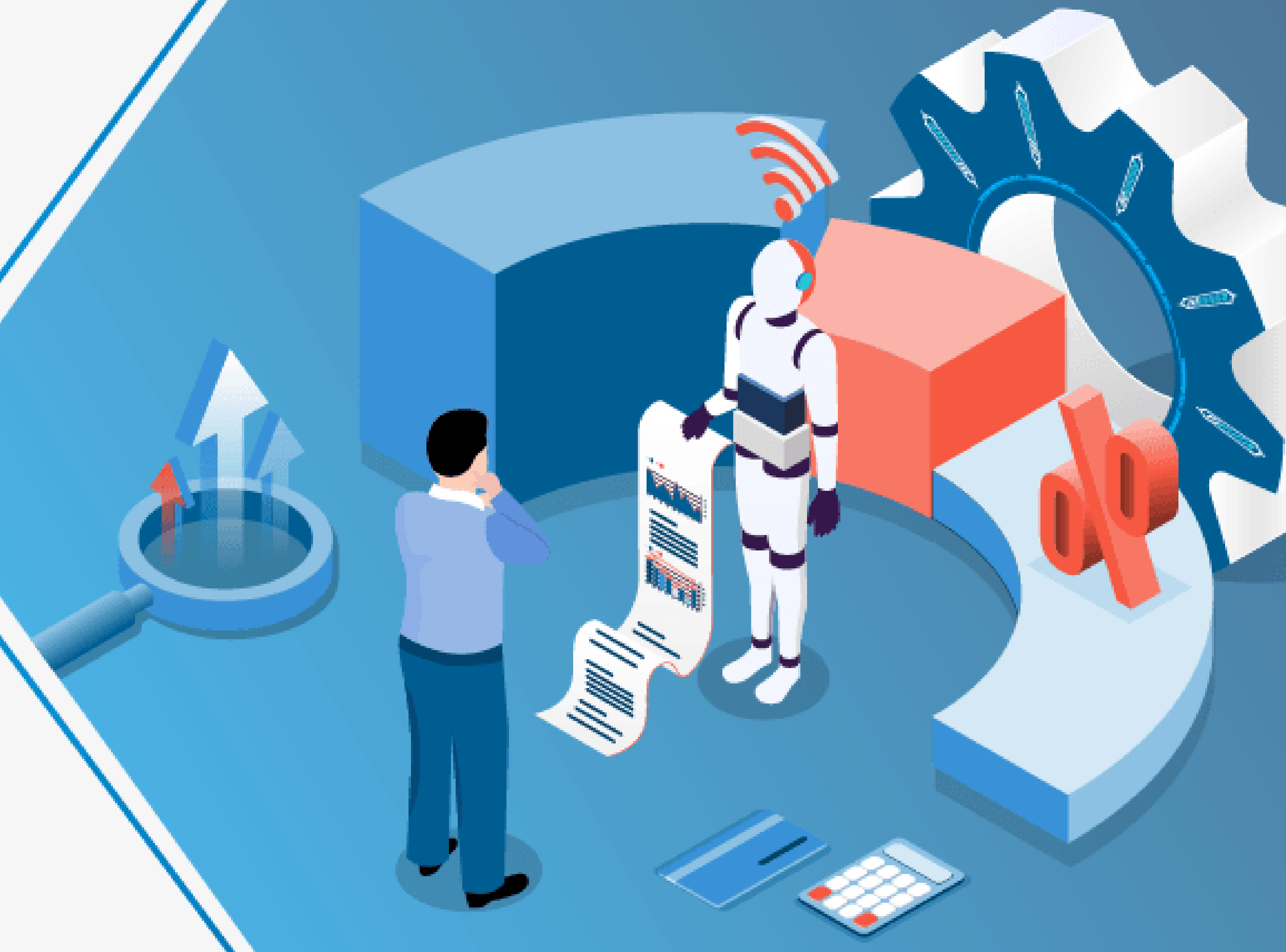
# Data Science Fundamentals

## Matrices 2



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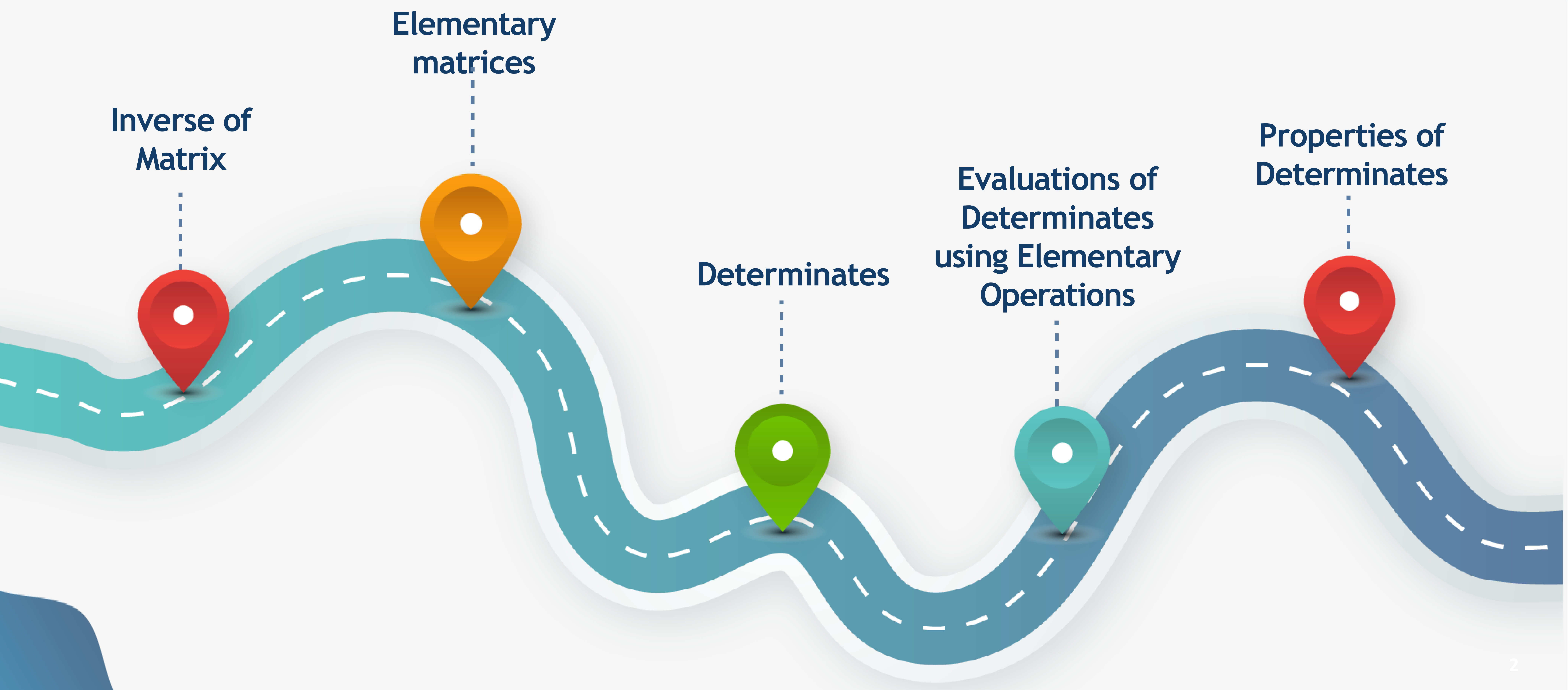

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# Inverse of Matrix

# Inverse of Matrix

## Definition of the inverse of a Matrix

The matrix A m x m is invertible if there exists a matrix B m x m such that

$$AB=BA=Im$$

where $Im$ is the identity matrix of order m and matrix B is called the inverse of matrix A.

if a matrix does not have an inverse, then is called noninvertible

# Inverse of Matrix

**The Inverse of is denoted by $A^{-1}$**

- Non-square matrices do not have inverses
  - This is due to the non-commutative nature of matrix multiplication Non-square

- If A is an invertible matrix, then its inverse is unique.

