Assignment 5

Part 1

We choose as example problem the Lotka-Volterra equations - the differential equations describing predator-prey relations as follows:

1. A prey population x increases at a rate x'[t] = Ax (proportional to the number of prey) but is simultaneously destroyed by predators at a rate x'[t] = -Bxy (proportional to the product of the numbers of prey and predators).

2. A predator population y decreases at a rate y'[t] = -Cy (proportional to the number of predators), but increases at a rate y'[t] = Dxy (again proportional to the product of the numbers of prey and predators).

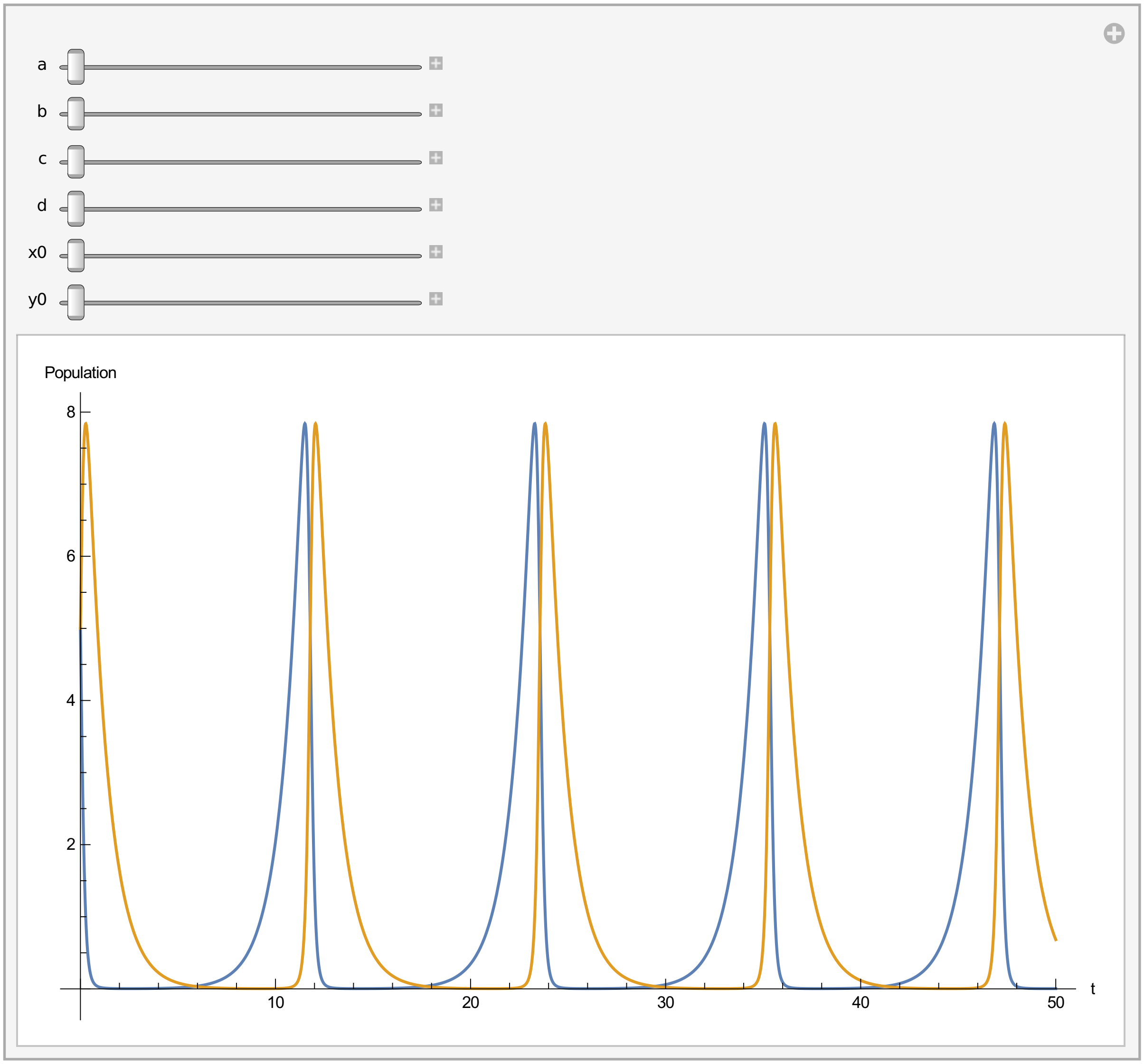
This gives the set of coupled equations:

$$x'[t] = Ax - Bxy$$
$$y'[t] = - Cy + Dxy$$

[http://mathworld.wolfram.com/Lotka-VolterraEquations.html]

