MS3

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Introduction to Mathematical Modeling

Mathematical modeling is a method used in science and engineering to describe and analyze the behavior of complex real-world systems using mathematical equations. These models can predict the system's future behavior or investigate how changes in system parameters affect its behavior. In electronics and mechanics, mathematical modeling can provide crucial insights into system dynamics and enable the development of control strategies and system optimization.

Techniques for Building Mathematical Models

Building a mathematical model requires an understanding of both the system you are trying to model and the mathematical tools available. The model should be as simple as possible while still capturing the essential features of the system.

Differential Equations

One of the primary tools for building mathematical models is differential equations. These are equations that describe the relationship between a function and its derivatives. Differential equations are particularly suitable for modeling systems where the rate of change of a quantity (e.g., position, velocity, current, voltage, etc.) is significant.

For example, in a simple electronic circuit containing a resistor and a capacitor (an RC circuit), the relationship between the voltage across the capacitor (V) and the current through the circuit (I) can be described by the differential equation

$$I = C\frac{dV}{dt}$$

where C is the capacitance of the capacitor, and dV/dt is the rate of change of voltage with respect to time.

Analogy Considerations

When building mathematical models, it can be helpful to draw on analogies between different physical systems. For example, electrical and mechanical systems often have analogous components. Resistors are analogous to dampers, capacitors to springs, and inductors to masses. These analogies can help in the formulation of mathematical models and can make complex systems more understandable.

Practical Applications

In practical applications, the establishment of differential equations often involves the application of fundamental laws of physics, such as Kirchhoff's laws in electrical circuits or Newton's laws in mechanics. The mathematical model can then be solved to obtain the system response. In some cases, the model's equations may be too complex to solve analytically, and numerical methods may be used. Python's SciPy library provides a wide range of tools for solving differential equations numerically.

Examples of Mathematical Modeling Using Differential Equations

Modeling an RC Circuit

Let's consider a simple RC circuit with a resistor and a capacitor connected in series. If we apply a voltage source V_s across this circuit, we can represent the behavior of this system with a first-order differential equation.

By Kirchhoff's voltage law, the voltage across the resistor V_R and the voltage across the capacitor V_C must add up to the source voltage:

$$V_s = V_R + V_C$$

The voltage across the resistor can be represented as $V_R = RI$, and the voltage across the capacitor as $V_C = \frac{1}{C} \int I, dt$. Substituting these expressions into the equation gives us:

$$V_s = RI + \frac{1}{C} \int I, dt$$

This is a first-order differential equation that can be solved to find the current $I(t)$ and voltage across the capacitor $V_C(t)$ as functions of time.

Modeling a Mass-Spring-Damper System

Consider a mass-spring-damper system with mass m, damping coefficient b, and spring constant k. If we denote the displacement of the mass from its equilibrium position as x, the system can be represented by the second-order differential equation:

$$m\frac{d^2x}{dt^2} + b\frac{dx}{dt} + kx = F$$

Here F is the external force applied to the system. This equation is derived from Newton's second law of motion ($F = ma$), where the forces acting on the mass are the spring force $-kx$, the damping force $-b\frac{dx}{dt}$, and the external force F.

These models can be numerically solved using Python's SciPy library, giving the behavior of these systems over time under given initial conditions and inputs.

Modeling an RL Circuit

An RL circuit consists of a resistor (R) and inductor (L) connected in series and driven by a voltage source $V_s(t)$. If we apply Kirchhoff's voltage law to this circuit, we obtain the following differential equation:

$$V_s(t) = RI(t) + L\frac{dI(t)}{dt}$$

In this equation, $I(t)$ is the current through the circuit as a function of time. $RI(t)$ represents the voltage drop across the resistor, and $L\frac{dI(t)}{dt}$ is the voltage drop across the inductor. Solving this first-order differential equation will provide the current $I(t)$ as a function of time.

