

# Ph21 Assignment 4 - Bayesian Analysis

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## 1 Coin Toss

We use the likelihood distribution for Binomial processes (given in the reading for this assignment). We then calculate the posterior by assuming a constant prior and a Gaussian prior. Here are the results of our simulations:

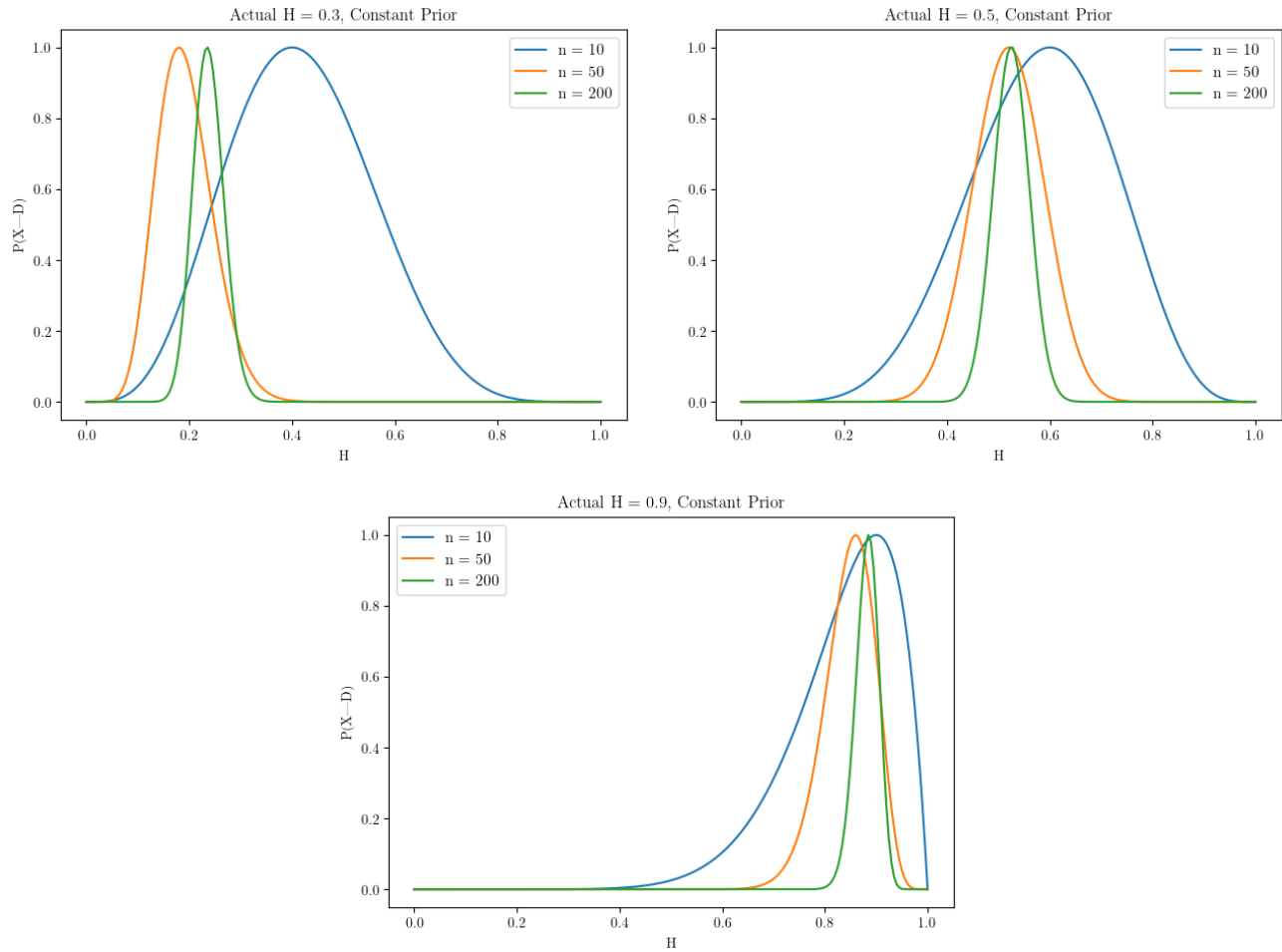


Figure 1: Posterior distributions for multiple  $n$ , for a uniform prior. Note that the plots have been normalized. Observe that the peak becomes narrower and closer to the actual  $H$  for larger  $n$ .

