Physics Lecture Notes - 6

Gravity

June 7, 2023

The Law of Gravity

Gravity is a force of attraction that exists between any two masses, any two bodies, any two particles. It is mathematically described by Newton's law of universal gravitation, which states:

$$F = G\frac{m_1 m_2}{r^2}$$

where F is the force between the masses, G is the gravitational constant, m_1 and m_2 are the two masses, and r is the distance between the centers of the two masses.

History

Gravitation, the force that brings apples down to the ground and governs the motions of celestial bodies, has been a subject of human curiosity for millennia. From the musings of ancient philosophers, to the quantitative insights of Isaac Newton, to the revolutionary understanding brought forth by Albert Einstein, our understanding of gravitation has significantly evolved over the centuries.

Ancient Civilizations and Gravity

In ancient times, people observed the effects of gravity, such as the fall of objects to the ground or the motion of the Sun across the sky, but did not have a scientific understanding of why these phenomena occurred. Ancient Greeks, including Aristotle, proposed that it was the natural state of objects to be at rest, and objects moved because of natural tendencies. Aristotle posited that heavier objects fall faster than lighter ones, an idea that held sway for nearly two millennia.

Middle Ages

In the Middle Ages, thinkers like John Philoponus began to question Aristotelian physics, leading to a slow shift in understanding. The groundwork for modern physics was further laid by medieval scholars working in the Islamic world, such as Ibn Al-Haytham, who developed an early version of the law of inertia.

The Copernican Revolution

In the 16th century, Nicolaus Copernicus proposed a heliocentric model of the solar system, challenging the then-dominant geocentric model. This marked a significant step towards understanding gravitation, as it suggested that the same force which caused apples to fall to the ground might also govern the motion of celestial bodies.

Galileo Galilei

Galileo Galilei, in the late 16th and early 17th centuries, conducted experiments demonstrating that all objects, regardless of their mass, fall at the same rate in the absence of air resistance, contradicting Aristotelian physics. His work on projectile motion laid the groundwork for the concept of inertia and the realization that forces, like gravity, affect how objects move.

Isaac Newton

The next great leap in understanding gravitation came with Isaac Newton in the 17th century. His formulation of the three laws of motion and the law of universal gravitation – that every pair of particles in the universe attract each other with a force directly proportional to the product of their masses and inversely proportional to the square of the distance between their centers – provided a comprehensive framework that could explain both terrestrial and celestial phenomena. Newton's laws enabled calculations of the orbits of planets and the trajectories of projectiles.

Albert Einstein and General Relativity

Despite the success of Newtonian physics, there were certain anomalies it could not explain, like the precession of the perihelion of Mercury. Albert Einstein, in the early 20th century, proposed a radical new theory of gravitation called General Relativity, in which gravity is not a force transmitted through space, but a curvature of spacetime caused by mass and energy. This theory successfully explained the anomalies and has been confirmed by many experiments since.

Conclusion

From the initial observations of the ancients, through the landmark contributions of Galileo, Newton, and Einstein, our understanding of gravitation has grown immensely. Today, researchers continue to explore the complexities of gravity, with topics like quantum gravity and the behavior of gravity in the extreme conditions around black holes being areas of active research.

Gravitational Constant

The gravitational constant (G) is an empirical physical constant involved in the calculation of gravitational effects in Sir Isaac Newton's law of universal gravitation and in Albert Einstein's general theory of relativity.

In Newton's law, it is the proportionality constant connecting the gravitational force between two bodies with the product of their masses and the inverse square of their distance. It is given by:

$$F = G\frac{m_1 m_2}{r^2}$$

where:

- F is the force between the masses,
- G is the gravitational constant,
- m_1 and m_2 are the two masses, and
- r is the distance between the centers of the two masses.

The value of the gravitational constant is approximately:
$6.67430(15) \times 10^{-11}\,\mathrm{m}^3\,\mathrm{kg}^{-1}\,\mathrm{s}^{-2}$.

The gravitational constant, G, is one of the most difficult constants to measure to high accuracy. This is due to the inherently weak nature of the gravitational force as compared with other fundamental forces. High precision measurements of G provide a stringent test of our understanding of the theory of general relativity and the fundamental laws of physics.

