又有 $X\langle Y,\omega\rangle = X\langle Y, g\,df\rangle = X(g\cdot Yf) = Xg\cdot Yf + g\cdot X(Yf)$

$Y\langle X,\omega\rangle = Y\langle X, g\,df\rangle = Y(g\cdot Xf) = Yg\cdot Xf + g\cdot Y(Xf)$

$\langle [X,Y],\omega\rangle = \langle [X,Y], g\,df\rangle = g(X(Yf) - Y(Xf))$

故 $X\langle Y,\omega\rangle - Y\langle X,\omega\rangle - \langle [X,Y],\omega\rangle = Xg\cdot Yf - Yg\cdot Xf = d\omega(X,Y)$

12. 练习2-8 设 R^3 中的光滑向量场 $X = y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y}$，$Y = z\frac{\partial}{\partial y} - y\frac{\partial}{\partial z}$，

$Z = \frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}$. 求 $[X,Y]$, $[Y,Z]$, ~~$[Z,X]$~~.

解. $[X,Y] = XY - YX = \left(y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y}\right)\left(z\frac{\partial}{\partial y} - y\frac{\partial}{\partial z}\right) - \left(z\frac{\partial}{\partial y} - y\frac{\partial}{\partial z}\right)\left(y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y}\right)$

$$= yz\frac{\partial^2}{\partial x\partial y} - y^2\frac{\partial^2}{\partial x\partial z} - xz\frac{\partial^2}{\partial y^2} + x\frac{\partial}{\partial z} + xy\frac{\partial^2}{\partial y\partial z}$$

$$-\left(z\frac{\partial}{\partial x} + zy\frac{\partial^2}{\partial y\partial x} - zx\frac{\partial^2}{\partial y^2} - y^2\frac{\partial^2}{\partial z\partial x} + yx\frac{\partial^2}{\partial z\partial y}\right)$$

$$= x\frac{\partial}{\partial z} - z\frac{\partial}{\partial x}$$

$[Y,Z] = YZ - ZY = \left(z\frac{\partial}{\partial y} - y\frac{\partial}{\partial z}\right)\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}\right) - \left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}\right)\left(z\frac{\partial}{\partial y} - y\frac{\partial}{\partial z}\right)$

$$= z\frac{\partial^2}{\partial y\partial x} + z\frac{\partial^2}{\partial y^2} + z\frac{\partial^2}{\partial y\partial z} - y\frac{\partial^2}{\partial z\partial x} - y\frac{\partial^2}{\partial z\partial y} - y\frac{\partial^2}{\partial z^2}$$

$$-\left(z\frac{\partial^2}{\partial x\partial y} - y\frac{\partial^2}{\partial x\partial z} + z\frac{\partial^2}{\partial y^2} - \frac{\partial}{\partial z} - y\frac{\partial^2}{\partial y\partial z} + \frac{\partial}{\partial y} + z\frac{\partial^2}{\partial z\partial y} - y\frac{\partial^2}{\partial z^2}\right)$$

$$= \frac{\partial}{\partial z} - \frac{\partial}{\partial y}$$

13. 练习2-11 设 $\omega = xy\,dx + z\,dy - y\,dz$，$\eta = x\,dx - yz^2\,dy - 2x\,dz$，求 $d\omega$；$d\eta$；$d\omega\wedge\eta - \omega\wedge d\eta$

解：$d\omega = d(xy)\wedge dx + dz\wedge dy - dy\wedge dz = x\,dy\wedge dx - 2\,dy\wedge dz$

$d\eta = dx\wedge dx - d(yz^2)\wedge dy - d(2x)\wedge dz = 2y\,dy\wedge dz + 2\,dz\wedge dx$

$d\omega\wedge\eta - \omega\wedge d\eta = (-x\,dx\wedge dy - 2\,dy\wedge dz)\wedge(x\,dx - yz^2\,dy - 2x\,dz)$

$$-(xy\,dx + z\,dy - y\,dz)\wedge(2y\,dy\wedge dz + 2\,dz\wedge dx)$$

$$= (2x^2 - 2x - 2xy^2 - 2z)\,dx\wedge dy\wedge dz$$

14. 叙述 Stokes 公式，并用实例说明.