C.W.Gardiner
QUANTUM NOISE
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In Memory of my Mother

Nellie Muriel Olive Gardiner
1908–1989

Preface

The term "Quantum Noise" covers a number of separate concepts, and has a variety of manifestations in different realms of physics. The principal experimental areas now covered by the term arise in quantum optics and in the study of Josephson junctions. Although there has been some interaction between these two fields, it is essentially true to say that the two fields have developed independently, and have their own body of techniques for theoretical investigation. This book is largely about the quantum optical field. By this I mean that the techniques presented here are based on the methods used in quantum optics—I certainly do not mean that these methods are applicable only to quantum optics.

My aim in this book is to give a systematic and consistent exposition of those quantum stochastic methods which have developed over the last thirty years in quantum optics. In my previous book, "A Handbook of Stochastic Methods", I concluded with a chapter on quantum mechanical Markov processes. This was only a brief outline of those methods which harmonised most naturally with the classical theory of Markov processes. Since that time the realisation has grown that the study of *squeezing* and other more exotic properties of light fields required a much more careful description of quantum noise, and in particular the relation between inputs and outputs to quantum optical systems. After Matthew Collett and myself worked out such a formalism for describing the production of squeezed light beams, we realised that the same basic approach had within it the foundation for a complete description of quantum noise. This book is the result of this realisation.

This book is not merely a collection of methods drawn from the literature—there is a significant amount of new work, and the interconnections between the various descriptions are new. The structure of the book is as illustrated in the following diagram.

The introduction and Chap. 2 on quantum statistics set the background for the real development of the theory, which takes place mainly in Chap. 3. This chapter develops the quantum Langevin equation of a system in interaction with a heat bath of harmonic oscillators in a form which is rather general, and which can be applied. The *adjoint equation*, which is analogous to a Schrödinger picture version of the quantum Langevin equation is also introduced, and it is shown how all the standard quantum optical techniques can be derived as limits of the adjoint equation. The great advantage of the adjoint equation comes from the remarkable fact that the quantum noise which arises can be exactly represented by a c-number stochastic process. To emphasise this point, a final section in Chap. 3 shows how to apply it to the problem of macroscopic quantum coherence in a low temperature two level system, which can perhaps be realised with a SQUID—that is, it is an application of a quantum optical method to a problem in superconductivity. This section, written

1. Introduction

2. Quantum Statistics

3. Quantum Langevin Equations

4. Phase Space Methods

5. Quantum Markov Processes

6. Applying the Master Equation

7. Amplifiers and Measurement

9. Interaction of Light with Atoms

8. Photon Counting

10. Squeezing
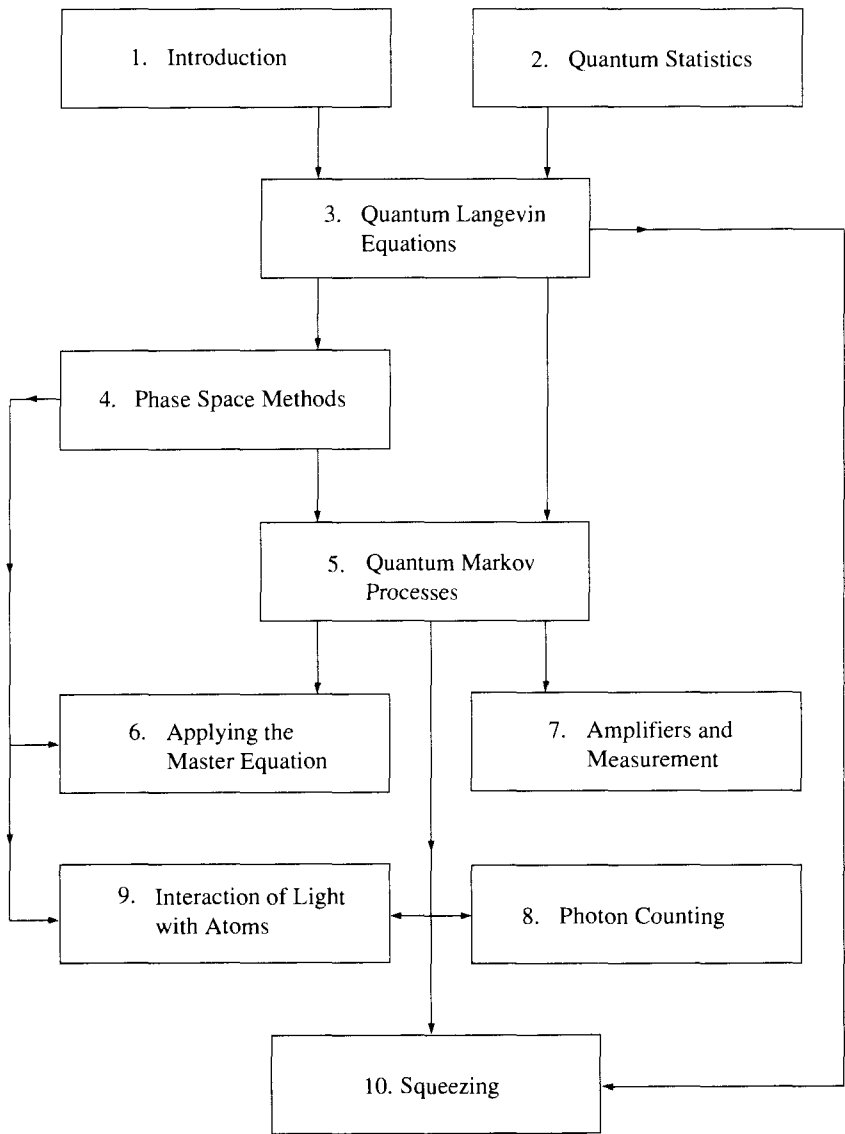
in collaboration with Scott Parkins, shows how quantum noise can be numerically simulated to give slightly more general results than are achievable from the path integral methods usually employed.

The fourth chapter is essentially a compendium of phase space methods, by which c-number representations of quantum processes can be developed. It contains a systematic explanation of coherent states, and the three principal phase space methods, the P-, Q- and Wigner representations.

Chapters 5, 6, and 7 form a group. Chapter 5 contains the theory of the Markovian limit of systems driven by quantum noise, and also shows that this limit occurs in a wider range of systems than those described in Chap. 3. It contains, for the first time in any book, a dual description similar to that which occurs in classical stochastic processes. Thus we have the master equation, which corresponds classically to the Fokker-Planck equation, and *quantum* stochastic differential equations, which are in exact correspondence to their classical counterparts. Basically, these are all results of a quantum white noise limit, though in the case of linear systems, non-white noise can be handled. Thus Chap. 5 contains all that is necessary for a theory of quantum Markov processes.

It is perhaps appropriate to add a word of caution here. The white noise limit is very simple and elegant and has led to some very rigorous mathematical theory. There have been claims that quantum mechanics as usually presented (i.e., using only unitary time evolution) is less general than the theory of quantum Markov processes, and even that because of this conventional quantum mechanics is incomplete. However there is no physical evidence that nature makes use of this more general theory—rather, it appears that the only way in which quantum Markov processes turn up in practice is as a *limiting* description of the motion of a subsystem of a larger system, from which all knowledge of a large number of degrees of freedom is eliminated. This is certainly the point of view I take throughout the book, though I acknowledge that in the field of quantum measurement theory there is a widespread opinion that the theory is not complete. To emphasise—*everything in this book can be understood as a reformulation or a limit of conventional quantum mechanics.*

Chapter 6 is very largely concerned with joining together the results of the previous two chapters, and in developing the ideas of generalised P-representations, introduced by Peter Drummond and myself ten years ago. These methods have gradually become standard techniques, even though there are still some fundamental existence questions to be answered.

On the other hand Chap. 7 depends largely on quantum stochastic differential equations, and presents a way of formulating amplifier theory with inputs and outputs, which is the logical continuation of the 1982 formulation of Carl Caves. I feel that this point of view can be developed much more, and could lead to a "quantum engineering" formulation of amplifiers and related quantum systems. The section on measurement is included here because it is another application of the quantum stochastic differential equation approach, and because, in the model described, it is shown that an understanding of the von Neumann collapse of the wavefunction can be understood as the result of quantum noise and amplifier gain.

My chapter on photon counting is an almost entirely new formulation, in which the techniques of quantum Markov processes are utilised to develop a fully spatially

dependent description of the photodetection process. Of course no new physical results arise from this, but I believe the derivation of the conditions under which the quantum Mandel formula is valid, and the relation to the theory of continuous measurements is very illuminating. The latter part of the chapter puts together a way of looking at photodetectors from the point of view of inputs and outputs—that is, we look at input photons and output electrons. Almost incidentally, a Fermionic form of quantum white noise is introduced to describe electron fields.

Chapter 9 is a rather brief summary of the interaction of light and the two level atom, including gas laser theory and optical bistability. The presentation is again unconventional, since it is an application of the quantum Markov process description developed in the earlier parts of the book.

The final chapter is a brief summary of some aspects of squeezed light. This field is still developing, and I did not feel a more intensive discussion would be appropriate.

In order to understand this book, it is necessary to have some knowledge of field quantisation, and a thorough knowledge of non-relativistic quantum mechanics. I have chosen not to put in a description of classical stochastic processes, since this is well covered in my previous book, "A Handbook of Stochastic Methods", to specific sections of which I shall frequently refer by using the abbreviation S.M. In that sense, this book does demand a lot of preparation, but I think that is unavoidable. To understand the full range of physical noise phenomena requires a thorough understanding of both the classical and the quantum fields, and I hope this book and my previous one will provide that.

I have not designed this book primarily as a textbook for a course, though I have included some exercises, which are not necessarily always easy. However, I have given a short course of eight two-hour lectures at the University of Linz, in which I covered the material in Chaps. 1,3 (two lectures), Chap. 4, Chap. 5 and the first three sections of Chap. 9. A very suitable course of about 24 one hour lectures could be made from selections from "Handbook of Stochastic Methods", say Chaps. 1–7, and Chap. 9, and from "Quantum Noise", Chaps. 3–7, and Chap. 9. Other material would be optional. For a balanced view, the student should also attend a course on quantum optics with a more applied point of view, including information on experiments.

Hamilton, New Zealand   *Crispin Gardiner*
May 1991

Acknowledgements

1.   A Historical Introduction

Quantum mechanics has had a statistical aspect since the formulation of the probability interpretation by *Born* [1.1] in 1926, and in spite of repeated attempts to re-interpret this intrinsic statistical aspect as the consequence of the observer's incomplete access to information which is in principle accessible (as is the case classically), Born's probability interpretation remains with us to this day. There is as yet no experiment which has shown any disagreement whatsoever with the basic principles of quantum mechanics, even though some of its predictions are quite counter intuitive.

    In simple experiments, such as scattering experiments, the quantum mechanics of few body problems—essentially Schrödinger's equation—is sufficient to give a complete description. But even in the archetypal quantum mechanical concept of atomic energy levels and spectral lines the simple few body view is no longer adequate. Spectral lines are not sharp. The origin of the spectral line width is the coupling of the atom to the electromagnetic field, which has infinitely many degrees of freedom, and it is only because of this infinity that the irreversible phenomenon of atomic decay takes place.

    Thus the description of atomic decay embodies the two fundamental building blocks of the theory of quantum noise—the intrinsic statistical aspect of quantum mechanics, and the statistical aspect which arises from our inability to specify each of all of the infinite number of degrees of freedom of the electromagnetic field. *Weisskopf* and *Wigner* [1.2] gave the first description of atomic decay and the consequent existence of a spectral line width as long ago as 1930, and their paper can be regarded as the beginning of the theory of quantum noise, though the concept of noise *per se* did not really enter their formulation. The purely quantum mechanical aspect of noise arises from Heisenberg's uncertainty principle, which we shall now investigate in some detail.

1.1   Heisenberg's Uncertainty Principle

Heisenberg's uncertainty principle states that it is impossible to measure simultaneously two canonically conjugate variables such as position x and momentum p, with arbitrary precision. Explicitly,

$$\Delta x \Delta p \geq \hbar/2.$$

(1.1.1)

From Heisenberg's principle alone, it is clear that we simply cannot measure all of the variables of a system precisely. Repeated measurements on the same system will

yield values of x and p which fluctuate about certain mean values, with uncertainties Δx and Δp. It is instructive to derive (1.1.1), for this yields a connection with classical noise theory. Let us consider two variables x and y, and define

$$\delta x = x - \langle x \rangle$$
$$\delta y = y - \langle y \rangle \qquad\qquad (1.1.2)$$

where x and y are quantum mechanical operators, and $\langle \ \rangle$ is the average over quantum wavefunctions as well as any other statistical elements. Then we know that for any operator A, with hermitian conjugate $A^\dagger$

$$\langle AA^\dagger \rangle \geq 0. \qquad\qquad (1.1.3)$$

Let us put $A = \delta x + \lambda e^{i\theta} \delta y$ and substitute in (1.1.3). We get

$$\langle \delta x^2 \rangle + \lambda(\cos\theta \langle [\delta x, \delta y]_+ \rangle - i \sin\theta \langle [\delta x, \delta y] \rangle) + \lambda^2 \langle \delta y^2 \rangle \geq 0. \qquad (1.1.4)$$

The criterion for (1.1.4), regarded as a quadratic in λ, to be non-negative is

$$\left(\cos\theta \langle [\delta x, \delta y]_+ \rangle - i \sin\theta \langle [\delta x, \delta y] \rangle \right)^2 \leq 4 \langle \delta x^2 \rangle \langle \delta y^2 \rangle. \qquad (1.1.5)$$

(Note that the mean of a commutator is imaginary, so that $i \langle [\delta x, \delta y] \rangle$ is real). This must be true for all θ, so maximising the left hand side yields

$$\langle \delta x^2 \rangle \langle \delta y^2 \rangle \geq \tfrac{1}{4} | \langle [\delta x, \delta y] \rangle |^2 + \tfrac{1}{4} \langle [\delta x, \delta y]_+ \rangle^2. \qquad (1.1.6)$$

We can now see a classical term and a quantum term on the right hand side. The second term involves the covariance

$$\sigma_{xy} = \tfrac{1}{2} \langle \delta x \delta y + \delta y \delta x \rangle. \qquad\qquad (1.1.7)$$

In a classical theory the products would commute, and we would recover the usual definition of a covariance. In the classical case we would then have

$$\langle \delta x^2 \rangle \langle \delta y^2 \rangle \geq \sigma_{xy}^2 \qquad\qquad (1.1.8)$$

and this expresses the fact that if x and y are correlated, then the product of the variances must satisfy (1.1.8) as illustrated in Fig.1.1. This second term, being one which arises classically, has an origin in the coupling of the simple system to an external reservoir, as mentioned in the introduction to this chapter. Quantum effects can still be manifested in this term through the quantum mechanical nature of the reservoir, so although it has a classical analogue, it cannot be called a purely classical term. The first term is purely quantum, since it vanishes if the operators commute. We can see that even if x and y are uncorrelated, there is a minimum product of their variances. In the case of position x and momentum p we know

$$[x, p] = i\hbar \qquad\qquad (1.1.9)$$

so that the inequality (1.1.6) becomes in this case

$$\Delta x \Delta p \geq \sqrt{\tfrac{1}{4}\hbar^2 + \sigma_{xp}^2}. \qquad\qquad (1.1.10)$$

If the correlation σ_{xp} vanishes then we recover the Heisenberg principle, as stated in (1.1.1).
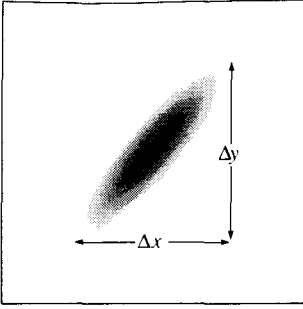
Fig. 1.1 Diagram illustrating how the existence of a covariance requires that the product of variances not be zero

1.1.1 The Equation of Motion and Repeated Measurements

We can now ask how Heisenberg's principle will manifest itself as quantum noise. Like all noise, quantum noise is a time dependent phenomenon. Without noise, repeated measurements of a physical quantity would be expected to yield a smoothly varying function of time. To analyse such a time dependent situation one must have some equations of motion, so we consider a specific example; repeated measurements at time intervals t of the position of a free particle, in the absence of any thermal noise. The Heisenberg equations of motion are of the same form as the classical Hamilton equations :

$$\left. \begin{array}{l} m\dot{x}(t) = p(t) \\ \dot{p}(t) \quad = 0 \end{array} \right\} \tag{1.1.11}$$

so that

$$x(t + \tau) = p(t)\frac{\tau}{m} + x(t) \tag{1.1.12}$$

from which one can deduce

$$\langle \delta x(t+\tau)^2 \rangle = \langle \delta x(t)^2 \rangle + \left(\frac{\tau}{m}\right)^2 \langle \delta p(t)^2 \rangle + \left(\frac{\tau}{m}\right) \langle [\delta x, \delta p]_+ \rangle. \tag{1.1.13}$$

We note some interesting consequences of (1.1.13).

i) A precise measurement of $x(t)$ results, from Heisenberg's principle, in

$$\langle \delta p(t)^2 \rangle \to \infty \tag{1.1.14}$$

and hence also

$$\langle \delta x(t+\tau)^2 \rangle \to \infty. \tag{1.1.15}$$

The measurement acts back on the system in such a way as to make any further precise measurement of x impossible. The infinite uncertainty in $p(t)$ means that the velocity of the particle is infinitely arbitrary, and hence so is the future position.

ii) The best we could do would perhaps be to measure with uncertainties in $x(t)$ and $p(t)$ which were comparable with each other; for example Heisenberg's principle permits

$$\langle \delta x(t)^2 \rangle = \hbar\tau/(2m), \qquad \langle \delta p(t)^2 \rangle = m\hbar/(2\tau) \tag{1.1.16}$$

which yields

$$\langle \delta x(t + \tau)^2 \rangle = \hbar \tau / m. \tag{1.1.17}$$

Hence the measurement at time $t + \tau$ has twice the variance of that at time t.

iii) It should be borne in mind that to measure with the uncertainties given by (1.1.16) requires that the initial state be a pure quantum-mechanical state. How to carry out any of these measurements in practice is non-trivial, though it is theoretically possible.

iv) Not all quantum measurements have this property. For example, (1.1.11) clearly shows that

$$p(t + \tau) = p(t) \tag{1.1.18}$$

and hence repeated measurements of the momentum do not introduce noise. This is the basis of back action evading or quantum non-demolition measurements [1.3].

1.2   The Spectrum of Quantum Noise

Nyquist's theorem [1.4] as experimentally verified by *Johnson* [1.5] established that a resistor R develops a noise voltage $E(t)$ across its ends, whose value can be written classically as

$$\langle E(t + \tau) E(t) \rangle = \int\limits_{-\infty}^{\infty} e^{i\omega\tau} S(\omega) \, d\omega \tag{1.2.1}$$

where

$$S(\omega) = RkT/\pi. \tag{1.2.2}$$

(k = Boltzmann's constant, T = absolute temperature.)

The origin of these fluctuations is well understood: they are simply the fluctuations inherent in maintaining a Boltzmann distribution of the appropriate canonical variables in the electric circuit. A stochastic analysis is given in S.M.5.3.6(d).†
Even in 1928 Nyquist considered what should happen at high frequencies, such that $\hbar \omega \gg kT$, since it was known that in the case of the black body radiation, the effect of quantisation was to yield the Planck spectrum, which would be equivalent in this case to

$$S(\omega) = \frac{R\hbar\omega}{\pi[\exp(\hbar\omega/kT) - 1]}. \tag{1.2.3}$$

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† The abbreviation S.M. is used for the author's book, "A Handbook of Stochastic Methods", as noted in the preface.

This function is flat for $\hbar\omega \ll kT$, but for $\hbar\omega > kT$, it rapidly approaches zero. If (1.2.3) is used instead of (1.2.2), the correlation function (1.2.1) becomes a well-behaved approximation to a delta function. Thus Nyquist concluded that the physically unrealistic flat noise spectrum would be eliminated by the incorporation of quantum mechanics.

A more serious quantum treatment did not come until 1951, when *Callen* and *Welton* [1.6] gave a form of reasoning which suggested that the true correlation function should have a quantum mechanical spectral density given by

$$S(\omega) = \frac{R\hbar\omega}{2\pi}\coth\left(\frac{\hbar\omega}{2kT}\right) \tag{1.2.4}$$

$$= \frac{R}{\pi}\left[\tfrac{1}{2}\hbar\omega + \frac{\hbar\omega}{\exp(\hbar\omega/kT) - 1}\right]. \tag{1.2.5}$$

This result shows that Nyquist was correct in surmising that the thermal spectrum would be cut off when $\hbar\omega > kT$. However the total spectrum is not cut off—rather, it rises linearly with increasing ω because of the first term in (1.2.5), which arises from the zero point fluctuations in the harmonic oscillators which model the microscopic structure of the resistive element. It might be thought that effects arising from zero point fluctuations would be unobservable, since it is well attested by experiment that black body radiation is observed to have a Planck spectrum, corresponding only to the second term in (1.2.5)—we cannot detect the zero point spectrum when we measure the spectrum by means involving the absorption of photons from a radiation field.

