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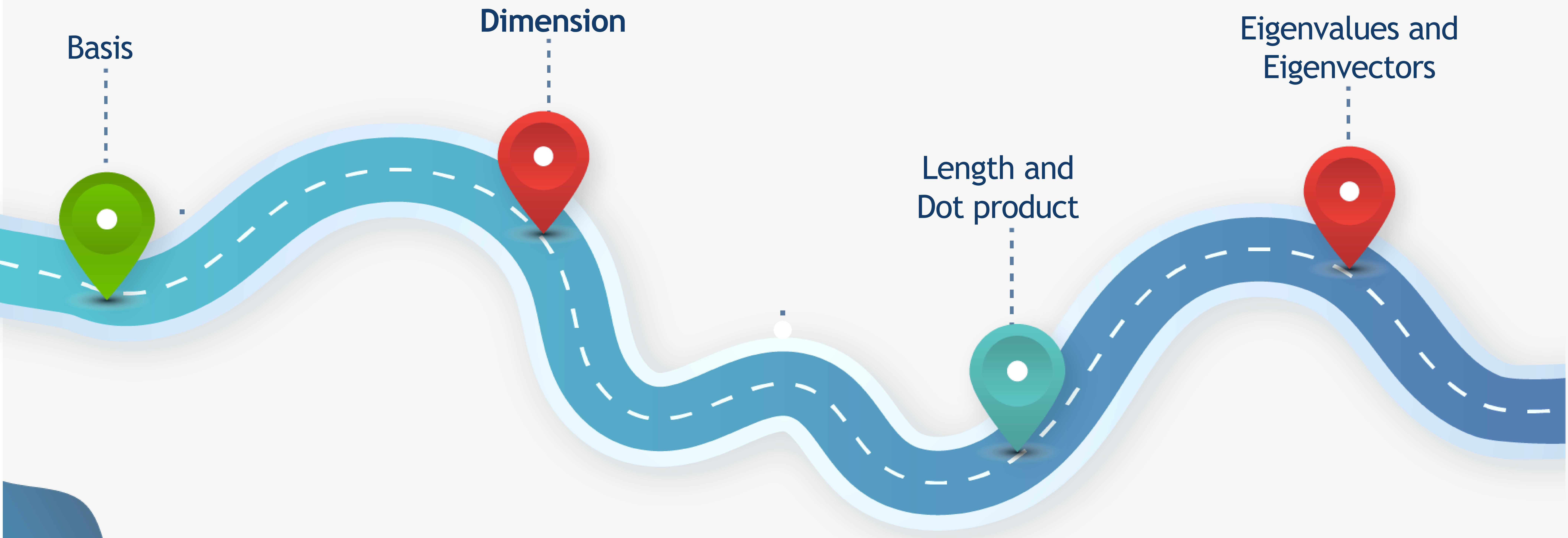
# Data Science Fundamentals
## Vectors 2


Basis
Dimension
Length and
Dot product
Eigenvalues and
Eigenvectors

# Basis

# Basis

## Definition of Basis

A set of vectors S={$v_1$, $v_2$ ...., $v_k$} in a vector space V is called a **basis** for V if the following conditions are true.

1. S is linearly independent

2. S spans V.

- A basis must have enough vectors to span V but not so many vectors that one of them could be written as a linear combination of the other vectors in S.
- If a vector space V has a basis consisting of a finite number of vectors, then V is finite-dimensional. Otherwise, V is called infinite-dimensional.

# Basis

- S={(1,0,0),(0,1,0),(0,0,1)} is a basis for $R^3$ , This is called the standard basis of $R^3$

The basis S={1, $x$,$x^2$,$x^3$} is called the standard basis for $P_3$. Similarly, the **standard basis** for $P_n$ is

$$S = \{1, x, x^2, \ldots, x^n\}.$$

**$P_n$ is a polynomial function of n degrees**

# Basis

## Uniqueness of basis Representation

If S={$v_1$, $v_2$ …., $v_n$} is a basis for a vector space V, then every vector in V can be written in one and only one way as a linear combination of vectors in S.

