MATH10101, optional exercises on linear and non-linear congruences. Will not be discussed in the supervisions

Opt7. Alice comes home from school and tells her baby brother Bob: "Our mum's age, expressed in *months*, is congruent to 20 modulo 17 and is congruent to 17 modulo 20." What is most likely the age of their mother?

Opt8. a) By using the method of *successive squaring*, find the remainders of the following numbers on dividing by 41: (i) 5^4, (ii) 5^{16}, (iii) 5^{64}.

In particular, check that 5^4 and 5^{64} leave the same remainder when divided by 41.

b) Use the answers to part (a) to find an $n \in \mathbb{N}$ such that $5^n \equiv 1 \mod 41$.

c) Use part (b) to solve $25x \equiv 7 \mod 41$.

Opt9. What are the remainders when 3^{40} and 40^{35} are divided by 11? Prove that $3^{40} + 40^{35}$ is divisible by 11.

Opt10. Show that $7x^4 + 2y^3 = 3$ has no integer solutions.

Opt11. Show that $2x^3 + 27y^4 = 21$ has no integer solutions.

Opt12. Show that $7x^5 + 3y^4 = 2$ has no integer solutions.

Opt13. Show that 7 never divides $n^4 + n^2 + 2$ for $n \in \mathbb{Z}$.

Opt14. (i) Show that $n^2 - n + 41$ is never divisible by 2, 3, 4, 5, 6, 7, 8, 9 nor by 10.

(ii) If you have time, show that $n^2 - n + 41$ is never divisible by any integer from $\{10, 11, \ldots, 40\}$. (*Warning*: (ii) can be done by "brute force" which is time-consuming. A conceptual solution which shows that (i) implies (ii) is beyond the scope of this course.)