Digital Signal Processing - Lecture Notes - 4

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Basis Vectors of DFT

In the context of the Discrete Fourier Transform (DFT), the basis vectors are complex exponentials, which are sequences of complex numbers. For a sequence of N points, the k-th basis vector v_k is defined as:

$$v_k[n] = e^{i2\pi nk/N} \tag{1}$$

for $n = 0, 1, 2, ..., N-1$. Here, i is the imaginary unit, and e is the base of the natural logarithm. Each basis vector corresponds to a different frequency. The frequency is determined by the parameter k.

The k-th basis vector represents a sinusoid that completes k full cycles over N points. For example, if $N = 8$ and $k = 1$, then the basis vector v_1 completes one full cycle over eight points. If $k = 2$, then the basis vector v_2 completes two full cycles over eight points, and so on.

Each basis vector is orthogonal to all the other basis vectors, which means that the dot product of any two different basis vectors is zero. This property is a key feature of the DFT and provides a powerful way of analyzing signals.

Basis Vectors of DFT - Simplified Explanation

Let's think of Discrete Fourier Transform (DFT) as a way of showing us what "ingredients" make up our signal. These ingredients are the basis vectors. In DFT, the basis vectors are certain patterns, called complex exponentials. We can imagine them like musical notes.

$$v_k[n] = e^{i2\pi nk/N} \tag{2}$$

This equation is a recipe for each basis vector or "note". The term n is the point in time we're at (from 0 to $N-1$), k is which "note" we're making, and N is the total number of points in our sequence. The whole expression $e^{i2\pi nk/N}$ is a complex number that represents a point on a circular path, and as n changes, we go around this path k times.

If k is bigger, we go around the circle more times, which is like a higher pitch note. Each different k value gives us a different note, and our signal is made up of these notes.

The great thing is, each "note" or basis vector is orthogonal to the others. This means they don't interfere with each other. It's as if each note is played on a different instrument, so we can hear each one separately, which makes it easier to see what our signal is made up of!

DFT Basis Vector Examples

Example 1

Determine the DFT basis vector $v_k[n]$ for $N = 4$ and $k = 1$.

Solution:
We use the formula for the DFT basis vector:

$$v_k[n] = e^{i2\pi nk/N} \tag{3}$$

For $N = 4$ and $k = 1$, this gives us

$$\begin{aligned}
v_1[0] &= e^{i2\pi(1)(0)/4} = 1 \\
v_1[1] &= e^{i2\pi(1)(1)/4} = i \\
v_1[2] &= e^{i2\pi(1)(2)/4} = -1 \\
v_1[3] &= e^{i2\pi(1)(3)/4} = -i
\end{aligned}$$

So, the DFT basis vector $v_1[n]$ for $N = 4$ and $k = 1$ is $[1, i, -1, -i]$.

Example 2

Show that the DFT basis vectors $v_0[n]$ and $v_1[n]$ for $N = 4$ are orthogonal.

Solution:
We calculate the dot product of the two vectors:

$$v_0[n] \cdot v_1[n] = \sum_{n=0}^{N-1} v_0[n]\overline{v_1[n]} \tag{4}$$

For $N = 4$, the basis vectors are:

$$\begin{aligned}
v_0[n] &= [1, 1, 1, 1] \\
v_1[n] &= [1, i, -1, -i]
\end{aligned}$$

Their dot product is:

$$\begin{aligned}v_0[n] \cdot v_1[n] &= 1 \cdot 1 + 1 \cdot -i + 1 \cdot -1 + 1 \cdot i \\ &= 1 - i - 1 + i = 0\end{aligned}$$

Since the dot product is zero, the vectors are orthogonal.

Example 3

Calculate the DFT of the sequence $x[n] = [1, 2, 3, 4]$ using the DFT basis vectors.

Solution:
The DFT of a sequence $x[n]$ is given by:

$$X[k] = \sum_{n=0}^{N-1} x[n]\overline{v_k[n]} \tag{5}$$

We can calculate each $X[k]$ using the DFT basis vectors $v_k[n]$:

$$\begin{aligned}X[2] &= \sum_{n=0}^{3} x[n]\overline{v_2[n]} = 1 \cdot 1 + 2 \cdot -1 + 3 \cdot 1 + 4 \cdot -1 = -2 \\ X[3] &= \sum_{n=0}^{3} x[n]\overline{v_3[n]} = 1 \cdot 1 + 2 \cdot i + 3 \cdot -1 + 4 \cdot -i = 2 - 2i\end{aligned}$$

So, the DFT of the sequence $x[n] = [1, 2, 3, 4]$ is $X[k] = [10, -2 + 2i, -2, 2 - 2i]$.

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When you perform a Discrete Fourier Transform (DFT) on a signal and then plot the results, you are viewing the frequency components of the original signal. This is often called the frequency spectrum of the signal. The frequency spectrum tells us which frequencies are present in the signal and at what intensities.

In the context of the DFT, the "frequencies" are the cycles or repetitions that occur in the signal over a fixed interval. For example, if the signal represents a sound wave, these frequencies correspond to the pitches of the sound.

Now let's interpret the DFT plot:

1. **X-Axis (Frequency bins):** Each point on the x-axis represents a particular frequency. The frequency is given in terms of "bins". These bins

are integer multiples of the fundamental frequency. The fundamental frequency depends on the total sampling duration of the time-domain signal. For a signal of length N, there will be N bins in the DFT, each corresponding to a frequency of $k * (F_s/N)$, where F_s is the sampling frequency and k is the bin number.

2. **Y-Axis (Amplitude):** The y-axis shows the amplitude of the signal at each frequency. In other words, the y-axis tells us how "strong" each frequency is in the original signal. The magnitude of the DFT output is proportional to the amount of the frequency present in the original signal.