However, serious doubts can be raised about the status of this quantum mechanical version of $S(\omega)$. Callen and Welton's derivation is not flawless, and furthermore they computed only the mean square voltage $\langle E(t)^2\rangle$ in the case that one has a frequency dependent resistor $R(\omega)$, so that

$$\langle E(t)^2\rangle = \frac{1}{2\pi}\int_{-\infty}^{\infty} d\omega\,\hbar\,\omega R(\omega)\coth\left(\frac{\hbar\omega}{2kT}\right) \tag{1.2.6}$$

and if $R(\omega)$ is sufficiently rapidly decreasing, this integral converges. But, most importantly, the meaning of the product specified classically in (1.2.1) for the quantum mechanical situation, where $E(t)$ and $E(t+\tau)$ are operators which cannot be expected to commute, has not been specified. *Ford* et al. [1.7] carried out a more well defined procedure, and brought up this matter of the choice of quantum mechanical product. They suggested the use of the *normal product*, (see Sect.4.3.1) which has the effect of omitting the linearly rising term in (1.2.5), and therefore seems more physical.

However this can only be a guess; a correct answer to the problem can only come by a detailed modelling of what is actually measured. We consider therefore two pieces of experimental evidence.

a) Measurement of the Spectrum of Black Body Radiation by Absorption: As just mentioned, in this we get a Planck spectrum. Furthermore, we will show later in

Chap.8 that in absorption measurements, it is the normal product that is measured, and hence the Planck spectrum is expected.

b) Heterodyne Measurement of the Spectrum of Quantum Noise: *Koch* et al. [1.8] chose to carry out measurements without a complete theory. They assumed that an electric circuit behaves classically, except that the classical flat spectrum is replaced by the expression (1.2.4), and they rather convincingly showed that this expression is correct—that the linearly rising term, which Clarke has termed "f-noise", does exist and is measurable, as is shown in Fig.1.2. The observed spectral density therefore does actually possess both terms. The first term, proportional to $\hbar\omega$, is temperature independent and is therefore purely quantum mechanical—it is truly "quantum noise".
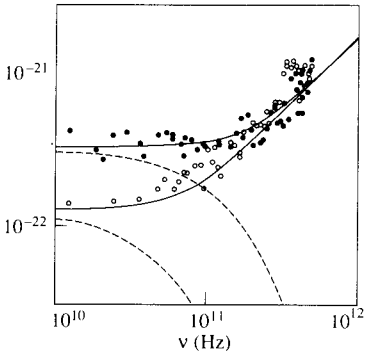
Fig. 1.2 Measured spectral density of noise current in the experiment of *Koch* et al at 4.2 K (solid circles) and 1.6 K (hollow circles). The solid lines are the prediction of (1.2.5), while the dashed lines correspond to the Planck spectrum (1.2.3)

However, the second "thermal" term is also influenced by quantum mechanics, since it does contain Planck's constant. There is no escape from the details of quantum mechanics. Pragmatically, the results of this experiment show that there is considerable validity in a treatment based on a classical noise theory with a modified "quantum" spectrum. For example, according to this kind of treatment, one would treat an LRC circuit by the equations

$$\frac{dQ}{dt} = I \tag{1.2.7}$$

$$L\frac{dI}{dt} = -\frac{Q}{C} - IR + E(t) \tag{1.2.8}$$

in which

$$\langle E(t)E(t')\rangle = \frac{R}{2\pi} \int\limits_{-\infty}^{\infty} e^{i\omega(t-t')}\hbar\omega \coth\left(\frac{\hbar\omega}{2kT}\right) d\omega \tag{1.2.9}$$

and where Q, the charge on the capacitor C, I, the current through the circuit, and $E(t)$ are viewed as classical non-operator quantities. This kind of theory is widely used in the treatment of macroscopic superconductivity problems [1.9].

In the limit of high temperature it reduces to the classical white noise theories, as long as one is not interested in very high frequency phenomena. This is an example of a damped harmonic oscillator, and we will show in Sect.3.4.3 that a

true quantum theory of the harmonic oscillator can be cast into this form exactly. But the introduction of any amount of anharmonicity destroys this rosy picture and serious discrepancies arise. The major difference that arises is one noted by *Dirac* [1.10]. The motion of a harmonic oscillator has only one frequency—no harmonics occur. Quantum mechanically, the harmonic oscillator has evenly spaced energy levels such that in a transition from one to the next, radiation of a corresponding frequency is emitted. There is therefore no a priori contradiction between the two viewpoints. However in the case of an anharmonic oscillator there is classically a fundamental frequency and an infinite number of harmonics. These do *not* match up with the frequencies of the possible transitions between the energy levels of the quantum-mechanical version—there is no way in which in the small noise limit one will obtain agreement between these two pictures.

1.3 Emission and Absorption of Light

Having introduced the subject of energy levels in the last section, it now seems appropriate to ask what is the relationship between these energy levels and quantum noise. Historically, energy levels occur in their most easily observable form in atoms and molecules, which interact with the radiation field, experiencing transitions from one energy level to the other as they absorb and emit light. This picture has the essentials given in the beginning of this chapter, of a small system with few degrees of freedom (an atom) interacting with a system having very many degrees of freedom (the radiation field).

In 1917 *Einstein* [1.11] introduced a stochastic method of treating emission and absorption of *thermal* light, with the principal result that he was able to show that the Planck distribution formula was necessarily required if molecules which emitted and absorbed the light had a Maxwellian velocity distribution. Einstein's equation remains with us today, though in a more general form.

We suppose that the molecules have (internal) energy levels $\epsilon_1, \epsilon_2, \epsilon_3, \cdots$. Statistical mechanics says that in equilibrium, the relative frequency of the state n is given by

$$w_n = p_n e^{-\epsilon_n/kT} \tag{1.3.1}$$

where k is Boltzmann's constant, T is the temperature, and p_n represents the "weight" of the state, i.e., the number of different states with the same energy ϵ_n.

Suppose that n and m are two states whose energies satisfy

$$\epsilon_m > \epsilon_n. \tag{1.3.2}$$

The transition $m \rightarrow n$ is possible provided energy is emitted into the radiation field, and the transition $n \rightarrow m$ is possible if energy is absorbed from the radiation field. Einstein made two hypotheses :

a) Emitted Radiation: On the basis that it is known that oscillators radiate when excited, whether or not a radiation field is present, one assumes that the probability per unit time of making the transition $m \to n$ can be written

$$dW = A_m^n \, dt \tag{1.3.3}$$

where A_m^n are certain coefficients. This is also in exact correspondence with the law of radioactive decay.

b) Effect of Incident Radiation: Suppose a molecule is in the presence of a radiation field. Then the energy of the molecule can either increase or decrease, depending on the relative phase of the oscillator and the radiation field. Correspondingly, two hypotheses are introduced: The probability per unit time for the transition $n \to m$ (absorption) is

$$dW = B_n^m \bar{\rho} \, dt \tag{1.3.4}$$

and for the transition $m \to n$ (emission)

$$dW = B_m^n \bar{\rho} \, dt \tag{1.3.5}$$

where $\bar{\rho}$ is the energy density (per unit volume per unit frequency) of the radiation field, and B_m^n, B_n^m are certain constants, which we do not specify for the moment.

One can now derive the Planck radiation formula. In equilibrium, the rate of transitions $n \to m$ must balance that of the transitions $m \to n$. Using the expressions (1.3.1) and (1.3.3–5) one sees that

$$p_n e^{-\epsilon_n/kT} B_n^m \bar{\rho} = p_m e^{-\epsilon_m/kT} \left(B_m^n \bar{\rho} + A_m^n \right) . \tag{1.3.6}$$

If we assume that $\bar{\rho}$ can become arbitrarily large by increasing the temperature, then we may derive from (1.3.6) (by noting that the exponentials then approach unity)

$$p_n B_n^m = p_m B_m^n . \tag{1.3.7}$$

Using (1.3.7) we can solve (1.3.6) for $\bar{\rho}$, to find

$$\bar{\rho} = \frac{A_m^n / B_m^n}{\exp\left[(\epsilon_m - \epsilon_n)/kT\right] - 1} \tag{1.3.8}$$

which is almost Planck's radiation law. To complete the result, Einstein used Wien's displacement law, in the form that says that $\bar{\rho}$ can be written in the form

$$\bar{\rho} = \nu^3 f(\nu/T), \tag{1.3.9}$$

from which it is clear that

$$A_m^n / B_m^n = \alpha \nu^3 \tag{1.3.10}$$

and

$$\epsilon_m - \epsilon_n = h\nu. \tag{1.3.11}$$

The constants α and h are not determined by this reasoning—their determination requires both experimental information and a more precise theory of radiation. Even less so are the coefficients A_m^n and B_m^n determined. However identifying h with Planck's constant we find that

$$\bar{\rho}(\nu) = \alpha\nu^3/(\exp(h\nu/kT) - 1) \tag{1.3.12}$$

which is of course Planck's radiation law.

c) Commentary: Einstein went on further to show that the conservation of momentum requires that the molecules have the Maxwellian distribution of velocities, if it is assumed that *each photon is emitted in a particular direction*, rather than as a spherical wave, as might have been expected classically. From a modern quantum-mechanical point of view this is not acceptable. The emission of photons can be computed either as emission of plane waves (corresponding to quanta with a definite direction) or as emission of spherical waves. Since each is a superposition of the others, there is no difference. However the concept of superposition of probability amplitudes, which is essential for this equivalence, plays no part in Einstein's formulation; only probabilities enter into the argument, and the specific quantum-mechanical connection only arises from the assumption that energy levels *exist* and that the transitions between energy levels occur when electromagnetic energy is absorbed or emitted. This means that Einstein's derivation is consistent with a purely probabilistic quantum theory, in which, however, photons are to be regarded as *particles* travelling in a definite direction. The question of wave-particle duality, which is such a central part of quantum mechanics, is nowhere to be seen.

This does not invalidate Einstein's arguments, but the question of why photons *must* be treated as particles travelling in a definite direction is not answered. We will see in Sect.3.6.3 that purely quantum-mechanical arguments give the same results, though not with the simplicity of Einstein's methods.

There is a long tradition of this kind of "probabilistic" quantum mechanics. After the development of modern quantum theory in the 1920s, the use of time-dependent perturbation theory as developed by *Dirac* [1.10] and transformed by *Fermi* into his "golden rule number two" [1.12] was used as the basis for multitudes of calculations by probabilistic methods. The basic technique was to use quantum mechanics to compute probabilities for transitions (such as absorption and emission of photons) and then to use these probabilities in classical stochastic equations for the probability of occupation of quantum mechanical energy levels. Using such methods, it is not at all difficult to compute the coefficients A_m^n and B_m^n in Einstein's equations, and indeed to derive the whole equation, subject to one major assumption. This assumption was first made explicit by *Pauli* [1.13], and it was called the "repeated random phase assumption". It is assumed that the phase relations between wavefunctions are always (repeatedly) randomised, so that all one has to deal with are the probabilities, which are given by the squares of the wavefunctions. How this is achieved is not explained, but the success of many calculations based on it shows that it often has considerable validity.

The arrival of the laser in the 1960s changed this picture. The repeated random phase assumption does not hold for the highly coherent fields that can be produced

in a laser, and the whole field of quantum noise, emission and absorption theory, and optical coherence was transformed under its influence. A highly effective theory of "quantum Markov processes" was developed for application in the field of quantum optics, and has generated a renaissance in the theory of stochastic processes. The development of this point of view will occupy much of this book.

1.4   Consistency Requirements for Quantum Noise Theory

In setting up theories of noise and damping one does not have complete freedom— to be acceptable, theories must be consistent with the laws of physics, and in this field there are two principal consistency requirements which have to be dealt with. Firstly, there is the requirement that the well verified results of statistical mechanics be reproduced, and secondly, the requirement that canonical commutation relations be preserved by any equation of motion.

1.4.1   Consistency with Statistical Mechanics.

Methods based on generalisations of the "quasiclassical" approach of Sect. 1.2 and of the stochastic approach of Sect. 1.3 must be compatible with each other. In particular, the requirement of consistency with statistical mechanics, as enunciated by Mark Kac must be shown to be satisfied. Kac observed that in the classical theory of noise, we can show that the motion of a particle in a potential governed by the Langevin equation

$$m\ddot{x} = -V'(x) - \gamma\dot{x} + \sqrt{2\gamma kT}\,\xi(t) \tag{1.4.1}$$

in which

$$\langle \xi(t)\xi(t')\rangle = \delta(t - t') \tag{1.4.2}$$

yields the stationary distribution function

$$p_s(x, \dot{x}) = \mathcal{N}\exp\left(-\frac{m\dot{x}^2}{2kT} - \frac{V(x)}{kT}\right) \tag{1.4.3}$$

which is the familiar Boltzmann distribution of equilibrium statistical mechanics.

Any satisfactory theory of quantum noise must give the corresponding quantum result, though some care is needed in formulating what the corresponding quantum result should be: In the words of *Kac* and *Benguria* [1.14],

"The position of a quantum mechanical particle subject to an external potential $V(x)$ and in equilibrium with a heat bath of absolute temperature T should be distributed according to the probability density

$$\frac{\sum_{n=1}^{\infty}\exp\left(-E_n/kT\right)|\psi_n(x)|^2}{\sum_{n=1}^{\infty}\exp\left(-E_n/kT\right)} \tag{1.4.4}$$

where the E_n and ψ_n are respectively the eigenvalues and the normalised eigenfunctions of the Schrödinger equation with potential $V(x)$. One might thus suspect that as $t \to \infty$ the probability density of $x(t)$ should approach the canonical density (1.4.4). But this cannot be so for $\gamma \neq 0$, because for nonvanishing friction one would expect shifting and broadening of the spectral lines. What one should therefore expect is that only in the additional limit $\gamma \to 0$ the limiting distribution of $x(t)$ as $t \to \infty$ should be (1.4.4)."

In a limited form, Benguria and Kac were able to prove the above result by using a model of quantum noise developed previously by *Ford* et al. [1.7] which we shall describe in Chap.3. There we will see that the canonical distribution can be derived directly from a quantum Langevin equation of the kind (1.4.1), by means of a route which yields a generalisation of the kind of method used by Einstein—but by a route quite different from that used by Kac and Benguria.

The essence of this model is the rather simple result that, appropriately generalised, (1.4.1) represents a satisfactory quantum theory of Brownian motion of a particle in a potential, provided that:

i)   The quantities x, $\dot{x}$, $\ddot{x}$, are regarded as Heisenberg operators, obeying the appropriate canonical commutation relations.

ii)   The noise quantity $\xi(t)$ is regarded as an operator, with the commutator

$$[\xi(t), \xi(t')] = 2i\hbar\gamma\frac{d}{dt}\delta(t-t') \tag{1.4.5}$$

and the mean anticommutator

$$\langle[\xi(t),\xi(t')]_+\rangle = \frac{\gamma\hbar}{\pi}\int_{-\infty}^{\infty}d\omega\,\omega\coth\left(\hbar\omega/2kT\right)e^{i\omega(t-t')}. \tag{1.4.6}$$

This latter quantity corresponds exactly to the spectrum of quantum noise (1.2.4). We neglect the technical problems associated with the rising spectrum, which we will go into in Sects. 3.4.2,3. The limit $\hbar \to 0$ clearly gives a smooth transition to the classical Langevin equation, in the sense that in this limit all commutators vanish and the equation of motion becomes an equation for c-numbers driven by a noise term. Because of the limit,

$$\lim_{\hbar\to 0}\hbar\omega\coth\left(\hbar\omega/2kT\right) = 2kT \tag{1.4.7}$$

this noise term possesses a flat spectrum.

1.4.2   Consistency with Quantum Mechanics

If (1.4.1), regarded as quantum equations with the noise $\xi(t)$ satisfying the conditions (1.4.5,6), are solved, we will obtain solutions for $x(t)$ and $p(t) \equiv m\dot{x}(t)$ which must obey the canonical commutation relations

$$[x(t), p(t)] = i\hbar \tag{1.4.8}$$

for all times. The method of derivation of the equations used by Ford et al. [1.7] makes it clear that this will be the case. At this stage, however, let us show how a

simplified model can illustrate how the existence of noise can ensure that the commutation relations are preserved.

If we consider a harmonic oscillator, of natural frequency ω, then the Heisenberg equations for the *destruction* and *creation* operators

$$a(t) = \frac{\omega x(t) + ip(t)}{\sqrt{2\hbar\omega}}, \qquad a^\dagger(t) = \frac{\omega x(t) - ip(t)}{\sqrt{2\hbar\omega}} \tag{1.4.9}$$

are, in the absence of noise and damping,

$$\dot{a}(t) = -i\omega a(t), \qquad \dot{a}^\dagger(t) = i\omega a^\dagger(t). \tag{1.4.10}$$

A simplified way to introduce damping and noise is to straightforwardly add appropriate terms to the equation, thus;

$$\dot{a}(t) = -i\omega a(t) - \gamma a(t) + \sqrt{2\gamma}\,\Gamma(t). \tag{1.4.11}$$

Correspondingly, we also assume some simple delta-correlated commutation relations and correlation function of the noise term $\Gamma(t)$, namely

$$\begin{aligned}
[\Gamma(t), \Gamma^\dagger(t')] &= D\delta(t - t') \\
\langle \Gamma^\dagger(t)\Gamma(t') \rangle &= \bar{N}\delta(t - t').
\end{aligned} \tag{1.4.12}$$

Here the delta functions indicate that some kind of quantum generalisation of the classical delta correlated white noise process is being contemplated, and D and $\bar{N}$ are numbers which remain to be determined. Solving (1.4.11), we get

$$a(t) = e^{-(i\omega+\gamma)t}a(0) + \sqrt{2\gamma}\int_0^t dt'\, e^{-(i\omega+\gamma)(t-t')}\Gamma(t') \tag{1.4.13}$$

and if we assume that

$$[a(0), \Gamma^\dagger(t)] = [a^\dagger(0), \Gamma(t)] = 0 \tag{1.4.14}$$

and of course

$$[a(0), a^\dagger(0)] = 1, \tag{1.4.15}$$

then the commutator can be evaluated as

$$[a(t), a^\dagger(t)] = e^{-2\gamma t} + D\left(1 - e^{-2\gamma t}\right) \tag{1.4.16}$$

so the commutator will be preserved if

$$D = 1. \tag{1.4.17}$$

The mean energy of the oscillator is similarly found to be

$$\hbar\omega\langle a^\dagger(t)a(t) \rangle = \hbar\omega e^{-2\gamma t} + \hbar\omega\bar{N}\left(1 - e^{-2\gamma t}\right) \tag{1.4.18}$$

so that the quantity $\bar{N}$ represents the eventual mean occupation number of the oscillator, and for consistency with the Planck distribution this must require

$$\bar{N} = 1/[\exp(\hbar\omega/kT) - 1]. \tag{1.4.19}$$

It can be seen then that, subject to a set of rather sweeping assumptions (1.4.12,14), a quantum mechanically consistent picture can be built up. Although this is only intended as an illustration and cannot pretend to give a fully realistic picture of the physics, we will show in Sect.3.4.5 that it can be derived as a high frequency limit of a more physical model, namely, the quantum version of (1.4.1), and indeed from the quantum electrodynamics of an atom interacting with the radiation field. In such a model $\Gamma(t)$ is essentially the operator of the electric field evaluated at the position of the atom. It is therefore quite an appropriate model for quantum optics, for which it was in fact first developed.

However the philosophy originally used was much more like that presented in this section—to assume a damping-noise equation like (1.4.11) and to deduce noise and damping parameters from the preservation of commutation relations and the requirement that the mean energy of the oscillator be that given by statistical mechanics. At these high frequencies this technique works well because there are considerable simplifications available in high frequency systems, as will be shown in Chap.3 and Chap.4. What it shows is what all physicists know—physics has sufficient internal consistency to enable one quite often to guess the right answer to a problem in favourable circumstances. Such guesses are often illuminating, but there is more to physics than guessing.

1.5   Quantum Stochastic Processes and the Master Equation

The approach to quantum noise based on quantum Langevin equations has its roots in the Heisenberg equations of motion. It is well known that in all quantum mechanical situations the Heisenberg equations of motion are not easy to deal with, mainly because of their non-linear operator nature. The Schrödinger equation is always the preferred alternative for practical calculations. The major advantage of the Heisenberg equations arises from their closer resemblance to the corresponding classical equations, which can be of advantage in trying to understand particular problems and in general theoretical formulations. This situation is amply illustrated by the material in this chapter—for example the quantum version of the Langevin equation (1.4.1) is at least easily described. In contrast, it does not seem at all obvious how to generalise the Schrödinger equation to take account of noise.

The appropriate approach was developed in the late 1960s by quantum optical workers, in particular by *Haken, Lax* and *Gordon* et al. [1.15], and this amounts to a generalisation of the approach of Einstein discussed in Sect. 1.3. From the rate laws (1.3.3–5), it is clear that there is a probabilistic law governing the probability

function $P_n(t)$ which gives the probability that the atom is in the state n; this will be

$$\dot{P}_n(t) = \sum_{m>n}(A_m^n + B_m^n\bar{\rho})P_m(t) + \sum_{m<n}B_m^n\bar{\rho}P_m(t)$$
$$- \sum_{m<n}(A_n^m + B_n^m\bar{\rho})P_n(t) - \sum_{m>n}B_n^m\bar{\rho}P_n(t) \tag{1.5.1}$$

which is in the form of the classical master equation, S.M.3.5.1, in which the positive terms represent gain of probability from transitions into the state n, and the negative terms represent loss of probability by transitions from the state n. (Here, as in Sect. 1.3, the symbol $\bar{\rho}$ is used for the energy density of the electromagnetic field to avoid confusion with the use of the symbol ρ for the density operator in the remainder of this book.)

This kind of equation is called the *Pauli master equation*, after Pauli who first introduced it [1.13]. However it has the serious defect of dealing only with probabilities, whereas a proper quantum mechanical theory should deal with the *density operator* for the system as described in Sect.2.1. The density operator was introduced by *von Neumann* [1.16] in order to treat situations where there is a statistical element to the physics other than that which arises directly from quantum mechanics, and is thus appropriate in the kind of situation relevant to emission and absorption of light, where the light field is in a thermal state. We will describe how this is to be done in a simplified derivation of the generalised master equation for the interaction of light with an atom.

