MATH10101, for supervision in week 13. — SOLUTIONS

This problem sheet will not be discussed in the weekly supervision classes. The students are nevertheless advised to attempt all questions and then to work through the model solutions available via the course website. The material covered by the questions below is examinable.

Q40. Count the number of elements in the following subsets of S_8.

(i) The set of cycles of length 2.

(ii) The set of cycles of length 3.

(iii) The set of permutations that fix a given element.

Which of the above subsets are closed under composition?

Q40 - solution. (i) For a cycle of length 2 we have $(a,b) = (b,a)$, hence cycles of length 2 are in bijective correspondence with (unordered) *subsets* of $\{1,2,3,4,5,6,7,8\}$ of size 2. There are $\binom{8}{2} = 28$ of these.

(ii) For a 3-cycle we have $(a,b,c) = (b,c,a) = (c,a,b)$. So we need count the number of *ordered* 3-tuples, of which there are $8 \times 7 \times 6$ choices, but then divide this by 3. So there are 112 3-cycles.

Alternatively, for each 3-cycle (a,b,c), consider the set $Move((a,b,c)) = \{a,b,c\}$ which is a subset of $\mathbb{N}_8$ of size 3. Note that exactly two 3-cycles correspond to the same subset $\{a,b,c\}$: one cycle is $(a,b,c) = (b,c,a) = (c,a,b)$ and the other cycle is $(a,c,b) = (c,b,a) = (b,a,c)$. Therefore, the number of 3-cycles is two times the number of subsets of size 3 in $\mathbb{N}_8$. The answer is $2 \times \binom{8}{3} = 2\frac{8\times7\times6}{1\times2\times3} = 112$.

(iii) If the permutations fix 8 say, we are counting the number of elements that permute $\{1,2,3,4,5,6,7\}$, which is the same as the cardinality of S_7, i.e. $7! = 5040$.

Out of (i), (ii), (iii), only the set of permutations that fix a given element is closed under composition. A product of two 2-cycles may not be a 2-cycle in S_8, and a product of two 3-cycles may not be a 3-cycle: consider $(12)(34)$ which moves more than 2 numbers, and $(123)(456)$ which moves more than 3 numbers.

Q41. Calculate the orders of the following permutations in S_{11} :

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 \\ 8 & 5 & 10 & 11 & 7 & 4 & 9 & 1 & 2 & 3 & 6 \end{pmatrix},$$

$$\tau = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 \\ 2 & 4 & 6 & 8 & 10 & 5 & 7 & 9 & 11 & 1 & 3 \end{pmatrix} \circ \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 \\ 3 & 6 & 9 & 1 & 4 & 7 & 10 & 2 & 5 & 8 & 11 \end{pmatrix}.$$

Q41 - solution. $\sigma = (1,8) \circ (2,5,7,9) \circ (3,10) \circ (4,11,6)$, a product of disjoint cycles. So the order of σ is $\mathrm{lcm}(2,4,2,3) = 12$.

$$\begin{aligned} \tau &= \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 \\ 6 & 5 & 11 & 2 & 8 & 7 & 1 & 4 & 10 & 9 & 3 \end{pmatrix} \\ &= (1,6,7) \circ (2,5,8,4) \circ (3,11) \circ (9,10), \end{aligned}$$

a product of disjoint cycles. So the order of τ is $\mathrm{lcm}(3,4,2,2) = 12$.

Q42. Calculate the orders of the following permutations in S_5 :

(i) $(1,2,3) \circ (1,3,4) \circ (1,3,5)$,

(ii) $(1,2) \circ (1,3) \circ (1,4) \circ (1,5)$,

(iii) $(2,3,5) \circ (1,2) \circ (2,4) \circ (1,2)$.

Q42 - solution. Be careful, the cycles in these compositions are not disjoint. You should first write them out as compositions of disjoint cycles.

(i) $(1,2,3) \circ (1,3,4) \circ (1,3,5) = (1,4,2,3,5)$ which has order 5,

(ii) $(1,2) \circ (1,3) \circ (1,4) \circ (1,5) = (1,5,4,3,2)$ which has order 5,

(iii) $(2,3,5) \circ (1,2) \circ (2,4) \circ (1,2) = (1,4) \circ (2,3,5)$ which has order 6.

Q43. (*not quite easy*) Give an example of a permutation in S_{13} with the largest possible order.

Q43 - solution. Recall that a permutation in S_n is a product of disjoint cycles whose lengths sum up to n (where we allow cycles of length 1). Hence the largest possible order of a permutation in S_n is equal to the maximum possible value of $\mathrm{lcm}(a,b,c,\ldots)$ over all lists of positive integers $a,b,c,\ldots$ which satisfy the constraint $a+b+c+\ldots = n$. Denote such a maximum possible value of $\mathrm{lcm}(a,b,c,\ldots)$ by $g(n)$.

