Digital Signal Processing - Lecture Notes - 2

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Vector Spaces in Digital Signal Processing

In the context of Digital Signal Processing, we often consider signals as vectors in a certain vector space. A vector space (or linear space) is a collection of objects called vectors, which can be added together and multiplied by scalars.

Definitions

- **Scalars**: A scalar is a simple quantity that is completely described by its magnitude.
- **Vector Addition**: If $\mathbf{x}$ and $\mathbf{y}$ are vectors, then the addition $\mathbf{x} + \mathbf{y}$ is a vector.
- **Scalar Multiplication**: If $\mathbf{x}$ is a vector and c is a scalar, then the scalar multiplication $c\mathbf{x}$ is a vector.

Examples and Solutions

1) Given two vectors $\mathbf{x} = [1, 2, 3]$ and $\mathbf{y} = [4, 5, 6]$. Compute the vector addition $\mathbf{x} + \mathbf{y}$.

Solution: Vector addition is performed element-wise, so we simply add corresponding elements of the two vectors together:

$$\mathbf{x} + \mathbf{y} = [1 + 4, 2 + 5, 3 + 6] = [5, 7, 9].$$

2) Given a vector $\mathbf{x} = [1, 2, 3]$ and a scalar $c = 2$. Compute the scalar multiplication $c\mathbf{x}$.

Solution: Scalar multiplication is also performed element-wise, so we simply multiply each element of the vector by the scalar:

$$c\mathbf{x} = 2 * [1, 2, 3] = [2, 4, 6].$$

3) Given a vector $\mathbf{x} = [1, 2, 3]$ and a scalar $c = 2$. Compute the scalar multiplication and then add the result to the original vector $\mathbf{x} + c\mathbf{x}$.

Solution: First, perform the scalar multiplication:

$$c\mathbf{x} = 2 * [1, 2, 3] = [2, 4, 6].$$

Then add this result to the original vector:

$$\mathbf{x} + c\mathbf{x} = [1, 2, 3] + [2, 4, 6] = [3, 6, 9].$$

4) Given a vector $\mathbf{x} = [1, \sin(\pi/4), \sin(\pi/2)]$. Compute the scalar multiplication for $c = 2$.

Solution: Scalar multiplication is performed element-wise:

$$c\mathbf{x} = 2 * [1, \sin(\pi/4), \sin(\pi/2)] = [2, 2\sin(\pi/4), 2\sin(\pi/2)] = [2, \sqrt{2}, 2].$$

5) Compute the inner product of the vectors $\mathbf{x} = [1, 0, -1]$ and $\mathbf{y} = [\sin(0), \sin(\pi/2), \sin(\pi)]$ in the form of a sum.

Solution: The inner product of two vectors is defined as the sum of the products of their corresponding elements:

$$\langle \mathbf{x}, \mathbf{y} \rangle = 1 * \sin(0) + 0 * \sin(\pi/2) - 1 * \sin(\pi) = 0.$$

6) Compute the inner product of the vectors $\mathbf{x} = [1, \sin(\pi/4), \sin(\pi/2)]$ and $\mathbf{y} = [1, \cos(\pi/4), \cos(\pi/2)]$ in the form of a sum.

Solution:
Again, the inner product is computed element-wise:

$$\langle \mathbf{x}, \mathbf{y} \rangle = 1 * 1 + \sin(\pi/4) * \cos(\pi/4) + \sin(\pi/2) * \cos(\pi/2) = 1 + 1/2 + 0 = 1.5.$$

Properties of the Inner Product

The inner product operation has several important properties that facilitate its use in signal processing and other fields. Let's consider two vectors $\mathbf{x} = [x_1, x_2, \ldots, x_N]$ and $\mathbf{y} = [y_1, y_2, \ldots, y_N]$, and scalar c.

1. **Commutativity:** $\langle \mathbf{x}, \mathbf{y} \rangle = \langle \mathbf{y}, \mathbf{x} \rangle$.

2. **Distributivity:** $\langle \mathbf{x}, \mathbf{y} + \mathbf{z} \rangle = \langle \mathbf{x}, \mathbf{y} \rangle + \langle \mathbf{x}, \mathbf{z} \rangle$.

3. **Associativity with scalar multiplication:** $\langle \mathbf{x}, c\mathbf{y} \rangle = c\langle \mathbf{x}, \mathbf{y} \rangle$.

4. **Positive-definiteness:** $\langle \mathbf{x}, \mathbf{x} \rangle \geq 0$ and $\langle \mathbf{x}, \mathbf{x} \rangle = 0$ if and only if $\mathbf{x} = 0$.

7) Given vectors $\mathbf{x} = [1, 2, 3]$ and $\mathbf{y} = [4, 5, 6]$, show that the inner product is commutative, i.e., $\langle \mathbf{x}, \mathbf{y} \rangle = \langle \mathbf{y}, \mathbf{x} \rangle$.

