Exercises to Chapter 4

Exercise 4.1 (important fact about perfect linear codes — needed for exam)**.** Let C be a linear $[n, k, d]_q$-code. As usual, let $t = \left[\frac{d-1}{2}\right]$. Show:

(a) Every vector in $\mathbb{F}_q^n$ of weight $\le t$ is a **unique coset leader** of its coset (*i.e., is a coset leader, and its coset has no other coset leaders*).

(*Hint*: if $\underline{a}_1, \underline{a}_2$ are coset leaders of a coset, then $\underline{a}_1 - \underline{a}_2$ is a codevector of weight $\le w(\underline{a}_1) + w(\underline{a}_2) = 2w(\underline{a}_1)$.)

$w(\underline{a}) \le t$ Other vectors in the coset $\underline{a} + C$ are $\underline{a} + \underline{c}$ with $\underline{c} \in C$, $\underline{c} \neq 0$ (so: $w(\underline{c}) \geq d$)

$w(\underline{a}+\underline{c}) = w(\underline{a} - (-\underline{c})) = d(\underline{a}, -\underline{c})$

$d(\underline{a}, -\underline{c}) + \underbrace{d(\underline{a}, \underline{0})}_{w(\underline{a})} \geq d(-\underline{c}, \underline{0})$ || $w(\underline{a}+\underline{c}) + t \geq d$

so $w(\underline{a}+\underline{c}) \geq d - t > t$

(b) If C is perfect, the number of distinct cosets equals $\#S_t(\underline{0})$.

(*Hint.* By the Hamming bound, $M \times \#S_t(\underline{0}) \le q^n$, or is it $= q^n$?)

(c) Deduce that if C is perfect, every coset has a unique coset leader, all coset leaders are of weight $\le t$, and the set of all coset leaders is $S_t(\underline{0})$.