

Macro1: Solow, RCK, and Diamond OLG

1) Solow Growth Model (exogenous saving)

Core idea. Long-run growth in *output per worker* is driven by exogenous technological progress; saving affects levels, not long-run growth.

Environment. Continuous time. Aggregate production

$$Y(t) = F(K(t), A(t)L(t)), \quad \text{CRS, focus on labor-augmenting } A(t).$$

Exogenous growth:

$$\frac{\dot{L}}{L} = n, \quad \frac{\dot{A}}{A} = g.$$

Capital accumulation (saving is non-strategic):

$$\dot{K} = sY - \delta K, \quad s \in (0, 1).$$

Per effective worker. Define $k = \frac{K}{AL}$ and $y = \frac{Y}{AL} = f(k)$. Key law of motion:

$$\dot{k} = sf(k) - (n + g + \delta)k.$$

Steady state / BGP. k^* solves $sf(k^*) = (n + g + \delta)k^*$, so $\dot{k} = 0$ and $y^* = f(k^*)$ constant. Growth on BGP:

$$\text{growth of } (Y/L) = g, \quad \text{growth of } Y = n + g.$$

Saving-rate change. A higher s raises k^* and y^* (a *level effect*), but does not change long-run growth of Y/L (still g).

2) Ramsey–Cass–Koopmans Model (endogenous saving via optimization)

Core idea. “Solow + microfoundations”: saving is chosen optimally by households; equilibrium is efficient (under standard conditions).

Technology and prices. Same $Y = F(K, AL)$, CRS, and (n, g) as above (lecture notes often set $\delta = 0$ for simplicity). Let $r(t)$ be the real rental rate of capital and $w(t)$ the real wage. In per effective worker terms:

$$r(t) = f'(k(t)), \quad w(t) = f(k(t)) - k(t)f'(k(t)).$$

Households. Infinitely-lived representative household chooses consumption to maximize discounted utility (CRRA shown in notes):

$$\max_{\{C(t)\}} \int_0^\infty e^{-\rho t} u(C(t)) \frac{L(t)}{H} dt, \quad u(C) = \frac{C^{1-\theta} - 1}{1-\theta}, \quad \theta > 0.$$

In per effective worker variables $c = C/A$ and $k = K/(AL)$, the resource/budget dynamics imply:

$$\dot{k} = f(k) - c - (n + g)k \quad (\text{if } \delta = 0; \text{ replace } n + g \text{ by } n + g + \delta \text{ if } \delta > 0).$$

Euler equation (consumption growth). Using the notes’ derivation,

$$\frac{\dot{c}}{c} = \frac{r(t) - \rho - \theta g}{\theta} = \frac{f'(k(t)) - \rho - \theta g}{\theta}.$$

Steady state / saddle path. The system in (k, c) has a saddle path; k^* satisfies

$$f'(k^*) = \rho + \theta g \quad (\text{and } + \delta \text{ if depreciation is included}).$$

Welfare. Competitive equilibrium coincides with the social planner allocation (Pareto efficient) in this setup.

3) Diamond Overlapping-Generations (OLG) Model (finite lives, turnover)

Core idea. Individuals live two periods (young/old). Turnover can break the RCK efficiency result; equilibria may be Pareto inefficient (dynamic inefficiency possible).

Timing. Discrete time $t = 0, 1, 2, \dots$. Population and technology grow:

$$L_t = (1+n)L_{t-1}, \quad A_t = (1+g)A_{t-1}.$$

Households (born at t). Young earn wage W_t , choose saving S_t :

$$W_t = C_{1t} + S_t, \quad C_{2,t+1} = (1+r_{t+1})S_t,$$

equivalently the lifetime budget:

$$C_{1t} + \frac{C_{2,t+1}}{1+r_{t+1}} = W_t.$$

Utility in the notes is CRRA across the two periods (and a log-utility special case is shown).

Firms and factor prices. CRS production each period:

$$Y_t = F(K_t, A_t L_t), \quad r_t = f'(k_t), \quad w_t = f(k_t) - k_t f'(k_t),$$

where $k_t = \frac{K_t}{A_t L_t}$ and $w_t = \frac{W_t}{A_t}$.

Capital accumulation and law of motion. Saving by the young becomes next period's capital:

$$K_{t+1} = L_t S_t.$$

In per effective worker terms (as in the notes),

$$k_{t+1} = \frac{1}{(1+n)(1+g)} s(r_{t+1}) w_t,$$

where $s(\cdot)$ is the (endogenous) saving share implied by the household Euler equation.

Closed-form special case (log + Cobb–Douglas). If $u = \ln C_1 + \frac{1}{1+\rho} \ln C_2$ and $f(k) = k^\alpha$,

$$k_{t+1} = \frac{1}{(1+n)(1+g)} \cdot \frac{1}{2+\rho} (1-\alpha) k_t^\alpha.$$

Efficiency and policy. Unlike RCK, OLG equilibria can be Pareto inefficient (dynamic inefficiency). A pay-as-you-go social security transfer can raise welfare in such cases.

Quick comparison (what changes across the three)

	Solow	RCK	Diamond OLG
Time	continuous	continuous	discrete
Lives	—	infinite-lived	two-period lives
Saving	exogenous s	optimal via Euler	optimal (young save)
State dynamics	$\dot{k} = sf(k) - (n+g+\delta)k$	(\dot{k}, \dot{c}) saddle path	$k_{t+1} = G(k_t)$
Long-run Y/L growth	g	g	g
Welfare	not explicit	Pareto efficient	may be inefficient