Physics Lecture Notes - 4

Momentum and Harmonic Oscillation

11 May 2023

Momentum and Impulse

Momentum is the product of an object's mass and velocity. It is a vector quantity, and its SI unit is kilogram meter per second (kg m/s). The momentum of an object is given by:

$$p = mv$$

where p is momentum, m is mass, and v is velocity.

Impulse is the product of the force acting on an object and the time interval over which the force is applied. Impulse is also a vector quantity, and its SI unit is newton second (N s). The impulse experienced by an object is given by:

$$J = F\Delta t$$

where J is impulse, F is force, and Δt is the time interval.

According to Newton's second law, the net force acting on an object is equal to the rate of change of its momentum:

$$F = \frac{\Delta p}{\Delta t}$$

Therefore, the impulse experienced by an object is equal to the change in its momentum:

$$J = \Delta p$$

Example 1: A 0.5 kg ball is moving horizontally at a speed of 4 m/s. It is struck by a bat and then moves in the opposite direction with a speed of 6 m/s. Calculate the impulse experienced by the ball.

Solution: First, we find the initial and final momenta of the ball:

$$p_i = mv_i = (0.5\,\text{kg})(4\,\text{m/s}) = 2\,\text{kg m/s}$$

$$p_f = mv_f = (0.5\,\text{kg})(-6\,\text{m/s}) = -3\,\text{kg m/s}$$

Next, we calculate the change in momentum:

$$\Delta p = p_f - p_i = -3\,\text{kg m/s} - 2\,\text{kg m/s} = -5\,\text{kg m/s}$$

The impulse experienced by the ball is -5 kg m/s.

Conservation of Momentum

The principle of conservation of momentum states that the total momentum of a system of particles remains constant, provided no external forces are acting on the system. Mathematically, this can be expressed as:

$$m_1 v_{1i} + m_2 v_{2i} = m_1 v_{1f} + m_2 v_{2f}$$

where the subscripts 1 and 2 denote the two objects, and the subscripts i and f denote initial and final velocities, respectively.

Example 2: A 3 kg object moving at a velocity of 5 m/s collides with a 2 kg object moving in the opposite direction with a velocity of 4 m/s. If the objects stick together after the collision, what is their final velocity?

Solution: We will use the conservation of momentum principle:

$$m_1 v_{1i} + m_2 v_{2i} = (m_1 + m_2) v_f$$

Substitute the given values:

$$(3\,\mathrm{kg})(5\,\mathrm{m/s}) + (2\,\mathrm{kg})(-4\,\mathrm{m/s}) = (3\,\mathrm{kg} + 2\,\mathrm{kg}) v_f$$

Solve for the final velocity:

$$15\,\mathrm{kg\ m/s} - 8\,\mathrm{kg\ m/s} = 5\,\mathrm{kg} v_f$$

$$v_f = \frac{7\,\mathrm{kg\ m/s}}{5\,\mathrm{kg}} = 1.4\,\mathrm{m/s}$$

The final velocity of the combined objects is 1.4 m/s.

Example 3: A force of 12 N is applied to a 4 kg object for 3 seconds. What is the change in velocity of the object?

Solution: We will use the impulse-momentum theorem:

$$J = F \Delta t = \Delta p$$

$$J = (12\,\mathrm{N})(3\,\mathrm{s}) = 36\,\mathrm{N\ s}$$

Now, we have the impulse, which is equal to the change in momentum:

$$\Delta p = m \Delta v$$

Substitute the values:

$$36\,\mathrm{N\ s} = (4\,\mathrm{kg}) \Delta v$$

Solve for the change in velocity:

$$\Delta v = \frac{36\,\mathrm{N\ s}}{4\,\mathrm{kg}} = 9\,\mathrm{m/s}$$

The change in velocity of the object is 9 m/s.

Collisions and Explosions

Collisions can be classified into two types: elastic and inelastic. In an elastic collision, both momentum and kinetic energy are conserved. In an inelastic collision, momentum is conserved, but kinetic energy is not. In a perfectly inelastic collision, the objects stick together after the collision.

Explosions are the opposite of collisions. In an explosion, an object breaks into two or more pieces, and each piece moves away from the others with some velocity. In explosions, momentum is also conserved.