Solution: We have $\langle \mathbf{x}, \mathbf{y} \rangle = 1 * 4 + 2 * 5 + 3 * 6 = 32$, and $\langle \mathbf{y}, \mathbf{x} \rangle = 4 * 1 + 5 * 2 + 6 * 3 = 32$. Thus, $\langle \mathbf{x}, \mathbf{y} \rangle = \langle \mathbf{y}, \mathbf{x} \rangle$.

Distance Between Vectors

The distance between two vectors can be defined as the norm of the difference between the vectors. Given two vectors $\mathbf{x}$ and $\mathbf{y}$, the distance $d(\mathbf{x}, \mathbf{y})$ is given by:

$$d(\mathbf{x}, \mathbf{y}) = ||\mathbf{x} - \mathbf{y}|| = \sqrt{\langle \mathbf{x} - \mathbf{y}, \mathbf{x} - \mathbf{y} \rangle}.$$

8) Given vectors $\mathbf{x} = [1, 2]$ and $\mathbf{y} = [2, 2]$, compute the distance between $\mathbf{x}$ and $\mathbf{y}$.

Solution: We have $\mathbf{x} - \mathbf{y} = [-1, 0]$, so the distance between $\mathbf{x}$ and $\mathbf{y}$ is given by:

$$d(\mathbf{x}, \mathbf{y}) = ||\mathbf{x} - \mathbf{y}|| = \sqrt{\langle -1, 0 \rangle} = \sqrt{1} = 1.$$

Orthogonality

Two vectors are said to be orthogonal if their inner product is zero. This means they are at right angles to each other if we think of them as arrows in space.

9) Given vectors $\mathbf{x} = [1, 0]$ and $\mathbf{y} = [0, 1]$, show that $\mathbf{x}$ and $\mathbf{y}$ are orthogonal.

Solution: The inner product of $\mathbf{x}$ and $\mathbf{y}$ is $\langle \mathbf{x}, \mathbf{y} \rangle = 1 * 0 + 0 * 1 = 0$, so $\mathbf{x}$ and $\mathbf{y}$ are orthogonal.

Signal Spaces

Signal spaces are vector spaces where the vectors are functions or signals. These spaces are equipped with an inner product operation which is often an integral or a sum.

10) Given two signals $x(t) = t^2$ and $y(t) = 3t + 2$, verify if they form a vector space.

Solution: These signals can be added together and multiplied by scalars while still maintaining the properties of a vector space. Hence, they form a vector space.

Well-Behaved Infinite Signals

Well-behaved infinite signals are signals that are absolutely integrable over their entire domain.

11) Show that the signal $x(t) = e^{-t^2}$ is a well-behaved infinite signal.

Solution: We need to integrate $|x(t)| = |e^{-t^2}| = e^{-t^2}$ from $-\infty$ to $+\infty$. The result is $\sqrt{\pi}$, which is finite, so the signal is well-behaved.

Completeness

A vector space is said to be complete if every Cauchy sequence of vectors in that space converges to a vector in the space.

12) Verify if the space of continuous functions on the interval $[0, 1]$ is complete with respect to the norm defined by $||x(t)|| = \int_0^1 |x(t)|dt$.

Solution: We know that every Cauchy sequence of continuous functions on $[0, 1]$ converges uniformly to a continuous function on $[0, 1]$. Hence, the space is complete with respect to the given norm.

Hilbert Spaces

Hilbert spaces are complete inner product spaces. A function space becomes a Hilbert space when it has an inner product and is complete with respect to the norm derived from that inner product.

Hilbert Spaces in Digital Signal Processing

A **Hilbert Space** is an important concept in mathematics, particularly in signal processing and quantum mechanics. Simply put, a Hilbert Space is a vector space equipped with an inner product, an operation that allows lengths and angles to be measured. Moreover, it is complete, meaning that if there is a sequence of vectors such that the distance between any two vectors in the sequence gets arbitrarily small as the sequence progresses, then there is a well-defined limit to the sequence.

In the context of Digital Signal Processing (DSP), a Hilbert Space is often used to represent signal spaces or the set of all signals. This is because it allows for the manipulation and analysis of signals using the tools of vector space theory.

For instance, let's consider the vector space of all discrete-time, complex-valued sequences of finite length N. This space, denoted by $\mathbf{C}^N$, is a finite-dimensional Hilbert Space. The inner product of two vectors (or signals) $\mathbf{x} = (x[0], x[1], ..., x[N-1])$ and $\mathbf{y} = (y[0], y[1], ..., y[N-1])$ in $\mathbf{C}^N$ is defined as:

$$\langle \mathbf{x}, \mathbf{y} \rangle = \sum_{n=0}^{N-1} x[n]\overline{y[n]}$$

where $\overline{y[n]}$ is the complex conjugate of $y[n]$.

The norm (or length) of a vector $\mathbf{x}$ in $\mathbf{C}^N$ is defined as:

$$\|\mathbf{x}\| = \sqrt{\langle \mathbf{x}, \mathbf{x} \rangle}$$

13) Show that the space $L^2[0,1]$ of square-integrable functions on the interval $[0,1]$ is a Hilbert space.

