Three hours

The number of marks available on this paper is 75.

THE UNIVERSITY OF MANCHESTER

FOUNDATIONS OF PURE MATHEMATICS A

17 January 2017

09:45 – 12:45

Answer **ALL FIVE** questions in Section A (25 marks in all) and
ALL FIVE questions in Section B (50 marks in all).

Electronic calculators may be used, provided they cannot store text

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SECTION A

Answer **ALL FIVE** questions

A1. Construct truth tables for the statements:

(i) $R \Leftarrow S$
(ii) R and (not R)
(iii) not (R or S)
(iv) (not R) or (not S)
(v) (Q and R) $\Rightarrow S$.

[5 marks]

A2. Prove or disprove each of the following statements:

(i) $\forall u \in \mathbb{R}^+$, $\exists v \in \mathbb{R}^+$, $u = 2v$
(ii) $\exists v \in \mathbb{R}^+$, $\forall u \in \mathbb{R}^+$, $u = 2v$
(iii) $\forall u \in \mathbb{R}^+$, $\forall v \in \mathbb{R}^+$, $u \geq 2v$
(iv) $\exists u \in \mathbb{R}^+$, $\exists v \in \mathbb{R}^+$, $u \geq 2v$
(v) $\exists v \in \mathbb{R}^{\geq}$, $\forall u \in \mathbb{R}^+$, $u > 2v$.

[5 marks]

A3.

(i) State the Division Theorem and explain what is meant by the quotient and the remainder when an integer a is divided by a positive integer b.
(ii) Let k be a non-negative integer. Find the quotient and the remainder of $5k + 4$ when divided by $3k + 2$, writing them in terms of k. Give your reasons.
(iii) If ℓ is a non-negative integer, find the quotient and the remainder when $-5\ell - 4$ is divided by $3\ell + 2$. Give your reasons.

[5 marks]

A4.

(i) Use Euclid's algorithm to find the greatest common divisor of 35 and 91.
(ii) Describe all solutions $(x, y) \in \mathbb{Z}^2$ of the Diophantine equation

$$35x + 91y = 28.$$

(iii) Determine whether 35 is invertible mod 91, and if so, find the inverse of 35 mod 91. Give your reasons.

[5 marks]

A5. Let permutations in S_9 be given by

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 8 & 6 & 2 & 9 & 1 & 3 & 5 & 7 & 4 \end{pmatrix}, \qquad \tau = (1, 5, 6, 4, 7, 8) \circ (4, 9, 6, 5).$$

(i) Write σ and τ as products of disjoint cycles. Find the orders of σ and τ.
(ii) Give an example of $k \in \mathbb{Z}$ such that $\sigma^k = \tau$.
(iii) Hence or otherwise, show that $\sigma \circ \tau = \tau \circ \sigma$.

[5 marks]

SECTION B

Answer **ALL FIVE** questions

B6.

(i) (a) Explain how the real numbers x^n are defined inductively, for any $x \in \mathbb{R}$ and any $n \in \mathbb{Z}^+$;
(b) describe the *induction principle* for statements $P(n)$, where $n \in \mathbb{Z}^+$.

[5 marks]

(ii) Prove by induction on n that

$$\sum_{j=1}^{n} j \;=\; \frac{1}{2}n(n+1)$$

for every positive integer n.

[5 marks]

B7.

(i) Define the *Cartesian product* $A \times B$ of two sets A and B, and describe the plane $\mathbb{R}^2$ as a Cartesian product. List the elements of the set $X \times Y$ when $X = \{1, 2, 3\}$ and $Y = \{3, 4\}$, and deduce that the Cartesian product operation is not necessarily commutative.

[5 marks]

(ii) For any sets C, D, and E, prove that

$$(C \cup D) \times E \;=\; (C \times E) \cup (D \times E),$$

and explain how this equation simplifies (a) when $D = \varnothing$ and (b) when $E = \varnothing$.

[5 marks]

B8.

(i) For non-negative integers k, n with $k \leq n$, give the definition of

$$\binom{n}{k}$$

in terms of subsets of a finite set A of cardinality n. If $\mathcal{P}(A)$ denotes the power set of A, write down a bijection from $\mathcal{P}(A)$ to $\mathcal{P}(A)$ which maps subsets of A with k elements to subsets of A with $n - k$ elements. Hence conclude that

$$\binom{n}{k} = \binom{n}{n-k}.$$

[5 marks]

(ii) Give a formula for $\binom{n}{k}$ using factorials. Assuming that $n \geq 2$, write down explicit formulae for $\binom{n}{0}$, $\binom{n}{1}$ and $\binom{n}{2}$ without factorials. Use these formulae to show that

$$\binom{n}{0} - \binom{n}{1} + 2\binom{n}{2} = (n-1)^2.$$

[5 marks]

B9.

(i) Determine all possible remainders that $x^5 - y^5$ (where $x, y \in \mathbb{Z}$) can leave when divided by 11. Hence or otherwise, show that the equation $x^5 - y^5 = 10101$ has no integer solutions.

[5 marks]

(ii) State Fermat's Little Theorem for calculating the residues of a^p and of a^{p-1} modulo prime p. Use the Theorem to show that if $x^5 \equiv y^5 \bmod 5$, then $x \equiv y \bmod 5$. Prove that if this holds, $x^5 - y^5$ is divisible by 25. Deduce that the equation $x^5 - y^5 = 10110$ has no integer solutions.

[5 marks]

B10.

(i) Define Euler's phi-function $\phi(n)$ for $n \geq 1$. Let p be a prime number; show that $\phi(p) = p - 1$ and calculate $\phi(p^k)$ for any $k \geq 1$, giving your reasons.

[5 marks]

(ii) Prove that if $n \geq 1$ is such that $\phi(n) = n - 1$, then n is a prime number. Give an example of a positive integer m such that $\phi(m) = 10100$; is m prime?

[5 marks]

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END OF EXAMINATION PAPER