Example 4: Elastic Collision Two balls of equal mass, m = 2 kg, are involved in an elastic collision. Ball A is moving at 6 m/s, and ball B is initially at rest. What are the final velocities of the balls after the collision?

Solution: For an elastic collision, both momentum and kinetic energy are conserved. We start with the conservation of momentum:

$$m_A v_{Ai} + m_B v_{Bi} = m_A v_{Af} + m_B v_{Bf}$$

Since the masses are equal and ball B is initially at rest, the equation becomes:

$$v_{Ai} = v_{Af} + v_{Bf}$$

Using the conservation of kinetic energy:

$$\frac{1}{2} m_A v_{Ai}^2 + \frac{1}{2} m_B v_{Bi}^2 = \frac{1}{2} m_A v_{Af}^2 + \frac{1}{2} m_B v_{Bf}^2$$

Since ball B is initially at rest, the equation becomes:

$$v_{Ai}^2 = v_{Af}^2 + v_{Bf}^2$$

Now we have two equations:
(1) $v_{Ai} = v_{Af} + v_{Bf}$
(2) $v_{Ai}^2 = v_{Af}^2 + v_{Bf}^2$
Substitute the initial velocity of ball A (6 m/s):
(1) $6\,\mathrm{m/s} = v_{Af} + v_{Bf}$
(2) $(6\,\mathrm{m/s})^2 = v_{Af}^2 + v_{Bf}^2$
Solving these equations simultaneously, we get:
$v_{Af} = 0\,\mathrm{m/s}$ and $v_{Bf} = 6\,\mathrm{m/s}$
After the elastic collision, ball A comes to rest, and ball B moves with a velocity of 6 m/s.

Example 5: Inelastic Collision A 5 kg object moving at 4 m/s collides with a 3 kg object moving at 2 m/s in the same direction. The objects stick together after the collision. What is their final velocity?

Solution: For an inelastic collision, only momentum is conserved:

$$m_1 v_{1i} + m_2 v_{2i} = (m_1 + m_2) v_f$$

Substitute the given values:

$$(5\,\mathrm{kg})(4\,\mathrm{m/s}) + (3\,\mathrm{kg})(2\,\mathrm{m/s}) = (5\,\mathrm{kg} + 3\,\mathrm{kg}) v_f$$

Solve for the final velocity:

$$20\,\mathrm{kg\ m/s} + 6\,\mathrm{kg\ m/s} = 8\,\mathrm{kg} v_f$$

$$v_f = \frac{26\,\text{kg m/s}}{8\,\text{kg}} = 3.25\,\text{m/s}$$

The final velocity of the combined objects after the inelastic collision is 3.25 m/s.

Example 6: Explosion A 4 kg object at rest explodes into two fragments, one with a mass of 1.5 kg and the other with a mass of 2.5 kg. The 1.5 kg fragment moves with a velocity of 6 m/s to the right. What is the velocity of the 2.5 kg fragment?

Solution: Since the initial momentum of the system is zero, the final momentum of the system must also be zero. We can write:

$$m_1 v_1 + m_2 v_2 = 0$$

Substitute the given values:

$$(1.5\,\text{kg})(6\,\text{m/s}) + (2.5\,\text{kg})v_2 = 0$$

Solve for the velocity of the 2.5 kg fragment:

$$v_2 = \frac{-(1.5\,\text{kg})(6\,\text{m/s})}{2.5\,\text{kg}} = -3.6\,\text{m/s}$$

The 2.5 kg fragment moves with a velocity of -3.6 m/s (to the left).

Example 7: Momentum and Impulse A 0.5 kg ball is thrown horizontally at a wall with an initial velocity of 12 m/s. It rebounds from the wall with a velocity of -8 m/s. If the ball is in contact with the wall for 0.4 seconds, what is the average force exerted by the wall on the ball?

Solution: We can use the impulse-momentum theorem:

$$J = F\Delta t = \Delta p$$

First, we need to calculate the change in momentum:

$$\Delta p = m\Delta v = m(v_f - v_i)$$

Substitute the given values:

$$\Delta p = (0.5\,\text{kg})(-8\,\text{m/s} - 12\,\text{m/s}) = -10\,\text{kg m/s}$$

Now, we can find the average force exerted by the wall:

$$F = \frac{J}{\Delta t} = \frac{-10\,\text{kg m/s}}{0.4\,\text{s}} = -25\,\text{N}$$

The average force exerted by the wall on the ball is -25 N (in the opposite direction to the initial velocity of the ball).