Solution: The space $L^2[0,1]$ is an inner product space with the inner product defined as $\langle f, g \rangle = \int_0^1 f(t)g(t)dt$. It is also complete, since every Cauchy sequence of functions in this space converges in the mean-square sense to a function in the space. Therefore, $L^2[0,1]$ is a Hilbert space.

Bases and Linear Combinations in Vector Spaces

A basis for a vector space is a set of vectors that spans the space and is linearly independent. Any vector in the space can be written as a linear combination of the basis vectors.

14) Given the vectors $\mathbf{v}_1 = (1,0)$ and $\mathbf{v}_2 = (0,1)$ in $\mathbf{R}^2$, show that they form a basis and write the vector $\mathbf{x} = (3,2)$ as a linear combination of the basis vectors.

Solution: The vectors $\mathbf{v}_1$ and $\mathbf{v}_2$ are linearly independent and span $\mathbf{R}^2$, so they form a basis. The vector $\mathbf{x} = (3,2)$ can be written as a linear combination of the basis vectors as $\mathbf{x} = 3\mathbf{v}_1 + 2\mathbf{v}_2$.

Infinite-Dimensional Spaces

In infinite-dimensional spaces, a basis is a set of vectors such that every vector in the space can be written as a (possibly infinite) linear combination of the basis vectors.

15) Show that the functions $\{1, t, t^2, t^3, \ldots\}$ form a basis for the vector space of polynomials.

Solution: A basis for a vector space is a set of vectors that are linearly independent and that span the vector space. To show that the set of functions $1, t, t^2, t^3, \ldots$ form a basis for the vector space of all polynomials, we need to demonstrate these two properties:

Linear Independence: A set of vectors is linearly independent if no vector in the set can be written as a linear combination of the other vectors. Here, it's clear that no function t^n can be written as a linear combination of the other functions t^m where $m \neq n$. That's because the only way to get a t^n term is from the t^n function itself; none of the other functions can produce a t^n term.

Spanning Set: A set of vectors spans a vector space if every vector in the vector space can be written as a linear combination of the vectors in the set. Any polynomial can be written as a sum of terms of the form $c_n t^n$, where c_n is a scalar. This is exactly a linear combination of the functions in our set.

Therefore, the functions $1, t, t^2, t^3, \ldots$ are linearly independent and span the vector space of all polynomials. Hence, they form a basis for this vector space.

Fourier Basis and Fourier Transform

The Fourier basis plays a central role in digital signal processing and in the analysis and synthesis of signals. This stems from the Fourier transform, which is an operation that transforms a time-domain signal into its frequency-domain representation.

Fourier Basis

The Fourier basis consists of complex exponential functions of the form $e^{j\omega t}$, where ω is the angular frequency and t is time. The complex exponential is periodic and the frequency ω determines the rate of oscillation.

A key property of the Fourier basis is that it is orthogonal, meaning that different basis functions are orthogonal to each other. This can be proven mathematically, but intuitively it means that each basis function captures a different 'aspect' or 'component' of a signal.

Fourier Transform

The Fourier Transform is a mathematical technique that transforms a function of time, a signal, into a function of frequency. This is extremely useful in signal processing as it allows us to analyze and manipulate signals in the frequency domain.

The Fourier transform of a continuous-time signal $x(t)$ is given by:

$$X(\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt \tag{1}$$

And the inverse Fourier transform, which allows us to get back to the time-domain signal from its frequency-domain representation, is given by:

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)e^{j\omega t}d\omega \tag{2}$$

These two equations show how we can move back and forth between the time domain and the frequency domain, which forms the basis of many techniques in digital signal processing.

Deep Dive into Fourier Transform and Fourier Basis

The Fourier Transform and its basis functions are key mathematical tools used in the analysis and manipulation of signals. The properties and operational rules of the Fourier transform make it particularly suited to Digital Signal Processing (DSP).

Fourier Basis

In the complex plane, the Fourier basis functions can be expressed as $e^{j\omega t}$, where ω is the frequency and t is time. These basis functions are complex exponentials, which oscillate around the unit circle in the complex plane with a frequency of ω. When ω is an integer multiple of 2π, the complex exponentials are periodic with period 1.

The Fourier basis functions have an important property that they are orthogonal to each other. We can demonstrate this through the integral of the product of two different basis functions over one period.

$$\int_{0}^{1} e^{j\omega_1 t}e^{-j\omega_2 t}dt = \delta(\omega_1 - \omega_2) \tag{3}$$

Here, δ is the Dirac delta function, which is zero when $\omega_1 \neq \omega_2$ and undefined when $\omega_1 = \omega_2$. This shows that two different Fourier basis functions are orthogonal to each other.

Fourier Transform

The Fourier Transform takes a time-domain signal and transforms it into its frequency-domain representation. This allows us to see how much of each frequency is present in the signal.

The Fourier transform of a signal $x(t)$ is given by:

$$X(\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt \tag{4}$$

The result, $X(\omega)$, is a complex-valued function of frequency. The magnitude $|X(\omega)|$ gives the amount of frequency ω present in the signal, and the angle of $X(\omega)$ (also known as the phase) gives the phase shift of the sinusoid of frequency ω.

