Two Endogenous Growth Models

Focus: how technological progress / knowledge $A(t)$ is generated and how this changes long-run growth.

Endogenous Growth Model (Knowledge Accumulation with Exogenous R&D Shares)

Core idea. Interpret Solow's $A(t)$ as *knowledge* and specify a technology for producing new knowledge. A fixed share of resources is allocated to R&D (exogenous allocation), so the model explains $\dot{A}$ but does not fully microfound the R&D share.

Baseline setup (illustrative, no-capital version).

- Labor is split: production uses $(1 - a_L)L(t)$ and R&D uses $a_L L(t)$, where $a_L \in (0, 1)$ is exogenous.
- Output (final good):
$$Y(t) = A(t)\,(1 - a_L)L(t).$$
- Knowledge production:
$$\dot{A}(t) = B\,[a_L L(t)]^{\gamma}\,A(t)^{\theta}, \qquad B > 0,\ \gamma > 0.$$
- Knowledge growth rate:
$$g_A(t) \equiv \frac{\dot{A}(t)}{A(t)} = B\,[a_L L(t)]^{\gamma}\,A(t)^{\theta-1}.$$

Balanced-growth implications (key regimes).

- If $\theta < 1$, $g_A(t)$ converges to a constant; long-run growth is tied to population growth:
$$g_A^* = \frac{\gamma}{1-\theta}\,n \quad \text{when} \quad \frac{\dot{L}}{L} = n.$$
In this case, a_L mainly affects *levels* (transition and scale), not the asymptotic growth rate.
- If $\theta = 1$, then $g_A(t) = B[a_L L(t)]^{\gamma}$; with $n = 0$, growth is constant immediately.
- If $\theta > 1$, knowledge growth can accelerate over time (non-constant growth rates).

Generalization with capital (typical extended form).

- Output with resource splits:
$$Y(t) = \big[(1 - a_K)K(t)\big]^{\alpha}\big[A(t)(1 - a_L)L(t)\big]^{1-\alpha}, \qquad 0 < \alpha < 1.$$
- Knowledge production:
$$\dot{A}(t) = B\,[a_K K(t)]^{\beta}[a_L L(t)]^{\gamma}A(t)^{\theta}.$$
- Capital accumulation (often in the notes $\delta = 0$ for simplicity):
$$\dot{K}(t) = sY(t) - \delta K(t).$$

Takeaway. This model endogenizes technology by specifying $\dot{A}$, but keeps the R&D allocation rules (a_L, a_K) exogenous.

Romer Model (Patents, Variety Expansion, and Endogenous R&D Labor)

Core idea. Endogenize the *incentive* to conduct R&D. Ideas are patented (property rights), intermediate-goods producers earn monopoly profits, and free entry into R&D pins down the equilibrium amount of R&D labor. Growth arises from expansion in the number of varieties $A(t)$.

Structure.

- $A(t)$ is the mass/number of patented varieties.
- Final-good production aggregates intermediate inputs $\{y(i,t)\}_{i\in[0,A(t)]}$ with CES:

$$Y(t) = \left(\int_0^{A(t)} y(i,t)^{\phi}\, di\right)^{1/\phi}, \qquad 0 < \phi < 1.$$

- Each intermediate good i is produced using labor one-for-one:

$$y(i,t) = L(i,t).$$

- Labor market clearing (R&D plus production labor equals total labor $\bar{L}$):

$$L_A(t) + \int_0^{A(t)} L(i,t)\, di = \bar{L} \quad \Longleftrightarrow \quad L_A(t) + A(t)y(t) = \bar{L},$$

  under symmetry $y(i,t) = y(t)$.

Monopoly pricing (markup).

- Intermediate producers set prices; with CES demand, optimal pricing implies a constant markup over the wage:

$$p(i,t) = \frac{1}{\phi} W(t).$$

Knowledge accumulation (variety expansion).

- New varieties are produced by R&D labor:

$$\dot{A}(t) = B\, L_A(t)\, A(t), \qquad B > 0.$$

  Thus, the growth rate of varieties is

$$\frac{\dot{A}(t)}{A(t)} = B\, L_A(t).$$

Households and Euler equation.

- With log utility (representative household):

$$\int_0^{\infty} e^{-\rho t} \ln C(t)\, dt,$$

  the Euler equation is

$$\frac{\dot{C}(t)}{C(t)} = r(t) - \rho.$$

Free entry into R&D and equilibrium R&D labor.

- Entrepreneurs enter R&D until the value of a new patent equals its creation cost, implying an interior equilibrium of the form

$$L_A = \max\left\{(1-\phi)\bar{L} - \frac{\phi\rho}{B},\, 0\right\}.$$

- Then balanced growth features constant growth rates:

$$\frac{\dot{A}}{A} = BL_A, \qquad \frac{\dot{Y}}{Y} = \frac{\dot{C}}{C} = \frac{\dot{W}}{W} = \frac{1-\phi}{\phi}\, BL_A,$$

  and a constant interest rate consistent with household optimality.

Efficiency benchmark (planner vs market).

- The decentralized equilibrium can underinvest in R&D relative to the planner due to monopoly distortions / externalities; a common comparison is

$$L_A^{\text{EQ}} = (1-\phi)\, L_A^{\text{OPT}},$$

  so equilibrium R&D labor is below the socially optimal level.

Takeaway. The model provides a market mechanism that pins down L_A and generates sustained growth through idea accumulation with patent-driven incentives.

Side-by-side comparison (what to remember)

Feature	**Endogenous Growth Model**	**Romer Model**
What is $A(t)$?	Knowledge / effectiveness of labor	Number (mass) of patented varieties
How does A grow?	$\dot{A} = B(\text{R\&D inputs}) \cdot A^{\theta}$	$\dot{A} = BL_A A$
R&D allocation	Exogenous shares (a_L, a_K given)	Endogenous L_A from profits + free entry
Source of sustained growth	Knowledge accumulation; regime depends on θ	Variety expansion with patent incentives
Welfare / efficiency	Typically focused on growth mechanics; allocation taken as given	Equilibrium can underinvest in R&D vs planner