Modeling the Spread of a Disease

Mathematical modeling is extensively used in epidemiology to predict the spread of diseases. The SIR model, for instance, divides the population into three categories: Susceptible (S), Infected (I), and Recovered (R). The transitions between these states can be represented by the following set of differential equations:

$$\frac{dS}{dt} = -\beta SI \; \frac{dI}{dt} \qquad = \beta SI - \gamma I \; \frac{dR}{dt} = \gamma I$$

Here, β is the transmission rate of the disease and γ is the recovery rate. S, I, and R represent the number of susceptible, infected, and recovered individuals, respectively. By solving these equations, we can predict the number of people in each category over time, and thus understand how the disease might spread.

Both of these examples can be numerically solved using libraries in Python like SciPy, providing valuable insights into the dynamics of these systems. They demonstrate how mathematical modeling is not just limited to physical systems but can be extended to many fields, including public health.

NOTE: These models are based on certain assumptions. For instance, the RL circuit model assumes that the circuit components are ideal and that there are no other energy losses in the circuit. The SIR model assumes that the population is mixed uniformly and that there are no births or deaths. It's essential to keep these assumptions in mind when interpreting the results of the models.

Application of Mathematical Models in Electronics and Mechanics

Mathematical models form the backbone of our understanding of many systems in electronics and mechanics. These models allow us to predict system behavior, optimize system performance, and investigate the impact of design changes.

Electronics

In electronics, mathematical models are used to describe and analyze circuits, components, and systems. For example, consider an RC circuit consisting of a resistor (R) and capacitor (C) connected in series to a voltage source. This system can be described by a simple differential equation derived from Kirchhoff's circuit laws. Once this equation is solved, we can predict how the voltage across the capacitor will change over time in response to a given input voltage.

Similarly, electronic components like transistors, diodes, and operational amplifiers can all be modeled using mathematical equations. These equations capture the essential characteristics of the devices and allow us to predict their behavior in a circuit.

Mathematical models also play a critical role in the design of digital systems. Boolean algebra, for instance, provides a mathematical model for digital logic and is used extensively in the design of digital circuits.

Diodes

Diodes are semiconductor devices that allow current to flow in one direction only. The ideal diode can be described by a simple mathematical model, where the current through the diode is zero for a reverse bias, and infinite for a forward bias. In the real world, however, diodes do not abruptly turn on and off, and their behavior is better described by the Shockley diode equation:

$$I = I_S(e^{\frac{V}{nV_T}} - 1) \tag{1}$$

Where I is the diode current, I_S is the reverse bias saturation current, V is the voltage across the diode, n is the ideality factor (typically close to 1), and V_T is the thermal voltage.

Transistors

Transistors are semiconductor devices used to amplify or switch electronic signals. A common type of transistor, the Bipolar Junction Transistor (BJT), can be mathematically modelled by the Ebers-Moll equations. For the NPN transistor in the active mode, the collector current I_C is given by:

$$I_C = I_S(e^{\frac{V_{BE}}{V_T}} - 1) - I_S(e^{\frac{V_{BC}}{V_T}} - 1) \tag{2}$$

Diode Example

Suppose we have a diode with a saturation current $I_S = 10^{-12}$ A and a voltage across the diode $V = 0.7$ V. We want to find the current flowing through the diode. We can use the Shockley diode equation:

$$I = I_S(e^{\frac{V}{nV_T}} - 1) \tag{3}$$

Assuming the diode is silicon (thus $n = 2$) and room temperature (so $V_T \approx 0.026$ V), we can substitute the given values into the equation to find the current:

$$I = 10^{-12}(e^{\frac{0.7}{2*0.026}} - 1) \tag{4}$$

```
import numpy as np

# Constants
I_S = 10**-12
V = 0.7
n = 2
V_T = 0.026

# Compute current
I = I_S * (np.exp(V / (n*V_T)) - 1)

print(f"The current through the diode is {I} A.")
```

