MATH10101 JANUARY 2017: SOLUTIONS+ FEEDBACK FOR A3–A5, B8–B10

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`* Feedback for A1-A2, B6-B7 (Nige Ray): see pages 13-14 *`

see pages 13-14

Feedback: ***General comments:*** *this was the first MATH10101 exam paper in the new format where the students had to do all the questions. It was pleasing to see so many high quality scripts. Question B10 turned out to be the most challenging, as expected.*

A1. Construct truth tables for the statements:

(i) $R \Leftarrow S$
(ii) R and (not R)
(iii) not (R or S)
(iv) (not R) or (not S)
(v) (Q and R) $\Rightarrow S$.

[5 marks]

A1. Solution (Similar to, or the same as classwork and homework)

The required truth tables are:

(i)

R	S	$R \Leftarrow S$
T	T	T
T	F	T
F	T	F
F	F	T

(ii)

R	not R	R and (not R)
T	F	F
F	T	F

(iii)

R	S	R or S	not(R or S)
T	T	T	F
T	F	T	F
F	T	T	F
F	F	F	T

(iv)

R	S	not R	not S	(not R) or (not S)
T	T	F	F	F
T	F	F	T	T
F	T	T	F	T
F	F	T	T	T

(v)

Q	R	S	Q and R	$(Q$ and $R) \Rightarrow S$
T	T	T	T	T
T	T	F	T	F
T	F	T	F	T
T	F	F	F	T
F	T	T	F	T
F	T	F	F	T
F	F	T	F	T
F	F	F	F	T

[5 marks]

A2. Prove or disprove each of the following statements:

(i) $\forall u \in \mathbb{R}^+, \exists v \in \mathbb{R}^+, u = 2v$
(ii) $\exists v \in \mathbb{R}^+, \forall u \in \mathbb{R}^+, u = 2v$
(iii) $\forall u \in \mathbb{R}^+, \forall v \in \mathbb{R}^+, u \geq 2v$
(iv) $\exists u \in \mathbb{R}^+, \exists v \in \mathbb{R}^+, u \geq 2v$
(v) $\exists v \in \mathbb{R}^{\geq}, \forall u \in \mathbb{R}^+, u > 2v$.

[5 marks]

A2. Solution (Similar to classwork and homework)

(i) The statement $\forall u \in \mathbb{R}^+, \exists v \in \mathbb{R}^+, u = 2v$ is true, by choosing $v = u/2$
(ii) The statement $\exists v \in \mathbb{R}^+, \forall u \in \mathbb{R}^+, u = 2v$ is false, because $1 = 2v$ and $2 = 2v$ cannot hold simultaneously for any $v \in \mathbb{R}^+$
(iii) The statement $\forall u \in \mathbb{R}^+, \forall v \in \mathbb{R}^+, u \geq 2v$ is false, because $u = v = 1$ is a counterexample
(iv) The statement $\exists u \in \mathbb{R}^+, \exists v \in \mathbb{R}^+, u \geq 2v$ is true, for example by choosing $u = 3$ and $v = 1$
(v) The statement $\exists v \in \mathbb{R}^{\geq}, \forall u \in \mathbb{R}^+, u > 2v$ is true, by choosing $v = 0$.

[5 marks]

A3.

(i) State the Division Theorem and explain what is meant by the quotient and the remainder when an integer a is divided by a positive integer b.
(ii) Let k be a non-negative integer. Find the quotient and the remainder of $5k+4$ when divided by $3k+2$, writing them in terms of k. Give your reasons.
(iii) If ℓ is a non-negative integer, find the quotient and the remainder when $-5\ell - 4$ is divided by $3\ell + 2$. Give your reasons.

[5 marks]

A3. Solution (Bookwork, classwork and a specific example from homework)

(i) Given an integer a and a positive integer b, there exist unique integers q, r such that

$$a = bq + r \text{ and } 0 \leq r < b.$$

These unique integers q, r are referred to as the quotient and the remainder, respectively.