# Inverse of Matrix

- For 2x2 matrices, we can use a simple formula to determine the inverse

If $A$ is a $2 \times 2$ matrix represented by

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix},$$

then $A$ is invertible if and only if $ad - bc \neq 0$. Moreover, if $ad - bc \neq 0$, then the inverse is represented by

$$A^{-1} = \frac{1}{ad - bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}.$$

$$A^{-1} = \frac{1}{|A|} \cdot \text{Adj } A$$

# Inverse of Matrix

❖ **Example : Show that B is the inverse of A**

$$A = \begin{bmatrix} -1 & 2 \\ -1 & 1 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 1 & -2 \\ 1 & -1 \end{bmatrix}.$$

➢ **Solution:**

$$AB = \begin{bmatrix} -1 & 2 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} 1 & -2 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} -1+2 & 2-2 \\ -1+1 & 2-1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$BA = \begin{bmatrix} 1 & -2 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} -1 & 2 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} -1+2 & 2-2 \\ -1+1 & 2-1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

# Finding Inverse of Matrix

Let A be a square matrix of order n

1. Write the n x 2n matrix that consists of the given matrix A on the left and the n x n identity matrix $I$ on the right to obtain $[A:\ I]$.

2. If possible , reduce A to $I$ using elementary row operations on the entire matrix $[A:\ I]$, and the result will be the matrix $[I: A^{-1}]$.

If this is not possible, then is a noninvertible (or singular) Matrix.

# Finding Inverse of Matrix

3. Check your work by multiplying $A\,A^{-1}$ and $A^{-1}A$ to see that $AA^{-1} = I = A^{-1}A$

Note: when you separate the matrices by dotted line like this $[A\!:I]$.

This process is called adjoining matrix to matrix

# Finding Inverse of Matrix

❖ **Example:** Find the inverse of matrix A

$$A = \begin{bmatrix} 1 & -1 & 0 \\ 1 & 0 & -1 \\ -6 & 2 & 3 \end{bmatrix}$$

$$[A \vdots I] = \left[\begin{array}{rrr|rrr} 1 & -1 & 0 & 1 & 0 & 0 \\ 1 & 0 & -1 & 0 & 1 & 0 \\ -6 & 2 & 3 & 0 & 0 & 1 \end{array}\right].$$

# Inverse of Matrix

- We then use row operations to transform the matrix into its inverse

$$\left[\begin{array}{rrr|rrr} 1 & -1 & 0 & 1 & 0 & 0 \\ 0 & 1 & -1 & -1 & 1 & 0 \\ -6 & 2 & 3 & 0 & 0 & 1 \end{array}\right] \quad R_2 + (-1)R_1 \to R_2$$

$$\left[\begin{array}{rrr|rrr} 1 & -1 & 0 & 1 & 0 & 0 \\ 0 & 1 & -1 & -1 & 1 & 0 \\ 0 & -4 & 3 & 6 & 0 & 1 \end{array}\right] \quad R_3 + (6)R_1 \to R_3$$

$$\left[\begin{array}{rrr|rrr} 1 & -1 & 0 & 1 & 0 & 0 \\ 0 & 1 & -1 & -1 & 1 & 0 \\ 0 & 0 & -1 & 2 & 4 & 1 \end{array}\right] \quad R_3 + (4)R_2 \to R_3$$

$$\left[\begin{array}{rrr|rrr} 1 & -1 & 0 & 1 & 0 & 0 \\ 0 & 1 & -1 & -1 & 1 & 0 \\ 0 & 0 & 1 & -2 & -4 & -1 \end{array}\right] \quad (-1)R_3 \to R_3$$

$$\left[\begin{array}{rrr|rrr} 1 & -1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & -3 & -3 & -1 \\ 0 & 0 & 1 & -2 & -4 & -1 \end{array}\right] \quad R_2 + R_3 \to R_2$$

$$\left[\begin{array}{rrr|rrr} 1 & 0 & 0 & -2 & -3 & -1 \\ 0 & 1 & 0 & -3 & -3 & -1 \\ 0 & 0 & 1 & -2 & -4 & -1 \end{array}\right] \quad R_1 + R_2 \to R_1$$

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The right-hand side of the matrix is A inverse

# Inverse of Matrix

- A matrix with no inverse is called a singular matrix

- For 2x2 matrices, we can use a simple formula to determine the inverse

If $A$ is a $2 \times 2$ matrix represented by

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix},$$

then $A$ is invertible if and only if $ad - bc \neq 0$. Moreover, if $ad - bc \neq 0$, then the inverse is represented by

$$A^{-1} = \frac{1}{ad - bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}.$$

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# Inverse of Matrix

❖ **Exercise:** Find the inverse of matrix A and B:

$$A = \begin{bmatrix} 3 & -1 \\ -2 & 2 \end{bmatrix}$$

$$B = \begin{bmatrix} 3 & -1 \\ -6 & 2 \end{bmatrix}$$

# Properties of Inverse Matrices

- If is an invertible matrix, j is a positive integer and c is a scalar not equal to zero, then $A^{-1}$, $A^k$ ,cA and $A^T$ are invertible and the following are true.

