8. Dynamic Games: Model and Metho

8.1 Introduction

The following chapter deals with methods and applications of empirical dynamic games of oligopoly competition. More generally, some of the methods that we will describe can be applied other applied fields such as political economy (for instance, competition between political parties), or international economics (for instance, ratification of international treaties), among others.

Dynamic games are powerful tools for the analysis of phenomena characterized by **dynamic strategic interactions** between multiple agents. By *dynamic strategic interactions* we mean that:

> **(a)** players' current decisions affect their own and other players' payoffs in the future (that is, *multi-agent dynamics*);
>
> **(b)** players' decisions are forward-looking in the sense that they take into account the implications on their own and on their rivals' future behavior and how this behavior affects future payoffs (that is, *strategic behavior*).

Typical sources of dynamic strategic interactions are decisions that are partially irreversible (costly to reverse) or that involve sunk costs. Some examples in the context of firm oligopoly competition: **(1)** entry-exit in markets; **(2)** introduction of a new product; timing of the release of a new movie; **(3)** repositioning of product characteristics; **(4)** investment in capacity, or equipment, or R&D, or quality, or goodwill, or advertising; **(5)** pricing of a durable good; pricing when demand is characterized by consumer switching costs; **(6)** production when there is learning-by-doing.

Taking into account dynamic strategic interactions may change substantially our interpretation of some economic phenomena or the implications of some public policies. We have already discussed some examples from recent applications in IO: **(1)** Short-run and long-run responses to changes in industry regulations (**ryan_2006** ,**ryan_2006**); **(2)** Product repositioning in differentiated product markets (Sweeting ,2007); **(3)** Dynamic aspects of network competition (**aguirregabiria_ho_2008** ,**aguirregabiria_ho_2008**).

Road Map: 1. Structure of empirical dynamic games; 2. Identification; 3. Estimation; 4. Dealing with unobserved heterogeneity; 5. Empirical Applications; 5.1. Dynamic effects of industry regulation (**ryan_2006** ,**ryan_2006**); 5.2. Product repositioning in differentiated product markets (Sweeting ,2007); 5.3. Dynamic aspects of network competition (**aguirregabiria_ho_2008** ,**aguirregabiria_ho_2008**).

8.2 Dynamic version of Bresnahan-Reiss model

The following section is based on Bresnahan and Reiss (Bresnahan and Reiss):

Complete information; homogeneous firms

8.2.1 Motivation

Suppose that we have panel data of M markets over T periods of time.

$$\text{Data} = \{ n_{mt}, X_{mt} : m = 1,2,...,M; t = 1,2,...,T \}$$

In these data, we observe how the number of firms grows or declines in the market. Suppose that we do not know the gross changes in the number of firms, that is, we do not observe the number of new entrants, en_{mt}, and number of exits, ex_{mt}. We only observe the net change $n_{mt} - n_{mt-1} = en_{mt} - ex_{mt}$.

To explain the observed variation, across markets and over time, in the number of firms, we could estimate the BR static model that we have considered so far. The only difference is that now we have multiple realizations of the game both because the game is played at different locations and because it is played at different periods of time.

However, the static BR model imposes a strong and unrealistic restriction on this type of panel data. According to the static model, the number of firms in the previous period, n_{mt-1}, does not play a role in the determination of the current number of firms n_{mt}. This is because the model considers that the profit of an active firm is the same regardless of whether it was active in the previous period or not. That is, the model assumes that either there are no entry costs, or that entry costs are paid every period in which the firm is active such that both new entrants and incumbents should pay these costs. Of course, this assumption is very unrealistic for most industries.

Bresnahan and Reiss (1994) propose and estimate a dynamic extension of their static model of entry. This dynamic model distinguishes between incumbents and potential entrants and takes into account the existence of sunk entry costs. The model is simple but interesting, and useful because of its simplicity. We could call it a "semi-structural" model. It is structural in the sense that it is fully consistent with dynamic game of entry-exit in an oligopoly industry. But it is only "semi" in the sense that it does not model explicitly how the future expected value function of an incumbent firm depends on the sunk-cost. Ignoring this relationship has clear computational advantages in the estimation of the model. However it has also limitations in terms of the type of counterfactuals and empirical questions that can be studied using this model.

8.2.2 Model

Let n_t be the number of active firms in the market at period t. n_t belongs to the set $\{0, 1, ..., N\}$ where N is a large but finite number. Let $V(n_t, X_t) - \varepsilon_t$ be the value function of an active firm in a market with exogenous characteristics (X_t, ε_t) and number of firms n_t. The additive error term ε_t can be interpreted as an iid shock in the fixed cost of being active in the market. The function $V(n,X)$ is strictly decreasing in n.

This value function does not include the cost of entry. Let EC be the entry cost that a new entrant should pay to be active in the market at period t. Let SV be the scrapping value of a firm that decides to exit from the market. For the moment, we consider that EC and SV are constant parameters but we will discuss later how this assumption can be relaxed.

An important and obvious condition is that $SV \leq EC$. That is, firms cannot make profits by constantly entering and exiting in a market. It is an obvious arbitrage condition. The parameter $EC - SV$ is called the sunk entry cost, that is, it is the part of the entry cost that is sunk and cannot be recovered upon exit. This can include administrative costs, costs of market research, and in general any investment in capital that is firm specific and therefore will not have market value when the firm exits the market.

The values or payoffs of incumbents and potential entrants are: Incumbent that decides to stay: $V(n_t, X_t) - \varepsilon_t$; Incumbent that exits: SV; New entrant: $V(n_t, X_t) - \varepsilon_t - EC$; Potential entrant stays out: 0.

Below, we describe the entry-exit equilibrium conditions that determine the equilibrium number of firms n_t as a function of (X_t, ε_t).

Regime 1: Exit. Suppose that $n_{t-1} > 0$ and $V(n_{t-1}, X_t) - \varepsilon_t < SV$. That is, at the beginning of period t, the values of the exogenous variables X_t and ε_t are realized, and the incumbent firms from the previous period find out that the value of being active in the market is smaller than the scrapping value of the firm. Therefore, these firms want to exit.

It should be clear that under this regime there is no entry. Since $SV \leq EC$, we have that $V(n_{t-1}, X_t) - \varepsilon_t < EC$ and therefore $V(n_{t-1}+1, X_t) - \varepsilon_t < EC$. The value for a new entrant is smaller than the entry cost and therefore there is no entry.

Therefore, incumbent firms will start exiting the market up to the point where either: (a) there are no more firms in the market, that is, $n_t = 0$; or (b) there are still firms in the market and the value of an active firm is greater or equal to the scrapping value. The equilibrium number of firms in this regime is given by the conditions:

$$\begin{cases} n_t = 0 & \text{if} \quad V(1, X_t) - \varepsilon_t < SV \\ \text{OR} & \\ n_t = n > 0 & \text{if} \quad \{V(n, X_t) - \varepsilon_t \geq SV\} \quad \text{AND} \quad \{V(n+1, X_t) - \varepsilon_t < SV\} \end{cases}$$

The condition $\{V(n_t, X_t) - \varepsilon_t \geq SV\}$ says that an active firm in the market does not want to exit. Condition $\{V(n_t+1, X_t) - \varepsilon_t < SV\}$ establishes that if there were any number of firms in the market greater than n_t, firms would prefer to exit.

Summarizing, **Regime 1 [Exit]** is described by the following condition on exogenous variables $\{n_{t-1} > 0\}$ and $\{\varepsilon_t > V(n_{t-1}, X_t) - SV\}$, and this condition implies that:

$$n_t < n_{t-1}$$

and n_t is determined by

$$\begin{cases} n_t = 0 & \text{if} \quad V(1, X_t) - \varepsilon_t < SV \\ \text{OR} & \\ n_t = n > 0 & \text{if} \quad V(n+1, X_t) - SV < \varepsilon_t \leq V(n, X_t) - SV \end{cases}$$

Regime 2: Entry. Suppose that $n_{t-1} < N$ and $V(n_{t-1}+1, X_t) - \varepsilon_t \geq EC$. At the beginning of period t, potential entrants realize that the value of being active in the market is greater than the entry cost. Therefore, potential entrants want to enter in the market.

It should be clear that under this regime there is no exit. Since $SV \leq EC$ and $V(n_{t-1}+1, X_t) < V(n_{t-1}, X_t)$, we have that the condition $\{V(n_{t-1}+1, X_t) - \varepsilon_t \geq EC\}$ implies that $\{V(n_{t-1}, X_t) - \varepsilon_t > SV\}$. The value of an incumbent is greater than the scrapping value and therefore there is no exit.

Therefore, new firms will start entering the market up to the point where either: (a) there are no more potential entrants to enter in the market, that is, $n_t = N$; or (b) there are still potential entrants that may enter the market but the value of an active firm goes down to a level such that there are no more incentives for additional entry. The equilibrium number of firms in this regime is given by the conditions:

$$\begin{cases} n_t = N & \text{if} \quad V(N, X_t) - \varepsilon_t \geq EC \\ \text{OR} & \\ n_t = n < N & \text{if} \quad \{V(n, X_t) - \varepsilon_t \geq EC\} \quad \text{AND} \quad \{V(n+1, X_t) - \varepsilon_t < EC\} \end{cases}$$

Condition $\{V(n_t, X_t) - \varepsilon_t \geq EC\}$ says that the last firm that entered the market had an incentive to enter. Condition $\{V(n_t+1, X_t) - \varepsilon_t < EC\}$ establishes that the next firm entering the market would not get enough value to cover the entry cost.

Summarizing, **Regime 2 [Entry]** is described by the following condition on exogenous variables $\{n_{t-1} < N\}$ and $\{\varepsilon_t \leq V(n_{t-1}+1, X_t) - EC\}$, and this condition implies that:

$$n_t > n_{t-1}$$

and n_t is determined by

$$\begin{cases} n_t = N & \text{if} \quad V(N, X_t) - \varepsilon_t \geq EC \\ \text{OR} & \\ n_t = n < N & \text{if} \quad V(n+1, X_t) - EC < \varepsilon_t \leq V(n, X_t) - EC \end{cases}$$

Regime 3: Inaction. The third possible regime is given by the complementary conditions to those that define regimes 1 and 2. Under these conditions, incumbent firms do not want to exit and potential entrants do not want to enter.

$$\{n_t = n_{t-1}\} \text{ iff} \begin{cases} & \{n_{t-1} = 0\} \text{ AND } \{V(1, X_t) - \varepsilon_t < EC\} \\ \text{OR} & \\ & \{n_{t-1} = N\} \text{ AND } \{V(N, X_t) - \varepsilon_t \geq SV\} \\ \text{OR} & \\ & \{0 < n_{t-1} < N\} \text{ AND } \{V(n_{t-1}+1, X_t) - \varepsilon_t < EC\} \text{ AND } \{V(n_{t-1}, X_t) - \varepsilon_t \geq SV\} \end{cases}$$

Putting the three regimes together, we can obtain the probability distribution of the endogenous n_t conditional on (n_{t-1}, X_t). Assume that ε_t is i.i.d. and independent of X_t

with CDF F_ε. Then:

$$\Pr(n_t = n \mid n_{t-1},\ X_t) = \begin{cases} F_\varepsilon\left(\frac{V(n,X_t)-SV}{\sigma}\right) - F_\varepsilon\left(\frac{V(n+1,X_t)-SV}{\sigma}\right) & \text{if} \quad n < n_{t-1} \\ F_\varepsilon\left(\frac{V(n_{t-1},X_t)-SV}{\sigma}\right) - F_\varepsilon\left(\frac{V(n_{t-1}+1,X_t)-EC}{\sigma}\right) & \text{if} \quad n = n_{t-1} \\ F_\varepsilon\left(\frac{V(n,X_t)-EC}{\sigma}\right) - F_\varepsilon\left(\frac{V(n+1,X_t)-EC}{\sigma}\right) & \text{if} \quad n > n_{t-1} \end{cases}$$

It is interesting to compare this probability distribution of the number of firms with the one from the static BR model. In the static BR model:

$$\Pr(n_t = n \mid n_{t-1},\ X_t) = F_\varepsilon\left(\frac{V(n,X_t)}{\sigma}\right) - F_\varepsilon\left(\frac{V(n+1,X_t)}{\sigma}\right)$$

This is exactly the distribution that we get in the dynamic model when $EC = SV$. Note that under $EC = SV$, the sunk cost $EC - SV$ is zero and firms' entry-exit decisions are static.

When $EC > SV$ (positive sunk cost), the dynamic model delivers different predictions than the static model. There are two main differences. First, the number of firms is more persistent over time, that is, there is "structural state dependence" in the number of firms.

$$\Pr(n_t = n_{t-1} \mid n_{t-1},\ X_t) = \begin{cases} F_\varepsilon\left(\frac{V(n_{t-1},X_t)-SV}{\sigma}\right) - F_\varepsilon\left(\frac{V(n_{t-1}+1,X_t)-EC}{\sigma}\right) & \text{if} \quad EC > SV \\ F_\varepsilon\left(\frac{V(n_{t-1},X_t)}{\sigma}\right) - F_\varepsilon\left(\frac{V(n_{t-1}+1,X_t)}{\sigma}\right) & \text{if} \quad EC = SV \end{cases}$$

In the static model, all the persistence in the number of firms is because this variable is indivisible - it is an integer. However, in the dynamic model, sunk entry costs introduce more persistence. A purely transitory shock (in X_t or in ε_t) that increases the number of firms at some period t will have a persistent effect for several periods in the future.

Second, the number of firms responds asymmetrically to positive and negative shocks. Given $EC > SV$, it is possible to show that the upward response is less elastic than the downward response.

8.2.3 Identification

It is interesting to explore the identification of the model. With this model and data, we cannot identify nonparametrically the distribution of ε_t. So we make a parametric assumption on this distribution. For instance, we assume that ε_t has a $N(0,\sigma^2)$ distribution.

Define $P_{entry}(n_{t-1}, X_t)$ and $P_{exit}(n_{t-1}, X_t)$ as the probabilities of positive (entry) and negative (exit) changes in the number of firms, respectively. That is, $P_{entry}(n_{t-1}, X_t) \equiv \Pr(n_t > n_{t-1} \mid n_{t-1}, X_t)$ and $P_{exit}(n_{t-1}, X_t) \equiv \Pr(n_t < n_{t-1} \mid n_{t-1}, X_t)$. These probability functions are nonparametrically identified from our panel data on $\{n_t, X_t\}$.

The model predicts the following structure for the probabilities of entry and exit:

$$\begin{aligned} P_{entry}(n_{t-1}, X_t) &= \Pr(V(n_{t-1}+1, X_t) - \varepsilon_t > EC \mid X_t) = \\ &= \Phi\left(\frac{V(n_{t-1}+1, X_t) - EC}{\sigma}\right) \end{aligned}$$

and:

$$
\begin{aligned}
P_{exit}(n_{t-1}, X_t) &= \Pr\left(V(n_{t-1}, X_t) - \varepsilon_t < SV \mid X_t\right) \\
&= 1 - \Phi\left(\frac{V(n_{t-1}, X_t) - SV}{\sigma}\right)
\end{aligned}
$$

Using these expressions, it is simple to obtain that, for any (n_{t-1}, X_t):

$$
\frac{EC - SV}{\sigma} = \Phi^{-1}\left(1 - P_{exit}(n_{t-1}, X_t)\right) - \Phi^{-1}\left(P_{entry}(n_{t-1} - 1, X_t)\right)
$$

where Φ^{-1} is the inverse function of the CDF of ε_t.

Therefore, even with a nonparametric specification of the value function $V(n,X)$, we can identify the sunk cost up to scale. Note that this expression provides a clear intuition about the source of identification of this parameter. The magnitude of this parameter is identified by "a distance" between the probability of entry of potential entrants and the probability of staying for incumbents $(1 - P_{exit})$. In a model without sunk costs, both probabilities should be the same. In a model with sunk costs, the probability of staying in the market should be greater than the probability of entry.

