32031 Feedback Quiz, 2022/23, Week 08: Hamming codes

Open-book. 10–15 minutes. Not for credit. To be marked in class.

You are given a check matrix for a Hamming code over the field $\mathbb{F}_7$. Some entries are replaced by letters:

$$H = \begin{bmatrix} 1 & 1 & 2 & 3 & 4 & 3 & A & 4 \\ 2 & 5 & 6 & B & 4 & 0 & 3 & 2 \end{bmatrix}$$

Question 1 What is the code for which H is a check matrix?

○ Ham(7,2) ○ Ham(8,7) ○ Ham(8,2) ○ Ham(7,5) ○ Ham(7,7)
○ Ham(2,7) ○ Ham(2,8) ○ Ham(7,8) ○ Ham(5,7)

Question 2 Entry A is

○ 0 ○ 1 ○ 2 ○ 3 ○ 4 ○ 5 ○ 6

Question 3 Entry B is

○ 0 ○ 1 ○ 2 ○ 3 ○ 4 ○ 5 ○ 6

Question 4 Eve wrote down the syndrome, $S(\underline{y}) = \underline{y}H^T$, of every vector $\underline{y} \in \mathbb{F}_7^8$. How many *distinct* syndromes did Eve obtain?

○ 56 ○ 7 ○ 48 ○ 2^7 ○ 7^5 ○ 49 ○ 8 ○ 7^8 ○ 42

CORRECTED

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⊗ Ham(2,7) ◯ Ham(2,8) ◯ Ham(7,8) ◯ Ham(5,7)

Explanation: A Ham(r,q) code is a linear code over the field $\mathbb{F}_q$ which has a check matrix with r rows, satisfying certain further constraints. We are given that $q = 7$, and the matrix H has 2 rows, hence the code is Ham(2,7).

Question 2 Entry A is

⊗ 0 ◯ 1 ◯ 2 ◯ 3 ◯ 4 ◯ 5 ◯ 6

Explanation: A check matrix for Ham(2,7) has the following property: its columns are representatives of all lines (one-dimensional subspaces) in $\mathbb{F}_7^2$. There is a line in $\mathbb{F}_7^2$ which consists of column vectors with first (topmost) entry equal to zero: namely, the line which consists of $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$, $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$, $\begin{bmatrix} 0 \\ 2 \end{bmatrix}$, $\ldots$, $\begin{bmatrix} 0 \\ 6 \end{bmatrix}$. One representative of this line must be a column in H. Therefore, there must be a zero in the top row of H. The only place where zero can be is entry A. Hence $A = 0$.

Question 3 Entry B is

CORRECTED

○ 0 ○ 1 ○ 2 ○ 3 ⊠ 4 ○ 5 ○ 6

Explanation: We need to find the value of B such that the column $\begin{bmatrix} 3 \\ B \end{bmatrix}$ does not represent the same line as any other column of H. Observe that two columns, $\begin{bmatrix} a \\ b \end{bmatrix}$ and $\begin{bmatrix} c \\ d \end{bmatrix}$, represent the same line, if and only if $a^{-1}b = c^{-1}d$ or $a = c = 0$. Let us write the value of $a^{-1}b$ under each column $\begin{bmatrix} a \\ b \end{bmatrix}$ of H:

$$\begin{array}{rc} a = & \left[\begin{array}{cccccccc} 1 & 1 & 2 & 3 & 4 & 3 & 0 & 4 \\ 2 & 5 & 6 & B & 4 & 0 & 3 & 2 \end{array}\right] \\ b = & \\ a^{-1}b = & \begin{array}{cccccccc} 2 & 5 & 3 & 3^{-1}B & 1 & 0 & * & 4 \end{array} \end{array}$$

where we write $*$ under the column where $a = 0$. All the values in the bottom row must be distinct, since no two columns belong to the same line. We conclude that $3^{-1}Y = 6 = -1$ so $Y = -3 = 4$.

Question 4 Eve wrote down the syndrome, $S(\underline{y}) = \underline{y}H^T$, of every vector $\underline{y} \in \mathbb{F}_7^8$. How many *distinct* syndromes did Eve obtain?

○ 56 ○ 7 ○ 48 ○ 2^7 ○ 7^5 ⊠ 49 ○ 8 ○ 7^8 ○ 42

Explanation: The number of possible syndromes is equal to the number of cosets of the code, which is q^{n-k}. Incidentally, this shows (because syndromes are vectors of size $n-k$) that all vectors of size $n-k$ are syndromes. In this case, $n-k = r = 2$ so there are $7^2 = 49$ possible syndromes.