Transistor Example

Consider a NPN transistor in active mode with $I_S = 10^{-14}$ A, $V_{BE} = 0.7$ V, and $V_{BC} = 0$. We can find the collector current I_C using the Ebers-Moll equations:

$$I_C = I_S(e^{\frac{V_{BE}}{V_T}} - 1) - I_S(e^{\frac{V_{BC}}{V_T}} - 1) \tag{5}$$

Assuming room temperature, we substitute the given values into the equation:

$$I_C = 10^{-14}(e^{\frac{0.7}{0.026}} - 1) - 10^{-14}(e^{\frac{0}{0.026}} - 1) \tag{6}$$

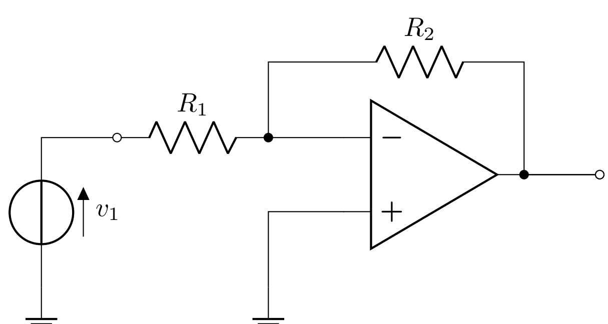
```
# Constants
I_S = 10**-14
V_BE = 0.7
V_BC = 0
V_T = 0.026

# Compute collector current
I_C = I_S * (np.exp(V_BE / V_T) - 1) - I_S * (np.exp(V_BC / V_T) - 1)

print(f"The collector current is {I_C} A.")
```

Inverting Amplifier

An inverting amplifier using an operational amplifier is one of the simplest amplifiers.

An inverting amplifier is a specific configuration of an operational amplifier (op-amp) that inverts the phase of the input signal at the output, while amplifying its absolute value. The output signal is a scaled, inverted version of the input.

Here is the basic configuration of an inverting amplifier:



$$A = -\frac{R_2}{R_1} \tag{7}$$

The relationship between the input and output voltage is given by the equation

$$V_{\text{out}} = -\frac{R_f}{R_i} V_{\text{in}}$$

where V_{out} is the output voltage, R_f is the feedback resistor, R_i is the input resistor and V_{in} is the input voltage.

The inverting amplifier is governed by the laws of an operational amplifier, which dictate that the voltage difference between the two inputs (inverting "-" and non-inverting "+") is zero when the op-amp is in a state of "negative feedback" (which is the case for this configuration).

Thus, the voltage at the inverting input (the junction of resistors R_1 and R_2) is zero. This condition is known as a virtual short circuit.

The input voltage v_1 causes a current to flow through R_1. Since no current enters the op-amp (due to its very high input impedance), the same current flows through R_2.

This can be written in terms of the input and output voltages and the resistances R_1 and R_2:

$$I_{R1} = I_{R2} \ \frac{v_1}{R_1} = \frac{v_1 - V_{out}}{R_2}$$

Solving the above equation for V_{out}, we get the gain of the inverting amplifier:

$$V_{out} = -\frac{R_2}{R_1} v_1$$

This shows that the output is an inverted (due to the negative sign) version of the input, and the gain of the amplifier is $\frac{R_2}{R_1}$.

Python Simulation

Given the input voltage and the resistance values, this function will calculate the output voltage. Here's how we could do it:

```
import numpy as np
import matplotlib.pyplot as plt

def inverting_amplifier(v_in, R1, R2):
    """Function to simulate an inverting amplifier."""
    return -R2 / R1 * v_in

# Define the input voltage as a sine wave
```

```
t = np.linspace(0, 1, 500, endpoint=False) # time variable
v_in = np.sin(2 * np.pi * 5 * t)

# Use resistances R1 = 1000 Ohms and R2 = 10000 Ohms
R1 = 1000
R2 = 10000

# Compute the output voltage
v_out = inverting_amplifier(v_in, R1, R2)

# Plot the input and output voltages
plt.figure(figsize=(10, 6))
plt.plot(t, v_in, label="Input")
plt.plot(t, v_out, label="Output")
plt.title("Inverting Amplifier")
plt.xlabel("Time [s]")
plt.ylabel("Voltage [V]")
plt.legend()
plt.grid(True)
plt.show()
```

This script creates a time variable t and uses it to define an input voltage v_in as a sine wave. The inverting amplifier function is then used to compute the output voltage. The input and output voltages are then plotted as a function of time.