Since we do not know the value of σ, the value of the parameter $\frac{EC-SV}{\sigma}$ is not meaningful from an economic point of view. However, based on the identification of $\frac{EC-SV}{\sigma}$ and the identification up to scale of the value function $V(n,X)$ (that we show below), it is possible to get an economically meaningful estimate of the importance of the sunk cost. Suppose that $V(n,X)/\sigma$ is identified. Then, we can identify the ratio:

$$
\frac{EC - SV}{V(n,X)} = \frac{\frac{EC-SV}{\sigma}}{\frac{V(n,X)}{\sigma}}
$$

For instance, we have $\frac{EC-SV}{V(1,X)}$ which is the percentage of the sunk cost over the value of a monopoly in a market with characteristics X.

Following the same argument as for the identification of the constant parameter $\frac{EC-SV}{\sigma}$, we can show the identification of a sunk cost that depends nonparametrically on the state variables (n_{t-1}, X_t). That is, we can identify a sunk cost function $\frac{EC(n_{t-1},X_t)-SV(n_{t-1},X_t)}{\sigma}$. This has economic interest. In particular, the dependence of the sunk cost with respect to the number of incumbents n_{t-1} is evidence of endogenous sunk costs (see Sutton's book titled *"Sunk Costs and Market Structure,"* MIT Press, 1991). Therefore, we can test nonparametrically for the existence of endogenous sunk costs by testing the dependence of the estimated function $\frac{EC(n_{t-1},X_t)-SV(n_{t-1},X_t)}{\sigma}$ with respect to n_{t-1}.

We can also use the probabilities of entry and exit to identify the value function $V(n,X)$. The model implies that:

$$
\begin{aligned}
\Phi^{-1}\left(P_{entry}(n_{t-1} - 1, X_t)\right) &= \tfrac{V(n_{t-1},X_t)-EC}{\sigma} \\
\Phi^{-1}\left(1 - P_{exit}(n_{t-1}, X_t)\right) &= \tfrac{V(n_{t-1},X_t)-SV}{\sigma}
\end{aligned}
$$

The left-hand-side of these equations is identified from the data. From these expressions, it should be clear that we cannot identify EC/σ separately from a constant term in the value function (a fixed cost), and we cannot identify SV/σ separately from a constant term in the value function.

Let $-FC$ be the constant term or fixed cost in the value function. More formally, define the parameter FC as the expected value:

$$FC \equiv -\mathbb{E}\left(V(n_{t-1}, X_t)\right)$$

Also define the function $V^*(n_{t-1}, X_t)$ as the deviation of the value function with respect to its mean:

$$\begin{aligned} V^*(n_{t-1}, X_t) &\equiv V(n_{t-1}, X_t) - -\mathbb{E}\left(V(n_{t-1}, X_t)\right) \\ &= V(n_{t-1}, X_t) + FC \end{aligned}$$

Also, define $EC^* \equiv EC + FC$, and $SV^* \equiv SV + FC$ such that, by definition, $V(n_{t-1}, X_t) - EC = V^*(n_{t-1}, X_t) - EC^*$, and $V(n_{t-1}, X_t) - SV = V^*(n_{t-1}, X_t) - SV^*$.

Then, $\frac{EC^*}{\sigma}$, $\frac{SV^*}{\sigma}$, and $V^*(n_{t-1}, X_t)/\sigma$ are identified nonparametrically from the following expressions:

$$\begin{aligned} \frac{EC^*}{\sigma} &= E\left(\Phi^{-1}\left(P_{entry}(n_{t-1} - 1, X_t)\right)\right) \\ \frac{SV^*}{\sigma} &= \mathbb{E}\left(\Phi^{-1}\left(1 - P_{exit}(n_{t-1}, X_t)\right)\right) \end{aligned}$$

And

$$\begin{aligned} \frac{V^*(n_{t-1}, X_t)}{\sigma} &= \Phi^{-1}\left(P_{entry}(n_{t-1} + 1, X_t)\right) - \mathbb{E}\left(\Phi^{-1}\left(P_{entry}(n_{t-1} + 1, X_t)\right)\right) \\ &\text{and} \\ \frac{V^*(n_{t-1}, X_t)}{\sigma} &= \Phi^{-1}\left(1 - P_{exit}(n_{t-1}, X_t)\right) - \mathbb{E}\left(\Phi^{-1}\left(1 - P_{exit}(n_{t-1}, X_t)\right)\right) \end{aligned}$$

In fact, we can see that the function $V^*(.,.)$ is over identified: it can be identified either from the probability of entry or from the probability of exit. This provides over-identification restrictions that can be used to test the restrictions or assumptions of the model.

Again, one of the main limitations of this model is the assumption of homogeneous firms. In fact, as an implication of that assumption, the model predicts that there should not be simultaneous entry and exit. This prediction is clearly rejected in many panel datasets on industry dynamics.

8.2.4 Estimation of the model

Given a parametric assumption about the distribution of ε_t, and a parametric specification of the value function $V(n, X)$, we can estimate the model by conditional maximum likelihood. For instance, suppose that ε_t is i.i.d. across markets and over time with a distribution $N(0, \sigma^2)$, and the value function is linear in parameters:

$$V(n_t, X_t) = g(n_t, X_t)'\beta - FC$$

where $g(.,.)$ is a vector of known functions, and β is a vector of unknown parameters.

Let θ be the vectors of parameters to estimate:

$$\theta = \{ \beta/\sigma, EC^*/\sigma, SV/\sigma\}$$

Then, we can estimate θ using the conditional maximum likelihood estimator:

$$\hat{\theta} = \arg\max_{\theta} \sum_{m=1}^{M} \sum_{t=1}^{T} \sum_{n=0}^{N} 1\{n_{mt} = n\} \log \Pr(n \mid n_{mt-1}, X_{mt}; \theta)$$

where:

$$\Pr(n_t = n \mid n_{t-1}, X_t) = \begin{cases} \Phi\left(g(n,X_t)'\frac{\beta}{\sigma} - \frac{SV}{\sigma}\right) - \Phi\left(g(n+1,X_t)'\frac{\beta}{\sigma} - \frac{SV}{\sigma}\right) & \text{if} \quad n < n_{t-1} \\ \Phi\left(g(n,X_t)'\frac{\beta}{\sigma} - \frac{SV}{\sigma}\right) - \Phi\left(g(n+1,X_t)'\frac{\beta}{\sigma} - \frac{EC}{\sigma}\right) & \text{if} \quad n = n_{t-1} \\ \Phi\left(g(n,X_t)'\frac{\beta}{\sigma} - \frac{EC}{\sigma}\right) - \Phi\left(g(n+1,X_t)'\frac{\beta}{\sigma} - \frac{EC}{\sigma}\right) & \text{if} \quad n > n_{t-1} \end{cases}$$

Based on the previous identification results, we can also construct a simple least squares estimator of θ. Let $\hat{P}_{mt}^{entry}$ and $\hat{P}_{mt}^{exit}$ be nonparametric Kernel estimates of $P_{entry}(n_{mt-1}+1, X_{mt})$ and $P_{exit}(n_{mt-1}, X_{mt})$, respectively. The model implies that:

$$\begin{aligned} \Phi^{-1}\left(\hat{P}_{mt}^{entry}\right) &= \left(-\frac{EC^*}{\sigma}\right) + g(n_{mt-1}, X_{mt})'\frac{\beta}{\sigma} + e_{mt}^{entry} \\ \Phi^{-1}\left(1 - \hat{P}_{mt}^{exit}\right) &= \left(-\frac{SV^*}{\sigma}\right) + g(n_{mt-1}, X_{mt})'\frac{\beta}{\sigma} + e_{mt}^{exit} \end{aligned}$$

where e_{mt}^{entry} and e_{mt}^{exit} are error terms that come from the estimation error in $\hat{P}_{mt}^{entry}$ and $\hat{P}_{mt}^{exit}$. We can put together these regression equations in a single regression as:

$$Y_{dmt} = D_{dmt}\left(-\frac{EC^*}{\sigma}\right) + (1 - D_{dmt})\left(-\frac{SV^*}{\sigma}\right) + g(n_{mt-1}, X_{mt})'\frac{\beta}{\sigma} + e_{mt}$$

where $Y_{dmt} \equiv D_{dmt}\, \Phi^{-1}\left(\hat{P}_{mt}^{entry}\right) + (1 - D_{dmt})\Phi^{-1}\left(1 - \hat{P}_{mt}^{exit}\right)$; the subindex d represents the "regime", $d \in \{entry, exit\}$, and D_{dmt} is a dummy variable that is equal to one when $d = entry$ and it is equal to zero when $d = exit$.

OLS estimation of this linear regression equation provides a consistent estimator of θ. This estimator is not efficient but we can easily obtain an asymptotically efficient estimator by making one Newton-Raphson iteration in the maximization of the likelihood function.

8.2.5 Structural model and counterfactual experiments

This dynamic model is fully consistent with a dynamic game of entry-exit. However, the value function $V(n,X)$ is not a primitive or a structural function. It implicitly depends on the one-period profit function, on the entry cost EC, on the scrapping value SV, and on the equilibrium of the model (that is, on equilibrium firms' strategies).

The model and the empirical approach that we have described above does not make explicit the relationship between the primitives of the model and the value function, or how this value function depends on the equilibrium transition probability of the number of firms, $P^*(n_{t+1}|n_t,X_t)$. This "semi-structural" approach has clear advantages in terms of computational and conceptual simplicity. However, it also has its limitations. We discuss here its advantages and limitations.

Similar approaches have been proposed and applied for the estimation of dynamic models of occupational choice by **geweke_keane_2001** (**geweke_keane_2001**) and **hoffmann_2009** (**hoffmann_2009**). This type of approach is different to other methods that have been proposed and applied to the estimation of dynamic structural models and that also try to reduce the computational cost in estimation, such as Hotz and Miller (1993 and 1994) and Aguirregabiria and Mira (2002 and 2007).

To understand the advantages and limitations of Bresnahan and Reiss' "semi-structural" model of industry dynamics, it is useful to relate the value function $V(n_t,X_t)$ with the actual primitives of the model. Let $\pi(n_t,X_t,\varepsilon_t)$ be the profit function of an incumbent firm that stays in the market. Therefore:

$$V(n_t,X_t) = \mathbb{E}\left(\sum_{j=0}^{\infty} \delta^j \,\left[(1-Exit_{t+j})\;\pi(n_{t+1},X_{t+j},\varepsilon_{t+j}) + Exit_{t+j}SV\right]\;|\; n_t,X_t\right)$$

where δ is the discount factor, and $Exit_{t+j}$ is a binary variable that indicates if the firm exits from the market at period $t+j$ (that is, $Exit_{t+j}=1$) or stays in the market (that is, $Exit_{t+j}=0$). The expectation is taken over all future paths of the state variables $\{n_{t+1},X_{t+j},\varepsilon_{t+j}\}$. In particular, this expectation depends on the stochastic process that follows the number of firms in equilibrium and that is governed by the transition probability $\Pr(n_{t+1}|n_t,X_t)$.

The transition probability $\Pr(n_{t+1}|n_t,X_t)$ is determined in equilibrium and it depends on all the structural parameters of the model. More specifically, this transition probability can be obtained as the solution of a fixed point problem. Solving this fixed point problem is computationally demanding. The "semi-structural" approach avoids this computational cost by ignoring the relationship between the value function $V(n_t,X_t)$ and the structural parameters of the model. This can provide huge computational advantages, especially when the dimension of the state space of (n_t,X_t) is large and/or when the dynamic game may have multiple equilibria.

These significant computational gains come with a cost. The range of predictions and counterfactual experiments that we can make using the estimated "semi-structural" model is very limited. In particular, we cannot make predictions about how the equilibrium transition $\Pr(n_{t+1}|n_t,X_t)$ (or the equilibrium steady-state distribution of n_t) changes when we perturb one the parameters in θ.

There are two types of problems in this model associated with implementing the predictions of counterfactual experiments. First, the parameters β are not structural such that we cannot change one of these parameters and assume that the rest will stay constant. In other words, we do not know what that type of experiment means.

Second, though EC^* and SV^* are structural parameters, the parameters β in the value function should depend on EC^* and SV^*, but we do not know the form of that relationship. We cannot assume that EC^* or SV^* and β remains constant. In other words, that type of experiment does not have a clear interpretation or economic interest.

For instance, suppose that we want to predict how a 20% increase in the entry cost would affect the transition dynamics and the steady state distribution of the number of firms. If $\lambda_0 = EC^*/\sigma$ is our estimate of the value of the parameter in the sample, then its counterfactual value is $\lambda_1 = 1.2\lambda_0$. However, we also know that the value function should change. In particular, the value of an incumbent firm increases when the entry costs increases.

The "semi-structural" model ignores that the value function V will change as the result of the change in the entry cost. Therefore, it predicts that entry will decline, and that the exit/stay behavior of incumbent firms will not be affected because V and SV have not changed.

There are two errors in the prediction of the "semi-structural" model. First, it overestimates the decline in the amount of entry because it does not take into account that being an incumbent in the market now has more value. And second, it ignores that, for the same reason, exit of incumbent firms will also decline.

Putting these two errors together, we have that this counterfactual experiment using the "semi-structural" model can lead to a serious under-estimate of the number of firms in the counterfactual scenario.

Later in the chapter we will study other methods for the estimation of structural models of industry dynamics that avoid the computational cost of solving for the equilibrium of the game but that do not have the important limitations, in terms of counterfactual experiments, of the semi-structural model here.

Nevertheless, it is difficult to overemphasize the computational advantages of Bresnahan-Reiss' empirical model of industry dynamics. It is a useful model to obtain a first cut of the data, and to answer empirical questions that do not require the implementation of counterfactual experiments. For instance, we can test for endogenous sunk costs, or measure the magnitude of sunk costs relative to the value of an incumbent firm.

8.3 The structure of dynamic games of oligopoly competition

8.3.1 Basic Framework and Assumptions

Time is discrete and indexed by t. The game is played by N firms that we index by i. Following the standard structure in the Ericson and Pakes (1995) framework, firms compete in two different dimensions: a static dimension and a dynamic dimension. We denote the dynamic dimension as the "investment decision". Let a_{it} be the variable that represents the investment decision of firm i at period t. This investment decision can be an entry/exit decision, a choice of capacity, investment in equipment, R&D, product quality, other product characteristics, etc. Every period, given their capital stocks that can affect demand and/or production costs, firms compete in prices or quantities in a static Cournot or Bertand model. Let p_{it} be the static decision variables (for instance, price) of firm i at period t.

For simplicity and concreteness, we start by presenting a simple dynamic game of market entry-exit where every period incumbent firms compete à la Bertrand. In this entry-exit model, the dynamic investment decision a_{it} is a binary indicator of the event "firm i is active in the market at period t". The action is taken to maximize the expected and discounted flow of profits in the market, $\mathbb{E}_t\left(\sum_{r=0}^{\infty}\delta^r\ \Pi_{it+r}\right)$ where $\delta\in(0,1)$ is the discount factor, and Π_{it} is firm i's profit at period t. The profits of firm i at time t are given by

$$\Pi_{it} = VP_{it} - FC_{it} - EC_{it}$$

where VP_{it} represents variable profits, FC_{it} is the fixed cost of operating, and EC_{it} is a one time entry cost. We now describe these different components of the profit function.

