Stage 1
A Coherent Narrative: Consumer Choice → KKT → Geometry → Utility Classes

1 The Consumer Problem (One Statement)

A consumer chooses a bundle $x = (x_1, \ldots, x_n) \in \mathbb{R}^n_+$ to maximize utility subject to a budget:

$$\max_{x \geq 0} \; u(x) \quad \text{s.t.} \quad p \cdot x \leq m, \tag{1}$$

where prices $p = (p_1, \ldots, p_n) \gg 0$ and income $m > 0$.

Objects and definitions

- **Budget set:** $B(p, m) = \{x \in \mathbb{R}^n_+ : \; p \cdot x \leq m\}$.
- **Marshallian demand:** $x(p, m)$ is the optimizer(s) of (1).
- **Indirect utility:** $v(p, m) = \max_{x \in B(p,m)} u(x)$.
- **Ordinal utility:** if u represents preferences, then any strictly increasing transform $f(u)$ represents the same preferences.

2 KKT as the Universal Solver (Conditions + Interpretation)

Consumer problems involve inequality constraints ($p \cdot x \leq m$) and nonnegativity ($x \geq 0$). KKT is the standard optimality system.

2.1 General KKT form (maximization with inequalities)

Consider:

$$\max_{x \in \mathbb{R}^n} \; f(x) \quad \text{s.t.} \quad g_j(x) \geq 0, \; j = 1, \ldots, J. \tag{2}$$

A point x^* is a **KKT point** if there exist multipliers $\lambda_j^* \geq 0$ such that:

(1) **Primal feasibility:** $g_j(x^*) \geq 0$ for all j.

(2) **Dual feasibility:** $\lambda_j^* \geq 0$ for all j.

(3) **Complementary slackness:** $\lambda_j^* \, g_j(x^*) = 0$ for all j.

(4) **Stationarity:** $\nabla f(x^*) + \sum_{j=1}^J \lambda_j^* \nabla g_j(x^*) = 0$.

2.2 Interpretation (how KKT explains consumer solutions)

- If a constraint is *slack* (not binding), its multiplier is 0.
- If a multiplier is > 0, that constraint must be *binding*.
- This is exactly the algebra behind **interior vs corner vs kink** outcomes.

3 Geometry Equivalence: Tangency / Corner / Kink (Three Solution Types)

In two goods (x_1, x_2), the same consumer problem typically lands in one of three geometric types:

3.1 Type I: Interior optimum (tangency)

- $x_1^* > 0$ and $x_2^* > 0$ (nonnegativity constraints slack).
- Budget binds under monotonicity: $p_1 x_1^* + p_2 x_2^* = m$.
- Tangency condition:
$$\mathrm{MRS}(x^*) = \frac{p_1}{p_2}.$$

3.2 Type II: Corner optimum (boundary)

- One good hits zero, e.g. $x_2^* = 0$.
- A nonnegativity constraint binds, so its multiplier can be positive.
- No interior tangency; optimality is characterized by case comparisons ("bang-per-buck").

3.3 Type III: Kink optimum (nondifferentiable / complements)

- Utility is not differentiable at the optimum (e.g. Leontief kink).
- Optimality often comes from eliminating "waste": equalize the arguments inside a $\min\{\cdot,\cdot\}$ whenever feasible.

4 Applications by Utility Class (Four Standard Utilities)

Now we apply the same steps to standard utility families. The only thing that changes is the *shape of indifference curves*, which determines whether we get tangency, corners, or kinks.

4.1 Cobb–Douglas: interior/tangency → demands

Let
$$u(x_1, x_2) = x_1^a x_2^{1-a}, \quad a \in (0,1). \tag{3}$$

Because $\log(\cdot)$ is strictly increasing, we can maximize:
$$\max_{x_1, x_2 \geq 0} \; a \log x_1 + (1-a) \log x_2 \quad \text{s.t.} \quad p_1 x_1 + p_2 x_2 \leq m.$$

This yields an interior tangency solution and Marshallian demands:

$$x_1^*(p,m) = \frac{a\,m}{p_1}, \qquad x_2^*(p,m) = \frac{(1-a)\,m}{p_2}. \tag{4}$$

Interpretation: fixed budget shares (a on good 1 and $1-a$ on good 2).

4.2 Linear: corner solutions → cases

Let

$$u(x_1,x_2) = \alpha x_1 + x_2, \quad \alpha > 0. \tag{5}$$

Compare utility-per-dollar:

$$\frac{\alpha}{p_1} \text{ vs } \frac{1}{p_2}.$$

The solution is typically a corner:

$$(x_1^*, x_2^*) = \begin{cases} \left(\dfrac{m}{p_1}, 0\right), & \text{if } \dfrac{\alpha}{p_1} > \dfrac{1}{p_2}, \\ \left(0, \dfrac{m}{p_2}\right), & \text{if } \dfrac{\alpha}{p_1} < \dfrac{1}{p_2}, \\ \text{any bundle on the budget line,} & \text{if } \dfrac{\alpha}{p_1} = \dfrac{1}{p_2}. \end{cases} \tag{6}$$

Interpretation: buy only the good with higher bang-per-buck (unless tied).

4.3 Leontief: kink → proportion + budget

Let

$$u(x_1,x_2) = \min\{\alpha x_1,\ x_2\}, \quad \alpha > 0. \tag{7}$$

At the optimum, avoid waste by matching:

$$\alpha x_1 = x_2.$$

Combine with budget exhaustion $p_1x_1 + p_2x_2 = m$ to get:

$$x_1^* = \frac{m}{p_1 + \alpha p_2}, \qquad x_2^* = \frac{\alpha m}{p_1 + \alpha p_2}. \tag{8}$$

Interpretation: fixed proportions; the optimum is at the kink.

4.4 CES: unifying family + limits (one spectrum)

A CES family (for $\rho \neq 0$) is:

$$u(x_1,x_2) = \left(a x_1^\rho + (1-a)x_2^\rho\right)^{1/\rho}, \quad a \in (0,1). \tag{9}$$

Interpretation: one parameter controls substitutability and connects the three geometric types:

- high substitutability ⇒ behavior approaches **linear** (perfect substitutes, corners),
- intermediate substitutability ⇒ smooth **interior/tangency** (Cobb–Douglas-like),
- low substitutability ⇒ behavior approaches **Leontief** (perfect complements, kink).

This is why CES is a useful "unifying spectrum."

5 Consistency Checks (Walras, Homogeneity, Intuition)

After solving any consumer problem, check these to ensure coherence:

(1) Walras / budget exhaustion

If u is monotone, then the optimum uses all income:

$$p \cdot x^*(p, m) = m.$$

(2) Homogeneity of degree 0

Scaling prices and income by the same factor does not change demand:

$$x(\gamma p, \gamma m) = x(p, m) \quad \text{for any } \gamma > 0.$$

(3) Geometry matches the utility class

- Cobb–Douglas $\Rightarrow$ interior tangency (as in (4)).
- Linear $\Rightarrow$ corners (as in (6)).
- Leontief $\Rightarrow$ kink (as in (8)).
- CES $\Rightarrow$ moves smoothly along the substitutes–complements spectrum (9).

(4) Basic intuition (comparative statics)

- Higher income should expand the affordable set (demands typically increase for normal goods).
- Price increases tend to reduce demand for that good (except special cases like Giffen).

Summary in one line: one problem $\rightarrow$ one method (KKT) $\rightarrow$ three solution types $\rightarrow$ four utilities $\rightarrow$ one unifying spectrum (CES).