The Learning Hub

Data Science Fundamentals

Vectors

Vectors in R^n

Vector Spaces

Subspaces of Vector Spaces

Spanning sets and Linear independence

The Learning Hub

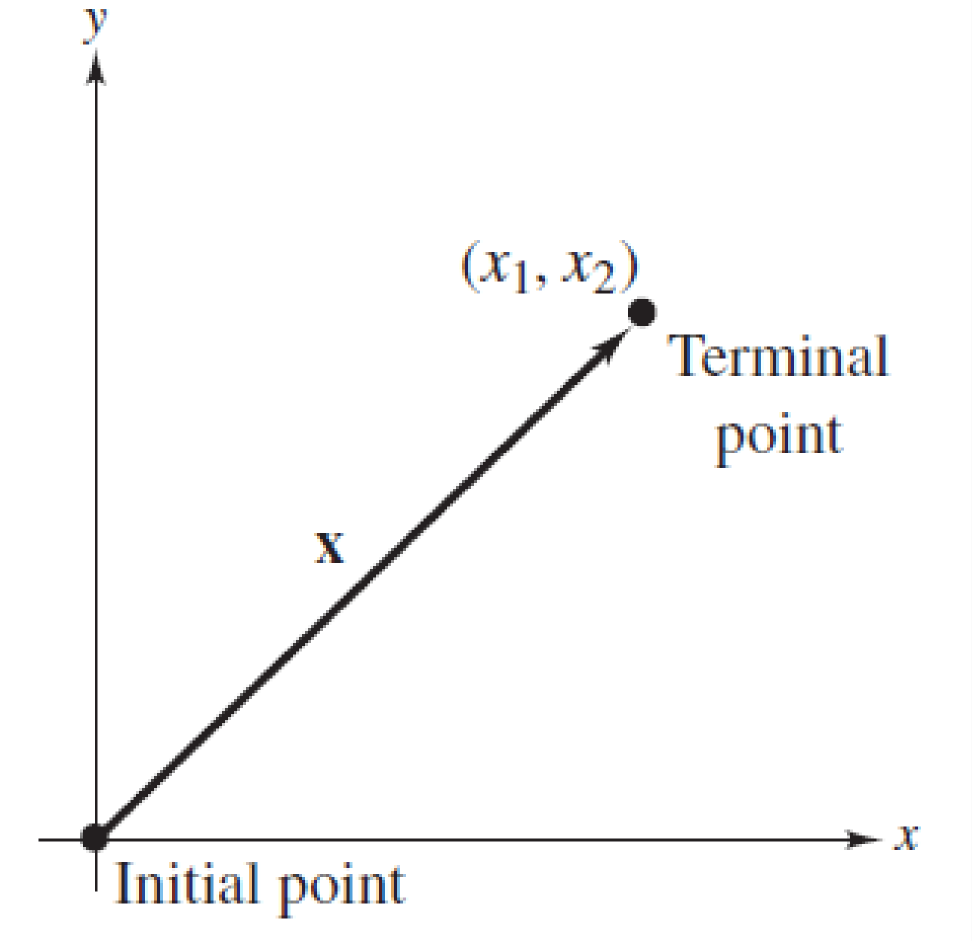
Vectors in R^n

Vectors in $\boldsymbol{R}$^$\boldsymbol{n}$

- Vectors are geometrical entities that have magnitude and direction. A vector is represented by a line with an arrow indicating its direction, and its length denotes the magnitude of the vector.
- Vectors have multiple applications in math, physics, engineering, and different other fields.
- A vector is characterized by two quantities: Length and direction and is represented by a directed line segment

Vectors in R^n

- Vectors in the plane: The initial point of a vector is the origin point and its terminal point is (x_1, x_2)