1.5.1   The Two Level Atom in a Thermal Radiation Field

The physics of the problem of a thermal radiation field interaction with an atom can be viewed as that of an atom interacting with a *random electric field*—but this field must be *operator valued*. Let us consider the simplest possible atom—an atom with only two energy levels. This can be described simply in terms of matrix wavefunctions, labelled by a for the lower level, and b for the upper level. The energy eigenstates are

$$u(a) = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad , \quad u(b) = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \tag{1.5.2}$$

and the Hamiltonian is (in the Schrödinger picture)

$$H_{\text{Atom}} = \tfrac{1}{2}\hbar\,\Omega \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \equiv \tfrac{1}{2}\hbar\,\Omega\sigma_z. \tag{1.5.3}$$

We have set the zero of energy to be midway between the upper and lower levels, so that Ω is the frequency of the radiation that the atom will emit or absorb. The description of the interaction with a radiation field can be most simply dealt with by introducing the interaction Hamiltonian

$$H_{\text{Int}} = g(\sigma^+ + \sigma^-)E. \tag{1.5.4}$$

Here E is an operator representing the electric field at the position of the atom, and

$$\sigma^+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \qquad \sigma^- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \tag{1.5.5}$$

are operators which allow this field to induce transitions between levels a and b. Finally there must be a Hamiltonian for the radiation field, H_{Rad}, which we shall not specify in detail. The total Hamiltonian is

$$H = H_{\text{Atom}} + H_{\text{Int}} + H_{\text{Rad}}. \tag{1.5.6}$$

The interaction is assumed to be weak, so that to a first approximation, the atom and field do not interact. The appropriate method of solution to von Neumann's equation is to go to the *interaction picture* whose density operator is in terms of the Schrödinger picture density operator for the atom-field system

$$\rho_{\text{Int}}(t) = \exp\left(-\frac{H_{\text{Atom}} + H_{\text{Rad}}}{i\hbar}t\right) \rho_S(t) \exp\left(\frac{H_{\text{Atom}} + H_{\text{Rad}}}{i\hbar}t\right). \tag{1.5.7}$$

From von Neumann's equation for the Schrödinger picture density operator

$$i\hbar \dot{\rho}_S(t) = [H, \rho_{\text{Int}}(t)] \tag{1.5.8}$$

follows the interaction picture equation of motion

$$i\hbar \dot{\rho}_{\text{Int}}(t) = [H_{\text{Int}}(t), \rho_S(t)] \tag{1.5.9}$$

where

$$H_{\text{Int}}(t) = \exp\left(-\frac{H_{\text{Atom}} + H_{\text{Rad}}}{i\hbar}t\right) H_{\text{Int}} \exp\left(\frac{H_{\text{Atom}} + H_{\text{Rad}}}{i\hbar}t\right). \tag{1.5.10}$$

Since H_{Atom} and H_{Rad} represent different degrees of freedom, they commute, and we can then write

$$H_{\text{Int}}(t) = g\left(\sigma^+ e^{i\Omega t} + \sigma^- e^{-i\Omega t}\right) E(t), \tag{1.5.11}$$

where $E(t)$ is defined by

$$E(t) = \exp\left(-\frac{H_{\text{Rad}}}{i\hbar}t\right) E \exp\left(\frac{H_{\text{Rad}}}{i\hbar}t\right). \tag{1.5.12}$$

At this stage we can make a "randomness" ansatz for $E(t)$, the electric field operator in the interaction picture. This operator will contain a wide range of frequencies, but the only ones which will give any significant effect in the equation of motion (1.5.9) will be those which almost match the frequencies $\pm\Omega$, so that there will be a part of $H_{\text{Int}}(t)$ which is sufficiently slowly varying for a significant change to $\rho_{\text{Int}}(t)$ to be built up. The terms corresponding to frequencies significantly different from $\pm\Omega$ will oscillate so rapidly that their net effect will be almost zero. We can even evaluate the correlations of $E(t)$, $E(t')$, under this approximation. Let us take the correlations to be given by the same formula as for $\xi(t)$ in (1.4.5,6), but

omit the factor γ, which refers specifically to the interaction. Thus we assume that the electric field operator behaves like our "quantum noise". Then we can write the correlation function in terms of the sum of the commutator and anticommutator terms given by (1.4.5,6) as

$$\langle E(t)E(t')\rangle = \frac{\hbar}{\pi} \int\limits_{-\infty}^{\infty} d\omega \left(-\omega + \omega \coth\left(\frac{\hbar\omega}{2kT}\right) \right) e^{i\omega(t-t')} \qquad (1.5.13)$$

Since only the components near $\pm\Omega$ interact significantly, we may approximate the spectral factors depending on ω by their values at $\pm\Omega$, so

$$e^{i\Omega(t-t')}\langle E(t)E(t')\rangle \sim \frac{\hbar}{\pi} \left(\Omega + \Omega \coth\left(\frac{\hbar\Omega}{2kT}\right) \right) \int\limits_{-\infty}^{\infty} d\omega\, e^{i\omega(t-t')}$$

$$= 4\hbar\Omega \left(\bar{N}(\Omega) + 1 \right) \delta(t-t') \qquad (1.5.14)$$

and similarly

$$e^{-i\Omega(t-t')}\langle E(t)E(t')\rangle \sim 4\hbar\Omega\bar{N}(\Omega)\delta(t-t') \qquad (1.5.15)$$

where $\bar{N}(\Omega)$ is the Planck function (1.4.19). These correlation functions are of exactly the same form as used in the arguments for quantum mechanical consistency of Sect. 1.4.2 so we may make the replacements

$$E(t)e^{i\Omega t} \rightarrow 2\sqrt{\hbar\Omega}\,\Gamma(t), \qquad E(t)e^{-i\Omega t} \rightarrow 2\sqrt{\hbar\Omega}\,\Gamma^{\dagger}(t), \qquad (1.5.16)$$

where $\Gamma(t), \Gamma^{\dagger}(t)$ are the quantities of Sect. 1.4.2 with the properties (1.4.12).

The equation of motion for the interaction picture density operator thus becomes

$$i\hbar\,\dot{\rho}_{\text{Int}} = 2g\sqrt{\hbar\Omega}\,\left[\sigma^{+}\Gamma(t) + \sigma^{-}\Gamma^{\dagger}(t), \rho_{\text{Int}}\right]. \qquad (1.5.17)$$

This is now a kind of quantum white noise equation, but it contains more information than we actually want, since ρ_{Int} is the density operator for the radiation field as well as the atom: we want to obtain an equation for a *reduced* density operator, which refers only to atomic variables—this would be defined by tracing out over the radiation variables i.e.,

$$\tilde{\rho}_{\text{Int}} \equiv \text{Tr}_{\text{Rad}}\{\rho_{\text{Int}}\}. \qquad (1.5.18)$$

To find an equation for $\tilde{\rho}_{\text{Int}}$ we need to trace out (1.5.17) over the radiation field, and will hence need to evaluate terms like

$$\text{Tr}_{\text{Rad}}\{\Gamma(t)\rho_{\text{Int}}\}. \qquad (1.5.19)$$

In the classical theory of white noise differential equations, the very irregularity of the white noise functions (which arises directly from the idea that the noise at one

instant of time is absolutely independent of the noise at any other instant, no matter how short the time interval between the two instants) leads to considerable technical difficulties if we assume the equations are truly *differential* equations. The reason is quite straightforward—what is usually assumed to be white noise is the derivative of a non-differentiable, but continuous, random function; the Wiener process. The way out of this logical anomaly in the classical case is to integrate the differential equation and turn it into an integral equation. To do the same thing here we define the integrals of the quantum white noise quantities by

$$B(t, t_0) = \int_{t_0}^{t} \Gamma(t)\, dt, \qquad B^{\dagger}(t, t_0) = \int_{t_0}^{t} \Gamma^{\dagger}(t)\, dt. \qquad (1.5.20)$$

The differential equation (1.5.17) is then written as the integral equation

$$\rho_{\mathrm{Int}}(t) - \rho_{\mathrm{Int}}(t_0) = -2\mathrm{i} g \sqrt{\frac{\Omega}{\hbar}} \int_{t_0}^{t} \left[\sigma^{+} dB(t) + \sigma^{-} dB^{\dagger}(t), \rho_{\mathrm{Int}}(t) \right]. \qquad (1.5.21)$$

The definition of the integral is taken in exactly the same form as that of the classical Stratonovich integral, S.M.4.3.6, which leads to an approximate form for the situation where

$$t \to t + \delta t, \qquad t_0 \to t - \delta t \qquad (1.5.22)$$

and

$$\delta B(t) \equiv B(t + \delta t, t - \delta t), \qquad \delta B^{\dagger}(t) \equiv B^{\dagger}(t + \delta t, t - \delta t) \qquad (1.5.23)$$

in the form

$$\rho_{\mathrm{Int}}(t + \delta t) - \rho_{\mathrm{Int}}(t - \delta t)$$
$$= -\mathrm{i} g \sqrt{\frac{\Omega}{\hbar}} [\sigma^{+} \delta B(t) + \sigma^{-} \delta B^{\dagger}(t), \rho_{\mathrm{Int}}(t + \delta t) + \rho_{\mathrm{Int}}(t - \delta t)]. \qquad (1.5.24)$$

We now want to take the trace over the radiation field. Some terms will obviously vanish, for example $\mathrm{Tr}_{\mathrm{Rad}}\{\delta B(t)\rho_{\mathrm{Int}}(t - \delta t)\}$, since from the equation of motion in the form (1.5.24) it is clear that $\rho_{\mathrm{Int}}(t - \delta t)$ must be independent of $\delta B(t)$. The same is not the case with terms involving $\rho(t + \delta t)$, so we can substitute for these using (1.5.24) itself, which yields

$$\rho_{\mathrm{Int}}(t + \delta t) - \rho_{\mathrm{Int}}(t - \delta t) = -2\mathrm{i} g \sqrt{\frac{\Omega}{\hbar}} \left[\sigma^{+} \delta B(t) + \sigma^{-} \delta B^{\dagger}(t), \rho_{\mathrm{Int}}(t - \delta t) \right]$$
$$- \frac{g^2 \Omega}{\hbar} \left[\sigma^{+} \delta B(t) + \sigma^{-} \delta B^{\dagger}(t), \left[\sigma^{+} \delta B(t) + \sigma^{-} \delta B^{\dagger}(t), \right. \right.$$
$$\left. \left. \rho_{\mathrm{Int}}(t + \delta t) + \rho_{\mathrm{Int}}(t - \delta t) \right] \right]. \qquad (1.5.25)$$

Taking the trace over the radiation field, all terms on the right hand side of the first line vanish, because of the independence of the noise terms of the density operator at the time $t - \delta t$. In the next line we still have the same problem, but we can substitute again, and without much difficulty show that the resulting higher order terms vanish sufficiently rapidly as $\delta t \to 0$ to be neglected. Thus we are left to evaluate terms like

$$\text{Tr}_{\text{Rad}} \left\{ \delta B(t) \rho_{\text{Int}}(t - \delta t) \delta B^{\dagger}(t) \right\} \tag{1.5.26}$$

Since the interaction is assumed weak, we approximate

$$\rho_{\text{Int}}(t - \delta t) \approx \bar{\rho}_{\text{Int}}(t - \delta t) \otimes \rho_{\text{Rad}}(t - \delta t) \approx \bar{\rho}_{\text{Int}}(t) \otimes \rho_{\text{Rad}}(t). \tag{1.5.27}$$

Thus we can evaluate as follows;

$$\text{Tr}_{\text{Rad}} \left\{ \delta B(t) \rho_{\text{Int}}(t) \delta B^{\dagger}(t) \right\}$$

$$= \bar{\rho}_{\text{Int}}(t) \int\limits_{t-\delta t}^{t+\delta t} dt' \int\limits_{t-\delta t}^{t+\delta t} dt'' \text{Tr}_{\text{Rad}} \left\{ \Gamma(t') \rho_{\text{Rad}}(t) \Gamma^{\dagger}(t'') \right\} \tag{1.5.28}$$

$$= \bar{\rho}_{\text{Int}}(t) \int\limits_{t-\delta t}^{t+\delta t} dt' \int\limits_{t-\delta t}^{t+\delta t} dt'' \langle \Gamma^{\dagger}(t'') \Gamma(t') \rangle \tag{1.5.29}$$

$$= 2 \bar{\rho}_{\text{Int}}(t) \bar{N}(\Omega) \delta t. \tag{1.5.30}$$

Putting all these together in (1.5.25), and taking the limit $\delta t \to 0$, we finally obtain the *master equation* for the reduced density matrix in the form

$$\frac{d\bar{\rho}_{\text{Int}}(t)}{dt} = \frac{2g^2 \Omega}{\hbar} \left(\bar{N}(\Omega) + 1 \right) \left\{ 2\sigma^- \bar{\rho}_{\text{Int}} \sigma^+ - \bar{\rho}_{\text{Int}} \sigma^+ \sigma^- - \sigma^+ \sigma^- \bar{\rho}_{\text{Int}} \right\}$$

$$+ \frac{2g^2 \Omega}{\hbar} \bar{N}(\Omega) \left\{ 2\sigma^+ \bar{\rho}_{\text{Int}} \sigma^- - \bar{\rho}_{\text{Int}} \sigma^- \sigma^+ - \sigma^- \sigma^+ \bar{\rho}_{\text{Int}} \right\}. \tag{1.5.31}$$

1.5.2   Relationship to the Pauli Master Equation

The reduced density matrix has the property that its diagonal elements are the probabilities of occupying the relevant states a or b. It is not difficult, by using the definitions (1.5.5), to show that we have the two equations for the diagonal matrix elements

$$P_a(t) \equiv \langle a | \bar{\rho}_{\text{Int}}(t) | a \rangle, \qquad P_b(t) \equiv \langle b | \bar{\rho}_{\text{Int}}(t) | b \rangle, \tag{1.5.32}$$

$$\frac{dP_a}{dt} = \frac{4g^2 \Omega}{\hbar} \left\{ -\bar{N}(\Omega) P_a + \left(\bar{N}(\Omega) + 1 \right) P_b \right\} \tag{1.5.33}$$

$$\frac{dP_b}{dt} = \frac{4g^2 \Omega}{\hbar} \left\{ \bar{N}(\Omega) P_a - \left(\bar{N}(\Omega) + 1 \right) P_b \right\} \tag{1.5.34}$$

and these equations are exactly of the form of the Pauli master equation (1.5.1). There are a number of points to note.

a) Stationary Solutions: It is straightforward that these are

$$P_a^s = \frac{\bar{N}(\Omega) + 1}{2\bar{N}(\Omega) + 1} = \frac{1}{1 + \exp(-\hbar\Omega/kT)} \qquad (1.5.35)$$

$$P_b^s = \frac{\bar{N}(\Omega)}{2\bar{N}(\Omega) + 1} = \frac{\exp(-\hbar\Omega/kT)}{1 + \exp(-\hbar\Omega/kT)}. \qquad (1.5.36)$$

These are of course just the probabilities that Einstein used to derive the theory presented in Sect. 1.3

b) Stimulated and Spontaneous Processes: On comparing the Pauli master equation with (1.5.33,34), we can see that $\bar{\rho}$ and $\bar{N}(\Omega)$ must be proportional. Put more carefully, we can see that there are terms proportional to $\bar{N}(\Omega)$, identified as *stimulated emission and absorption terms* and there are also the *spontaneous emission terms*, which are present even when $\bar{N}(\Omega)$ is zero. These terms arise out of the non-commutativity of the noise terms, $\Gamma(t)$, $\Gamma^\dagger(t)$, which ultimately arises because it is assumed that $E(t)$ is an operator quantity whose commutation relations are given by a form like (1.4.5). Thus these terms, added almost arbitrarily by Einstein, are a natural consequence of the quantum nature of the effective noise. Indeed, one can assume that the radiation field is a classical random quantity with the proper spectrum of quantum noise (1.2.4,5) and carry out a similar analysis. The result is a similar equation to (1.5.33,34), but instead of $\bar{N}(\Omega)$ and $\bar{N}(\Omega) + 1$, one obtains in both cases $\bar{N}(\Omega) + \frac{1}{2}$, which is wrong physics. Einstein's result amounts to assuming classical noise, but with a Planck spectrum, and then arbitrarily adding the spontaneous terms.

c) Off-diagonal Elements: The density operator $\bar{\rho}$ has two off-diagonal elements, which are complex conjugates of each other. Following the same methods as before, this gives the equation

$$\frac{d}{dt}\langle a|\bar{\rho}_{\text{Int}}(t)|b\rangle = -\frac{2g^2\Omega}{\hbar}(2\bar{N} + 1)\langle a|\bar{\rho}_{\text{Int}}(t)|b\rangle \qquad (1.5.37)$$

so that in the stationary state, this off-diagonal element vanishes. However this does not mean that such off-diagonal elements have no physical relevance. They represent the quantum-mechanical coherence, and what (1.5.37) implies is that, under the influence of random noise, this dies off with time.

d) Application of a Classical Light Field: Suppose, however, we introduce a slightly more intricate situation—an atom in a thermal light field being illuminated by a laser, modelled by a classical field, which we introduce into the Hamiltonian (1.5.4) by

$$E \rightarrow E + \mathcal{E}e^{i\Omega t} + \mathcal{E}^* e^{-i\Omega t} \qquad (1.5.38)$$

so that E is the *operator* representing the quantised radiation field, while the remaining term is a classical c-number. (This procedure turns out to be in fact a perfectly

correct quantum mechanical way of representing an ideal laser field (see Chap. 4) and even in the absence of explicit justification, it is intuitively obvious that if the light field is strong enough, it will behave classically.)

The net effect of making this change is to give a master equation in the form

$$
\begin{aligned}
\frac{d\tilde{\rho}_{\text{Int}}(t)}{dt} &= \frac{2g^2\Omega}{\hbar}\left(\bar{N}(\Omega)+1\right)\{2\sigma^-\tilde{\rho}_{\text{Int}}\sigma^+ - \tilde{\rho}_{\text{Int}}\sigma^+\sigma^- - \sigma^+\sigma^-\tilde{\rho}_{\text{Int}}\} \\
&\quad + \frac{2g^2\Omega}{\hbar}\bar{N}(\Omega)\{2\sigma^+\tilde{\rho}_{\text{Int}}\sigma^- - \tilde{\rho}_{\text{Int}}\sigma^-\sigma^+ - \sigma^-\sigma^+\tilde{\rho}_{\text{Int}}\} \\
&\quad - \frac{ig}{\hbar}\left[(\mathcal{E}\sigma^+ + \mathcal{E}^*\sigma^-), \tilde{\rho}_{\text{Int}}\right]
\end{aligned}
\tag{1.5.39}
$$

where terms oscillating with frequencies $\pm 2\Omega$ have been omitted to obtain the last line. This equation cannot be reduced to diagonal and off-diagonal equations because the last term couples diagonal and off-diagonal terms. The simplest method of solution is to note that $\tilde{\rho}_{\text{Int}}$ is completely specified by the quantities

$$
\left.
\begin{aligned}
\langle\sigma^\pm\rangle &\equiv \text{Tr}\left\{\tilde{\rho}_{\text{Int}}\sigma^\pm\right\} \\
\langle\sigma_z\rangle &\equiv \text{Tr}\left\{\tilde{\rho}_{\text{Int}}\sigma_z\right\}
\end{aligned}
\right\}
\tag{1.5.40}
$$

and to use the algebra of the matrices to derive

$$
\frac{d\langle\sigma^+\rangle}{dt} = -\frac{2g^2\Omega}{\hbar}\left(2\bar{N}(\Omega)+1\right)\langle\sigma^+\rangle - \frac{ig}{\hbar}\mathcal{E}^*\langle\sigma_z\rangle
\tag{1.5.41}
$$

$$
\frac{d\langle\sigma^-\rangle}{dt} = -\frac{2g^2\Omega}{\hbar}\left(2\bar{N}(\Omega)+1\right)\langle\sigma^-\rangle + \frac{ig}{\hbar}\mathcal{E}\langle\sigma_z\rangle
\tag{1.5.42}
$$

$$
\frac{d\langle\sigma_z\rangle}{dt} = -\frac{4g^2\Omega}{\hbar}\left(2\bar{N}(\Omega)+1\right)\langle\sigma_z\rangle - \frac{4g^2\Omega}{\hbar} - \tfrac{1}{2}i\frac{g}{\hbar}\left(\mathcal{E}\langle\sigma^+\rangle - \mathcal{E}^*\langle\sigma^-\rangle\right).
\tag{1.5.43}
$$

We can solve this equation in the stationary situation, to obtain

$$
\langle\sigma_z\rangle_s = \frac{-\left(2\bar{N}(\Omega)+1\right)}{\left(2\bar{N}(\Omega)+1\right)^2 + 2|\mathcal{E}|^2/g^2\Omega^2}
\tag{1.5.44}
$$

$$
\langle\sigma^+\rangle_s = \langle\sigma^-\rangle_s^* = \frac{2i\,\mathcal{E}/g\Omega}{\left(2\bar{N}(\Omega)+1\right)^2 + 2|\mathcal{E}|^2/g^2\Omega^2}
$$

from which we can reconstruct $\tilde{\rho}_{\text{Int}}^s$ using

$$
\tilde{\rho}_{\text{Int}}^s = \tfrac{1}{2} + \tfrac{1}{2}\langle\sigma_z\rangle_s\sigma_z + \langle\sigma^+\rangle_s\sigma^- + \langle\sigma^-\rangle_s\sigma^+.
\tag{1.5.45}
$$

From this it is clear that there are non-diagonal terms—these represent quantum mechanical coherence, and arise from the coherent driving field.