1. $(A^{-1})^{-1} = A$
2. $(A^k)^{-1} = A^{-1}A^{-1} \cdots A^{-1} = (A^{-1})^k$
3. $(cA)^{-1} = \frac{1}{c}A^{-1}, c \neq 0$
4. $(A^T)^{-1} = (A^{-1})^T$

- If A and B are invertible matrices of size n , then AB is invertible and

$$(AB)^{-1} = B^{-1}A^{-1}.$$


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# Properties of Inverse

- If C is an invertible matrix, then the following properties still hold

1. If $AC = BC$, then $A = B$. ➔ **Right Cancellation Property**
2. If $CA = CB$, then $A = B$. ➔ **Left Cancellation Property**

- If A is an invertible matrix, the system of linear equations Ax=b has one unique solution given by

$$x = A^{-1}b$$

# Elementary matrices

# Elementary matrices

## Definition of Elementary Matrices

An n x n matrix is an elementary matrix if it can be obtained (which differs ) from the identity matrix *In* by a single elementary row operation.

**Exercise:** Which of the following matrices are elementary Matrices?

(a) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ (b) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$ (c) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

(d) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$ (e) $\begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}$ (f) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -1 \end{bmatrix}$

# Elementary matrices

- The Identity matrix is, by definition, an elementary matrix. where it can be obtained from itself by multiplying any one of its rows by 1

- Elementary matrices allow us to use matrix multiplication to do elementary row operations.

- Let E be the elementary matrix obtained by performing an elementary row operation on $Im$ . If that same elementary row operation is performed on an m x n matrix then the resulting matrix is given by the product EA.

# Elementary matrices

❖ **Example:** Find a sequence of elementary matrices of Matrix A that can be used to write the matrix in row-echelon form.

$$A = \begin{bmatrix} 0 & 1 & 3 & 5 \\ 1 & -3 & 0 & 2 \\ 2 & -6 & 2 & 0 \end{bmatrix}$$

| A Matrix | Elementary row operation | Elementary Matrix |
|---|---|---|
| $\begin{bmatrix} 1 & -3 & 0 & 2 \\ 0 & 1 & 3 & 5 \\ 2 & -6 & 2 & 0 \end{bmatrix}$ | **R1↔R2** | $E_1 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ |

# Elementary matrices - Example

| A Matrix | Elementary row operation | Elementary Matrix |
| --- | --- | --- |
| $\begin{bmatrix} 1 & -3 & 0 & 2 \\ 0 & 1 & 3 & 5 \\ 0 & 0 & 2 & -4 \end{bmatrix}$ | R3+(-2)R1 → R3 | $E_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix}$ |
| $\begin{bmatrix} 1 & -3 & 0 & 2 \\ 0 & 1 & 3 & 5 \\ 0 & 0 & 1 & -2 \end{bmatrix}$ | ½ R3 → R3 | $E_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \frac{1}{2} \end{bmatrix}$ |

- The three elementary matrices E1,E2, and E3 and can be used to perform the same elimination

# Elementary matrices - Example

$$B = E_3E_2E_1A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \frac{1}{2} \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 & 3 & 5 \\ 1 & -3 & 0 & 2 \\ 2 & -6 & 2 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \frac{1}{2} \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & -3 & 0 & 2 \\ 0 & 1 & 3 & 5 \\ 2 & -6 & 2 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \frac{1}{2} \end{bmatrix} \begin{bmatrix} 1 & -3 & 0 & 2 \\ 0 & 1 & 3 & 5 \\ 0 & 0 & 2 & -4 \end{bmatrix} = \begin{bmatrix} 1 & -3 & 0 & 2 \\ 0 & 1 & 3 & 5 \\ 0 & 0 & 1 & -2 \end{bmatrix}$$