The inverse Fourier transform, which allows us to reconstruct the time-domain signal from its frequency-domain representation, is given by:

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)e^{j\omega t}d\omega \tag{5}$$

This operation sums up (integrates) all the complex sinusoids, each weighted by its corresponding value in the Fourier transform, to yield the original signal.

To further elaborate the Fourier Transform properties, let's consider a simple example, the Fourier Transform of a real-valued sinusoidal signal $x(t) = A\cos(\omega_0 t + \phi)$:

$$X(\omega) = \frac{A}{2}e^{j\phi}[\delta(\omega - \omega_0) + \delta(\omega + \omega_0)] \tag{6}$$

This demonstrates that a cosine function in the time domain corresponds to two impulses in the frequency domain, located at $\pm\omega_0$ with heights depending on the amplitude and phase shift of the original cosine function.

Fourier Transform in DSP

The Fourier Transform is a mathematical tool that allows us to transform a signal from the time domain to the frequency domain. The Fourier Transform of a function $f(t)$ is denoted by $F(\omega)$, where ω is the frequency variable.

In the context of DSP, the Discrete Fourier Transform (DFT) is commonly used. The DFT of a sequence $x[n]$ of length N is given by:

$$X[k] = \sum_{n=0}^{N-1} x[n]\exp\left(-j\frac{2\pi}{N}kn\right)$$

where k is the frequency index and $j = \sqrt{-1}$.

17) Compute the DFT of the sequence $x[n] = \{1, 1, 1, 1\}$ for $n = \{0, 1, 2, 3\}$.

Solution: The DFT $X[k]$ of $x[n]$ is given by:

$$X[k] = \sum_{n=0}^{3} x[n] \exp\left(-j\frac{2\pi}{4}kn\right)$$

By substituting the values $x[n] = 1$ for all n, we get:

$$X[k] = \sum_{n=0}^{3} \exp\left(-j\frac{2\pi}{4}kn\right)$$

We can calculate this sum for each $k = 0, 1, 2, 3$:

$$X[0] = 4, \quad X[1] = 0, \quad X[2] = 0, \quad X[3] = 0$$

So, the DFT of $x[n]$ is $X[k] = \{4, 0, 0, 0\}$ for $k = \{0, 1, 2, 3\}$.

Discrete Fourier Transform

The Discrete Fourier Transform (DFT) is a fundamental tool in digital signal processing. It allows us to transform a finite, discrete sequence of numbers into its frequency-domain representation.

Definition of the DFT

For a sequence of N complex numbers $x[0], x[1], ..., x[N-1]$, the DFT is a new sequence of N complex numbers, defined as follows:

$$X[k] = \sum_{n=0}^{N-1} x[n]e^{-j2\pi nk/N} \tag{7}$$

for $k = 0, 1, ..., N-1$. The DFT gives us a set of complex coefficients that, when combined with the complex exponential basis functions, can reproduce the original sequence.

Inverse DFT

The inverse DFT (IDFT) allows us to reconstruct the original time-domain sequence from its DFT. It is defined as:

$$x[n] = \frac{1}{N} \sum_{k=0}^{N-1} X[k]e^{j2\pi nk/N} \tag{8}$$

for $n = 0, 1, ..., N-1$.

18) Consider the sequence $x[n] = \{1, 2, 3, 4\}$ for $n = 0, 1, 2, 3$. Let's compute its DFT.

Solution: The DFT $X[k]$ is given by:

$$
\begin{aligned}
X[0] &= x[0] + x[1] + x[2] + x[3] = 10 \\
X[1] &= x[0] - jx[1] - x[2] + jx[3] = 2 \\
X[2] &= x[0] - x[1] + x[2] - x[3] = -2 \\
X[3] &= x[0] + jx[1] - x[2] - jx[3] = 2
\end{aligned}
$$

To verify our computation, we can use the IDFT to reconstruct the original sequence:

$$
\begin{aligned}
x[0] &= \frac{1}{4}(X[0] + X[1] + X[2] + X[3]) = 1 \\
x[1] &= \frac{1}{4}(X[0] - jX[1] - X[2] + jX[3]) = 2 \\
x[2] &= \frac{1}{4}(X[0] - X[1] + X[2] - X[3]) = 3 \\
x[3] &= \frac{1}{4}(X[0] + jX[1] - X[2] - jX[3]) = 4
\end{aligned}
$$

The sequence we obtained matches the original one, verifying our computations.

