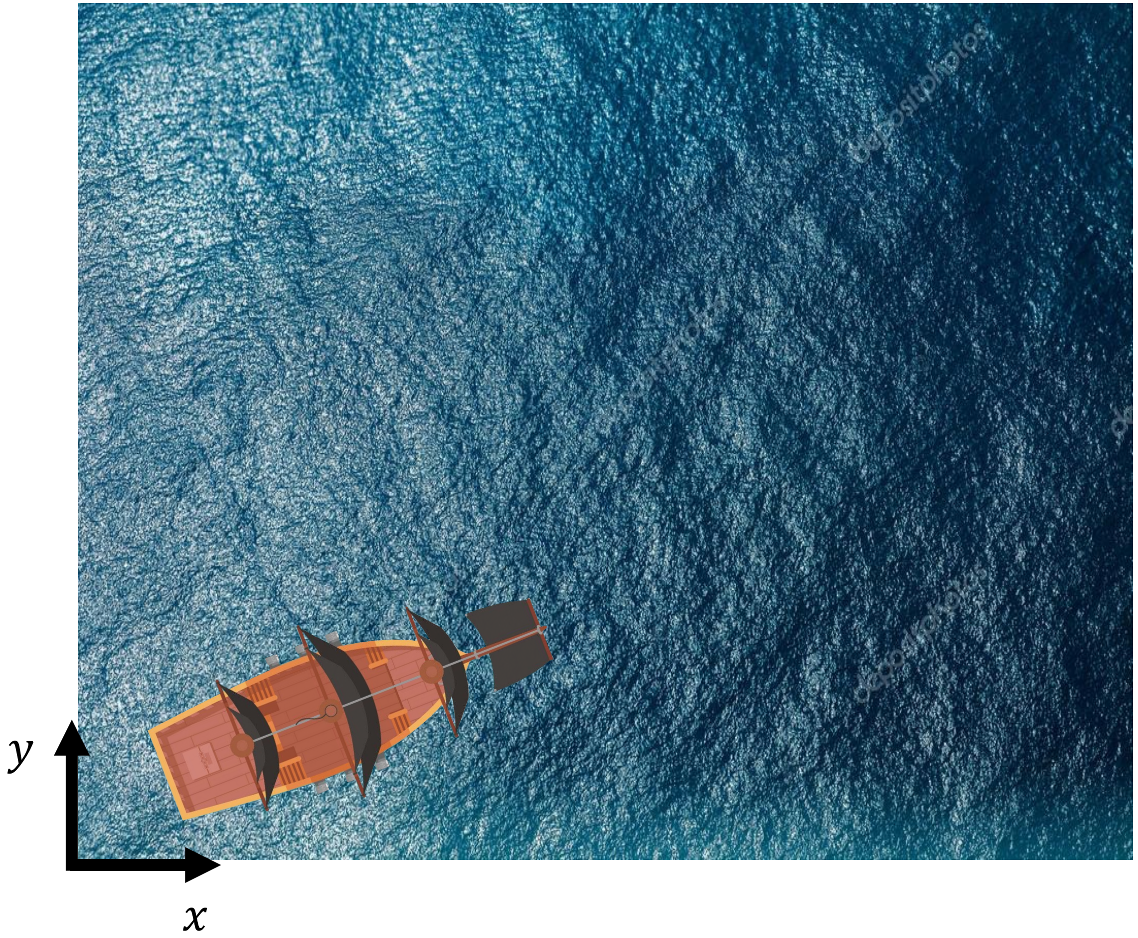
LINMA1731 - Stochastic Processes

Project 2023: Tracking a ship at see

Tracking a ship at see

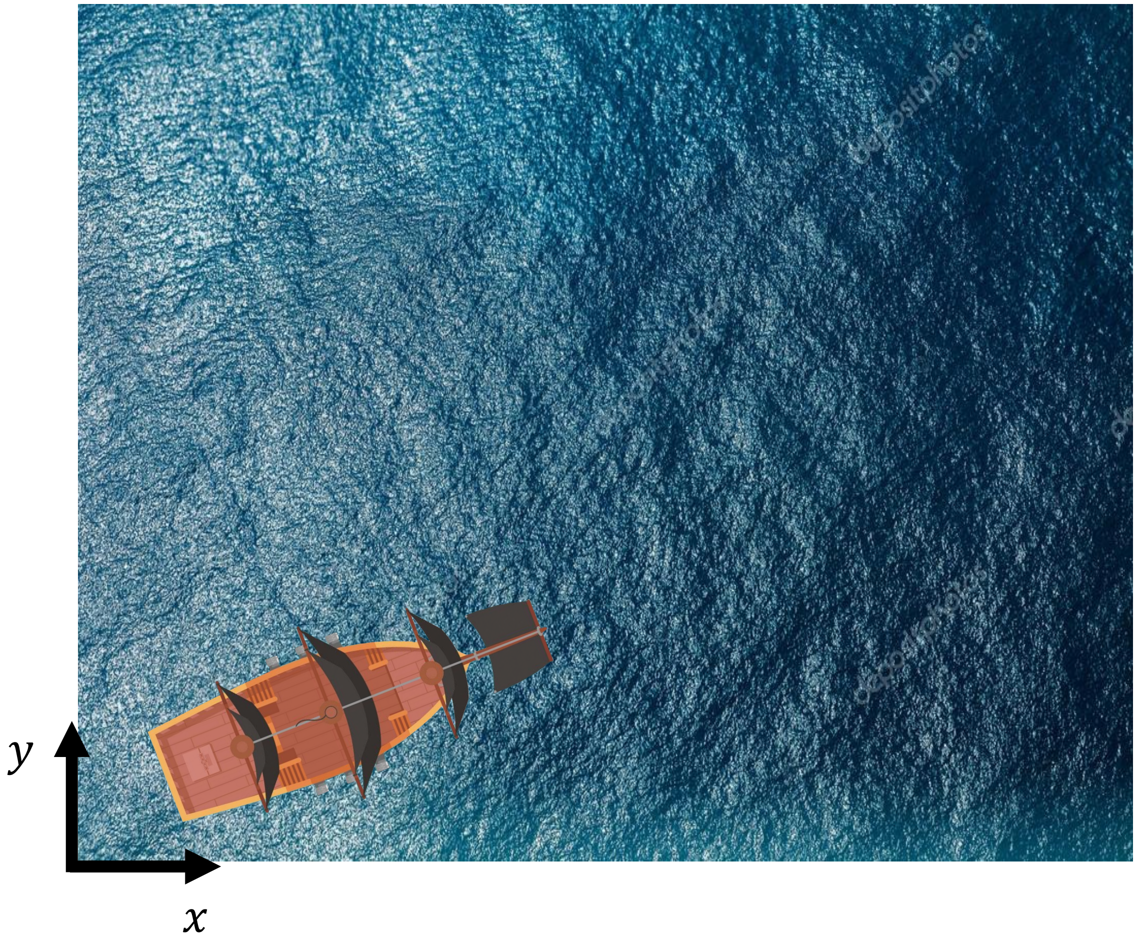


True 2D position of the ship at time k is s_k

Access to only noisy observations $Y_k = s_k + v_k$

$v_k \sim \mathcal{N}(0, \sigma_v^2)$ is a white noise (e.g. sextant positioning error)

Dynamics of the ship



Position s_k
Velocity $\dot{s}_k$
Acceleration a_k

$\Longrightarrow$ Linked thanks to the Uniformly Accelerated Motion equations

$$\begin{cases} s_{k+1} = s_k + \dot{s}_k dt + \dfrac{1}{2} a_k dt^2 \\ \dot{s}_{k+1} = \dot{s}_k + a_k dt \end{cases}$$

with dt the time increment.

$$\begin{cases} s_{k+1} = s_k + \dot{s}_k dt + \frac{1}{2} a_k dt^2 \\ \dot{s}_{k+1} = \dot{s}_k + a_k dt \end{cases}$$

Focus on the acceleration

$$\boxed{a_k = u_k + w_k}$$

Manoeuvring acceleration
Input of the system (controlled)

Random acceleration
Perturbation of the system (not controlled)

$$\begin{cases} s_{k+1} = s_k + \dot{s}_k dt + \frac{1}{2} a_k dt^2 \\ \dot{s}_{k+1} = \dot{s}_k + a_k dt \end{cases}$$

Focus on the acceleration

$$a_k = u_k + w_k$$

Manoeuvring acceleration
Input of the system (controlled)

Random acceleration
Perturbation of the system (not controlled)

$u_k \sim \mathcal{N}(\mu_N, \sigma^2)$

$w_k \sim Gumbel(\mu, \beta)$

Part 1: Acceleration Noise Estimation

$$\boxed{w_k \sim Gumbel(\mu, \beta)}$$

Maximum Likelihood Estimator (MLE)