The function $g(n)$ of a positive integer n is known in the literature as **Landau's function**. The infinite sequence $g(1)$, $g(2)$, $g(3), \ldots$ has been studied in the literature and features in the *The On-Line Encyclopedia of Integer Sequences* (*OEIS*) under the reference A000793. Whereas $g(n)$ is straightforward to compute for small n (e.g., by considering all possible *partitions* of n, i.e., all possible ways to write n as a sum of several positive integers, calculating the lowest common multiple of the parts in each partition, and taking the maximum over all partitions), there is no known formula which would give the value of $g(n)$ for all n. See https://oeis.org/A000793 and references therein.

To find $g(13)$, we can approach calculation of $g(n)$ systematically and observe that

$$g(n) = \max\{\mathrm{lcm}(k, g(n-k)) : 1 \leq k \leq n\}.$$

This formula is justified by noting that the maximum LCM of the parts of a partition of n must be attained on a partition where one part is k and the LCM of the other parts, which sum up to

$n-k$, is maximised. We also need a convention $g(0)=1$ to ensure that the partition with only one part, n, is correctly counted: $n = \mathrm{lcm}(n, g(0)) = \mathrm{lcm}(n, 1)$.

It is obvious that $g(1) = 1$. Then, $g(2) = \max\{\mathrm{lcm}(1, g(1)), \mathrm{lcm}(2, g(0))\} = \max\{1, 2\} = 2$. Furthermore, $g(3) = \max\{\mathrm{lcm}(3, g(0)), \mathrm{lcm}(2, g(1)), \mathrm{lcm}(1, g(2))\} = 3$. Continuing in the same fashion, one obtains $g(4) = 4$, $g(5) = \mathrm{lcm}(2, g(3)) = 6$, $g(6) = \mathrm{lcm}(1, g(5)) = 6$, $g(7) = \mathrm{lcm}(3, g(4)) = 12$, $g(8) = \mathrm{lcm}(5, g(3)) = 15$, $g(9) = \mathrm{lcm}(5, g(4)) = 20$, $g(10) = \mathrm{lcm}(5, g(5)) = 30$, $g(11) = \mathrm{lcm}(5, g(6)) = 30$, $g(12) = \mathrm{lcm}(5, g(7)) = 60$. (Finding $g(n)$ involves comparing n values; we omit explicit comparisons.)

Finally, $\boldsymbol{g(13) = \mathrm{lcm}(1, g(12)) = 60}$. This answer agrees with the value of $g(13)$ given in the OEIS (see above).

An example of a permutation of order 60 in S_{13} is $(1)(2,3,4,5,6)(7,8,9,10)(11,12,13)$.

Q44. Define $*$ on $\mathbb{Q}$ by

$$\frac{a}{b} * \frac{c}{d} = \frac{a+c}{b+d},$$

where $a, c \in \mathbb{Z}$ and $b, d \in \mathbb{N}$. Is this a binary operation?

Q44 - solution. No. It is not well defined on $\mathbb{Q}$. For instance,

$$\frac{1}{2} * \frac{1}{1} = \frac{2}{3}.$$

But in $\mathbb{Q}$ we have $\frac{1}{2} = \frac{2}{4}$ yet

$$\frac{2}{4} * \frac{1}{1} = \frac{3}{5} \neq \frac{2}{3}.$$

Q45. Are the following operations closed?

(i) Addition on the set of odd integers,

(ii) Multiplication on the set of even integers,

(iii) $a \circ b = a + b - ab$ on the set of odd integers.

Q45 - solution. (i) No. $1 + 1 = 2$ which is not odd.

(ii) Yes. If a is an even integer then $a = 2k$ for some integer k. But then $ab = (2k)b = 2(kb)$ is even.

(iii) Yes. If a and b are odd integers then $a+b$ is even and ab is odd, so $a+b-ab$ is even$-$odd $=$ odd.

Or in more detail, $a = 2k+1$ and $b = 2\ell+1$ for some integers k, ℓ. Thus

$$\begin{aligned} a \circ b &= a + b - ab \\ &= 2k+1+2\ell+1-(2k+1)(2\ell+1) \\ &= 2(-2k\ell)+1 \end{aligned}$$

which is odd.

Q46. Which of the following binary operations are associative and which are commutative? **Give your reasons**.

(i) $x * y = 2(x+y)$ on $\mathbb{R}$,

(ii) $x * y = x|y|$ on $\mathbb{R}$,

(iii) $x * y = \dfrac{x+y}{xy}$ on $\mathbb{R}\diagdown\{0\}$,

(iv) $x * y = x + y - xy$ on $\mathbb{Z}$,

(v) $x * y = \max(x, y)$ on $\mathbb{N}$,

Q46 - solution. (i) Is commutative. Proof: addition on $\mathbb{R}$ is commutative, so $\forall x, y \in \mathbb{R}$ $y * x = 2(y+x) = 2(x+y) = x * y$.