## Bases and linear dependence

If S={$v_1$, $v_2$ …., $v_k$} is a basis for a vector space V, then every set containing more than n vectors in V is linearly dependent.

## Number of vectors in a basis

If a vector space V has one basis with n vectors, then every basis has n vectors

# Dimension

# Dimension

## Definition of Dimension of a vector space

If a vector space V has a basis consisting of n vectors, then the number n is called the **dimension** of V, denoted by dim(V)=n.

The dimension of V is defined as zero If consists of the zero vector alone.

## Characteristics of dimensions

- The dimension of $R^n$ with the standard operations is n
- The dimension of $P_n$ with the standard operations is n+1
- The dimension of $M_{m,n}$ with the standard operations is *mn*

# Dimension

## ➢ Basis tests in an *n*-dimensional space

Let V be a vector space of dimension n.

1. If $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ is a linearly independent set of vectors in $V$, then $S$ is a basis for $V$.
2. If $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ spans $V$, then $S$ is a basis for $V$.

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# Length and Dot product

# Length and Dot product

## Length of a vector

The length, or magnitude, of a vector v= $(v_1, v_2 ..., v_n)$ in $R^n$ is given by

$$\|\mathbf{v}\| = \sqrt{v_1^2 + v_2^2 + \cdot\cdot\cdot + v_n^2}.$$

- This is also called the norm. If the norm of a vector = 1, then the vector is called a unit vector.
- This definition shows that the length of a vector cannot be negative
- We can also update the definition for scalar multiples

# Length and Dot product

- **Length of a vector**

Let v be a vector in $R^n$ and let c be a scalar. Then

$$\|c\mathbf{v}\| = |c|\,\|\mathbf{v}\|,$$

where $|c|$ is the absolute value of $c$.

# Length and Dot product

- **Unit vectors in the direction of v**

  If v is a nonzero vector in $R^n$, then the vector

$$\mathbf{u} = \frac{\mathbf{v}}{\|\mathbf{v}\|}$$

has length 1 and has the same direction v . This vector is called the **unit vector in the direction of v.**

# Length and Dot product

- The process of finding the unit vector in the direction of v is called normalizing the vector v

  - **Distance between two vectors**

The distance between two vectors **u** and **v** in $R^n$ is

$$d(\mathbf{u}, \mathbf{v}) = \|\mathbf{u} - \mathbf{v}\|.$$

# Length and Dot product

## ➢ Dot product

The dot product of $\mathbf{u} = (u_1, u_2, \ldots, u_n)$ and $\mathbf{v} = (v_1, v_2, \ldots, v_n)$ is the *scalar* quantity

$$\mathbf{u} \cdot \mathbf{v} = u_1v_1 + u_2v_2 + \cdots + u_nv_n.$$

- The dot product of two vectors is scalar, not another vector

# Length and Dot product

## ➢ Properties of the dot product

If $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ are vectors in $R^n$ and $c$ is a scalar, then the following properties are true.

1. $\mathbf{u} \cdot \mathbf{v} = \mathbf{v} \cdot \mathbf{u}$
2. $\mathbf{u} \cdot (\mathbf{v} + \mathbf{w}) = \mathbf{u} \cdot \mathbf{v} + \mathbf{u} \cdot \mathbf{w}$
3. $c(\mathbf{u} \cdot \mathbf{v}) = (c\mathbf{u}) \cdot \mathbf{v} = \mathbf{u} \cdot (c\mathbf{v})$
4. $\mathbf{v} \cdot \mathbf{v} = \|\mathbf{v}\|^2$
5. $\mathbf{v} \cdot \mathbf{v} \geq 0$, and $\mathbf{v} \cdot \mathbf{v} = 0$ if and only if $\mathbf{v} = \mathbf{0}$.