We conduct the same simulation again, but this time we assume a Gaussian prior. The results are given below for different means and standard deviations of the priors:

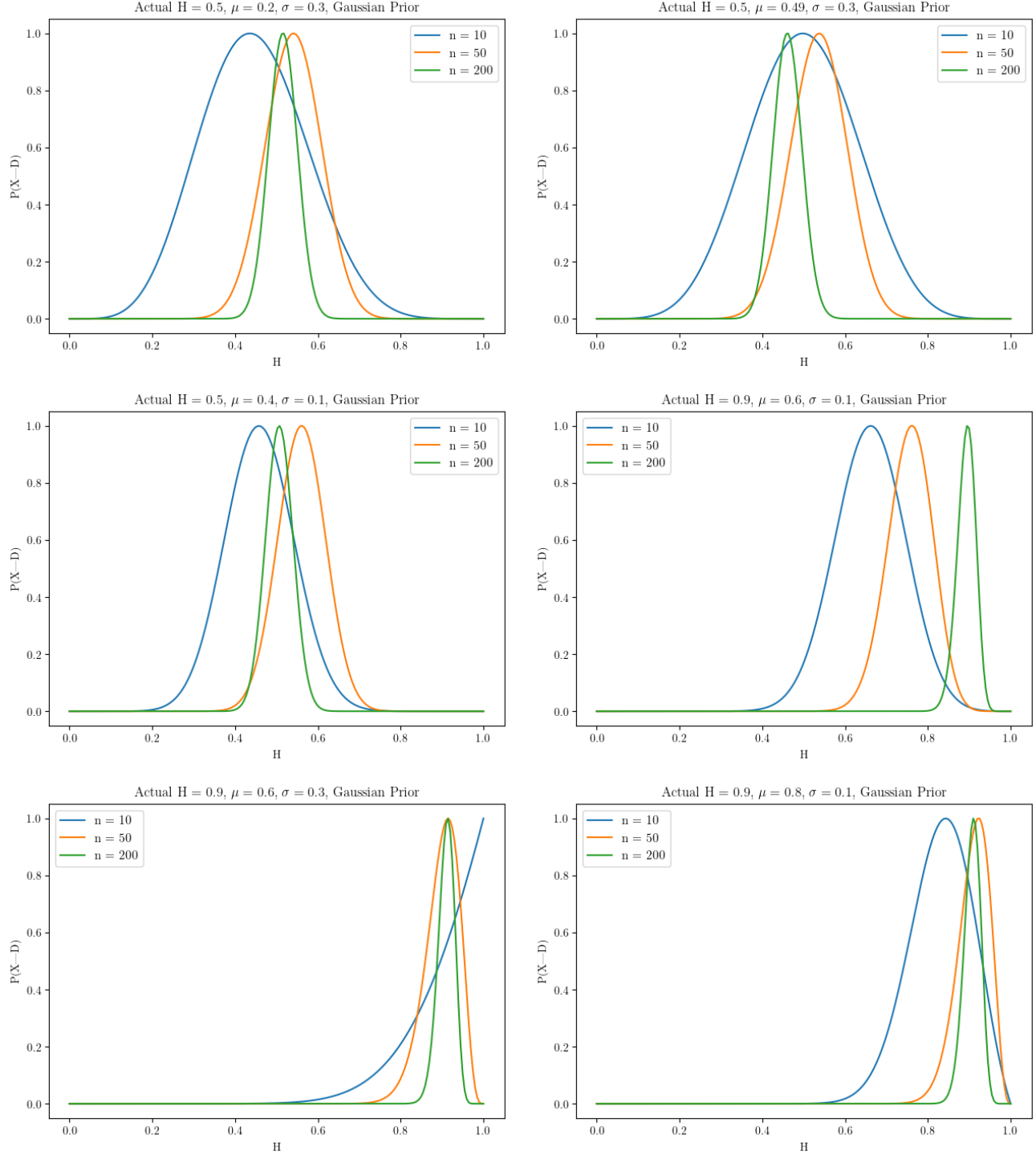


Figure 2: Posterior distributions for multiple  $n$ , for a Gaussian prior. Again, the plots have been normalized. Observe that the peak becomes narrower and closer to the actual  $H$  for larger  $n$ , and that the peak converges much faster to the actual  $H$  for  $\mu$  within small  $\sigma$  of  $H$ .