2. Quantum Statistics

The very concept of "quantum noise" implies some kind of a unification of statistics and quantum mechanics. It is therefore essential to set out the background to this unification in order to establish both notation and some less widely known matters relevant to the subject.

There are two rather separate subjects which are relevant to the subject "Quantum Statistics". The first of these is the aspect of statistics which arises in the very process of measurement, and this really amounts to the quantum theory of measurement. The usual treatments of the quantum theory of measurement found in texts on quantum mechanics concentrate on developing the formulation of a single measurement at a precise time on a system in a certain quantum state. Such a formulation is in principle sufficient for our needs, but in fact falls rather short of what is in practice done. A typical measurement in a noisy system of any kind (quantum mechanical or otherwise) will amount to the possibly continuous monitoring of a small number of simultaneously measurable quantities, which will probably be multiplied together with time delays to get correlation functions, or be Fourier analysed (either computationally by means of the fast Fourier transform, or physically with a spectrum analyser such as a Fabry-Pérot interferometer) to produce spectra.

Thus, in Sect. 2.2 of this chapter, a kind of quantum measurement theory is introduced which can generate the results of arbitrarily many measurements of the most general kind, and the correlation functions which ensue. The end result is that any time ordered correlation function can be generated by means of sequences of possible measurements.

The formalism of measurement theory is essential—but it must be combined with experience of the everyday world—which is that there is a great deal of randomness in the world as we see it, and this is well described by the quantum mechanical adaptation of the subject of statistical mechanics. Thus the final section of this chapter gives an outline of quantum statistical mechanics in a form suitable for the remainder of the book

2.1 The Density Operator

If a system has a state vector (in the Schrödinger picture) $|\psi, t\rangle$, then it is possible to define the *density operator* for the system by the outer product

$$\rho(t) = |\psi, t\rangle\langle\psi, t|. \tag{2.1.1}$$

The quantum mechanical formula for the mean of an operator A is then

$$\langle A \rangle = \langle \psi, t | A | \psi, t \rangle = \mathrm{Tr}\, \{A\, \rho(t)\}. \tag{2.1.2}$$

It is not difficult to see that all of the *measurable* information in the wavefunction is contained in the density operator. Thus a density operator of the kind (2.1.1) is essentially an object equivalent to the state vector.

However, the density operator has a more general formulation which is of particular importance in the case that the state of the system is not known with the precision implicit in the specification of a state vector $|\psi, t\rangle$. We can consider the situation in which the system is prepared with probabilities $P(a)$ in various states $|\psi_a\rangle$ (which need not be orthogonal to each other). The simplest example is the possibility of preparing a beam of particles in polarisation states which could be oriented in a number of directions. The average of a quantum mechanical operator A is given by the probabilistic average over $P(a)$ of the quantum mechanical averages $\langle \psi_a | A | \psi_a \rangle$, thus

$$\langle A \rangle = \sum_a P(a) \langle \psi_a | A | \psi_a \rangle. \tag{2.1.3}$$

If we now define the density operator ρ in this case, to be

$$\rho = \sum_a P(a) |\psi_a\rangle \langle \psi_a|, \tag{2.1.4}$$

it is clear that again

$$\langle A \rangle = \mathrm{Tr}\, \{A\rho\}. \tag{2.1.5}$$

In this form the density operator contains in a rather compact form both the statistical and quantum mechanical information about the system.

2.1.1   Density Operator Properties

There are a number of properties of the density operator which are true in all situations, most of which depend on the representation (2.1.4):

a) $\mathrm{Tr}\, \{\rho\} = 1$ $\hspace{8cm}$ (2.1.6)

$\quad$ for $\quad$ $\mathrm{Tr}\, \{\rho\} = \sum_a P_a \langle \psi_a | \psi_a \rangle = \sum_a P_a = 1.$ $\hspace{3cm}$ (2.1.7)

b) ρ is positive semidefinite: for any state $|A\rangle$

$$\langle A | \rho | A \rangle = \sum_a P_a |\langle A | \psi_a \rangle|^2 \geq 0. \tag{2.1.8}$$

c)   If the ensemble of $|\psi_a\rangle$ has only one member, then

$$\rho = |\psi_a\rangle \langle \psi_a| \tag{2.1.9}$$

and this is said to be a *pure state*, since the system is always in the state $|\psi_a\rangle$.

In this case

$$\rho^2 = |\psi_a\rangle\langle\psi_a|\psi_a\rangle\langle\psi_a| = \rho. \tag{2.1.10}$$

Conversely, if $\rho^2 = \rho$, we can write ρ in the form (2.1.9). We note that ρ is Hermitian, and hence write a spectral representation

$$\rho = \sum_i \lambda_i |i\rangle\langle i| \tag{2.1.11}$$

where λ_i are the eigenvalues of ρ, and $|i\rangle$ is the corresponding eigenvector. Because in this case $\rho^2 = \rho$, only $\lambda_i = 0, 1$ are allowable, and because $\text{Tr}\{\rho\} = 1$, only one eigenvalue, say λ_m is 1. Hence

$$\rho = |m\rangle\langle m| \tag{2.1.12}$$

which is a pure state form, like (2.1.9).

d) $\quad \text{Tr}\{\rho^2\} \leq 1 \tag{2.1.13}$

with equality only for a pure state. For

$$\text{Tr}\{\rho^2\} = \sum_{a,b} P_a P_b |\langle\psi_a|\psi_b\rangle|^2. \tag{2.1.14}$$

Since

$$|\langle\psi_a|\psi_b\rangle|^2 \leq 1, \qquad \text{and} \qquad \sum P_b = 1 \tag{2.1.15}$$

$$\sum_b P_b |\langle\psi_a|\psi_b\rangle|^2 \leq 1 \qquad \text{for any } a \tag{2.1.16}$$

hence

$$\text{Tr}\{\rho^2\} \leq \sum_a P_a = 1. \tag{2.1.17}$$

Furthermore, if

$$\text{Tr}\{\rho^2\} = 1, \tag{2.1.18}$$

we can write (2.1.14) in the form

$$\sum_a P_a \phi_a = 1 \tag{2.1.19}$$

where

$$\phi_a = \sum_b P_b |\langle\psi_a|\psi_b\rangle|^2 \leq 1 \tag{2.1.20}$$

from (2.1.16). The only way (2.1.19) can be satisfied is if

$$P_a = \delta_{a,a_0} \quad \text{for some } a_0. \tag{2.1.21}$$

In this case $\phi_a = \delta_{a,a_0}$, and hence $\text{Tr}\{\rho^2\} = 1$ and

$$\rho = |\psi_{a_0}\rangle\langle\psi_{a_0}| \tag{2.1.22}$$

that is, a pure state.

2.1.2   Von Neumann's Equation

The equation of motion for the density operator in the form (2.1.4) is obtained directly from the Schrödinger equation for the time development of the state vector,

$$H |\psi_a, t\rangle = i\hbar \frac{d}{dt} |\psi_a, t\rangle \qquad (2.1.23)$$

as

$$\dot{\rho}(t) = -\frac{i}{\hbar} [H, \rho(t)] \qquad (2.1.24)$$

where H is the Hamiltonian for the system. This equation has the formal solution

$$\rho(t) = \exp\left(-\frac{i}{\hbar} Ht\right) \rho(0) \exp\left(\frac{i}{\hbar} Ht\right). \qquad (2.1.25)$$

In contrast to von Neumann's equation, Heisenberg's equation of motion for dynamical operators (in the Heisenberg picture) is

$$\dot{A}(t) = \frac{i}{\hbar} [H, A(t)] \qquad (2.1.26)$$

which differs in sign from (2.1.24).

2.2   Quantum Theory of Measurement

From the early days of quantum mechanics the contradiction between our experience that quantities can be measured and the quantum mechanical formulation of physics only in terms of probability amplitudes has given the field of quantum measurement theory a rather metaphysical character. There is a certain way in which the theory of quantum noise can clarify matters, as will be shown in Chap.7.

The original formulation of measurement theory is that of *von Neumann* [2.1], and in spite of its rather artificial appearance, it does seem to be in accord with reality. Thus, in this section, this basic and traditional formalism will be outlined and developed along the lines of *Caves* [2.2] to give us in the end a theory of repeated measurements on the same system. From this we can develop the theory of *correlation functions* as the basic measurable quantities of quantum theory.

This section will discuss only the basic and traditional formalism of quantum measurement theory. There are some mysteries in this formalism, which impinge directly on the field of quantum noise, and these will be discussed in Chap.7.

2.2.1   Precise Measurements

Von Neumann's theory of measurement is very simple in the case that the specification of the eigenvalue of one variable A specifies the state of the system—thus we can label the states as $|a\rangle$ where a is the eigenvalue of A. The measurement postulate states that if the system is originally in the state $|\psi, t\rangle$ and the value a of A is measured then

i)   The value a is found with probability $|\langle a|\psi, t\rangle|^2$

ii)   The state after the measurement is $|a\rangle$.

Thus the state vector is "instantaneously" reduced from $|\psi, t\rangle$ to $|a\rangle$. (A more modest interpretation is that the state transforms from $|\psi, t\rangle$ to $|a\rangle$ in the time taken to make the measurement.) This may be symbolised as

$$|\psi, t\rangle \rightarrow P_a|\psi, t\rangle \equiv \big(|a\rangle\langle a|\big)|\psi, t\rangle \tag{2.2.1}$$

Note that P_a is a *projector*, i.e. $P_a^2 = P_a$, so that the evolution implied by (2.2.1) is not unitary, as is obvious from the fact that the norm of the right hand side is $|\langle a|\psi, t\rangle|^2$, and not one.

The measurement postulate as it stands is a postulate on its own, which has been shown to be *compatible* with quantum mechanics, but has not yet been satisfactorily derived from the dynamical postulates of quantum mechanics. (See however [2.3] for an interesting attempt.) Most physicists are uncomfortable with the idea of measurement as being in some way in addition to dynamics—it is felt that we are all part of this physical world, and any conclusions we draw about it should arise from the same dynamical description that we apply to it.

2.2.2   Imprecise Measurements

It is not possible to build a measurement theory solely on the basis of precise measurements, for two reasons:

i)   No perfect measuring apparatus has ever been built, so there is always some intrinsic error to any measurement in addition to that required by quantum mechanics.

ii)   Usually a complete description of a system is effected most naturally by the use of more than one operator, e.g., the operators for x, y and z coordinates are necessary to specify the position of a particle.

We therefore have to work out a theory of measurement in which we do not measure sufficient variables to specify precisely what the final state is.

We will consider a system describable as above, by the eigenstates $|a\rangle$, and assume that these have a discrete spectrum. It is obviously possible to approximate any situation by this kind of description, since a discrete set can always be ordered, and hence described by a single operator whose eigenvalues represent the position in order.

It is experimentally true that a measurement of any kind appears to leave the system in a final state which is a *linear* function of the initial state, in the sense implied by (2.2.1), that the final state is given up to a normalisation factor by a linear operator acting on it.

With an imprecise measurement there is some arbitrariness in defining what the measured value obtained by a measurement actually is. An imprecise measurement necessarily will sometimes give a wrong answer! A more balanced point of view is perhaps that the value we infer from a measurement of A has an error associated with it, and we can only say that the "true value" lies within some range of the measured value.

All of this can be compactly put together into a rather elegant formalism. Suppose $\bar{a}$ are the possible "measured values" of A which result as the end result of a measurement process. For each $\bar{a}$ we introduce the amplitudes $\Upsilon_{\bar{a}a}$ and the operator

$$\hat{\Upsilon}_{\bar{a}} \equiv \sum_a \Upsilon_{\bar{a}a} |a\rangle\langle a| = \sum_a \Upsilon_{\bar{a}a} P_a. \tag{2.2.2}$$

This definition of the operator $\hat{\Upsilon}_{\bar{a}}$ actually means that $\hat{\Upsilon}_{\bar{a}}$ is a function of the operator A which is being measured.

We assume that the amplitudes are normalised

$$\sum_{\bar{a}} |\Upsilon_{\bar{a}a}|^2 = 1. \tag{2.2.3}$$

This normalisation property is related to the completeness of the possible measured values $\bar{a}$, and can be interpreted as meaning that the probability of obtaining one of all the possible values $\bar{a}$ is one—there are therefore no other possible values.

We can also introduce another operator, called an *effect*, $\hat{F}_{\bar{a}}$, given by

$$\hat{F}_{\bar{a}} = \hat{\Upsilon}_{\bar{a}}^\dagger \hat{\Upsilon}_{\bar{a}} = \sum_a |\Upsilon_{\bar{a}a}|^2 P_a. \tag{2.2.4}$$

Notice that there is a completeness relation

$$\sum_{\bar{a}} \hat{F}_{\bar{a}} = \sum_{a\bar{a}} |\Upsilon_{\bar{a}a}|^2 P_a = \sum_a P_a = 1, \tag{2.2.5}$$

which follows by use of (2.2.3).

We can now put all of this together to give a theory of imprecise measurement, via two postulates.

i) The probability of obtaining the value $\bar{a}$ as a result of this kind of measurement is

$$P(\bar{a}) = \langle \psi, t | \hat{F}_{\bar{a}} | \psi, t \rangle = \sum_a |\Upsilon_{\bar{a}a}|^2 |\langle a | \psi, t \rangle|^2. \tag{2.2.6}$$

ii) If the measured result is $\bar{a}$, then the state of the system immediately after the measurement is

$$|\psi_{\bar{a}}, t\rangle = \frac{\hat{\Upsilon}_{\bar{a}}|\psi, t\rangle}{\sqrt{P(\bar{a})}} = \sum_a |a\rangle \frac{\Upsilon_{\bar{a}a}\langle a|\psi, t\rangle}{\sqrt{P(\bar{a})}}. \qquad (2.2.7)$$

Comments

i)   Notice that this theory reduces to the precise measurement theory in the case that $\Upsilon_{\bar{a}a}$ is zero unless $\bar{a} = a$.

ii)   A particular case of great interest occurs when $\Upsilon_{\bar{a}a}$ is either 1 or 0, so that

$$\hat{\Upsilon}_{\bar{a}}^2 = \hat{\Upsilon}_{\bar{a}} \qquad (2.2.8)$$

and thus $\hat{\Upsilon}_{\bar{a}}$ is a projection operator. If we further consider the situation that each value of $\bar{a}$ corresponds to a range of values of a which does not overlap with that for any other value of $\bar{a}$, we have the orthogonality condition,

$$\hat{\Upsilon}_{\bar{a}}\hat{\Upsilon}_{\bar{b}} = \delta_{\bar{a}\bar{b}}\hat{\Upsilon}_{\bar{a}}. \qquad (2.2.9)$$

In this case, the measurement of A is essentially equivalent to a precise measurement of the reduced operator $\tilde{A} = \sum_{\bar{a}} \hat{\Upsilon}_{\bar{a}}$ in which the unmeasured range is not affected. This happens in practice when the system can be described by two commuting operators, A_1 and A_2, and we make a measurement simply by applying the precise measurement theory to one of them, so that the relevant projection operator takes the form

$$\hat{\Upsilon}_{\bar{a}_1} = \sum_{a_2} |\bar{a}_1, a_2\rangle\langle\bar{a}_1, a_2|. \qquad (2.2.10)$$

iii) In general, however, the imprecise measurement does not correspond to the precise measurement of any operator.

iv)   The construction of $\hat{F}_{\bar{a}}$ shows that there are a large number of $\hat{\Upsilon}_{\bar{a}}$ which give the same effect, $\hat{F}_{\bar{a}}$, and from (2.2.7) it is clear that all the information about the measured values obtained from the measurement is contained in the effect $\hat{F}_{\bar{a}}$. The phases of the $\Upsilon_{\bar{a}a}$ are essential only for the elucidation of the final state of the system. However this information is very important if we wish to carry out a further measurement after the first.

v)   In principle, any $\Upsilon_{\bar{a}a}$ are allowed for any possible basis set $|a\rangle$ in the quantum mechanical Hilbert space.

2.2.3   The Quantum Bayes Theorem

Caves [2.2] has pointed out that the measurement formulation has a very suggestive interpretation as a kind of quantum Bayes theorem. From (2.2.7) we can derive the result

$$|\langle a|\psi, t\rangle|^2 |\Upsilon_{\bar{a}a}|^2 = P(\bar{a})|\langle a|\psi_{\bar{a}}, t\rangle|^2 \qquad (2.2.11)$$

$$\equiv P(a; \bar{a}). \qquad (2.2.12)$$

We introduce the quantity $P(a;\bar{a})$ as a joint probability that the system had the eigenvalue a before measurement and the result $\bar{a}$ was obtained as a result of the measurement, for we can make the two interpretations, depending on whether we look at the LHS or the RHS of (2.2.11) :

LHS;   Refers to the situation before measurement.

$P(a;\bar{a}) =$ *(probability that the system is in state a before measurement)*
         × *(probability of getting the value $\bar{a}$ given that it was in the state a*
         before measurement)

RHS;   Refers to the situation after measurement.

$P(a;\bar{a}) =$ *(probability that the result of measurement was $\bar{a}$)*
         × *(probability that the system was in the state a after measurement)*

If we now consider two different reference states, $|a\rangle$ and $|b\rangle$, we can cancel out $P(\bar{a})$, to get

$$\frac{|\langle a|\psi_{\bar{a}},t\rangle|^2}{|\langle b|\psi_{\bar{a}},t\rangle|^2} = \frac{|\langle a|\psi,t\rangle|^2\,|\Upsilon_{\bar{a}a}|^2}{|\langle b|\psi,t\rangle|^2\,|\Upsilon_{\bar{a}b}|^2} \tag{2.2.13}$$

or, one sees that (2.2.13) can be written as

$$|\langle a|\psi_{\bar{a}},t\rangle|^2 \propto |\langle a|\psi,t\rangle|^2\,|\Upsilon_{\bar{a}a}|^2 \qquad \text{for fixed } \bar{a}, \tag{2.2.14}$$

which means, in words,

Probability the system is in the state $|a\rangle$, given that the result $\bar{a}$ has been measured

$\propto$ *Probability that the system was in state $|a\rangle$ before measurement*

                × *probability that the result is $\bar{a}$ given that the state is $|a\rangle$.*

This is in exact analogy to the Bayes theorem concepts of classical probability theory. The term $|\langle a|\psi,t\rangle|^2$ is the a priori probability of the value a. The term $|\Upsilon_{\bar{a}a}|^2$ is a conditional probability for getting the result $\bar{a}$, given that we start with $|a\rangle$—this can be seen directly from (2.2.6), where we only need to set $|\psi,t\rangle = |a\rangle$ to get

$$\{P(\bar{a}) \text{ given the initial state is } |a\rangle\} = |\Upsilon_{\bar{a}a}|^2. \tag{2.2.15}$$

Thus, like Bayes theorem, this result takes the conditional probabilities for a result, given an initial value, and the a priori probabilities of the initial values, and yields a result for the probability of a given value, given a particular result.

Finally we note that there is a more powerful result here, for probability amplitudes, following from (2.2.7). We see that

$$|\langle a|\psi_{\bar{a}},t\rangle|P(\bar{a})^{\frac{1}{2}} = \langle a|\psi,t\rangle\Upsilon_{\bar{a}a}. \tag{2.2.16}$$

Using different $|a\rangle$, $|b\rangle$, and cancelling $P(a)^{\frac{1}{2}}$ as above, we deduce that

$$\langle a|\psi_{\bar{a}},t\rangle \propto \langle a|\psi,t\rangle\Upsilon_{\bar{a}a} \qquad \text{for fixed} \quad \bar{a}, \tag{2.2.17}$$

which gives an amplitude interpretation of Bayes theorem. This is a much stronger result than the form (2.2.14), and fully embodies the quantum mechanical centrality of the concept of a probability amplitude. The formulation (2.2.17) is fully equivalent to the original postulates (2.2.6,7), but presents a much more natural way of looking at them.

2.2.4   More General Kinds of Measurements

We have to ask what kind of operations $\hat{\varUpsilon}_{\bar{a}}$ are allowed by the theoretical development of Sect. 2.2.2. This is easily seen from (2.2.2), which implies that $\hat{\varUpsilon}_{\bar{a}}$ is a function of the operator A which is itself Hermitian and an observable. This is of course quite a restrictive requirement: that there must exist some observable A of which $\hat{\varUpsilon}_{\bar{a}}$ is a function, and indeed this is the observable which is being measured.

Is this assumption needed? The reason for it lies in our concept of what we would like to think a measurement is. This kind of measurement has the property that if the initial state is the eigenstate $|a_0\rangle$ of A, then the state after measurement is also $|a_0\rangle$. In this sense we can say that the system is not disturbed by the measurement when the initial value is precise. This arises basically by assumption, and as a consequence, $\hat{\varUpsilon}_{\bar{a}}$ is a function of A. It is conventional to call this *a measurement of the first kind*.

