Three hours

THE UNIVERSITY OF MANCHESTER

FOUNDATIONS OF PURE MATHEMATICS A

16 January 2018

09:45 – 12:45

Answer **ALL FIVE** questions in Section A (25 marks in all) and
ALL FIVE questions in Section B (50 marks in all).

Electronic calculators may be used, provided they cannot store text

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SECTION A

Answer **ALL FIVE** questions

A1. Construct truth tables for the statements:

(i) $P \Leftarrow Q$
(ii) P or (not Q)
(iii) (not P) and (not Q)
(iv) $P \nRightarrow$ (not P)
(v) $P \Rightarrow$ (Q or R).

[5 marks]

A2. Prove or disprove each of the following statements:

(i) $\forall m \in \mathbb{Z}$, $\forall n \in \mathbb{Z}^+$, $mn \geq m$
(ii) $\exists m \in \mathbb{Z}^+$, $\exists n \in \mathbb{Z}^+$, $mn \geq m$
(iii) $\forall m \in \mathbb{Z}$, $\exists n \in \mathbb{Z}$, $m = 2n$
(iv) $\exists n \in \mathbb{Z}$, $\forall m \in \mathbb{Z}$, $m > 2n$
(v) $\exists n \in \mathbb{Z}^+$, $\forall m \in \mathbb{Z}^+$, $mn > 2m$.

[5 marks]

A3.

(i) State without proof the *factorial formula* for the binomial coefficient $\binom{n}{r}$.
(ii) State without proof the *Binomial Theorem*.
(iii) Let $n \in \mathbb{Z}^+$. Show that $2^n n = \displaystyle\sum_{r=0}^{n} 2r\binom{n}{r}$ and verify this result for $n = 4$ by direct calculation.

[5 marks]

A4.

(i) Let a, b, c be integers. State the criterion for the equation $ax + by = c$ to have at least one solution $(x, y) \in \mathbb{Z}^2$.
(ii) Describe all the solutions $(x, y) \in \mathbb{Z}^2$ of the equation

$$19x + 8y = 3.$$

(iii) Now consider the equation $19x+by = 3$. List all the integer values of b such that this equation has *no integer solutions* and $0 \leq b \leq 100$. You do not have to justify your answer.

[5 marks]

A5.

(i) Find all the possible remainders that $a^6 - 3a^2 + 2$, where $a \in \mathbb{Z}$, can leave when divided by 9. Show that if a is not divisible by 3 then $a^6 - 3a^2 + 2$ is divisible by 9.
(ii) Use the result of (i) to find all $n \in \mathbb{Z}^+$ such that $(n! + 1)^{6n} - 3(n! + 1)^{2n} + 2$ is divisible by 9. Explain how you arrived at the answer.

[5 marks]

SECTION B

Answer **ALL FIVE** questions

B6.

(i) Given disjoint finite sets A and B, state the *Addition Principle* for the cardinality of $A \cup B$. Explain the modification required when $A \cap B \neq \emptyset$. By substituting $A = C \cup D$ and $B = E$ into your formula, prove that

$$|C \cup D \cup E| \;=\; |C| + |D| + |E| - |C \cap D| - |C \cap E| - |D \cap E| + |C \cap D \cap E|$$

for any finite sets C, D and E. [*You may assume the distributive law for* $\cup$ *and* $\cap$ *without proof*].

[5 marks]

(ii) Each of a collection of 145 pullovers is either long or short, green or black, and woollen or cotton. There are 75 long pullovers, 69 green pullovers, 60 woollen pullovers, 38 long green pullovers, 40 long woollen pullovers, 36 green woollen pullovers, and 23 long green woollen pullovers. Use part (i) to calculate the number of short black cotton pullovers.

[5 marks]

B7.

(i) Express the *decimal* $0 \cdot a_1 a_2 \ldots a_n$ as a rational number, where $0 \leq a_k \leq 9$ for each integer $a_1, \ldots, a_n$. Explain how *infinite decimals* may be used to represent real numbers r in the interval $(0,1) \subset \mathbb{R}$, and compute the infinite decimal representing the rational number $3/7$.

[5 marks]

(ii) Define the term *uncountable set*, and prove that the interval $(0,1)$ is uncountable.

[5 marks]

B8. Let a and b be integers, at least one of which is not 0.

(i) Give the definition of the *greatest common divisor*, $\gcd(a,b)$, of a and b. Prove the lemma which says that if $a = bq + r$ with $q, r \in \mathbb{Z}$, then $\gcd(a,b) = \gcd(b,r)$. Give an example which shows that in this situation $\gcd(a,b)$ may not be equal to $\gcd(a,r)$.

[5 marks]

(ii) *In the second part of the question, you are allowed to use the fact that there exist integers* m, n *such that* $\gcd(a,b) = am + bn$, *and you do not have to prove it.*

State what is meant by saying that a, b are coprime. Prove that a, b are coprime, if and only if $ap + bq = 1$ for some integers p, q. Now assume that a is odd and $b = a + 2$; show that $a + b$ and ab are coprime.

[5 marks]

B9.

(i) Give the definition of a *prime number*. Assuming that p is a prime number, show that for any integer a, if p divides a^2 then p^2 divides a^2. (*You may use any facts from the course provided that you carefully state them.*) Now assume that an integer $n \geq 2$ is such that for any integer a, if n divides a^2 then n^2 divides a^2. Is n necessarily prime? Justify your answer.

[5 marks]

(ii) Show that there exist integers $a, b, c, d, e > 1$ such that $ab = cde$, $ac = bd^{10101}$. Do such integers exist if it is further required that e is a prime number? Justify your answer.

[5 marks]

B10.

(i) Define what is meant by a *permutation* of the set $\mathbb{N}_n = \{1, 2, \ldots, n\}$. What is meant by saying that permutations σ, τ of $\mathbb{N}_n$ are *disjoint*? Prove the lemma which says that if σ and τ are disjoint, then $\sigma \circ \tau = \tau \circ \sigma$.

[5 marks]

(ii) Assume that π, ρ are permutations of $\mathbb{N}_{20}$ such that π has order 21 and $\pi \circ \rho$ has order 24.

(a) Prove that π and ρ are not disjoint.

(b) Prove that π and $\pi \circ \rho$ are not disjoint.

(c) Write down an example of π and ρ such that the orders are as required above, or give a convincing explanation of how such an example can be constructed.

[*Any facts from the course can be used without particular comment.*]

[5 marks]

END OF EXAMINATION PAPER