The plot shows that the output voltage is indeed an inverted version of the input, as expected for an inverting amplifier. The gain of the amplifier is $-R_2/R_1$, as reflected in the amplitude of the output voltage.

Mechanics

In the field of mechanics, mathematical models are used to describe the motion and forces in mechanical systems. One of the most common mathematical models in mechanics is Newton's second law, which states that the force acting on an object is equal to its mass times its acceleration. This law can be used to model a wide range of systems, from a falling apple to a satellite orbiting the Earth.

In more complex systems, such as a car suspension or a robotic arm, the forces and motions involved may be described by a set of differential equations. These equations can be derived from principles of mechanics and often involve parameters like mass, stiffness, and damping.

Once these mathematical models are established, they can be solved to predict system behavior. For example, in designing a car suspension, engineers might use a mathematical model to predict how the car will respond to bumps in the road.

Simple Harmonic Oscillator - Mass-Spring System

A classic example of mathematical modeling in mechanics is the simple harmonic oscillator. Consider a spring-mass system with a mass m attached to a spring of spring constant k. This system can be modeled by Hooke's Law, which states that the force exerted by a spring is proportional to the displacement of the spring from its equilibrium position. Mathematically, this can be expressed as:

$$F = -kx$$

where: - F is the force, - k is the spring constant, and - x is the displacement.

Using Newton's second law, $F = ma$, where a is acceleration, we can substitute F with ma to get a second-order differential equation representing the motion of the mass:

$$m\frac{d^2x}{dt^2} = -kx$$

or simplifying:

$$\frac{d^2x}{dt^2} = -\frac{k}{m}x$$

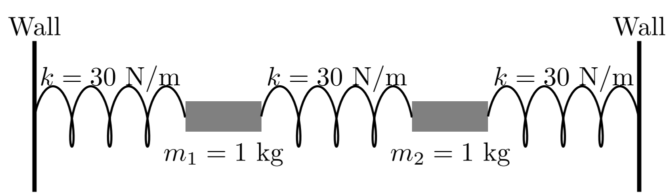
This differential equation is a mathematical model that describes the behavior of the mass-spring system over time given initial conditions (e.g., initial position and velocity of the mass).

```
import numpy as np
from scipy.integrate import odeint
import matplotlib.pyplot as plt

# define model
def model(y, t, k, m):
    x, v = y
    dydt = [v, -(k/m)*x]
    return dydt

# parameters
k = 0.5  # spring constant
m = 1.0  # mass

# initial condition
```

```
y0 = [1.0, 0.0]  # initial displacement and velocity

# time points
t = np.linspace(0, 20, 500)

# solve ODE
y = odeint(model, y0, t, args=(k, m))

# plot results
plt.figure(figsize=(10, 5))
plt.plot(t, y[:, 0], label='x (displacement)')
plt.plot(t, y[:, 1], label='v (velocity)')
plt.legend()
plt.xlabel('Time')
plt.ylabel('Values')
plt.title('Mass-Spring System')
plt.grid(True)
plt.show()
```

Example: Spring - Mass System

Let's consider a system of two identical masses, m_1 and m_2, each with a mass of 1 kg, connected by three springs to two rigid walls. Each spring has a spring constant k of 30 N/m. The system can be visualized as:



We want to calculate the normal modes of this system. The equations of motion, derived from Newton's second law, for the two masses are:

$$m_1 \frac{d^2 x_1}{dt^2} = -kx_1 + k(x_2 - x_1)$$

$$m_2 \frac{d^2 x_2}{dt^2} = -kx_2 + k(x_1 - x_2)$$

To solve the above equations, we can seek solutions in the form $x_i = A_i \cos(\omega t + \phi_i)$ where A_i, ω, and ϕ_i are constants to be determined. Substituting this ansatz into the above equations and canceling out common factors, we can find the normal modes by setting the determinant of the resulting system of equations to zero.