Feedback: *Some students omitted the word* ***unique*** *and/or the condition* $0 \leq r < b$. *The statement is not valid if any one of these is omitted.*

(ii) Observe that

$$5k + 4 = (3k + 2) \times 1 + (2k + 2).$$

If $\mathbf{k > 0}$, then $0 \leq 2k + 2 < 3k + 2$ hence **1 is the quotient, $\mathbf{2k + 2}$ is the remainder**.

If $k = 0$, then $2k + 2$ is not the remainder as it is equal to $3k + 2$. Instead, $4 = 2 \times 2 + 0$, so:

If k = 0, 2 is the quotient and 0 is the remainder.

Feedback: *Many students did not consider the case $k = 0$ hence lost some marks. In this case, the inequality $2k+2 < 3k+2$ does not hold, hence this case must be considered separately.*

(iii) Observe that

$$\begin{aligned} -5\ell - 4 &= (3\ell + 2) \times (-1) + (-2\ell - 2) \\ &= (3\ell + 2) \times (-2) + \ell, \end{aligned}$$

hence $\mathbf{-2}$ **is the quotient and** ℓ **is the remainder**, as $0 \le \ell < 3\ell + 2$.

Feedback: *Generally done well except in cases where students gave a negative number as the remainder. Remember, the remainder is non-negative even if a is negative!*

[5 marks]

A4.

(i) Use Euclid's algorithm to find the greatest common divisor of 35 and 91.

(ii) Describe all solutions $(x, y) \in \mathbb{Z}^2$ of the Diophantine equation

$$35x + 91y = 28.$$

(iii) Determine whether 35 is invertible mod 91, and if so, find the inverse of 35 mod 91. Give your reasons.

[5 marks]

A4. Solution (A standard example as in classwork/homework.)

(i) Euclid's algorithm:

$$\begin{aligned} 91 &= 35 \times 2 + 21, \\ 35 &= 21 \times 1 + 14, \\ 21 &= 14 \times 1 + 7, \\ 14 &= 7 \times 1 + 0. \end{aligned}$$

Hence $\mathbf{gcd(35, 91) = 7}$.

Feedback: *Done very well.*

(ii) The equation has solutions as 7 divides 28.

To find a particular solution, we first write 7 as an integral linear combination of 35 and 91, working back up Euclid's algorithm:

$$\begin{aligned} 7 &= 21 - 14 = 21 - (35 - 21) = 35 \times (-1) + 21 \times 2 \\ &= 35 \times (-1) + (91 - 35 \times 2) \times 2 \\ &= 35 \times (-5) + 91 \times 2. \end{aligned}$$

Multiply through by 4 to obtain the particular solution

$$(\mathbf{x_0}, \mathbf{y_0}) = (\mathbf{-20}, \mathbf{8}).$$

Hence the general solution is

$$(\mathbf{x}, \mathbf{y}) = (-20 - \frac{91}{7}t, 8 + \frac{35}{7}t) = (\mathbf{-20 - 13t}, \mathbf{8 + 5t}), \qquad \mathbf{t} \in \mathbb{Z}.$$

Remark. It is possible to find a different general solution by inspection, not using the result of Euclid's algorithm: for example, $35 \times 6 + 91 \times (-2) = 28$ so $(6, -2)$ is a particular solution.

Feedback: *A common mistake was to find a particular solution for $35x + 91y = 7$ and then to forget to multiply through by 4.*

(iii) No, 35 is **not invertible** mod 91 because 35 is **not coprime** to 91.

Feedback: *Done well.*

[5 marks]

A5. Let permutations in S_9 be given by

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 8 & 6 & 2 & 9 & 1 & 3 & 5 & 7 & 4 \end{pmatrix}, \qquad \tau = (1, 5, 6, 4, 7, 8) \circ (4, 9, 6, 5).$$

(i) Write σ and τ as products of disjoint cycles. Find the orders of σ and τ.
(ii) Give an example of $k \in \mathbb{Z}$ such that $\sigma^k = \tau$.
(iii) Hence or otherwise, show that $\sigma \circ \tau = \tau \circ \sigma$.

