Chapter 2 Exercises

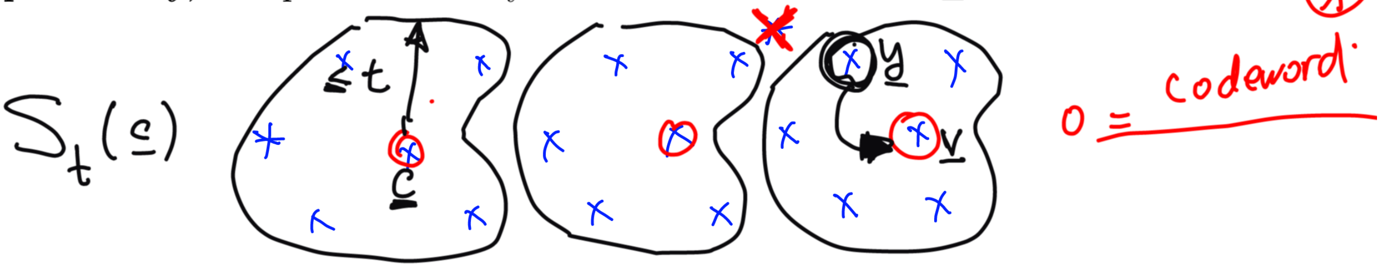
Exercise 2.1. Consider the trivial code F^n, the Manchester code and the Luhn code. For each of these codes, determine the parameters $[n, k, d]_q$ of the code; state how many errors the code can detect and how many errors the code can correct; determine if the code is perfect and/or MDS.

Exercise 2.2 (important part of the theory; you will need these facts for the exam). **Definition (repeated from the lecture notes):** a code C is ***perfect*** if C **attains** the Hamming bound, meaning that

$$\#C = \frac{q^n}{\sum_{i=0}^{t} \binom{n}{i}(q-1)^i} \qquad \text{where } n \text{ is the length of the code and } t = [(d(C)-1)/2].$$

(a) Use the proof of Theorem 2.3 to show that a code $C \subseteq F^n$ is perfect, if and only if the (disjoint) spheres of radius t, centred at codewords of C, fill up the set F^n of all words.

Equivalently, C is perfect iff every word in F^n is at distance $\leq t$ from some codeword.



Inside the Hamming spheres $S_t(\text{codeword})$:

$$q^n = \#F^n \overset{\text{if perfect}}{=} \#C \times \underbrace{\#S_t(\cdot)}_{} = \sum_{i=0}^{t} \binom{n}{i}(q-1)^i$$

(b) Prove that a perfect code has odd minimum distance d. (*Hint*: if d is even, construct a word at distance $d/2$ from a codeword and show that it is not at distance $\leq t$ from any codeword.)

Even $d \Rightarrow C$ not perfect.

Suppose d is even. Take $\underline{c} \in C$.

$t = \left[\frac{d-1}{2}\right] = \frac{d}{2} - 1$ if d is even

$\underline{c} \xleftrightarrow{d/2} \underline{y}$, $\quad d(\underline{y}, \underline{v}) \leq t$, $\quad \underline{v} \in C$

Changing $d/2$ symbols in $\underline{c}$, get $\underline{y}$: $d(\underline{y}, \underline{c}) = \frac{d}{2}$.

So $\underline{y} \notin S_t(\underline{c})$

$d(\underline{c}, \underline{v}) \leq d(\underline{c}, \underline{y}) + d(\underline{y}, \underline{v}) \leq \frac{d}{2} + \frac{d}{2} - 1 < d$ and $\underline{v} \neq \underline{c}$

contradiction.

(c) Show that binary repetition codes of odd lengths are perfect.

$t = \frac{n-1}{2}$, n is odd, $d = n$

0000000 $\leq t$ $\underline{y}$ — #1's in $\underline{y}$ less than $n/2$

OR: $\leq t$ if #1's $> \frac{n}{2}$ — 1111111

(d) Show that $\text{Rep}(n, F)$ is not perfect if $q = \#F > 2$. (*Hint*: using three different symbols, write down a word at distance $> n/2$ from each codeword.)

00000 11111 2

$> n/2$: 000000 00000

$> \frac{n}{2}$: 11111 111 111