Exercises to Chapter 10

Exercise 10.1 (the extended binary Golay code)**.** The code $\boxed{G_{24}}$ is defined as $\widehat{G}_{23}$, that is, by *extending* the binary Golay code defined earlier.

$$\underline{x} \in \mathbb{F}_2^n \mapsto \hat{\underline{x}} \in \mathbb{F}_2^{n+1}, \quad \hat{x}_i = x_i \ (i=1,\dots,n),$$
$$\hat{x}_{n+1} = x_1 + \dots + x_n$$

(a) Determine the parameters $[n,k,d]_q$ of G_{24}. State how many bit errors per codevector is the code guaranteed to *detect*. Same for *correct*. Find the rate of G_{24}.

$n = 24 \quad \# G_{24} = \# G_{23} = 2^{12} \Rightarrow k = 12 = \log_2 \# G_{24}.$

$w(G_{23}) = 7 \Rightarrow w(G_{24}) = 8 = d(G_{24}) = 8$ detects ≤ 7 bit errors, corrects $\leq \lfloor (8-1)/2 \rfloor = 3$ errors.

(b) A codevector of G_{24} is transmitted, and thirteen bit errors occur. Will an error be detected?

even weight $\underline{c} \xrightarrow[\text{13 errors}]{} $ odd weight $\underline{y}$ DETECT!

(c) Prove that G_{24} is a self-dual code. The proof may involve calculations, but they should not be computer-aided — it should be possible to do them by hand in a reasonable amount of time.

$G_{24} = \{ ----, 1010\,111\,0011\,0\dots01, \dots$

$\downarrow$

$\Lambda = (a_1, \dots, a_{24}) \in \mathbb{Z}^{24}$ | the Leech lattice

$(a_1 \bmod 2, \dots, a_{24} \bmod 2) \in G_{24}$

Proof: self-dual: $\overset{24}{n} = 2\overset{12}{k}$

$GG^T = 0$

$$\begin{array}{cl} r_1 & \\ r_2 & \end{array} \begin{bmatrix} 1 & 0 & 1 & 0 & 1 & 1 & 1 & 0 & 0 & 1 & 1 & 0 \dots 0 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 & 1 & 1 & 0 & 0 & 1 & 1\,0 \dots 0 & 1 \\ & & & & & & & & & & & & \end{bmatrix}$$

$r_1 \cdot r_1 = 0$

$r_2 \cdot r_2 = 0 \quad r_1 \cdot r_2 = 0$

Exercises to Chapter 11

Exercise 11.1 (identification of the Reed-Muller codes with $m = 3$)**.** Let $m = 3$. Write down the value vectors (in $\mathbb{F}_2^8$) of all the monomials in the Boolean algebra. Hence find generator matrices of the codes $R(r, 3)$, $0 \le r \le 3$. Try to recognise the codes obtained.

m

$n = 2^m = 2^3 = 8$

Partial answer. We use a slightly unconventional ordering of binary words in V^3. The value vectors of all the monomials in the Boolean algebra with $m = 3$:

$R(1,3)$ $r=1$ $m=3$

	001	010	011	100	101	110	111	000
1	1	1	1	1	1	1	1	1
v_1	0	0	0	1	1	1	1	0
v_2	0	1	1	0	0	1	1	0
v_3	1	0	1	0	1	0	1	0
v_1v_2	0	0	0	0	0	1	1	0
v_1v_3	0	0	0	0	1	0	1	0
v_2v_3	0	0	1	0	0	0	1	0
$v_1v_2v_3$	0	0	0	0	0	0	1	0

$\mathrm{Ham}(3,2)$

G for $R(1,3) = \widehat{\mathrm{Ham}}(3,2)$ self-dual

$\Lambda = E_8$

Exercise 11.2 ("the Mariner 9 code")**.** Check that $R(1,5)$ is a $[32, 6, 16]_2$ code and detects up to 15 errors in a 32-bit codeword.