MATH10101 Foundations of Pure Mathematics A

Solutions to Exercise Sheet 5

1. Write $A \cup B \cup C = A \cup (B \cup C)$ and apply the Inclusion-Exclusion Principle to get

$$|A \cup (B \cup C)| = |A| + |B \cup C| - |A \cap (B \cup C)|$$
$$= |A| + |B| + |C| - |B \cap C| - |(A \cap B) \cup (A \cap C)|$$
$$= |A| + |B| + |C| - |B \cap C| - (|(A \cap B)| + |(A \cap C)| - |(A \cap B) \cap (A \cap C)|$$
$$= |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|.$$

2. Let U be the set of all tiles. Let T be the set of triangular tiles, R denote the set of red tiles and W denote the set of wooden tiles. The number we require is $|U \backslash (T \cup R \cup W)| = 144 - |T \cup R \cup W|$. By Ex 1,

$$|T \cup R \cup W| = |T| + |R| + |W| - |T \cap R| - |T \cap W| - |R \cap W| + |T \cap R \cap W|$$

$$= 68 + 69 + 75 - 36 - 40 - 38 + 23 = 121.$$

Therefore the number of blue, plastic, square tiles is 23.

3. (i) When $A = \{1\}$ we have $\mathcal{P}(A) = \{\emptyset, \{1\}\}$ and $\mathcal{P}(\mathcal{P}(A)) = \{\emptyset, \{\emptyset\}, \{\{1\}\}, \{\emptyset, \{1\}\}\}$.

   (ii) The graph G_c of $c\colon \mathcal{P}(A) \to \mathbb{Z}^{\geq}$ is the subset of $\mathcal{P}(A) \times \mathbb{Z}^{\geq}$ consisting of the ordered pairs $(X, |X|)$ where $X \subset A$. In the case $A = \{1\}$, we get $G_c = \{(\emptyset, 0), (\{1\}, 1)\}$.

   (iii) The image of $c\colon \mathcal{P}(\mathcal{P}(A)) \to \mathbb{Z}^{\geq}$ is the subset of $\mathbb{Z}^{\geq}$ whose elements are the cardinalities of elements of $\mathcal{P}(\mathcal{P}(A))$. In the case $A = \{1\}$, we get Im$c = \{0, 1, 2\}$.

4. By long division of 7 into 2 we get $2/7 = 0.\overline{285714}$.

Let $x = 1.778\overline{2345}$. Then $10000x = 17782.345\overline{2345}$ and so

$$9999x = 10000x - x = 17780.567 = 17780567/1000.$$

Therefore $x = 17780567/9999000$.

5. We use proof by contradiction. Assume that $X\backslash A$ is countable, ie. finite or denumerable. Since $X = A \cup (X\backslash A)$ we can apply Proposition 5.12 to show that X is denumerable. (Prop 5.12 states that the union of two denumerable sets is denumerable but the proof can easily be modified to show that the union of a denumerable set and a finite set is denumerable.) This contradicts the fact that X is uncountable. Therefore $X\backslash A$ is uncountable.

6. (i) We need to show that f is a bijection. The easiest way to do this is to show it has an inverse. Let $y = f(x) = x/(1-x)$. Rearranging we get $x = y/(1+y)$. Therefore the inverse of f is $f^{-1} : \mathbb{R}^+ \to (0,1)$, defined by $f^{-1}(x) = x/(1+x)$. Therefore $\mathbb{R}^+$ and $(0,1)$ are equipotent.

   (ii) The function $g : (1,0) \to (10,0)$ defined by $g(x) = 10x$ is a bijection and so $(1,0)$ and $(10,0)$ are equipotent. We can interpret this in terms of the decimal representation. Every infinite decimal of the form $0.a_1a_2a_3\ldots$ (where $a_i \in \{0,1,2,\ldots,9\}$ for all $i \in \mathbb{N}$ and the digits are not all 0 and not all 9) corresponds bijectively to the infinite decimal $a_1.a_2a_3\ldots$. This shows there is a one-to-one correspondence between the elements of $(0,1)$ and $(0,10)$.

   We can generalize the argument in (ii) to show that $(0,1)$ and $(0,10^n)$ are equipotent for any $n \in \mathbb{Z}$ by shifting the decimal point to the right or left by $|n|$ places, depending on whether n is positive or negative. Using (i) and the fact that the composite of two bijections is a bijection, $\mathbb{R}^+$ is equipotent to $(0,10^n)$ for all $n \in \mathbb{Z}$.

7. (i) For every integer $n \geq 2$, $2^{1/n}$ satisfies the equation $x^n - 2 = 0$ and so it is algebraic.

   Suppose, for a contradiction, that $2^{1/n}$ is rational and write it as a/b for positive integers a, b with no common factors except 1. Then $2 = a^n/b^n$ and so $a^n = 2b^n$. Therefore $2|a^n$ and so $2|a$ because 2 is prime. Let $a = 2k$ for some integer k. Then $2^nk^n = 2b^n$ and so $b^n = 2^{n-1}a^n$. Since $n \geq 2$ this means that $2|b^n$ and therefore $2|b$. This contradicts our choice of a/b in its lowest terms. Hence $2^{1/n}$ is irrational.

   (ii) Suppose, for a contradiction, that $2e$ is algebraic. Then by definition we can find integers $a_0, a_1, \ldots, a_n$, with $n \geq 1$ and $a_n > 0$, such that

$$a_0 + a_1(2e) + \ldots + a_n(2e)^n = 0.$$

Rewriting this as $a_0+2a_1e+\ldots+2^na_ne^n = 0$ implies that e is algebraic which contradicts the fact that e is transcendental. Therefore $2e$ must be transcendental.

(iii) Suppose, for a contradiction, that π^3 is algebraic. Then we can find integers $a_0, a_1, \ldots, a_n$, with $n \geq 1$ and $a_n > 0$, such that

$$a_0 + a_1(\pi^3) + \ldots + a_n(\pi^3)^n = 0.$$

This means that $a_0+a_1(\pi)^3+\ldots+a_n(\pi)^{3n} = 0$ and so π is algebraic. This contradicts the fact that π is transcendental and therefore π^3 is transcendental.