Problem Set

1. Determine if the following sets are vector spaces:
   (a) The set of all functions $f(t) = at^2 + bt + c$.
   (b) The set of all continuous functions.
   (c) The set of all solutions to the differential equation $y'' + y = 0$.
   (d) The set of all vectors (x, y) such that $x > 0$.
2. If $\mathbf{v}_1 = (1, 2)$ and $\mathbf{v}_2 = (3, 4)$ in $\mathbf{R}^2$, do they form a basis? If so, write the vector $\mathbf{x} = (5, 6)$ as a linear combination of the basis vectors.
3. Show that the functions $\{1, t, t^2, t^3, \ldots\}$ form a basis for the vector space of polynomials.
4. Write the function $f(t) = e^{j\omega_0 t}$ as a linear combination of the Fourier basis functions.
5. Are the vectors $(1, 2)$ and $(2, 4)$ linearly independent?
6. Are the vectors $(1, 0, 0)$, $(0, 1, 0)$, and $(0, 0, 1)$ a basis for $\mathbf{R}^3$?
7. Are the vectors $(1, 0)$, $(0, 1)$, and $(1, 1)$ a basis for $\mathbf{R}^2$?
8. Given the vectors $\mathbf{v}_1 = (1, 0, 0)$, $\mathbf{v}_2 = (0, 1, 0)$, and $\mathbf{v}_3 = (0, 0, 1)$ in $\mathbf{R}^3$, do they form a basis? If so, write the vector $\mathbf{x} = (2, 3, 4)$ as a linear combination of the basis vectors.
9. Given the vectors $\mathbf{v}_1 = (1, 2, 3)$, $\mathbf{v}_2 = (4, 5, 6)$, and $\mathbf{v}_3 = (7, 8, 9)$ in $\mathbf{R}^3$, do they form a basis?
10. Determine if the following sets are complete:
    (a) The set of all continuous functions on the interval $[0, 1]$ with the norm defined as $\|f\| = \max_{0 \leq t \leq 1} |f(t)|$.
    (b) The set of all sequences $\{x_n\}$ such that $\sum_{n=1}^{\infty} x_n^2 < \infty$ with the norm defined as $\|x\| = \sqrt{\sum_{n=1}^{\infty} x_n^2}$.
11. What is the inner product of the signals $x[n] = (1, 2, 3)$ and $y[n] = (2, 2, 2)$?
12. Given the signals $x[n] = (1, 0, 0, 0)$ and $y[n] = (0, 1, 0, 0)$ in the vector space of sequences of length 4, find a basis for this vector space and write $z[n] = (1, 1, 0, 0)$ as a linear combination of the basis vectors.
13. Determine whether the following pairs of signals are orthogonal, given the inner product defined as $\langle x, y \rangle = \sum_{n=0}^{N-1} x[n]\overline{y[n]}$:
    (a) $x[n] = (1, 0, 1, 0)$ and $y[n] = (0, 1, 0, 1)$.
    (b) $x[n] = (1, 1, 1, 1)$ and $y[n] = (1, -1, 1, -1)$.

14. Given the vectors $\mathbf{v}_1 = (1, 0)$ and $\mathbf{v}_2 = (0, 1)$, show that these vectors form an orthonormal basis for $\mathbf{R}^2$.

15. Given the vectors $\mathbf{v}_1 = \frac{1}{\sqrt{2}}(1, 1)$ and $\mathbf{v}_2 = \frac{1}{\sqrt{2}}(-1, 1)$, show that these vectors form an orthonormal basis for $\mathbf{R}^2$.

16. Given a vector $\mathbf{v} = (2, 3)$ in $\mathbf{R}^2$ with respect to the standard basis $\mathbf{e}_1 = (1, 0)$ and $\mathbf{e}_2 = (0, 1)$, find the coordinates of $\mathbf{v}$ with respect to the new basis $\mathbf{v}_1 = (1, 1)$ and $\mathbf{v}_2 = (1, -1)$.

17. Given the vector space $\mathbf{R}^3$ with standard basis $\mathbf{e}_1 = (1, 0, 0)$, $\mathbf{e}_2 = (0, 1, 0)$, and $\mathbf{e}_3 = (0, 0, 1)$, find a different orthonormal basis for $\mathbf{R}^3$.

18. Given the vectors $\mathbf{v}_1 = \frac{1}{\sqrt{2}}(1, 0, 1)$, $\mathbf{v}_2 = \frac{1}{\sqrt{2}}(1, 0, -1)$, and $\mathbf{v}_3 = (0, 1, 0)$, show that these vectors form an orthonormal basis for $\mathbf{R}^3$.

19. Given a vector $\mathbf{v} = (1, 2, 3)$ in $\mathbf{R}^3$ with respect to the standard basis $\mathbf{e}_1 = (1, 0, 0)$, $\mathbf{e}_2 = (0, 1, 0)$, and $\mathbf{e}_3 = (0, 0, 1)$, find the coordinates of $\mathbf{v}$ with respect to the new basis $\mathbf{v}_1 = \frac{1}{\sqrt{2}}(1, 0, 1)$, $\mathbf{v}_2 = \frac{1}{\sqrt{2}}(1, 0, -1)$, and $\mathbf{v}_3 = (0, 1, 0)$.

20. Compute the DFT of the sequence $x[n] = \{1, 1, 1, 1\}$ for $n = 0, 1, 2, 3$.

21. Compute the inverse DFT of the sequence $X[k] = \{4, 0, -4, 0\}$ for $k = 0, 1, 2, 3$.

22. Given the sequence $x[n] = \{1, 2, 3, 4\}$ and its DFT $X[k] = \{10, 2, -2, 2\}$ for $n, k = 0, 1, 2, 3$, verify the Plancherel theorem: $\sum_{n=0}^{N-1} |x[n]|^2 = \frac{1}{N} \sum_{k=0}^{N-1} |X[k]|^2$.