Is not associative. Counterexample: $1 * (2 * 3) = 1 * 10 = 22$ while $(1 * 2) * 3 = 18$.

(ii) Is not commutative. Counterexample: $1 * -1 = 1$ while $-1 * 1 = -1$.

Is associative. Proof: $a * (b * c) = a * (b|c|) = a|b|c|| = a|b||c|$ and $(a * b) * c = (a|b|) * c = a|b||c|$.

(iii) Is commutative. Proof: both addition and multiplication are commutative on $\mathbb{R}$.

Is not associative. Counterexample:

$$\begin{aligned} 1 * (2 * 3) &= \frac{1 + (2 * 3)}{1 \times (2 * 3)} = \frac{1 + \frac{2+3}{2\times 3}}{\frac{2+3}{2\times 3}} = \frac{11}{5}. \\ (1 * 2) * 3 &= \frac{(1 * 2) + 3}{(1 * 2) \times 3} = \frac{\frac{1+2}{2} + 3}{\frac{1+2}{2}} = \frac{9}{3}. \end{aligned}$$

(iv) Is commutative Proof: addition and multiplication are commutative on $\mathbb{Z}$.

Is associative. Proof:

$$\begin{aligned} a * (b * c) &= a * (b + c - bc) \\ &= a + (b + c - bc) - a(b + c - bc) \\ &= a + b + c - bc - ab - ac + abc \\ &= a + b - ab + c - ac - bc + abc \\ &= (a + b - ab) + c - (a + b - ab)c \\ &= (a * b) + c - (a * b)c \\ &= (a * b) * c. \end{aligned}$$

(v) Is commutative, Proof: $x * y = \max(x, y) = \max(y, x) = y * x$.

Is associative. Proof:

$$\begin{aligned} x * (y * z) &= \max(x, y * z) = \max(x, \max(y, z)) \\ &= \max(x, y, z) = \max(\max(x, y), z) \\ &= \max(x * y, z) = (x * y) * z. \end{aligned}$$

Q47. Which of the following binary operations have identities:

(i) $x * y = \max(x, y)$ on $\{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$,

(ii) $x * y = \max(x, y)$ on $\mathbb{Z}$,

(iii) $x * y = x + y - xy$ on $\mathbb{Q}$,

(iv) matrix multiplication on the set $\left\{ \begin{pmatrix} a & a \\ a & a \end{pmatrix} : a \in \mathbb{R} \setminus \{0\} \right\}$?

Q47 - solution. (i) Yes. $\max(1, n) = \max(n, 1) = n$ for any $n \geq 1$ and so 1 is the identity.

(ii) No. *If* $e \in \mathbb{Z}$ is an identity, then we can choose an integer $x < e$ and for this integer we find that $x * e = \max(x, e) = e \neq x$.

(iii) Yes. 0. We have seen that this $x * y$ is commutative so we need only examine $x * 0 = x + 0 - x \times 0 = x$.

(iv) Yes. $\left(\begin{array}{cc} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{array} \right)$

Q48. Why is $(\mathbb{Z}, -)$ not a group?

Q48 - solution. Many reasons. Perhaps because subtraction is not associative.

$$1 - (2 - 3) = 2 \quad \text{but} \quad (1 - 2) - 3 = -4.$$

Q49. a) Draw up the multiplication table for $(\{4, 8, 12, 16, 20, 24\}, \times_{28})$. Find the identity element and the inverse of each element. (Here $\times_{28}$ denotes multiplication mod 28, in other words the operation of multiplyimng two integers and then taking the remainder on division by 28.)

b) Can you find a **proper** subset of $\{4, 8, 12, 16, 20, 24\}$ which is closed under $\times_{28}$?

Q49 - solution. a)

$\times_{28}$	4	8	12	16	20	24
4	16	4	20	8	24	12
8	4	8	12	16	20	24
12	20	12	4	24	16	8
16	8	16	24	4	12	20
20	24	20	16	12	8	4
24	12	24	8	20	4	16

The identity element is 8.

$4^{-1} = 16$, $8^{-1} = 8$, $12^{-1} = 24$, $16^{-1} = 4$, $20^{-1} = 20$ and $24^{-1} = 12$.

b) $(\{4, 8, 16\}, \times_{28})$ is a closed subset.

$\times_{28}$	4	8	16
4	16	4	8
8	4	8	16
16	8	16	4

$(\{8, 20\}, \times_{28})$ is a closed subset:

$\times_{28}$	8	20
8	8	20
20	20	8