由(1)有 $R(X, fY) = -R(fY, X) = -fR(Y, X) = fR(X, Y)$

(3) $R(X, Y)(fZ) = fR(X, Y)Z$

$R(X, Y)(fZ) = \nabla_X \nabla_Y (fZ) - \nabla_Y \nabla_X (fZ) - \nabla_{[X, Y]}(fZ)$

$= \nabla_X(Y(f)Z + f\nabla_Y Z) - \nabla_Y(X(f)Z + f\nabla_X Z) - [X, Y](f)Z - f\nabla_{[X, Y]}Z$

$= X(Y(f))Z + Y(f)\nabla_X Z + X(f)\nabla_Y Z + f\nabla_X\nabla_Y Z$

$\quad - (Y(X(f))Z + X(f)\nabla_Y Z + Y(f)\nabla_X Z + f\nabla_Y\nabla_X Z)$

$\quad - ([X, Y](f))Z - f\nabla_{[X, Y]}Z$

$= fR(X, Y)Z$

(4) 无挠联络, 即 $\nabla_X Y - \nabla_Y X = [X, Y]$ 时,

$R(X, Y)Z + R(Z, X)Y + R(Y, Z)X = \nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{[X, Y]}Z$

$\quad + \nabla_Z\nabla_X Y - \nabla_X\nabla_Z Y - \nabla_{[Z, X]}Y \quad + \nabla_Y\nabla_Z X - \nabla_Z\nabla_Y X - \nabla_{[Y, Z]}X$

$= \nabla_X(\nabla_Y Z - \nabla_Z Y) - \nabla_{[Y, Z]}X \quad + \nabla_Y(\nabla_Z X - \nabla_X Z) - \nabla_{[Z, X]}Y$

$\quad + \nabla_Z(\nabla_X Y - \nabla_Y X) - \nabla_{[X, Y]}Z$

$= \nabla_X[Y, Z] - \nabla_{[Y, Z]}X + \nabla_Y[Z, X] - \nabla_{[Z, X]}Y + \nabla_Z[X, Y] - \nabla_{[X, Y]}Z$

$= [X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]] = 0$

22. 曲率张量场的性质

曲率张量场: $R(X, Y, Z, W) = g(R(Z, W)X, Y)$

$\quad - R(X, Y, W, Z)$

(1) $R(X, Y, Z, W) = \overline{-R(Y, X, Z, W)}$ (由21(1)即得)

$R(X, Y, Z, W) = -R(Y, X, Z, W)$

$R(X, Y, Z, W) = g(R(Z, W)X, Y) = g(\nabla_Z\nabla_W X, Y) - g(\nabla_W\nabla_Z X, Y) - g(\nabla_{[Z, W]}X, Y)$

(相容)
$= Z(g(\nabla_W X, Y)) - g(\nabla_W X, \nabla_Z Y) - (W(g(\nabla_Z X, Y)) - g(\nabla_Z X, \nabla_W Y)) - ([Z, W](g(X, Y)) - g(X, \nabla_{[Z, W]}Y))$