Three hours

THE UNIVERSITY OF MANCHESTER

FOUNDATIONS OF PURE MATHEMATICS A

January 2016 (mock paper prepared 2018-01-06)

12.00 – 15.00

Answer **ALL FIVE** questions in Section A (25 marks in all) and
ALL FIVE questions in Section B (50 marks in all).

Electronic calculators may be used, provided they cannot store text

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SECTION A

Answer **ALL FIVE** questions

A1. Construct truth tables for the statements:

(i) Q and R
(ii) $P \not\Rightarrow Q$
(iii) (not P) or Q
(iv) not (P or (not Q))
(v) $P \Rightarrow$ (Q and R).

[5 marks]

A2. Prove or disprove each of the following statements:

(i) $\exists p \in \mathbb{Q}$, $\forall q \in \mathbb{Q}$, $p + q = 1/2$
(ii) $\forall q \in \mathbb{Q}$, $\exists p \in \mathbb{Q}$, $p + q = 1/2$
(iii) $\forall q \in \mathbb{Q}$, $\exists p \in \mathbb{Q}$, $p + q \neq 1/2$
(iv) $\exists p \in \mathbb{Q}$, $\exists q \in \mathbb{Q}$, $p + q < 1/2$
(v) $\forall p \in \mathbb{Q}$, $\forall q \in \mathbb{Q}$, $p + q \notin \mathbb{Z}$.

[5 marks]

A3.

(i) Explain why the Diophantine equation

$$6x + 10y = 90$$

has infinitely many solutions $(x, y) \in \mathbb{Z}^2$, and describe them all.

(ii) Solve the same equation subject to the additional constraints $x > 3$ and $y > 3$.

[5 marks]

A4.

(i) Find the multiplicative inverse of 13 mod 31.
(ii) Hence or otherwise, solve the congruence

$$13x \equiv 7 \bmod 31.$$

(iii) Use modular arithmetic and the method of successive squaring to calculate the least positive residue of

$$37^{514} \bmod 7.$$

[5 marks]

A5.

(i) Define what is meant by a *permutation* of the finite set $X = \{1, 2, \ldots, n\}$.
(ii) Write each of the following three permutations in disjoint cycle form:

$$(3\,4\,2)(2\,5\,3), \qquad \Big((2\,4\,3)(1\,5\,6\,7)\Big)^{-1}, \qquad (1\,2)(2\,3)(3\,4).$$

(iii) Determine the order of the second permutation in part (ii).

[5 marks]

SECTION B

Answer **ALL FIVE** questions

B6.

(i) For any sets A and B, define the sets $A \cap B$ and $A \cup B$. For any set C, prove that

$$(A \cap B) \cup C \;\supseteq\; (A \cap C) \cup (B \cap C)\,,$$

and explain how this statement simplifies when $C = \emptyset$. Under what circumstances does your simplification give equality?

[5 marks]

(ii) Given disjoint finite sets D and E, state the *Addition Principle* for the cardinality of $D \cup E$. Explain the modification required when $D \cap E \neq \emptyset$. By substituting $D = A \cup B$ and $E = C$ into your formula, prove that

$$|A \cup B \cup C| \;=\; |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|$$

for any finite sets A, B and C. [*You may use the fact that* $(A \cup B) \cap C = (A \cap C) \cup (B \cap C)$ *without proof*].

[5 marks]

B7.

(i) Explain what is meant by an *inverse* of a function $g\colon X \to Y$, and prove that if g has an inverse, then it is a bijection. Is the converse true or false?

[5 marks]

(ii) Let $h\colon \mathbb{R} \to \mathbb{R}$ be defined by $h(x) = \cos x$, forall $x \in \mathbb{R}$, and show that h is neither an injection nor a surjection. Find closed intervals I, $J \subset \mathbb{R}$ for which the restriction $h|_I\colon I \to J$ is a bijection. In this case, describe the inverse function.

[5 marks]

B8.

(i) For non-negative integers k, n such that $k \leq n$, define

$$\binom{n}{k}$$

in terms of subsets of a finite set of size n, and give an explicit formula for it in terms of factorials. State the *Binomial Theorem* for expanding $(a+b)^n$ for any positive integer n and real numbers a and b; deduce that

$$\sum_{k=0}^{n} (-1)^k \binom{n}{k} \;=\; 0.$$

[5 marks]

(ii) Compute

$$\sum_{k=0}^{5} \frac{2^{3k}(-2)^{7-k}}{k!\,(5-k)!}\,.$$

[5 marks]

B9.

(i) Explain what is meant by a *relation* $\sim$ on a set X, and describe the properties required for $\sim$ to be *reflexive*, *symmetric*, and *transitive*. Determine whether

$$m \sim n \iff m|n$$

defines an equivalence relation on $\mathbb{Z}$. [5 marks]

(ii) Given an equivalence relation $\sim$ on X, define the *equivalence class* $[x]$ for any $x \in X$, and prove that either $[x] = [y]$ or $[x] \cap [y] = \emptyset$ for any $y \in X$. [5 marks]

B10.

(i) Explain what it means for a positive integer $p \in \mathbb{Z}^+$ to be *prime*, and prove that there are infinitely many primes in $\mathbb{Z}^+$. [*You may use the fact that every positive integer factorises into a product of primes in a unique way*] [5 marks]

(ii) Let [0], [1], [2], [3], [4], [5] be the six congruence classes of the integers modulo 6; explain why there are only two primes in the set

$$[0] \cup [2] \cup [3] \cup [4] \subset \mathbb{Z},$$

and show that x, $y \in [1]$ implies $xy \in [1]$. By considering integers of the form $6P - 1$, deduce that [5] contains infinitely many primes. [5 marks]