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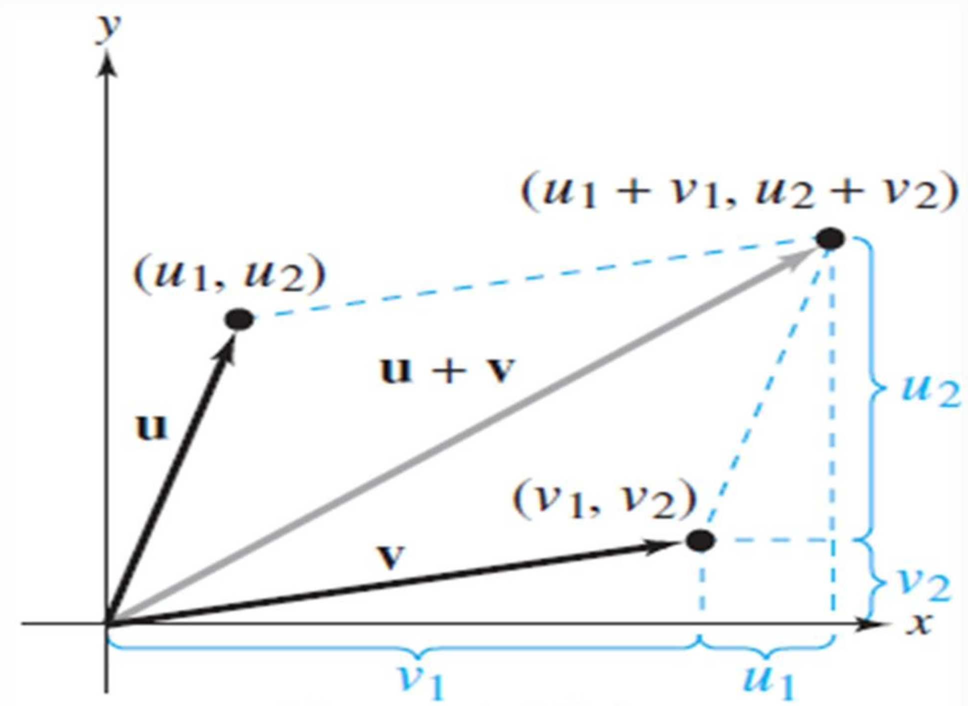
Vectors in $\boldsymbol{R}$^$\boldsymbol{n}$

- x_1 and x_2 are called the components of the vector **x** which is represented as an ordered pair $\mathbf{x} = (x_1, x_2)$
- Two vectors in the plane u and v where $\mathbf{u} = (u_1, u_2)$ and $\mathbf{v} = (v_1, v_2)$ are equal if and only if $u_1 = v_1$ and $u_2 = v_2$
- A vector is represented in lowercase boldface letters

Vectors in $\boldsymbol{R}$^$\boldsymbol{n}$

❖Vector addition :

- To add two vectors in the plane, add their corresponding components
- $\mathbf{u} + \mathbf{v} = (u_1, u_2) + (v_1, v_2) = (u_1 + v_1, u_2 + v_2)$
- The sum of two vectors in the plane is the parallelogram having the two vectors as its adjacent sides