- The matrices B and A Matrices are row equivalent because you can obtain B by performing a sequence of row operations on A

# Elementary matrices

## Definition of A Row Equivalence

Let A and B be m x n matrices. Matrix B is row-equivalent to A if there exists a finite number of elementary matrices $E_1, E_2, \ldots, E_k$ such that

$$\boldsymbol{B} = E_k E_{k-1} \ldots E_2 E_1 \boldsymbol{A}.$$

- If E is an elementary matrix, then $E^{-1}$ exists and is an elementary matrix

# Elementary matrices

## A Property of Invertible Matrices

A square matrix A is invertible if and only if it can be written as the product of elementary matrices

**Exercises:** Find a sequence of elementary matrices whose product is

$$A = \begin{bmatrix} -1 & -2 \\ 3 & 8 \end{bmatrix}.$$

# Elementary matrices

## Equivalent Conditions

**If A is an n x n matrix, then the following statements are equivalent**

1. $A$ is invertible.
2. $A\mathbf{x} = \mathbf{b}$ has a unique solution for every $n \times 1$ column matrix $\mathbf{b}$.
3. $A\mathbf{x} = O$ has only the trivial solution.
4. $A$ is row-equivalent to $I_n$.
5. $A$ can be written as the product of elementary matrices.


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# LU-Factorization

## Definition of LU-Factorization

Nowadays, the most efficient algorithm for solving linear systems is the LU-factorization,

Where If the n x n matrix A can be written as the product of a lower triangular matrix L

and an upper triangular matrix U, then A=LU is an **LU-factorization of A**.

# LU-Factorization

The square matrix is expressed as a product Where may be:

1. The square matrix is lower triangular, which means all the entries above the main diagonal are zero.
2. The square matrix is upper triangular, which means all the entries below the main diagonal are zero.

$$\begin{bmatrix} a_{11} & 0 & 0 \\ a_{21} & a_{22} & 0 \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \qquad \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{22} & a_{23} \\ 0 & 0 & a_{33} \end{bmatrix}$$

$3 \times 3$ lower triangular matrix $\qquad$ $3 \times 3$ upper triangular matrix

# LU-Factorization

❖ **Example**: Find the LU factorization of matrix A

$$A = \begin{bmatrix} 1 & -3 & 0 \\ 0 & 1 & 3 \\ 2 & -10 & 2 \end{bmatrix}.$$

| A Matrix | Elementary row operation | Elementary Matrix |
|---|---|---|
| $\begin{bmatrix} 1 & -3 & 0 \\ 0 & 1 & 3 \\ 0 & -4 & 2 \end{bmatrix}$ | $\boldsymbol{R3 + (-2)\,R1 \rightarrow R3}$ | $E_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix}$ |

# LU-Factorization

$$\begin{bmatrix} 1 & -3 & 0 \\ 0 & 1 & 3 \\ 0 & 0 & 14 \end{bmatrix}$$

$$\boldsymbol{R3} + (4)\,\boldsymbol{R2} \rightarrow \boldsymbol{R3}$$

$$E_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 4 & 1 \end{bmatrix}$$

The matrix is an upper triangular U

A lower triangular matrix L

$$E_1^{-1}E_2^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -4 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & -4 & 1 \end{bmatrix}$$

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# Determinates

# Determinates

- They arose from the recognition of special patterns in the solutions of systems of linear equations
- Every square matrix has a real number associated with it called a determinant
- The determinant is the difference between the products of the two diagonals of the matrix

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# Determinates

The **determinant** of the matrix

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

is given by

$$\det(A) = |A| = a_{11}a_{22} - a_{21}a_{12}.$$

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$$|A| = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{21}a_{12}$$

# Determinates

- **Example**: Find the determinant of matrix A

$$A = \begin{bmatrix} 2 & -3 \\ 1 & 2 \end{bmatrix}$$

$$|A| = \begin{vmatrix} 2 & -3 \\ 1 & 2 \end{vmatrix} = 2(2) - 1(-3) = 4 + 3 = 7$$

- **Exercises**: Find the determinant of matrix B

$$B = \begin{bmatrix} 2 & 1 \\ 4 & 2 \end{bmatrix}$$

# Determinates

- The determinant of a matrix can be one of the following values:
  1. positive
  2. zero
  3. negative

