EIOBook-8: Dynamic Games (Model & Methods)

Fast, intuitive, and highly coherent review notes (with inner connections)

Coherence spine (read this first)

Dynamic games extend dynamic programming by adding strategic interaction. Once today's action changes tomorrow's state, firms are forward-looking; once rivals' actions affect payoffs and state transitions, each firm's best response depends on rivals' strategies. Equilibrium therefore becomes a **fixed point in strategy space**. The chapter progresses by systematically reducing this fixed-point problem into tractable empirical objects: it uses Markov Perfect Equilibrium (MPE) to restrict strategies to depend only on payoff-relevant states; it replaces high-dimensional strategies with Conditional Choice Probabilities (CCPs) that can be estimated from data; it develops valuation formulas that compute continuation values given CCPs; it presents estimation approaches that differ by how explicitly they enforce the equilibrium fixed point; and it addresses practical obstacles—unobserved heterogeneity, curse of dimensionality, and equilibrium selection in counterfactuals—because these issues break naive likelihood and make counterfactual equilibria ambiguous.

The one mental picture

Dynamic games = (state today) + (my action today) + (rivals' actions today) $\rightarrow$ (future state), with forward-looking firms. Empirically:

data on states and actions $\Rightarrow$ estimate CCPs $\Rightarrow$ recover primitives $\Rightarrow$ compute counterfactual equilibria.

The "missing object" logic is: dynamics introduces continuation values; strategic interaction turns continuation values into a fixed point.

Connected chain of ideas (each point motivates the next)

1) Dynamic BR as the entry-to-dynamics bridge

Connection tag: static entry/count models miss sunk costs $\Rightarrow$ market structure becomes state-dependent.

What it solves. Dynamic Bresnahan–Reiss (BR) illustrates why dynamics matters: with entry cost EC and scrap value SV, sunk cost $EC - SV$ generates **persistence** in the number of firms n_t (incumbency matters).

Missing object it reveals. A payoff-relevant **state variable** (e.g., incumbency/previous structure) must enter profits. Static models that condition only on current demand shifters cannot capture this.

Why the next step is needed. Once state dependence is admitted, we need a general dynamic-game framework for multiple firms and richer actions than "enter or not."

2) General dynamic oligopoly game: static competition + dynamic actions

Connection tag: define payoffs and transitions ⇒ need an equilibrium concept for forward-looking strategic interaction.

Each period typically contains:

- a **static stage game** (e.g., Bertrand/Cournot) that maps the current state into variable profits, and
- a **dynamic action** (entry/exit, capacity, quality, R&D) that changes the next state.

Write profits schematically as

$$\Pi_{it} \;=\; VP_{it}(x_t, a_t) \;-\; FC_{it}(x_t, a_t) \;-\; EC_{it}(x_t, a_t),$$

and a controlled transition

$$x_{t+1} \sim F(\cdot \mid x_t, a_t).$$

Missing object it creates. A firm's continuation value depends on rivals' future behavior, so we need an equilibrium notion for intertemporal strategic interaction.

3) Markov Perfect Equilibrium (MPE)

Connection tag: restrict strategies to payoff-relevant states ⇒ still too high-dimensional for estimation.

What it solves. MPE restricts strategies to depend only on payoff-relevant state variables x (not the full history). With private shocks ε_i, firm i uses a Markov strategy $\alpha_i(x, \varepsilon_i)$ and is optimal given rivals' strategies.

Missing object it creates. Even with Markov restrictions, strategy functions $\alpha_i(x, \varepsilon_i)$ are extremely high-dimensional objects and are not directly estimable from typical datasets.

Why the next step is needed. We need a lower-dimensional object that both summarizes strategies and is directly learnable from observed behavior.

4) Conditional Choice Probabilities (CCPs)

Connection tag: compress strategies into estimable objects ⇒ still need continuation values to connect to primitives.

Define the CCP:

$$P_i(a \mid x) \;=\; \Pr\big(\alpha_i(x, \varepsilon_i) = a \mid x\big).$$

CCPs integrate out private shocks and can be estimated from the empirical frequency of actions conditional on states.

Missing object it creates. To recover primitives θ (profit and cost parameters), CCPs must be linked to continuation values. That requires a valuation step: compute value objects given a candidate CCP vector P.

5) Valuation given CCPs (continuation values for arbitrary P)

Connection tag: compute continuation values from CCPs $\Rightarrow$ exposes the equilibrium fixed-point constraint.