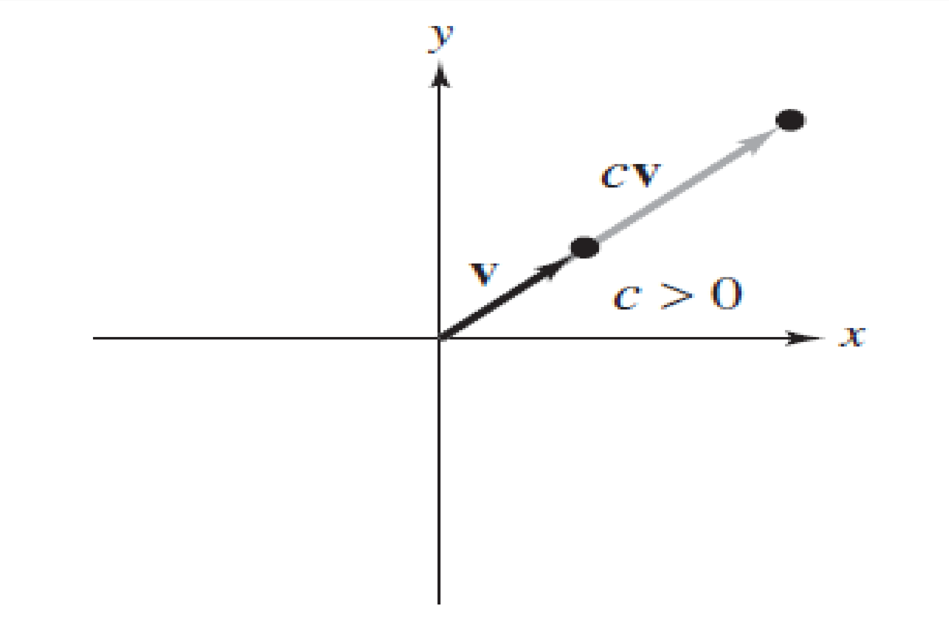


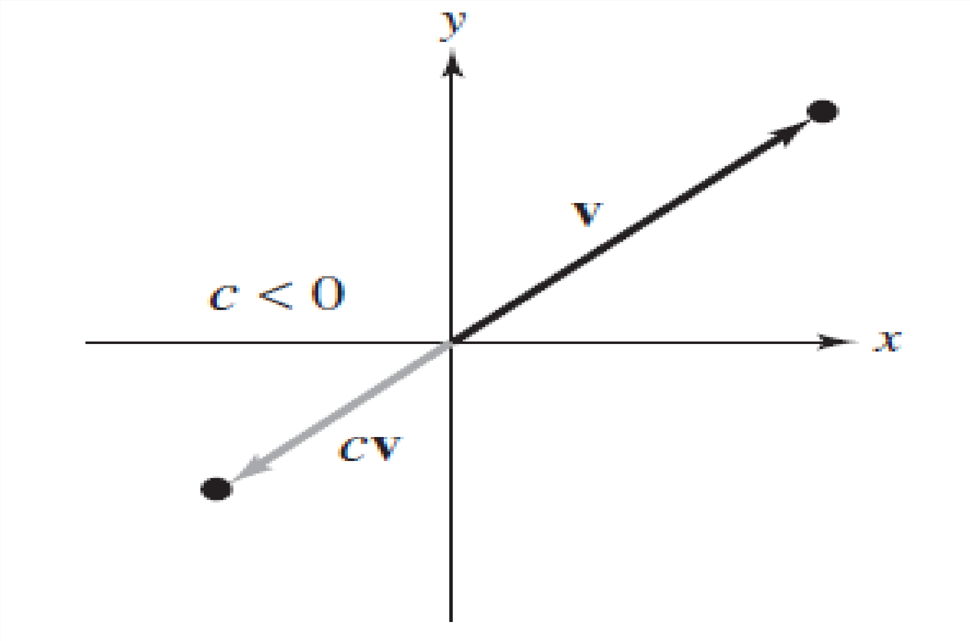
Vector Addition

Vectors in R^n

❖ Scalar multiplication :

- To multiply a vector **v** by a scalar *c*, multiply each of the components of the vector by the scalar
- $c\mathbf{v} = c(v_1, v_2) = (cv_1, cv_2)$





- Subtraction is defined as a combination of scalar multiplication and addition

Vectors in R^n

➢ Properties of vector addition and scalar multiplication :

Let u, v, and w be vectors in the plane, and let c and d be scalars.

1. $\mathbf{u} + \mathbf{v}$ is a vector in the plane.	Closure under addition
2. $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$	Commutative property of addition
3. $(\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w})$	Associative property of addition
4. $\mathbf{u} + \mathbf{0} = \mathbf{u}$	Additive identity property
5. $\mathbf{u} + (-\mathbf{u}) = \mathbf{0}$	Additive inverse property
6. $c\mathbf{u}$ is a vector in the plane.	Closure under scalar multiplication
7. $c(\mathbf{u} + \mathbf{v}) = c\mathbf{u} + c\mathbf{v}$	Distributive property
8. $(c + d)\mathbf{u} = c\mathbf{u} + d\mathbf{u}$	Distributive property
9. $c(d\mathbf{u}) = (cd)\mathbf{u}$	Associative property of multiplication
10. $1(\mathbf{u}) = \mathbf{u}$	Multiplicative identity property

Vectors in R^n

- Let's now extend the definition from vectors in the plane to vectors in R^n, A vector in *n*-space is represented by an ordered ***n*-tuple**

$$
\begin{aligned}
R^1 &= \text{1-space} = \text{set of all real numbers} \\
R^2 &= \text{2-space} = \text{set of all ordered pairs of real numbers} \\
R^3 &= \text{3-space} = \text{set of all ordered triples of real numbers} \\
R^4 &= \text{4-space} = \text{set of all ordered quadruples of real numbers} \\
&\vdots \\
R^n &= n\text{-space} = \text{set of all ordered } n\text{-tuples of real numbers}
\end{aligned}
$$