## 2 The lighthouse problem

### 2.1 PDF of the lighthouse problem

Let the azimuth of the  $k^{th}$  pulse as measured clockwise from the normal from the lighthouse to the shore be given by  $\theta_k$ . We assume that  $\theta_k \sim \text{Uniform}(0, 2\pi)$ .

Using the geometry of the problem, we can see that the CDF, say  $F_k = P(x_k \leq x)$  where  $x$  is some position on the shore, of the distribution of  $x_k$  is given by:

$$F_k = \frac{1}{4} + \frac{1}{2\pi} \arctan\left(\frac{x - \alpha}{\beta}\right)$$

Then we differentiate w.r.t.  $x$  and find the PDF, say  $p_k(x)$  of  $x_k$ :

$$p_k(x) = \frac{\beta/2\pi}{(x - \alpha)^2 + \beta^2}$$

This is known as a Lorentzian or Cauchy distribution, and is our likelihood function.

### 2.2 Posterior for known $\beta$

For this section and for the next, we choose "true" values for our parameters of  $\alpha = 1.0$  and  $\beta = 1.5$ . We write a script to generate  $x_k$  data for a total number of pulses  $n$ , and use the data to calculate the posterior assuming a Gaussian prior (since this showed much better convergence than a constant prior, for instance). We start by assuming  $\beta$  is known, and plot the posterior for  $\alpha$ . The results are shown below:

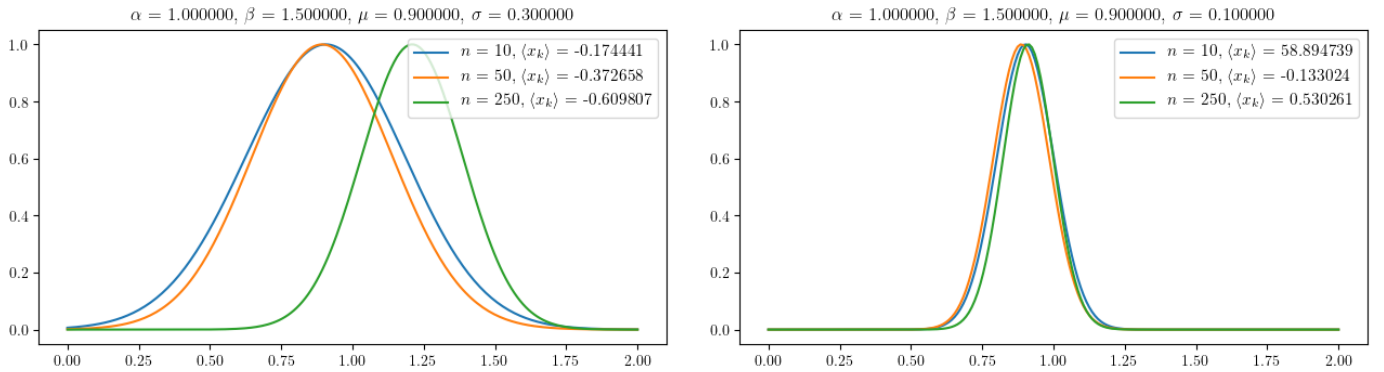


Figure 3: Posterior distributions for multiple  $n$ , for a Gaussian prior and known  $\alpha$ . Note that the plots have been normalized. Observe that the peak becomes narrower and closer to the actual  $\alpha$  for larger  $n$ .

Interestingly enough, we can observe that  $\langle x_k \rangle$  is a very poor estimator of  $\alpha$ . This is to be expected: from our  $p_k(x)$ , we can observe that the Cauchy distribution has no finite second moment (and therefore no finite variance), violating one of the assumptions of the Central Limit Theorem.

For instance, we can use a change in variables to transform our PDF to one of the form  $p_k(x) = \frac{1}{1+x^2}$ . Then,

$$\begin{aligned}
\mathbb{E}[X^2] &\propto \int_{-\infty}^{\infty} \frac{x^2}{1+x^2} dx \\
&= \int_{-\infty}^{\infty} 1 - \frac{1}{1+x^2} dx \\
&= \int_{-\infty}^{\infty} dx - \int_{-\infty}^{\infty} \frac{1}{1+x^2} dx \\
&= \int_{-\infty}^{\infty} dx - \pi \\
&= \infty
\end{aligned}$$

### 2.3 Posterior for unknown $\alpha$ and $\beta$

We calculate the posterior distribution over a grid of  $\alpha$  and  $\beta$  values, assuming a Gaussian prior for both parameters. We make use of the a dataset  $x_k$  generated in the same way as before, with the same "true" values of  $\alpha$  and  $\beta$ . The results are shown below on a color map:

