Physics Lecture Notes - 2

Classical Mechanics

Kinematics

Kinematics is a subfield of physics, developed in classical mechanics, that describes the motion of points, bodies (objects), and systems of bodies (groups of objects) without considering the forces that cause them to move. Kinematics, as a field of study, is often referred to as the "geometry of motion" and is occasionally seen as a branch of mathematics.

A kinematics problem begins by describing the geometry of the system and declaring the initial conditions of any known values of position, velocity and/or acceleration of points within the system. Then, using arguments from geometry, the position, velocity and acceleration of any unknown parts of the system can be determined.

Scalars and Vectors

Scalars are quantities that have only magnitude, such as distance, speed, and mass. Vectors, on the other hand, have both magnitude and direction, like displacement, velocity, and force. Vectors can be represented graphically as arrows with length proportional to their magnitude and direction indicated by the arrowhead.

Scalar quantities have magnitude only. Examples of scalar quantities in physics include:

- Mass
- Temperature
- Speed
- Energy

Vector quantities have both magnitude and direction. Examples of vector quantities in physics include:

- Displacement

- Velocity
- Acceleration
- Force

Vector and Scalar Quantity Algebra

Scalar Addition and Subtraction
Scalar quantities can be easily added or subtracted using basic arithmetic. For example, if we have two masses, $m_1 = 3\,\mathrm{kg}$ and $m_2 = 4\,\mathrm{kg}$, their total mass is:

$$m_{\text{total}} = m_1 + m_2 = 3\,\mathrm{kg} + 4\,\mathrm{kg} = 7\,\mathrm{kg}$$

Vector Addition and Subtraction
Vector quantities require special consideration when performing arithmetic operations. To add or subtract vectors, we must add or subtract their components separately.

Example:
$\vec{A} = (3\,\mathrm{m}, 4\,\mathrm{m})$ and $\vec{B} = (1\,\mathrm{m}, -2\,\mathrm{m})$ are two displacement vectors. Find $\vec{A} + \vec{B}$ and $\vec{A} - \vec{B}$.
Solution:
To add vectors, we add their components separately:

$$\begin{aligned} \vec{A} + \vec{B} &= (3\,\mathrm{m} + 1\,\mathrm{m}, 4\,\mathrm{m} + (-2\,\mathrm{m})) \\ &= (4\,\mathrm{m}, 2\,\mathrm{m}) \end{aligned}$$

To subtract vectors, we subtract their components separately:

$$\begin{aligned} \vec{A} - \vec{B} &= (3\,\mathrm{m} - 1\,\mathrm{m}, 4\,\mathrm{m} - (-2\,\mathrm{m})) \\ &= (2\,\mathrm{m}, 6\,\mathrm{m}) \end{aligned}$$

Scalar Multiplication
To multiply a vector by a scalar, we multiply each component of the vector by the scalar. For example, if we have a displacement vector $\vec{C} = (2\,\mathrm{m}, 3\,\mathrm{m})$ and a scalar $k = 3$, the product is:

$$k\vec{C} = 3(2\,\mathrm{m}, 3\,\mathrm{m}) = (6\,\mathrm{m}, 9\,\mathrm{m})$$

Dot Product and Cross Product

Dot Product
The dot product (or scalar product) of two vectors is a scalar quantity. It is defined as the product of the magnitudes of the two vectors and the cosine of the angle between them.

$$\vec{A} \cdot \vec{B} = |\vec{A}||\vec{B}| \cos\theta$$

Physical Example:
Work done (W) by a constant force ($\vec{F}$) on an object that moves along a straight path with displacement ($\vec{d}$) is given by the dot product of the force and displacement vectors:

$$W = \vec{F} \cdot \vec{d}$$

Cross Product
The cross product (or vector product) of two vectors is a vector quantity. It is defined as the product of the magnitudes of the two vectors and the sine of the angle between them, with the direction given by the right-hand rule.

$$\vec{A} \times \vec{B} = |\vec{A}||\vec{B}| \sin\theta \, \hat{n}$$

Here, $\hat{n}$ is a unit vector perpendicular to both $\vec{A}$ and $\vec{B}$, and its direction is determined by the right-hand rule.

Physical Example:
Torque ($\vec{\tau}$) is the cross product of the position vector ($\vec{r}$) and the force vector ($\vec{F}$) applied to an object:

$$\vec{\tau} = \vec{r} \times \vec{F}$$

This gives the direction and magnitude of the torque that causes the object to rotate about an axis perpendicular to the plane formed by $\vec{r}$ and $\vec{F}$.

Problem 1:*An object moves 5 meters north and then 3 meters east. Calculate its total displacement and average velocity if it took 4 seconds for the entire motion.*

Solution:
Displacement is a vector quantity, so we need to find the magnitude and direction of the resulting displacement vector.

$$\begin{aligned}
\text{Total displacement} &= \sqrt{(5\,\text{m})^2 + (3\,\text{m})^2} \\
&= \sqrt{34}\,\text{m} \\
\text{Direction} &= \tan^{-1}\left(\frac{3\,\text{m}}{5\,\text{m}}\right) \\
&\approx 31^\circ \text{ east of north}
\end{aligned}$$

Average velocity is the total displacement divided by the time:

$$\text{Average velocity} = \frac{\sqrt{34}\,\text{m}}{4\,\text{s}} = \frac{\sqrt{34}}{4} \frac{\text{m}}{\text{s}}$$

Problem 2:
Calculate the average speed of a car that travels 100 km in 2 hours.

Solution:
Average speed is a scalar quantity and is calculated as the total distance divided by the time taken.

$$\text{Average speed} = \frac{\text{Total distance}}{\text{Time taken}} = \frac{100\,\text{km}}{2\,\text{h}} = 50\,\text{km/h}$$

Problem 3:
A plane is flying at a velocity of 300 m/s east. A wind blows with a velocity of 50 m/s towards the north. Determine the plane's resultant velocity.

Solution:
The plane's velocity and wind velocity are vector quantities. We need to find the magnitude and direction of the resultant velocity vector. We can use the Pythagorean theorem and the tangent function to solve this problem.

$$\begin{aligned}
\text{Resultant velocity} &= \sqrt{(300\,\text{m/s})^2 + (50\,\text{m/s})^2} \\
&= \sqrt{90{,}000\,\text{m}^2/\text{s}^2 + 2{,}500\,\text{m}^2/\text{s}^2} \\
&= \sqrt{92{,}500\,\text{m}^2/\text{s}^2} = 305\,\text{m/s} \\
\text{Direction} &= \tan^{-1}\left(\frac{50\,\text{m/s}}{300\,\text{m/s}}\right) \\
&\approx 9.5^\circ \text{ north of east}
\end{aligned}$$

Motion in One Dimension

One-dimensional motion is motion along a straight line. The key quantities in one-dimensional motion are displacement, velocity, and acceleration.

- Displacement (x) is the change in position, which can be positive or negative.
- Velocity (v) is the rate of change of displacement with respect to time, which can also be positive or negative.
- Acceleration (a) is the rate of change of velocity with respect to time, and it can be positive (increasing velocity) or negative (decreasing velocity).

Acceleration, Velocity, and Displacement

Acceleration (a) is the rate of change of velocity (v) with respect to time (t). Velocity is the rate of change of displacement (s) with respect to time. The equations of motion for constant acceleration are:

$$v = v_0 + at \tag{1}$$

$$s = v_0 t + \frac{1}{2}at^2 \tag{2}$$

$$v^2 = v_0^2 + 2as \tag{3}$$

where v_0 is the initial velocity.

Example:

A car accelerates from rest at a constant rate of 2 m/s² for 5 seconds. Calculate its final velocity and the distance traveled.

Solution:

Since the car starts from rest, its initial velocity (v_0) is 0 m/s. We're given the acceleration ($a = 2\,\mathrm{m/s}^2$) and the time taken ($t = 5\,\mathrm{s}$). We can use the following equations of motion to find the final velocity and the distance traveled:

$$v = v_0 + at$$

$$x = v_0 t + \frac{1}{2}at^2$$

$$\begin{aligned} v &= 0\,\mathrm{m/s} + (2\,\mathrm{m/s}^2)(5\,\mathrm{s}) \\ &= (2\,\mathrm{m/s}^2)(5\,\mathrm{s}) \\ &= 10\,\mathrm{m/s} \end{aligned}$$

Now, we can find the distance traveled:

$$\begin{aligned} x &= 0\,\mathrm{m}(5\,\mathrm{s}) + \frac{1}{2}(2\,\mathrm{m/s}^2)(5\,\mathrm{s})^2 \\ &= \frac{1}{2}(2\,\mathrm{m/s}^2)(25\,\mathrm{m}^2/\mathrm{s}^2) \\ &= (1\,\mathrm{m/s}^2)(25\,\mathrm{m}^2/\mathrm{s}^2) \\ &= 25\,\mathrm{m} \end{aligned}$$

So, the car's final velocity is 10 m/s, and it travels a distance of 25 meters.

One-Dimensional Motion in Perspective of Derivatives

In one-dimensional motion, calculus can be used to describe the relationship between displacement, velocity, and acceleration.

Derivatives and Motion

- **Velocity:** Velocity (v) is the first derivative of displacement (x) with respect to time (t):
$$v(t) = \frac{dx}{dt}$$
- **Acceleration:** Acceleration (a) is the first derivative of velocity or the second derivative of displacement with respect to time:
$$a(t) = \frac{dv}{dt} = \frac{d^2x}{dt^2}$$

Example:

An object is moving along a straight line with a displacement function given by $x(t) = 3t^2 - 2t + 1$, where x is in meters and t is in seconds. Find the velocity and acceleration functions, and determine the object's velocity and acceleration at $t = 3$ s.

Solution:

To find the velocity function, take the first derivative of the displacement function with respect to time:

$$\begin{aligned} v(t) &= \frac{d}{dt}(3t^2 - 2t + 1) \\ &= 6t - 2 \end{aligned}$$

To find the acceleration function, take the first derivative of the velocity function with respect to time:

$$\begin{aligned} a(t) &= \frac{d}{dt}(6t - 2) \\ &= 6 \end{aligned}$$

Now, we can find the velocity and acceleration at $t = 3\,\mathrm{s}$:

$$\begin{aligned} v(3) &= 6(3) - 2 \\ &= 16\,\mathrm{m/s} \\ a(3) &= 6 \\ &= 6\,\mathrm{m/s}^2 \end{aligned}$$

The object's velocity at $t = 3\,\mathrm{s}$ is 16 m/s, and its acceleration is constant at 6 m/s**²**.

Example A car accelerates uniformly from rest. Its acceleration function is given by $a(t) = 4\,m/s^2$. Find the velocity and displacement functions, and the car's velocity and displacement after 5 seconds.

Solution:
To find the velocity function, integrate the acceleration function with respect to time:

$$\begin{aligned} v(t) &= \int a(t)\,dt \\ &= \int 4\,\mathrm{m/s^2}\,dt \\ &= 4t + C_1 \end{aligned}$$

Since the car starts from rest, $v(0) = 0$, and therefore $C_1 = 0$. So the velocity function is:

$$v(t) = 4t$$

To find the displacement function, integrate the velocity function with respect to time:

$$\begin{aligned} x(t) &= \int v(t)\,dt \\ &= \int 4t\,dt \\ &= 2t^2 + C_2 \end{aligned}$$

Assuming the car starts at $x = 0$, $C_2 = 0$, and the displacement function is:

$$x(t) = 2t^2$$

Now, we can find the velocity and displacement after 5 seconds:

$$\begin{aligned} v(5) &= 4(5) \\ &= 20\,\mathrm{m/s} \\ x(5) &= 2(5)^2 \\ &= 50\,\mathrm{m} \end{aligned}$$

The car's velocity after 5 seconds is 20 m/s, and its displacement is 50 meters.

Two-Dimensional Motion

Examples for Two-Dimensional Motion

Example 1: *A projectile is launched with an initial velocity of $20\,m/s$ at an angle of $30°$ above the horizontal. Determine its horizontal and vertical components of velocity, and its initial position vector.*

Solution:
To find the horizontal and vertical components of velocity, we use the trigonometric functions:

$$v_x = v_0 \cos\theta = 20\,\mathrm{m/s} \cdot \cos(30^\circ) = 20\,\mathrm{m/s} \cdot \frac{\sqrt{3}}{2} = 10\sqrt{3}\,\mathrm{m/s}$$
$$v_y = v_0 \sin\theta = 20\,\mathrm{m/s} \cdot \sin(30^\circ) = 20\,\mathrm{m/s} \cdot \frac{1}{2} = 10\,\mathrm{m/s}$$

Assuming the projectile is launched from the origin, its initial position vector is $\vec{r_0} = (0, 0)$.

Example 2: *A ball is thrown horizontally from a cliff 80 meters high with an initial speed of 15 m/s. Calculate the time it takes for the ball to reach the ground and its horizontal distance from the base of the cliff.*

Solution:
The vertical motion is independent of the horizontal motion. We can use the kinematic equation to find the time it takes for the ball to hit the ground:

$$y = y_0 + v_{y0}t + \frac{1}{2}at^2$$

Here, $y_0 = 80\,\mathrm{m}$, $v_{y0} = 0$, $a = -9.8\,\mathrm{m/s}^2$ (downward acceleration due to gravity), and $y = 0$ (ground level). So,

$$0 = 80 - \frac{1}{2}(9.8)t^2$$

Solving for t, we get $t \approx 4.04\,\mathrm{s}$.
Now, we can calculate the horizontal distance using the horizontal velocity:

$$x = x_0 + v_{x0}t$$

Here, $x_0 = 0$, $v_{x0} = 15\,\mathrm{m/s}$, and $t = 4.04\,\mathrm{s}$. Thus,

$$x \approx 15(4.04) = 60.6\,\mathrm{m}$$

The ball takes approximately 4.04 seconds to reach the ground and travels 60.6 meters horizontally.

Example 3: *A projectile is launched with an initial velocity of $25\,m/s$ at an angle of 60° above the horizontal. Determine the maximum height reached and the range of the projectile.*

Solution:
First, find the horizontal and vertical components of the initial velocity:

$$v_x = v_0 \cos\theta = 25\,\mathrm{m/s} \cdot \cos(60^\circ) = 25\,\mathrm{m/s} \cdot \frac{1}{2} = 12.5\,\mathrm{m/s}$$
$$v_y = v_0 \sin\theta = 25\,\mathrm{m/s} \cdot \sin(60^\circ) = 25\,\mathrm{m/s} \cdot \frac{\sqrt{3}}{2} \approx 21.65\,\mathrm{m/s}$$

To find the maximum height, we use the kinematic equation when the vertical velocity becomes zero:

$$v_y^2 = v_{y0}^2 + 2a(y - y_0)$$

Here, $v_y = 0$, $v_{y0} = 21.65\,\mathrm{m/s}$, $a = -9.8\,\mathrm{m/s}^2$, and $y_0 = 0$. So,

$$0 = (21.65)^2 - 2(9.8)(y - 0)$$

Solving for y, we get $y \approx 23.91\,\mathrm{m}$.

To find the range, we first calculate the time of flight. The time of flight is twice the time it takes for the projectile to reach its maximum height:

$$t_{flight} = 2\frac{v_{y0}}{g} \approx 2\frac{21.65\,\mathrm{m/s}}{9.8\,\mathrm{m/s}^2} \approx 4.42\,\mathrm{s}$$

Now, we can calculate the range using the horizontal velocity and the time of flight:

$$R = v_x t_{flight} \approx 12.5\,\mathrm{m/s} \cdot 4.42\,\mathrm{s} \approx 55.25\,\mathrm{m}$$

The maximum height reached by the projectile is approximately 23.91 meters, and its range is approximately 55.25 meters.

Problem Set - 2

1. Calculate the sum, difference, and scalar multiplication of the following vectors:

$$\vec{A} = 3\hat{\imath} - 4\hat{\jmath} + 2\hat{k}$$
$$\vec{B} = -\hat{\imath} + 5\hat{\jmath} + 3\hat{k}$$

2. Compute the dot product and cross product of the vectors $\vec{A}$ and $\vec{B}$ given in Problem 1.

3. An object moves along the x-axis with a position function given by $x(t) = 4t^3 - 6t^2 + 2t$, where x is in meters and t is in seconds. Find the velocity and acceleration functions, and determine the object's velocity and acceleration at $t = 2\,\mathrm{s}$.

4. A car accelerates uniformly from rest with an acceleration of $5\,\mathrm{m/s^2}$. Determine the car's velocity after 8 seconds and the distance it traveled during this time.

5. A projectile is launched with an initial velocity of $30\,\mathrm{m/s}$ at an angle of $45°$ above the horizontal. Calculate the maximum height reached by the projectile and its range.

6. A stone is thrown horizontally with a speed of $10\,\mathrm{m/s}$ from the top of a 50-meter-high cliff. Calculate the time it takes for the stone to hit the ground and the horizontal distance it travels.

7. Two vectors $\vec{A}$ and $\vec{B}$ have magnitudes $|\vec{A}| = 5$ and $|\vec{B}| = 8$, and the angle between them is $60°$. Calculate the magnitude of their sum $\vec{C} = \vec{A} + \vec{B}$.

8. A particle moves in a plane with position vector $\vec{r}(t) = (3t^2 - 4t)\hat{\imath} + (5t - 2)\hat{\jmath}$, where t is in seconds. Determine the particle's velocity and acceleration vectors at $t = 3\,\mathrm{s}$.

9. A ball is kicked with an initial velocity of $22\,\mathrm{m/s}$ at an angle of $30°$ above the horizontal on a flat field. Calculate the time of flight, maximum height reached, and horizontal range of the ball.

10. A car traveling at $25\,\mathrm{m/s}$ decelerates uniformly to a stop in 8 seconds. Calculate the car's acceleration and the distance it traveled during this time.

11. A projectile is launched horizontally from the edge of a cliff with an initial speed of $15\,\mathrm{m/s}$. If the cliff is 90 meters high, calculate the time it takes for the projectile to hit the ground and its horizontal distance from the base of the cliff.

12. A cyclist accelerates uniformly from rest to a speed of 7.2 m/s over a distance of 20 m. Calculate the cyclist's acceleration and the time taken to cover this distance.

13. A boat travels in a river with a velocity of 8 m/s at an angle of 30° upstream relative to the river's flow. The river flows at a constant speed of 4 m/s eastward. Determine the boat's velocity vector with respect to an observer on the shore.

14. A particle moves in a plane with a position vector given by $\vec{r}(t) = (6t - 2)\hat{\imath} + (3t^2 + 1)\hat{\jmath}$, where t is in seconds. Calculate the particle's velocity and acceleration vectors at $t = 4\,\mathrm{s}$, and determine the angle between the velocity and acceleration vectors at this time.