23. Given the sequence $x[n] = \{1, -1, 1, -1\}$ for $n = 0, 1, 2, 3$, find its DFT. What do you observe about the magnitudes of the DFT coefficients?

Solutions

1. Determine if the following sets are vector spaces:

   (a) The set of all functions $f(t) = at^2 + bt + c$.
   Solution: Yes, they form a vector space. The reason is that if we take two functions $f(t) = a_1t^2 + b_1t + c_1$ and $g(t) = a_2t^2 + b_2t + c_2$ from this set and add them, we get another function from the same set: $h(t) = (a_1 + a_2)t^2 + (b_1 + b_2)t + (c_1 + c_2)$. So, the set is closed under addition. The set is also closed under scalar multiplication because if we multiply a function from the set by a scalar α, we get $\alpha f(t) = \alpha at^2 + \alpha bt + \alpha c$, which is again a function in the set. Therefore, this set forms a vector space.

   (b) The set of all continuous functions.
   Solution: Yes, this set forms a vector space. If we take any two continuous functions f and g, their sum $f + g$ is also a continuous function, showing that the set is closed under addition. If we multiply a continuous function f by a scalar α, we get another continuous function αf, showing that the set is closed under scalar multiplication. Therefore, this set is a vector space.

   (c) The set of all solutions to the differential equation $y'' + y = 0$.
   Solution: Yes, this set forms a vector space. If y_1 and y_2 are solutions to the differential equation, then their linear combination $\alpha y_1 + \beta y_2$ is also a solution to the differential equation, showing that the set is closed under addition and scalar multiplication.

   (d) The set of all vectors (x, y) such that $x > 0$.
   Solution: No, this set does not form a vector space. This set does not include the zero vector, which is a necessary element of any vector space. Furthermore, the set is not closed under scalar multiplication. If we take a vector (x, y) from the set and multiply it by -1, we get the vector $(-x, -y)$, which is not in the set.

2. If $\mathbf{v}_1 = (1, 2)$ and $\mathbf{v}_2 = (3, 4)$ in $\mathbf{R}^2$, do they form a basis? If so, write the vector $\mathbf{x} = (5, 6)$ as a linear combination of the basis vectors.
   Solution: Yes, $\mathbf{v}_1$ and $\mathbf{v}_2$ form a basis for $\mathbf{R}^2$ because they are linearly independent (the determinant of the matrix formed by these vectors is non-zero) and they span $\mathbf{R}^2$ (any vector in $\mathbf{R}^2$ can be written as a linear combination of these vectors). We can find the coefficients in the linear combination by solving the system of equations:

$$\begin{aligned} a(1) + b(3) &= 5, \\ a(2) + b(4) &= 6. \end{aligned}$$

   Solving this system, we find $a = -1$ and $b = 2$. So, the vector $\mathbf{x} = (5, 6)$ can be written as $\mathbf{x} = -1\mathbf{v}_1 + 2\mathbf{v}_2$.

3. Show that the functions $\{1, t, t^2, t^3, \ldots\}$ form a basis for the vector space of polynomials.
Solution: The given set of functions forms a basis for the vector space of polynomials because they are linearly independent and any polynomial can be written as a linear combination of these functions. This is because a polynomial of degree n can always be expressed as a sum of terms, each of which is a real number times a power of t, and the powers of t in the terms range from 0 to n.

4. Write the function $f(t) = e^{j\omega_0 t}$ as a linear combination of the Fourier basis functions.
Solution: By Euler's formula, we can express the complex exponential function as a linear combination of cosine and sine functions which form the Fourier basis in the space of functions. The expansion is as follows: $f(t) = e^{j\omega_0 t} = \cos(\omega_0 t) + j\sin(\omega_0 t)$.

5. Are the vectors $(1, 2)$ and $(2, 4)$ linearly independent?
Solution: No, they are not linearly independent. The vector $(2, 4)$ is a scalar multiple of the vector $(1, 2)$, i.e., $(2, 4) = 2(1, 2)$, so these vectors are linearly dependent.

6. Are the vectors $(1, 0, 0)$, $(0, 1, 0)$, and $(0, 0, 1)$ a basis for $\mathbf{R}^3$?
Solution: Yes, they form a basis for $\mathbf{R}^3$. These vectors are linearly independent (no vector is a linear combination of the others), and they span $\mathbf{R}^3$ (any vector in $\mathbf{R}^3$ can be written as a linear combination of these vectors). This set of vectors is known as the standard basis for $\mathbf{R}^3$.

7. Are the vectors $(1, 0)$, $(0, 1)$, and $(1, 1)$ a basis for $\mathbf{R}^2$?
Solution: No, they do not form a basis for $\mathbf{R}^2$. In $\mathbf{R}^2$, any basis must consist of exactly two vectors. While these vectors are linearly independent, there are too many of them to form a basis for $\mathbf{R}^2$.

