MATH10101, for supervision in week 09. Euclid's algorithm. Diophantine equations

($\star$) **Q9**. Let a, b be integers. Use Bezout's Lemma to prove that every common divisor of a and b divides $\gcd(a, b)$.

Q10. Find the greatest common divisors of the following pairs of integers. In each case write the greatest common divisor as an integral linear combination of the two initial numbers.

(i) $(97, 157)$; (ii) $(2323, 1679)$; (iii) $(10^{10} - 1, 10^9 - 1)$.

($\star$) **Q11**. (i) Use Euclid's algorithm to show that $\gcd(589, 779) = 19$.

($\star$) (ii) Write 19 as an integral linear combination of 589 and 779.

($\star$) (iii) Find all solutions $(x, y) \in \mathbb{Z}^2$ to the homogeneous equation $589x + 779y = 0$.

($\star$) (iv) Find all solutions $(x, y) \in \mathbb{Z}^2$ to the equation $589x + 779y = 19$.

($\star$) (v) Find all solutions $(x, y) \in \mathbb{Z}^2$ to the equation $589x + 779y = -190$.

($\star$) (vi) Find all solutions $(x, y) \in \mathbb{Z}^2$ to the equation $589x + 779y = 119$.

Q12. Find the greatest common divisors of (i) 15691 and 44517, (ii) 173417 and 159953.

Q13. For further practice, find **all** solutions $(x, y) \in \mathbb{Z}^2$ to the following equations.

Reminder *In each case, start by finding a particular solution, either by inspection or by Euclid's algorithm. Then write down the general solution. You should* **check** *your answer.*

(i) $3x + 5y = 1$;
(ii) $2x + 15y = 4$;
(iii) $31x + 385y = 1$;
(iv) $41x + 73y = 20$;
(v) $93x + 81y = 3$;
(vi) $533x + 403y = 52$.

Q14. Let $a, b \in \mathbb{Z}$. Prove formally: a, b are coprime $\iff \exists m, n \in \mathbb{Z}$: $am + bn = 1$.

Q15. Continuing on from the previous question, find m and n to show that (i) 41 and 68 are coprime; (ii) 71 and 118 are coprime.

More generally, prove that $3k + 2$ and $5k + 3$ are coprime for all $k \in \mathbb{Z}$.

Q16. (*Important*) Prove that if $\gcd(a, c) = 1$ and $\gcd(b, c) = 1$ then $\gcd(ab, c) = 1$.

Q17. Alison spends £11.00 on sweets for prizes in a contest. If a large box of sweets costs 90p and a small box 70p, how many boxes of each size did she buy?