There are physical observations which do not have this property. For example in photon counting we count photons by absorbing them, and no matter what state the system is initially in, the final state will have fewer photons. The measurement destroys the state being measured. We are all aware that this kind of measurement is quite important in everyday life. For example, one usually tests the quality of an apple by eating it. At the end of the experiment we have no apple, but consider the information gained to be relevant to similarly prepared apples, which others may choose to eat. Classical sampling theory does not normally take into account that sampling may destroy the sample, even though it is obvious that such destructive sampling is very common. It is conventional to say that these measurements are *measurements of the second kind*. In quantum mechanics we can conceptualise such a measurement very similarly to the way we have already done. The simplest thing to assume is that the operator $\hat{\varUpsilon}_{\bar{a}}$, which is a function of the observable A, is replaced by a general operator $\hat{\varPhi}(\bar{a})$, which can be represented as

$$\hat{\varPhi}(\bar{a}) = \sum_{a,b} \varPhi(\bar{a})_{ab} |a\rangle\langle b|. \tag{2.2.18}$$

We now define the effect corresponding to the operator $\hat{\varPhi}(\bar{a})$ by

$$\hat{G}(\bar{a}) = \hat{\varPhi}^{\dagger}(\bar{a})\hat{\varPhi}(\bar{a}) \tag{2.2.19}$$

and the assumption of completeness is

$$\sum_{\bar{a}} \hat{G}(\bar{a}) = 1. \tag{2.2.20}$$

The measurement postulates are now framed from the point of view that $\bar{a}$ is not so much the value of an operator A as a label of a possible outcome of the measurement. We then postulate

i)   The probability of obtaining the outcome $\bar{a}$ as a result of the measurement is

$$P(\bar{a}) = \langle\psi, t|\hat{G}(\bar{a})|\psi, t\rangle. \tag{2.2.21}$$

ii) If the outcome is $\bar{a}$, then the state of the system immediately after the measurement is

$$|\psi_{\bar{a}}, t\rangle = \frac{\hat{\Phi}(\bar{a})|\psi, t\rangle}{\sqrt{P(\bar{a})}} = \sum_{a,b} \frac{|a\rangle\Phi(\bar{a})_{a,b}\langle b|\psi, t\rangle}{\sqrt{P(\bar{a})}}. \tag{2.2.22}$$

The definition (2.2.18) of $\hat{\Phi}(\bar{a})$ means that it is a completely general linear operator; thus (2.2.22) will give the most general *linear* functional of the initial state which results from the process that we view as the measurement giving the result $\bar{a}$.

Notice that the completeness assumption (2.2.20) means that

$$\sum_{\bar{a}} P(\bar{a}) = 1 \tag{2.2.23}$$

as required for a probability

Example: Let us consider a two level atom, described by the two states

$$u(+) = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad , \quad u(-) = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{2.2.24}$$

and assume that we wish to test whether the atom is in the excited state, $u(+)$, by absorbing the energy in it, and thus leaving it in the lower state. Let us also allow the method of measurement to be somewhat imperfect, so that even if the atom is in the upper state, we will not necessarily absorb the energy—however we do assume that if the energy is absorbed, then we do detect it.

The first outcome is that the energy is detected, and the atom left in the ground state; thus

$$\hat{\Phi}(1) = \begin{pmatrix} 0 & 0 \\ \lambda & 0 \end{pmatrix} \tag{2.2.25}$$

is a possible choice. The only possible other outcome is that the energy is not detected, in which case (because we assume that if the energy is released, it is detected) the atom, if it is in either the ground or the excited state, remains there. The only reasonable choice for the other operation is

$$\hat{\Phi}(2) = \begin{pmatrix} \mu & 0 \\ 0 & 1 \end{pmatrix} \tag{2.2.26}$$

where

$$|\mu|^2 = 1 - |\lambda|^2 \tag{2.2.27}$$

which ensures that

$$\hat{\Phi}(2)^\dagger\hat{\Phi}(2) + \hat{\Phi}(1)^\dagger\hat{\Phi}(1) = 1. \tag{2.2.28}$$

We can then see that on a general state $\begin{pmatrix} u \\ v \end{pmatrix}$ the final state is

$$\begin{pmatrix} 0 & 0 \\ \lambda & 0 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 0 \\ \lambda u \end{pmatrix} \qquad \text{if the energy is detected,} \tag{2.2.29}$$

and

$$\begin{pmatrix} \mu & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} \mu u \\ \lambda v \end{pmatrix} \qquad \text{if the energy is not detected.} \tag{2.2.30}$$

2.2.5   Measurements and the Density Operator

The general measurement theory can be quite simply expressed in terms of the density operator as follows. If the density operator is initially ρ, then the probability of the outcome $\bar{a}$ is

$$P(\bar{a}) = \mathrm{Tr}\left\{ \hat{\Phi}(\bar{a})\rho\,\hat{\Phi}^{\dagger}(\bar{a}) \right\} \tag{2.2.31}$$

and after the measurement, the density operator has the form

$$\rho(\bar{a}) = \hat{\Phi}(\bar{a})\rho\hat{\Phi}^{\dagger}(\bar{a})/P(\bar{a}). \tag{2.2.32}$$

This expression in terms of the density operator is merely a transcription from the formulation in terms of state vectors. But there is a real generalisation of the concept of a measurement which can only be expressed using the density operator.

Let us return to the example of Sect. 2.2.4, and consider the situation in which the energy may be absorbed from the atom, but yet not be detected, because the detector is not perfect. It is natural to try to remedy this by including in $\hat{\Phi}(2)$ an amplitude to de-excite the atom, thus we might modify the expression (2.2.26) for $\hat{\Phi}(2)$ to $\begin{pmatrix} \mu & 0 \\ \tau & 1 \end{pmatrix}$, where the term τ allows this possibility. However to preserve the completeness (2.2.28), we find we have to modify both $\hat{\Phi}(1)$ and $\hat{\Phi}(2)$ to

$$\hat{\Phi}(1) = \begin{pmatrix} 0 & 0 \\ \lambda & -\tau^{*}/\sqrt{1 - |\mu|^{2}} \end{pmatrix} \tag{2.2.33}$$

$$\hat{\Phi}(2) = \begin{pmatrix} \mu & 0 \\ \tau & \lambda^{*}/\sqrt{1 - |\mu|^{2}} \end{pmatrix}. \tag{2.2.34}$$

where $|\lambda|^{2} + |\mu|^{2} + |\tau|^{2} = 1$. The new expression for $\hat{\Phi}(1)$ means that there is a probability of detecting the energy even when the atom is in the ground state. While this is not impossible, it is certainly not experienced in all detectors. It is a particular (an unlikely) way in which inefficiency can arise.

A more natural way is to introduce a probability of detection of the energy released; i.e. we separate the detection of the energy once it has been released, from the release of the energy by the atom. We can thus isolate three possibilities. If ϵ is the efficiency of detecting the energy once it is released, the operations are:

$$\hat{\Phi}(1) \quad = \begin{pmatrix} 0 & 0 \\ \lambda & 0 \end{pmatrix}$$

The atom is put into the ground state, and the energy is detected, with probability ϵ.

$$\hat{\Phi}(2, \alpha) = \begin{pmatrix} \mu & 0 \\ 0 & 1 \end{pmatrix}$$

The atom remains in the ground or the excited state, and no energy is detected.

$$\hat{\Phi}(2, \beta) = \begin{pmatrix} 0 & 0 \\ \lambda & 0 \end{pmatrix}$$

The atom is put into the ground state, and no energy is detected, with probability $1 - \epsilon$.

The observable outcomes of the $\hat{\Phi}(2,\alpha)$ and $\hat{\Phi}(2,\beta)$ are the same, and the density operator after performing the detection, and observing outcome 2 (no energy detected) is the average of the two. Thus, the probability of outcome 2 is

$$P(2) \equiv \text{Tr} \left\{ \hat{\Phi}(2,\alpha) \rho \hat{\Phi}^\dagger(2,\alpha) + (1-\epsilon) \hat{\Phi}(2,\beta) \rho \hat{\Phi}^\dagger(2,\beta) \right\} \qquad (2.2.35)$$

and the density operator after measurement is given by

$$\rho(2) = P(2)^{-1} \left\{ \hat{\Phi}(2,\alpha) \rho \hat{\Phi}^\dagger(2,\alpha) + (1-\epsilon) \hat{\Phi}(2,\beta) \rho \hat{\Phi}^\dagger(2,\beta) \right\}. \quad (2.2.36)$$

On the other hand the probability of outcome 1 is

$$P(1) = \text{Tr} \left\{ \epsilon \hat{\Phi}(1) \rho \hat{\Phi}^\dagger(1) \right\} \qquad (2.2.37)$$

and

$$\rho(1) = P(1)^{-1} \epsilon \hat{\Phi}(1) \rho \hat{\Phi}^\dagger(1). \qquad (2.2.38)$$

Notice that the requirement $P(1) + P(2) = 1$ is ensured by the completeness sum in the form

$$\epsilon \hat{\Phi}^\dagger(1) \hat{\Phi}(1) + (1-\epsilon) \hat{\Phi}^\dagger(2,\beta) \hat{\Phi}(2,\beta) + \hat{\Phi}^\dagger(2,\alpha) \hat{\Phi}(2,\alpha) = 1. \quad (2.2.39)$$

General Formulation: We can now formulate this kind of generalised detection as follows. We assume that the outcomes of a given detection process are labelled by the index $\bar{a}$, as before, and that each outcome can arise by a number of different operations $\hat{\Phi}(\bar{a},\alpha)$. Then the probability of measuring the outcome $\bar{a}$ is

$$P(\bar{a}) = \sum_\alpha \text{Tr} \left\{ \hat{\Phi}(\bar{a},\alpha) \rho \hat{\Phi}^\dagger(\bar{a},\alpha) \right\} \qquad (2.2.40)$$

and the density operator after the measurement is

$$\rho(\bar{a}) = P(\bar{a})^{-1} \sum_\alpha \hat{\Phi}(\bar{a},\alpha) \rho \hat{\Phi}^\dagger(\bar{a},\alpha). \qquad (2.2.41)$$

Notice that the efficiency ϵ of the example has been absorbed into the definition of the effect; thus the completeness takes the form

$$\sum_{\bar{a},\alpha} \hat{\Phi}^\dagger(\bar{a},\alpha) \hat{\Phi}(\bar{a},\alpha) = 1. \qquad (2.2.42)$$

The type of mapping $\rho \to \rho(\bar{a})$ specified by (2.2.41) is known as a *completely positive map*, since if ρ is itself a positive definite operator, i.e. one such that $\langle u|\rho|u\rangle \geq 0$ for all vectors $|u\rangle$, then so is $\rho(\bar{a})$ positive definite.

   This is the most general kind of formulation of a measurement which is usually made.

2.3 Multitime Measurements

We now want to extend the measurement theory developed for measurements at a single time to a theory of measurements at several times. The basic process is quite easy to formulate. A measurement at an initial time instantaneously transforms the state of the system, which then evolves according to a Hamiltonian time development until the next measurement which instantaneously transforms the state of the system. The Hamiltonian evolution continues until the next measurement and so on. Eventually we obtain the multitime joint probability for obtaining a sequence of measured values corresponding to the measurements carried out sequentially.

We will be able to show that by means of such a process we can, in principle, measure any time ordered correlation function of any quantum mechanical operators.

2.3.1 Sequences of Measurements

Let us, for simplicity, consider the process of carrying out a sequence of measurements on a *well defined initial quantum state* $|\psi, t_0\rangle$. We must first take account of the fact that during the times between successive measurements the system will evolve unitarily. Thus, provided no disturbance or measurement happens to the system between t and $t + \tau$,

$$|\psi, t + \tau\rangle = U(t + \tau, t)|\psi, t\rangle. \tag{2.3.1}$$

Suppose we will make measurements at times $t_q, q = 1, 2, 3, \ldots$, where $t_q > t_{q-1}$. We label the possible results of the qth measurement by $\bar{a}_q$. At some initial time $t_0 < t_1$ we assume the state of the system is $|\psi_0, t_0\rangle$.

The result can be developed by iteration. We assume that the results of the first $q - 1$ samples are $\bar{a}_1, \bar{a}_2, \ldots \bar{a}_{q-1}$. The state of the system just after the $(q-1)$th measurement depends, in general, on the results of all previous measurements, so we will denote it by $|\psi_{\bar{a}_1, \bar{a}_2, \ldots \bar{a}_{q-1}}, t_{q-1}\rangle$. The state just before the qth measurement is given by the unitary time evolution from the state just after the $(q-1)$th measurement, namely

$$|\psi_{\bar{a}_1 \bar{a}_2, \ldots \bar{a}_{q-1}}, t_q\rangle = U(t_q, t_{q-1})|\psi_{\bar{a}_1 \bar{a}_2, \ldots \bar{a}_{q-1}}, t_{q-1}\rangle. \tag{2.3.2}$$

We now use (2.2.6) to compute that the probability of obtaining the result $\bar{a}_q$ at the qth measurement, given the results of all previous samplings, is

$$P(\bar{a}_q|\bar{a}_1, \bar{a}_2, \ldots, \bar{a}_{q-1}) = \langle \psi_{\bar{a}_1, \ldots \bar{a}_{q-1}}, t_q | \hat{F}_{\bar{a}_q} | \psi_{\bar{a}_1, \ldots \bar{a}_{q-1}}, t_q \rangle \tag{2.3.3}$$

and the wavefunction is

$$|\psi_{\bar{a}_1, \ldots \bar{a}_q}, t_q\rangle = \frac{\hat{\Upsilon}_{\bar{a}_q} |\psi_{\bar{a}_1, \ldots \bar{a}_{q-1}}, t_q\rangle}{\sqrt{P(\bar{a}_q|\bar{a}_1, \bar{a}_2, \ldots, \bar{a}_{q-1})}}. \tag{2.3.4}$$

We can now iterate these equations to get

$$|\psi_{\bar{a}_1,\ldots\bar{a}_q}, t_q\rangle = \frac{\left[\prod_{r=1}^{q} \Upsilon_{\bar{a}_r} U(t_r, t_{r-1})\right] |\psi_0, t_0\rangle}{\sqrt{P(\bar{a}_1; \ldots; \bar{a}_q)}}, \tag{2.3.5}$$

where the product in the numerator is ordered so that the times increase to the left, and

$$P(\bar{a}_1; \ldots; \bar{a}_q) = \prod_{r=1}^{q} P(\bar{a}_r | \bar{a}_1, \ldots, \bar{a}_{r-1}). \tag{2.3.6}$$

Since the state is normalised, we deduce that

$$P(\bar{a}_1; \ldots; \bar{a}_q) = \langle \psi_0, t_0 | \left[\prod_{r=1}^{q} \hat{\Upsilon}_{\bar{a}_r} U(t_r, t_{r-1})\right]^{\dagger} \prod_{r=1}^{q} \hat{\Upsilon}_{\bar{a}_r} U(t_r, t_{r-1}) |\psi_0, t_0\rangle. \tag{2.3.7}$$

2.3.2   Expression as a Correlation Function

If we define the Heisenberg picture operators corresponding to the $\hat{\Upsilon}_{\bar{a}}$ by

$$\hat{\Upsilon}_{\bar{a}}(t) = U^{\dagger}(t, t_0) \hat{\Upsilon}_{\bar{a}} U(t, t_0) \tag{2.3.8}$$

the formula (2.3.7) can be written

$$P(\bar{a}_1; \bar{a}_2; \ldots \bar{a}_q) = \langle \psi_0, t_0 | \hat{\Upsilon}_{\bar{a}_1}^{\dagger}(t_1) \ldots \hat{\Upsilon}_{\bar{a}_q}^{\dagger}(t_q) \hat{\Upsilon}_{\bar{a}_q}(t_q) \ldots \hat{\Upsilon}_{\bar{a}_1}(t_1) |\psi_0, t_0\rangle. \tag{2.3.9}$$

This is a kind of correlation function of the operators $\hat{\Upsilon}_{\bar{a}}(t)$ with each other.

2.3.3   General Correlation Functions

There is in principle no reason why the measurement cannot be a completely arbitrary operation $\hat{\Phi}(\bar{a})$, apart from the normalisation necessary to make the completeness assumption (2.2.20) possible. This means that the arguments of Sect. 2.3.1 can be carried through to show that the joint probability for a sequence of outcomes $\bar{a}_1, \bar{a}_2, \ldots$ can be related directly to the correlation functions of the $\hat{\Phi}(\bar{a}_i)$, which are essentially arbitrary; thus following (2.3.9)

$$P(\bar{a}_1; \bar{a}_2; \ldots; \bar{a}_q)$$
$$= \langle \psi_0, t_0 | \hat{\Phi}^{\dagger}(\bar{a}_1, t_1) \ldots \hat{\Phi}^{\dagger}(\bar{a}_q, t_q) \hat{\Phi}(\bar{a}_q, t_q) \ldots \hat{\Phi}(\bar{a}_1, t_1) |\psi_0, t_0\rangle. \tag{2.3.10}$$

We can use the polarisation identity

$$A^\dagger M B = \tfrac{1}{4}\left\{(A+B)^\dagger M(A+B) - (A-B)^\dagger M(A-B)\right.$$
$$\left.-\mathrm{i}(A+\mathrm{i}B)^\dagger M(A+\mathrm{i}B) + \mathrm{i}(A-\mathrm{i}B)^\dagger M(A-\mathrm{i}B)\right\} \qquad (2.3.11)$$

to generate more general correlations by linear combinations of probabilities, and by assuming that the $\hat{\Phi}(\bar{a},t)$ are complete in the sense that any operator including the identity, can be expressed in terms of a linear combination of them, we can generate correlation functions of the kind

$$\langle\psi_0,t_0|A_1(t_1)A_2(t_2)\ldots A_n(t_n)B_m(s_m)\ldots B_2(s_2)B_1(s_1)|\psi_0,t_0\rangle \quad (2.3.12)$$

where the A_i and the B_i are arbitrary operators, and s_i and t_i are times such that

$$t_1 \le t_2 \le \ldots \le t_n, \qquad s_1 \le s_2 \le \ldots \le s_m. \qquad (2.3.13)$$

Such correlation functions are generally called *time ordered correlation functions*, and are the only kind of correlation functions susceptible to direct measurement in this form.

2.4   Quantum Statistical Mechanics

The introduction of the density operator as a means of describing quantum mechanical states in the absence of complete information is the first step in formulating quantum statistics. While the technique is very flexible, there is still a lack of structure—we can conceive of large numbers of ways in which information may be incomplete, but these may not be very relevant to real life. The contact with reality comes from the introduction into quantum mechanics of the concepts of temperature, entropy, and statistical ensembles, which were so successful in the nineteenth century development of statistical mechanics. The formal method of doing this is via the concept of entropy maximisation, subject to constraints.

2.4.1   Entropy

The concept of a macrostate, defined by the values of certain macroscopic variables, such as temperature, pressure, momentum, etc., of a system composed of a very large number of particles is fundamental to statistical mechanics. A macrostate is represented microscopically by myriads of possible configurations of the microscopic variables, (energy, momentum, etc., of the individual particles)—there is no unique microscopic configuration corresponding to a macrostate (other than in very exceptional cases).

The *entropy* of a macrostate is introduced as a measure of how many different microstates represent the same macrostate. In a quantum mechanical system, this

gives a particularly simple formula for entropy. If we consider a density operator ρ which is diagonal in a certain basis, then ρ may be written as a matrix

$$\rho = \begin{pmatrix} p_1 & 0 & 0 & 0 & \ldots \\ 0 & p_2 & 0 & 0 & \ldots \\ 0 & 0 & p_3 & 0 & \ldots \\ 0 & 0 & 0 & p_4 & \ldots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}. \tag{2.4.1}$$

Suppose N different quantum states contribute to this matrix in equal proportions: then the logical measure of entropy is

$$S = k \log N \tag{2.4.2}$$

where k is Boltzmann's constant. In this case, we have also

$$\left. \begin{aligned} p_i &= \frac{1}{N} & i \leq N \\ &= 0 & i > N \end{aligned} \right\} \tag{2.4.3}$$

and we can use this to write the entropy in a form which does not depend on the particular basis, for it is clear that in this situation

$$\mathrm{Tr}\{\rho \log \rho\} = \sum p_i \log p_i = -\log N \tag{2.4.4}$$

where we have used $x \log x = 0$ for $x = 0$.

A change of basis is a unitary transformation, which leaves the trace invariant; hence as a general form of entropy S when (2.4.3) is satisfied,

$$S = -k \mathrm{Tr}\{\rho \log \rho\}. \tag{2.4.5}$$

Obviously the form of distribution (2.4.3) is not always satisfied: in this case one cannot easily say how many quantum states contribute to the density matrix, but (2.4.5) gives a measure of the *average* number of states which contribute to the density matrix.

2.4.2   Thermodynamic Equilibrium

The concept that thermodynamic equilibrium is obtained by maximising entropy is introduced by many compelling and ingenious arguments, which I do not choose to repeat here. Excellent discussions are given by *Tolman* [2.4] and it Landau and *Lifshitz* [2.5].