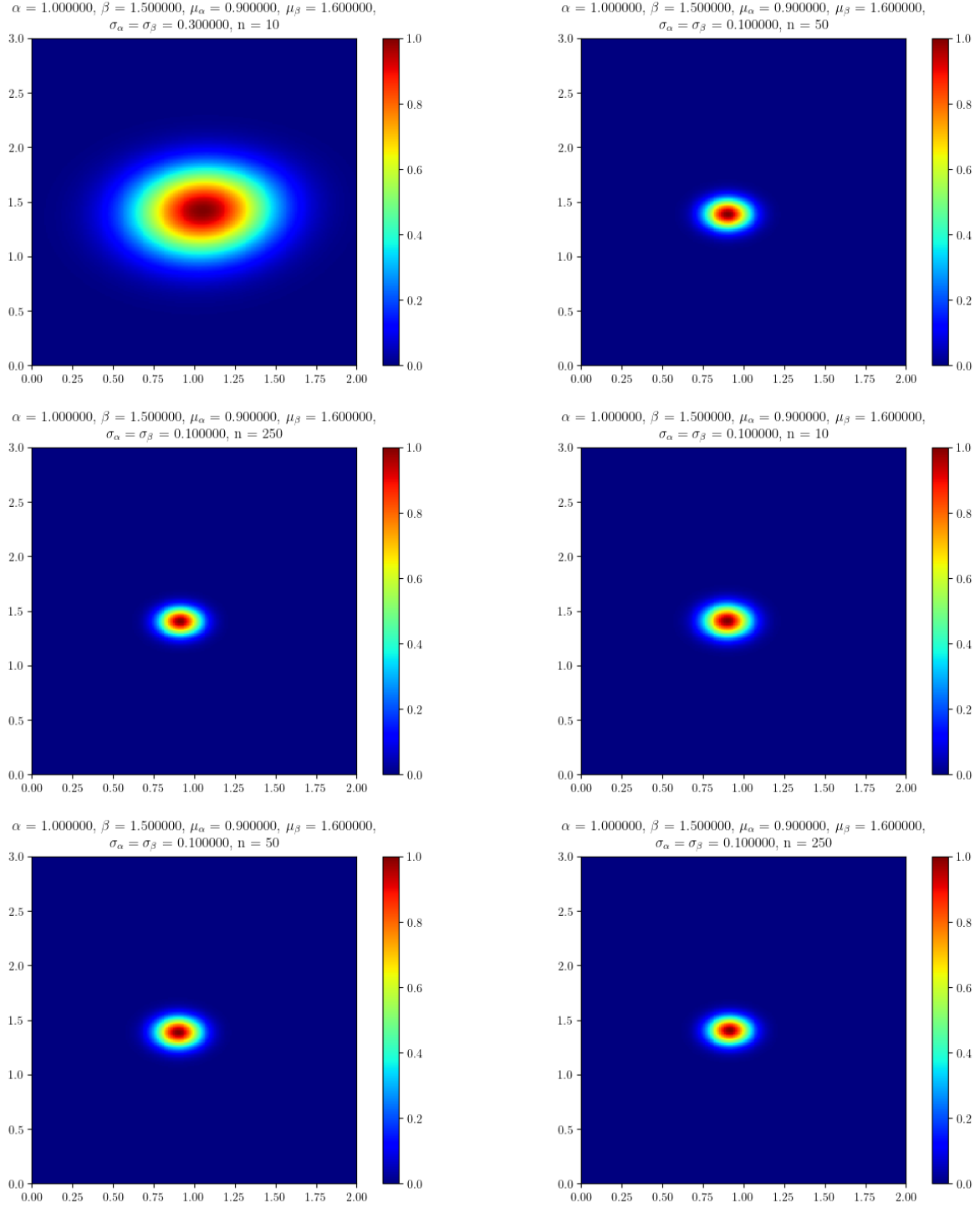


Figure 4: Colormaps of posteriors over a grid of  $(\alpha, \beta)$  values for different Gaussian priors and at different progressions in time (the lighthouse emits pulses regularly spaced in time, so the total number of pulses  $n$  corresponds to time progression). As before, the posterior converges much faster to a point in parameter-space for Gaussian priors with  $\mu$  within a small  $\sigma$  of the corresponding parameter.