- The determinant of a matrix of order 1 is defined as the entry of the matrix.

$$A = [3] \ , \det(A) = 3$$

- What if we want to define the determinant of a matrix with higher order than 2?
  - We need to define minors and cofactors

# Determinates

- If A is a square matrix, then the minor Mij of the element aij is the determinant of the matrix obtained by deleting the ith row and jth column of A. The cofactor Cij is given by

$$C_{ij} = (-1)^{i+j} M_{ij}.$$


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# Determinates

- **Let's look at a 3x3 matrix A the minors and cofactors of a21 and a22 are as below:**

*Minor of $a_{21}$*

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}, \quad M_{21} = \begin{vmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{vmatrix}$$

Delete row 2 and column 1.

*Cofactor of $a_{21}$*

$$\begin{aligned} C_{21} &= (-1)^{2+1} M_{21} \\ &= -M_{21} \end{aligned}$$

*Minor of $a_{22}$*

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}, \quad M_{22} = \begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix}$$

Delete row 2 and column 2.

*Cofactor of $a_{22}$*

$$\begin{aligned} C_{22} &= (-1)^{2+2} M_{22} \\ &= M_{22} \end{aligned}$$

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# Determinates

- Minors and cofactors differ only in sign
- To obtain the cofactors of a matrix, find the minors and then apply a checkerboard pattern of +'s and –'s where odd positions (i+j is odd) have a negative sign and even positions (i+j is even) have a positive sign.


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# Determinates

**Example**: Find the minors and cofactors of A, where

$$A = \begin{bmatrix} 0 & 2 & 1 \\ 3 & -1 & 2 \\ 4 & 0 & 1 \end{bmatrix}.$$

Find minor $M_{11}$

$$\begin{bmatrix} 0 & 2 & 1 \\ 3 & -1 & 2 \\ 4 & 0 & 1 \end{bmatrix}, \quad M_{11} = \begin{vmatrix} -1 & 2 \\ 0 & 1 \end{vmatrix} = -1(1) - 0(2) = -1$$

# Determinates

❖ **Exercise**: Find the determinate of A,

where A=

$$\begin{bmatrix} 1 & 0 & 3 & 1 \\ 0 & 2 & -1 & 0 \\ 1 & -3 & 2 & 2 \\ -1 & 0 & 1 & -2 \end{bmatrix}$$


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# Determinates-Example

Find minor $M_{12}$

$$\begin{bmatrix} 0 & 2 & 1 \\ 3 & -1 & 2 \\ 4 & 0 & 1 \end{bmatrix}, \quad M_{12} = \begin{vmatrix} 3 & 2 \\ 4 & 1 \end{vmatrix} = 3(1) - 4(2) = -5$$

➢ **Continuing with the same pattern we get**

$$\begin{aligned} M_{11} &= -1 & M_{12} &= -5 & M_{13} &= 4 \\ M_{21} &= 2 & M_{22} &= -4 & M_{23} &= -8 \\ M_{31} &= 5 & M_{32} &= -3 & M_{33} &= -6. \end{aligned}$$

# Determinates

- **Cofactors are**

$$
\begin{array}{lll}
C_{11} = -1 & C_{12} = 5 & C_{13} = 4 \\
C_{21} = -2 & C_{22} = -4 & C_{23} = 8 \\
C_{31} = 5 & C_{32} = 3 & C_{33} = -6.
\end{array}
$$

- Let's redefine the determinant of a matrix using our cofactor knowledge
- **Definition of the determinant of a matrix**

If A is a square matrix (of order 2 or greater), then the determinant of A is the sum of the entries in the first row of A multiplied by their cofactors. That is:

$$\det(A) = |A| = \sum_{j=1}^{n} a_{1j}C_{1j} = a_{11}C_{11} + a_{12}C_{12} + \cdot \cdot \cdot + a_{1n}C_{1n}.$$

# Determinates

- **Example**: Let's look at A, where

$$A = \begin{bmatrix} 0 & 2 & 1 \\ 3 & -1 & 2 \\ 4 & 0 & 1 \end{bmatrix}.$$

- We already found the cofactors to be

$$\begin{aligned} C_{11} &= -1 & C_{12} &= 5 & C_{13} &= 4 \\ C_{21} &= -2 & C_{22} &= -4 & C_{23} &= 8 \\ C_{31} &= 5 & C_{32} &= 3 & C_{33} &= -6. \end{aligned}$$