# Length and Dot product

## The Cauchy-Shwarz Inequality

If $\mathbf{u}$ and $\mathbf{v}$ are vectors in $R^n$, then

$$|\mathbf{u} \cdot \mathbf{v}| \leq \|\mathbf{u}\| \, \|\mathbf{v}\|,$$

where $|\mathbf{u} \cdot \mathbf{v}|$ denotes the *absolute value* of $\mathbf{u} \cdot \mathbf{v}$.

## Definition of the angle between two vectors

The **angle** $\theta$ between two nonzero vectors in $R^n$ is given by

$$\cos\theta = \frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{u}\| \, \|\mathbf{v}\|}, \quad 0 \leq \theta \leq \pi.$$

## Orthogonal vectors

Two vectors $\mathbf{u}$ and $\mathbf{v}$ in $R^n$ are **orthogonal** if

$$\mathbf{u} \cdot \mathbf{v} = 0.$$

# Length and Dot product

## ➢ The triangle inequality

If $\mathbf{u}$ and $\mathbf{v}$ are vectors in $R^n$, then

$$\|\mathbf{u} + \mathbf{v}\| \leq \|\mathbf{u}\| + \|\mathbf{v}\|.$$

## ➢ The Pythagorean Theorem

If $\mathbf{u}$ and $\mathbf{v}$ are vectors in $R^n$, then $\mathbf{u}$ and $\mathbf{v}$ are orthogonal if and only if

$$\|\mathbf{u} + \mathbf{v}\|^2 = \|\mathbf{u}\|^2 + \|\mathbf{v}\|^2.$$

# Eigenvalues and Eigenvectors

# Eigenvalues and Eigenvectors

## What is the Eigenvalues and Eigenvectors?

Eigenvectors help understand linear transformations easily.

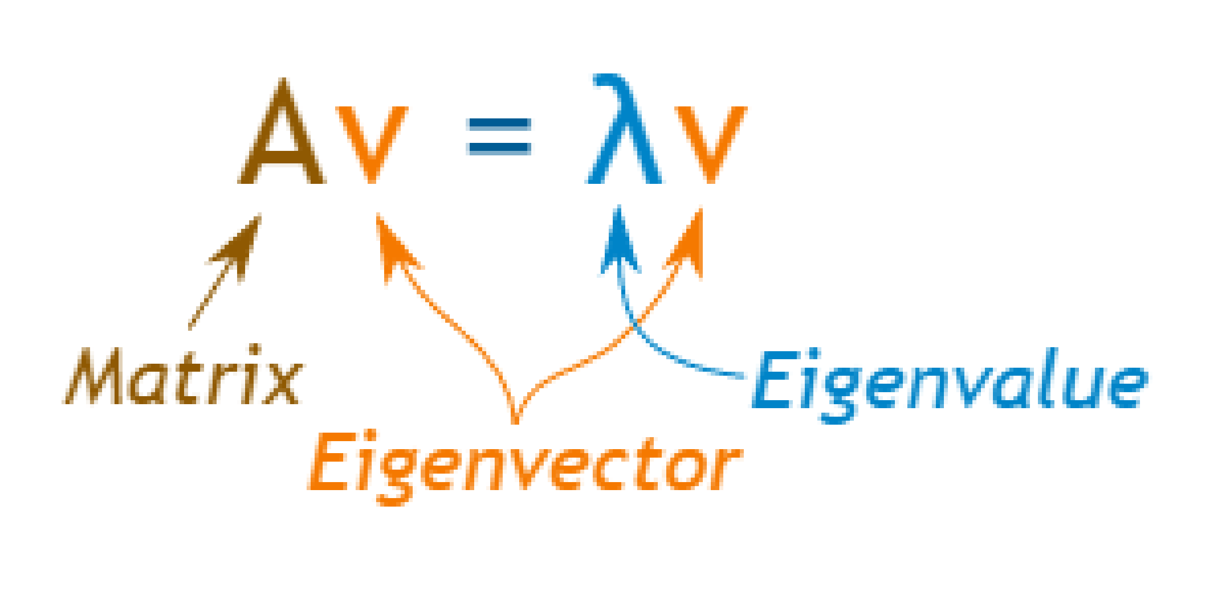
- Eigenvectors are the axes or directions along which a linear transformation acts by stretching or compressing or flipping on an X-Y line chart without changing their direction.

