MATH10101, for supervision in week 08. Counting. Definition of GCD

Q1.

($\star$) (i) Write down one example of a function $f\colon \{1,2,3,4\} \to \{a,b,c,d\}$ such that f is not injective but the restriction, $f|_{\{1,2,3\}}$, of f onto the subset $\{1,2,3\}$ of the domain is injective.

(ii) Prove that there are $n(n!)$ functions $f\colon \mathbb{N}_n \to \mathbb{N}_n$ such that the restriction $f|_{\mathbb{N}_{n-1}}$ is injective.

($\star$) **Q2**. Find the number of subsets of $\{1,2,\ldots,10\}$ which contain 1 and do not contain 10.

Q3. (i) Prove, using the Binomial Theorem, that for all $n \in \mathbb{Z}^{\geq}$, $\sum_{r=0}^{n} \binom{n}{r} = 2^n$.

(ii) Now prove the same statement *without* using the Binomial Theorem by considering a set A with $|A| = n$ and calculating the cardinality of $\bigcup_{r=0}^{n} \mathcal{P}_r(A)$.

(iii) Check that the statement in (i) is true for $n = 5$ by direct evaluation of both sides.

($\star$) **Q4**. (i) Using the Binomial Theorem, prove that $\displaystyle\sum_{r=0}^{n} (-1)^r \binom{n}{r} = 0$.

($\star$) (ii) Use the result of (i) along with **Q3**(i) to evaluate the sums $\displaystyle\sum_{\substack{r=0\\ r \text{ even}}}^{n} \binom{n}{r}$ and $\displaystyle\sum_{\substack{r=1\\ r \text{ odd}}}^{n} \binom{n}{r}$.

($\star$) (iii) Check that both of your answers in (ii) are correct for $n = 4$ by direct calculation.

Q5. Use the Binomial Theorem to calculate $\displaystyle\sum_{r=0}^{100} 4^{2r} 5^{100-2r} \binom{100}{r}$. [*Answer*: 8.2^{100}]

Q6. Let A be a finite set and let $\mathcal{Q}(A) = \{(C,D) \in \mathcal{P}(A) \times \mathcal{P}(A) : C \subseteq D\}$. Prove that $|\mathcal{Q}(A)| = 3^{|A|}$.

($\star$) **Q7**. Find the quotient q and remainder r on dividing the following numbers by 17:

(i) 1; (ii) -1; (iii) 100; (iv) -100.

Q8. Let a, b be integers such that $b \mid a$.

($\star$) (i) Carefully prove the following proposition: $\forall c \in \mathbb{Z}$, $\big((c \mid b) \implies (c \mid a)\big)$.

($\star$) (ii) Assume $b \geq 0$. Use the definition of $\gcd$ to prove that $\gcd(a,b) = b$.