What it solves. Given a CCP vector P, one can compute expected present values without repeatedly solving the full dynamic game from scratch. In many setups, expected values can be computed by solving a linear system, and conditional choice values become linear in parameters:

$$v_i^P(a,x) \;=\; \tilde{z}_i^P(a,x)\,\theta_i.$$

Missing object it creates. Not every P is an equilibrium. Equilibrium CCPs satisfy a fixed-point mapping

$$P \;=\; \Psi(P,\theta).$$

This is the central computational/econometric challenge.

Why the next step is needed. Estimation methods differ precisely in how they handle the fixed-point constraint $P = \Psi(P,\theta)$.

6) Estimation methods: ways to handle the fixed point $P = \Psi(P,\theta)$

Connection tag: equilibrium is a fixed point $\Rightarrow$ choose an estimator by computational strategy.

1. **Full ML / constrained ML (including MPEC).**
   Solve
   $$(\hat{\theta},\hat{P}) = \arg\max_{\theta,P} Q(\theta,P) \quad \text{s.t.} \quad P = \Psi(P,\theta).$$
   Tradeoff: statistically attractive, computationally heavy when P is huge.

2. **Two-step CCP estimators.**
   First estimate $\hat{P}_0(a \mid x)$ from data; then estimate
   $$\hat{\theta} = \arg\max_{\theta} Q(\theta,\hat{P}_0).$$
   Tradeoff: feasible and fast, but finite-sample bias can arise from noisy $\hat{P}_0$.

3. **NPL / K-step methods (Aguirregabiria–Mira).**
   Iteratively update (P,θ) by alternating best-response updates; convergence requires local stability (Lyapunov stability).

4. **Moment inequalities (BBL-style).**
   Use "equilibrium dominates deviations" inequalities without solving for P exactly.
   Tradeoff: flexible (can handle continuous actions), but often set identification and sensitivity to deviation choices.

Missing object it creates. Realistic environments introduce unobserved heterogeneity and high-dimensional states, which can break naive estimation and make equilibrium computation infeasible at scale.

7) Real-world obstacles and fixes

Connection tag: heterogeneity + dimensionality break naive estimation ⇒ impose structure or use mixture models.

(i) Unobserved heterogeneity. Markets differ in persistent unobservables; ignoring these can confound competitive effects. A common fix is finite mixtures / random effects ("market types"), often estimated with EM-style updates combined with CCP/NPL logic.

(ii) Curse of dimensionality. State spaces grow rapidly with the number of heterogeneous firms and state variables. Fixes include symmetry/homogeneity restrictions, state compression (inclusive-value-type reductions), and simulation (with careful control of simulation error).

Missing object it creates. Multiple equilibria create ambiguity in counterfactual predictions: which equilibrium is selected after a policy change?

8) Counterfactuals under multiple equilibria (equilibrium selection)

Connection tag: multiplicity ⇒ counterfactual equilibria require a selection rule or a stable continuation path.

Even if the equilibrium observed in the data is identified, under new parameters the selected equilibrium may differ. One approach is to assume smooth selection locally and use a Taylor/homotopy approximation around the observed equilibrium, then iterate the equilibrium mapping toward a stable counterfactual equilibrium.

Connection tags (one-line memory anchors)

- **Dynamic BR** → sunk costs create state dependence (need dynamics).
- **Framework** → defines states/actions/payoffs/transitions (need equilibrium concept).
- **MPE** → restricts strategies to Markov states (still high-dimensional).
- **CCPs** → compress strategies into estimable probabilities (need valuation).
- **Valuation** → computes continuation values given CCPs (reveals fixed-point constraint).
- **Estimation** → differs by how it enforces $P = \Psi(P, \theta)$.
- **Obstacles** → heterogeneity + dimensionality require structure/mixtures.
- **Counterfactuals** → multiplicity requires equilibrium selection/stability.

Five-minute self-test

1. Why do sunk costs create persistence in market structure relative to static entry models?
2. What is an MPE, and why does it improve tractability relative to history-dependent strategies?
3. Define a CCP $P_i(a \mid x)$ and explain why it is lower-dimensional than a strategy.
4. Why does equilibrium estimation involve a fixed-point constraint $P = \Psi(P, \theta)$?

5. Intuitively, when can NPL iterations converge (what does local stability mean)?
6. Why do multiple equilibria complicate counterfactuals even if the data equilibrium is known?