Vectors in R^n

Scalar multiplication and vector addition in R^n :

Let $\mathbf{u} = (u_1, u_2, u_3, . . . , u_n)$ and $\mathbf{v} = (v_1, v_2, v_3, . . . , v_n)$ be vectors in R^n and let c be a real number. Then the sum of $\mathbf{u}$ and $\mathbf{v}$ is defined as the vector

$$\mathbf{u} + \mathbf{v} = (u_1 + v_1, u_2 + v_2, u_3 + v_3, . . . , u_n + v_n),$$

and the **scalar multiple** of $\mathbf{u}$ by c is defined as the vector

$$c\mathbf{u} = (cu_1, cu_2, cu_3, . . . , cu_n).$$

This is the same as in the plane but with higher orders

Vectors in $\boldsymbol{R^n}$

➢ Properties of vector addition and scalar multiplication in R^n :

Let $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ be vectors in R^n, and let c and d be scalars.

1. $\mathbf{u} + \mathbf{v}$ is a vector in R^n.	Closure under addition
2. $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$	Commutative property of addition
3. $(\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w})$	Associative property addition
4. $\mathbf{u} + \mathbf{0} = \mathbf{u}$	Additive identity property
5. $\mathbf{u} + (-\mathbf{u}) = \mathbf{0}$	Additive inverse property
6. $c\mathbf{u}$ is a vector in R^n.	Closure under scalar multiplication
7. $c(\mathbf{u} + \mathbf{v}) = c\mathbf{u} + c\mathbf{v}$	Distributive property
8. $(c + d)\mathbf{u} = c\mathbf{u} + d\mathbf{u}$	Distributive property
9. $c(d\mathbf{u}) = (cd)\mathbf{u}$	Associative property of multiplication
10. $1(\mathbf{u}) = \mathbf{u}$	Multiplicative identity property

- ➢ The zero vector is defined as $\mathbf{0} = (0, 0,.. 0)$
  - ▪ The zero vector is called the additive identity in R^n
- ➢ The vector **-v** is called the additive inverse of **v**

Vectors in R^n

➢ Properties of additive inverse and additive identity

Let v be a vector in R^n and let c be a scalar. Then the following properties are true

1. The additive identity is unique. That is, if $\mathbf{v} + \mathbf{u} = \mathbf{v}$, then $\mathbf{u} = \mathbf{0}$.
2. The additive inverse of $\mathbf{v}$ is unique. That is, if $\mathbf{v} + \mathbf{u} = \mathbf{0}$, then $\mathbf{u} = -\mathbf{v}$.
3. $0\mathbf{v} = \mathbf{0}$
4. $c\mathbf{0} = \mathbf{0}$
5. If $c\mathbf{v} = \mathbf{0}$, then $c = 0$ or $\mathbf{v} = \mathbf{0}$.
6. $-(-\mathbf{v}) = \mathbf{v}$

Vectors in $\boldsymbol{R}$^$\boldsymbol{n}$

- **Writing a vector as a linear combination of other vectors :**

Provided that x=(-1,-2,-2) , u=(0,1,4), v=(-1,1,2)and w=(3,1,2) in R^3 , find scalars a,b ,and c such that

$$X = au + bv + cw$$

- **By Writing**

$$\overbrace{(-1, -2, -2)}^{\mathbf{x}} = a\overbrace{(0, 1, 4)}^{\mathbf{u}} + b\overbrace{(-1, 1, 2)}^{\mathbf{v}} + c\overbrace{(3, 1, 2)}^{\mathbf{w}}$$
$$= (-b + 3c, a + b + c, 4a + 2b + 2c),$$

Vectors in R^n

- you can equate corresponding components so that they form the system of three linear equations in a, b, and c and shown below

$$\begin{aligned} -b + 3c &= -1 && \text{Equation from first component} \\ a + b + c &= -2 && \text{Equation from second component} \\ 4a + 2b + 2c &= -2 && \text{Equation from third component} \end{aligned}$$

- Using any of the techniques before, solve for a, b, and c to get

$$a = 1 \,, b = -2 \,, c = -1$$

- X can be written as a linear combination of u, v and w.

$$X = u - 2v - w$$

Vectors in $\boldsymbol{R}$^$\boldsymbol{n}$

- Vector in R^n can be represented as either a 1 x *n* row matrix or an *n* x 1 column matrix

$$\mathbf{u} = \begin{bmatrix} u_1 & u_2 & \cdots & u_n \end{bmatrix},$$

$$\mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix}.$$

Vector Spaces

Vector Spaces

- The properties (axioms) we defined for vectors seem to be shared with many other mathematical quantities like matrices, polynomials, and functions.

