non-zero triple contributes exactly

$$w(001) + w(010) + w(100) + w(011) + w(110) + w(111) + w(101) \quad = \quad 12$$

to that sum. The same argument applies to the second, ..., wth non-zero symbol in $\underline{v}$. We obtain

$$\begin{aligned} & w((\overline{001\underline{v}})^{\text{bin}}) + w((\overline{010\underline{v}})^{\text{bin}}) + w((\overline{100\underline{v}})^{\text{bin}}) \\ & \qquad + w((\overline{011\underline{v}})^{\text{bin}}) + w((\overline{110\underline{v}})^{\text{bin}}) + w((\overline{111\underline{v}})^{\text{bin}}) + w((\overline{101\underline{v}})^{\text{bin}}) = 12w. \end{aligned}$$

But if the sum of weights of the given seven vectors in $C^{\text{bin}} \setminus \{\underline{0}\}$ is $12w$, then **at least one** of those vectors will have weight **less than or equal to** $12w/7$. Q.E.D.

D. An interesting weight enumerator [20 marks] Show that there is no linear code over $\mathbb{F}_8$ with weight enumerator $x^9 + 16x^5y^4 + 16x^4y^5 + 256y^9$. Does a linear code with such weight enumerator exist over any other field? Justify your answer.

D: answer

Suppose a linear code has weight enumerator $x^9 + 16x^5y^4 + 16x^4y^5 + 256y^9$. This means that the code contains one codevector of weight 0, sixteen codevectors of weight 4, sixteen codevectors of weight 5 and 256 codevectors of weight 9. The total number of codevectors is then $1 + 16 + 16 + 256 = 289$. Since 289 is not a power of 8, the code cannot be a linear code over $\mathbb{F}_8$: such codes always contain 8^k elements where $k = \dim C$ is an integer.

Since $289 = 17^2$, such a code could in principle exist over $\mathbb{F}_{17}$ or over $\mathbb{F}_{289}$. But saying that a code *could* exist is not a proof that it exists — we need to construct the code. Consider the 17-ary code generated by $G = \begin{bmatrix} 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}$. It has 16 codevectors of weight 4 and of the form $\lambda\lambda\lambda\lambda 00000$ with $\lambda \in \mathbb{F}_{17} \setminus \{0\}$; 16 codevectors of weight 5 of the form $0000\mu\mu\mu\mu\mu$ with $\mu \in \mathbb{F}_{17} \setminus \{0\}$; and the rest of its 289 codevectors are of the form $\lambda\lambda\lambda\lambda\mu\mu\mu\mu\mu$, of weight 9. Hence the code has the required weight enumerator.