This yields a characteristic equation which can be solved to find the normal mode frequencies ω. The normal mode shapes (given by A_1 and A_2) can be found by substituting the frequencies back into the system of equations.

The full derivation is beyond the scope of this document and is a standard result in many textbooks on classical mechanics or vibrations.

```
import numpy as np
from scipy.integrate import odeint
import matplotlib.pyplot as plt

# parameters
m1 = m2 = 1  # kg
k = 30  # N/m
b = 0.20  # damping coefficient

# system of equations
def system(y, t):
    x1, v1, x2, v2 = y
    dx1dt = v1
    dv1dt = (-k*x1 + k*(x2 - x1) - b*v1) / m1
    dx2dt = v2
    dv2dt = (-k*x2 + k*(x1 - x2) - b*v2) / m2
    return [dx1dt, dv1dt, dx2dt, dv2dt]

# initial conditions
x1_0 = 0.0  # cm
v1_0 = 0.0  # cm/s
x2_0 = 10.0  # cm
v2_0 = 0.0  # cm/s
y0 = [x1_0, v1_0, x2_0, v2_0]

# time grid
t = np.linspace(0, 10, 1000)  # 10 seconds, 1000 points

# solve
solution = odeint(system, y0, t)

# plot
plt.figure(figsize=(8, 6))
plt.plot(t, solution[:, 0], label='x1 (cm)')
plt.plot(t, solution[:, 2], label='x2 (cm)')
plt.legend()
```

```
plt.xlabel('Time (s)')
plt.ylabel('Position (cm)')
plt.title('Motion of the Masses')
plt.grid(True)
plt.show()
```

Exploring the Analogy Considerations Between Different Physical Systems

Often, when developing mathematical models for physical systems, we find that there are analogies between seemingly different systems. By identifying and exploiting these analogies, we can often simplify the modeling process and gain deeper insight into the behavior of the systems.

One of the most commonly encountered analogies is between electrical and mechanical systems. Consider, for instance, an electrical circuit consisting of a resistor (R), inductor (L), and capacitor (C). The behavior of this circuit is described by a second-order differential equation derived from Kirchhoff's laws.

Similarly, a mechanical system consisting of a mass (M), spring (K), and damper (B) is also described by a second-order differential equation, but this time derived from Newton's second law. Even though the two systems are fundamentally different—one being electrical and the other mechanical—the differential equations that describe their behavior have the same form, and so we can analyze them using the same mathematical techniques.

$$\begin{aligned} \text{Electrical:} \quad & L\frac{d^2q}{dt^2} + R\frac{dq}{dt} + \frac{1}{C}q = E(t) \\ \text{Mechanical:} \quad & M\frac{d^2x}{dt^2} + B\frac{dx}{dt} + Kx = F(t) \end{aligned}$$

In the electrical equation, q is the charge, $E(t)$ is the electromotive force, and the terms on the left-hand side represent the inductive, resistive, and capacitive effects respectively. In the mechanical equation, x is the displacement, $F(t)$ is the external force, and the terms on the left-hand side represent the inertia, damping, and stiffness effects respectively.

This kind of analogy can often provide us with intuitive insights into the behavior of a system, and guide us in the development of mathematical models.