[5 marks]

A5. Solution (As in classwork and homework)

(i) $\sigma = (1, 8, 7, 5) \circ (2, 6, 3) \circ (4, 9)$ of order $\mathrm{lcm}(4, 3, 2) = 12$;

$\tau = (1, 5, 7, 8) \circ (4, 9)$ of order $\mathrm{lcm}(4, 2) = 4$.

Feedback: *Done well, but some students had trouble calculating the LCM of the very small integers involved. Pay attention to your arithmetic skills!*

(ii) For example, $k = 3$. In general, $k \equiv 3 \bmod 12$.

Feedback: *Done well. No explanation was required.*

(iii) $\sigma \circ \tau = \sigma \circ \sigma^3 = \sigma^4 = \sigma^3 \circ \sigma = \tau \circ \sigma$.

Feedback: *Done well, but many opted to fully calculate $\sigma\tau$ and $\tau\sigma$ in two-row notation rather than use the more general idea above. This was correct but cost students time in the exam.*

[5 marks]

B6.

(i) (a) Explain how the real numbers x^n are defined inductively, for any $x \in \mathbb{R}$ and any $n \in \mathbb{Z}^+$;
(b) describe the *induction principle* for statements $P(n)$, where $n \in \mathbb{Z}^+$.

[5 marks]

(ii) Prove by induction on n that

$$\sum_{j=1}^{n} j \;=\; \frac{1}{2}n(n+1)$$

for every positive integer n.

[5 marks]

B6. Solution (Bookwork and classwork)

(i) (a) The real numbers x^n are defined inductively by $x^1 = x$ and $x^{n+1} = x \cdot x^n$ for every integer $n \geq 1$.
(b) The induction principle for $P(n)$, $n \in \mathbb{Z}^+$, states that $P(n)$ is true for all positive integers n if and only if (a) $P(1)$ is true (the base case), and (b) $P(k) \Rightarrow P(k+1)$ for all integers $k \geq 1$.

[5 marks]

(ii) Let $P(n)$ be the statement that $\sum_{j=1}^{n} j = \frac{1}{2}n(n+1)$. This is clearly true for $n = 1$, the base case, because $\sum_{j=1}^{1} j = 1 = \frac{1}{2} \cdot 1 \cdot 2$.

Now assume that $P(k)$ is true for some positive integer k; thus $\sum_{j=1}^{k} j = \frac{1}{2}k(k+1)$. Adding $k+1$ to both sides gives

$$\begin{aligned}\sum_{j=1}^{k+1} j &= \frac{1}{2}k(k+1) + k + 1 \\ &= \frac{1}{2}\big(k(k+1) + 2(k+1)\big) \\ &= \frac{1}{2}(k+1)(k+2).\end{aligned}$$

Thus $P(k+1)$ is true. It follows that $P(k) \Rightarrow P(k+1)$ for any positive integer k, so $P(n)$ is true for all positive integers n by induction.

[5 marks]

B7.

(i) Define the *Cartesian product* $A \times B$ of two sets A and B, and describe the plane $\mathbb{R}^2$ as a Cartesian product. List the elements of the set $X \times Y$ when $X = \{1, 2, 3\}$ and $Y = \{3, 4\}$, and deduce that the Cartesian product operation is not necessarily commutative.

[5 marks]

(ii) For any sets C, D, and E, prove that

$$(C \cup D) \times E \;=\; (C \times E) \cup (D \times E),$$

and explain how this equation simplifies (a) when $D = \varnothing$ and (b) when $E = \varnothing$.