- Let's find the determinant

$$|A| = a_{21}C_{21} + a_{22}C_{22} + a_{23}C_{23} = 3(-2) + (-1)(-4) + 2(8) = 14$$

# Determinates

- The determinant of A can be evaluated by expanding any row or column. Not just the first row. Try it

Let A be a square matrix of order n Then the determinant of A is:

➢ Expansion by cofactors

$$\det(A) = |A| = \sum_{j=1}^{n} a_{ij}C_{ij} = a_{i1}C_{i1} + a_{i2}C_{i2} + \cdot \cdot \cdot + a_{in}C_{in}$$

or

$$\det(A) = |A| = \sum_{i=1}^{n} a_{ij}C_{ij} = a_{1j}C_{1j} + a_{2j}C_{2j} + \cdot \cdot \cdot + a_{nj}C_{nj}.$$


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# Determinates

## ➢ Triangular matrices

- Finding the determinant of a matrix of order 4 or higher is a tedious job.
- A special case arises in a triangular matrix (Lower and Upper) where all entries above and below the diagonal are zero. This is called a diagonal matrix.

  ➢ The determinant of a triangular matrix is the product of all entries on the diagonal

# Determinates

## Determinates of a Triangular matrices

If A is a triangular matrix of order then its determinant is the product of the entries on the main diagonal. where :

$$\det(A) = |A| = a_{11}a_{22}a_{33} \cdot \cdot \cdot a_{nn}.$$

*Upper Triangular Matrix*

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ 0 & a_{22} & a_{23} & \cdots & a_{2n} \\ 0 & 0 & a_{33} & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & a_{nn} \end{bmatrix}$$

*Lower Triangular Matrix*

$$\begin{bmatrix} a_{11} & 0 & 0 & \cdots & 0 \\ a_{21} & a_{22} & 0 & \cdots & 0 \\ a_{31} & a_{32} & a_{33} & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & a_{n3} & \cdots & a_{nn} \end{bmatrix}$$

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# Evaluations of Determinates using Elementary Operations

# Evaluations of determinates using elementary operations

## ➢ Elementary Row Operations and Determinants

Let $A$ and $B$ be square matrices.

1. If $B$ is obtained from $A$ by interchanging two rows of $A$, then

$$\det(B) = -\det(A).$$

2. If $B$ is obtained from $A$ by adding a multiple of a row of $A$ to another row of $A$, then

$$\det(B) = \det(A).$$

3. If $B$ is obtained from $A$ by multiplying a row of $A$ by a nonzero constant $c$, then

$$\det(B) = c\det(A).$$

➢ This theorem allows us to transform the matrix into a triangular one using elementary row operations and then simply calculate the determinant by getting the product of all entries on the diagonal

# Evaluations of determinates using elementary operations-Example

# Evaluations of Determinates using Elementary Operations

❖ **Example:** Find the determinant of A,

where

$$A = \begin{bmatrix} 2 & -3 & 10 \\ 1 & 2 & -2 \\ 0 & 1 & -3 \end{bmatrix}.$$

# Evaluations of Determinates using Elementary Operations

$$\begin{vmatrix} 2 & -3 & 10 \\ 1 & 2 & -2 \\ 0 & 1 & -3 \end{vmatrix} = -\begin{vmatrix} 1 & 2 & -2 \\ 2 & -3 & 10 \\ 0 & 1 & -3 \end{vmatrix}$$

Interchange the first two rows

$$= -\begin{vmatrix} 1 & 2 & -2 \\ 0 & -7 & 14 \\ 0 & 1 & -3 \end{vmatrix}$$

Add -2 times the first row to the second row to produce a new second row.

# Evaluations of Determinates using Elementary Operations

$$= 7\begin{vmatrix} 1 & 2 & -2 \\ 0 & 1 & -2 \\ 0 & 1 & -3 \end{vmatrix}$$

Factor out of the second row.

$$= 7\begin{vmatrix} 1 & 2 & -2 \\ 0 & 1 & -2 \\ 0 & 0 & -1 \end{vmatrix}$$

Add times the second row to the third row to produce a new third row.