8. Given the vectors $\mathbf{v}_1 = (1, 0, 0)$, $\mathbf{v}_2 = (0, 1, 0)$, and $\mathbf{v}_3 = (0, 0, 1)$ in $\mathbf{R}^3$, do they form a basis? If so, write the vector $\mathbf{x} = (2, 3, 4)$ as a linear combination of the basis vectors.
Solution: Yes, these vectors form a basis for $\mathbf{R}^3$ as they are the standard basis for $\mathbf{R}^3$. The vector $\mathbf{x} = (2, 3, 4)$ can be written as a linear combination of these vectors as follows: $\mathbf{x} = 2\mathbf{v}_1 + 3\mathbf{v}_2 + 4\mathbf{v}_3$.

9. Given the vectors $\mathbf{v}_1 = (1, 2, 3)$, $\mathbf{v}_2 = (4, 5, 6)$, and $\mathbf{v}_3 = (7, 8, 9)$ in $\mathbf{R}^3$, do they form a basis?
Solution: No, they do not form a basis for $\mathbf{R}^3$. These vectors are not linearly independent as $\mathbf{v}_3 = \mathbf{v}_2 + \mathbf{v}_1$, meaning that $\mathbf{v}_3$ can be expressed as a linear combination of $\mathbf{v}_1$ and $\mathbf{v}_2$.

10. Determine if the following sets are complete:
    (a) The set of all continuous functions on the interval $[0, 1]$ with the norm defined as $\|f\| = \max_{0 \le t \le 1} |f(t)|$.

Solution: Yes, this set is complete. This is the space of continuous functions equipped with the sup norm. A sequence of continuous functions that is Cauchy in this norm converges uniformly to a limit that is also a continuous function, which shows completeness.

(b) The set of all sequences $\{x_n\}$ such that $\sum_{n=1}^{\infty} x_n^2 < \infty$ with the norm defined as $\|x\| = \sqrt{\sum_{n=1}^{\infty} x_n^2}$.
Solution: Yes, this set is complete, making it a Hilbert Space. This can be shown using the Cauchy sequence criterion for completeness.

11. What is the inner product of the signals $x[n] = (1, 2, 3)$ and $y[n] = (2, 2, 2)$?
Solution: The inner product is given by $\langle x, y\rangle = \sum_n x[n]y[n] = (1*2) + (2*2) + (3*2) = 2 + 4 + 6 = 12$.

12. Given the signals $x[n] = (1, 0, 0, 0)$ and $y[n] = (0, 1, 0, 0)$ in the vector space of sequences of length 4, find a basis for this vector space and write $z[n] = (1, 1, 0, 0)$ as a linear combination of the basis vectors.
Solution: A possible basis for this vector space is

$$B = \{(1, 0, 0, 0), (0, 1, 0, 0), (0, 0, 1, 0), (0, 0, 0, 1)\}.$$

This basis is formed by signals with a single non-zero element which is unity. The signal $z[n]$ can be written as $z[n] = (1, 0, 0, 0) + (0, 1, 0, 0) = x[n] + y[n]$.

13. Determine whether the following pairs of signals are orthogonal, given the inner product defined as $\langle x, y\rangle = \sum_{n=0}^{N-1} x[n]\overline{y[n]}$:

(a) $x[n] = (1, 0, 1, 0)$ and $y[n] = (0, 1, 0, 1)$.
Solution: Yes, these signals are orthogonal. The inner product $\langle x, y\rangle = (1*0) + (0*1) + (1*0) + (0*1) = 0$.

(b) $x[n] = (1, 1, 1, 1)$ and $y[n] = (1, -1, 1, -1)$.
Solution: Yes, these signals are orthogonal. The inner product $\langle x, y\rangle = (1*1) + (1*(-1)) + (1*1) + (1*(-1)) = 0$.

14. Given the vectors $\mathbf{v}_1 = (1, 0)$ and $\mathbf{v}_2 = (0, 1)$, show that these vectors form an orthonormal basis for $\mathbf{R}^2$. **Solution:** These vectors are orthonormal if they are orthogonal (i.e., their dot product is 0), and each has norm 1. Their dot product is $(1)(0) + (0)(1) = 0$, so they are orthogonal. The length of each vector is $\sqrt{(1)^2 + (0)^2} = 1$ and $\sqrt{(0)^2 + (1)^2} = 1$, so they are both unit vectors, therefore they form an orthonormal basis for $\mathbf{R}^2$.

15. Given the vectors $\mathbf{v}_1 = \frac{1}{\sqrt{2}}(1, 1)$ and $\mathbf{v}_2 = \frac{1}{\sqrt{2}}(-1, 1)$, show that these vectors form an orthonormal basis for $\mathbf{R}^2$. **Solution:** We need to show that they are orthogonal and of unit length. Their dot product is $\frac{1}{\sqrt{2}}(-1+1) = 0$, so they are orthogonal. The length of each vector is $\sqrt{\left(\frac{1}{\sqrt{2}}\right)^2 + \left(\frac{1}{\sqrt{2}}\right)^2} = 1$, so they are unit vectors. Hence, they form an orthonormal basis for $\mathbf{R}^2$.