Experiments to measure G have been performed at least since the time of Henry Cavendish in 1798. Modern experiments, while using Cavendish's torsion balance methodology, have improved upon his results with more accurate measuring equipment and better experimental procedures. However, the value of G remains one of the least accurately measured among the fundamental constants.

Examples

1. The force of gravity acting on an object near the Earth's surface is given by $F = mg$, where m is the mass of the object and g is the acceleration due to gravity.
2. The gravitational force between the Earth and the Moon causes the tides on Earth.

1. **Example 1:** How much force does the Earth exert on a 70 kg person?

   Solution: To solve this, we use the formula for the force of gravity: $F = G\frac{m_1 m_2}{r^2}$. Here, $m_1 = 70\,\mathrm{kg}$ (the person's mass), $m_2 = 5.972 \times 10^{24}\,\mathrm{kg}$

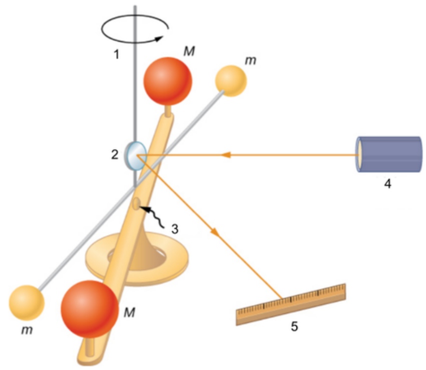

Figure 1: Illustration of the Cavendish Experiment

(Earth's mass), $r = 6.371 \times 10^6\,\mathrm{m}$ (Earth's radius), and $G = 6.674 \times 10^{-11}\,\mathrm{N\ m^2/kg^2}$. Plugging these values in, we get:

$$F = 6.674 \times 10^{-11}\,\mathrm{N\ m^2/kg^2} \times \frac{70\,\mathrm{kg} \times 5.972 \times 10^{24}\,\mathrm{kg}}{(6.371 \times 10^6\,\mathrm{m})^2} \approx 686.5\,\mathrm{N}$$

So, the force that the Earth exerts on the person is approximately 686.5 N.

2. **Example 2:** What is the gravitational force between two 1 kg masses that are 1 m apart from each other?

   Solution: Here, $m_1 = m_2 = 1\,\mathrm{kg}$, $r = 1\,\mathrm{m}$, and $G = 6.674 \times 10^{-11}\,\mathrm{N\ m^2/kg^2}$. Substituting these values into the formula, we get:

$$F = 6.674 \times 10^{-11}\,\mathrm{N\ m^2/kg^2} \times \frac{1\,\mathrm{kg} \times 1\,\mathrm{kg}}{1\,\mathrm{m}^2} = 6.674 \times 10^{-11}\,\mathrm{N}$$

   So, the gravitational force between the two 1 kg masses is 6.674×10^{-11} N. This is a very small force, which shows why we don't normally notice the gravitational forces between everyday objects around us.

Gravitational Potential Energy

The gravitational potential energy of an object at height h is given by $U = mgh$, where m is the mass of the object, g is the acceleration due to gravity, and h is the height above the reference point.

Formula Proof

To derive this formula, we consider the work done against gravity to lift an object from a reference point to a height h. This work is stored in the object as gravitational potential energy.

$$U = \int_0^h F\,dh = \int_0^h mg\,dh = mgh$$

Examples

1. A 1 kg object at a height of 10 m has a gravitational potential energy of $U = mgh = (1\,\mathrm{kg})(9.8\,\mathrm{m/s}^2)(10\,\mathrm{m}) = 98\,\mathrm{J}$.

Problems and Solutions

1. **Problem 1:** What is the work done by gravity to bring a 1 kg object from a distance of 3 Earth radii from the Earth's center to 2 Earth radii from the center?