The maximisation is carried out subject to constraints, which specify the macroscopic knowledge we have of the system. Fundamental to the idea of equilibrium is the idea that the system is stationary, i.e., the density operator has no time dependence. From von Neumann's equation (2.1.24), in the case that there is no explicit time dependence of the Hamiltonian, one sees immediately that in equilibrium the Hamiltonian and the density operator commute with each other. Maximisation of

entropy can be carried out on the assumption that ρ is diagonal in an energy representation, so that

$$\rho \quad = \sum p_i |i\rangle\langle i| \tag{2.4.6}$$

$$H \quad = \sum E_i |i\rangle\langle i| \tag{2.4.7}$$

$$S \quad = -k \sum p_i \log p_i. \tag{2.4.8}$$

One is now in a position to specify exactly what is known about the system. The most common assumption is that the *mean energy content* (essentially the temperature) is known, i.e., that

$$\langle E \rangle = \sum p_i E_i \tag{2.4.9}$$

must be specified as a constraint when maximising the entropy. Another constraint is, of course, that

$$\text{Tr}\,\{\rho\} = \sum p_i = 1. \tag{2.4.10}$$

This means that we must maximise using Lagrange multipliers: namely, one requires

$$\delta \left\{ \sum p_i \log p_i + \alpha \sum p_i + \beta \sum E_i p_i \right\} = 0 \tag{2.4.11}$$

for which the solution is

$$p_i = \exp(-\alpha - \beta E_i) \tag{2.4.12}$$

which can be written as

$$\rho = \exp(-\alpha - \beta H). \tag{2.4.13}$$

Using (2.4.10) we can solve for α to get the *canonical density operator*

$$\rho = Z(\beta)^{-1} \exp(-\beta H) \tag{2.4.14}$$

where $Z(\beta)$ is the *canonical partition function*, defined by

$$Z(\beta) = \text{Tr}\,\{\exp(-\beta H)\}. \tag{2.4.15}$$

The most useful way of looking at β is to use it to define the temperature T through

$$\beta = 1/kT \tag{2.4.16}$$

where k is Boltzmann's constant.

Other ensembles can be useful. There is always the possibility that certain other constants of the motion may be specified; for the most general stationary solution of von Neumann's equation (2.1.24) is a density operator which is an arbitrary function of any constants of the motion, and these may be included as additional constraints. The most well known is the grand canonical ensemble, in which the particle number operator, $\hat{N}$, is assumed to be a constant of the motion, and entropy is maximised subject to a fixed $\langle \hat{N} \rangle$. This leads to the *grand canonical density operator*

$$\rho = Z(\beta, \mu)^{-1} \exp\left[-\beta(\hat{H} - \mu\hat{N})\right] \tag{2.4.17}$$

where μ is known as the *chemical potential*, and

$$Z(\beta, \mu) = \text{Tr}\,\left\{ \exp\left[-\beta(\hat{H} - \mu\hat{N})\right] \right\} \tag{2.4.18}$$

is the *grand canonical partition function*.

2.4.3   The Bose Einstein Distribution

a) A Single Harmonic Oscillator: A harmonic oscillator has states which can be specified by the "number of quanta" of energy $\hbar \omega$; namely the states can be written $|n\rangle$, each of which has energy $(n + \frac{1}{2})\hbar \omega$. The canonical density operator in this case is given by

$$\langle n|\rho|m\rangle = \delta_{nm}\mathrm{e}^{-\beta(n+\frac{1}{2})\hbar\omega}/Z(\beta) \tag{2.4.19}$$

and

$$Z(\beta) = \sum_{n=0}^{\infty}\mathrm{e}^{-\beta(n+\frac{1}{2})\hbar\omega} = \mathrm{e}^{-\beta\hbar\omega/2}/(1 - \mathrm{e}^{-\beta\hbar\omega}). \tag{2.4.20}$$

The mean occupation of the n th state is

$$\langle N\rangle = \sum n\langle n|\rho|n\rangle = 1/(\mathrm{e}^{\hbar\omega\beta} - 1). \tag{2.4.21}$$

Note that setting $\beta = 1/kT$ and taking the limit $T \rightarrow \infty$, we find

$$\langle N\rangle \rightarrow \frac{kT}{\hbar\omega} \tag{2.4.22}$$

which is obviously to be interpreted as meaning that the mean occupation is the mean energy available divided by the energy of a quantum.

This is the prototype of the Bose-Einstein distribution. If we have many oscillators, then the distribution is obtained simply by multiplying many copies of (2.4.19) together for each oscillator, provided there are no constraints, in which case it is more appropriate to use the grand canonical density operator.

b) General Bose Einstein Distribution: This kind of distribution is most generally applied to assemblies of atoms or molecules, such as ^{4}He atoms, which have integer spin, and therefore obey Bose statistics. A state of the system can therefore be written as the direct product of harmonic oscillator states, one for each mode. Thus the states are written $|n_1, n_2, n_3, \ldots n_k, \ldots\rangle$, and the energy of one quantum of the state k is $\hbar\omega_k$. The grand canonical density operator will be given in this case by

$$\langle n_1, n_2, \ldots |\rho|m_1, m_2, \ldots\rangle = \prod_{k}\delta_{n_k,m_k}\mathrm{e}^{-\beta[(n_k+\frac{1}{2})\hbar\omega_k - \mu n_k]}. \tag{2.4.23}$$

The grand canonical partition function is

$$Z(\beta, \mu) = \prod_{k}\left\{\frac{\mathrm{e}^{-\beta\hbar\omega_k/2}}{1 - \mathrm{e}^{-\beta(\hbar\omega_k - \mu)}}\right\} \tag{2.4.24}$$

and

$$\langle N_k\rangle = \frac{1}{\mathrm{e}^{\beta(\hbar\omega_k - \mu)} - 1}. \tag{2.4.25}$$

The chemical potential μ is determined from the constraint that the total number of particles is N, so that

$$N = \sum_k \langle N_k \rangle. \tag{2.4.26}$$

c) Fermi-Dirac Distribution: Here the particles under discussion obey the exclusion principle, so that the states are written $|n_1, n_2, \ldots, n_k, \ldots\rangle$, where each of the n_k can only be 0 or 1, and the term $(n_k + \frac{1}{2})\hbar\omega$ is merely $n_k \hbar\omega$. Thus the grand canonical partition function is

$$Z(\beta, \mu) = \sum_k \left\{ 1 + e^{-\beta(\hbar\omega_k - \mu)} \right\} \tag{2.4.27}$$

with

$$\langle N_k \rangle = \frac{1}{e^{\beta(\hbar\omega_k - \mu)} + 1} \tag{2.4.28}$$

and μ is determined by the same constraint, (2.4.26).

2.5   System and Heat Bath

Statistical mechanics often makes use of the idea of a rather small system which is brought into contact with a very large system, usually held at some definite temperature. The origin of this concept is presumably the study of the Carnot cycle, where a finite volume of gas is transferred from an environment at one temperature to another environment at a different temperature. This could be achieved with a small piston assembly of high conductivity metal, which is transferred from a large tank of water at one temperature to another tank at another temperature—hence the idea of a *"heat bath"*.

   In more general situations the heat bath need not be so explicitly constructed. For example, in a chemical reaction taking place in solution the "system" consists of the molecules of the particular species which react, while the "bath" consists of the non-reacting degrees of freedom of the solution as a whole. The "system" and the "bath" occupy the same space! Indeed, it may be possible, if the reaction does not liberate any significant amount of heat, to operate the reaction without a solution, e.g., a gas phase reaction, and in this case the "bath" consists of the kinetic degrees of freedom only.

2.5.1   Density Operators for "System" and "Heat Bath"

We shall therefore consider the situation in which the physics as a whole does admit a rather clear separation into "system" and "bath", but will not enquire exactly as

to how this is achieved. We will therefore suppose that the states of the physical system as a whole can be written

$$|x, a\rangle \equiv |x\rangle |a\rangle \qquad (2.5.1)$$

where

  a represents the bath degrees of freedom,
  x represents the system degrees of freedom.

For simplicity, let us assume the ranges of both a and x are discrete sets. We can consider now operators which act in the system only, and operators which act in the bath only. Of greatest interest are those of the system, since this is what is under study—the bath is merely an environment.

Suppose we are interested in the mean of a system operator, M. Since this is an operator in the system space only, we can write:

$$\langle x'a'|M|xa\rangle = \delta_{aa'}\langle x'|M|x\rangle. \qquad (2.5.2)$$

If $|\psi\rangle$ is any quantum state, the mean of M is

$$\langle \psi|M|\psi\rangle \qquad = \sum_{xx'}\sum_{aa'}\langle \psi|xa\rangle\langle xa|M|x'a'\rangle\langle x'a'|\psi\rangle \qquad (2.5.3)$$

and using (2.5.2)

$$= \sum_{xx'}\sum_{a}\langle \psi|xa\rangle\langle x'a|\psi\rangle\langle x|M|x'\rangle \qquad (2.5.4)$$

$$= \text{Tr}_{\text{sys}}\left\{\rho_{\text{sys}}M\right\} \qquad (2.5.5)$$

where

$$\rho_{\text{sys}} = \sum_{a}\sum_{xx'}|x'\rangle\langle x'a|\psi\rangle\langle \psi|xa\rangle\langle x| \qquad (2.5.6)$$

is called the *reduced density operator* for the system. This expression makes it clear that

$$\rho_{\text{sys}} = \text{Tr}_{\text{B}}\left\{|\psi\rangle\langle \psi|\right\}, \qquad (2.5.7)$$

that is, it is the partial trace of the density operator for the whole physical system taken over the bath variables only. Alternatively, we can write

$$\rho_{\text{sys}} = \sum_{a}P(a)|\psi_a\rangle\langle \psi_a| \qquad (2.5.8)$$

where

$$P(a) = \sum_{x}|\langle xa|\psi\rangle|^2 \qquad (2.5.9)$$

and $|\psi_a\rangle$ is a *system state*, given by

$$|\psi_a\rangle = \sum_{x}|x\rangle\langle xa|\psi\rangle/\sqrt{P(a)}. \qquad (2.5.10)$$

The form (2.5.8) makes it clear that ρ_{sys} is a density operator such as was conceived in Sect. 2.1, and it is positive definite. It shows that a subsystem of a system described by a pure state density operator $|\psi\rangle\langle\psi|$ must be described by a mixed state density operator ρ_{sys}. Of course if the whole physical system is itself described by a density operator ρ, then the generalisation of (2.5.8) is

$$\rho_{sys} = \text{Tr}_B \{\rho\}. \tag{2.5.11}$$

2.5.2   Mutual Influence of "System" and "Bath"

The separation into "system" and "bath" is usually quite clear where it is appropriate, for example in quantum optics, the system could be the atom, and the heat bath the radiation field. We are often interested in the atoms only, and use the radiation field merely as a probe to investigate the atom, but there are situations in which the precise state of the radiation field which arises can be of interest.

The separation into the "system" and the "bath" then brings to mind the matter of their interaction with each other. The general supposition is that the bath is so large that the system can only have a negligible influence on it, and it is therefore possible to make approximations which depend on this assumption. This will be done in Chap. 5. However, the influence of the system on the bath can be quite significant, for example, in the case of an atom in a radiation field, the atom may fluoresce, and the fluorescent light is itself the effect on the radiation field, without which the atom could not be detected.

The influence of the "bath" on the "system" is rather more obvious. The system and the bath can exchange energy, and this leads to *dissipation*, and to *fluctuations*. An excited atom will lose its energy by radiation into the bath, but it will also feel the random fluctuations of the electromagnetic field which pervades all of space—even at zero temperature there are still vacuum fluctuations. How to write and compute these effects is the subject of this book.

3. Quantum Langevin Equations

The mutual influence of a "system", with few degrees of freedom, and a "heat bath", with many degrees of freedom, on each other is the central concept in the physics of noise, both quantum and classical. The action of the many variables of the bath on the system is to modify the equations of motion of the system by the inclusion of apparently random terms. The archetypal version of this concept dates back to Langevin's introduction of the kind of equation which now bears his name: the equation for a Brownian particle moving in a viscous fluid under the influence of a potential in the form

$$m\ddot{x} = -V'(x) - \gamma\dot{x} + \sqrt{2\gamma kT}\,\xi(t).$$

The aim of this chapter is to derive analogous equations for quantum systems, and to develop some of the basic methods of using them in situations of physical interest.

No derivation can be carried out without some basic assumptions, and so far the only usable basis for a derivation has been the assumption that the heat bath consists of an assembly of harmonic oscillators. This is not a bad assumption, since the most useful form of quantum noise theory is that developed for quantum optics, in which the relevant heat bath is the electromagnetic field, which is exactly equivalent to such an assembly of oscillators. In other systems, arguments can be made that the bath can be so approximated, but there will always remain some doubts, based on the knowledge that the harmonic oscillator is special, so that generalisations to other systems can be misleading.

We will also find that the assembly of harmonic oscillators must have some rather special properties. In particular:

i) There must be a smooth dense spectrum of oscillator frequencies.

ii) The coupling of the system to the bath operators must be *linear* in the bath harmonic oscillator operators.

iii) The coupling constants of the system to the bath operators must be a smooth function of the frequency of the oscillators.

These properties are all provided rather naturally by a *quantum field*, of which the electromagnetic field is the foremost example. The oscillators which arise as the normal modes of a field in a large (or possibly infinite) volume do have a smooth dense frequency spectrum. Non-trivial couplings do arise which are linear in the field variables. Finally, the most natural coupling of a small system to a field is a *local* coupling, in which the coupling depends only on the value of the field at a single point. The fact that the normal mode variables of the field are essentially spatial Fourier transforms of the field variables means that such a local coupling

corresponds to taking a coupling to the normal modes which is in fact independent of mode frequency.

In fact it will turn out that a field interpretation of the heat bath is almost always possible—indeed it may be possible in several ways. The most important conclusion to be drawn from this can be seen by considering the case of a single atom in the electromagnetic field. When viewed from the position of the atom there are two kinds of field modes, incoming and outgoing. The incoming modes influence the motion of the atom, whereas the outgoing modes are produced by the atom—they affect the atom because they carry away energy, and thus give rise to damping. The effect of the incoming modes is to produce random noise effects and to feed energy into the atom, which is carried away by the outgoing modes. Thus the kind of damping produced is *radiation* damping. However this view will always be true whenever a field interpretation of the bath is possible, and in my experience this is always possible.

We are then led to the concepts of noise *inputs* and *outputs*. Every system which can be modelled as being coupled to a bath of harmonic oscillators can be considered as being driven by a *noise input* and as radiating a *noise output*. In optical and transmission line systems the noise outputs may be directed into other systems for which they become the inputs. However, the inputs may consist of both noise and possibly a well defined coherent part which is perhaps best called a *signal*. Reverting to the case of an atom in an electromagnetic field, we can identify three parts in the input:

i)   A *signal* which could arise from shining light from a light source such as a laser on the atom.

ii) *Noise* which arises from the fluctuations in the laser light, and is produced by random effects in the laser. Such noise could also arise from other sources of light, such as sunlight, which impinge on the atom.

iii) *Vacuum noise*, which arises even when there is no light source. The vacuum state of a quantum field is a state in which the mean square field is non-zero. Thus, these vacuum fluctuations will always be present, and must satisfy the equations of motion. Even in the vacuum—that is, in a completely dark room at absolute zero—the atom experiences a vacuum noise input, and radiates a vacuum noise output.

Mathematically, the effect of the vacuum noise is to ensure that the Langevin equations *always* have a driving term as well as a damping term. As explained in Chap. 1, without some such term the canonical commutation relations would decay to zero, violating quantum mechanics. Physically the effect is slightly alarming, since the effect of this vacuum noise can be viewed as rather small, or infinite, depending on one's point of view.

In one sense the noise is infinite, because, as mentioned in Chap. 1, the spectrum of quantum noise rises in proportion to the mode frequency, so that the total power is infinite. However, the net effect is to *renormalise* the constants involved. As a result the vacuum noise shifts the energy levels of an atom to give the Lamb shift, which is dealt with by the renormalisation techniques of quantum electrodynamics. The net effect is very small. In dealing with lumped systems, such as the Josephson junction,

the finite size translates into a high frequency cutoff, which makes the effect of the rising noise spectrum finite, and able to be calculated. However, this does not mean that it is easy to calculate in all cases, and considerable care must always be taken. For example, it is natural to assume that we can use an initial condition for the system-bath density operator which is the direct product of an operator for the system and one for the bath. In general this is true, but it may be a very exceptional kind of state, since the effect of the high frequency modes is very strong, and thus such a state will experience a very large transient as the bath and system come to equilibrium with each other. For example, a harmonic oscillator coupled to such a bath experiences a very large input of energy, which effectively raises the energies of all the energy levels. Nevertheless, measurable energy differences between the levels are only slightly affected.

All of this is rather daunting, but fortunately, we can get away with a drastic approximation which eliminates all these effects, known as the *rotating wave approximation*. This eliminates all the terms in the Hamiltonian which cause the undesired effects, while leaving the measurable effects much the same. It is only valid if the damping is very weak, but that is very often true. However, this does not eliminate vacuum noise—it merely eliminates the infinite effects, and is equivalent to a kind of renormalisation. The remaining part of the vacuum noise is still there, and serves to preserve the commutation relations in this approximation.

3.1   The Harmonic Oscillator Heat Bath

The model of a heat bath as an assembly of harmonic oscillators has a Hamiltonian which can be written in the form

$$H_B = \sum_n \left\{ \frac{p_n^2}{2m_n} + \frac{k_n q_n^2}{2} \right\}. \tag{3.1.1}$$

The system Hamiltonian can be left arbitrary. There must, however, be a number of system variables which we can write as a vector $\boldsymbol{Z}$, with a finite number of elements Z_i. Thus the system Hamiltonian is called

$$H_{\text{sys}}(\boldsymbol{Z}). \tag{3.1.2}$$

This kind of heat bath could be a model of an elastic solid, or indeed of the electromagnetic field (since both of these can be viewed as assemblies of harmonic oscillators, one for each normal mode). The *system* could be an atom, as in quantum optics, or something like a macroscopic LC circuit. Examples will be treated later.

a) Coupling Between System and Bath: We can get a coupling between bath and system which yields an exact quantum Langevin equation, by writing

$$H = H_{\text{sys}}(\boldsymbol{Z}) + \sum_n \left\{ \frac{1}{2m_n}p_n^2 + \frac{k_n}{2}(q_n - X)^2 \right\}, \tag{3.1.3}$$

where X is a particular one of the system operators $\boldsymbol{Z}$. The coupling is physically very simple—it makes the potential energy depend on the deviation of X from all the q_n; in other words, it is as if each co-ordinate q_n is harmonically bound to a "position" X. Conventionally, the interaction term is often written as $-\sum_n k_n q_n X$ omitting the term $\frac{1}{2}\sum_n k_n X^2$. This second term is a function only of the system operators and therefore can be absorbed into the system Hamiltonian, so the two formulations are physically equivalent. However the formulation in terms of (3.1.3) turns out to be simpler, since the observed systematic motion is in fact given by H_{sys}, without the need for any correction. The Hamiltonian (3.1.3) can be simplified considerably by the canonical transformation

$$
\begin{aligned}
q_n &\rightarrow p_n / \sqrt{k_n} \\
p_n &\rightarrow -q_n \sqrt{k_n} \\
k_n / m_n &\rightarrow \omega_n^2 \\
\sqrt{k_n} &\rightarrow \kappa_n
\end{aligned}
\tag{3.1.4}
$$

which gives the Hamiltonian

$$
H = H_{\text{sys}}(\boldsymbol{Z}) + \tfrac{1}{2}\sum_n \left\{ (p_n - \kappa_n X)^2 + \omega_n^2 q_n^2 \right\}.
\tag{3.1.5}
$$

For future reference, the equal times commutation relations implicit in this Hamiltonian are

$$
\begin{aligned}
[\boldsymbol{Z}, p_n] &= [\boldsymbol{Z}, q_n] = 0 \\
[p_n, p_m] &= [q_n, q_m] = 0 \\
[q_n, p_m] &= i\hbar\,\delta_{nm},
\end{aligned}
\tag{3.1.6}
$$

but no particular commutation relations are specified between the different components of $\boldsymbol{Z}$—these will depend on the system under consideration.

b) Non-Linear Couplings: This kind of coupling to the bath is linear in the bath variables, and corresponds to energy being transferred to and from the bath by the absorption or emission of bath quanta. This is certainly what happens in the case of electromagnetic radiation, but in something like the theory of electrical resistance in a metal, a different formulation is appropriate, since it is known that electrical resistance arises from scattering of electrons by phonons and impurities. The scattering by phonons gives a temperature dependent resistance which arises because scattering depends on the average number of bath phonons, which of course depends on temperature. This cannot be reproduced by the linear coupling assumed. Non-linear couplings are not easy to create by the quantum Langevin equation, but can be dealt with by master equation methods, as will be shown in Chap.5. The residual scattering by impurities does not have this temperature dependence, but it is not really an incoherent effect, as was pointed out many years ago by *Landauer* [3.1]. The scattering randomly changes the phase of the electron wavefunction, but if the scatterer is fixed, this does not change the coherence, and interference between electron waves can be observed. However, this phenomenon is outside the scope of this book.

3.1.1   Derivation of the Langevin Equation

The Langevin equation can be derived by solving the equations of motion for the oscillators in terms of the variables Z, and substituting these back into the equations of motion for the Z. The result is an equation involving only the system variables and an operator function of time, $\xi(t)$, which is determined by the initial values of the bath variables.