(a) Variable Profit Function. The variable profit VP_{it} is an "indirect" variable profit function that comes from the equilibrium of a static Bertrand game with differentiated product. Consider the simplest version of this type of model. Suppose that all firms have the same marginal cost c, and product differentiation is symmetric. Consumer utility of buying product i is $u_{it} = v - \alpha p_{it} + \varepsilon_{it}$, where v and α are parameters, and ε_{it} is a consumer-specific i.i.d. extreme value type 1 random variable. Under these conditions, the equilibrium variable profit of an active firm depends only on the number of firms active in the market.

$$VP_{it} = (p_{it} - c)q_{it}$$

where p_{it} and q_{it} represent the price and the quantity sold by firm i at period t, respectively. According to this model, the quantity is:

$$q_{it} = H_t\ \frac{a_{it}\ \exp\{v-\alpha\ p_{it}\}}{1+\sum_{j=1}^{N} a_{jt}\ \exp\{v-\alpha\ p_{jt}\}} = H_t\ s_{it}$$

where H_t is the number of consumers in the market (market size) and s_{it} is the market share of firm i. Under the Nash-Bertrand assumption, the first order conditions for profit maximization are:

$$q_{it} + (p_{it} - c)\ (-\alpha)\ q_{it}\ (1 - s_{it}) = 0$$

or

$$p_{it} = c + \frac{1}{\alpha\,(1-s_{it})}$$

Since all firms are identical, we consider a symmetric equilibrium, $p_t^* = p_{it}^*$, for every firm i. Therefore, $s_{it} = a_{it}\ s_t^*$, and:

$$s_t^* = \frac{\exp\{v-\alpha\ p_t^*\}}{1+n_t\ \exp\{v-\alpha\ p_t^*\}}$$

where $n_t \equiv \sum_{j=1}^{N} a_{jt}$ is the number of active firms at period t. Then, it is simple to show that the equilibrium price p_t^* is implicitly defined as the solution to the following fixed point problem:

$$p_t^* = \left(c+\frac{1}{\alpha}\right) + \frac{1}{\alpha}\left(\frac{\exp\{v-\alpha\ p_t^*\}}{1+(n_t-1)\ \exp\{v-\alpha\ p_t^*\}}\right)$$

It is simple to show that an equilibrium always exists. The equilibrium price depends on the number of firms active in the market, but in this model it does not depend on market size: $p_t^* = p^*(n_t)$. Similarly, the equilibrium market share s_t^* is a function of the number of active firms: $s_t^* = s^*(n_t)$. Therefore, the indirect or equilibrium variable profit of an active firm is:

$$\begin{aligned} VP_{it} &= a_{it}\ H_t\ (p^*(n_t) - c)\ s^*(n_t) \\ &= a_{it}\ H_t\ \theta^{VP}(n_t) \end{aligned}$$

where $\theta^{VP}$is a function that represents variable profits per capita.

For most of the analysis below, we will consider that the researcher does not have access to information on prices and quantities. Therefore, we will treat $\{\theta^{VP}(1), \theta^{VP}(2),$ ..., $\theta^{VP}(N)\}$ as parameters to estimate from the structural dynamic game.

Of course, we can extend the previous approach to incorporate richer forms of product differentiation. In fact, product differentiation can be endogenous. Suppose that the quality parameter v in the utility function can take A possible values: $v(1) < v(2) <$ ... $< v(A)$. And suppose that the investment decision a_{it} combines an entry/exit decision with a "quality" choice decision. That is, $a_{it} \in \{0, 1, ..., A\}$ where $a_{it} = 0$ represents firm i not being active in the market, and $a_{it} = a > 0$ implies that firm i is active in the market with a product of quality a. It is straightforward to show that, in this model, the equilibrium variable profit of an active firm is:

$$VP_{it} = \sum_{a=1}^{A} 1\{a_{it} = a\}\ H_t\ \theta^{VP}(a, n_t^{(1)}, n_t^{(2)}, ..., n_t^{(A)})$$

where $\theta^{VP}$is the variable profit per capita that now depends on the firm's own quality, and on the number of competitors at each possible level of quality.

(b) Fixed Cost. The fixed cost is paid every period that the firm is active in the market, and it has the following structure:

$$FC_{it} = a_{it}\ \left(\theta_i^{FC} + \varepsilon_{it}\right)$$

θ_i^{FC} is a parameter that represents the mean value of the fixed operating cost of firm i. ε_{it} is a zero-mean shock that is private information to firm i. There are two main reasons why we incorporate these private information shocks in the model. First, as shown in **doraszelski_satterthwaite_2007** (**doraszelski_satterthwaite_2007**) it is a way to guarantee that the dynamic game has at least one equilibrium in pure strategies. And second, they are convenient econometric errors. If private information shocks are independent over time and over players, and unobserved to the researcher, they can 'explain' players heterogeneous behavior without generating endogeneity problems.

We will see later that the assumption of the private information shocks being the only unobservables for the researcher can be too restrictive. We will study how to incorporate richer forms of unobserved heterogeneity.

For the model with endogenous quality choice, we can generalize the structure of fixed costs:

$$FC_{it} = \sum_{a=1}^{A} 1\{a_{it} = a\}\ \left(\theta_i^{FC}(a) + \varepsilon_{it}(a)\right)$$

where now the mean value of the fixed cost, $\theta_i^{FC}(a)$, and the private information shock, $\varepsilon_{it}(a)$, depend on the level quality.

(c) Entry Cost and Repositioning costs. The entry cost is paid only if the firm was not active in the market at the previous period:

$$EC_{it} = a_{it}\ (1 - x_{it})\ \theta_i^{EC}$$

where x_{it} is a binary indicator that is equal to 1 if firm i was active in the market in period $t-1$, that is, $x_{it} \equiv a_{i,t-1}$, and θ_i^{EC} is a parameter that represents the entry cost of firm i. For the model with endogenous quality, we can also generalize this entry cost to also incorporate costs of adjusting the level of quality, or repositioning product characteristics. For instance,

$$\begin{aligned} EC_{it} &= 1\{x_{it} = 0\} \left(\sum_{a=1}^{A} 1\{a_{it} = a\}\ \theta_i^{EC}(a)\right) \\ &+ 1\{x_{it} > 0\} \left(\theta_i^{AC(+)} 1\{a_{it} > x_{it}\} + \theta_i^{AC(-)} 1\{a_{it} < x_{it}\}\right) \end{aligned}$$

Now, $x_{it} = a_{i,t-1}$ again represents the firm's quality at previous period, $\theta_i^{EC}(a)$ is the cost of entry with quality a, and $\theta_i^{AC(+)}$ and $\theta_i^{AC(-)}$ represent the costs of increasing and reducing quality, respectively, once the firm is active.

The payoff relevant state variables of this model are: (1) market size H_t; (2) the incumbent status (or quality levels) of firms at previous period $\{x_{it} : i = 1, 2, ..., N\}$; and (3) the private information shocks $\{\varepsilon_{it} : i = 1, 2, ..., N\}$. The specification of the model is completed with the transition rules of these state variables. Market size follows an exogenous Markov process with transition probability function $F_H(H_{t+1}|H_t)$. The transition of the incumbent status is trivial: $x_{it+1} = a_{it}$. Finally, the private information shock ε_{it} is i.i.d. over time and independent across firms with CDF G_i.

Note that in this example, we consider that firms' dynamic decisions are made at the beginning of period t and they are effective during the same period. An alternative timing that has been considered in some applications is that there is a one-period time-to-build. That is, the decision is made at period t, and entry costs are paid at period t, but the firm is not active in the market until period $t+1$. This is in fact the timing of decisions in Ericson and Pakes (1995). All the results below can be generalized in a straightforward way to that case, and we will see empirical applications with that timing assumption.

8.3.2 Markov Perfect Equilibrium

Most of the recent literature in IO studying industry dynamics focuses on studying a Markov Perfect Equilibrium (MPE), as defined by **maskin_tirole_1988** (**maskin_tirole_1988**). The key assumption in this solution concept is that players' strategies are functions of only payoff-relevant state variables. We use the vector $\mathbf{x}_t$ to represent all the common knowledge state variables at period t, that is, $\mathbf{x}_t \equiv (H_t, x_{1t}, x_{2t}, ..., x_{Nt})$. In this model, the payoff-relevant state variables for firm i are $(\mathbf{x}_t, \varepsilon_{it})$.

Note that if private information shocks are serially correlated, the history of previous decisions contains useful information to predict the value of a player's private information, and it should be part of the set of payoff relevant state variables. Therefore, the assumption that private information is independently distributed over time has implications for the set of payoff-relevant state variables.

Let $\alpha = \{\alpha_i(\mathbf{x}_t, \varepsilon_{it}) : i \in \{1, 2, ..., N\}\}$ be a set of strategy functions, one for each firm. A MPE is a set of strategy functions α^* such that every firm is maximizing its value given the strategies of the other players. For given strategies of the other firms, the decision problem of a firm is a single-agent dynamic programming (DP) problem. Let $V_i^{\alpha}(\mathbf{x}_t, \varepsilon_{it})$ be the value function of this DP problem. This value function is the unique solution to the Bellman equation:

$$V_i^{\alpha}(\mathbf{x}_t, \varepsilon_{it}) = \max_{a_{it}} \left\{ \Pi_i^{\alpha}(a_{it}, \mathbf{x}_t) - \varepsilon_{it}(a_{it}) + \delta \int V_i^{\alpha}(\mathbf{x}_{t+1}, \varepsilon_{it+1})\, dG_i(\varepsilon_{it+1})\, F_i^{\alpha}(\mathbf{x}_{t+1}|a_{it}, \mathbf{x}_t) \right\} \tag{8.1}$$

where $\Pi_i^{\alpha}(a_{it}, \mathbf{x}_t)$ and $F_i^{\alpha}(\mathbf{x}_{t+1}|a_{it}, \mathbf{x}_t)$ are the expected one-period profit and the expected transition of the state variables, respectively, for firm i given the strategies of the other firms. For the simple entry/exit game, the expected one-period profit $\Pi_i^{\alpha}(a_{it}, \mathbf{x}_t)$ is:

$$\Pi_i^{\alpha}(a_{it}, \mathbf{x}_t) = a_{it} \left[H_t \sum_{n=0}^{N-1} \Pr\left(\sum_{j \neq i} \alpha_j(\mathbf{x}_t, \varepsilon_{jt}) = n \mid \mathbf{x}_t \right) \theta^{VP}(n+1) - \theta_i^{FC} - (1 - x_{it})\theta_i^{EC} \right]$$

And the expected transition of the state variables is:

$$F_i^{\alpha}(\mathbf{x}_{t+1}|a_{it}, \mathbf{x}_t) \quad = \quad 1\{x_{it+1} = a_{it}\} \left[\prod_{j \neq i} \Pr\left(x_{j,t+1} = \alpha_j(\mathbf{x}_t, \varepsilon_{jt}) \mid \mathbf{x}_t \right) \right] F_H(H_{t+1} \mid H_t)$$

A player's best response function gives her optimal strategy if the other players behave, now and in the future, according to their respective strategies. In this model, the best response function of player i is:

$$\alpha_i^*(\mathbf{x}_t, \varepsilon_{it}) = \arg\max_{a_{it}} \{ v_i^{\alpha}(a_{it}, \mathbf{x}_t) - \varepsilon_{it}(a_{it}) \}$$

where $v_i^{\alpha}(a_{it}, \mathbf{x}_t)$ is the conditional choice value function that represents the value of firm i if: (1) the firm chooses alternative a_{it} today and then behaves optimally forever in the future; and (2) the other firms behave according to their strategies in α. By definition,

$$v_i^{\alpha}(a_{it}, \mathbf{x}_t) \equiv \Pi_i^{\alpha}(a_{it}, \mathbf{x}_t) + \delta \int V_i^{\alpha}(\mathbf{x}_{t+1}, \varepsilon_{it+1})\, dG_i(\varepsilon_{it+1})\, F_i^{\alpha}(\mathbf{x}_{t+1}|a_{it}, \mathbf{x}_t)$$

A Markov perfect equilibrium (MPE) in this game is a set of strategy functions α^* such that for any player i and for any $(\mathbf{x}_t, \varepsilon_{it})$ we have that:

$$\alpha_i^*(\mathbf{x}_t, \varepsilon_{it}) = \arg\max_{a_{it}} \left\{ v_i^{\alpha^*}(a_{it}, \mathbf{x}_t) - \varepsilon_{it}(a_{it}) \right\}$$

8.3.3 Conditional Choice Probabilities

Given a strategy function $\alpha_i(\mathbf{x}_t, \varepsilon_{it})$, we can define the corresponding *Conditional Choice Probability (CCP)* function as :

$$\begin{aligned} P_i(a|\mathbf{x}) &\equiv \Pr(\alpha_i(\mathbf{x}_t, \varepsilon_{it}) = a \mid \mathbf{x}_t = \mathbf{x}) \\ &= \int 1\{\alpha_i(\mathbf{x}_t, \varepsilon_{it}) = a\}\, dG_i(\varepsilon_{it}) \end{aligned}$$

Since choice probabilities are integrated over the continuous variables in ε_{it}, they are lower dimensional objects than the strategies α. For instance, when both a_{it} and $\mathbf{x}_t$ are discrete, CCPs can be described as vectors in a finite dimensional Euclidean space. In our entry-exit model, $P_i(1|\mathbf{x}_t)$ is the probability that firm i is active in the market given the state $\mathbf{x}_t$. Under standard regularity conditions, it is possible to show that there is a one-to-one relationship between strategy functions $\alpha_i(\mathbf{x}_t, \varepsilon_{it})$ and CCP functions $P_i(a|\mathbf{x}_t)$. From now on, we use CCPs to represent players' strategies, and use the terms 'strategy' and 'CCP' as interchangeable. We also use $\Pi_i^{\mathbf{P}}$ and $F_i^{\mathbf{P}}$ instead of Π_i^{α} and F_i^{α} to represent the expected profit function and the transition probability function, respectively.

Based on the concept of CCP, we can represent the equilibrium mapping and a MPE in a way that is particularly useful for the econometric analysis. This representation has two main features:

(1) a MPE is a vector of CCPs;

(2) a player's best response is an optimal response not only to the other players' strategies but also to her own strategy in the future.

A MPE is a vector of CCPs, $\mathbf{P} \equiv \{P_i(a|\mathbf{x}) : \text{for any } (i,a,x)\}$, such that for every firm and any state $\mathbf{x}$ the following equilibrium condition is satisfied:

$$P_i(a|\mathbf{x}) = \Pr\left(a = \arg\max_{a_i}\left\{v_i^{\mathbf{P}}(a_i,\mathbf{x}) - \varepsilon_i(a_i)\right\} \mid \mathbf{x}\right)$$

The right hand side of this equation is a best response probability function. $v_i^{\mathbf{P}}(a_i,\mathbf{x})$ is a conditional choice probability function, but it has a slightly different definition than before. Now, $v_i^{\mathbf{P}}(a_i,\mathbf{x})$ represents the value of firm i if: (1) the firm chooses alternative a_i today; and (2) **all the firms**, including firm i, behave according to their respective CCPs in $\mathbf{P}$. The Representation Lemma in **aguirregabiria_ mira_2007** (**aguirregabiria_ mira_2007**) shows that every MPE in this dynamic game can be represented using this mapping. In fact, this result is a particular application of the so called "one-period deviation principle".

The form of this equilibrium mapping depends on the distribution of ε_i. For instance, in the entry/exit model, if ε_i is $N(0,\sigma_\varepsilon^2)$:

$$P_i(1|\mathbf{x}) = \Phi\left(\frac{v_i^{\mathbf{P}}(1,\mathbf{x}) - v_i^{\mathbf{P}}(0,\mathbf{x})}{\sigma_\varepsilon}\right)$$

In the model with endogenous quality choice, if $\varepsilon_i(a)$'s are extreme value type 1 distributed:

$$P_i(a|\mathbf{x}) = \frac{\exp\left\{\frac{v_i^{\mathbf{P}}(a,\mathbf{x})}{\sigma_\varepsilon}\right\}}{\sum_{a'=0}^{A}\exp\left\{\frac{v_i^{\mathbf{P}}(a',\mathbf{x})}{\sigma_\varepsilon}\right\}}$$

8.3.4 Computing $v_i^{\mathbf{P}}$ for arbitrary $\mathbf{P}$

Now, we describe how to obtain the conditional choice value functions $v_i^{\mathbf{P}}$. Since $v_i^{\mathbf{P}}$ is not based on the optimal behavior of firm i in the future, but just in an arbitrary behavior described by $P_i(.|.)$, calculating $v_i^{\mathbf{P}}$ does not require solving a DP problem, and it only implies a valuation exercise.