Example 8: Elastic Collision A 3 kg object moving at 5 m/s to the right collides elastically with a 2 kg object moving at 3 m/s to the left. What are their velocities after the collision?

Solution: For an elastic collision, both momentum and kinetic energy are conserved. We start with the conservation of momentum:

$$m_A v_{Ai} + m_B v_{Bi} = m_A v_{Af} + m_B v_{Bf}$$

Substitute the given values:

$$(3\,\text{kg})(5\,\text{m/s}) + (2\,\text{kg})(-3\,\text{m/s}) = (3\,\text{kg})v_{Af} + (2\,\text{kg})v_{Bf}$$

Using the conservation of kinetic energy:

$$\frac{1}{2}m_A v_{Ai}^2 + \frac{1}{2}m_B^2 = \frac{1}{2}m_A v_{Af}^2 + \frac{1}{2}m_B v_{Bf}^2$$

Substitute the given values:

$$\frac{1}{2}(3\,\text{kg})(5\,\text{m/s})^2 + \frac{1}{2}(2\,\text{kg})(-3\,\text{m/s})^2 = \frac{1}{2}(3\,\text{kg})v_{Af}^2 + \frac{1}{2}(2\,\text{kg})v_{Bf}^2$$

Now, we have a system of two equations with two unknowns, which can be solved using substitution or elimination. In this case, let's solve for v_{Af} in the first equation and substitute it into the second equation:

$$v_{Af} = \frac{(3\,\text{kg})(5\,\text{m/s}) + (2\,\text{kg})(-3\,\text{m/s}) - (2\,\text{kg})v_{Bf}}{3\,\text{kg}}$$

Substitute into the second equation:

$$\frac{1}{2}(3\,\text{kg})\left(\frac{(3\,\text{kg})(5\,\text{m/s}) + (2\,\text{kg})(-3\,\text{m/s}) - (2\,\text{kg})v_{Bf}}{3\,\text{kg}}\right)^2 + \frac{1}{2}(2\,\text{kg})v_{Bf}^2 = \frac{1}{2}(3\,\text{kg})(5\,\text{m/s})^2 + \frac{1}{2}(2\,\text{kg})(-3\,\text{m/s})^2$$

After solving this equation, we find:

$$v_{Af} \approx 1.16\,\text{m/s}$$
$$v_{Bf} \approx 0.49\,\text{m/s}$$

So after the collision, the 3 kg object has a velocity of approximately 1.16 m/s to the right, and the 2 kg object has a velocity of approximately 0.49 m/s to the right.

Simple Harmonic Motion

Simple harmonic motion (SHM) is a type of periodic motion where the restoring force is directly proportional to the displacement from the equilibrium position and acts in the opposite direction of the displacement. Mathematically, this can be expressed as:

$$F = -kx$$

where F is the restoring force, (k is the spring constant, and x is the displacement from the equilibrium position.

The motion of an object in SHM can be described by the following equation:

$$x(t) = A\cos(\omega t + \phi)$$

where $x(t)$ is the displacement as a function of time, A is the amplitude, ω is the angular frequency, t is time, and ϕ is the phase angle.

The period T of an object in SHM is the time it takes to complete one full oscillation. The period is related to the angular frequency by:

$$\omega = \frac{2\pi}{T}$$

Springs and Pendulums

Springs and pendulums are common examples of systems that exhibit SHM. For a spring, Hooke's law gives the restoring force:

$$F = -kx$$

The period of oscillation for a mass-spring system is given by:

$$T = 2\pi\sqrt{\frac{m}{k}}$$

where m is the mass of the object and k is the spring constant.

For a simple pendulum, the restoring force is provided by the gravitational force, and the period of oscillation is given by:

$$T = 2\pi\sqrt{\frac{l}{g}}$$

where l is the length of the pendulum and g is the acceleration due to gravity.

Derivation of the Period of a Simple Pendulum

The formula for the period of a simple pendulum, $T = 2\pi\sqrt{\frac{l}{g}}$, can be derived from the equation of motion for a pendulum under the small angle approximation. This approximation allows us to simplify the motion of the pendulum to simple harmonic motion. Here is a brief overview of the derivation:

1. For a pendulum of length l and displacement angle θ from the vertical, the restoring force due to gravity is $F = -mg\sin\theta$, where m is the mass of the bob and g is the acceleration due to gravity. The negative sign indicates that the force is directed towards the equilibrium position.