We therefore first write the Heisenberg equations of motion for the oscillator variables,

$$\dot{q}_n = \frac{i}{\hbar}[H, q_n] = p_n - \kappa_n X$$
$$\dot{p}_n = \frac{i}{\hbar}[H, p_n] = -\omega_n^2 q_n. \tag{3.1.7}$$

The solution of these is most easily accomplished in terms of the creation and destruction operators, $a_n^\dagger$ and a_n, defined for this system by

$$a_n = \frac{\omega_n q_n + i p_n}{\sqrt{2\hbar\omega_n}} \qquad a_n^\dagger = \frac{\omega_n q_n - i p_n}{\sqrt{2\hbar\omega_n}} \tag{3.1.8}$$

whose equations of motion are

$$\dot{a}_n = -i\omega_n a_n - \kappa_n \sqrt{\frac{\omega_n}{2\hbar}} X \tag{3.1.9}$$

with the solution

$$a_n(t) = e^{-i\omega_n(t-t_0)} a_n(t_0) - \kappa_n \sqrt{\frac{\omega_n}{2\hbar}} \int_{t_0}^{t} e^{-i\omega_n(t-t')} X(t') dt' \tag{3.1.10}$$

and the corresponding Hermitian conjugate equations for $a_n^\dagger$. The equation of motion for an arbitrary system operator, Y, is obtained by commuting with H, (and using the commutation relations (3.1.6)) in the two alternative forms

$$\dot{Y} = \frac{i}{\hbar}[H, Y] = \frac{i}{\hbar}[H_{\text{sys}}, Y] + \frac{i}{2\hbar}\sum_n \left[[Y, p_n - \kappa_n X]_+, \kappa_n X\right]$$
$$= \frac{i}{\hbar}[H_{\text{sys}}, Y] + \frac{i}{2\hbar}\sum_n \left[[Y, \kappa_n X], p_n - \kappa_n X\right]_+. \tag{3.1.11}$$

We now substitute for p_n using the definition (3.1.8) and the solution (3.1.10). After a partial integration with respect to t', we can collect everything into one equation, which can be correspondingly written in two forms :

$$\dot{Y} = \frac{i}{\hbar}[H_{sys}, Y]$$

$$-\frac{i}{2\hbar}\left[X, [Y, \xi(t) - \int_{t_0}^{t} f(t-t')\dot{X}(t')dt' - f(t-t_0)X(t_0)]_+\right] \tag{3.1.12}$$

$$= \frac{i}{\hbar}[H_{sys}, Y]$$

$$-\frac{i}{2\hbar}\left[[X,Y], \xi(t) - \int_{t_0}^{t} f(t-t')\dot{X}(t')dt' - f(t-t_0)X(t_0)\right]_+ \tag{3.1.13}$$

in which

$$\xi(t) = i\sum_n \kappa_n \sqrt{\frac{\hbar\,\omega_n}{2}}\left[-a_n(t_0)e^{-i\omega_n(t-t_0)} + a_n^+(t_0)e^{i\omega_n(t-t_0)}\right] \tag{3.1.14}$$

and

$$f(t) = \sum_n \kappa_n^2 \cos(\omega_n t). \tag{3.1.15}$$

The equations (3.1.12,13) will be called the *quantum Langevin equations*, provided the function $f(t)$ satisfies appropriate conditions which make the terminology reasonable.

a) A Particle Moving in a Potential: In order to see what these conditions are, consider the quantum version of Langevin's original equation for the motion of a particle in a potential under the influence of viscous drag and a fluctuating force. The system then has canonical co-ordinate q and canonical momentum p, and the operator X which couples to the heat bath will be chosen to be q—this is exactly the model of *Ford* et al. [3.2]. Thus,

$$H_{SYS} = \frac{p^2}{2m} + V(q). \tag{3.1.16}$$

There are then two quantum Langevin equations obtained by explicit substitution in (3.1.13):

$$\dot{q}(t) = p(t)/m \tag{3.1.17}$$

$$\dot{p}(t) = -V'(q(t)) - \int_{t_0}^{t} f(t-t')\dot{q}(t')dt' - f(t-t_0)q(t_0) + \xi(t). \tag{3.1.18}$$

The function $f(t)$ can be seen to have the form of a *memory function*, since it makes the equations of motion at time t depend on the values of $\dot{q}(t)$ for previous times. If $f(t)$ is a rapidly decaying function, the equations can be said to have a short memory. More precisely, if $f(t)$ goes to zero in a time scale which is much less that the time over which $\dot{q}(t)$ changes, then we can replace $\dot{q}(t')$ by $\dot{q}(t)$, and for t not close to t_0, the initial time, we can drop $f(t - t_0)q(t_0)$.

These equations then take on exactly the form of Langevin's original equation. Formally, we can assume the function $f(t)$ has the delta function form

$$f(t) = 2\gamma\delta(t). \tag{3.1.19}$$

This will be achieved if the coupling constants have the form described in the following.

b) The Nature of the Coupling Constants: In principle we know nothing about the coupling of the system to the heat bath. However, to ensure that the energy is dissipated among the many modes, there must be some significant coupling to all of the modes. Further, the only distinguishing feature between the different modes is their frequency. There may, however, be a number of modes which all have the same frequency. We can take account of this by introducing a parameter g_n, which is the number of modes of frequency ω_n, and introducing another index m which distinguishes the different modes of frequency ω_n. In this case, we deduce that instead of (3.1.14,15), we can write

$$\xi(t) = i\sum_n \sqrt{\frac{\hbar\omega_n}{2}} \sum_{m=0}^{g_n} \kappa_{n,m}\left[-a_{n,m}(t_0)e^{-i\omega_n(t-t_0)} + a_{n,m}^{\dagger}(t_0)e^{i\omega_n(t-t_0)}\right]$$

$$\tag{3.1.20}$$

and

$$f(t) = \sum_n \cos(\omega_n t)\left\{\sum_{m=0}^{g_n}\kappa_{n,m}^2\right\}. \tag{3.1.21}$$

Finally, we have to consider what frequencies ω_n might be available. If these are phonon modes, or photon modes, we would expect the spectrum to be given by the normal modes of a large cavity, and if the modes are very close together in frequency, i.e., the cavity is very large, then we might approximate the spectrum by a continuum. Writing

$$\sum_{m=0}^{g_n}\kappa_{n,m}^2 = G(\omega_n) \tag{3.1.22}$$

we arrive at

$$f(t) = \sum_n \cos(\omega_n t)G(\omega_n) \rightarrow \int_0^{\infty}\cos(\omega t)G(\omega)\frac{dn(\omega)}{d\omega}d\omega \tag{3.1.23}$$

in a continuum limit where only $G(\omega)dn(\omega)/d\omega$ is of real significance, and is the factor which governs all the main damping features.

c) The First Markov Approximation: The simplest situation, and one which is considered the ideal situation, occurs when $G(\omega)dn(\omega)/d\omega$ is *constant*, i.e.,

$$G(\omega)\frac{dn}{d\omega} = \frac{2\gamma}{\pi} \tag{3.1.24}$$

so that

$$f(t) = \frac{2\gamma}{\pi} \int\limits_0^\infty \cos(\omega t)d\omega = 2\gamma\delta(t) \tag{3.1.25}$$

which is the form required to give the Langevin equation in the form

$$\dot{q}(t) = p(t)/m \tag{3.1.26}$$
$$\dot{p}(t) = -V'\big(q(t)\big) - \gamma\dot{q}(t) + \xi(t) \tag{3.1.27}$$

which is the same as Langevin's original formulation, but is an operator equation, and is thus fully quantum mechanical.

d) General Quantum Langevin Equation in the First Markov Approximation: The forms (3.1.20,25) for $f(t)$ and $\xi(t)$ can be assumed for the more general form of the quantum Langevin equation (3.1.13), and this yields the results

$$\dot{Y} = \frac{i}{\hbar}[H_{\text{sys}}, Y] - \frac{i}{2\hbar}[[X, Y], \xi(t) - \gamma\dot{X}]_+ \tag{3.1.28}$$

(where only half the weight of the $\delta(t - t')$ has been counted since it occurs at the end of the interval of integration.)

The most important result of the assumption (3.1.24) is the fact that the equation is now a simple first order differential equation. This means that the future time development of any operator is determined simply by the knowledge of all the system operators in the present. It is for this reason that it is called the first Markov approximation. However this differential equation does not represent a Markovian *physics*, since the physics also depends on the choice of physical state vector or density operator, and we will see later that this usually requires the operator $\xi(t)$ to have a stochastic nature with a non-zero correlation time. This means that the equations of motion for any averages are not simple first order differential equations, and the future time development of them depends on both their present values and their values in the past.

3.1.2   Commutation Relations for Noise Sources

The quantity $\xi(t)$ takes the part of noise in (3.1.12,13), which are equations for system operators driven by this operator. However, $\xi(t)$ is itself not a system operator, but rather an externally specified operator—it can be seen from (3.1.14) that

$\xi(t)$ in fact depends only on the *initial* bath operators—its time development is then implicit in its definition, and is *not* given by a Heisenberg equation of motion.

The most important quantum-mechanical aspect to be borne in mind is the matter of commutation relations. Thus, we can compute that

$$[\xi(t), \xi(t')] = i\hbar \frac{d}{dt} f(t - t') \tag{3.1.29}$$

as a straightforward consequence of the definitions (3.1.14,15) and the creation and destruction operator commutation relations.

A more interesting commutation relation is that between the noise operator and the system variables. From (3.1.10) we can note that $a_n(t)$ is related to the canonical operators $p_n(t)$ and $q_n(t)$ at the time t, and therefore commutes with all the system operators at the same time. Hence we deduce the general result, for any system operator $Y(t)$, that

$$[Y(t), a_n(t_0)] = \kappa_n \sqrt{\frac{\omega_n}{2\hbar}} \int_{t_0}^{t} e^{i\omega_n(t' - t_0)} [Y(t), X(t')] \, dt' \tag{3.1.30}$$

and by using the definitions (3.1.14,15) of $\xi(t)$ and $f(t)$, we find

$$[Y(t), \xi(s)] = \int_{t_0}^{t} [Y(t), X(t')] \frac{d}{ds} f(s - t') dt'. \tag{3.1.31}$$

This relation will be of great utility in formulating the theory of *noise inputs* and *outputs* in Sect. 3.2. From its derivation it is clear that (3.1.31) must be consistent with the equation of motion in either of the forms (3.1.12,13), though this is not at all obvious at first glance.

a) Commutation Relations in the First Markov Approximation: Corresponding to the first Markov Approximation, the commutation relations for the noise operators, (3.1.29,31) become (assuming $s, t > t_0$),

$$[\xi(t), \xi(t')] = 2i\hbar\gamma\frac{d}{dt}\delta(t - t') \tag{3.1.32}$$

$$[Y(t), \xi(s)] = 2\gamma\frac{d}{ds}\{u(t - s)[Y(t), X(s)]\} \tag{3.1.33}$$

where

$$u(x) = \begin{cases} 1, & x > 0 \\ \frac{1}{2}, & x = 0 \\ 0, & x < 0. \end{cases} \tag{3.1.34}$$

Apart from the rather singular nature of the commutator at $t = s$, (3.1.33) has a very straightforward interpretation. Namely, for s in the future of t, this commutator vanishes, which means that one can specify the system variables independent of the *future* behaviour of the driving force. But the past behaviour of $\xi(t)$ does affect $Y(t)$, so this commutator does not vanish.

3.2   The Field Interpretation—Noise Inputs and Outputs

In this section we will consider a particular implementation of the harmonic oscillator heat bath, in the form of a semi-infinite transmission line. Let us consider the system to be described by a Lagrangian $L_{sys}(Z)$ interacting with a one dimensional electromagnetic field, $A(t, x)$. The full Lagrangian is then

$$L = L_{sys}(Z) + \tfrac{1}{2} \int_0^\infty dx \left\{ \dot{A}(t, x)^2 - c^2 [\partial_x A(t, x)]^2 \right\} + X \int_0^\infty dx\, \kappa(x) \dot{A}(t, x).$$

(3.2.1)
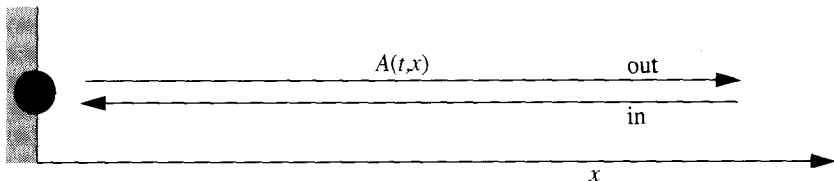
Fig. 3.1   The field interpretation of a system interacting with a heat bath. The system is a rather small localised object which interacts with a one sided field which streams in from the right, interacts with the system, and streams out to the right. Damping appears as *radiation damping*

Here the notation is as in Sect. 3.1; Z are the system operators, and X is the particular operator which interacts with the heat bath. The function $\kappa(x)$ determines the range of the interaction, and will be considered to be zero except for a small region in the vicinity of $x = 0$. Thus, the system can be viewed as being localised near $x = 0$, where it interacts only with the field $A(t, x)$ in that vicinity. The Hamiltonian corresponding to (3.2.1) is

$$H = H_{sys}(Z) + \tfrac{1}{2} \int_0^\infty dx \left\{ [\pi(t, x) - X\kappa(x)]^2 + c^2 [\partial_x A(t, x)]^2 \right\}$$

(3.2.2)

where the canonical momentum $\pi(t, x)$ is given by

$$\pi(t, x) = \dot{A}(t, x) + X\kappa(x),$$

(3.2.3)

and $H_{sys}(Z)$ is the Hamiltonian corresponding to the Lagrangian $L_{sys}(Z)$, since the Lagrangian L contains no terms which depend on derivatives of Z other than those which already occur in $L_{sys}(Z)$. We can define some Fourier transform variables by

$$A(t, x) = \sqrt{\frac{2}{\pi c}} \int_0^\infty q(\omega, t) \cos(\omega x / c)\, d\omega$$

(3.2.4)

$$\pi(t, x) = \sqrt{\frac{2}{\pi c}} \int_0^\infty p(\omega, t) \cos(\omega x / c)\, d\omega$$

(3.2.5)

and

$$\kappa(x) = \sqrt{\frac{2}{\pi c}} \int\limits_0^\infty \tilde{\kappa}(\omega)\, \cos(\omega x/c)\, d\omega, \tag{3.2.6}$$

in terms of which the Hamiltonian takes the form

$$H = H_{\text{sys}}(\boldsymbol{Z}) + \tfrac{1}{2} \int\limits_0^\infty d\omega \left\{ [p(\omega,t) - \tilde{\kappa}(\omega)X]^2 + \omega^2 q(\omega)^2 \right\} \tag{3.2.7}$$

which is a continuum version of the Hamiltonian chosen in Sect. 3.1, and given in (3.1.5) in which $dn(\omega)/d\omega = 1$. We make the following comments.

a) Damping and Locality: The assumption made in Sect. 3.1.1c that gives rise to the first Markov approximation amounts to assuming that

$$\tilde{\kappa}(\omega) = \sqrt{2\gamma/\pi} \tag{3.2.8}$$

since $dn/d\omega = 1$ has been assumed. From the definition of $\tilde{\kappa}(\omega)$ in terms of $\kappa(x)$, (given in (3.2.6)), this corresponds to

$$\kappa(x) = 2\sqrt{\gamma c}\,\delta(x), \tag{3.2.9}$$

so that the assumption of constancy of $\tilde{\kappa}(\omega)$ is equivalent to the locality of the interaction in the x space. Although the three dimensional aspect of an atom is important, it is easy to see that this is the kind of interaction that an atom has with the radiation field, since the size of an atom is very small compared to the wavelength of the light with which it interacts. This is known in atomic physics as the electric dipole approximation.

b) The Process of Damping: Viewed from the perspective of a localised system interacting with a semi-infinite field, the process of damping is essentially that of *radiation*. The system may start with some energy, but this will induce outgoing waves which take the energy away. The ensuing wavetrain proceeds out to infinity, and as far as the system is concerned, the process is irreversible. However, at any finite time, it is possible to reflect the outgoing wave back and reverse the motion of the system.

c) Renormalisation: The term $\tfrac{1}{2} \int\limits_0^\infty d\omega \kappa(\omega)^2 X^2$ appears quite naturally as a result of the assumption of the straightforward Lagrangian (3.2.1), whereas in Sect. 3.1 it appeared almost arbitrary. The Lagrangian assumed here is a simplification of a full three dimensional electromagnetic Lagrangian, so that this "renormalisation" of the system Hamiltonian can be considered quite natural.

3.2.1   Input and Output Fields

Using Lagrangian methods, the equation of motion for the field $A(t, x)$ is the wave equation in the presence of a source; namely

$$\ddot{A}(t, x) - c^2 \partial_x^2 A(t, x) = -\kappa(x)\dot{X}. \tag{3.2.10}$$

This equation can be solved by using the kind of solutions we have already derived in Sect. 3.1.1 (3.1.10), which in this case take the form

$$a(\omega, t) = \mathrm{e}^{-\mathrm{i}\omega(t - t_0)} a(\omega, t_0) - \kappa(\omega) \sqrt{\frac{\omega}{2\hbar}} \int_{t_0}^{t} dt' \mathrm{e}^{-\mathrm{i}\omega(t - t')} X(t') \tag{3.2.11}$$

in which $t_0 < t$ is an initial time, usually assumed to be in the remote past, and of course

$$a(\omega, t) = \frac{\omega q(\omega, t) + \mathrm{i} p(\omega, t)}{\sqrt{2\hbar\omega}}. \tag{3.2.12}$$

We can also consider the possibility of specifying a solution in terms of a *final condition* $t_1 > t$, which can usually be assumed to be in the remote future. Thus

$$a(\omega, t) = \mathrm{e}^{-\mathrm{i}\omega(t - t_1)} a(\omega, t_1) + \kappa(\omega) \sqrt{\frac{\omega}{2\hbar}} \int_{t}^{t_1} dt' \mathrm{e}^{-\mathrm{i}\omega(t - t')} X(t'). \tag{3.2.13}$$

We will now define *input* and *output* field operators in terms of these initial and final conditions, thus

$$A_{\mathrm{in}}(t) = \frac{1}{2} \int_0^\infty d\omega \sqrt{\frac{\hbar}{\pi\omega c}} \left\{ a(\omega, t_0) \mathrm{e}^{-\mathrm{i}\omega(t - t_0)} + a^\dagger(\omega, t_0) \mathrm{e}^{\mathrm{i}\omega(t - t_0)} \right\} \tag{3.2.14}$$

$$A_{\mathrm{out}}(t) = \frac{1}{2} \int_0^\infty d\omega \sqrt{\frac{\hbar}{\pi\omega c}} \left\{ a(\omega, t_1) \mathrm{e}^{-\mathrm{i}\omega(t - t_1)} + a^\dagger(\omega, t_1) \mathrm{e}^{\mathrm{i}\omega(t - t_1)} \right\}. \tag{3.2.15}$$

By substituting these solutions for $q(\omega, t)$ implicit in (3.2.13), into the definition of $A(t, x)$ in terms of $q(\omega, t)$, (3.2.4), we can obtain, after some careful manipulation

$$A(t, x) = A_{\mathrm{in}}\left(t + \frac{x}{c}\right) + A_{\mathrm{in}}\left(t - \frac{x}{c}\right) - \frac{1}{2} \int_{\frac{x}{c} - (t - t_0)}^{\frac{x}{c} + (t - t_0)} d\tau\, \kappa(c\tau) X\left(t - \left|\tau - \frac{x}{c}\right|\right) \tag{3.2.16}$$

$$= A_{\mathrm{out}}\left(t + \frac{x}{c}\right) + A_{\mathrm{out}}\left(t - \frac{x}{c}\right) + \frac{1}{2} \int_{\frac{x}{c} + (t - t_1)}^{\frac{x}{c} - (t - t_1)} d\tau\, \kappa(c\tau) X\left(t + \left|\tau - \frac{x}{c}\right|\right). \tag{3.2.17}$$

These two solutions are general solutions for arbitrary $\kappa(x)$. However, in all cases of interest $\kappa(x)$ is zero for x outside some finite range, even if it does not take the idealised delta function form (3.2.9). If we take x to be outside this finite range, then in the integrals in (3.2.16,17) τ is always less than x/c for nonzero $\kappa(c\tau)$, and we can write

$$|\tau - x/c| = x/c - \tau \tag{3.2.18}$$

in both cases. If t_0 is in the remote past, and t_1 in the remote future, we can also set the limits on the integrals to $\pm\infty$. It is then apparent that in both of these equations, the right hand sides can be expressed as a sum of a function of $t - x/c$ and a function of $t + x/c$. We can equate these functions in (3.2.16) with the corresponding functions in (3.2.17), and we find

$$A_{\text{in}}(t + x/c) = A_{\text{out}}(t + x/c) + \tfrac{1}{2} \int\limits_{-\infty}^{\infty} d\tau\, \kappa(c\tau) X(t + x/c - \tau). \tag{3.2.19}$$

Since this is true for *any* t, even though x must be outside the range of $\kappa(x)$, we can see that

$$A_{\text{out}}(t) = A_{\text{in}}(t) - \tfrac{1}{2} \int\limits_{-\infty}^{\infty} d\tau\, \kappa(c\tau) X(t - \tau) \tag{3.2.20}$$

is generally true for all finite t. (Equating the functions of $t - x/c$ gives the same result). Finally, using this result, we see that (3.2.16) gives, for x outside the range of $\kappa(x)$

$$A(t,x) = A_{\text{in}}(t + x/c) + A_{\text{out}}(t - x/c). \tag{3.2.21}$$

This means that outside the range of interaction, the total field is given by the sum of an incoming "in" field, and an outgoing "out" field. The expression of the "out" field by (3.2.20) shows that the "out" field is itself given by the sum of a *reflection* of the "in" field and a *radiated field*, $\tfrac{1}{2} \int d\tau\, \kappa(c\tau) X(t - \tau)$, whose behaviour is determined by the equation of motion for the system operator X.

   This point of view makes very obvious sense in the field of quantum optics, where the input and output fields are very apparent as light beams. However, it can be seen here that even the vacuum is an input—one can view the zero temperature quantum noise as arising from the inward propagation of vacuum fluctuations, and it is these vacuum fluctuations which preserve the commutation relations from decay. Thus, there is *always* an input, even when the light beams are turned off.