- **Electrical and Thermal Systems:**

  Electrical current flow is analogous to heat flow in a thermal system. Ohm's law, which states that the current (I) through a conductor between two points is directly proportional to the voltage (V) across the two points, and inversely proportional to the resistance (R), has a direct analogy in the law of heat conduction. The heat flow (Q) is proportional to the temperature difference (δT) and inversely proportional to the thermal resistance (R).

$$\begin{aligned} \text{Electrical:} \quad & V = IR \\ \text{Thermal:} \quad & Q = \Delta T / R \end{aligned}$$

- **Mechanical and Acoustic Systems:**

  The analogy between mechanical systems and acoustic systems is another interesting example. The displacement of an object in a mechanical vibrating system is analogous to the pressure variation in an acoustic wave. The mass in the mechanical system is analogous to the inertia of the air, the spring to the elasticity of the air, and the damper to the air's resistance to flow.

$$\begin{aligned} \text{Mechanical:} \quad & F = m \cdot a \\ \text{Acoustic:} \quad & p = Z \cdot u \end{aligned}$$

  Here, F is force, m is mass, a is acceleration, p is pressure, Z is acoustic impedance, and u is particle velocity.

- **Electrical and Fluid Systems:**

  Lastly, the flow of electrical current in a circuit can be compared with the flow of fluid in a pipe. The voltage difference in an electrical circuit is analogous to the pressure difference in a fluid system. The electrical resistance to the flow of current in a conductor is analogous to the frictional resistance to the flow of fluid in a pipe.

$$\begin{aligned} \text{Electrical:} \quad & I = \frac{V}{R} \\ \text{Fluid:} \quad & Q = \frac{\Delta P}{R} \end{aligned}$$

  Here, I is the current, V is the voltage, R is the resistance, Q is the flow rate, ΔP is the pressure difference, and R is the fluid resistance.

Example:

Let's consider an example using an analogy between an electrical system (a simple RC circuit) and a thermal system. A resistor-capacitor (RC) circuit is a simple electrical circuit that can be used to demonstrate this analogy.

An RC circuit consists of a resistor and a capacitor connected in series. The behavior of this circuit can be described using a simple first-order differential equation. Similarly, the behavior of heat flow across a body (like a metal bar) can also be described by a similar differential equation.

Let's consider a case where we are charging a capacitor in an RC circuit with a resistor of 1 Ohm and a capacitor of 1 Farad. The input voltage is a step input of 1 Volt at t=0. This is similar to suddenly introducing heat at one end of a metal bar and observing how the temperature changes with time.

The voltage across the capacitor (or the temperature of the bar) as a function of time can be found by solving the differential equation:

$$\dot{V} = \frac{1 - V}{RC} \tag{8}$$

where:

- $\dot{V}$ is the derivative of the voltage with respect to time (rate of change of voltage),
- V is the voltage across the capacitor.

To solve the above first-order differential equation analytically, you would integrate it to find the equation for voltage (V) across the capacitor as a function of time (t):

- Separate the variables:

  $$RC\frac{dV}{dt} = 1 - V \tag{9}$$

  Move variables to opposite sides:

  $$\frac{dV}{1 - V} = \frac{dt}{RC} \tag{10}$$

  Integrate both sides:

  $$-RC \ln |1 - V| = t + C \tag{11}$$

  Solve for V:

  $$V(t) = 1 - e^{-\frac{t}{RC}} \tag{12}$$

  where RC is the time constant of the system and e is the base of natural logarithm.

Let's find the numerical solution using Python. We will be using the scipy library to solve the differential equation. Here is a Python code snippet that solves the differential equation using the odeint function:

```
import numpy as np
from scipy.integrate import odeint
import matplotlib.pyplot as plt

# define the ODE
def rc_circuit(V, t):
    R = 1
    C = 1
    dVdt = (1 - V) / (R*C)
    return dVdt

# initial condition
V0 = 0

# time points
t = np.linspace(0,5)

# solve ODE
V = odeint(rc_circuit,V0,t)

# plot results
plt.plot(t,V)
plt.xlabel('time')
plt.ylabel('V(t)')
plt.show()
```