By definition:

$$v_i^{\mathbf{P}}(a_i,\mathbf{x}) = \Pi_i^{\mathbf{P}}(a_i,\mathbf{x}) + \delta \sum_{\mathbf{x}'} V_i^{\mathbf{P}}(\mathbf{x}')\ F_i^{\mathbf{P}}(\mathbf{x}'|a_i,\mathbf{x})$$

$\Pi_i^{\mathbf{P}}(a_i,\mathbf{x})$ is the expected current profit. In the entry/exit example:

$$\begin{aligned}\Pi_i^{\mathbf{P}}(a_i,\mathbf{x}) &= a_i \left[H \sum_{n=0}^{N-1} \Pr(n_{-i} = n \mid \mathbf{x}, \mathbf{P})\, \theta^{VP}(n+1) - \theta_i^{FC} - (1-x_i)\theta_i^{EC} \right] \\ &= a_i \left[\mathbf{z}_i^{\mathbf{P}}(\mathbf{x})\ \theta_i \right]\end{aligned}$$

where θ_i is the vector of parameters:

$$\theta_i = \left(\theta^{VP}(1),\, \theta^{VP}(2),\, ...,\theta^{VP}(N),\, \theta_i^{FC},\, \theta_i^{EC} \right)'$$

and $\mathbf{z}_i^{\mathbf{P}}(\mathbf{x})$ is the vector that depends only on the state $\mathbf{x}$ and on the CCPs at state $\mathbf{x}$, but not on structural parameters:

$$\mathbf{z}_i^{\mathbf{P}}(\mathbf{x}) = (H \Pr(n_{-i} = 1|\mathbf{x},\mathbf{P}), ..., H \Pr(n_{-i} = N-1|\mathbf{x},\mathbf{P}),\ -1,\ -(1-x_i))$$

For the dynamic game with endogenous quality choice, we can also represent the expected current profit $\Pi_i^{\mathbf{P}}(a_i,\mathbf{x})$ as:

$$\Pi_i^{\mathbf{P}}(a_i,\mathbf{x}) = \mathbf{z}_i^{\mathbf{P}}(a_i,\mathbf{x})\ \theta_i$$

The value function $V_i^{\mathbf{P}}$represents the value of firm i if all the firms, including firm i, behave according to their CCPs in $\mathbf{P}$. We can obtain $V_i^{\mathbf{P}}$ as the unique solution of the recursive expression:

$$V_i^{\mathbf{P}}(\mathbf{x}) = \sum_{a_i=0}^{A} P_i(a_i|\mathbf{x}) \left[\mathbf{z}_i^{\mathbf{P}}(a_i,\mathbf{x})\theta_i + \delta \sum_{\mathbf{x}'} V_i^{\mathbf{P}}(\mathbf{x}')\ F^{\mathbf{P}}(\mathbf{x}'|a_i,\mathbf{x}) \right]$$

When the space $\mathscr{X}$ is discrete and finite, we can obtain $V_i^{\mathbf{P}}$ as the solution of a system of linear equations of dimension $|\mathscr{X}|$. In vector form:

$$\begin{aligned}\mathbf{V}_i^{\mathbf{P}} &= \left[\sum_{a_i=0}^{A} P_i(a_i) * \mathbf{z}_i^{\mathbf{P}}(a_i) \right] \theta_i + \delta \left[\sum_{a_i=0}^{A} P_i(a_i) * \mathbf{F}_i^{\mathbf{P}}(a_i) \right] \mathbf{V}_i^{\mathbf{P}} \\ &= \bar{\mathbf{z}}_i^{\mathbf{P}}\ \theta_i + \delta\ \bar{\mathbf{F}}^{\mathbf{P}} \mathbf{V}_i^{\mathbf{P}}\end{aligned}$$

where $\bar{\mathbf{z}}_i^{\mathbf{P}} = \sum_{a_i=0}^{A} P_i(a_i) * \mathbf{z}_i^{\mathbf{P}}(a_i)$, and $\bar{\mathbf{F}}^{\mathbf{P}} = \sum_{a_i=0}^{A} P_i(a_i) * \mathbf{F}_i^{\mathbf{P}}(a_i)$. Then, solving for $\mathbf{V}_i^{\mathbf{P}}$, we have:

$$\begin{aligned}\mathbf{V}_i^{\mathbf{P}} &= \left(\mathbf{I} - \delta\ \bar{\mathbf{F}}^{\mathbf{P}} \right)^{-1} \bar{\mathbf{z}}_i^{\mathbf{P}}\ \theta_i \\ &= \mathbf{W}_i^{\mathbf{P}}\ \theta_i\end{aligned}$$

where $\mathbf{W}_i^{\mathbf{P}} = \left(\mathbf{I} - \delta\ \bar{\mathbf{F}}^{\mathbf{P}}\right)^{-1} \bar{\mathbf{z}}_i^{\mathbf{P}}$ is a matrix that only depends on CCPs and transition probabilities but not on θ.

Solving these expressions into the formula for the conditional choice value function, we have that:

$$v_i^{\mathbf{P}}(a_i, \mathbf{x}) = \tilde{z}_i^{\mathbf{P}}(a_i, \mathbf{x})\ \theta_i$$

where:

$$\tilde{z}_i^{\mathbf{P}}(a_i, \mathbf{x}) = z_i^{\mathbf{P}}(a_i, \mathbf{x}) + \delta \sum_{\mathbf{x}'} F_i^{\mathbf{P}}(\mathbf{x}'|a_i, \mathbf{x})\ \mathbf{W}_i^{\mathbf{P}}$$

Finally, the equilibrium or best response mapping in the space of CCPs becomes:

$$P_i(a|\mathbf{x}) = \Pr\left(a = \arg\max_{a_i} \left\{\tilde{z}_i^{\mathbf{P}}(a_i, \mathbf{x})\ \theta_i - \varepsilon_i(a_i)\right\}\ |\ \mathbf{x}\right)$$

For the entry/exit model with $\varepsilon_i \sim N(0, \sigma_\varepsilon^2)$:

$$P_i(1|\mathbf{x}) = \Phi\left(\left[\tilde{z}_i^{\mathbf{P}}(1, \mathbf{x}) - \tilde{z}_i^{\mathbf{P}}(0, \mathbf{x})\right] \frac{\theta_i}{\sigma_\varepsilon}\right)$$

In the model with endogenous quality choice with $\varepsilon_i(a)$'s extreme value type 1 distributed:

$$P_i(a|\mathbf{x}) = \frac{\exp\left\{\tilde{z}_i^{\mathbf{P}}(a, \mathbf{x}) \dfrac{\theta_i}{\sigma_\varepsilon}\right\}}{\sum_{a'=0}^{A} \exp\left\{\tilde{z}_i^{\mathbf{P}}(a', \mathbf{x}) \dfrac{\theta_i}{\sigma_\varepsilon}\right\}}$$

Identification

First, let's summarize the structure of the dynamic game of oligopoly competition.

Let θ be the vector of structural parameters of the model, where $\theta = \{\theta_i : i = 1, 2, ..., N\}$ and θ_i includes the vector of parameters in the variable profit, fixed cost, and entry cost of firm i: for instance, in the entry-exit example, $\theta_i = (\theta^{VP}(1)$, $\theta^{VP}(2)$, ...,$\theta^{VP}(N)$, θ_i^{FC}, $\theta_i^{EC})'$. Let $\mathbf{P}(\theta) = \{P_i(a|\mathbf{x}, \theta) : \text{for any } (i, a, \mathbf{x})\}$ be a MPE of the model associated with θ. $\mathbf{P}(\theta)$ is a solution to the following equilibrium mapping: for any $(i, a_i, \mathbf{x})$:

$$P_i(a_i|\mathbf{x}, \theta) = \frac{\exp\left\{\tilde{z}_i^{\mathbf{P}}(a_i, \mathbf{x}) \dfrac{\theta_i}{\sigma_\varepsilon}\right\}}{\sum_{a'=0}^{A} \exp\left\{\tilde{z}_i^{\mathbf{P}}(a', \mathbf{x}) \dfrac{\theta_i}{\sigma_\varepsilon}\right\}}$$

where the vector of values $\tilde{z}_i^{\mathbf{P}}(a, \mathbf{x})$ are

$$\tilde{z}_i^{\mathbf{P}}(a_i, \mathbf{x}) = z_i^{\mathbf{P}}(a_i, \mathbf{x}) + \delta \sum_{\mathbf{x}'} F_i^{\mathbf{P}}(\mathbf{x}'|a_i, \mathbf{x})\ \mathbf{W}_i^{\mathbf{P}}$$

and $\mathbf{W}_i^{\mathbf{P}} = \mathbf{W}_i^{\mathbf{P}} = \left(\mathbf{I} - \delta\ \bar{\mathbf{F}}^{\mathbf{P}}\right)^{-1} \bar{\mathbf{z}}_i^{\mathbf{P}}$, and $z_i^{\mathbf{P}}(a_i, \mathbf{x})$ is a vector with the different components of the current expected profit. For instance, in the entry-exit example:

$$\begin{aligned}
\mathbf{z}_i^{\mathbf{P}}(0, \mathbf{x}) &= (0,\ 0,\ 0,\ ...0) \\
\mathbf{z}_i^{\mathbf{P}}(1, \mathbf{x}) &= (H \Pr(n_{-i} = 1|\mathbf{x}, \mathbf{P}), ..., H \Pr(n_{-i} = N-1|\mathbf{x}, \mathbf{P}),\ -1,\ -(1 - x_i))
\end{aligned}$$

That is, $\tilde{z}_i^{\mathbf{P}}(a_i, \mathbf{x})$ represents the expected present value of the different components of the current profit of firm i if it chooses alternative a_i today, and then all the firms, including firm i, behave in the future according to their CCPs in the vector $\mathbf{P}$.

In general, we will use the function $\Psi_i(a_i, \mathbf{x}; \mathbf{P}, \theta)$ to represent the best response or equilibrium function that in our example is $\frac{\exp\left\{\tilde{z}_i^{\mathbf{P}}(a_i,\mathbf{x}) \frac{\theta_i}{\sigma_\varepsilon}\right\}}{\sum_{a'=0}^{A} \exp\left\{\tilde{z}_i^{\mathbf{P}}(a',\mathbf{x}) \frac{\theta_i}{\sigma_\varepsilon}\right\}}$. Then, we can represent in a compact form a MPE as:

$$\mathbf{P} = \Psi(\mathbf{P}, \theta)$$

where $\Psi(\mathbf{P}, \theta) = \{\Psi_i(a_i, \mathbf{x}; \mathbf{P}, \theta) : \text{for any } (i, a, \mathbf{x})\}$.

Our first goal is to use data on firms' investment decisions $\{a_{it}\}$ and state variables $\{x_{it}\}$ to estimate the parameters θ.

Our second goal is to use the estimated model to perform counterfactual analysis/experiments that will help us understand competition in this industry and to evaluate the effects of public policies or/and changes in structural parameters.

Data

In most applications of dynamic games in empirical IO, the researcher observes a random sample of M markets, indexed by m, over T periods of time, where the observed variables consists of players' actions and state variables. In the standard application in IO, the values of N and T are small, but M is large. Two aspects of the data deserve some comments. For the moment, we consider that the industry and the data are such that: (a) each firm is observed making decisions in every of the M markets; and (b) the researcher knows all the payoff relevant market characteristics that are common knowledge to the firms. We describe condition (a) as a data set with *global players*. For instance, this is the case in a retail industry characterized by competition between large retail chains which are potential entrants in any of the local markets that constitute the industry. With this type of data we can allow for rich firm heterogeneity that is fixed across markets and time by estimating firm-specific structural parameters, θ_i. This 'fixed-effect' approach to deal with firm heterogeneity is not feasible in data sets where most of the competitors can be characterized as *local players*, that is, firms specialized in operating in a few markets. Condition (b) rules out the existence of unobserved market heterogeneity. Though it is a convenient assumption, it is also unrealistic for most applications in empirical IO. Later we present estimation methods that relax conditions (a) and (b) and deal with unobserved market and firm heterogeneity.

Suppose that we have a random sample of M local markets, indexed by m, over T periods of time, where we observe:

$$Data = \{a_{mt},\ \mathbf{x}_{mt} : m = 1, 2, ..., M;\ t = 1, 2, ..., T\}$$

We want to use these data to estimate the model parameters in the population that has generated this data: $\theta^0 = \{\theta_i^0 : i \in I\}$.

Identification

A significant part of this literature has considered the following identification assumptions.

Assumption (ID 1): Single equilibrium in the data. Every observation in the sample comes from the same Markov Perfect Equilibrium, that is, for any observation (m,t), $\mathbf{P}_{mt}^0 = \mathbf{P}^0$.

Assumption (ID 2): No unobserved common-knowledge variables. The only unobservables for the econometrician are the private information shocks ε_{imt} and the structural parameters θ.

Comments on these assumptions: .... The assumption of no unobserved common knowledge variables (for instance, no unobserved market heterogeneity) is particularly strong.

It is possible to relax these assumptions. We will see later identification and estimation when we relax assumption ID 2. The following is a standard regularity condition.

Assumption (ID 3): For some benchmark choice alternative, say $a_i = 0$, define $Z_{imt} \equiv \tilde{z}_i^{\mathbf{P}^0}(a_{imt}, \mathbf{x}_{mt}) - \tilde{z}_i^{\mathbf{P}^0}(0, \mathbf{x}_{mt})$. Then, $\mathbb{E}(Z_{imt}' Z_{imt})$ is a non-singular matrix.

Under assumptions ID-1 to ID-3, the proof of identification is straightforward. First, under assumptions ID-1 and ID-2, the equilibrium that has generated the data, $\mathbf{P}^0$, can be estimated consistently and nonparametrically from the data. For any $(i, a_i, \mathbf{x})$:

$$P_i^0(a_i|\mathbf{x}) = \Pr(a_{imt} = a_i \mid \mathbf{x}_{mt} = \mathbf{x})$$

For instance, we can estimate consistently $P_i^0(a_i|\mathbf{x})$ using the following simple kernel estimator:

$$P_i^0(a_i|\mathbf{x}) = \frac{\sum_{m,t} 1\{a_{imt} = a_i\}\, K\left(\dfrac{\mathbf{x}_{mt} - \mathbf{x}}{b_n}\right)}{\sum_{m,t} K\left(\dfrac{\mathbf{x}_{mt} - \mathbf{x}}{b_n}\right)}$$

Second, given that $\mathbf{P}^0$ is identified, we can identify also the expected present values $\tilde{z}_i^{\mathbf{P}^0}(a_i, \mathbf{x})$ at the "true" equilibrium in the population. Third, we know that $\mathbf{P}^0$ is an equilibrium associated to θ^0. Therefore, the following equilibrium conditions should hold: for any $(i, a_i, \mathbf{x})$,

$$P_i^0(a_i|\mathbf{x}) = \frac{\exp\left\{\tilde{z}_i^{\mathbf{P}^0}(a_i, \mathbf{x}) \dfrac{\theta_i^0}{\sigma_\varepsilon^0}\right\}}{\sum_{a'=0}^{A} \exp\left\{\tilde{z}_i^{\mathbf{P}^0}(a', \mathbf{x}) \dfrac{\theta_i^0}{\sigma_\varepsilon^0}\right\}}$$

It is straightforward to show that under Assumption ID-3, these equilibrium conditions identify $\dfrac{\theta_i^0}{\sigma_\varepsilon^0}$. For instance, in this logit example, we have that for $(i, a_i, \mathbf{x})$,

$$\ln\left(\frac{P_i^0(a_i|\mathbf{x})}{P_i^0(0|\mathbf{x})}\right) = \left[\tilde{z}_i^{\mathbf{P}^0}(a_i, \mathbf{x}) - \tilde{z}_i^{\mathbf{P}^0}(0, \mathbf{x})\right] \frac{\theta_i^0}{\sigma_\varepsilon^0}$$

Define $Y_{imt} \equiv \ln\left(\frac{P_i^0(a_{imt}|\mathbf{x}_{mt})}{P_i^0(0|\mathbf{x}_{mt})}\right)$ and $Z_{imt} \equiv \tilde{z}_i^{\mathbf{P}^0}(a_{imt}, \mathbf{x}_{mt}) - \tilde{z}_i^{\mathbf{P}^0}(0, \mathbf{x}_{mt})$. Then,

$$Y_{imt} = Z_{imt}\, \frac{\theta_i^0}{\sigma_\varepsilon^0}$$

And we can also write this system as, $\mathbb{E}(Z'_{imt}Y_{imt}) = \mathbb{E}(Z'_{imt}Z_{imt})\ \frac{\theta_i^0}{\sigma_\varepsilon^0}$. Under assumption ID-3:

$$\frac{\theta_i^0}{\sigma_\varepsilon^0} = \mathbb{E}(Z'_{imt}Z_{imt})^{-1}\mathbb{E}(Z'_{imt}Y_{imt})$$

and $\frac{\theta_i^0}{\sigma_\varepsilon^0}$ is identified.