In[113]:=
```
Manipulate[Plot[Evaluate[{x[t], y[t]} /. NDSolve[{x'[t] == a x[t] - b x[t] y[t],
      y'[t] == -c y[t] + d x[t] y[t], x[0] == x0, y[0] == y0}, {x[t], y[t]}, {t, 0, 50}]],
   {t, 0, 50}, PlotRange → All, AxesLabel → {"t", "Population"}, ImageSize → Large],
  {a, 1, 4}, {b, 1, 4}, {c, 1, 4}, {d, 1, 4}, {x0, 5, 20}, {y0, 5, 20}]
```

Out[113]=



Part 2

In[127]:=
```
SerCos[x_, n_] := Normal[Series[Cos[x1], {x1, 0, n}]] /. x1 → x
SerSin[x_, n_] := Normal[Series[Sin[x2], {x2, 0, n}]] /. x2 → x

(* Difference between SerSin and Sin for n=3 *)
```

In[141]:=
```
N[SerSin[2, 3] - Sin[2]]
```

Out[141]= -0.242631

In[133]:=
```
N[SerSin[1, 3] - Sin[1]]
```
Out[133]= -0.00813765

In[142]:=
```
N[SerSin[0.5, 3] - Sin[0.5]]
```
Out[142]= -0.000258872

In[148]:=
```
N[SerSin[0.01, 3] - Sin[0.01]]
```
Out[148]= -8.33332×10^{-13}

```
(* Difference between SerSin and Sin for n=5 *)
```

In[144]:=
```
N[SerSin[2, 5] - Sin[2]]
```
Out[144]= 0.0240359

In[145]:=
```
N[SerSin[1, 5] - Sin[1]]
```
Out[145]= 0.000195682

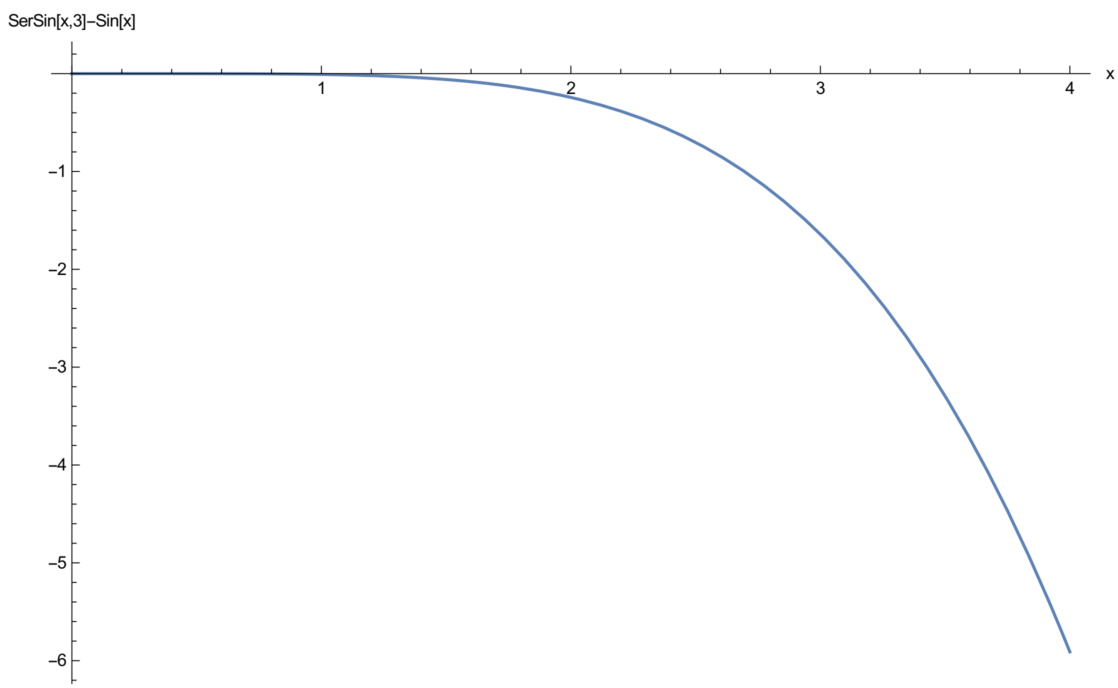
In[146]:=
```
N[SerSin[0.5, 5] - Sin[0.5]]
```
Out[146]= 1.54473×10^{-6}