- Any set that satisfies our defined axioms is called a vector space and the objects in it are called vectors

- A vector space consists of four entities
  - A set of vectors
  - A set of scalars
  - Two operations: vector addition and scalar multiplication

Vector Spaces

➢ Definition of a vector space :

Let V be a set on which two operations (vector addition and scalar multiplication) are defined. If the listed axioms are satisfied for every u , v, and w and every scalar (real number) c and, then V is called a vector space

- Addition:

1. $\mathbf{u} + \mathbf{v}$ is in V. — Closure under addition
2. $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$ — Commutative property
3. $\mathbf{u} + (\mathbf{v} + \mathbf{w}) = (\mathbf{u} + \mathbf{v}) + \mathbf{w}$ — Associative property
4. V has a zero vector $\mathbf{0}$ such that for every $\mathbf{u}$ in V, $\mathbf{u} + \mathbf{0} = \mathbf{u}$. — Additive identity
5. For every $\mathbf{u}$ in V, there is a vector in V denoted by $-\mathbf{u}$ such that $\mathbf{u} + (-\mathbf{u}) = \mathbf{0}$. — Additive inverse

Vector Spaces

➢ Scalar Multiplication:

6. $c\mathbf{u}$ is in V.	Closure under scalar multiplication
7. $c(\mathbf{u} + \mathbf{v}) = c\mathbf{u} + c\mathbf{v}$	Distributive property
8. $(c + d)\mathbf{u} = c\mathbf{u} + d\mathbf{u}$	Distributive property
9. $c(d\mathbf{u}) = (cd)\mathbf{u}$	Associative property
10. $1(\mathbf{u}) = \mathbf{u}$	Scalar identity

Vector Spaces

❖ Examples :

R^2 with the Standard Operations Is a vector space

- The set of all ordered pairs of real numbers R^2 with the standard operations is a vector space

R^n with the Standard Operations Is a vector space

- The set of all ordered *n*-tuples of real numbers R^n with the standard operations is a vector space

Vector Spaces

❖ Examples :

The Vector Space of All $2 \; x \; 3$ Matrices

- Show that the set of all 2 x 3 matrices with matrix scalar multiplication and addition operations is a vector space.

- If A and B are 2 x 3 matrices and c is a scalar, then *c*A and A + B are also 2 x 3 matrices. The set is closed under matrix scalar multiplication and addition. Moreover, the other eight vector space axioms follow directly. You can conclude that the set is a vector space.

Vector Spaces

➢ Summary of important vector spaces

R=set of all real numbers

R^2 set of all ordered pairs

R^3= set of all ordered triples

R^n= set of all n-tuples

(-∞,∞)=set of all continuous functions defined on the real number line

C[a, b]= set of all continuous functions defined on a closed interval [a,b]

P=set of all polynomials

P_n= set of all polynomials of degree ≤ n

$M_{m,n}$ =set of all m x n matrices

$M_{n,n}$ =set of all n x n square matrices

Vector Spaces

➢ Properties of scalar multiplication

Let v be any element of a vector space V ,and let c be any scalar. Then the following properties are true.

1. $0\mathbf{v} = \mathbf{0}$
2. $c\mathbf{0} = \mathbf{0}$
3. If $c\mathbf{v} = \mathbf{0}$, then $c = 0$ or $\mathbf{v} = \mathbf{0}$.
4. $(-1)\mathbf{v} = -\mathbf{v}$

- The set of integers is not a vector space
  - Because it is not closed under scalar multiplication
    - ½(1) = ½

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Subspaces of Vector Spaces

Subspaces of Vector Spaces

- A subset of a vector space is a subspace if its is a vector space

A nonempty subset W of a vector space V is called a **subspace** of V if W is a vector space under the operations of scalar multiplication and addition defined in V

Note: If W is a subspace of V, then it must be closed under the operations inherited from V

- **Test for a subspace**

If W is a nonempty subset of a vector space V, then W is a subspace of V if and only if the following closure conditions hold

1. If $\mathbf{u}$ and $\mathbf{v}$ are in W, then $\mathbf{u} + \mathbf{v}$ is in W.
2. If $\mathbf{u}$ is in W and c is any scalar, then $c\mathbf{u}$ is in W.