[5 marks]

B7. Solution (Bookwork, classwork, and similar to homework)

(i) The Cartesian product $A \times B$ is the set $\{(a, b) : a \in A, b \in B\}$. Thus

$$\mathbb{R}^2 \;:=\; \{(x, y) : x, y \in \mathbb{R}\} \;=\; \mathbb{R} \times \mathbb{R}.$$

For $X = \{1, 2, 3\}$ and $Y = \{3, 4\}$,

$$X \times Y = \{(1, 3), (1, 4), (2, 3), (2, 4), (3, 3), (3, 4)\}.$$

Since $(1, 3)$ lies in $X \times Y$ but not in $Y \times X$, it follows that $X \times Y \neq Y \times X$. So the operation $\times$ is not necessarily commutative.

[5 marks]

(ii) If x lies in $(C \cup D) \times E$ then $x = (a, e)$, where a lies in $C \cup D$, and e lies in E. Thus a lies in C or D, and e lies in E, so (a, e) lies in $C \times E$ or in $D \times E$. Hence (a, e) lies in $(C \times E) \cup (D \times E)$.

Vice versa, if y lies in $(C \times E) \cup (D \times E)$ then y lies in $C \times E$ or in $D \times E$. Thus $y = (b, f)$, where b lies in C or D, and f in E. Hence (b, f) lies in $(C \cup D) \times E$.

Finally, we deduce that $(C \cup D) \times E \;=\; (C \times E) \cup (D \times E)$, as required.

If $A = \varnothing$ then $A \times B = \varnothing$ for any set B, since there is no available choice for first coordinate. So (a) when $D = \varnothing$, the equation gives $C \times E = C \times E$, and (b) when $E = \varnothing$ it gives $\varnothing = \varnothing$.

[5 marks]

B8.

(i) For non-negative integers k, n with $k \leq n$, give the definition of

$$\binom{n}{k}$$

in terms of subsets of a finite set A of cardinality n. If $\mathcal{P}(A)$ denotes the power set of A, write down a bijection from $\mathcal{P}(A)$ to $\mathcal{P}(A)$ which maps subsets of A with k elements to subsets of A with $n-k$ elements. Hence conclude that

$$\binom{n}{k} = \binom{n}{n-k}.$$

[5 marks]

(ii) Give a formula for $\binom{n}{k}$ using factorials. Assuming that $n \geq 2$, write down explicit formulae for $\binom{n}{0}$, $\binom{n}{1}$ and $\binom{n}{2}$ without factorials. Use these formulae to show that

$$\binom{n}{0} - \binom{n}{1} + 2\binom{n}{2} = (n-1)^2.$$

[5 marks]

B8. Solution (Straight bookwork and homework; formulas for $\binom{n}{k}$ for small k are classwork.)

(i) Let A be a set of cardinality n, and let $\mathcal{P}_k(A)$ denote the set of all subsets $X \subseteq A$ such that $|X| = k$. By definition,

$$\binom{n}{k} = |\mathcal{P}_k(A)|.$$

Consider the following function:

$$f\colon \mathcal{P}(A) \to \mathcal{P}(A), \qquad X \mapsto A \setminus X.$$

Then f is a bijection from $\mathcal{P}(A)$ to itself, and f restricts to a bijection

$$\mathcal{P}_k(A) \to \mathcal{P}_{n-k}(A).$$

The cardinalities of these two sets are therefore equal, giving $\binom{n}{k} = \binom{n}{n-k}$.

Feedback: *Done correctly by many, however, some students stated that the bijection exists but failed to write it down. Also, some gave the factorial formula for* $\binom{n}{k}$ *as the definition; it is not the definition but rather a theorem which needs to be proved, and was proved in the course. There were students who confused a set with its cardinality; e.g. incorrectly writing "*$\binom{n}{k} = \mathcal{P}_k(A)$*". Marks were deducted for this.*

(ii) One has $\binom{n}{k} = \dfrac{n!}{(n-k)!\,k!}$. Explicitly,

$$\binom{n}{0} = 1, \qquad \binom{n}{1} = n, \qquad \binom{n}{2} = \frac{n(n-1)}{2}.$$

Substitute to obtain

$$\begin{aligned}\binom{n}{0}-\binom{n}{1}+2\binom{n}{2} &= 1-n+2\times\frac{n(n-1)}{2}\\ &= 1-n+n^2-n\\ &= n^2-2n+1\\ &= (n-1)^2.\end{aligned}$$

[5 marks]

Feedback: *Part (ii) was done quite well.*

B9.

