Problem 1:

Leapfrog scheme conserves energy as long as the Courant-Friedrichs-Lewy Condition is preserved.

Leapfrog scheme: $$\frac{f(t+dt,x)-f(t-dt,x)}{2dt} = -v\,\frac{f(t,x+dx)-f(t,x-dx)}{2dx}$$

Sol^{n} will look smthn like $\rightarrow$ $f(x,t) = \varepsilon^{t}\exp(ikx)$ ← let's plug the sol^{n} into the scheme...

$$\hookrightarrow \frac{\varepsilon^{t+dt}\exp[ikx]-\varepsilon^{t-dt}\exp[ikx]}{2dt} = -v\,\frac{\varepsilon^{t}\exp[ik(x+dx)]-\varepsilon^{t}\exp[ik(x-dx)]}{2dx}$$

$$\Leftrightarrow \frac{\varepsilon^{dt}-\varepsilon^{-dt}}{2dt} = -v\,\frac{\exp[ikdx]-\exp[-ikdx]}{2dx} = -i\left(\frac{v}{dx}\right)\sin(kdx)$$

$$\Leftrightarrow \varepsilon^{dt}-\varepsilon^{-dt} = -2iv\left(\frac{dt}{dx}\right)\sin(kdx) \Leftrightarrow \frac{\varepsilon^{2dt}-1}{\varepsilon^{dt}} = -i\,\frac{2v(dt)}{dx}\sin(kdx)$$

$$\Leftrightarrow \varepsilon^{2dt}-1 = -i\,\varepsilon^{dt}\,\frac{2v(dt)}{dx}\sin(kdx)$$

$\left\{\begin{array}{l}\text{let } \varepsilon^{dt}\rightarrow\varepsilon \\ \text{as } dt\rightarrow 1\end{array}\right.$

We get $$\varepsilon^{2}+\left(i\,\frac{2v(dt)}{dx}\sin(kdx)\right)\varepsilon-1=0$$ $\rightsquigarrow$ Quadratic Formula

$$\Rightarrow \varepsilon = \frac{-b\pm\sqrt{b^{2}-4ac}}{2a}$$

$$\Rightarrow \varepsilon = -i\,\frac{v(dt)}{dx}\sin(kdx) \pm \frac{\sqrt{4-4\left(\frac{vdt}{dx}\sin(kdx)\right)^{2}}}{2}$$

$$\varepsilon = -i\,\frac{v(dt)}{dx}\sin(kdx) \pm \sqrt{1-\left(\frac{vdt}{dx}\sin(kdx)\right)^{2}}$$

To satisfy the CFL condition, $|\varepsilon| = 1 \Rightarrow |\varepsilon|^2 = \varepsilon \cdot \mathrm{conj}(\varepsilon) =$

$$= \underbrace{\left(\frac{v\,dt}{dx}\right)\sin(k\,dx)}_{=\alpha} + 1 - \left(\frac{v\,dt}{dx}\sin(k\,dx)\right)^2$$

$$\Rightarrow |\varepsilon|^2 = 1 = \alpha + 1 - \alpha^2$$

$$\alpha - \alpha^2 = 0$$

$$\alpha(1-\alpha) = 0$$

(I) $\frac{v\,dt}{dx} = 0$

(II) $1 - \alpha = 0$

$$\Rightarrow 1 = \frac{v\,dt}{dx}\sin(k\,dx)$$

Consequently from (II), we must have $\boxed{\frac{v\,dt}{dx} \geq 1}$ which satisfies our CFL condition...

There is zero amplitude dissipation using a leapfrog method...