Exercises to Chapter 7

Notation: let $\mathcal{H}$ denote a $\mathrm{Ham}(3,2)$ code. It is a $[7,4,3]_2$ linear code.

Exercise 7.1. (a) Use a parity check matrix of $\mathcal{H}$ to show: $\mathcal{H} \ni 1111111$.

$$H = \begin{bmatrix} 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 \end{bmatrix}$$

$$S(1111111) = 1111111\,H^T = h_1 + h_2 \cdots + h_7 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Exercise 7.2.

(a) Construct a generator matrix for $\mathcal{H}$. Hence write all the codevectors of $\mathcal{H}$ and find the weight enumerator $W_{\mathcal{H}}(x,y)$ of $\mathcal{H}$.

$$H' = \left[\begin{array}{cccc|ccc} 1 & 1 & 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 1 & 0 & 0 & 1 \end{array}\right] \qquad -A^T \quad I_3$$

$$\longmapsto G = \begin{array}{c} \left[\begin{array}{cccc|ccc} 1 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 & 1 \end{array}\right] \begin{array}{l} r_1 \\ r_2 \\ r_3 \\ r_4 \end{array} \end{array} \qquad I_4 \quad A$$

0	0000000
r_1	
r_2	
r_3	
r_4	
r_1+r_2	1100001
r_1+r_3	1010010
r_1+r_4	1001100
r_2+r_4	0101101
r_2+r_3	0110011
r_3+r_4	0011110
$r_1+r_2+r_3$	$= 1111111 - r_4 = 1110100$
$r_1+r_3+r_4$	$= 1011001$
$\overline{r_3}$	$= 1101010$
$\overline{r_1}$	$= 0111000$
$r_1+r_2+r_3+r_4$	1111111

$$W_{\mathcal{H}}(x,y) = x^7 + 7x^4y^3 + 7x^3y^4 + y^7$$

Exercise 7.3. (a) If $\underline{v} = (x_1, x_2, \ldots, x_n)$ is a binary vector, we extend $\underline{v}$ to obtain the vector $\widehat{\underline{v}} = (x_1, \ldots, x_n, x_{n+1})$ where $x_{n+1} = x_1 + \ldots + x_n$ in $\mathbb{F}_2$. That is, a vector is extended by appending one bit so that the resulting vector has even weight.

If C is a binary linear code, we define the **extended code** $\widehat{C} = \{\widehat{\underline{c}} : \underline{c} \in C\}$.

By looking at $W_{\mathcal{H}}$, the weight enumerator of the *extended Hamming code* $\widehat{\mathcal{H}}$ Determine the length, dimension and weight of $\widehat{\mathcal{H}}$.

$$\widehat{\mathcal{H}} \qquad W_{\widehat{\mathcal{H}}} = x^8 + 14\,x^4y^4 + y^8$$

(c) Deduce from (a) and (b) that $\widehat{\mathcal{H}}$ is a self-dual code.