(i) Determine all possible remainders that x^5-y^5, where $x,y\in\mathbb{Z}$, can leave when divided by 11. Hence or otherwise, show that the equation $x^5-y^5=10101$ has no integer solutions.

[5 marks]

(ii) State Fermat's Little Theorem for calculating the residues of a^p and of a^{p-1} modulo prime p. Use the Theorem to show that if $x^5\equiv y^5 \bmod 5$, then $x\equiv y \bmod 5$. Prove that if this holds, x^5-y^5 is divisible by 25. Deduce that the equation $x^5-y^5=10110$ has no integer solutions.

[5 marks]

B9. Solution ((i) is standard as in classwork/homework; in (ii) divisibility by 25 is more challenging but a similar example with 3 and $3^2=9$ appeared in homework)

(i) Let us find possible remainders of x^5 when divided by 11:

$x \bmod 11$	0	1	2	3	4	5	6	7	8	9	10
$x^5 \bmod 11$	0	1	10	1	1	1	10	10	10	1	10

These can be found using successive squaring where needed:

$$\begin{array}{ll} 0^5=0, & 3^5=81\times 3\equiv 4\times 3\equiv 1,\\ 1^5=1, & 4^5=(2^5)^2\equiv 10^2\equiv 1,\\ 2^5=32\equiv 10, & 5^5=25^2\times 5\equiv 3^2\times 5=45\equiv 1 \bmod 11,\end{array}$$

and observing that $(11-x)^5\equiv -x^5 \bmod 11$.

Since x^5 is always congruent to 0, 1 or -1 mod 11, it is easy to see that

the possible remainders of $\mathbf{x^5-y^5}$ mod 11 are 0, 1, 2, $-1\equiv$10, $-2\equiv$9.

The number **10101 is congruent to 3 mod 11:** e.g., $10101=10^4+10^2+1\equiv(-1)^4+(-1)^2+1=3$ mod 11. Since 3 is not one of the remainders found above, 10101 cannot be written as x^5-y^5.

[5 marks]

Feedback: *Generally done well. Some students calculated remainders of x^5 using large numbers such as $9^5 = 59049$. This was correct yet unnecessary, and may have cost the students time in the exam — see above. Some did not include 0 in the list of possible remainders of $x^5 - y^5$, or wrote negative remainders — marks were deducted for this.*

(ii) **Fermat's Little Theorem:**

Modulo prime p, $a^p \equiv a$ for all $a \in \mathbb{Z}$ and $a^{p-1} \equiv 1$ whenever a is coprime to p.

Putting $p = 5$ gives $x^5 \equiv x$ and $y^5 \equiv y$ mod 5 for all integers x, y, hence $x^5 \equiv y^5$ implies $x \equiv y$.

Now assume that $x \equiv y$ mod 5, meaning that $5 \mid (x - y)$. Then $x - y = 5k$ for some integer k, hence by the Binomial Theorem

$$\begin{aligned} x^5 - y^5 &= (y + 5k)^5 - y^5 \\ &= y^5 + 5y^4 \times 5k + 10y^3 \times 25k^2 + 10y^2 \times 5^3k^3 + 5y \times 5^4k^4 + 5^5k^5 - y^5 \end{aligned}$$

where y^5 cancels, and each of the remaining terms is divisible by 5^2 showing that $25 \mid (x^5 - y^5)$.

Finally, 10110 is divisible by 5 but not by 25, hence cannot be written as $x^5 - y^5$ due to the above.