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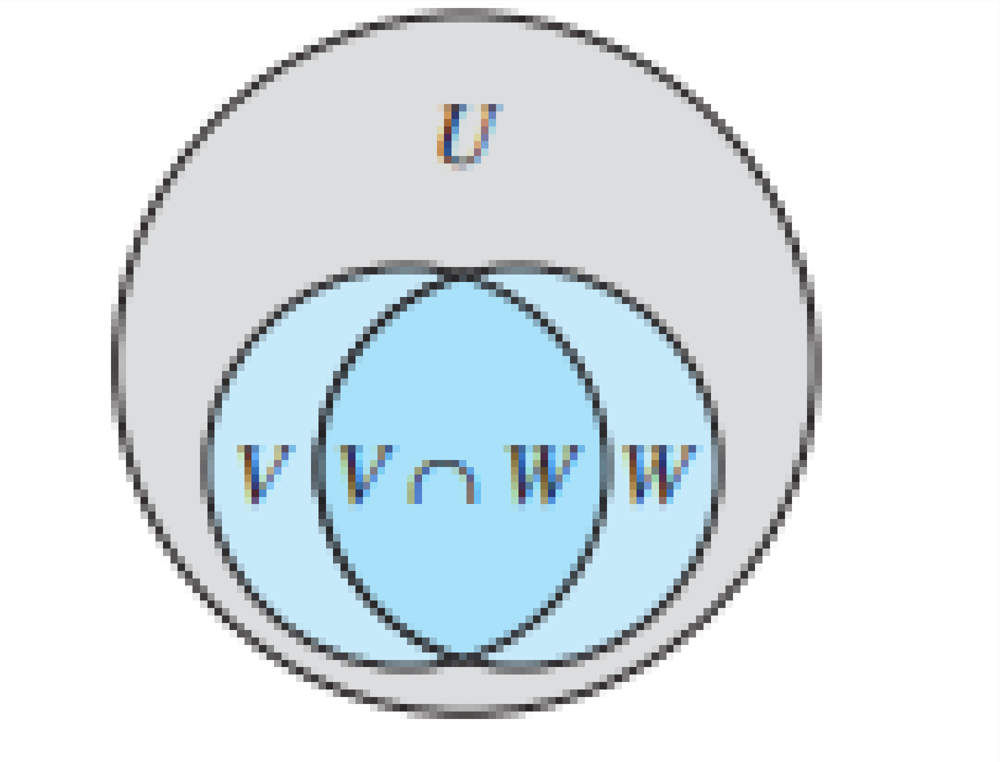
Subspaces of Vector Spaces

- both W and V must have the same zero vector **0**, If W is a subspace of a vector space V.
- The simplest subspace of a vector space is the one consisting of only the zero vector W = {0}, It is called the zero subspace.
- Another subspace of V is V itself.
- The zero subspace and self-subspace are contained in every vector space and are called trivial subspaces. Other subspaces are called proper subspaces or nontrivial subspaces.

Subspaces of Vector Spaces

- The intersection of two subspaces is a subspace



- If V and W are both subspaces of a vector space U then the intersection of V and W (denoted by V ∩ W) is also a subspace of U.

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Spanning sets and Linear independence

Spanning sets and Linear independence

➢ **Linear combination of vectors :**

A vector v in a vector space V is called a **linear combination** of the vectors u1,u2,.....,uk in V if v can be written in the form

$$\mathbf{v} = c_1\mathbf{u}_1 + c_2\mathbf{u}_2 + \cdot \cdot \cdot + c_k\mathbf{u}_k,$$

where c1,c2,....,ck are scalars

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Spanning sets and Linear independence