μ unknown, β known

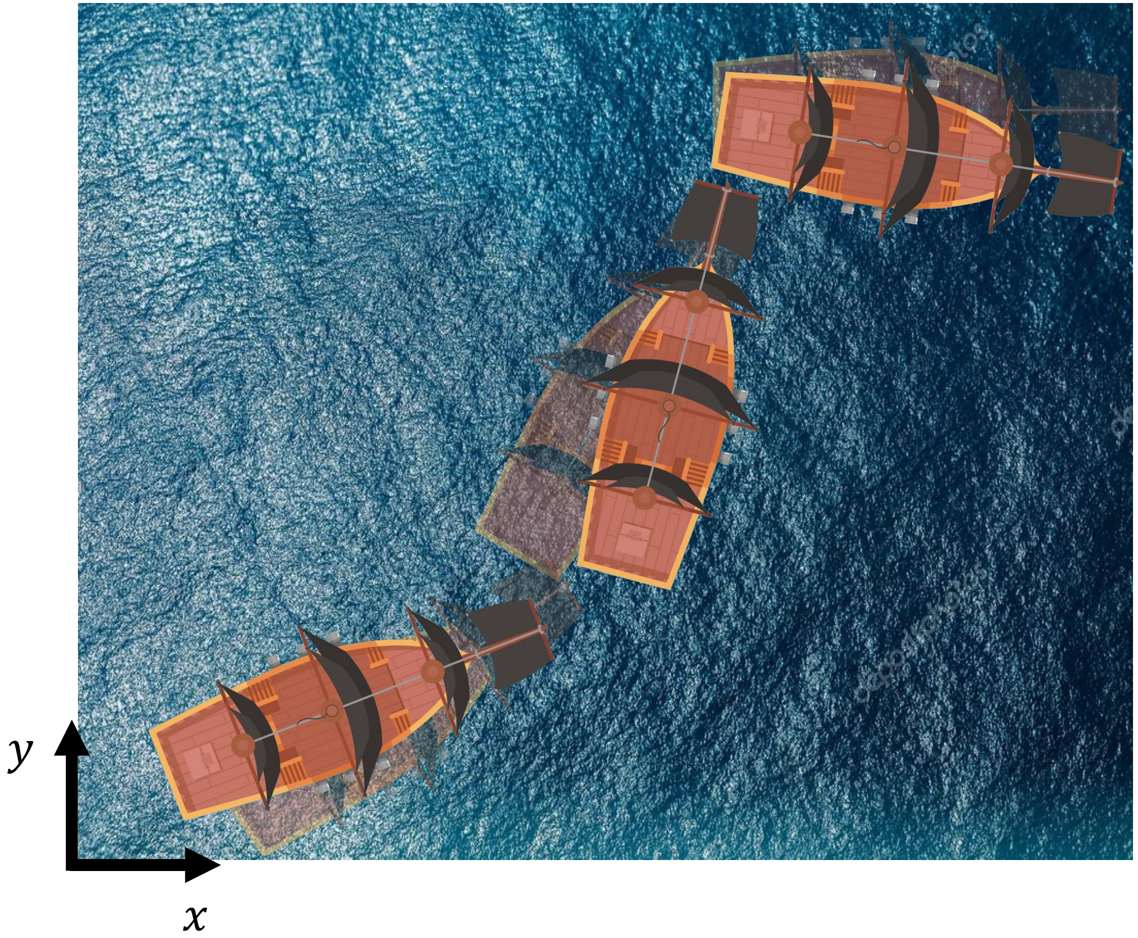
Both μ and β unknown

Theoretical properties of the estimators (consistency, Cramér-Rao bound, …)

Numerical verification of the properties

Comparison with a Normal Distribution

Part 2: Kalman Filter vs. Particle Filter

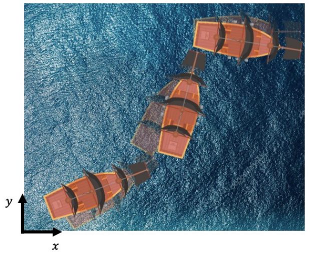


Estimation of the position of the ship based on noisy observations

Implementation in Python

- Kalman filter
- Particle filter

Comparison of the performances



State space equations

Position s_k
Velocity $\dot{s}_k$
Acceleration a_k
Time increment dt

The state X_k of the system at time k is defined as the position and speed of the ship