2. Under the small angle approximation (θ is small), we can approximate $\sin\theta \approx \theta$ (when θ is in radians). Therefore, the restoring force becomes $F = -mg\theta$.

3. According to Newton's second law, $F = ma$, where a is the acceleration of the bob. The acceleration is related to the displacement by $a = -\omega^2\theta$, where ω is the angular frequency. The negative sign indicates that the acceleration is directed towards the equilibrium position.

4. Equating the two expressions for the force gives $mg\theta = m\omega^2\theta$. The mass and displacement angle cancel out, leaving $\omega^2 = g/l$.

5. The angular frequency is related to the period by $\omega = 2\pi/T$. Substituting this into the previous equation gives $4\pi^2/T^2 = g/l$.

6. Solving for T gives the formula for the period of a simple pendulum: $T = 2\pi\sqrt{\frac{l}{g}}$.

This derivation assumes that the oscillations are small, which allows us to use the small angle approximation, and that there is no damping, which would cause the amplitude and energy to decrease over time.

Damping and Resonance

Damping is a phenomenon that causes a decrease in the amplitude of oscillations over time due to the dissipation of energy, often as a result of friction or other resistive forces. In a damped harmonic oscillator, the equation of motion becomes:

$$m\frac{d^2x}{dt^2} + c\frac{dx}{dt} + kx = 0$$

where c is the damping coefficient.

Resonance occurs when an external force is applied to an oscillating system at a frequency that matches its natural frequency. At resonance, the amplitude of the oscillations becomes larger, and the system can absorb energy from the external force efficiently.

Example 9: Simple Harmonic Motion of a Mass-Spring System A 0.2 kg mass is attached to a spring with a spring constant of 50 N/m. The mass is displaced 0.1 m from its equilibrium position and released. What is the speed of the mass when it passes through the equilibrium position?

Solution: When the mass is at its equilibrium position, all of its energy is kinetic. We can use the conservation of energy to solve for the speed. The total mechanical energy of the system is constant and is given by:

$$E_{\text{total}} = \frac{1}{2}kA^2 = \frac{1}{2}mv^2$$

where k is the spring constant, A is the amplitude of the motion, m is the mass, and v is the velocity. Solving for v, we get:

$$v = \sqrt{\frac{kA^2}{m}}$$

Substituting the given values:

$$v = \sqrt{\frac{50\,\mathrm{N/m}(0.1\,\mathrm{m})^2}{0.2\,\mathrm{kg}}}$$

$$v = 1\,\mathrm{m/s}$$

So the speed of the mass when it passes through the equilibrium position is 1 m/s.

Example 10: Simple Harmonic Motion of a Pendulum A pendulum of length 1 m swings back and forth. What is its period of oscillation? (Assume small oscillations and ignore air resistance.)

Solution: For small oscillations, the period T of a pendulum is given by the formula:

$$T = 2\pi\sqrt{\frac{l}{g}}$$

where l is the length of the pendulum and g is the acceleration due to gravity. Substituting the given values:

$$T = 2\pi\sqrt{\frac{1\,\mathrm{m}}{9.8\,\mathrm{m/s}^2}} \approx 2.0\,\mathrm{s}$$

So the period of oscillation is approximately 2.0 seconds.

Example 11: Simple Harmonic Motion of a Mass-Spring System A 0.5 kg mass attached to a spring oscillates with a frequency of 2 Hz. If the amplitude of the motion is 0.1 m, what is the maximum speed of the mass?

Solution: The maximum speed $v_{\max}$ of an object in simple harmonic motion is given by the formula:

$$v_{\max} = \omega A$$

where ω is the angular frequency and A is the amplitude. The angular frequency is related to the frequency f by the equation $\omega = 2\pi f$. Substituting the given values:

$$v_{\max} = 2\pi(2\,\mathrm{Hz})(0.1\,\mathrm{m}) = 1.26\,\mathrm{m/s}$$

So the maximum speed of the mass is approximately 1.26 m/s.

Example 12: Damping Consider a damped harmonic oscillator with a damping constant $b = 4\,\mathrm{kg/s}$, mass $m = 1\,\mathrm{kg}$, and spring constant $k = 16\,\mathrm{N/m}$. Find the damping ratio and determine whether the motion is underdamped, overdamped, or critically damped.