➢ **Example:**

- For the set of vectors in R^3, $S=\{(1,3,1),(0,1,2),(1,0,-5)\}$ where S is composed of v1, v2, and v3 respectively.
- v_1 is a linear combination of v_2 and v_3 because

$$\mathbf{v1=3v2+v3=3(0,1,2)+(1,0,-5)=(1,3,1)}$$

Spanning sets and Linear independence

- **How to find linear combinations**

S={(1,2,3),(0,1,2),(−1,0,1)}

- **Need to find scalar c_1, c_2 and c_3 such that**

(1,1,1)=c1(1,2,3)+c2(0,1,2)+c3(−1,0,1)=(c1−c3, 2c1+c2,3c1+2c2+c3)

- **By using Gauss Jordan elimination, we get:**

$c_1 = 2$, c_2=-3, c_3=1

Spanning sets and Linear independence

➢ Spanning sets :

If every vector in a vector space can be written as a linear combination of vectors in a set S then S is called a spanning set of the vector space.

Let S ={v1,v2,......,vk}be a subset of a vector space V. The set S is called a **spanning set** of V if every vector in V can be written as a linear combination of vectors in S In such cases, it is said that S **spans V.**

➢ Example

(a) The set $S = \{(1, 0, 0), (0, 1, 0), (0, 0, 1)\}$ spans R^3 because any vector $\mathbf{u} = (u_1, u_2, u_3)$ in R^3 can be written as

$$\mathbf{u} = u_1(1, 0, 0) + u_2(0, 1, 0) + u_3(0, 0, 1) = (u_1, u_2, u_3).$$

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Spanning sets and Linear independence

➢ The span of a set :

If S={v1,v2,....,vk} is a set of vectors in a vector space V, then the **span of S** is the set of all linear combinations of the vectors in S.

$$\text{span}(S) = \{c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdot\cdot\cdot + c_k\mathbf{v}_k : c_1, c_2, \ldots, c_k \text{ are real numbers}\}.$$

The span of is denoted by span(S) or span{v1,v2,...,vk}. If span(S)=V ,it is said that V is spanned by {v1,v2,.....,vk} or v that S spans V.

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Spanning sets and Linear independence

If S={v1,v2,....,vk} is a set of vectors in a vector space V, then span(S) is a subspace of V

Moreover, span(S) is the smallest subspace of V that contains S, in the sense that every other subspace of V that contains S must contain span(S) .

Any finite nonempty subset of a vector space V is a subspace of V

Spanning sets and Linear independence

➢ Linear dependence and Linear independence :

If a given set of vectors S has only the trivial solution, it is called linearly independent. If S has proper solutions, then it is called a linearly dependent set.

A set of vectors $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$ in a vector space V is called **linearly independent** if the vector equation

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_k\mathbf{v}_k = \mathbf{0}$$

has only the trivial solution, $c_1 = 0, c_2 = 0, \ldots, c_k = 0$. If there are also nontrivial solutions, then S is called **linearly dependent.**

Spanning sets and Linear independence

❖ Example :

(a) The set $S = \{(1, 2), (2, 4)\}$ in R^2 is linearly dependent because

$$-2(1, 2) + (2, 4) = (0, 0).$$

(b) The set $S = \{(1, 0), (0, 1), (-2, 5)\}$ in R^2 is linearly dependent because

$$2(1, 0) - 5(0, 1) + (-2, 5) = (0, 0).$$

(c) The set $S = \{(0, 0), (1, 2)\}$ in R^2 is linearly dependent because

$$1(0, 0) + 0(1, 2) = (0, 0).$$

Spanning sets and Linear independence

- **Testing for linear dependence and independence :**

Let $S=\{v1,v2,....,vk\}$ be a set of vectors in a vector space V. To determine whether S is v linearly dependent or linearly independent, Do the below steps:

1. From the vector equation $c1v1+c2v2+......+ckvk = 0$, write a homogeneous system of linear equations in the variables c1,c2,.....,and ck.
2. To determine whether the system has a unique solution use Gaussian elimination.
3. The set S is linearly independent If the system has only the trivial solution $c1=0,c2=0,....,ck=0$. The S is linearly dependent If the system also has nontrivial

Spanning sets and Linear independence

- **Properties of linearly dependent sets :**

  A set S={v1,v2,....,vk} .k≥2 , is linearly dependent if and only if at least one of the Vectors vj can be written as a linear combination of the other vectors in S.

- Two vectors u and v in a vector space V are linearly dependent if and only if one is a scalar multiple of the other.

  Note : The zero vector is always a scalar multiple of another vector in a vector space

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THANK YOU

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