[5 marks]

Feedback: *In stating Fermat's Little Theorem, some students lost marks for failing to specify that a was an integer and p was a prime number. The implication $x^5 \equiv y^5$ mod 5 $\implies x^5 \equiv y^5$ mod 25 was only demonstrated by a minority of students. Finally, some incorrectly implied that $x^5 - y^5$ is always divisible by 25. The correct way was to assume that a solution to $x^5 - y^5 = 10110$ exists, **then** to say that $x^5 - y^5$ is divisible by 5, **hence** is divisible by 25, a contradiction as 10110 is not divisible by 25. In other words, without pointing out that 10110 is divisible by 5, the argument is not valid.*

B10.

(i) Define Euler's phi-function $\phi(n)$ for $n \geq 1$. Let p be a prime number; show that $\phi(p) = p - 1$ and calculate $\phi(p^k)$ for any $k \geq 1$, giving your reasons.

[5 marks]

(ii) Prove that if $n \geq 1$ is such that $\phi(n) = n - 1$, then n is a prime number. Give an example of a positive integer m such that $\phi(m) = 10100$; is m prime?

[5 marks]

B10. Solution ((i) is pure bookwork; (ii) is is based on an exercise in lectures but not straight bookwork/homework.)

(i) By definition,

$$\textbf{if } n \geq 1 \textbf{ then } \phi(n) \textbf{ is the cardinality of the set}$$
$$\{\mathbf{r} \in \mathbb{Z} : \mathbf{0} \leq \mathbf{r} \leq \mathbf{n} - \mathbf{1},\ \mathbf{r} \textbf{ is coprime to } \mathbf{n}\}.$$

Now let p be a prime and $n = p^k$ where $k \geq 1$. The integers between 0 and $n - 1$ that are **not** coprime to p^k are exactly the multiples of p in this interval, because they must have a prime factor in common with n, and such a prime factor can only be p, the only prime which divides n. These are ℓp for $0 \leq \ell < p^{k-1}$ and there are p^{k-1} of them, hence

$$\phi(\mathbf{p^k}) = \mathbf{p^k - p^{k-1}}.$$

In particular,

$$\phi(\mathbf{p}) = \mathbf{p - 1}.$$

[5 marks]

Feedback: *Many people tried to define $\phi(n)$ as the cardinality of the set where $1 \leq r \leq n-1$. This is incorrect because $\phi(1) = 1 \neq 0$. You can use either $0 \leq r \leq n-1$ or $1 \leq r \leq n$. Further, some students claimed that p is coprime to all positive integers less than p by definition of a prime number. No, this is not the definition of prime number, and must be explained. Finally, many miscounted the number of integers from the relevant set which are not coprime to p^k, or failed to explain why those, and only those, integers are not coprime to p^k.*

(ii) Let $n \geq 1$ and $\phi(n) = n - 1$. Assume for contradiction that n is not prime. There are

two cases: $\mathbf{n = 1}$ or $\mathbf{n}$ is composite.

- If $n = 1$, one has $\phi(1) = 1 \neq 0$, a contradiction.
- If n is composite, then $n > 1$ and $n = ab$ with $1 < a, b < n$. Then a is an integer between 0 and $n-1$ and a is not coprime to n, since $\gcd(a, n) = a > 1$. But $\phi(n) = n - 1$ implies that all integers between 0 and $n-1$, except 0, are coprime to n, a contradiction.

Contradictions in both cases show that the assumption "n is not a prime" was false, and n is a prime.

