$g(\nabla_X Y, Z) = X(g(Y,Z)) - g(Y, \nabla_X Z)$ ①

(相容)

$g(\nabla_Y X, Z) = Y(g(X,Z)) - g(X, \nabla_Y Z)$

(无挠)

$g(\nabla_X Y, Z) = g([X,Y],Z) + g(\nabla_Y X, Z) = g([X,Y],Z) + Y(g(X,Z)) - g(X,\nabla_Y Z)$ ②

(相容)

$0 = -Z(g(X,Y)) + g(\nabla_Z X, Y) + g(X, \nabla_Z Y)$

(无挠)

$= -Z(g(X,Y)) + g(\nabla_X Z + [Z,X], Y) + g(X, \nabla_Y Z - [Y,Z])$ ③

①+②+③ 得

$2g(\nabla_X Y,Z) = X(g(Y,Z)) + Y(g(X,Z)) - Z(g(X,Y)) + g([X,Y],Z) + g([Z,X],Y) - g([Y,Z],X)$.

故 $g(\nabla_X Y,Z) = \frac{1}{2}(\cdots)$

20. 黎曼几何基本定理(定理4.8. 黎曼联络的唯一性)

$g(\nabla_X Y,Z) = \frac{1}{2}\big(X(g(Y,Z)) + Y(g(X,Z)) - Z(g(X,Y)) + g([X,Y],Z) + g([Z,X],Y) - g([Y,Z],X)\big)$

$g(\nabla_Y X,Z) = \frac{1}{2}\big(Y(g(X,Z)) + X(g(Y,Z)) - Z(g(Y,X)) + g([Y,X],Z) + g([Z,Y],X) - g([X,Z],Y)\big)$

$g(\nabla_X Z,Y) = \frac{1}{2}\big(X(g(Z,Y)) + Z(g(X,Y)) - Y(g(X,Z)) + g([X,Z],Y) + g([Y,X],Z) - g([Z,Y],X)\big)$

故 $g(\nabla_X Y,Z) - g(\nabla_Y X,Z) = \frac{1}{2}g([X,Y],Z) - \frac{1}{2}g([Y,X],Z) = g([X,Y],Z)$ (无挠)

$g(\nabla_X Y,Z) + g(\nabla_X Z,Y) = \frac{1}{2}X(g(Y,Z)) + \frac{1}{2}X(g(Z,Y)) = X(g(Y,Z))$ (相容)

21. 曲率算子的性质

曲率算子: $R(X,Y)Z = \nabla^2_{X,Y}Z - \nabla^2_{Y,X}Z = \nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{[X,Y]}Z$

(1) $R(X,Y) = -R(Y,X)$

(2) $R(fX,Y) = R(X,fY) = fR(X,Y)$

$R(fX,Y)Z = \nabla_{fX}\nabla_Y Z - \nabla_Y\nabla_{fX}Z - \nabla_{[fX,Y]}Z$

$= f\nabla_X\nabla_Y Z - \nabla_Y(f\nabla_X Z) - \nabla_{f[X,Y]-Y(f)X}Z$

$= f\nabla_X\nabla_Y Z - Y(f)\nabla_X Z - f\nabla_Y\nabla_X Z - f\nabla_{[X,Y]}Z + Y(f)\nabla_X Z$

$= fR(X,Y)Z$

注:(定理2.15) $[fX,Y] = f[X,Y] - Y(f)\cdot X$

$[X,fY] = f[X,Y] + X(f)\cdot Y$