$|A| = 7(1)(1)(-1) = -7$

# Evaluations of determinates using elementary operations

- The same theorem applies to column operations which are operations done on columns instead of rows
- If one column is a scalar multiple of another, then the determinant of the matrix is zero
- Conditions that yield a zero determinant


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# Evaluations of determinates using elementary operations

- The det(A)=0 If A is a square matrix and any one of the following conditions is true:

1. An entire column (or an entire row) consists of zeros.
2. Two columns (or rows) are equal.
3. One column (or one row) is a multiple of another column (or row).


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# Evaluations of determinates using elementary operations

- **Example:** Find the determinant of

$$A = \begin{bmatrix} 1 & 4 & 1 \\ 2 & -1 & 0 \\ 0 & 18 & 4 \end{bmatrix}.$$

- Adding times the first row to the second row produces

$$|A| = \begin{vmatrix} 1 & 4 & 1 \\ 2 & -1 & 0 \\ 0 & 18 & 4 \end{vmatrix} = \begin{vmatrix} 1 & 4 & 1 \\ 0 & -9 & -2 \\ 0 & 18 & 4 \end{vmatrix}.$$

- The determinant is zero because the second and third rows are multiples of each other

# Evaluations of determinates using elementary operations

- The number of operations needed for each of the two determinant evaluations methods based on order

| *Order n* | *Cofactor Expansion* *Additions* | *Cofactor Expansion* *Multiplications* | *Row Reduction* *Additions* | *Row Reduction* *Multiplications* |
|---|---|---|---|---|
| 3 | 5 | 9 | 5 | 10 |
| 5 | 119 | 205 | 30 | 45 |
| 10 | 3,628,799 | 6,235,300 | 285 | 339 |


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# Properties of Determinates

# Properties of Determinates

## ➢ Determinant of a matrix product

If $A$ and $B$ are square matrices of order $n$, then

$$\det(AB) = \det(A)\det(B).$$

## ➢ Determinant of a scalar multiple of a matrix

If $A$ is an $n \times n$ matrix and $c$ is a scalar, then the determinant of $cA$ is given by

$$\det(cA) = c^n \det(A).$$


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# Properties of Determinates

## ❖ Example:

Find $|A|$, $|B|$, and $|AB|$ for the matrices

$$A = \begin{bmatrix} 1 & -2 & 2 \\ 0 & 3 & 2 \\ 1 & 0 & 1 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 2 & 0 & 1 \\ 0 & -1 & -2 \\ 3 & 1 & -2 \end{bmatrix}.$$

$$|A| = \begin{vmatrix} 1 & -2 & 2 \\ 0 & 3 & 2 \\ 1 & 0 & 1 \end{vmatrix} = -7 \quad \text{and} \quad |B| = \begin{vmatrix} 2 & 0 & 1 \\ 0 & -1 & -2 \\ 3 & 1 & -2 \end{vmatrix} = 11.$$

# Properties of Determinates

The matrix product $AB$ is

$$AB = \begin{bmatrix} 1 & -2 & 2 \\ 0 & 3 & 2 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 0 & 1 \\ 0 & -1 & -2 \\ 3 & 1 & -2 \end{bmatrix} = \begin{bmatrix} 8 & 4 & 1 \\ 6 & -1 & -10 \\ 5 & 1 & -1 \end{bmatrix}.$$

$$|AB| = \begin{vmatrix} 8 & 4 & 1 \\ 6 & -1 & -10 \\ 5 & 1 & -1 \end{vmatrix} = -77.$$

$$|AB| = |A||B|$$
$$-77 = (-7)(11).$$

[3]

# Properties of Determinates

- Note: The sum of the determinants of two matrices is not usually equal to the determinant of their sum.

det(A) + det(B)!=det(A+B)

## ➢ Determinant of an inverse matrix

A square matrix A is invertible (nonsingular) if and only if $\det(A) \neq 0.$

If Ais invertible, then $\det(A^{-1}) = \dfrac{1}{\det(A)}.$


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# Properties of Determinates

- Determinant of a Transpose

  - If is a square matrix, then

$$\det(A) = \det(A^T).$$


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# THANK YOU