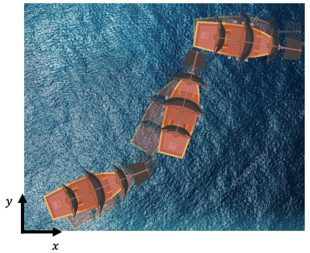
$$X_k = \begin{pmatrix} s_k \\ \dot{s}_k \end{pmatrix}$$

Hence, the state equation is

$$\begin{pmatrix} s_{k+1} \\ \dot{s}_{k+1} \end{pmatrix} = \begin{pmatrix} 1 & dt \\ 0 & 1 \end{pmatrix} \begin{pmatrix} s_k \\ \dot{s}_k \end{pmatrix} + \begin{pmatrix} \frac{1}{2} dt^2 \\ dt \end{pmatrix} u_k + \begin{pmatrix} \frac{1}{2} dt^2 \\ dt \end{pmatrix} w_k$$

while the observation equation is

$$Y_k = \begin{pmatrix} 1 & 0 \end{pmatrix} \begin{pmatrix} s_k \\ \dot{s}_k \end{pmatrix} + v_k$$



Extended model [1]

Position $s_k = (x_k\ , y_k)$
Velocity $\dot{s}_k = (\dot{x}_k\ , \dot{y}_k)$
Time increment dt

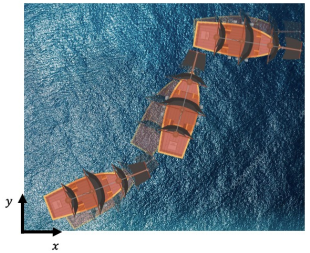
Developing the position and velocity, the extended model becomes

$$\begin{pmatrix} x_{k+1} \\ y_{k+1} \\ \dot{x}_{k+1} \\ \dot{y}_{k+1} \end{pmatrix} = \begin{pmatrix} 1 & 0 & dt & 0 \\ 0 & 1 & 0 & dt \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_k \\ y_k \\ \dot{x}_k \\ \dot{y}_k \end{pmatrix} + \begin{pmatrix} \frac{1}{2}dt^2 & 0 \\ 0 & \frac{1}{2}dt^2 \\ dt & 0 \\ 0 & dt \end{pmatrix} u_k + \begin{pmatrix} \frac{1}{2}dt^2 & 0 \\ 0 & \frac{1}{2}dt^2 \\ dt & 0 \\ 0 & dt \end{pmatrix} w_{k,(2\times 1)}$$

and

$$Y_k = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_k \\ y_k \\ \dot{x}_k \\ \dot{y}_k \end{pmatrix} + v_{k,(2\times 1)}$$

[1] Wanjin Xu et al, Ship tracking based on the fusion of Kalman filter and particle filter, J. Phys.: Conf. Ser. 2113 012017, 2021



Extended model [1]

Gauss-Markov model

$$\begin{cases} x_{k+1} = Ax_k + Bu_k + Gw_k \\ y_k = Cx_k + v_k \end{cases}$$

Developing the position and velocity, the extended model becomes

$$\begin{pmatrix} x_{k+1} \\ y_{k+1} \\ \dot{x}_{k+1} \\ \dot{y}_{k+1} \end{pmatrix} = \boxed{\begin{pmatrix} 1 & 0 & dt & 0 \\ 0 & 1 & 0 & dt \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}} \begin{pmatrix} x_k \\ y_k \\ \dot{x}_k \\ \dot{y}_k \end{pmatrix} + \boxed{\begin{pmatrix} \frac{1}{2}dt^2 & 0 \\ 0 & \frac{1}{2}dt^2 \\ dt & 0 \\ 0 & dt \end{pmatrix}} u_k + \boxed{\begin{pmatrix} \frac{1}{2}dt^2 & 0 \\ 0 & \frac{1}{2}dt^2 \\ dt & 0 \\ 0 & dt \end{pmatrix}} w_{k,(2\times 1)}$$

and

$$Y_k = \boxed{\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}} \begin{pmatrix} x_k \\ y_k \\ \dot{x}_k \\ \dot{y}_k \end{pmatrix} + v_{k,(2\times 1)}$$

[1] Wanjin Xu et al, Ship tracking based on the fusion of Kalman filter and particle filter, J. Phys.: Conf. Ser. 2113 012017, 2021

Kalman Filter

Gauss-Markov model:

$$\begin{cases} x_{k+1} = \boxed{A}x_k + \boxed{B}u_k + \boxed{G}w_k \\ y_k = \boxed{C}x_k + v_k, \end{cases}$$

where $x_0 \sim \mathcal{N}(\bar{x}_0, P_0)$, $w_k \sim \mathcal{N}(0, Q)$, $v_k \sim \mathcal{N}(0, R)$.