- Eigenvalues are the factors by which this transformation happens.

# Eigenvalues and Eigenvectors

**An eigenvector** is a non-zero vector that changes by a scalar element which is an **eigenvalue** when applied a linear transformation to it.

**A is** a square matrix ,and an Eigenvector and Eigenvalue make this equation true




The vector x is called an eigenvector of A corresponding to $\lambda$.

# Eigenvalues and Eigenvectors

## Example:

For the matrix

$$A = \begin{bmatrix} 2 & 0 \\ 0 & -1 \end{bmatrix},$$

verify that $\mathbf{x}_1 = (1, 0)$ is an eigenvector of $A$ corresponding to the eigenvalue $\lambda_1 = 2$,

Multiplying $\mathbf{x}_1$ by $A$ produces

$$A\mathbf{x}_1 = \begin{bmatrix} 2 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \end{bmatrix} = \underbrace{2}_{\text{Eigenvalue}}\underbrace{\begin{bmatrix} 1 \\ 0 \end{bmatrix}}_{\text{Eigenvector}}.$$

# Eigenvalues and Eigenvectors

## ➢ Notes:

1. Eigenvalues and Eigenvectors are only for square matrices.
2. Eigenvectors are by definition nonzero.
3. Eigenvalues may be equal to zero.
4. An $n \times n$ matrix A has at most n eigenvalues.

# Finding Eigenvalues and Eigenvectors

To find the eigenvalues and eigenvectors of an n x n matrix A ,let $I$ be the n x n identity matrix. Writing the equation Ax= λx in the form $\lambda Ix = Ax$ then produces

$$(\lambda I - A)\mathbf{x} = \mathbf{0}.$$

The equation is called the **Characteristic Equation**

# Finding Eigenvalues and Eigenvectors

Let $A$ be an $n \times n$ matrix.

1. An eigenvalue of $A$ is a scalar $\lambda$ such that

$$\det(\lambda I - A) = 0.$$

2. The eigenvectors of $A$ corresponding to $\lambda$ are the nonzero solutions of

$$(\lambda I - A)\mathbf{x} = \mathbf{0}.$$

# Finding Eigenvalues and Eigenvectors-Example

❖ Example: Find the eigenvalues and corresponding eigenvectors of

$$A = \begin{bmatrix} 2 & -12 \\ 1 & -5 \end{bmatrix}.$$

❖ Solution:

$$\begin{aligned} |\lambda I - A| &= \begin{vmatrix} \lambda - 2 & 12 \\ -1 & \lambda + 5 \end{vmatrix} \\ &= (\lambda - 2)(\lambda + 5) - (-12) \\ &= \lambda^2 + 3\lambda - 10 + 12 \\ &= \lambda^2 + 3\lambda + 2 \\ &= (\lambda + 1)(\lambda + 2). \end{aligned}$$

# Finding Eigenvalues and Eigenvectors-Example

The equation is $(\lambda+1)(\lambda+2)=0$, which gives $\lambda 1 = -1$ and $\lambda 2 = -2$ as the eigenvalues of A.

