32031 Feedback Quiz, 2022/23, Week 07: The check matrix and the dual code

Open-book. 10–15 minutes. Not for credit. To be marked in class.

Also at `https://is.gd/math32031`

Recall that "*H is a check matrix for C*" means the same as "*H is a generator matrix for $C^\perp$*".

Question 1 ♣ Select all statements which are true for *all* linear codes C. If false, think of a counterexample:

- ◯ For all matrices H, if H is a check matrix for C, then $\underline{c}H^T = \underline{0}$ for all $\underline{c} \in C$
- ◯ For all matrices H, if $\underline{c}H^T = \underline{0}$ for all $\underline{c} \in C$, then H is a check matrix for C
- ◯ For all matrices H, if H is a check matrix of C, then H is of the form $[-A^T | I_{n-k}]$ for some matrix A

Now consider the ternary linear code C generated by the matrix $G = \begin{bmatrix} 1 & 0 & 2 & 2 \\ 0 & 1 & 1 & 2 \end{bmatrix}$.

Question 2 ♣ Find an example of a check matrix H for the code C:

$$H = \begin{bmatrix} \square & \square & \square & \square \\ \square & \square & \square & \square \end{bmatrix}.$$

Bring H to standard form to obtain H':

$$H' = \begin{bmatrix} \square & \square & \square & \square \\ \square & \square & \square & \square \end{bmatrix}.$$

Calculate the following matrix products:

$$GH^T = \begin{bmatrix} \square & \square \\ \square & \square \end{bmatrix}, \qquad GG^T = \begin{bmatrix} \square & \square \\ \square & \square \end{bmatrix}.$$

Now select all the statements and explanations that you agree with.

- ◯ The fact that $GH^T = 0$ tells us that C is self-orthogonal (and self-dual, because $n = 2k$)
- ◯ The fact that $GG^T = 0$ tells us that C is self-orthogonal (and self-dual, because $n = 2k$)
- ◯ Since the check matrix H found above is not equal to G, the code C is not self-dual
- ◯ H' is also a check marix for G, and $H' = G$ which tells us that $C^\perp = C$

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CORRECTED

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Recall that "*H is a check matrix for C*" means the same as "*H is a generator matrix for $C^\perp$*".

Question 1 ♣ Select all statements which are true for *all* linear codes C. If false, think of a counterexample:

- ⊗ For all matrices H, if H is a check matrix for C, then $\underline{c}H^T = \underline{0}$ for all $\underline{c} \in C$

  Explanation: True — this is shown in the course and allows us to **check** if a given $\underline{c}$ belongs to the code C!

- ○ For all matrices H, if $\underline{c}H^T = \underline{0}$ for all $\underline{c} \in C$, then H is a check matrix for C

  Explanation: False — e.g., the zero matrix cannot be a check matrix.

- ○ For all matrices H, if H is a check matrix of C, then H is of the form $[-A^T | I_{n-k}]$ for some matrix A

  Explanation: False — C may have check matrices not in this form, as a matrix row equivalent to a check matrix is again a check matrix. E.g., $\mathrm{Rep}(3, \mathbb{F}_2)$ has check matrix $\begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} = [-A^T | I_2]$ as well as $\begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$.

Now consider the ternary linear code C generated by the matrix $G = \begin{bmatrix} 1 & 0 & 2 & 2 \\ 0 & 1 & 1 & 2 \end{bmatrix}$.

Question 2 ♣ Find an example of a check matrix H for the code C:

$$H = \begin{bmatrix} \boxed{\mathbf{1}} & \boxed{\mathbf{2}} & \boxed{\mathbf{1}} & \boxed{\mathbf{0}} \\ \boxed{\mathbf{1}} & \boxed{\mathbf{1}} & \boxed{\mathbf{0}} & \boxed{\mathbf{1}} \end{bmatrix}.$$

Explanation: The generator matrix G is in standard form, $G = [I_2 \mid A]$ where $A = \begin{bmatrix} 2 & 2 \\ 1 & 2 \end{bmatrix}$, so by a theorem from the course, a check matrix can be taken to be $H = [-A^T \mid I_2]$ which gives the answer above.

Bring H to standard form to obtain H':

$$H' = \begin{bmatrix} \boxed{\mathbf{1}} & \boxed{\mathbf{0}} & \boxed{\mathbf{2}} & \boxed{\mathbf{2}} \\ \boxed{\mathbf{0}} & \boxed{\mathbf{1}} & \boxed{\mathbf{1}} & \boxed{\mathbf{2}} \end{bmatrix}.$$

CORRECTED

Explanation: $H \xrightarrow{\texttt{r2-=r1}} \begin{bmatrix} 1 & 2 & 1 & 0 \\ 0 & 2 & 2 & 1 \end{bmatrix} \xrightarrow{\texttt{r1-=r2}} \begin{bmatrix} 1 & 0 & 2 & 2 \\ 0 & 2 & 2 & 1 \end{bmatrix} \xrightarrow{\texttt{r2*=2}} \begin{bmatrix} 1 & 0 & 2 & 2 \\ 0 & 1 & 1 & 2 \end{bmatrix}$

Calculate the following matrix products:

$$GH^T = \begin{bmatrix} \boxed{\mathbf{0}} & \boxed{\mathbf{0}} \\ \boxed{\mathbf{0}} & \boxed{\mathbf{0}} \end{bmatrix}, \qquad GG^T = \begin{bmatrix} \boxed{\mathbf{0}} & \boxed{\mathbf{0}} \\ \boxed{\mathbf{0}} & \boxed{\mathbf{0}} \end{bmatrix}.$$

Now select all the statements and explanations that you agree with.

◯ The fact that $GH^T = 0$ tells us that C is self-orthogonal (and self-dual, because $n = 2k$)

Explanation: $GH^T = 0$ for **all** linear codes, because every row $\underline{r}$ of G is a codevector and so $\underline{r}H^T = \underline{0}$. Hence this fact does not tell us anything about C

⊗ The fact that $GG^T = 0$ tells us that C is self-orthogonal (and self-dual, because $n = 2k$)

Explanation: This is correct, as seen in exercises earlier

◯ Since the check matrix H found above is not equal to G, the code C is not self-dual

Explanation: A generator matrix of a linear code is not unique. We cannot reject the possibility that $C = C^\perp$ by looking at just one possible check matrix.

⊗ H' is also a check marix for G, and $H' = G$ which tells us that $C^\perp = C$

Explanation: True. Indeed, H' is a check matrix and generates $C^\perp$. So $C^\perp = C$.

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