In[147]:=
```
N[SerSin[0.01, 5] - Sin[0.01]]
```
Out[147]= 1.73472×10^{-18}

We now plot the SerSin[x,3] - Sin[x] against x.

In[150]:=
```
Plot[SerSin[x, 3] - Sin[x], {x, 0, 4}, PlotRange → All,
 ImageSize → Large, AxesLabel → {"x", "SerSin[x,3]-Sin[x]"}]
```

Out[150]=



We now investigate the behavior of $(\text{SerSin}[x, n]^2 + \text{SerCos}[x, n]^2)$. First, consider algebraically:

In[163]:=
```
SerSinSq[x_, n_] := Normal[Series[Sin[x1]^2, {x1, 0, n}]] /. x1 → x
SerCosSq[x_, n_] := Normal[Series[Cos[x2]^2, {x2, 0, n}]] /. x2 → x
```

In[171]:=
```
SerSin[x, 5]
```

Out[171]= $x - \frac{x^3}{6} + \frac{x^5}{120}$

In[169]:=
```
(* This is the expansion of SerSin[x,3]^2. *)
```

In[172]:=
```
Normal[Series[SerSin[x, 5]^2, {x, 0, 10}]]
```

Out[172]= $x^2 - \frac{x^4}{3} + \frac{2 x^6}{45} - \frac{x^8}{360} + \frac{x^{10}}{14400}$

In[170]:=
```
(* This is the power series of Sin[x]^2. *)
```

In[173]:=
```
SerSinSq[x, 10]
```

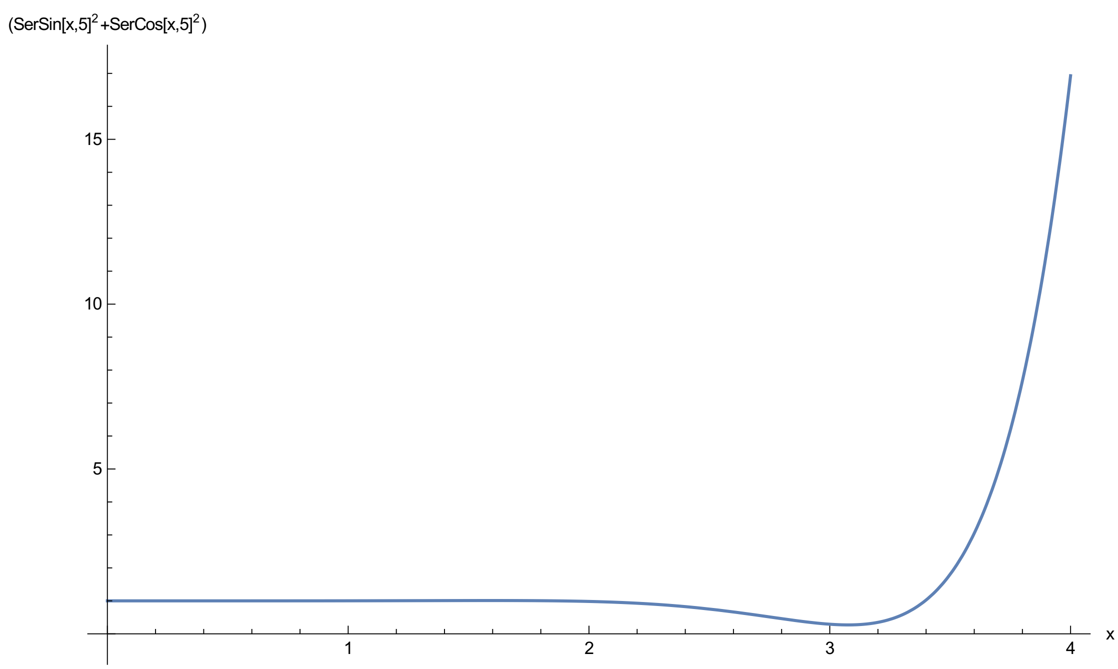
Out[173]= $x^2 - \frac{x^4}{3} + \frac{2 x^6}{45} - \frac{x^8}{315} + \frac{2 x^{10}}{14175}$