3. **Peaks (Frequency components):** Peaks or spikes in the DFT plot represent dominant frequency components. The location of the peak gives the frequency, and the height of the peak indicates the amplitude of that frequency in the signal.

In the case of a real-valued signal, the DFT is symmetric around the middle of the spectrum. The first half corresponds to the positive frequencies, and the second half corresponds to the negative frequencies. Generally, we only plot the first half for real-valued signals, because the second half doesn't provide additional information.

It's also important to note that if the original signal is a pure sine wave of a single frequency, the DFT will result in a single peak at that frequency. But most signals aren't pure sine waves; they are combinations of different sine waves of different frequencies and amplitudes. The DFT reveals these individual sine waves.

Examples:

1. Compute the DFT of the sequence $x[n] = 1, 2, 3, 4, 5$.
2. Compute the DFT of the sequence $x[n] = 0, 1, 0, 0$.
3. For a sequence $x[n] = 1, -1, 1, -1$, find the DFT.
4. Given the sequence $x[n] = 2, 3, -1, 4$, find the DFT.
5. Compute the inverse DFT of the sequence $X[k] = 4, -j, 2, j$.

Solutions:

1. Solution 1:

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X[k] = DFT(x[n]) = [15, -2.5+3.44j, -2.5+0.87j, -2.5-0.87j, -2.5-3.44j]
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2. Solution 2:

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X[k] = DFT(x[n]) = [1, -j, -1, j]
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3. Solution 3:

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X[k] = DFT(x[n]) = [0, 0, 4, 0]
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4. Solution 4:

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X[k] = DFT(x[n]) = [8, 3-5j, 2, 3+5j]
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5. Solution 5:

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x[n] = IDFT(X[k]) = [2, 1-j, 0, 1+j]
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Problem Set - 4

Numerical Problems for Python:

1. Create a sequence $x[n]$ of 1000 random real values between -1 and 1. Compute and plot its DFT.
2. Generate a sinusoidal signal of frequency 5 Hz, sampled at 100 Hz for 1 second. Compute and plot its DFT.
3. For a signal $x(t) = \sin(2\pi f t)$, where $f = 10Hz$, sampled at 1000 Hz for 1 second, compute and plot the magnitude of its DFT. What do you observe?
4. Generate a signal which is the sum of two sinusoids with frequencies 5Hz and 20Hz, sampled at 100 Hz for 2 seconds. Compute and plot the DFT of the signal.
5. Generate a noisy signal by adding Gaussian noise to the signal in problem 4. Compute and plot the DFT of the noisy signal. Can you still identify the frequencies of the original sinusoids?

Analytical Problems:

1. Prove that the set of vectors $e^{j2\pi kn/N}$ for $k = 0, 1, ..., N-1$ and $n = 0, ..., N-1$ forms an orthonormal basis for the space of sequences of length N over the complex numbers.
2. Given a sequence $x[n] = [1, 2, 3, 4, 5]$ for $n = 0, 1, ..., 4$. Express $x[n]$ as a linear combination of DFT basis vectors.
3. The kth basis vector of the DFT can be represented as $b_k[n] = e^{j2\pi kn/N}$ for $n = 0, 1, ..., N-1$. Show that the energy of each DFT basis vector $b_k[n]$ is equal to 1.
4. Show that the DFT basis vectors are orthogonal to each other.
5. Compute the 4-point DFT of the sequence $x[n] = [1, j, -1, -j]$. Express your answer in terms of the DFT basis vectors.

Solutions for Analytical Problems:

1. The set of vectors $e^{j2\pi kn/N}$ for $k = 0, 1, ..., N-1$ and $n = 0, ..., N-1$ form an orthonormal basis. This can be proven by taking the inner product of two different basis vectors:

$$\langle e^{j2\pi kn/N}, e^{j2\pi ln/N} \rangle = \frac{1}{N} \sum_{n=0}^{N-1} e^{j2\pi kn/N} e^{-j2\pi ln/N} = \frac{1}{N} \sum_{n=0}^{N-1} e^{j2\pi n(k-l)/N} \tag{6}$$

   If $k = l$, this sum equals 1, showing that the basis vectors are normalized. If $k \neq l$, this sum equals zero, showing that the basis vectors are orthogonal.

2. The sequence $x[n]$ can be expressed as a linear combination of DFT basis vectors by taking the DFT of $x[n]$:

$$X[k] = \sum_{n=0}^{N-1} x[n] e^{-j2\pi kn/N} =? \tag{7}$$

   Thus, $x[n] = IDFT\{X[k]\} = \frac{1}{N} \sum_{k=0}^{N-1} X[k] e^{j2\pi kn/N}$.

3. The energy of each DFT basis vector $b_k[n]$ can be calculated as follows:

$$E = \sum_{n=0}^{N-1} |b_k[n]|^2 = \sum_{n=0}^{N-1} |e^{j2\pi kn/N}|^2 = N \tag{8}$$

   Therefore, the energy of each DFT basis vector is equal to 1.

4. This can be shown by taking the inner product of any two different DFT basis vectors, as shown in the solution to problem 1.

5. We can compute the 4-point DFT of the sequence $x[n] = [1, j, -1, -j]$ using the definition of the DFT. For a sequence $x[n]$ of length N, its DFT $X[k]$ is given by:

$$X[k] = \sum_{n=0}^{N-1} x[n] e^{-j\frac{2\pi}{N} nk} \tag{9}$$

   We are computing a 4-point DFT, so $N = 4$. Therefore,

$$X[k] = \sum_{n=0}^{3} x[n] e^{-j\frac{2\pi}{4} nk} \tag{10}$$

   We compute each $X[k]$ for $k = 0, 1, 2, 3$: