Ph 20 - Assignment 2

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1 Derivation of Simpson's Formula

1.1 Extended Formula

We define $h_N = \frac{(b-a)}{N}, and x_0 = a, x_1 = a + h_N, x_2 = a + 2h_N, ..., x_N = b$. We then find, using Simpson's Rule, that:

$$\begin{aligned}
\int_a^b f(x)\,\mathrm{d}x &= \int_{x_0}^{x_1} f(x)\,\mathrm{d}x + \int_{x_1}^{x_2} f(x)\,\mathrm{d}x + ... + \int_{x_{N-1}}^{x_N} f(x)\,\mathrm{d}x \\
&\approx h_N\left(\frac{f(x_0)}{6} + \frac{4f(\frac{x_0+x_1}{2})}{6} + \frac{f(x_1)}{6}\right) \\
&+ h_N\left(\frac{f(x_1)}{6} + \frac{4f(\frac{x_1+x_2}{2})}{6} + \frac{f(x_2)}{6}\right) \\
&+ ... + h_N\left(\frac{f(x_{N-1})}{6} + \frac{4f(\frac{x_{N-1}+x_N}{2})}{6} + \frac{f(x_N)}{6}\right) \\
&= h_N\left(\frac{f(x_0)}{6} + \frac{4f(\frac{x_0+x_1}{2})}{6} + \frac{f(x_1)}{3} + \frac{4f(\frac{x_1+x_2}{2})}{6} + \frac{f(x_2)}{3}\right. \\
&\left. + ... + \frac{f(x_{N-1})}{3} + \frac{4f(\frac{x_{N-1}+x_N}{2})}{6} + \frac{f(x_N)}{6}\right)
\end{aligned}$$

The last line represents the extended Simpson's Formula.

1.2 Local Error

We write $f(x)$ as a Taylor sum centered at a:

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \frac{f''''(\eta)}{4!}(x-a)^4$$

where $x, \eta \in [a, b]$ and the last term is the Lagrange remainder.

By elementary integration, we have

$$I = f(a)H + f'(a)\frac{H^2}{2!} + f''(a)\frac{H^3}{3!} + f'''(a)\frac{H^4}{4!} + f''''(\eta)\frac{H^5}{5!}$$

Now we find the estimate of the integral using Simpson's formula, I_{simp}, estimating the $f(\frac{a+b}{2})$ and $f(c)$ terms using the Taylor sum above:

$$\begin{aligned}
I_{simp} &= H\Bigg(\frac{f(a)}{6} + \frac{4}{6}\big(f(a) + f'(a)\frac{H}{2} + \frac{f''(a)}{2!}(\frac{H}{2})^2 \\
&+ \frac{f'''(a)}{3!}(\frac{H}{2})^3 + \frac{f''''(\eta)}{4!}(\frac{H}{2})^4\big)\Bigg) \\
&= f(a)H + f'(a)\frac{H^2}{2} + f''(a)\frac{H^3}{6} + f'''(a)\frac{H^4}{24} \\
&+ f''''(\eta)\frac{5H^5}{24}
\end{aligned}$$

Therefore, we find

$$I_{simp} - I = f''''(\eta)\frac{H^5}{5}$$

and conclude that Simpson's Formula is locally of fifth order in H, since

$$I = I_{simp} + O(H^5).$$

The global error of Simpson's Formula is approximately

$$-f''''(\xi)\frac{h_N^5}{5} = -(b-a)f''''(\xi)\frac{h_N^4}{5}$$

where $\xi \in [a, b]$. Therefore the Extended Simpson Formula is globally of fourth order in h_N.