Measurements: $Z^k := (y_k^T, \dots, y_0^T)$.

Goal: compute $\hat{x}_{k|k} := \mathbf{E}\{x_k | Z^k\}$ and $\hat{x}_{k+1|k} := \mathbf{E}\{x_{k+1} | Z^k\}$ recursively.

$$\hat{x}_{k|k-1} := A\hat{x}_{k-1|k-1} + Bu_k, \quad \hat{x}_{0|-1} := \bar{x}_0 \qquad \text{(prediction equation)}$$

$$\hat{x}_{k|k} := \hat{x}_{k|k-1} + K_k^f (y_k - C\hat{x}_{k|k-1}) \qquad \text{(update equation)}$$

$$K_k^f := P_{k|k-1} C^T (C P_{k|k-1} C^T + R)^{-1} \qquad \text{(Kalman gain equation)}$$

$$P_{k|k-1} := A P_{k-1|k-1} A^T + Q, \quad P_{0|-1} := P_0 \qquad \text{(prediction MSE equation)}$$

$$P_{k|k} := P_{k|k-1} - K_k^f C P_{k|k-1} \qquad \text{(MSE equation)}$$

Cf. PAA06&07 (Slide 25)

Particle Filter

Gauss-Markov model
$x_{k+1} = Ax_k + Bu_k + Gw_k$

Algorithm 40 (Classical SMC) *1. Generate n samples $x_0^i \sim f(x_0)$. Set t=0.*

2. **Prediction***: Generate the prediction set using: $\boxed{\tilde{x}_{t+1}^i \sim f(x_{t+1}|x_t^i)}$, $i = 1, 2, \ldots, n$.*

3. **Update***: Compute the weights $w_{t+1}^i = f(y_{t+1}|\tilde{x}_{t+1})$, and normalize them using $\tilde{w}_{t+1}^i = \frac{w_{t+1}^i}{\sum_{j=1}^n w_{t+1}^j}$.*

