8. 李括号的坐标表示（P_{30} 最上；练习2-6）. 给定局部坐标系 $(U,\varphi;x^i)$,

设 $X=X^i\frac{\partial}{\partial x^i}$, $Y=Y^j\frac{\partial}{\partial x^j}$, $X^i,Y^j\in C^\infty(M)$. 计算 $[X,Y](f)$

解: $[X,Y](f)=(XY-YX)(f)=X(Y(f))-Y(X(f))=X\left(\sum_j Y^j\frac{\partial f}{\partial x^j}\right)-Y\left(\sum_i X^i\frac{\partial f}{\partial x^i}\right)$

$=\sum_{i,j}X^i\frac{\partial Y^j}{\partial x^i}\frac{\partial f}{\partial x^j}+\sum_{i,j}X^iY^j\frac{\partial^2 f}{\partial x^j\partial x^i}-\sum_{i,j}Y^j\frac{\partial X^i}{\partial x^j}\frac{\partial f}{\partial x^i}-\sum_{i,j}Y^jX^i\frac{\partial^2 f}{\partial x^i\partial x^j}$

$=\sum_{i,j}X^i\frac{\partial Y^j}{\partial x^i}\frac{\partial f}{\partial x^j}-\sum_{i,j}Y^i\frac{\partial X^j}{\partial x^i}\frac{\partial f}{\partial x^j}=\sum_{i,j}\left(X^i\frac{\partial Y^j}{\partial x^i}-Y^i\frac{\partial X^j}{\partial x^i}\right)\frac{\partial f}{\partial x^j}$

定理2.17
9. 设G是李群, 若X, Y是G上的左不变向量场, 则 $[X,Y]$ 也是G上的左不变向量场

证明: 设X, Y是G上两个左不变向量场, 则 $\forall\, g\in G$,

$(L_g)_*[X,Y]_{}(f)=[X,Y]_{}(f\circ L_g)=((XY-YX)f)\circ L_g=X(Y(f\circ L_g))-Y(X(f\circ L_g))$

$=X((L_g)_*Y(f))-Y((L_g)_*X(f))=(L_g)_*X((L_g)_*Y(f))-(L_g)_*Y((L_g)_*X(f))$

$=((L_g)_*X(L_g)_*Y)(f)-((L_g)_*Y(L_g)_*X)(f)=[(L_g)_*X,(L_g)_*Y](f)=[X,Y](f)$

10. 叙述 Poincaré 引理, 并证明古典场论公式

Poincaré 引理: 对任意的外微分式 ω, 有 $d(d\omega)=0$

$\mathrm{curl}(\mathrm{grad} f)=0$: 设 f 是 R^3 上的光滑函数, 则 $\mathrm{grad} f=df=\frac{\partial f}{\partial x}dx+\frac{\partial f}{\partial y}dy+\frac{\partial f}{\partial z}dz$

故 $\mathrm{curl}(\mathrm{grad} f)=d(df)=\left(\frac{\partial^2 f}{\partial y\partial x}-\frac{\partial^2 f}{\partial x\partial y}\right)dx\wedge dy+\left(\frac{\partial^2 f}{\partial z\partial y}-\frac{\partial^2 f}{\partial y\partial z}\right)dy\wedge dz+\left(\frac{\partial^2 f}{\partial x\partial z}-\frac{\partial^2 f}{\partial z\partial x}\right)dz\wedge dx$
$=0$

$\mathrm{div}(\mathrm{curl} X)=0$: 设 $X=(A,B,C)$ 是 R^3 上的向量值函数, 记 $\omega=Adx+Bdy+Cdz$.

则 $\mathrm{curl} X=d\omega=\left(\frac{\partial C}{\partial y}-\frac{\partial B}{\partial z}\right)dy\wedge dz+\left(\frac{\partial A}{\partial z}-\frac{\partial C}{\partial x}\right)dz\wedge dx+\left(\frac{\partial B}{\partial x}-\frac{\partial A}{\partial y}\right)dx\wedge dy$

故 $\mathrm{div}(\mathrm{curl} X)=d(d\omega)=\left(\frac{\partial^2 C}{\partial y\partial x}-\frac{\partial^2 B}{\partial z\partial x}+\frac{\partial^2 A}{\partial z\partial y}-\frac{\partial^2 C}{\partial x\partial y}+\frac{\partial^2 B}{\partial x\partial z}-\frac{\partial^2 A}{\partial y\partial z}\right)dx\wedge dy\wedge dz=0$

11. 定理2.20. 设 ω 是光滑流形M上的一次微分式, X和Y是M上的光滑

切向量场, 则 $d\omega(X,Y)=X\langle Y,\omega\rangle-Y\langle X,\omega\rangle-\langle[X,Y],\omega\rangle$

证明: 设 $\omega=g\,df$, 则 $d\omega=dg\wedge df$. 故 $d\omega(X,Y)=(dg\wedge df)(X,Y)=(dg\otimes df-df\otimes dg)(X,Y)$

$=\langle X,dg\rangle\langle Y,df\rangle-\langle X,df\rangle\langle Y,dg\rangle=Xg\cdot Yf-Xf\cdot Yg$