Note that under the single-equilibrium-in-the-data assumption, the multiplicity of equilibria in the model does not play any role in the identification of the structural parameters. The single-equilibrium-in-the-data assumption is sufficient for identification but it is not necessary. Sweeting (2009), **aguirregabiria_ mira_2009** (**aguirregabiria_ mira_2009**), and **paula_tang_2010** (**paula_tang_2010**) present conditions for the point-identification of games of incomplete information when there are multiple equilibria in the data.

Estimation

The use of an 'extended' or 'pseudo' likelihood (or alternatively GMM criterion) function plays an important role in the different estimation methods. For arbitrary values of the vector of structural parameters θ and firms' strategies $\mathbf{P}$, we define the following likelihood function of observed players' actions $\{a_{imt}\}$ conditional on observed state variables $\{\mathbf{x}_{mt}\}$:

$$Q(\theta,\mathbf{P}) = \sum_{i,m,t}\sum_{a_i=0}^{A} 1\{a_{imt} = a_i\}\ \ln\Psi_i(a_i,\mathbf{x}_{mt};\mathbf{P},\theta)$$

We call $Q(\theta,\mathbf{P})$ a 'Pseudo' Likelihood function because players' CCPs in $\mathbf{P}$ are arbitrary and do not represent the equilibrium probabilities associated with θ implied by the model.

An important implication of using arbitrary CCPs, instead of equilibrium CCPs, is that likelihood Q is a function and not a correspondence. To compute this pseudo likelihood, a useful construct is the representation of equilibrium in terms of CCPs, which we presented above.

We could also consider a Pseudo GMM Criterion function:

$$Q(\theta,\mathbf{P}) = -r(\theta,\mathbf{P})'\ \Omega\ r(\theta,\mathbf{P})$$

where Ω is the weighting matrix and $r(\theta,\mathbf{P})$ is the vector of moment conditions:

$$r(\theta,\mathbf{P}) = \frac{1}{MT}\sum_{m,t}\left[h(x_{mt})\otimes\begin{pmatrix} 1\{a_{imt} = a_i\} - \Psi_i(a_i,\mathbf{x}_{mt};\mathbf{P},\theta) \\ \cdots \\ \text{for any } (i,a_i) \end{pmatrix}\right]$$

and $h(x_{mt})$ is a vector of functions of x_{mt} (instruments).

Full Maximum Likelihood

The dynamic game imposes the restriction that the strategies in $\mathbf{P}$ should be in equilibrium. The ML estimator is defined as the pair $(\hat{\theta}_{MLE},\hat{\mathbf{P}}_{MLE})$ that maximizes the pseudo likelihood subject to the constraint that the strategies in $\hat{\mathbf{P}}_{MLE}$ are equilibrium strategies associated with $\hat{\theta}_{MLE}$. That is,

$$(\hat{\theta}_{MLE}, \hat{\mathbf{P}}_{MLE}) = \arg\max_{(\theta,\mathbf{P})} Q(\theta, \mathbf{P})$$

$$\text{s.t. } P_i(a_i|\mathbf{x}) = \Psi_i(a_i, \mathbf{x}; \mathbf{P}, \theta) \text{ for any } (i, a_i, \mathbf{x})$$

This is a constrained ML estimator that satisfies the standard regularity conditions for consistency, asymptotic normality and efficiency of ML estimation.

The numerical solution of the constrained optimization problem that defines these estimators requires one to search over an extremely large dimensional space. In the empirical applications of dynamic oligopoly games, the vector of probabilities $\mathbf{P}$ includes thousands or millions of elements. Searching for an optimum in that kind of space is computationally demanding. **su_judd_2008** (**su_judd_2008**) have proposed to use a MPEC algorithm, which is a general purpose algorithm for the numerical solution of constrained optimization problems. However, even using the most sophisticated algorithm such as MPEC, the optimization with respect to $(\mathbf{P}, \theta)$ can be extremely demanding when $\mathbf{P}$ has a high dimension.

Two-step methods

Let $\mathbf{P}^0$ be the vector with the population values of the probabilities $P_i^0(a_i|\mathbf{x}) \equiv \Pr(a_{imt} = a_i|\mathbf{x}_{mt} = \mathbf{x})$. Under the assumptions of "no unobserved common knowledge variables" and "single equilibrium in the data", the CCPs in $\mathbf{P}^0$ represent firms' strategies in the only equilibrium that is played in the data. These probabilities can be estimated consistently using standard nonparametric methods. Let $\hat{\mathbf{P}}^0$ be a consistent nonparametric estimator of $\mathbf{P}^0$. Given $\hat{\mathbf{P}}^0$, we can construct a consistent estimator of $\tilde{z}_i^{\mathbf{P}^0}(a_i, \mathbf{x})$. Then, the two-step estimator of θ^0 is defined as:

$$\hat{\theta}_{2S} = \arg\max_{\theta} Q(\theta, \hat{\mathbf{P}}^0)$$

After the computation of the expected present values $\tilde{z}_i^{\mathbf{P}^0}(a_i, \mathbf{x})$, this second step of the procedure is computationally very simple. It consists only in the estimation of a standard discrete choice model, for instance, a binary probit/logit in our entry-exit example, or a conditional logit in our example with quality choice. Under standard regularity conditions, this two-step estimator is root-M consistent and asymptotically normal.

This idea was originally exploited, for estimation of single agent problems, by Hotz and Miller (1993) and Hotz et al. (1994). It was expanded to the estimation of dynamic games by **aguirregabiria_ mira_2007** (**aguirregabiria_ mira_2007**), Bajari, Benkard, and Levin (2007), Pakes, Ostrovsky, and Berry (2007), and Pesendorfer and Schmidt-Dengler (2008).

The main advantage of these two-step estimators is their computational simplicity. The first step is a simple nonparametric regression, and the second step is the estimation of a standard discrete choice model with a criterion function that in most applications is globally concave (for instance, such as the likelihood of a standard probit model in our entry-exit example). The main computational burden comes from the calculation of the present values $W_i^{\hat{\mathbf{P}}}(\mathbf{x})$. Though the computation of these present values may be subject to a curse of dimensionality, the cost of obtaining a two-step estimator is several orders of magnitude smaller than solving (just once) for an equilibrium of the dynamic game. In most applications, this makes the difference between being able to estimate the model or not.

However, these two-step estimators have some important limitations. A first limitation is the restrictions imposed by the assumption of no unobserved common knowledge variables. Ignoring persistent unobservables, if present, can generate important biases in the estimation of structural parameters. We deal with this issue later.

A second problem is finite sample bias. The finite sample bias of the two-step estimator of θ^0 depends very importantly on the properties of the first-step estimator of $\mathbf{P}^0$. In particular, it depends on the rate of convergence and on the variance and bias of $\hat{\mathbf{P}}^0$. It is well-known that there is a **curse of dimensionality in the nonparametric estimation** of a regression function such as $\mathbf{P}^0$. The rate of convergence of the estimator (and its asymptotic variance) declines (increase) with the number of explanatory variables in the regression. The initial nonparametric estimator can be very imprecise in the samples available in actual applications, and this can generate serious finite sample biases in the two-step estimator of structural parameters.

In dynamic games with heterogeneous players, the number of observable state variables is proportional to the number of players, and therefore the so called *curse of dimensionality in nonparametric estimation* (and the associated bias of the two-step estimator) can be particularly serious. For instance, in our dynamic game of product quality choice, the vector of state variables contains the qualities of the N firms.

The source of this bias is well understood in two-step methods: $\hat{\mathbf{P}}$ enters nonlinearly in the sample moment conditions that define the estimator, and the expected value of a nonlinear function of $\hat{\mathbf{P}}$ is not equal to that function evaluated at the expected value of $\hat{\mathbf{P}}$. The larger the variance or the bias of $\hat{\mathbf{P}}$, the larger the bias of the two-step estimator of θ_0. To see this, note that the PML or GMM estimators in the second step are based on moment conditions at the true $\mathbf{P}^0$:

$$\mathbb{E}\left(\, h(x_{mt}) \ \left[1\{a_{imt} = a_i\} - \Psi_i(a_i, \mathbf{x}; \mathbf{P^0}, \theta)\right] \,\right) \ = \ 0$$

The same moment conditions evaluated at $\hat{\mathbf{P}}^0$ do not hold because of the estimation error:

$$\mathbb{E}\left(\, h(x_{mt}) \ \left[1\{a_{imt} = a_i\} - \Psi_i(a_i, \mathbf{x}; \hat{\mathbf{P}}^{\mathbf{0}}, \theta_0)\right] \,\right) \ \neq \ 0$$

This generates a finite sample bias. The best response function $\Psi_i(a_i, \mathbf{x}; \hat{\mathbf{P}}^{\mathbf{0}}, \theta_0)$ is a nonlinear function of the random vector $\hat{\mathbf{P}}_0$, and the expected value of a nonlinear function is not equal to the function evaluated at the expected value. The larger the finite sample bias or the variance of $\hat{\mathbf{P}}^{\mathbf{0}}$, the larger the bias of the two-step estimation of θ_0.

Recursive K-step estimators

To deal with finite sample bias, **aguirregabiria_ mira_2002** (**aguirregabiria_ mira_2002**,**aguirregabiria_** consider a recursive K-step extension. Given the two-step estimator $\hat{\theta}_{2S}$ and the initial nonparametric estimator of CCPs, $\hat{\mathbf{P}}^0$, we can construct a new estimator of CCPs, $\hat{\mathbf{P}}^1$, such that, for any $(i, a_i, \mathbf{x})$:

$$\hat{P}_i^1(a_i|\mathbf{x}) = \Psi_i(a_i, \mathbf{x}; \hat{\mathbf{P}}^{\mathbf{0}}, \hat{\theta}_{2S})$$

or in our example:

$$\hat{P}_i^1(a_i|\mathbf{x}) = \frac{\exp\left\{\tilde{z}_i^{\hat{\mathbf{P}}^0}(a_i,\mathbf{x})\ \hat{\theta}_{i,2S}\right\}}{\sum_{a'=0}^{A}\exp\left\{\tilde{z}_i^{\hat{\mathbf{P}}^0}(a',\mathbf{x})\ \hat{\theta}_{i,2S}\right\}}$$

This new estimator of CCPs exploits the parametric structure of the model, and the structure of best response functions. It seems intuitive that this new estimator of CCPs has better statistical properties than the initial nonparametric estimator, that is, smaller asymptotic variance, and smaller finite sample bias and variance. As we explain below, this intuition is correct as long as the equilibrium that generated the data is (Lyapunov) stable.

Under this condition, it seems natural to obtain a new two-step estimator by replacing $\hat{\mathbf{P}}^0$ with $\hat{\mathbf{P}}^1$ as the estimator of CCPs. Then, we can obtain the new estimator:

$$\hat{\theta} = \arg\max_{\theta} Q(\theta, \hat{\mathbf{P}}^1)$$

The same argument can be applied recursively to generate a sequence of $K-step$ estimators. Given an initial consistent nonparametric estimator $\hat{\mathbf{P}}^0$, the sequence of estimators $\{\hat{\theta}^K, \hat{\mathbf{P}}^K : K \geq 1\}$ is defined as:

$$\hat{\theta}^K = \arg\max_{\theta} Q(\theta, \hat{\mathbf{P}}^{K-1})$$

and

$$\hat{\mathbf{P}}^K = \Psi(\hat{\mathbf{P}}^{K-1}, \hat{\theta}^K)$$

Monte Carlo experiments in **aguirregabiria_ mira_2002** (**aguirregabiria_ mira_2002**,**aguirregabiri** and **kasahara_shimotsu_2008a** (**kasahara_shimotsu_2008a**,**kasahara_shimotsu_2009**) show that iterating in the NPL mapping can reduce significantly the finite sample bias of the two-step estimator. The Monte Carlo experiments in Pesendorfer and Schmidt-Dengler (2008) present a different, more mixed, picture. While for some of their experiments NPL iteration reduces the bias, in other experiments the bias remains constant or even increases. A closer look at the Monte Carlo experiments in Pesendorfer and Schmidt-Dengler shows that the NPL iterations provide poor results in those cases where the equilibrium that generates the data is not (Lyapunov) stable. As we explain below, this is not a coincidence. It turns out that the computational and statistical properties of the sequence of K-step estimators depend critically on the stability of the NPL mapping around the equilibrium in the data.

Convergence properties of recursive K-step estimators

To study the properties of these K-step estimators, it is convenient to represent the sequence $\{\hat{\mathbf{P}}^K : K \geq 1\}$ as the result of iterating in a fixed point mapping. For arbitrary $\mathbf{P}$, define the mapping:

$$\varphi(\mathbf{P}) \equiv \Psi(\mathbf{P},\ \hat{\theta}(\mathbf{P}))$$

where $\hat{\theta}(\mathbf{P}) \equiv \arg\max_{\theta}\ Q(\theta, \mathbf{P})$. The mapping $\varphi(\mathbf{P})$ is called the Nested Pseudo Likelihood (NPL) mapping.

The sequence of estimators $\{\hat{\mathbf{P}}^K : K \geq 1\}$ can be obtained by successive iterations in the mapping φ starting with the nonparametric estimator $\hat{\mathbf{P}}^0$, that is, for $K \geq 1$, $\hat{\mathbf{P}}^K = \varphi(\hat{\mathbf{P}}^{K-1})$.

Lyapunov stability. Let $\mathbf{P}^*$ be a fixed point of the NPL mapping such that $\mathbf{P}^* = \varphi(\mathbf{P}^*)$. We say that the mapping φ is Lyapunov-stable around the fixed point $\mathbf{P}^*$ if there is a neighborhood of $\mathbf{P}^*$, $\mathscr{N}$, such that successive iterations in the mapping φ starting at $\mathbf{P} \in \mathscr{N}$ converge to $\mathbf{P}^*$. A necessary and sufficient condition for Lyapunov stability is that the *spectral radius* of the Jacobian matrix $\partial\varphi(\mathbf{P}^*)/\partial\mathbf{P}'$ is smaller than one. The neighboring set $\mathscr{N}$ is denoted the dominion of attraction of the fixed point $\mathbf{P}^*$. The spectral radius of a matrix is the maximum absolute eigenvalue. If the mapping φ is twice continuously differentiable, then the spectral radius is a continuous function of $\mathbf{P}$. Therefore, if φ is Lyapunov stable at $\mathbf{P}^*$, for any $\mathbf{P}$ in the dominion of attraction of $\mathbf{P}^*$ we have that the spectral radius of $\partial\varphi(\mathbf{P})/\partial\mathbf{P}'$ is also smaller than one. Similarly, if $\mathbf{P}^*$ is an equilibrium of the mapping $\Psi(.,\theta)$, we say that this mapping is Lyapunov stable around $\mathbf{P}^*$ if and only if the *spectral radius* of the Jacobian matrix $\partial\Psi(\mathbf{P}^*,\theta)/\partial\mathbf{P}'$ is smaller than one.

There is a relationship between the stability of the NPL mapping and of the equilibrium mapping $\Psi\left(.,\theta^0\right)$ around $\mathbf{P}^0$ (that is, the equilibrium that generates the data). The Jacobian matrices of the NPL and equilibrium mapping are related by the following expression (see **kasahara_shimotsu_2009** ,**kasahara_shimotsu_2009**):

$$\frac{\partial\varphi\left(\mathbf{P}^0\right)}{\partial\mathbf{P}'} = M(\mathbf{P}^0)\,\frac{\partial\Psi(\mathbf{P}^0,\theta^0)}{\partial\mathbf{P}'}$$

where $M(\mathbf{P}^0)$ is an idempotent projection matrix $I - \Psi_\theta(\Psi'_\theta\ diag\{\mathbf{P}^0\}^{-1}\ \Psi_\theta)^{-1}\Psi'_\theta$ $diag\{\mathbf{P}^0\}^{-1}$, where $\Psi_\theta \equiv \partial\Psi(\mathbf{P}^0,\theta^0)/\partial\theta'$. In single-agent dynamic programming models, the Jacobian matrix $\partial\Psi\left(\mathbf{P}^0,\theta^0\right)/\partial\mathbf{P}'$ is zero (that is, zero Jacobian matrix property, **aguirregabiria_ mira_2002** ,**aguirregabiria_ mira_2002**). Therefore, for that class of models $\partial\varphi\left(\mathbf{P}^0\right)/\partial\mathbf{P}' = 0$ and the NPL mapping is Lyapunov stable around $\mathbf{P}^0$. In dynamic games, $\partial\Psi\left(\mathbf{P}^0,\theta^0\right)/\partial\mathbf{P}'$ is not zero. However, given that $M(\mathbf{P}^0)$ is an idempotent matrix, it is possible to show that the spectral radius of $\partial\varphi\left(\mathbf{P}^0\right)/\partial\mathbf{P}'$ is not larger than the spectral radius of $\partial\Psi(\mathbf{P}^0,\theta^0)/\partial\mathbf{P}'$. Therefore, Lyapunov stability of $\mathbf{P}^0$ in the equilibrium mapping implies stability of the NPL mapping.