Clearly the series for $\text{Sin}[x]^2$ does not match up with $\text{SerSin}[x, n]^2$. We can observe that this is because squaring the SerSin[x,n] takes care of all terms correctly up to the n^{th} term - but it also introduces higher order terms that are bound to be inaccurate, since SerSin[x,n] is missing some terms whose product would sum to such higher order terms in SerSinSq[x,n], as we can see in the examples above.

We can see the behavior that the error due to the incorrect higher order terms induces in $(\text{SerSin}[x, n]^2 + \text{SerCos}[x, n]^2)$:

In[176]:=
```
Plot[SerSin[x, 5]^2 + SerCos[x, 5]^2, {x, 0, 4}, PlotRange → All,
 ImageSize → Large, AxesLabel → {"x", "(SerSin[x,5]^2+SerCos[x,5]^2)"}]
```

Out[176]=

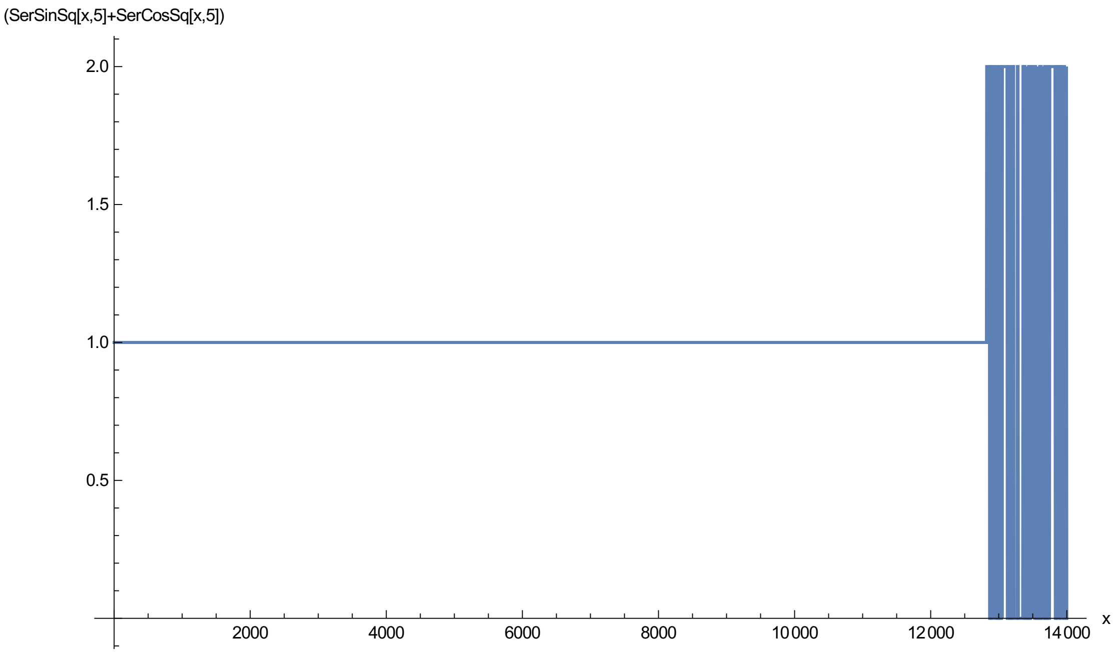


This confirms our suspicions: as x gets large, the incorrect higher order error terms generate a polynomially increasing error in (SerSin[x, n]^2 + SerCos[x, n]^2).

We now observe and compare the behavior of (SerSinSq[x, n] + SerCosSq[x, n]), where SerSinSq[x,n] and SerCosSq[x,n] are the correct power series for Sin^2[x] and Cos^2[x] respectively:

In[184]:=
```
Plot[SerSinSq[x, 5] + SerCosSq[x, 5], {x, 0, 14 000}, PlotRange → All,
 ImageSize → Large, AxesLabel → {"x", "(SerSinSq[x,5]+SerCosSq[x,5])"}]
```

Out[184]=



As we can see, this sum remains close to the expected value of 1 for a much longer time than the previous series.