Solution: The damping ratio ζ is given by $\zeta = \frac{b}{2\sqrt{mk}}$. Substituting the given values:

$$\zeta = \frac{4\,\mathrm{kg/s}}{2\sqrt{(1\,\mathrm{kg})(16\,\mathrm{N/m})}} = 0.5$$

Since $\zeta < 1$, the motion is underdamped.

Example 13: Resonance A child on a swing with a natural frequency of 0.5 Hz is pushed with a frequency of 0.5 Hz. What is the effect of the pushing frequency on the swing's motion?

Solution: When the pushing frequency equals the swing's natural frequency, the swing experiences resonance, leading to an increase in the amplitude of the swing's oscillations.

Example 12: Pendulum A simple pendulum with a length of 0.5 m is displaced by a small angle and released. What is the period of the pendulum's oscillations?

Solution: The period T of a simple pendulum is given by $T = 2\pi\sqrt{\frac{l}{g}}$, where l is the length of the pendulum and g is the acceleration due to gravity. Substituting the given values:

$$T = 2\pi\sqrt{\frac{0.5\,\mathrm{m}}{9.8\,\mathrm{m/s}^2}} \approx 1.4\,\mathrm{s}$$

Problem Set - 4

1. A 0.5 kg object moving at 2 m/s collides elastically with a 1 kg object at rest. After the collision, the 0.5 kg object moves at 1 m/s in the same direction. What is the velocity of the 1 kg object after the collision?

2. A bullet with a mass of 0.01 kg is fired horizontally at a block of wood with a mass of 4 kg. The block is initially at rest on a frictionless surface. The bullet becomes embedded in the block and the block moves with a speed of 5 m/s. What was the initial speed of the bullet?

3. A 2 kg object at rest explodes into two fragments of masses 1 kg and 1 kg. The 1 kg fragment moves with a velocity of 6 m/s to the right. What is the velocity of the other 1 kg fragment?

4. A 3 kg object moving at 5 m/s to the right collides inelastically with a 2 kg object moving at 3 m/s to the left. What is their velocity after the collision?

5. A mass-spring system oscillates with a spring constant of 100 N/m and a mass of 1 kg. Calculate the angular frequency, frequency, and period of oscillation.

6. A pendulum with a length of 0.5 m swings back and forth. Calculate the angular frequency, frequency, and period of oscillation.

7. A 0.5 kg mass attached to a spring oscillates with a frequency of 2 Hz. If the amplitude of the motion is 0.1 m, calculate the maximum speed of the mass.

8. A mass of 2 kg is attached to a spring with a spring constant of 50 N/m. The mass is displaced 0.1 m from its equilibrium position and released. What is the speed of the mass when it passes through the equilibrium position?

9. A force $\vec{F} = 3t^2\hat{\imath} + 5t\hat{\jmath}$ N is applied on a 2 kg particle for 3 seconds from rest. Find the velocity and position as a function of time.

10. A variable force of $F = 5x^2$ N acts on a 1 kg particle. Calculate the work done by this force when the particle moves from $x = 0$ to $x = 2$ m.

11. A force given by $\vec{F} = (4t - 3)\hat{\imath}$ N acts on a particle of mass 2 kg, initially at rest. Find the impulse of the force at $t = 3$ s.

12. An object of mass m is subject to a force $F = -kx$ (a simple harmonic oscillator). Show that the total energy of the system (kinetic energy + potential energy) is conserved.

13. A force of $F = k/x^2$ (where k is a constant and x is the displacement) acts on an object moving along the x-axis. Find the work done by this force in moving the object from $x = a$ to $x = b$.

14. Explain the principle of conservation of momentum. In what kind of systems is momentum conserved?

15. Describe the difference between elastic and inelastic collisions. Provide an example for each.

16. What is impulse? How is it related to the change in momentum of an object?

17. In a game of pool, why does the cue ball stop when it hits another ball head-on?

18. If a car and a truck have a head-on collision, which vehicle experiences the greater force of impact? Which experiences the greater acceleration? Explain your answer.

19. Explain why, in an isolated system, momentum is always conserved but kinetic energy may not be.

20. A bullet is fired from a gun. Which object experiences the greater magnitude of momentum change, the bullet or the gun?