Convergence of NPL iterations. Suppose that the true equilibrium in the population, $\mathbf{P}^0$, is Lyapunov stable with respect to the NPL mapping. This implies that with probability approaching one, as M goes to infinity, the (sample) NPL mapping is stable around a consistent nonparametric estimator of $\mathbf{P}^0$. Therefore, the sequence of K-step estimators converges to a limit $\hat{\mathbf{P}}^0_{\text{lim}}$ that is a fixed point of the NPL mapping, that is, $\hat{\mathbf{P}}^0_{\text{lim}} = \varphi(\hat{\mathbf{P}}^0_{\text{lim}})$. It is possible to show that this limit $\hat{\mathbf{P}}^0_{\text{lim}}$ is a consistent estimator of $\mathbf{P}^0$ (see **kasahara_shimotsu_2009** ,**kasahara_shimotsu_2009**). Therefore, under Lyapunov stability of the NPL mapping, if we start with a consistent estimator of $\mathbf{P}^0$ and iterate in the NPL mapping, we converge to a consistent estimator that is an equilibrium of the model. It is possible to show that this estimator is asymptotically more efficient than the two-step estimator (**aguirregabiria_ mira_2007** ,**aguirregabiria_ mira_2007**).

Pesendorfer and Schmidt-Dengler (2010) present an example where the sequence of K-step estimators converges to a limit estimator that is not consistent. As implied by the results presented above, the equilibrium that generates the data in their example is not Lyapunov stable. The concept of Lyapunov stability of the best response mapping at an equilibrium means that if we marginally perturb players' strategies, and then allow players to best respond to the new strategies, then we will converge to the original equilibrium. This seems like a plausible equilibrium selection criterion. Ultimately, whether an unstable equilibrium is interesting depends on the application and the researcher's taste. Nevertheless, at the end of this section we present simple modified versions of the NPL method that can deal with data generated from an equilibrium that is not stable.

Reduction of finite sample bias

kasahara_shimotsu_2008a (**kasahara_shimotsu_2008a**,**kasahara_shimotsu_2009**) derive a second order approximation to the bias of the K-step estimators. They show that the key component in this bias is the distance between the first step and the second step estimators of $\mathbf{P}^0$, that is, $\left\|\varphi\left(\hat{\mathbf{P}}^0\right)-\hat{\mathbf{P}}^0\right\|$. An estimator that reduces this distance is an estimator with lower finite sample bias. Therefore, based on our discussion in point (b) above, the sequence of K-step estimators are decreasing in their finite sample bias if and only if the NPL mapping is Lyapunov stable around $\mathbf{P}^0$.

The Monte Carlo experiments in Pesendorfer and Schmidt-Dengler (2008) illustrate this point. They implement experiments using different DGPs: in some of them the data is generated from a stable equilibrium, and in others the data come from a non-stable equilibrium. It is simple to verify (see **aguirregabiria_ mira_2010** ,**aguirregabiria_ mira_2010**) that the experiments where NPL iterations do not reduce the finite sample bias are those where the equilibrium that generates the data is not (Lyapunov) stable.

Modified NPL algorithms

Note that Lyapunov stability can be tested after obtaining the first NPL iteration. Once we have obtained the two-step estimator, we can calculate the Jacobian matrix $\partial\varphi(\hat{\mathbf{P}}^0)/\partial\mathbf{P}'$ and its eigenvalues, and then check whether Lyapunov stability holds at $\hat{\mathbf{P}}^0$.

If the applied researcher considers that her data may have been generated by an equilibrium that is not stable, then it will be worthwhile to compute this Jacobian matrix and its eigenvalues. If Lyapunov stability holds at $\hat{\mathbf{P}}^0$, then we know that NPL iterations reduce the bias of the estimator and converge to a consistent estimator.

When the condition does not hold, then the solution to this problem is not simple. Though the researcher might choose to use the two-step estimator, the non-stability of the equilibrium has also important negative implications on the properties of this simple estimator. Non-stability of the NPL mapping at $\mathbf{P}^0$ implies that the asymptotic variance of the two-step estimator of $\mathbf{P}^0$ is larger than the asymptotic variance of the nonparametric reduced form estimator. To see this, note that the two-step estimator of CCPs is $\hat{\mathbf{P}}^1 = \varphi(\hat{\mathbf{P}}^0)$, and applying the delta method we have that $Var(\hat{\mathbf{P}}^1) = [\partial\varphi\left(\mathbf{P}^0\right)/\partial\mathbf{P}']\ Var(\hat{\mathbf{P}}^{\mathbf{0}})$ $[\partial\varphi\left(\mathbf{P}^0\right)/\partial\mathbf{P}']'$. If the spectral radius of $\partial\varphi\left(\mathbf{P}^0\right)/\partial\mathbf{P}'$ is greater than 1, then $Var(\hat{\mathbf{P}}^1) > Var(\hat{\mathbf{P}}^0)$. This is a puzzling result because the estimator $\hat{\mathbf{P}}^0$ is nonparametric while the estimator $\hat{\mathbf{P}}^1$ exploits most of the structure of the model. Therefore, the non-stability of the equilibrium that generates the data is an issue for this general class of two-step or sequential estimators.

In this context, **kasahara_shimotsu_2009** (**kasahara_shimotsu_2009**) propose alternative recursive estimators based on fixed-point mappings other than the NPL that, by construction, are stable. Iterating in these alternative mappings is significantly more costly than iterating in the NPL mapping, but these iterations guarantee reduction of the finite sample bias and convergence to a consistent estimator.

aguirregabiria_ mira_2010 (**aguirregabiria_ mira_2010**) propose two modified versions of the NPL algorithm that are simple to implement and that always converge to a consistent estimator with better properties than two-step estimators. A *first modified-NPL-algorithm* applies to dynamic games. The first NPL iteration is standard but in every successive iteration best response mappings are used to update guesses of each player's own future behavior without updating beliefs about the strategies of the other players. This algorithm always converges to a consistent estimator even if the equilibrium generating the data is not stable and it reduces monotonically the asymptotic variance and the finite sample bias of the two-step estimator.

The *second modified-NPL-algorithm* applies to static games and it consists in the application of the standard NPL algorithm both to the best response mapping and to the inverse of this mapping. If the equilibrium that generates the data is unstable in the best response mapping, it should be stable in the inverse mapping. Therefore, the NPL applied to the inverse mapping should converge to the consistent estimator and should have a larger value of the pseudo likelihood than the estimator that we converge to when applying the NPL algorithm to the best response mapping. Aguirregabiria and Mira illustrate the performance of these estimators using the examples in Pesendorfer and Schmidt-Dengler (2008, 2010).

Estimation using Moment Inequalities

Bajari, Benkard, and Levin (2007) proposed a two-step estimator in the spirit of the ones described before but with two important differences:

(a) they use moment inequalities (instead of moment equalities);
(b) they do not calculate exactly the present value $\mathbf{W}_i^{\mathbf{P}}(\mathbf{x}_t)$ but they approximate them using Monte Carlo simulation.

(a) and (b) are two different ideas than can be applied separately. Both of these two ideas have different merits and therefore we will discuss them separately.

Estimation using Moment Inequalities. Remember that $V_i^{\mathbf{P}}(\mathbf{x}_t)$ is the value of player i at state $\mathbf{x}_t$ when all the players behave according to their strategies in $\mathbf{P}$. In a model where the one-period payoff function is multiplicatively separable in the structural parameters, we have that

$$V_i^{\mathbf{P}}(\mathbf{x}_t) \quad = \quad W_i^{\mathbf{P}}(\mathbf{x}_t)\ \theta_i$$

and the matrix of present values $\mathbf{W}_i^{\mathbf{P}} \equiv \{W_i^{\mathbf{P}}(\mathbf{x}_t) : \mathbf{x}_t \in X\}$ can be obtained exactly as:

$$\mathbf{W}_i^{\mathbf{P}} \equiv \left(\mathbf{I} - \beta\ \mathbf{F}_i^{\mathbf{P}}\right)^{-1} \bar{\mathbf{z}}_i^{\mathbf{P}}$$

For notational simplicity, we will use $W_{it}^{\mathbf{P}}$ to represent $W_i^{\mathbf{P}}(\mathbf{x}_t)$.

Let's split the vector of choice probabilities $\mathbf{P}$ into the sub-vectors $\mathbf{P}_i$ and $\mathbf{P}_{-i}$,

$$\mathbf{P} \equiv (\mathbf{P}_i\ ,\ \mathbf{P}_{-i})$$

where $\mathbf{P}_i$ are the probabilities associated to player i and $\mathbf{P}_{-i}$ contains the probabilities of players other than i. $\mathbf{P}^0$ is an equilibrium associated to θ^0. Therefore, $\mathbf{P}_i^0$ is firm i's best response to $\mathbf{P}_{-i}^0$, and for any $\mathbf{P}_i \neq \mathbf{P}_i^0$ the following inequality should hold:

$$W_{it}^{\left(\mathbf{P}_i^0, \mathbf{P}_{-i}^0\right)} \ \theta_i^0 \geq W_{it}^{\left(\mathbf{P}_i, \mathbf{P}_{-i}^0\right)} \ \theta_i^0$$

We can define an estimator of θ^0 based on these (moment) inequalities. There are infinite alternative policies $\mathbf{P}_i$, and therefore there are infinite moment inequalities. For estimation, we should select a finite set of alternative policies. This is a very important decision for this class of estimators (more below). Let H be a **(finite) set of alternative policies** for each player. Define the following criterion function:

$$R\left(\theta, \mathbf{P}^0\right) \equiv \sum_{i,m,t} \sum_{\mathbf{P} \in H} \left(\min \left\{ 0 \ ; \ \left[W_{imt}^{\left(\mathbf{P}_i^0, \mathbf{P}_{-i}^0\right)} - W_{imt}^{\left(\mathbf{P}_i, \mathbf{P}_{-i}^0\right)} \right] \theta_i \right\} \right)^2$$

This criterion function penalizes departures from the inequalities. Then, given an initial NP estimator of $\mathbf{P}^0$, say $\hat{\mathbf{P}}^0$, we can define the following estimator of θ^0 based on moment inequalities (MI):

$$\hat{\theta} = \arg\min_{\theta} \ R\left(\theta, \hat{\mathbf{P}}^0\right)$$

There are several relevant comments to make on this MI estimator: **(1)** Computational properties (relative to two-step ME estimators); **(2)** Point identification / Set identification; **(3)** How to choose the set of alternative policies?; **(4)** Statistical properties; **(5)** Continuous decision variables.

Computational Properties. **The two-step MI estimator is more computationally costly than a two-step ME estimator.** There at least three factors that contribute to this larger cost.

> **(a)** In both types of estimators, the main cost comes from calculating the present values $\mathbf{W}_i^{\mathbf{P}}$. In a 2-step ME estimator this evaluation is done once. In the MI estimator this is done as many times as there are alternative policies in the set H;
>
> **(b)** The ME criterion function $Q\left(\theta, \hat{\mathbf{P}}\right)$ is typically globally concave in θ, but $R\left(\theta, \hat{\mathbf{P}}\right)$ is not;
>
> **(c)** Set estimation versus point estimation. The MI estimator needs an algorithm for set optimization.

MI Estimator: Point / Set identification. This estimator is based on exactly the same assumptions as the 2-step moment equalities (ME) estimator. We have seen that θ^0 **is point identified** by the moment equalities of the ME estimators (for instance, by the pseudo likelihood equations). Therefore, if the set H of alternative policies is large enough, then θ^0 should be point identified as the unique minimizer of $R\left(\theta, \mathbf{P}^0\right)$. However, it is very costly to consider a set H with many alternative policies. For the type of H sets which are considered in practice, minimizing $R\left(\theta, \mathbf{P}^0\right)$ does not uniquely identify θ^0. Therefore, θ^0 **is set identified.**

How to choose the set of alternative policies? The choice of the alternative policies in the set H plays a key role in the statistical properties (for instance, precision, bias) of this estimator. However, there is no clear rule on how to select these policies.

Statistical properties of MI estimator (relative to ME). The MI estimator is not more 'robust' than the ME estimator. Both estimators are based on exactly the same model and assumptions. Set identification. **Asymptotically, the MI estimator is less efficient than the ME estimator.** The efficient 2-step Moment Equalities (ME) estimator has lower asymptotic variance than the MI estimator, even as the set H becomes very large.

Continuous decision variables. BBL show that, when combined with simulation techniques to approximate the values $\{W_{it}^{\mathbf{P}}\}$, the MI approach can be easily applied to the **estimation of dynamic games with continuous decision variables.** In fact, the BBL estimator of a model with continuous decision variable is basically the same as with a discrete decision variable**.** The ME estimator of models with continuous decision variable may be more complicated.

A different approach to construct inequalities in dynamic games. In a MPE a player's equilibrium strategy is her best response not only within the class of Markov strategies but also within the class of non Markov strategies: for instance, strategies that vary over time. Maskin and Tirole: if all the other players use Markov strategies, a player does not have any gain from using non Markov strategies.

Suppose that to construct the inequalities $W_{it}^{\left(\mathbf{P}_i^0,\mathbf{P}_{-i}^0\right)}\ \theta_i^0 \geq W_{it}^{\left(\mathbf{P}_i,\mathbf{P}_{-i}^0\right)}\ \theta_i^0$ we use alternative strategies which are non-Markov.

In a MPE a player equilibrium strategy is her best response not only within the class of Markov strategies but also within the class of non Markov strategies: for instance, strategies that vary over time. Maskin and Tirole: if all the other players use Markov strategies, a player does not have any gain from using non Markov strategies.

More specifically, suppose that the alternative strategy of player i is

$$\mathbf{P}_i = \{P_{it}(\mathbf{x}_t) : t = 1,2,3,....;\ \mathbf{x}_t \in X\}$$

with the following features.

(a) Two-periods deviation: $P_{it} \neq P_i^0$, $P_{it+1} \neq P_i^0$, but $P_{it+s} = P_i^0$for any $s \geq 2$;

(b) $P_{it+1}$is constructed in such a way that it compensates the effects of the perturbation P_{it}on the distribution of $\mathbf{x}_{t+2}$ conditional on $\mathbf{x}_t$, that is,

$$F_x^{\mathbf{P}^0(2)}(\mathbf{x}_{t+2} \mid \mathbf{x}_t) = F_x^{\mathbf{P}^0(2)}(\mathbf{x}_{t+2} \mid \mathbf{x}_t)$$

Given this type of alternative policies, we have that the value differences

$$W_{it}^{\left(\mathbf{P}_i^0,\mathbf{P}_{-i}^0\right)}\ \theta_i^0 \geq W_{it}^{\left(\mathbf{P}_i,\mathbf{P}_{-i}^0\right)}\ \theta_i^0$$

only depend on differences between expected payoffs at periods t and $t+1$. We do not have to use simulation, invert huge matrices, etc, and we can consider thousands (or even millions) of alternative policies.

Dealing with Unobserved Heterogeneity

So far, we have maintained the assumption that the only unobservables for the researcher are the private information shocks that are i.i.d. over firms, markets, and time. In most applications in IO, this assumption is not realistic and it can be easily rejected by the data. Markets and firms are heterogenous in terms of characteristics that are payoff-relevant for firms but unobserved to the researcher. Not accounting for this heterogeneity may generate significant biases in parameter estimates and in our understanding of competition in the industry.

