MATH10101, optional exercises on binomial coefficients. Will not be discussed in the supervisions

Opt1. In a card game you are dealt a hand of 13 cards from a normal playing deck of 52 cards.

i) How many different hands are possible?

ii) How many hands will contain all four aces?

iii) How many hands will contain no hearts?

iv) How many hands will contain at least one spade?

Opt2. Expand $(4x - 3y)^5$.

Opt3. Use the Binomial Theorem to calculate $\sum_{r=0}^{n} \frac{3^r 5^{n-r}}{r!\,(n-r)!}$.

Opt4. Calculate:

(i) $\binom{6}{0}2^{-0} + \binom{6}{1}2^{-1} + \cdots + \binom{6}{6}2^{-6}$;

(ii) $\binom{6}{0}(-2)^0 + \binom{6}{1}(-2)^1 + \cdots + \binom{6}{6}(-2)^6$.

Opt5. Find $x > 0$ that satisfy

(i) $x^2 = \sum_{r=0}^{4} 4^r \binom{4}{r}$;

(ii) $x^2 = \sum_{r=0}^{3} 3^r \binom{3}{r}$.

Opt6. Use the factorial formula for the binomial coefficient to prove that

$$r\binom{n}{r} = n\binom{n-1}{r-1}$$

for all $1 \leq r \leq n$.