16. Given a vector $\mathbf{v} = (2, 3)$ in $\mathbf{R}^2$ with respect to the standard basis $\mathbf{e}_1 = (1, 0)$ and $\mathbf{e}_2 = (0, 1)$, find the coordinates of $\mathbf{v}$ with respect to the new basis $\mathbf{v}_1 = (1, 1)$ and $\mathbf{v}_2 = (1, -1)$. **Solution:** We need to solve the equation $\mathbf{v} = a\mathbf{v}_1 + b\mathbf{v}_2$. Solving for a and b we get $a = \frac{5}{2}$ and $b = -\frac{1}{2}$.

17. Given the vector space $\mathbf{R}^3$ with standard basis $\mathbf{e}_1 = (1, 0, 0)$, $\mathbf{e}_2 = (0, 1, 0)$, and $\mathbf{e}_3 = (0, 0, 1)$, find a different orthonormal basis for $\mathbf{R}^3$. **Solution:** One possible different orthonormal basis for $\mathbf{R}^3$ is $\mathbf{v}_1 = \frac{1}{\sqrt{2}}(1, 1, 0)$, $\mathbf{v}_2 = \frac{1}{\sqrt{2}}(-1, 1, 0)$, $\mathbf{v}_3 = (0, 0, 1)$.

18. Given the vectors $\mathbf{v}_1 = \frac{1}{\sqrt{2}}(1, 0, 1)$, $\mathbf{v}_2 = \frac{1}{\sqrt{2}}(1, 0, -1)$, and $\mathbf{v}_3 = (0, 1, 0)$, show that these vectors form an orthonormal basis for $\mathbf{R}^3$. **Solution:** To check for orthonormality, we need to verify that each pair of vectors are orthogonal and each vector has length 1. Checking for orthogonality, we have $\mathbf{v}_1 \cdot \mathbf{v}_2 = \mathbf{v}_1 \cdot \mathbf{v}_3 = \mathbf{v}_2 \cdot \mathbf{v}_3 = 0$, so all vectors are orthogonal to each other. Checking for length, we have $||\mathbf{v}_1|| = ||\mathbf{v}_2|| = 1$ and $||\mathbf{v}_3|| = 1$, so all vectors have length 1. Therefore, the vectors form an orthonormal basis for $\mathbf{R}^3$.

19. Given a vector $\mathbf{v} = (1, 2, 3)$ in $\mathbf{R}^3$ with respect to the standard basis $\mathbf{e}_1 = (1, 0, 0)$, $\mathbf{e}_2 = (0, 1, 0)$, and $\mathbf{e}_3 = (0, 0, 1)$, find the coordinates of $\mathbf{v}$ with respect to the new basis $\mathbf{v}_1 = \frac{1}{\sqrt{2}}(1, 0, 1)$, $\mathbf{v}_2 = \frac{1}{\sqrt{2}}(1, 0, -1)$, and $\mathbf{v}_3 = (0, 1, 0)$. **Solution:** We need to solve the equation $\mathbf{v} = a\mathbf{v}_1 + b\mathbf{v}_2 + c\mathbf{v}_3$ for a, b, and c. This gives $a = \frac{1}{\sqrt{2}}(1 + 3) = 2$, $b = \frac{1}{\sqrt{2}}(1 - 3) = -1$, and $c = 2$.

20. The DFT $X[k]$ is given by:

$$
\begin{aligned}
X[0] &= x[0] + x[1] + x[2] + x[3] = 4 \\
X[1] &= x[0] - jx[1] - x[2] + jx[3] = 0 \\
X[2] &= x[0] - x[1] + x[2] - x[3] = 0 \\
X[3] &= x[0] + jx[1] - x[2] - jx[3] = 0
\end{aligned}
$$

21. The IDFT $x[n]$ is given by:

$$
\begin{aligned}
x[0] &= \frac{1}{4}(X[0] + X[1] + X[2] + X[3]) = 1 \\
x[1] &= \frac{1}{4}(X[0] - jX[1] - X[2] + jX[3]) = 1 \\
x[2] &= \frac{1}{4}(X[0] - X[1] + X[2] - X[3]) = 1 \\
x[3] &= \frac{1}{4}(X[0] + jX[1] - X[2] - jX[3]) = 1
\end{aligned}
$$

22. For the left-hand side, we compute $\sum_{n=0}^{N-1} |x[n]|^2 = |1|^2+|2|^2+|3|^2+|4|^2 = 30$. For the right-hand side, we compute $\frac{1}{N}\sum_{k=0}^{N-1} |X[k]|^2 = \frac{1}{4}(|10|^2 + |2|^2+|-2|^2+|2|^2) = 30$. The equality holds, thus verifying the Plancherel theorem.

23. The DFT $X[k]$ is given by:

$$
\begin{aligned}
X[0] &= x[0] + x[1] + x[2] + x[3] = 0 \\
X[1] &= x[0] - jx[1] - x[2] + jx[3] = 0 \\
X[2] &= x[0] - x[1] + x[2] - x[3] = 4 \\
X[3] &= x[0] + jx[1] - x[2] - jx[3] = 0
\end{aligned}
$$

We observe that the magnitudes of the DFT coefficients are the same for all k.