   Solution: To find the work done by gravity, we calculate the change in gravitational potential energy, which is defined as $U = -\frac{GMm}{r}$. The work done is then the difference in potential energy at the two distances:

   $$W = -\Delta U = U_{r2} - U_{r1} = -GMm\left(\frac{1}{r2} - \frac{1}{r1}\right)$$

   where $r1 = 3R$ and $r2 = 2R$. Plugging these values in, we get:

   $$W = -GMm\left(\frac{1}{2R} - \frac{1}{3R}\right) = \frac{GMm}{6R}$$

   Substituting the values $G = 6.67 \times 10^{-11}$ $\mathrm{m}^3\mathrm{kg}^{-1}\mathrm{s}^{-2}$, $M = 5.97 \times 10^{24}$ kg, $m = 1$ kg, and $R = 6.37 \times 10^6$ m, we obtain:

   $$W = \frac{6.67 \times 10^{-11} \times 5.97 \times 10^{24} \times 1}{6 \times 6.37 \times 10^6} \approx 1.75 \times 10^7\ \mathrm{J}$$

2. **Problem 2:** A small mass m is projected vertically upwards from the surface of the Earth with speed v. Assuming no air resistance, use calculus to derive an expression for the maximum height reached by the mass in terms of G, M, R (the mass, radius of Earth) and v.

   Solution: The kinetic energy of the object at the surface of the Earth is $\frac{1}{2}mv^2$. This will be converted to gravitational potential energy as the object ascends. At maximum height, the object will have no kinetic energy, so we can equate the kinetic energy at the surface to the increase in gravitational potential energy at maximum height h:

$$\frac{1}{2}mv^2 = G\frac{Mm}{R} - G\frac{Mm}{R+h}$$

Simplifying this gives:

$$\frac{v^2}{2} = G\frac{M}{R}\left(1 - \frac{R}{R+h}\right)$$

Then, assuming $h << R$, we use the binomial approximation $\frac{1}{1+x} \approx 1 - x$ for small x to find:

$$\frac{v^2}{2} \approx G\frac{M}{R^2}h$$

3. **Problem 3:** Calculate the escape velocity from Earth's surface.

   Solution: The escape velocity v from the surface of a planet is given by $v = \sqrt{2GM/r}$ where G is the gravitational constant, M is the mass of the planet, and r is the radius of the planet. For Earth, $G = 6.674 \times 10^{-11}$ $\mathrm{m^3kg^{-1}s^{-2}}$, $M = 5.972 \times 10^{24}$ kg, and $r = 6.371 \times 10^6$ m. Substituting these values gives $v = \sqrt{2 \times 6.674 \times 10^{-11} \times 5.972 \times 10^{24}/6.371 \times 10^6} \approx 1.12 \times 10^4$ m/s.

4. **Problem 4:** A rocket is launched from the surface of a planet with an escape velocity of 10^4 m/s. If the rocket is launched with a velocity of 1.5×10^4 m/s, what is its speed when it is far away from the planet?

   Solution: The speed v of the rocket when it is far away from the planet can be found by conserving mechanical energy. The kinetic energy of the rocket at launch must equal its kinetic energy plus gravitational potential energy when it is far away from the planet. Since the gravitational potential energy goes to zero at large distances, the kinetic energy at launch must equal the kinetic energy when far away. Therefore, $v = \sqrt{2GM/r + v_0^2}$, where v_0 is the launch velocity. Since $v_0 > \sqrt{2GM/r}$ (the condition for escape), the rocket will continue to move away from the planet with a speed of $v = \sqrt{v_0^2 - 2GM/r} = \sqrt{(1.5 \times 10^4)^2 - (10^4)^2} \approx 1.0 \times 10^4$ m/s.

Kepler's Laws of Planetary Motion

Johannes Kepler, a German mathematician and astronomer, formulated three laws of planetary motion that accurately describe the motion of planets around the Sun.

Kepler's First Law

Kepler's First Law states that planets move in elliptical orbits with the Sun at one of the focal points of the ellipse. This can be expressed mathematically as

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

where a and b are the semi-major and semi-minor axes of the ellipse, respectively, and (x, y) are the coordinates of the planet.

Kepler's Second Law

Kepler's Second Law, also known as the law of equal areas, states that the line joining the planet and the Sun sweeps out equal areas in equal times. Mathematically, this can be expressed as

$$\frac{dA}{dt} = \frac{1}{2}r^2\frac{d\theta}{dt} = constant$$

where dA is the area swept out by the planet in time dt, r is the distance from the Sun to the planet, and $d\theta$ is the change in angle during time dt.

