MATH10101, optional exercises on binomial coefficients. Will not be discussed in the supervisions — SOLUTIONS

Opt1. In a card game you are dealt a hand of 13 cards from a normal playing deck of 52 cards.

i) How many different hands are possible?

ii) How many hands will contain all four aces?

iii) How many hands will contain no hearts?

iv) How many hands will contain at least one spade?

Opt1 - solution. (i) $\binom{52}{13}$;

(ii) $\binom{48}{9}$ (a hand contains 4 aces and a further 9 cards from a set of $52-4=48$ cards);

(iii) $\binom{39}{13}$ (13-element subsets of the set of 39 cards which are not hearts);

(iv) $\binom{52}{13}-\binom{39}{13}$ (since there are $\binom{39}{13}$ hands which contain no spades).

NB: there is not much point in trying to write these numbers as decimals (but for reference, $\binom{52}{13}=635013559600>6\times 10^{11}$).

Opt2. Expand $(4x-3y)^5$.

Opt2 - solution.

$$\begin{aligned}(4x-3y)^5 &= (4x)^5+\binom{5}{1}(4x)^4(-3y)+\binom{5}{2}(4x)^3(-3y)^2\\ &\quad+\binom{5}{3}(4x)^2(-3y)^3+\binom{5}{4}(4x)(-3y)^4+(-3y)^5\\ &= 1024x^5-3840x^4y+5760x^3y^2-4320x^2y^3+1620xy^4-243y^5\end{aligned}$$

Opt3. Use the Binomial Theorem to calculate $\displaystyle\sum_{r=0}^{n}\frac{3^r5^{n-r}}{r!\,(n-r)!}$.

Opt3 - solution. $\displaystyle\sum_{r=0}^{n}\frac{3^r5^{n-r}}{r!(n-r)!}=\frac{1}{n!}\sum_{r=0}^{n}3^r5^{n-r}\frac{n!}{r!(n-r)!}=\frac{1}{n!}\sum_{r=0}^{n}3^r5^{n-r}\binom{n}{r}$

$=\dfrac{1}{n!}(3+5)^n=\dfrac{8^n}{n!}$.

Opt4. Calculate:

(i) $\binom{6}{0}2^{-0}+\binom{6}{1}2^{-1}+\cdots+\binom{6}{6}2^{-6}$;

(ii) $\binom{6}{0}(-2)^0+\binom{6}{1}(-2)^1+\cdots+\binom{6}{6}(-2)^6$.

Opt4 - solution. (i) The expression is a particular case of the expression $\binom{n}{0}a^nb^0+\binom{n}{1}a^{n-1}b^1+\cdots+\binom{n}{n}a^0b^n$ which by the Binomial Theorem is equal to $(a+b)^n$. Namely, to obtain the sum in (i), we put $n=6$, $a=1$ and $b=2^{-1}$. By the Binomial Theorem, the sum equals $(1+2^{-1})^6=(3/2)^6=729/64$.

(ii) Similarly we put $n=6$, $a=1$ and $b=-2$ in the Binomial Theorem. The sum is equal to $(1+(-2))^6=(-1)^6=1$.

Opt5. Find $x>0$ that satisfy

(i) $x^2=\sum_{r=0}^{4}4^r\binom{4}{r}$;

(ii) $x^2=\sum_{r=0}^{3}3^r\binom{3}{r}$.

Opt5 - solution. i) $\sum_{r=0}^{4}4^r\binom{4}{r}=(1+4)^4=25^2$, so $x=25$.

ii) $\sum_{r=0}^{3}3^r\binom{3}{r}=(1+3)^3=4^3=8^2$, so $x=8$.

Opt6. Use the factorial formula for the binomial coefficient to prove that

$$r\binom{n}{r}=n\binom{n-1}{r-1}$$

for all $1\le r\le n$.

Opt6 - solution. $r\binom{n}{r}=\dfrac{r\times n!}{r!\,(n-r)!}=\dfrac{r\times n!}{r\times(r-1)!\,(n-r)!}=\dfrac{n!}{(r-1)!\,(n-r)!}$

$=\dfrac{n\times(n-1)!}{(r-1)!\,((n-1)-(r-1))!}=n\binom{n-1}{r-1}$.