For instance, in the empirical applications in Aguirregabiria and Mira (2007) and Collard-Wexler (2006), the estimation of a model without unobserved market heterogeneity implies estimates of strategic interaction between firms (that is, competition effects) that are close to zero or even have the opposite sign to the one expected under competition. In both applications, including unobserved heterogeneity in the models results in estimates that show significant and strong competition effects.

Aguirregabiria and Mira (2007) and Arcidiacono and Miller (2008) have proposed methods for the estimation of dynamic games that allow for persistent unobserved heterogeneity in players or markets. Here we concentrate on the case of permanent unobserved market heterogeneity in the profit function.

$$\Pi_{imt} \quad = \quad z_i^{\mathbf{P}}(a_i, \mathbf{x}_{mt})\ \theta_{ii}^{EC} - \sigma_{\xi i}\ \xi_m - \varepsilon_{imt}$$

$\sigma_{\xi i}$ is a parameter, and ξ_m is a time-invariant 'random effect' that is common knowledge to the players but unobserved to the researcher.

The distribution of this random effect has the following properties: (A.1) it has a discrete and finite support $\{\xi^1, \xi^2, \ldots, \xi^L\}$, each value in the support of ξ represents a 'market type', and we index market types by $\ell \in \{1,2,...,L\}$; (A.2) it is i.i.d. over markets with probability mass function $\lambda_\ell \equiv \Pr(\xi_m = \xi^\ell)$; and (A.3) it does not enter into the transition probability of the observed state variables, that is, $\Pr(\mathbf{x}_{mt+1} \mid \mathbf{x}_{mt}, \mathbf{a}_{mt}, \xi_m) = F_x(\mathbf{x}_{mt+1} \mid \mathbf{x}_{mt}, \mathbf{a}_{mt})$. Without loss of generality, ξ_m has mean zero and unit variance because the mean and the variance of ξ_m are incorporated in the parameters θ_i^{FC} and $\sigma_{\xi i}$, respectively. Also, without loss of generality, the researcher knows the points of support $\{\xi^\ell : \ell = 1,2,\ldots,L\}$ though the probability mass function $\{\lambda_\ell\}$ is unknown.

Assumption (A.1) is common when dealing with permanent unobserved heterogeneity in dynamic structural models. The discrete support of the unobservable implies that the contribution of a market to the likelihood (or pseudo likelihood) function is a finite mixture of likelihoods under the different possible best responses that we would have for each possible market type. With continuous support we would have an infinite mixture of best responses and this could complicate significantly the computation of the likelihood. Nevertheless, as we illustrate before, using a pseudo likelihood approach and a convenient parametric specification of the distribution of ξ_m simplifies this computation such that we can consider many values in the support of this unobserved variable at a low computational cost. Assumption (A.2) is also standard when dealing with unobserved heterogeneity. Unobserved spatial correlation across markets does not generate inconsistency of the estimators that we present here because the likelihood equations that define the estimators are still valid moment conditions under spatial correlation. Incorporating spatial correlation in the model, if present in the data, would improve the efficiency of the estimator but at a significant computational cost. Assumption (A.3) can be relaxed, and in fact the method by Arcidiacono-Miller deals with unobserved heterogeneity both in payoffs and transition probabilities.

Each market type ℓ has its own equilibrium mapping (with a different level of profits given ξ^{ℓ}) and its own equilibrium. Let $\mathbf{P}_{\ell}$ be a vector of strategies (CCPs) in market-type ℓ: $\mathbf{P}_{\ell} \equiv \{P_{i\ell}(\mathbf{x}_t) : i = 1,2,...,N;\ \mathbf{x}_t \in \mathscr{X}\}$. The introduction of unobserved market heterogeneity also implies that we can relax the assumption of only ‘a single equilibrium in the data’ to allow for different market types having different equilibria. It is straightforward to extend the description of an equilibrium mapping in CCPs to this model. A vector of CCPs $\mathbf{P}_{\ell}$ is a MPE for market type ℓ if and only if for every firm i and every state $\mathbf{x}_t$ we have that: $P_{i\ell}(\mathbf{x}_t) = \Phi\left(\tilde{\mathbf{z}}_i^{\mathbf{P}_{\ell}}(\mathbf{x}_t, \xi^{\ell})\ \theta_i + \tilde{e}_i^{\mathbf{P}_{\ell}}(\mathbf{x}_t, \xi^{\ell})\right)$, where now the vector of structural parameters θ_i is $\left\{\theta_{i,0}^{VP},\ ...,\ \theta_{i,N-1}^{VP},\ \theta_i^{FC},\ \theta_i^{EC},\ \sigma_{\xi i}\right\}$ which includes $\sigma_{\xi i}$, and the vector $\tilde{\mathbf{z}}_i^{\mathbf{P}_{\ell}}(\mathbf{x}_t, \xi^{\ell})$ has a similar definition as before with the only difference being that it has one more component associated with $-\xi^{\ell}$. Since the points of support $\{\xi^{\ell} : \ell = 1,2,\ldots,L\}$ are known to the researcher, she can construct the equilibrium mapping for each market type.

Let λ be the vector of parameters in the probability mass function of ξ, that is, $\lambda \equiv \{\lambda_{\ell} : \ell = 1,2,...,L\}$, and let $\mathbf{P}$ be the set of CCPs for every market type, $\{\mathbf{P}_{\ell} : \ell = 1,2,...,L\}$. The (conditional) pseudo log likelihood function of this model is $Q(\theta,\lambda,\mathbf{P}) = \sum_{m=1}^{M} \log \Pr(\mathbf{a}_{m1},\ \mathbf{a}_{m2},\ ...,\ \mathbf{a}_{mT} \mid \mathbf{x}_{m1},\ \mathbf{x}_{m2},...,\ \mathbf{x}_{mT};\ \theta,\lambda,\mathbf{P})$. We can write this function as $\sum_{m=1}^{M} \log q_m(\theta,\lambda,\mathbf{P})$, where $q_m(\theta,\lambda,\mathbf{P})$ is the contribution of market m to the pseudo likelihood:

$$q_m(\theta,\lambda,\mathbf{P}) \quad = \quad \sum_{\ell=1}^{L} \lambda_{\ell|\mathbf{x}_{m1}} \left[\prod_{i,t} \Phi\left(\tilde{\mathbf{z}}_{im\ell t}^{\mathbf{P}_{\ell}}\ \theta_i + \tilde{e}_{im\ell t}^{\mathbf{P}_{\ell}}\right)^{a_{imt}} \Phi\left(-\tilde{\mathbf{z}}_{im\ell t}^{\mathbf{P}_{\ell}}\ \theta_i - \tilde{e}_{im\ell t}^{\mathbf{P}_{\ell}}\right)^{1-a_{imt}}\right]$$

where $\tilde{\mathbf{z}}_{im\ell t}^{\mathbf{P}_{\ell}} \equiv \tilde{\mathbf{z}}_i^{\mathbf{P}_{\ell}}(\mathbf{x}_{mt}, \xi^{\ell})$, $\tilde{e}_{im\ell t}^{\mathbf{P}_{\ell}} \equiv \tilde{e}_i^{\mathbf{P}_{\ell}}(\mathbf{x}_{mt}, \xi^{\ell})$, and $\lambda_{\ell|\mathbf{x}}$ is the conditional probability $\Pr(\xi_m = \xi^{\ell}|\mathbf{x}_{m1} = \mathbf{x})$. The conditional probability distribution $\lambda_{\ell|x}$ is different from the unconditional distribution λ_{ℓ}. In particular, ξ_m is not independent of the predetermined endogenous state variables that represent market structure. For instance, we expect a negative correlation between the indicators of incumbent status, s_{imt}, and the unobserved component of the fixed cost ξ_m, that is, markets where it is more costly to operate tend to have a smaller number of incumbent firms. This is the so called *initial conditions problem* (Heckman ,1981). In short panels (for T relatively small), not taking into account this dependence between ξ_m and $\mathbf{x}_{m1}$ can generate significant biases, similar to the biases associated with ignoring the existence of unobserved market heterogeneity. There are different ways to deal with the initial conditions problem in dynamic models (Heckman ,1981). One possible approach is to derive the joint distribution of $\mathbf{x}_{m1}$ and ξ_m implied by the equilibrium of the model. That is the approach proposed and applied in Aguirregabiria and Mira (2007) and Collard-Wexler (2006). Let $\mathbf{p}^{\mathbf{P}_{\ell}} \equiv \{p^{\mathbf{P}_{\ell}}(\mathbf{x}_t) : \mathbf{x}_t \in \mathscr{X}\}$ be the ergodic or steady-state distribution of $\mathbf{x}_t$ induced by the equilibrium $\mathbf{P}_{\ell}$ and the transition F_x. This stationary distribution can be simply obtained as the solution to the following system of linear equations: for every value $\mathbf{x}_t \in \mathscr{X}$, $p^{\mathbf{P}_{\ell}}(\mathbf{x}_t) = \sum_{\mathbf{x}_{t-1} \in \mathscr{X}} p^{\mathbf{P}_{\ell}}(\mathbf{x}_{t-1})\ F_x^{\mathbf{P}_{\ell}}(\mathbf{x}_t \mid \mathbf{x}_{t-1})$, or in vector form, $\mathbf{p}^{\mathbf{P}_{\ell}} = \mathbf{F}_x^{\mathbf{P}_{\ell}}\ \mathbf{p}^{\mathbf{P}_{\ell}}$ subject to $\mathbf{p}^{\mathbf{P}_{\ell}\prime}\mathbf{1} = 1$. Given the ergodic distributions for the L market types, we can apply Bayes’ rule to obtain:

$$\lambda_{\ell|\mathbf{x}_{m1}} = \frac{\lambda_\ell \, p^{\mathbf{P}_\ell}(\mathbf{x}_{m1})}{\sum_{\ell'=1}^{L} \lambda_{\ell'} \, p^{\mathbf{P}_{\ell'}}(\mathbf{x}_{m1})} \tag{8.2}$$

Note that given the CCPs $\{\mathbf{P}_\ell\}$, this conditional distribution does not depend on parameters in the vector θ, only on the distribution λ. Given this expression for the probabilities $\{\lambda_{\ell|\mathbf{x}_{m1}}\}$, we have that the pseudo likelihood in (**??**) only depends on the structural parameters θ and λ and the incidental parameters $\mathbf{P}$.

For the estimators that we discuss here, we maximize $Q(\theta,\lambda,\mathbf{P})$ with respect to (θ,λ) for given $\mathbf{P}$. Therefore, the ergodic distributions $p^{\mathbf{P}_\ell}$ are fixed during this optimization. This implies a significant reduction in the computational cost associated with the initial conditions problem. Nevertheless, in the literature of finite mixture models, it is well known that optimization of the likelihood function with respect to the mixture probabilities λ is a complicated task because the problem is plagued with many local maxima and minima. To deal with this problem, Aguirregabiria and Mira (2007) introduce an additional parametric assumption on the distribution of ξ_m that simplifies significantly the maximization of $Q(\theta,\lambda,\mathbf{P})$ for fixed $\mathbf{P}$. They assume that the probability distribution of unobserved market heterogeneity is such that the only unknown parameters for the researcher are the mean and the variance which are included in θ_i^{FC} and $\sigma_{\xi i}$, respectively. Therefore, they assume that the distribution of ξ_m (that is, the points of support and the probabilities λ_ℓ) are known to the researcher. For instance, we may assume that ξ_m has a discretized standard normal distribution with an arbitrary number of points of support L. Under this assumption, the pseudo likelihood function is maximized only with respect to θ for given $\mathbf{P}$. Avoiding optimization with respect to λ simplifies importantly the computation of the different estimators that we describe below.

NPL estimator. As defined above, the NPL mapping φ is the composition of the equilibrium mapping and the mapping that provides the maximand in θ to $Q(\theta,\mathbf{P})$ for given $\mathbf{P}$. That is, $\varphi(\mathbf{P}) \equiv \Psi(\hat{\theta}(\mathbf{P}),\mathbf{P})$ where $\hat{\theta}(\mathbf{P}) \equiv \arg\max_\theta Q(\theta,\mathbf{P})$. By definition, an NPL fixed point is a pair $(\hat{\theta},\hat{\mathbf{P}})$ that satisfies two conditions: (a) $\hat{\theta}$ maximizes $Q(\theta,\hat{\mathbf{P}})$; and (b) $\hat{\mathbf{P}}$ is an equilibrium associated to $\hat{\theta}$. The NPL estimator is defined as the NPL fixed point with the maximum value of the likelihood function. The NPL estimator is consistent under standard regularity conditions (Aguirregabiria and Mira ,2007, Proposition 2).

When the equilibrium that generates the data is Lyapunov stable, we can compute the NPL estimator using a procedure that iterates in the NPL mapping, as described in section 3.2 to obtain the sequence of K-step estimators (that is, NPL algorithm). The main difference is that now we have to calculate the steady-state distributions $\mathbf{p}(\mathbf{P}_\ell)$ to deal with the initial conditions problem. However, the pseudo likelihood approach also reduces significantly the cost of dealing with the initial conditions problem. This NPL algorithm proceeds as follows. We start with L arbitrary vectors of players' choice probabilities, one for each market type: $\{\hat{\mathbf{P}}_\ell^0 : \ell = 1,2,...,L\}$. Then, we perform the following steps. Step 1: For every market type we obtain the steady-state distributions and the probabilities $\{\lambda_{\ell|x_{m1}}\}$. Step 2: We obtain a pseudo maximum likelihood estimator of θ as $\hat{\theta}^1 = \arg\max_\theta Q(\theta,\hat{\mathbf{P}}^0\})$. Step 3: Update the vector of players' choice probabilities using the best response probability mapping. That is, for market type ℓ, firm i and state $\mathbf{x}$, $\hat{P}^1_{i\ell}(\mathbf{x}) = \Phi(\tilde{\mathbf{z}}_i^{\hat{\mathbf{P}}^0_\ell}(\mathbf{x},\xi^\ell)\hat{\theta}^1_i + \tilde{e}_i^{\hat{\mathbf{P}}^0_\ell}(\mathbf{x},\xi^\ell))$. If, for every type ℓ, $||\hat{\mathbf{P}}^1_\ell - \hat{\mathbf{P}}^0_\ell||$ is smaller than a predetermined small constant, then stop the iterative procedure and keep $\hat{\theta}^1$ as a candidate estimator. Otherwise, repeat steps 1 to 4 using $\hat{\mathbf{P}}^1$ instead of $\hat{\mathbf{P}}^0$.

The NPL algorithm, upon convergence, finds an NPL fixed point. To guarantee consistency, the researcher needs to start the NPL algorithm from different CCP's in case there are multiple NPL fixed points. This situation is similar to using a gradient algorithm, designed to find a local root, in order to obtain an estimator which is defined as a global root. Of course, this global search aspect of the method makes it significantly more costly than the application of the NPL algorithm in models without unobserved heterogeneity. This is the additional computational cost that we have to pay for dealing with unobserved heterogeneity. Note, however, that this global search can be parallelized in a computer with multiple processors.

Arcidiacono and Miller (2008) extend this approach in several interesting and useful ways. First, they consider a more general form of unobserved heterogeneity that may enter both in the payoff function and in the transition of the state variables. Second, to deal with the complexity in the optimization of the likelihood function with respect to the distribution of the finite mixture, they combine the NPL method with an EM algorithm. Third, they show that for a class of dynamic decision models, that includes but it is not limited to optimal stopping problems, the computation of the inclusive values $\tilde{\mathbf{z}}^{\mathbf{P}_\ell}_{im\ell t}$ and $\tilde{e}^{\mathbf{P}_\ell}_{im\ell t}$ is simple and it is not subject to a 'curse of dimensionality', that is, the cost of computing these value for given $\mathbf{P}_\ell$ does not increase exponentially with the dimension of the state space. Together, these results provide a relatively simple approach to estimate dynamic games with unobserved heterogeneity of finite mixture type. Note that Lyapunov stability of each equilibrium type that generates the data is a necessary condition for the NPL and the Arcidiacono-Miller algorithms to converge to a consistent estimator.