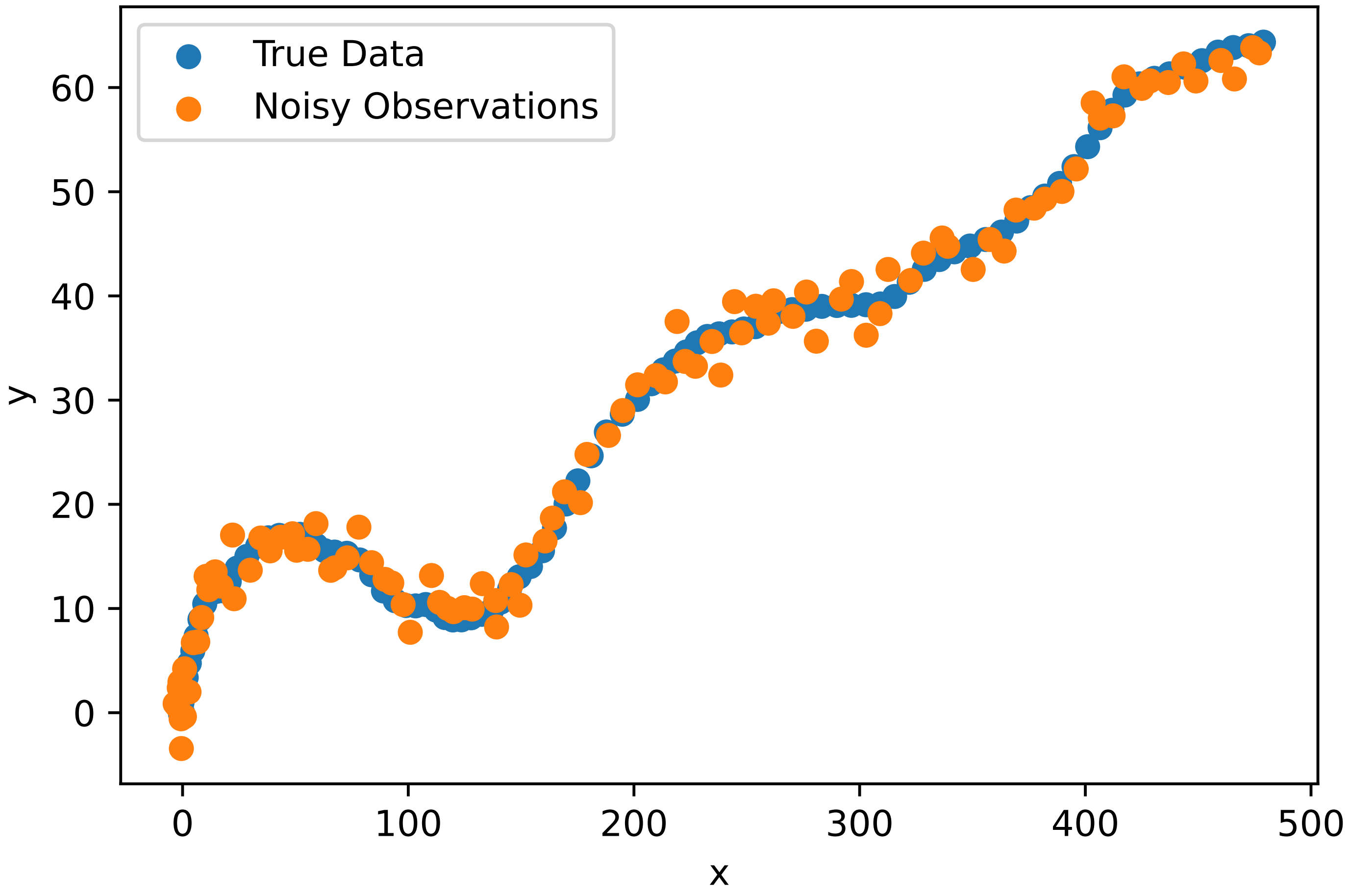
(a) Estimate θ_{t+1} using $\hat{\theta}_{t+1} = \sum_{i=1}^n g(\tilde{x}_{t+1}^i)\tilde{w}_{t+1}^i$.

(b) Resample from the set $\{\tilde{x}_{t+1}^1, \tilde{x}_{t+1}^2, \ldots, \tilde{x}_{t+1}^n\}$ with probabilities $\{\tilde{w}_{t+1}^1, \tilde{w}_{t+1}^2, \ldots, \tilde{w}_{t+1}^n\}$, n times to obtain the samples x_{t+1}^i, $i = 1, 2, \ldots, n$.

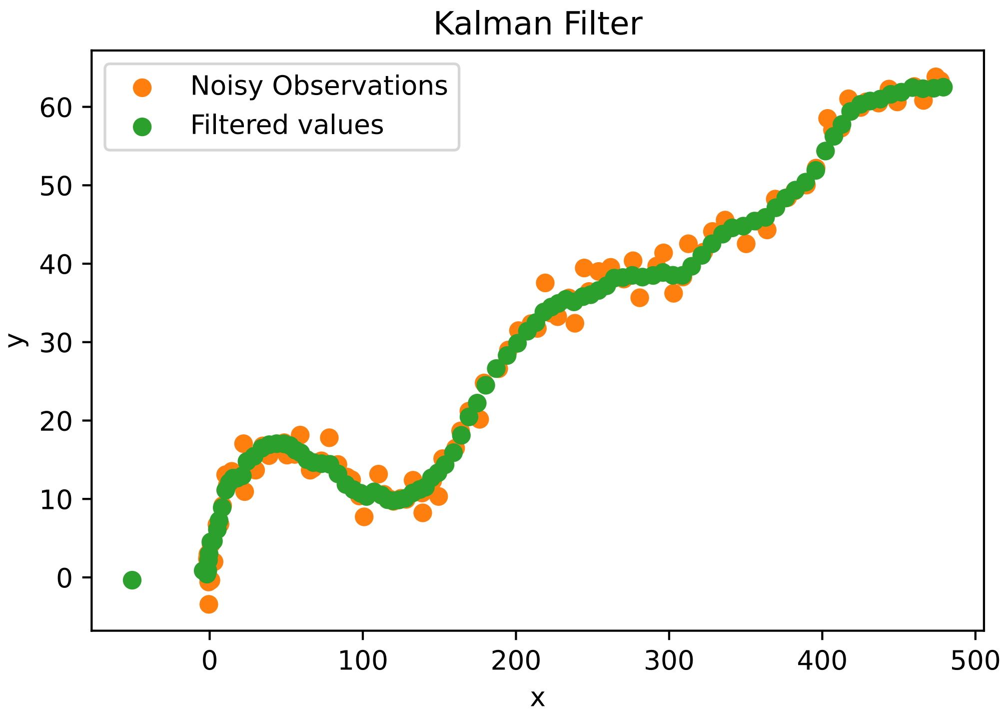
4. Set $t = t + 1$, and return to Step 2.

