MATH10101, for supervision in week 13.

This problem sheet will not be discussed in the weekly supervision classes. The students are nevertheless advised to attempt all questions and then to work through the model solutions available via the course website. The material covered by the questions below is examinable.

Q40. Count the number of elements in the following subsets of S_8.

(i) The set of cycles of length 2.

(ii) The set of cycles of length 3.

(iii) The set of permutations that fix a given element.

Which of the above subsets are closed under composition?

Q41. Calculate the orders of the following permutations in S_{11} :

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 \\ 8 & 5 & 10 & 11 & 7 & 4 & 9 & 1 & 2 & 3 & 6 \end{pmatrix},$$

$$\tau = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 \\ 2 & 4 & 6 & 8 & 10 & 5 & 7 & 9 & 11 & 1 & 3 \end{pmatrix} \circ \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 \\ 3 & 6 & 9 & 1 & 4 & 7 & 10 & 2 & 5 & 8 & 11 \end{pmatrix}.$$

Q42. Calculate the orders of the following permutations in S_5 :

(i) $(1,2,3) \circ (1,3,4) \circ (1,3,5)$,

(ii) $(1,2) \circ (1,3) \circ (1,4) \circ (1,5)$,

(iii) $(2,3,5) \circ (1,2) \circ (2,4) \circ (1,2)$.

Q43. (*not quite easy*) Give an example of a permutation in S_{13} with the largest possible order.

Q44. Define $*$ on $\mathbb{Q}$ by

$$\frac{a}{b} * \frac{c}{d} = \frac{a+c}{b+d},$$

where $a, c \in \mathbb{Z}$ and $b, d \in \mathbb{N}$. Is this a binary operation?

Q45. Are the following operations closed?

(i) Addition on the set of odd integers,

(ii) Multiplication on the set of even integers,

(iii) $a \circ b = a + b - ab$ on the set of odd integers.

Q46. Which of the following binary operations are associative and which are commutative? **Give your reasons**.

(i) $x * y = 2(x + y)$ on $\mathbb{R}$,

(ii) $x * y = x|y|$ on $\mathbb{R}$,

(iii) $x * y = \dfrac{x + y}{xy}$ on $\mathbb{R} \setminus \{0\}$,

(iv) $x * y = x + y - xy$ on $\mathbb{Z}$,

(v) $x * y = \max(x, y)$ on $\mathbb{N}$,

Q47. Which of the following binary operations have identities:

(i) $x * y = \max(x, y)$ on $\{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$,

(ii) $x * y = \max(x, y)$ on $\mathbb{Z}$,

(iii) $x * y = x + y - xy$ on $\mathbb{Q}$,

(iv) matrix multiplication on the set $\left\{ \begin{pmatrix} a & a \\ a & a \end{pmatrix} : a \in \mathbb{R} \setminus \{0\} \right\}$?

Q48. Why is $(\mathbb{Z}, -)$ not a group?

Q49. a) Draw up the multiplication table for $(\{4, 8, 12, 16, 20, 24\}, \times_{28})$. Find the identity element and the inverse of each element. (Here $\times_{28}$ denotes multiplication mod 28, in other words the operation of multiplyimng two integers and then taking the remainder on division by 28.)

b) Can you find a **proper** subset of $\{4, 8, 12, 16, 20, 24\}$ which is closed under $\times_{28}$?