Assume that $\phi(m) = 10100$. If m were a prime, m would have to be equal to 10101 — which is not a prime (is divisible by 3). Hence m must be composite. Observing that $10100 = 101 \times 100 = 101^2 - 101$ where 101 is a prime, we arrive at the following example:

$$\mathbf{m = 101^2 = 10201}.$$

[5 marks]

Remark. One can find further examples of m such that $\phi(m) = 10100$ (*this was not required in the exam*). For example, observing that 10201 is coprime to 2 and that $\phi(2) = 2 - 1 = 1$, we conclude that

$$\phi(20402) = \phi(2 \times 10201) = \phi(2)\phi(10201) = 1 \times 10100 = 10100.$$

There are three further solutions:

$$m = 15153 = 3 \times 5051, \qquad m = 20204 = 2^2 \times 5051, \qquad m = 30306 = 2 \times 3 \times 5051.$$

It is easy to check that $\phi(m) = 10100$ in each case, if one notices that 5051 is prime. However, finding those solutions and showing that there are no other solutions was beyond the scope of the exam paper.

Feedback: *A very popular mistake was to say "assume that n is not a prime. Then n is composite." This is wrong because the number* 1 *is neither prime nor composite. Some wrote that a composite n is not coprime to some positive integer less than n, without explaining this or claiming that this was the definition of composite. No, this is not the definition, and must be justified. Finally, the equation* $\phi(m) = 10100$ *proved to be difficult — as intended. Some incorrectly claimed that* 10101 *was a prime, despite the sum of digits being* 3 *hence* 10101 *being a multiple of* 3*. Well done to those who found a solution; some found the solutions* 15153 *and some other solutions as well, which was a pleasant surprise!*

FEEDBACK ON JANUARY 2017 EXAM; A1, A2, B6, B7

Because of the new examination format (the requirement to answer everything), I tried to make these four questions relatively straightforward. The majority of students answered them fairly well, and showed signs of beginning to appreciate the aims and objectives of the course.

Question A1

Generally well done, but

(ii) many students presented a truth table with propositions R and S, rather than just R; this caused a lot of confusion

(v) many students had difficulty listing all 8 possible combinations of Q, R and S, and produced a table with 4 (or sometimes 6) rows; a few simply began with a column for (Q and R) and a column for S, which is also not complete.

Question A2

Again well done, but with wobbles in parts (i) and (ii):

(i) a few students correctly identified the proposition as true, but attempted to prove it by disproving the contrapositive

(ii) many students correctly claimed that the statement is false, but attempted to disprove it by giving a counterexample for fixed v (such as $v = 1$).

Question B6

The marks for this question were generally quite high, in spite of the following.

(i)(a) A majority of students failed to understand what is meant by an inductive *definition*: there is no proposition $P(n)$, and nothing to prove!

(i)(b) A lot of students tried to *prove* the induction principle; it is an *axiom* that we always assume to be true.

(ii) $P(n)$ is *not* the formula $\frac{1}{2}n(n+1)$; it is the *statement* $\sum_{j=1}^{n} j = \frac{1}{2}n(n+1)$, which is very different (a statement always has a verb). For the base case several students wrote things such as "$P(1) \Rightarrow Q$ and Q is true, so $P(1)$ is true". That way madness lies, as I explained over and over again in lectures. A disappointingly large number also replaced "...assume that $P(k) \Rightarrow P(k+1)$" with "...assume that $P(k)$ and $P(k+1)$ are true"; the latter is different, and wrong. Finally, many students missed out logical connectors completely — yet they lie at the heart of the course!

Question B7

This was done less well than B6.

(i) This part was meant to offer some easy marks, but a lot of students seem not to understand Cartesian products. It is meaningless to write statements such as "$A \times B = (x, y)$", or "$A \times B = x \in A$ and $y \in B$". Many students also ignored the question about $\mathbb{R}^2$ — which is the motivating (and commonest) example!

(ii) Steps such as $(C \cup D) \times E \Rightarrow (C \times E) \cup (D \times E)$ appeared awfully often, even though the left hand side and right hand side are not statements, and cannot possibly be linked by $\Rightarrow$. It is like writing $11 \Rightarrow 37$. Other students offered no indication of logic, and seemed to be sprinkling unrelated (or perhaps related?) formulae around the page at random — that is not mathematics. Finally, $\{0\}$ is not the empty set!

12 February 2017