To find the corresponding eigenvectors, use Gauss-Jordan elimination to solve the homogeneous linear system represented by

$$(\lambda I - A)\mathbf{x} = \mathbf{0}.$$

**For each $\lambda$ solve $(A - \lambda I)x = 0$ or $Ax = \lambda x$ to find an eigenvector $x$.**

first for $\lambda = \lambda 1 = -1$ then for $\lambda = \lambda 2 = -2$

# Finding Eigenvalues and Eigenvectors-Example

For $\lambda_1 = -1$ the coefficient matrix is

$$(-1)I - A = \begin{bmatrix} -1-2 & 12 \\ -1 & -1+5 \end{bmatrix} = \begin{bmatrix} -3 & 12 \\ -1 & 4 \end{bmatrix}, \longrightarrow \begin{bmatrix} 1 & -4 \\ 0 & 0 \end{bmatrix},$$

For $\lambda_2 = -2$, you have

$$(-2)I - A = \begin{bmatrix} -2-2 & 12 \\ -1 & -2+5 \end{bmatrix} = \begin{bmatrix} -4 & 12 \\ -1 & 3 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & -3 \\ 0 & 0 \end{bmatrix}.$$

# Finding Eigenvalues and Eigenvectors-Example

- Exercise : Find the eigenvalues and corresponding eigenvectors of

$$\boldsymbol{A} = \begin{bmatrix} 4 & 2 \\ 1 & 3 \end{bmatrix}$$

# Finding Eigenvalues and Eigenvectors-Example

## Solution

$$p_{\boldsymbol{A}}(\lambda) = \det(\boldsymbol{A} - \lambda \boldsymbol{I}) \quad (4.29a)$$

$$= \det\left(\begin{bmatrix}4 & 2\\1 & 3\end{bmatrix} - \begin{bmatrix}\lambda & 0\\0 & \lambda\end{bmatrix}\right) = \begin{vmatrix}4-\lambda & 2\\1 & 3-\lambda\end{vmatrix} \quad (4.29b)$$

$$= (4-\lambda)(3-\lambda) - 2\cdot 1\,. \quad (4.29c)$$

We factorize the characteristic polynomial and obtain

$$p(\lambda) = (4-\lambda)(3-\lambda) - 2\cdot 1 = 10 - 7\lambda + \lambda^2 = (2-\lambda)(5-\lambda) \quad (4.30)$$

giving the roots $\lambda_1 = 2$ and $\lambda_2 = 5$.

**Step 3: Eigenvectors and Eigenspaces.** We find the eigenvectors that correspond to these eigenvalues by looking at vectors $\boldsymbol{x}$ such that

$$\begin{bmatrix}4-\lambda & 2\\1 & 3-\lambda\end{bmatrix}\boldsymbol{x} = \boldsymbol{0}\,. \quad (4.31)$$

For $\lambda = 5$ we obtain

$$\begin{bmatrix}4-5 & 2\\1 & 3-5\end{bmatrix}\begin{bmatrix}x_1\\x_2\end{bmatrix} = \begin{bmatrix}-1 & 2\\1 & -2\end{bmatrix}\begin{bmatrix}x_1\\x_2\end{bmatrix} = \boldsymbol{0}\,. \quad (4.32)$$

We solve this homogeneous system and obtain a solution space

$$E_5 = \text{span}[\begin{bmatrix}2\\1\end{bmatrix}]\,. \quad (4.33)$$

This eigenspace is one-dimensional as it possesses a single basis vector.

Analogously, we find the eigenvector for $\lambda = 2$ by solving the homogeneous system of equations

$$\begin{bmatrix}4-2 & 2\\1 & 3-2\end{bmatrix}\boldsymbol{x} = \begin{bmatrix}2 & 2\\1 & 1\end{bmatrix}\boldsymbol{x} = \boldsymbol{0}\,. \quad (4.34)$$

This means any vector $\boldsymbol{x} = \begin{bmatrix}x_1\\x_2\end{bmatrix}$, where $x_2 = -x_1$, such as $\begin{bmatrix}1\\-1\end{bmatrix}$, is an eigenvector with eigenvalue 2. The corresponding eigenspace is given as

$$E_2 = \text{span}[\begin{bmatrix}1\\-1\end{bmatrix}]\,. \quad (4.35)$$

# Finding Eigenvalues and Eigenvectors-Example

- Exercise: Find the eigenvalues and corresponding eigenvectors of

$$\boldsymbol{A} = \begin{bmatrix} 3 & 2 & 2 \\ 2 & 3 & 2 \\ 2 & 2 & 3 \end{bmatrix} .$$