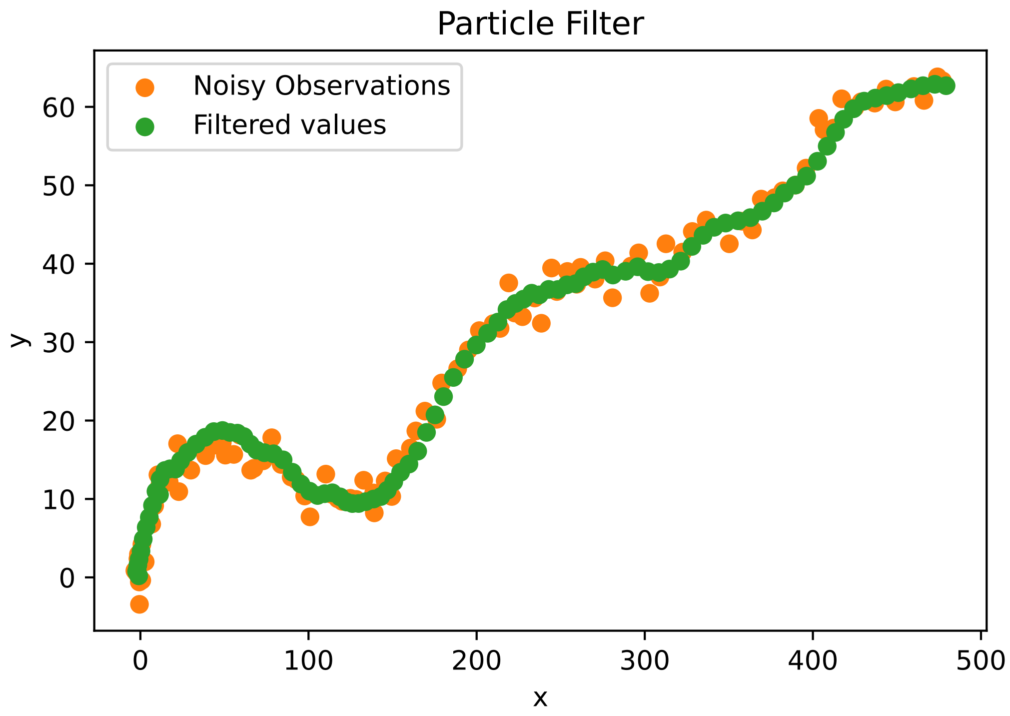
Section 10.4 of Srivastava, A. *Computational methods in statistics*. 2009.