Part 3

In[185]:=
```
Rx[θ_] := {{1, 0, 0}, {0, Cos[θ], -Sin[θ]}, {0, Sin[θ], Cos[θ]}}
Ry[ξ_] := {{Cos[ξ], 0, Sin[ξ]}, {0, 1, 0}, {-Sin[ξ], 0, Cos[ξ]}}
Rz[ϕ_] := {{Cos[ϕ], -Sin[ϕ], 0}, {Sin[ϕ], Cos[ϕ], 0}, {0, 0, 1}}
```

In[188]:=
```
Rot3[ψ_, θ_, ϕ_] := Rz[ψ].Rx[θ].Rz[ϕ]
```

In[198]:=
```
MatrixForm[Rot3[ψ, θ, ϕ] // Simplify, TableSpacing → {3, 3}]
```

Out[198]//MatrixForm=

$$\left(\begin{array}{ccc} \text{Cos}[\phi]\ \text{Cos}[\psi] - \text{Cos}[\theta]\ \text{Sin}[\phi]\ \text{Sin}[\psi] & -\text{Cos}[\psi]\ \text{Sin}[\phi] - \text{Cos}[\theta]\ \text{Cos}[\phi]\ \text{Sin}[\psi] & \text{Sin}[\theta]\ \text{Sin} \\ \text{Cos}[\theta]\ \text{Cos}[\psi]\ \text{Sin}[\phi] + \text{Cos}[\phi]\ \text{Sin}[\psi] & \text{Cos}[\theta]\ \text{Cos}[\phi]\ \text{Cos}[\psi] - \text{Sin}[\phi]\ \text{Sin}[\psi] & -\text{Cos}[\psi]\ \text{Sir} \\ \text{Sin}[\theta]\ \text{Sin}[\phi] & \text{Cos}[\phi]\ \text{Sin}[\theta] & \text{Cos}[\theta] \end{array}\right.$$

In[203]:=
```
Rot3Inverse[ψ_, θ_, ϕ_] := Rz[-ϕ].Rx[-θ].Rz[-ψ]
```

In[204]:=
```
MatrixForm[Rot3Inverse[ψ, θ, ϕ] // Simplify, TableSpacing → {3, 3}]
```

Out[204]//MatrixForm=

$$\begin{pmatrix} \text{Cos}[\phi]\,\text{Cos}[\psi] - \text{Cos}[\theta]\,\text{Sin}[\phi]\,\text{Sin}[\psi] & \text{Cos}[\theta]\,\text{Cos}[\psi]\,\text{Sin}[\phi] + \text{Cos}[\phi]\,\text{Sin}[\psi] & \text{Sin}[\theta]\,\text{Sin}[\\ -\text{Cos}[\psi]\,\text{Sin}[\phi] - \text{Cos}[\theta]\,\text{Cos}[\phi]\,\text{Sin}[\psi] & \text{Cos}[\theta]\,\text{Cos}[\phi]\,\text{Cos}[\psi] - \text{Sin}[\phi]\,\text{Sin}[\psi] & \text{Cos}[\phi]\,\text{Sin}[\\ \text{Sin}[\theta]\,\text{Sin}[\psi] & -\text{Cos}[\psi]\,\text{Sin}[\theta] & \text{Cos}[\theta] \end{pmatrix}$$

In[206]:=
```
Rot3[ψ, θ, ϕ].Rot3Inverse[ψ, θ, ϕ] // Simplify // MatrixForm
```

Out[206]//MatrixForm=

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

The product of Rot3 and Rot3Inverse gives us the identity transformation, as expected. We now try an alternative method for finding an inverse, and show that the inverse obtained is equivalent to the one we have found.

In[209]:=
```
Inverse[Rot3[ψ, θ, ϕ]].Rot3[ψ, θ, ϕ] // Simplify // MatrixForm
```

Out[209]//MatrixForm=

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

In[211]:=
```
Inverse[Rot3[ψ, θ, ϕ]] - Rot3Inverse[ψ, θ, ϕ] // Simplify // MatrixForm
```

Out[211]//MatrixForm=

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$