# Finding Eigenvalues and Eigenvectors-Example

## ❖ Solution

The characteristic polynomial of $\boldsymbol{A}$ is

$$p_{\boldsymbol{A}}(\lambda) = -(\lambda - 1)^2(\lambda - 7)\,, \tag{4.38}$$

so that we obtain the eigenvalues $\lambda_1 = 1$ and $\lambda_2 = 7$, where $\lambda_1$ is a repeated eigenvalue. Following our standard procedure for computing eigenvectors, we obtain the eigenspaces

$$E_1 = \operatorname{span}[\underbrace{\begin{bmatrix}-1\\1\\0\end{bmatrix}}_{=:\boldsymbol{x}_1}, \underbrace{\begin{bmatrix}-1\\0\\1\end{bmatrix}}_{=:\boldsymbol{x}_2}], \quad E_7 = \operatorname{span}[\underbrace{\begin{bmatrix}1\\1\\1\end{bmatrix}}_{=:\boldsymbol{x}_3}]\,. \tag{4.39}$$

We see that $\boldsymbol{x}_3$ is orthogonal to both $\boldsymbol{x}_1$ and $\boldsymbol{x}_2$. However, since $\boldsymbol{x}_1^\top \boldsymbol{x}_2 = 1 \neq 0$, they are not orthogonal. The spectral theorem (Theorem 4.15) states that there exists an orthogonal basis, but the one we have is not orthogonal. However, we can construct one.

To construct such a basis, we exploit the fact that $\boldsymbol{x}_1, \boldsymbol{x}_2$ are eigenvectors associated with the same eigenvalue $\lambda$. Therefore, for any $\alpha, \beta \in \mathbb{R}$ it holds that

$$\boldsymbol{A}(\alpha\boldsymbol{x}_1 + \beta\boldsymbol{x}_2) = \boldsymbol{A}\boldsymbol{x}_1\alpha + \boldsymbol{A}\boldsymbol{x}_2\beta = \lambda(\alpha\boldsymbol{x}_1 + \beta\boldsymbol{x}_2)\,, \tag{4.40}$$

i.e., any linear combination of $\boldsymbol{x}_1$ and $\boldsymbol{x}_2$ is also an eigenvector of $\boldsymbol{A}$ associated with $\lambda$. The Gram-Schmidt algorithm (Section 3.8.3) is a method for iteratively constructing an orthogonal/orthonormal basis from a set of basis vectors using such linear combinations. Therefore, even if $\boldsymbol{x}_1$ and $\boldsymbol{x}_2$ are not orthogonal, we can apply the Gram-Schmidt algorithm and find eigenvectors associated with $\lambda_1 = 1$ that are orthogonal to each other (and to $\boldsymbol{x}_3$). In our example, we will obtain

$$\boldsymbol{x}_1' = \begin{bmatrix}-1\\1\\0\end{bmatrix}, \quad \boldsymbol{x}_2' = \frac{1}{2}\begin{bmatrix}-1\\-1\\2\end{bmatrix}, \tag{4.41}$$

which are orthogonal to each other, orthogonal to $\boldsymbol{x}_3$, and eigenvectors of $\boldsymbol{A}$ associated with $\lambda_1 = 1$.

# Finding Eigenvalues and Eigenvectors-Summery

Let $A$ be an $n \times n$ matrix.

1. Form the characteristic equation $|\lambda I - A| = 0$. It will be a polynomial equation of degree $n$ in the variable $\lambda$.
2. Find the real roots of the characteristic equation. These are the eigenvalues of $A$.
3. For each eigenvalue $\lambda_i$, find the eigenvectors corresponding to $\lambda_i$ by solving the homogeneous system $(\lambda_i I - A)\mathbf{x} = \mathbf{0}$. This requires row reducing of an $n \times n$ matrix. The resulting reduced row-echelon form must have at least one row of zeros.

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# THANK YOU