Chapter 1

Exercises (answers at end)

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Exercise 1.1. The **Manchester code** was first used in the Manchester Mark 1 computer at the University of Manchester in 1949 and is still used in low-speed data transfer: e.g. TV remote sending signals via infrared. This binary code consists of two codewords: 10 and 01. The codeword 10 is interpreted by the recipient as the message 0, and 01 is understood to mean 1; whereas the received word 00 or 11 indicates a detected error.

The following error-free fragment of a bit stream encoded by Manchester code (that is: the stream is a sequence of codewords) had been intercepted: $\ldots 010101x01011010\ldots$ What was the bit x?

$\mathrm{Rep}(3, \mathbb{F}_2) = \{000, 111\}$; $E_3 = \{000, 011, 101, 110\}$

Manchester code error detection

$= \{01, 10\}$ detects 1 error:

$01 \to 11$ ✗

$01 \to 00$ ✗

$\ldots\; 0\,1\,0\,1\,0\,1\,x\,0\,1\,0\,1\,1\,0\,1\,0\; \ldots$

$x = 0$

Exercise 1.2. Consider the alphabet $\mathbb{Z}_{10} = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$. The **Luhn checksum** of a word $x_1x_2 \ldots x_{16} \in (\mathbb{Z}_{10})^{16}$ is $\pi(x_1)+x_2+\pi(x_3)+x_4+\pi(x_5)+\cdots+x_{16}$ mod 10, viewed as an element of $\mathbb{Z}_{10}$. Here $\pi\colon \mathbb{Z}_{10} \to \mathbb{Z}_{10}$ is defined by the rule "$\pi(a)$ is the sum of digits of $2a$". The **Luhn code** consists of all words in $(\mathbb{Z}_{10})^{16}$ whose Luhn checksum is 0.

1234 5678 9012 3456

2+2+6+4 + 1+6+5+8 + 9+0+2+2+6+4+1+6

π: 0 1 2 3 4 5 6 7 8 9
0 2 4 6 8 1 3 5 7 9

Luhn checksum = 4

The Luhn code detects 1 error.

checksum 0

Change 1 digit

(i) Write down all values of π and check that π is a permutation of the alphabet $\mathbb{Z}_{10}$. ✓

(ii) Find the total number of codewords of the Luhn code.

(iii) Prove that a single digit error is detected by the Luhn code.

(iv) Look at your 16-digit debit/credit card numbers. Are they codewords of the Luhn code? If you have a card with a number which is **not** a codeword of the Luhn code, can you bring it to the tutorial? Thanks!

Exercise 1.3. (*based on a question from a past exam. Not done in the tutorial.*). Alice transmitted the same binary word of length 6 to Bob three times, but Bob received three different words: 101010, 011100, 110001. Engineer Clara told Bob that at most two bit

$$(\underbrace{x_1, x_2, x_3, \ldots x_{15}}_{\text{arbitrary digits from } \{0,1,\ldots,9\}}, x_{16})$$

$$\pi(x_1) + x_2 + \pi(x_3) + \cdots + \pi(x_{15}) \equiv -x_{16} \bmod 10$$

$x_{16} :=$ remainder of $(-\pi(x_1) - x_2 \cdots - \pi(x_{15}))$ when divided by 10.

$$10 \times 10 \times \cdots \times 10 \times 1 = 10^{15}$$