K**kasahara_shimotsu_2008a (kasahara_shimotsu_2008a)**. The estimators of finite mixture models in Aguirregabiria and Mira (2007) and Arcidiacono and Miller (2008) consider that the researcher cannot obtain consistent nonparametric estimates of market-type CCPs $\{\mathbf{P}^0_\ell\}$. **kasahara_shimotsu_2008b** (**kasahara_shimotsu_2008b**), based on previous work by **hall_zhou_2003** (**hall_zhou_2003**), have derived sufficient conditions for the nonparametric identification of market-type CCPs $\{\mathbf{P}^0_\ell\}$ and the probability distribution of market types, $\{\lambda^0_\ell\}$. Given the nonparametric identification of market-type CCPs, it is possible to estimate structural parameters using a two-step approach similar to the one described above. However, this two-step estimator has three limitations that do not appear in two-step estimators without unobserved market heterogeneity. First, the conditions for nonparametric identification of $\mathbf{P}^0$ may not hold. Second, the nonparametric estimator in the first step is a complex estimator from a computational point of view. In particular, it requires the minimization of a sample criterion function with respect to the large dimensional object $\mathbf{P}$. This is in fact the type of computational problem that we wanted to avoid by using two-step methods instead of standard ML or GMM. Finally, the finite sample bias of the two-step estimator can be significantly more severe when $\mathbf{P}^0$ incorporates unobserved heterogeneity and we estimate it nonparametrically.

8.4 Reducing the State Space

Although two-step and sequential methods are computationally much cheaper than full solution-estimation methods, they are still impractical for applications where the dimension of the state space is large. The cost of computing exactly the matrix of present values $\mathbf{W}^{\mathbf{P}}_{z,i}$ increases cubically with the dimension of the state space. In the context of dynamic games, the dimension of the state space increases exponentially with the number of heterogeneous players. Therefore, the cost of computing the matrix of present values may become intractable even for a relatively small number of players.

A **simple approach to deal with this curse of dimensionality is to assume that** players are homogeneous and the equilibrium is symmetric. For instance, in our dynamic game of market entry-exit, when firms are heterogeneous, the dimension of the state space is $|H| * 2^N$, where $|H|$ is the number of values in the support of market size H_t. To reduce the dimensionality of the state space, we need to assume that: (a) only the number of competitors (and not their identities) affects the profit of a firm; (b) firms are homogeneous in their profit function; and (c) the selected equilibrium is symmetric.

Under these conditions, the payoff relevant state variables for a firm i are $\{H_t, s_{it}, n_{t-1}\}$ where s_{it} is its own incumbent status, and n_{t-1} is the total number of active firms at period $t-1$. The dimension of the state space is $|H| * 2 * (N+1)$ that increases only linearly with the number of players.[1] It is clear that the assumption of homogeneous firms and symmetric equilibrium can reduce substantially the dimension of the state space, and it can be useful in some empirical applications. Nevertheless, there are many applications where this assumption is too strong. For instance, in applications where firms produce differentiated products.

To deal with this issue, Hotz et al. (1994) proposed an estimator that uses Monte Carlo simulation techniques to approximate the values $\mathbf{W}^{\mathbf{P}}_{z,i}$. Bajari, Benkard, and Levin (2007) have extended this method to dynamic games and to models with continuous decision variables. This approach has proved useful in some applications. Nevertheless, it is important to be aware that in those applications with large state spaces, simulation error can be sizeable and it can induce biases in the estimation of the structural parameters. In those cases, it is worthwhile to reduce the dimension of the state space by making additional structural assumptions. That is the general idea in the inclusive-value approach that we have discussed in section 2 and that can be extended to the estimation of dynamic games. Different versions of this idea have been proposed and applied by Nevo and Rossi (2008), **maceriai_2007** (**maceriai_2007**), **rossi_2009** (**rossi_2009**), and **aguirregabiria_ho_2009** (**aguirregabiria_ho_2009**).

To present the main ideas, we consider here a dynamic game of quality competition in the spirit of Pakes and McGuire (1994), the differentiated product version of the Ericson-Pakes model. There are N firms in the market, that we index by i, and B brands or differentiated products, that we index by b. The set of brands sold by firm i is $\mathscr{B}_i \subset \{1, 2, ..., B\}$. Demand is given by a model similar to that of Section 2.1: consumers choose one of the B products offered in the market, or the outside good. The utility that consumer h obtains from purchasing product b at time t is $U_{hbt} = x_{bt} - \alpha\, p_{bt} + u_{hbt}$, where x_{bt} is the quality of the product, p_{bt} is the price, α is a parameter, and u_{hbt} represents consumer specific taste for product b. These idiosyncratic errors

[1] This is a particular example of the 'exchangeability assumption' proposed by Pakes and McGuire (2001).

are identically and independently distributed over (h,b,t) with type I extreme value distribution. If the consumer decides not to purchase any of the goods, she chooses the outside option that has a mean utility normalized to zero. Therefore, the aggregate demand for product b is $q_{bt} = H_t \exp\{x_{bt} - \alpha\ p_{bt}\}\ [1 + \sum_{b'=1}^{B} \exp\{x_{b't} - \alpha p_{b't}\}]^{-1}$, where H_t represents market size at period t. The market structure of the industry at time t is characterized by the vector $\mathbf{x}_t = (H_t, x_{1t}, x_{2t}, ..., x_{Bt})$. Every period, firms take as given current market structure and decide simultaneously their current prices and their investment in quality improvement. The one-period profit of firm i can be written as

$$\Pi_{it} = \sum\nolimits_{b \in \mathscr{B}_i} (p_{bt} - mc_b)\ q_{bt} - FC_b - (c_b + \varepsilon_{bt})\ a_{bt} \tag{8.3}$$

where $a_{bt} \in \{0,1\}$ is the binary variable that represents the decision to invest in quality improvement of product b; mc_b, FC_b, and c_b are structural parameters that represent marginal cost, fixed operating cost, and quality investment cost for product b, respectively; and ε_{bt} is an iid private information shock in the investment cost. Product quality evolves according to a transition probability $f_x(x_{bt+1}|a_{bt}, x_{bt})$. For instance, in the Pakes-McGuire model, $x_{bt+1} = x_{bt} - \zeta_t + a_{bt}\ v_{bt}$ where ζ_t and v_{bt} are two independent and non-negative random variables that are independently and identically distributed over (b,t).

In this model, price competition is static. The Nash-Bertrand equilibrium determines prices and quantities as functions of market structure $\mathbf{x}_t$, that is, $p_b^*(\mathbf{x}_t)$ and $q_b^*(\mathbf{x}_t)$. Firms' quality choices are the result of a dynamic game. The one-period profit function of firm i in this dynamic game is $\Pi_i(\mathbf{a}_{it}, \mathbf{x}_t) = \sum_{b \in \mathscr{B}_i} (p_i^*(\mathbf{x}_t) - mc_b)\ q_b^*(\mathbf{x}_t) - FC_b - (c_b + \varepsilon_{bt})$ a_{bt}, where $\mathbf{a}_{it} \equiv \{a_{bt} : b \in \mathscr{B}_i\}$. This dynamic game of quality competition has the same structure as the game that we have described in Section 3.1 and it can be solved and estimated using the same methods. However, the dimension of the state space increases exponentially with the number of products and the solution and estimation of the model becomes impractical even when B is not too large.

Define the *cost adjusted inclusive value* of firm i at period t as $\omega_{it} \equiv \log[\sum_{b \in \mathscr{B}_i} exp\{x_{bt} - \alpha\ mc_b\}]$.This value is closely related to the inclusive value that we have discussed in Section 2. It can be interpreted as the net quality level, or a value added of sort, that the firm is able to produce in the market. Under the assumptions of the model, the variable profit of firm i in the Nash-Bertrand equilibrium can be written as a function of the vector of inclusive values $\omega_t \equiv (\omega_{1t}, \omega_{2t}, ..., \omega_{Nt}) \in \Omega$, that is, $\sum_{b \in \mathscr{B}_i} (p_i^*(\mathbf{x}_t) - mc_b)\ q_b^*(\mathbf{x}_t) = vp_i(\omega_t)$. Therefore, the one-period profit Π_{it} is a function $\tilde{\Pi}_i(\mathbf{a}_{it}, \omega_t)$. The following assumption is similar to Assumption A2 made in Section 2 and it establishes that given vector ω_t the rest of the information contained in the in $\mathbf{x}_t$ is redundant for the prediction of future values of ω.

Assumption: The transition probability of the vector of inclusive values ω_t from the point of view a firm (that is, conditional on a firm's choice) is such that $\Pr(\omega_{t+1} \mid \mathbf{a}_{it}, \mathbf{x}_t) = \Pr(\omega_{t+1} \mid \mathbf{a}_{it}, \omega_t)$.

Under these assumptions, ω_t is the vector of payoff relevant state variables in the dynamic game. The dimension of the space Ω increases exponentially with the number of firms but not with the number of brands. Therefore, the dimension of Ω can be much smaller than the dimension of the original state space of $\mathbf{x}_t$ in applications where the number of brands is large relative to the number of firms.

Of course, the assumption of sufficiency of ω_t in the prediction of next period ω_{t+1} is not trivial. In order to justify it we can put quite strong restrictions on the stochastic process of quality levels. Alternatively, it can be interpreted in terms of limited information, and/or bounded rationality. For instance, a possible way to justify this assumption is that firms face the same type of computational burdens that we do. Limiting the information that they use in their strategies reduces a firm's computational cost of calculating a best response.

Note that the dimension of the space of ω_t still increases exponentially with the number of firms. To deal with this curse of dimensionality, **aguirregabiria_ho_2009** (**aguirregabiria_ho_2009**) consider a stronger inclusive value / sufficiency assumption. Let vp_{it} be the variable profit of firm i at period t. Assumption: $\Pr(\omega_{it+1}, vp_{it+1} \mid \mathbf{a}_{it}, \mathbf{x}_t) = \Pr(\omega_{it+1}, vp_{it+1} \mid \mathbf{a}_{it}, \omega_{it}, vp_{it})$. Under this assumption, the vector of payoff relevant state variables in the decision problem of firm i is (ω_{it}, vp_{it}) and the dimension of the space of (ω_{it}, vp_{it}) does not increase with the number of firms.

8.5 Counterfactual experiments with multiple equilibria

One of the attractive features of structural models is that they can be used to predict the effects of new counterfactual policies. This is a challenging exercise in a model with multiple equilibria. Under the assumption that our data has been generated by a single equilibrium, we can use the data to identify which of the multiple equilibria is the one that we observe. However, even under that assumption, we still do not know which equilibrium will be selected when the values of the structural parameters are different to the ones that we have estimated from the data. For some models, a possible approach to deal with this issue is to calculate all of the equilibria in the counterfactual scenario and then draw conclusions that are robust to whatever equilibrium is selected. However, this approach is of limited applicability in dynamic games of oligopoly competition because the different equilibria typically provide contradictory predictions for the effects we want to measure.

Here we describe a simple homotopy method that has been proposed in **aguirregabiria_2009** (**aguirregabiria_2009**) and applied in **aguirregabiria_ho_2009** (**aguirregabiria_ho_2009**). Under the assumption that the equilibrium selection mechanism, which is unknown to the researcher, is a smooth function of the structural parameters, we show how to obtain a Taylor approximation to the counterfactual equilibrium. Despite the equilibrium selection function being unknown, a Taylor approximation of that function, evaluated at the estimated equilibrium, depends on objects that the researcher knows.

Let $\Psi(\theta, \mathbf{P})$ be the equilibrium mapping such that an equilibrium associated with θ can be represented as a fixed point $\mathbf{P} = \Psi(\theta, \mathbf{P})$. Suppose that there is an equilibrium selection mechanism in the population under study, but we do not know that mechanism. Let $\pi(\theta)$ be the selected equilibrium given θ. The approach here is quite agnostic with respect to this equilibrium selection mechanism: it only assumes that there is such a mechanism, and that it is a smooth function of θ. Since we do not know the mechanism, we do not know the form of the mapping $\pi(\theta)$ for every possible θ. However, we know that the equilibrium in the population, $\mathbf{P}^0$, and the vector of the structural parameters in the population, θ^0, belong to the graph of that mapping, that is, $\mathbf{P}^0 = \pi(\theta^0)$.

Let θ^* be the vector of parameters under the counterfactual experiment that we want to analyze. We want to know the counterfactual equilibrium $\pi(\theta^*)$ and compare it to the factual equilibrium $\pi(\theta^0)$. Suppose that Ψ is twice continuously differentiable in θ and P. The following is the key assumption to implement the homotopy method that we describe here.

Assumption: The equilibrium selection mechanism is such that π is a continuous differentiable function within a convex subset of Θ that includes θ^0 and θ^*.

That is, the equilibrium selection mechanism does not "jump" between the possible equilibria when we move over the parameter space from θ^0 to θ^*. This seems a reasonable condition when the researcher is interested in evaluating the effects of a change in the structural parameters but "keeping constant" the same equilibrium type as the one that generates the data.

Under these conditions, we can make a Taylor approximation to $\pi(\theta^*)$ around θ^0 to obtain:

$$\pi(\theta^*) \quad = \quad \pi\left(\theta^0\right) + \frac{\partial \pi\left(\theta^0\right)}{\partial \theta'}\left(\theta^* - \theta^0\right) + O\left(\left\|\theta^* - \theta^0\right\|^2\right) \tag{8.4}$$

We know that $\pi\left(\theta^0\right) = \mathbf{P}^0$. Furthermore, by the implicit function theorem, $\partial \pi\left(\theta^0\right)/\partial \theta'$ $= \partial \Psi(\theta^0, \mathbf{P}^0)/\partial \theta'$ $+ \partial \Psi(\theta^0, \mathbf{P}^0)/\partial \mathbf{P}'$ $\partial \pi\left(\theta^0\right)/\partial \theta'$. If $\mathbf{P}^0$ is not a singular equilibrium then $I - \partial \Psi(\theta^0, \mathbf{P}^0)/\partial \mathbf{P}'$ is not a singular matrix and $\partial \pi\left(\theta^0\right)/\partial \theta' = (I - \partial \Psi(\theta^0, \mathbf{P}^0)/\partial \mathbf{P}')^{-1}$ $\partial \Psi(\theta^0, \mathbf{P}^0)/\partial \theta'$. Solving this expression into the Taylor approximation, we have the following approximation to the counterfactual equilibrium:

$$\hat{\mathbf{P}}^* = \hat{\mathbf{P}}^0 + \left(I - \frac{\partial \Psi(\hat{\theta}^0, \hat{\mathbf{P}}^0)}{\partial \mathbf{P}'}\right)^{-1} \frac{\partial \Psi(\hat{\theta}^0, \hat{\mathbf{P}}^0)}{\partial \theta'}\left(\theta^* - \hat{\theta}^0\right) \tag{8.5}$$

where $(\hat{\theta}^0, \hat{\mathbf{P}}^0)$ represents our consistent estimator of $(\theta^0, \mathbf{P}^0)$. It is clear that $\hat{\mathbf{P}}^*$ can be computed given the data and θ^*. Under our assumptions, $\hat{\mathbf{P}}^*$ is a consistent estimator of the linear approximation to $\pi(\theta^*)$.

As in any Taylor approximation, the order of magnitude of the error depends on the distance between the value of the structural parameters in the factual and counterfactual scenarios. Therefore, this approach can be inaccurate when the counterfactual experiment implies a large change in some of the parameters. For these cases, we can combine the Taylor approximation with iterations in the equilibrium mapping. Suppose that $\mathbf{P}^*$ is a (Lyapunov) stable equilibrium. And suppose that the Taylor approximation $\hat{\mathbf{P}}^*$ belongs to the dominion of attraction of $\mathbf{P}^*$. Then, by iterating in the equilibrium mapping $\Psi(\theta^*, .)$ starting at $\hat{\mathbf{P}}^*$ we will obtain the counterfactual equilibrium $\mathbf{P}^*$. Note that this approach is substantially different to iterating in the equilibrium mapping $\Psi(\theta^*, .)$ starting with the equilibrium in the data $\hat{\mathbf{P}}^0$. This approach will return the counterfactual equilibrium $\mathbf{P}^*$ *if and only if* $\hat{\mathbf{P}}^0$ belongs to the dominion of attraction of $\mathbf{P}^*$. This condition is stronger than the one establishing that the Taylor approximation $\hat{\mathbf{P}}^*$ belongs to the domination of attraction of $\mathbf{P}^*$.