MATH10101, optional exercises on proofs in elementary number theory. Will not be discussed in the supervisions

Opt15. Let a, b, c be integers. Prove that if $c \mid a$ and $c \nmid b$, then $c \nmid (a+b)$.

Opt16. Prove that the relation $\mid$ (divides) on $\mathbb{Z}$ is transitive.

Opt17. Let $a \equiv b \mod m$ and let $n \mid m$. Prove that $a \equiv b \mod n$.

Opt18. Let a, b, c be integers, $a \mid c$, $b \mid c$, $\gcd(a, b) = 1$. Prove that $ab \mid c$.

Opt19. Let a be an integer, p, q be primes, $p \neq q$. Prove: $(\, p \mid a \,) \wedge (\, q \mid a) \implies pq \mid a$.

Opt20. Let the relation $\sim$ on $\mathbb{Z}$ be defined by $a \sim b$ iff $5 \mid 9a^2 + b^2$.

a) Prove that $\sim$ is an equivalence relation.

b) Show that the equivalence class of 0 is the set of all integers divisible by 5.

c) Work out all the other equivalence classes induced by the relation $\sim$.

Opt21. Let the relation $\sim$ on the set $\mathbb{Z}_n$ be defined as follows: $[a]_n \sim [b]_n$ iff $\exists [x]_n \in \mathbb{Z}_n^*$, $[b]_n = [x]_n [a]_n$.

a) Use the properties of $\mathbb{Z}_n^*$, prove that $\sim$ is an equivalence relation on $\mathbb{Z}_n$.

b) Put $n = 14$. Find the equivalence classes induced on the set $\mathbb{Z}_{14}$ by the relation $\sim$.

Opt22. Let a, b be integers. Prove that the following two statements about a and b are equivalent:

(1) a and b are not coprime;
(2) there exists a prime p such that $p \mid a$ and $p \mid b$.

Opt23. Let $[0]_4$, $[1]_4$, $[2]_4$, $[3]_4$ be the congruence classes of the integers modulo 4; explain why there is only one prime in the set $[0]_4 \cup [2]_4 \subseteq \mathbb{Z}$ and show that $x, y \in [1]_4$ implies $xy \in [1]_4$. By considering integers of the form $4P - 1$, deduce that $[3]_4$ contains infinitely many primes.

Opt24. Prove that if $n \in \mathbb{N}$ and $n \geq 3$, then $\phi(n)$ is even. Here ϕ is Euler's phi-function.

Opt25. Prove Fermat's Little Theorem using Euler's theorem and properties of Euler's phi-function.