phi_0)/-R_0''(\phi_0)$$

$$= \left[-\frac{3\sqrt{2}}{8} + \frac{p}{16}\right]/4$$

$$\boxed{\phi_1 = -\frac{3\sqrt{2}}{32} + \frac{p}{64}}$$

$$\phi_2 = \left[\cancel{\tfrac{1}{2}\phi_1^2 R_0'''(\phi_0)} + \phi_1 R_1''(\phi_0) + R_2'(\phi_0)\right]/-R_0''(\phi_0)$$

$$= \left[\left(-\frac{3\sqrt{2}}{32} + \frac{p}{64}\right)\left(\frac{7\sqrt{2}}{4} + \frac{11p}{8}\right) - \frac{539}{480} + \frac{47p\sqrt{2}}{64} + \frac{3p^2}{256}\right]/4$$

$$= \left[\left(-\frac{3}{32}\cdot\frac{7}{4}\cdot 2 - \frac{539}{480}\right) + \left(-\frac{3}{32}\cdot\frac{11}{8} + \frac{1}{64}\cdot\frac{7}{4} + \frac{47}{64}\right)p\sqrt{2}\right.$$

$$\left. + \left(\frac{1}{64}\cdot\frac{11}{8} + \frac{3}{256}\right)p^2\right]/4$$

$$= \left[-\frac{1393}{960} + \frac{81p\sqrt{2}}{128} + \frac{17p^2}{512}\right]/4$$

$$\boxed{\phi_2 = -\frac{1393}{3840} + \frac{81p\sqrt{2}}{512} + \frac{17p^2}{2048}}$$

Result

To quadratic order in $B = \frac{bu^2}{mg}$, the optimal launch angle is

$$\phi = \phi_0 + B\phi_1 + B^2\phi_2,$$

where

$$\phi_0 = \frac{\pi}{4} = 45^\circ$$

$$\phi_1 = -\left(\frac{3\sqrt{2}}{32} - \frac{p}{64}\right)$$

$$= -\left(\frac{3\sqrt{2}}{32} - \frac{1}{64}\log\frac{1+1/\sqrt{2}}{1-1/\sqrt{2}}\right) = -6.02^\circ$$

$$\phi_2 = -\frac{1393}{3840} + \frac{81p\sqrt{2}}{512} + \frac{17p^2}{2048}$$

$$= -\frac{1393}{3840} + \frac{81\sqrt{2}}{512}\log\frac{1+1/\sqrt{2}}{1-1/\sqrt{2}} + \frac{17}{2048}\log^2\frac{1+1/\sqrt{2}}{1-1/\sqrt{2}}$$

$$= 3.29^\circ,$$

(In the above,

$$p = \log\frac{1+1/\sqrt{2}}{1-1/\sqrt{2}} = 2\tanh^{-1}\frac{1}{\sqrt{2}} = 2\coth^{-1}\sqrt{2} = 1.76.)$$

i.e.

$$\boxed{\phi = 45^\circ - 6.02^\circ B + 3.29^\circ B^2},$$

for $B = \frac{bu^2}{mg} = \frac{u^2}{c^2} \ll 1$.