Kepler's Third Law

Kepler's Third Law states that the square of the orbital period of a planet is directly proportional to the cube of the semi-major axis of its orbit. This can be written as

$$T^2 \propto a^3$$

or, equivalently,

$$\frac{T_1^2}{T_2^2} = \frac{a_1^3}{a_2^3}$$

where T_1 and T_2 are the periods of two planets and a_1 and a_2 are the semi-major axes of their orbits.

Problems and Solutions

Problem 1

Given that the Earth's orbital period is 1 year and its average distance from the Sun (semi-major axis) is 1 astronomical unit (AU), what is the orbital period of Mars, whose average distance from the Sun is 1.52 AU?

Solution 1

Using Kepler's Third Law,

$$\frac{T_{\text{Mars}}^2}{(1\text{ year})^2} = \frac{(1.52\text{ AU})^3}{(1\text{ AU})^3}$$

Solving for T_{Mars} yields $T_{\text{Mars}} = 1.88$ years.

Problem 2

Jupiter's orbital period is 11.86 years. What is its average distance from the Sun in AU?

Solution 2

Using Kepler's Third Law,

$$\frac{(11.86 \text{ years})^2}{(1 \text{ year})^2} = \frac{a^3_{\text{Jupiter}}}{(1 \text{ AU})^3}$$

Solving for a_{Jupiter} yields $a_{\text{Jupiter}} = 5.20$ AU.

Problem 3

A newly discovered exoplanet orbits its star every 100 days. If the star has the same mass as our Sun, what is the planet's average distance from the star in AU?

Solution 3

First, convert the orbital period to years: $T = 100 \text{ days} = 100/365.25 = 0.274$ years. Then, use Kepler's Third Law:

$$\frac{(0.274 \text{ years})^2}{(1 \text{ year})^2} = \frac{a^3_{\text{exoplanet}}}{(1 \text{ AU})^3}$$

Solving for $a_{\text{exoplanet}}$ yields $a_{\text{exoplanet}} = 0.63$ AU.

Problems and Solutions

1. **Problem 1:** Consider a planet in a circular orbit around the sun with an orbital radius of 1 AU (astronomical unit, the distance from the Earth to the Sun). If another planet is in an orbit with a radius of 4 AU, how does the period of the second planet relate to that of the first according to Kepler's Third Law?

   Solution: According to Kepler's Third Law, the square of the orbital period T is proportional to the cube of the semi-major axis r of the orbit (which reduces to the radius for a circular orbit). Thus, we can write

   $$\frac{T_2^2}{T_1^2} = \frac{r_2^3}{r_1^3}$$

   Substituting $r_2 = 4r_1$ gives

   $$\frac{T_2^2}{T_1^2} = \frac{(4r_1)^3}{r_1^3} = 64$$

Taking the square root of both sides gives $T_2 = 8T_1$, so the period of the second planet is 8 times longer than that of the first.

2. **Problem 2:** A newly discovered planet orbits a star with a period of 5 years. The semi-major axis of the planet's elliptical orbit is found to be 3 AU. How does the mass of the star compare to that of the Sun?

   Solution: According to Kepler's Third Law, $T^2 \propto r^3/M$, where T is the orbital period, r is the semi-major axis of the orbit, and M is the mass of the central star. We can compare the new star-planet system to the Sun-Earth system (which has $T = 1$ year and $r = 1$ AU) to find the mass of the star:

   $$\frac{T_2^2}{T_1^2} = \frac{r_2^3/M_2}{r_1^3/M_1}$$

   Simplifying gives

   $$M_2 = M_1 \frac{T_2^2 r_1^3}{T_1^2 r_2^3}$$

   Substituting M_1 as the mass of the Sun, $T_2 = 5$ years, $T_1 = 1$ year, $r_2 = 3$ AU, and $r_1 = 1$ AU gives

   $$M_2 = M_{\text{Sun}} \frac{5^2 \cdot 1^3}{1^2 \cdot 3^3} = \frac{25}{27} M_{\text{Sun}}$$

   So the mass of the star is approximately 0.93 times the mass of the Sun.