True Positions vs. Noisy Observations



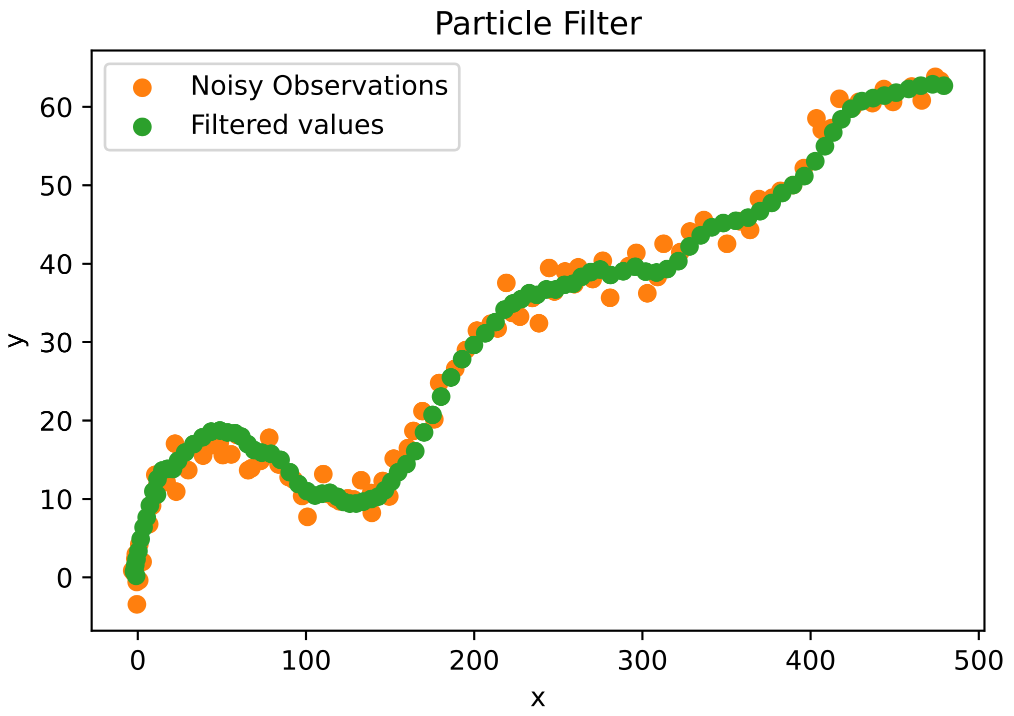
Spoiler Alert



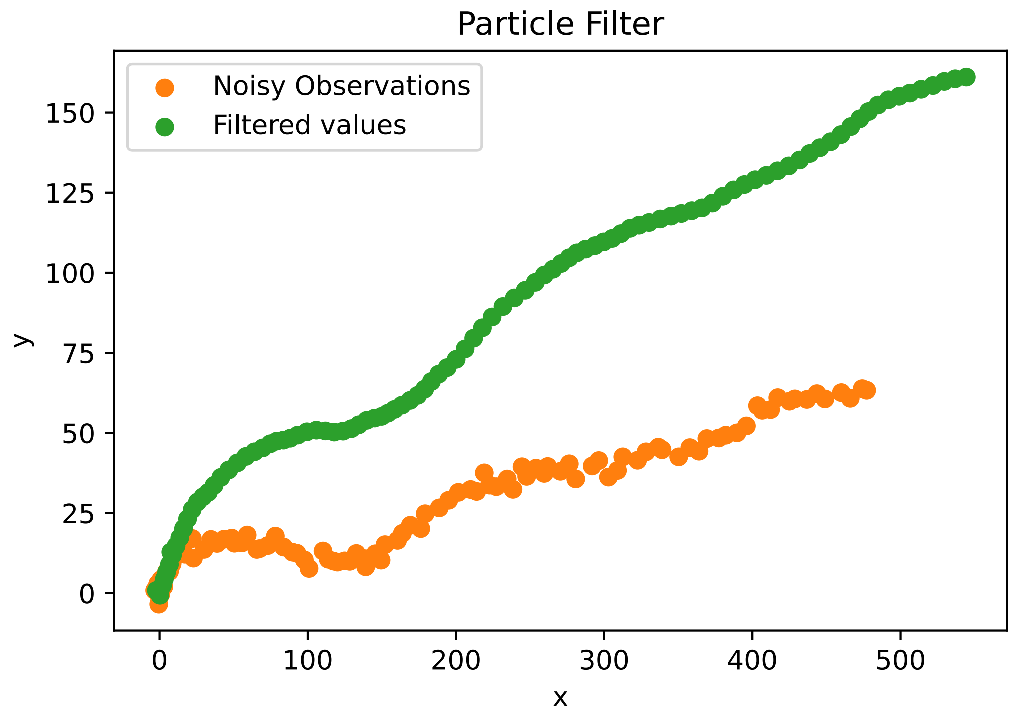


Stochastic Filtering: Yes, but…

Run 1



Run 2



Randomness issues $\Longrightarrow$ Repeat your experiments to draw consistent and relevant conclusions

Practical details

- Groups of two students
  - Register on Moodle before Friday 17th March
- Two parts
  - Part 1: end of week 10
  - Part 2: end of week 13
- Permanence
  - From week 7 to week 13
  - Tuesday 3-4 pm @Euler building (room A.010) with Philémon Beghin (rather for part 2)
  - Friday 2-3 pm @Teams with Jehum Cho (rather for part 1)
  - Mail to philemon.beghin@uclouvain.be or jehum.cho@uclouvain.be

Last advice: Don't wait for the last moment

Random MAP Student

During the semester (MAP Banquet ?)

At the end of the semester