3. **Problem 3:** A planet orbits a star in an elliptical orbit with semi-major axis a and semi-minor axis b. The star is at one of the foci of the ellipse. Show that the planet's orbital period T squares proportional to a^3 (Kepler's Third Law), given that the total energy of the planet (kinetic plus potential) is conserved.

   Solution: The semi-major axis a of the ellipse is the average distance of the planet from the star. The total energy of the planet is the sum of its kinetic and potential energy:

   $$E = \frac{1}{2}mv^2 - \frac{GmM}{r}$$

   where m is the planet's mass, v its speed, r its distance from the star, and M the mass of the star. The potential energy is negative because it takes work to separate the planet and star (to bring them from distance r to infinity).

   For a closed elliptical orbit, the total energy E must be negative: the kinetic energy is not enough to provide escape speed. Thus, for the entire orbit, the average kinetic energy is half (in absolute value) the total energy: $\langle \frac{1}{2}mv^2 \rangle = \frac{1}{2}|E|$. But the average kinetic energy is also $\frac{1}{2}m\langle v^2 \rangle = \frac{1}{2}mv_{avg}^2$, where v_{avg} is the average speed of the planet.

The average speed can be written as the total length of the orbit divided by the period: $v_{avg} = \frac{2\pi a}{T}$. Setting equal the two expressions for the average kinetic energy, we find:

$$\frac{2\pi a}{T} = \sqrt{\frac{2|E|}{m}}$$

Squaring both sides and solving for T^2, we get:

$$T^2 = \frac{4\pi^2 a^2 m}{2|E|} = \frac{2\pi^2 a^2 m}{|E|}$$

But by conservation of energy, the total energy E depends only on the semi-major axis and the masses of the planet and star: $|E| = \frac{GmM}{2a}$.

Plugging this into the previous equation gives Kepler's Third Law:

$$T^2 = \frac{4\pi^2 a^3}{GM}$$

4. **Problem 4:** A planet moves in an elliptical orbit with semi-major axis a and eccentricity e. At its farthest point from the star (the aphelion), it has speed v_a. Find an expression for the speed v_p of the planet at its closest point to the star (the perihelion).

   Solution: The distances of the planet from the star at the aphelion and perihelion are $r_a = a(1+e)$ and $r_p = a(1-e)$, respectively. By conservation of angular momentum, we have $mv_a r_a = mv_p r_p$, which gives:

$$v_p = \frac{a(1+e)}{a(1-e)} v_a = \frac{1+e}{1-e} v_a$$

Problem Set - 6

1. Calculate the gravitational force between the Earth (mass 5.98×10^{24} kg) and the Moon (mass 7.36×10^{22} kg) which are 3.84×10^{8} m apart.

2. A satellite of mass 500 kg is in a circular orbit 1000 km above the surface of the Earth. What is the gravitational force acting on the satellite?

3. A geostationary satellite orbits the Earth at a height of 36,000 km above the surface. What is the period of this satellite?

4. A 70 kg astronaut is on a space walk when the tether line to the shuttle breaks. The astronaut is able to throw a 10 kg oxygen tank in a direction away from the shuttle with a speed of 12 m/s. What would be the resulting speed of the astronaut?

5. What is the escape speed from the surface of the Sun ($mass_{sun} = 2.0 \times 10^{30}$ kg)?

6. Two stars of masses 2×10^{30} kg and 3×10^{30} kg are 3 million kilometers apart. Calculate the center of mass of the system.

7. An asteroid of mass 10^{6} kg moves in space at a speed of 20 km/s. A meteor of mass 10^{4} kg is stationary relative to the asteroid. The asteroid hits the meteor and they stick together. What is the final speed of the asteroid-meteor system?

8. A planet orbits a distant star in a circular path of radius 10^{11} m. The mass of the star is 2×10^{30} kg. Find the speed of the planet. Also, find the period of revolution of the planet.

9. The Moon's gravitational field strength is about 1/6th of the Earth's. If a person can throw a ball a maximum horizontal distance of 60 m on the Earth, how far could he throw it on the moon neglecting air resistance in both cases?

10. A car of mass 1500 kg is travelling around a roundabout of radius 50 m at a constant speed of 12 m/s. What is the frictional force between the car's tyres and the road surface that provides the necessary centripetal force?