3.2.2   Equations of Motion for System Operators

Although these have already been derived for the general case, it is interesting to see how they arise in the field context. Using the Hamiltonian (3.2.2) and the canonical commutation relations

$$[A(t,x), \pi(t,x')] = i\hbar\, \delta(x - x') \tag{3.2.22}$$

the equation of motion for any system operator is given by

$$\dot{Y} = \frac{i}{\hbar}[H_{\mathrm{sys}}, Y] - \frac{i}{2\hbar}\int_0^\infty \kappa(x)\,dx\,[\dot{A}(t, x), [X, Y]]_+. \tag{3.2.23}$$

For simplicity, let us assume that $\kappa(x)$ has the delta function form (3.2.9). In this case it is possible to substitute for $A(t, x)$ using (3.2.16), to derive

$$A(t, x) = A_{\mathrm{in}}\left(t + \frac{x}{c}\right) + A_{\mathrm{in}}\left(t - \frac{x}{c}\right) - \sqrt{\frac{\gamma}{c}}\,X\left(t - \frac{x}{c}\right) \tag{3.2.24}$$

and thus

$$\dot{Y} = \frac{i}{\hbar}[H_{\mathrm{sys}}, Y] + \frac{i}{2\hbar}[\gamma\dot{X} - 2\sqrt{\gamma c}\,\dot{A}_{\mathrm{in}}(t), [X, Y]]_+. \tag{3.2.25}$$

This is of course exactly the same as the quantum Langevin equation (3.1.28) in the first Markov approximation, as derived in Sect. 3.1.1, with the substitution

$$\xi(t) \to 2\sqrt{\gamma c}\,\dot{A}_{\mathrm{in}}(t). \tag{3.2.26}$$

a) Noise Inputs and Outputs: Thus the view of the quantum Langevin equation provided by this model is of a system connected to an infinite transmission line. This transmission line can support both incoming and outgoing modes, and there is a boundary condition at the $x = 0$ end of this transmission line which takes the form, obtained by setting $x = 0$ and $\kappa(x) = 2\sqrt{\gamma c}\,\delta(x)$ into (3.2.20),

$$A_{\mathrm{out}}(t) = A_{\mathrm{in}}(t) - \sqrt{\frac{\gamma}{c}}\,X(t). \tag{3.2.27}$$

The "in" field can be considered to be a thermal noise *input* if the statistics of the modes $a(\omega, t_0)$, $a^\dagger(\omega, t_0)$ are thermal. Since this specification is in terms of initial conditions in the remote past, we see that $A_{\mathrm{in}}(t)$ can be specified arbitrarily—it is in no way affected by the system operators. In contrast, $A_{\mathrm{out}}(t)$ is determined from the boundary condition (3.2.27), which requires the knowledge of $X(t)$, a system operator. Such operators are determined as solutions of the quantum Langevin equation (3.2.25). The damping, given by the term proportional to $\dot{X}$, is *radiation damping*—the quantum Langevin equation then displays two major terms, one which is the driving by the "in" field, and the other which represents the effect of loss of energy into the "out" field.

b) Causality: Finally, it is a straightforward matter to adapt the relationship (3.1.31) to yield in this case

$$[Y(t), A_{\mathrm{in}}(s)] = \sqrt{\frac{\gamma}{c}}\,u(t - s)[Y(t), X(s)]. \tag{3.2.28}$$

We can then use the boundary condition (3.2.27) to replace $A_{\mathrm{in}}(t)$ in terms of $A_{\mathrm{out}}(t)$ to obtain

$$[Y(t), A_{\mathrm{out}}(s)] = \sqrt{\frac{\gamma}{c}}\,u(s - t)[Y(t), X(s)]. \tag{3.2.29}$$

The causality interpretation is thus, that the system operators must be able to be specified independently of the *future* behaviour of the noise inputs, and hence $Y(t)$ commutes with $A_{in}(s)$ provided s is in the future of t. Conversely the output, once it leaves the system, is independent of $Y(t)$, i.e., $A_{out}(s)$ must be able to be specified independently of $Y(t)$ provided t is in the future of s. In fact, (3.2.28,29), follow simply from this causality condition and the boundary condition (3.2.27), provided we assume nothing singular happens at $s = t$.

c) Time Reversal: By substituting for $\dot{A}(t,x)$ in terms of $A_{out}(t)$, we can similarly derive a quantum Langevin equation in the form

$$\dot{Y} = \frac{i}{\hbar}[H_{sys}, Y] - \frac{i}{2\hbar}[\gamma \dot{X} + 2\sqrt{2\gamma c}\, \dot{A}_{out}(t), [X, Y]]_+. \tag{3.2.30}$$

Here the system is driven by the *output*, and the damping term has a reversed sign. Thus, we get a kind of "time reversed" quantum Langevin equation. In principle this need not be equivalent to time reversing the original equation, unless the Lagrangian assumed, (3.2.1), is itself time reversal invariant. If Θ is the antilinear time reversal operator, and we assume that

$$\Theta A(t,x)\Theta = A(-t,x) \tag{3.2.31}$$

$$\Theta L_{sys}[Z(t)]\Theta = L_{sys}[Z(-t)] \tag{3.2.32}$$

then time reversal of the Lagrangian (3.2.1) is assured if $\kappa(x)$ is real (which must be the case for a Hermitian Lagrangian) and

$$\Theta X(t)\Theta = -X(-t). \tag{3.2.33}$$

From (3.2.21) and (3.2.31), we note that

$$A(-t,x) = \Theta A_{in}(t+x/c)\Theta + \Theta A_{out}(t-x/c)\Theta$$
$$= A_{in}(-t+x/c) + A_{out}(-t-x/c) \tag{3.2.34}$$

and equating separately the functions of $t + x/c, t - x/c$, we see that

$$\Theta A_{in}(t)\Theta = A_{out}(-t)$$
$$\Theta A_{out}(t)\Theta = A_{in}(-t). \tag{3.2.35}$$

Using (3.2.33,34) it is then quite straightforward to show that the time reversed version of the quantum Langevin equation is indeed the "out" form (3.2.30).

d) Some Historical Comments: The model of a system in interaction with a bath of harmonic oscillators was first thoroughly investigated by *van Kampen* [3.3] in his doctoral thesis. His aim was to investigate the nature of quantum mechanical renormalisation in a simplified model of quantum electrodynamics. The use of the model to generate quantum mechanical Langevin equations is due to *Ford* et al. [3.2], who showed that the same procedure could be used for the derivation of both the classical and the quantum mechanical Langevin equations.

The introduction of "inputs" and "outputs" into quantum mechanical systems has its origins in the LSZ formulation of quantum field theory [3.4], in which scattering theory is formulated on the basis of the unitary transformation between asymptotic non-interacting "in" and "out" fields. However in quantum field theory there is no "system" with a few degrees of freedom—only *fields* occur in the formalism, which is an exact formulation of the underlying theory, unlike the model treated here, which is a simplified representation of reality.

Caves [3.5] was the first to introduce the concept of "inputs" and "outputs" into the theory of quantum amplifiers, and from this formulation was able to derive general relations for noise performance of linear quantum amplifiers by treating them as unitary scattering operators which transform "in" fields to "out fields". This procedure was generalised by *Yurke* and *Denker* [3.6] to a "Quantum Network Theory", adapted to the treatment of quantum microwave systems. Their treatment was explicitly based on electrodynamics, and did not have the generality of the methods used here. However, all the essential properties of "inputs" and "outputs" were present, including the identification of the noise term in the quantum Langevin equation with the "in" field.

The formulation presented here is due to *Collett* [3.7], who was the first to use the boundary condition (3.2.27) to obtain properties of the outputs in terms of those of the system and the inputs, and hence to use the quantum Langevin equation as a practical computational tool.

3.3   The Noise Interpretation

The quantity $\xi(t)$ is an operator, and the equations so far derived cannot be interpreted straightforwardly as noise equations unless some assumption is made about the statistics of $\xi(t)$. Since $\xi(t)$ is determined directly in terms of heat bath operators at the time t_0, the nature of the state of the heat bath at this initial time is significant.

The physical picture which gives the clearest understanding is obtained as follows. We assume the system and the bath are initially independent and non-interacting, and introduce the interaction between them at the time t_0. In the Schrödinger picture this would mean that the density operator could be written at time t_0 as a direct product

$$\rho(t_0) = \rho_{\text{sys}}(t_0) \otimes \rho_B(t_0). \tag{3.3.1}$$

We are working in the Heisenberg picture, however, in which the density operator is equal to that at time t_0, hence we assume

$$\rho = \rho_{\text{sys}} \otimes \rho_B. \tag{3.3.2}$$

In the Heisenberg picture the density operator is a direct product at all times. This will mean that any quantum mechanical average of the bath variables is simply able to be written as, for example

$$\langle \xi(t)\xi(t') \rangle = \text{Tr} \{\rho_B \xi(t)\xi(t')\}. \tag{3.3.3}$$

The assumption of an initially factorisable density operator is intuitively appealing, since it has a simple classical interpretation. For example, if we place a Brownian particle in a fluid at time t_0, then initially the particle and the fluid are quite independent, and quantum mechanically this would mean a factorised density operator.

In a quantum mechanical situation this is not always so straightforward. If we consider the case of a charged particle in an electromagnetic field, it is clear that there is no way of actually disconnecting the particle from the electromagnetic field, so that there is no natural way of generating a factorised density operator. This is a problem familiar in quantum electrodynamics, and is overcome by the concept of renormalisation. Unfortunately renormalisation often involves infinite quantities—the coupled system is infinitely different from the factorised system. The same problem of infinities occurs in quantum noise theory, and manifests itself in an infinitely rising noise spectrum.

If a strictly local theory is used, we find that the density operator of the damped system is infinitely different from a factorised density operator. In practice the only way of dealing with this problem is to introduce a high frequency cutoff. It is then found that measurable quantities are usually independent of the cutoff.

In some situations the cutoff actually has a physical meaning, for example in a Josephson junction oscillator, the main damping is by electromagnetic radiation. The high frequency cutoff is simply an expression of the finite size of the junction, which no longer can be treated as a lumped circuit for frequencies such that the wavelength is smaller than the diameter of the junction.

3.3.1   Thermal Statistics

We may assume that the bath is initially *thermal*, so that in the Heisenberg picture

$$\rho_B = Z \exp(-H_B/kT) \tag{3.3.4}$$

and hence (using the notation of Sect. 3.1.1b)

$$\langle a_{nm} \rangle \quad = \langle a_{nm}^\dagger \rangle = 0 \tag{3.3.5}$$

$$\langle a_{nm} a_{n'm'} \rangle = 0 \tag{3.3.6}$$

$$\langle a_{nm}^\dagger a_{n'm'} \rangle = \delta_{nn'} \delta_{mm'} \left\{ \exp(-\hbar \omega_n/kT) - 1 \right\}^{-1} \tag{3.3.7}$$

$$\equiv \delta_{nn'} \delta_{mm'} \bar{N}(\omega_n). \tag{3.3.8}$$

(Here, for brevity, we drop the t_0 dependence of $a, a^\dagger$.) We can straightforwardly compute that

$$\langle [\xi(t), \xi(t')]_+ \rangle = 2 \int_0^\infty d\omega \, \hbar \omega G(\omega) \frac{dn}{d\omega} \{ \bar{N}(\omega) + \tfrac{1}{2} \} \cos \omega(t - t') \tag{3.3.9}$$

and in the case that $G(\omega) \, dn/d\omega$ is a constant, as in (3.1.24), we may write

$$\langle [\xi(t), \xi(t')]_+ \rangle = \frac{2\gamma\hbar}{\pi} \int_0^\infty d\omega \, \omega \coth \left(\frac{\hbar \omega}{2 kT} \right) \cos \omega(t - t') \tag{3.3.10}$$

and, of course $\langle \xi(t) \rangle = 0$. The high frequency behaviour of the spectral function $\omega \coth(\hbar\omega/2kT)$ is proportional to $|\omega|$, which is very badly behaved—classical noise theory usually considers only asymptotically constant noise spectra, which corresponds to white noise. Even white noise is not straightforward to handle, since it is merely continuous, and not differentiable. To my knowledge, there is no way of handling noise with a linearly rising noise spectrum other than by introducing a cutoff at high frequencies. The physical justification for this comes from the form (3.3.9), where the factor $G(\omega)dn/d\omega$ comes into play. If this factor drops off sufficiently rapidly at high frequencies, the rising noise spectrum can be made to drop off, and thus give a well behaved correlation function. The exact nature of $G(\omega)dn/d\omega$ is model dependent.

3.3.2   The Classical Limit

If we let $\hbar \to 0$, then $\xi(t)$ commutes with $\xi(t')$, and on using the limit

$$\lim_{\hbar \to 0} \hbar \coth\left(\frac{\hbar\omega}{2kT}\right) = \frac{2kT}{\omega} \tag{3.3.11}$$

(3.3.10) becomes

$$\langle \xi(t)\xi(t') \rangle = 2\gamma kT\delta(t - t'). \tag{3.3.12}$$

This limit is, of course, rather naive, since for any finite $\hbar$ the hyperbolic cotangent becomes infinite as $\omega \to \infty$. In practice the factor $G(\omega)dn/d\omega$ is not constant, but drops off at high frequencies. The limit is then

$$\langle \xi(t)\xi(t') \rangle = kTf(t - t') \tag{3.3.13}$$

and this approaches the result (3.3.12) as $G(\omega)dn/d\omega$ approaches a constant. Thus the wide bandwidth limit of the classical limit is a delta correlated noise, but the limits cannot be taken in the reverse order.

3.3.3   Behaviour of the Langevin Correlation Function as a Function of Time

The anticommutator correlation function (3.3.10) is strictly speaking divergent. This means that it can only be interpreted as a *generalised function* or *distribution*. Nevertheless, it is necessary to evaluate it to understand the kind of noise it implies. We can do this as follows. It is first best to separate out the divergent part explicitly, thus

$$\langle [\xi(t), \xi(t')]_+ \rangle = \frac{2\gamma\hbar}{\pi} \int_0^\infty d\omega \left[\omega \coth\left(\frac{\hbar\omega}{kT}\right) - \omega\right] \cos\omega(t - t')$$

$$+ \frac{\gamma\hbar}{\pi} \int_{-\infty}^\infty d\omega \, |\omega| e^{i\omega(t-t')}. \tag{3.3.14}$$

The first term is convergent for all $t - t'$. It can be evaluated by differentiating with respect to a the result [3.8]

$$\int_0^\infty \sin ax (\coth \beta x - 1) dx = \frac{\pi}{2\beta} \coth \frac{a\pi}{2\beta} - \frac{1}{a} \qquad (3.3.15)$$

to obtain

$$\frac{2\gamma\hbar}{\pi} \left\{ -\left(\frac{\pi kT}{\hbar}\right)^2 \text{cosech}^2 \left(\frac{\pi kT}{\hbar}(t - t')\right) + \frac{1}{(t - t')^2} \right\}. \qquad (3.3.16)$$

The other term is only able to be evaluated as a distribution or by introducing a cutoff. By introducing an exponential cutoff

$$\frac{\gamma\hbar}{\pi} \int_{-\infty}^\infty d\omega \, |\omega| e^{i\omega(t-t')} e^{-\lambda\omega} = \frac{2\gamma\hbar}{\pi} \frac{\lambda^2 - (t - t')^2}{[\lambda^2 + (t - t')^2]^2} \qquad (3.3.17)$$

where λ is some very small time constant. The behaviour of (3.3.17) as $t \to t'$ is quite regular for any non-zero λ but diverges as $\lambda \to 0$. Formally, one can write

$$\langle [\xi(t), \xi(t')]_+ \rangle = -\frac{2\gamma\pi k^2 T^2}{\hbar} \text{cosech}^2 \left(\frac{\pi kT}{\hbar}(t - t')\right) \qquad (3.3.18)$$

when $\lambda \to 0$, on the understanding that the behaviour near $t \to t'$ is given by adding (3.3.16) and (3.3.17).

This correlation displays a *thermal correlation time* which arises from the large $|t - t'|$ behaviour

$$\langle [\xi(t), \xi(t')]_+ \rangle \approx -\frac{8\gamma\pi k^2 T^2}{\hbar} \exp\left(-\frac{2\pi kT}{\hbar}|t - t'|\right) \quad \text{as } |t - t'| \to \infty. \qquad (3.3.19)$$

The thermal correlation time is given by

$$\tau_T = \hbar / 2\pi kT. \qquad (3.3.20)$$

This time becomes infinite when $T \to 0$; in fact the $T \to 0$ limit of the correlation function is no longer exponential, but inverse quadratic; i.e.,

$$\langle [\xi(t), \xi(t')]\rangle_+ \approx -2\gamma\hbar / \pi(t - t')^2 \quad \text{for } t \neq t'. \qquad (3.3.21)$$

The slightly surprising thing about these results is the fact that the long time behaviour of the correlation function is *negative*—at long time differences there is an *anticorrelation*. To visualise what happens, it is helpful to graph the correlation function for some finite, but very small cutoff λ, and this is shown in Fig.3.4.

Notice that, from (3.3.10)

$$\int_{-\infty}^\infty dt \langle [\xi(t), \xi(t')]_+ \rangle = \pi \lim_{\omega \to 0} \left\{ \frac{2\gamma\hbar\omega}{\pi} \coth \frac{\hbar\omega}{2kT} \right\} = 4\gamma kT \qquad (3.3.22)$$
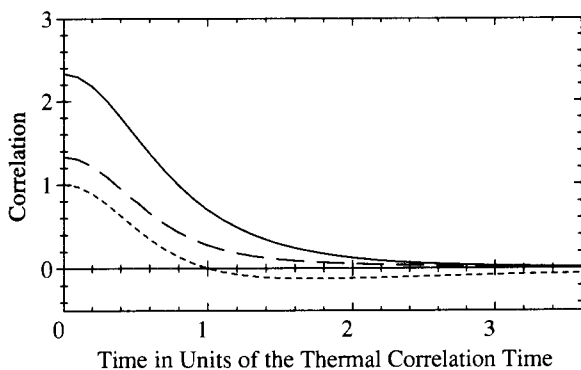
Fig. 3.2   Plot of the correlation function (3.3.14) against time, for the cutoff λ, equal to one thermal correlation time, and for temperatures 4.6pK (*solid line*), 2.3pK (*dashed line*), and 0.23 pK (*dotted line*)

so that the overall effect is for the cutoff dependent positive term to more than cancel the asymptotic negative term, except at $T = 0$, where the two effects cancel. This correlation function is a far more complicated object than the corresponding classical delta correlated result. The existence of the thermal correlation time τ_T does mean however that for any time scale significantly longer than this one can expect a description in terms of a kind of "quantum white noise", or a "quantum Markov process". The thermal correlation time is typically quite small; numerically $\tau_T = 1.27 \times 10^{-12}/T$ sec, so that at a temperature of 1K , τ_T is about a picosecond. There are time scales which are significantly faster than this; for example, the period of the light emitted in the Na_D transition is about 2×10^{-13} sec. However the time scale of an atomic decay is much slower, typically many thousands of periods. This means that we can expect a Markovian description of atomic decays, but not of the more detailed process of evolution over the time scale of the transition frequency.

3.3.4   Other Noise Statistics

The quantum Langevin equations derived in the preceding sections are not themselves specifically stochastic equations; they are valid independent of the statistics assumed for $\xi(t)$, and can be viewed simply as a convenient way of writing the Heisenberg equations of motion. Therefore other statistics of $\xi(t)$ are possible, such as *squeezed* statistics, which will be investigated in Chap.9. But the high frequency problems are not generally avoidable, since they arise from the commutation relations for the bath operators, which are always the same.

3.4   Examples and Applications

In this section a few of the more natural examples will be investigated, and in particular, the only soluble model, that is a free or harmonically bound particle, will be investigated.

3.4.1   A Particle Moving in a Potential

We have already considered this problem in Sect. 3.1.1a, and have found that the quantum Langevin equation corresponding to a particle with canonical co-ordinate and momentum q and p, and mass m, moving in potential $V(q)$ in which the operator X which couples to the heat bath is chosen to be q is, in the first Markov approximation,

$$m\ddot{q}(t) = -V'\big(q(t)\big) - \gamma\dot{q}(t) + \xi(t). \tag{3.4.1}$$

Apart from the cases of quadratic, linear, and constant potentials, this operator equation has not been solved.

Exercise. Suppose we make the replacement $X \to p$. Show the equation of motion is then very different.

The equation (3.4.1) is of exactly the same form as that studied in Sect. 1.4.1, and the commutator and mean anticommutator of $\xi(t)$ are as specified for a quantum Langevin equation. The requirement that the stationary distribution should be the Boltzmann distribution is not straightforward to prove, and will be left for Sect. 3.6.3

3.4.2   The Brownian Particle Langevin Equation

The case that $V(q) = 0$ represents a quantum analogue of the classical problem of Brownian motion, and is soluble by exactly the same method. The equation becomes

$$\dot{q} \;= p/m \tag{3.4.2}$$
$$\dot{p} \;= -\gamma p/m + \xi(t) \tag{3.4.3}$$

with the solution

$$p = e^{-\gamma(t-t_0)/m}p_0 + \int_{t_0}^{t} dt' e^{-\gamma(t-t')/m}\xi(t'). \tag{3.4.4}$$

a) Asymptotic Mean Energy: This is given by

$$\langle E(t)\rangle = \frac{1}{2m}\langle p(t)^2\rangle$$

$$= \frac{1}{2m}e^{-2\gamma t/m}\langle p_0^2\rangle + \frac{1}{2m}\int_0^t\int_0^t dt'dt'' e^{-\gamma(2t-t'-t'')/m}\langle \xi(t')\xi(t'')\rangle \tag{3.4.5}$$

where the cross correlation term has been set to zero because of the assumed initial independence of p and the heat bath, and $t_0 = 0$ is chosen for convenience. Using the spectrum formula (3.3.10), we find, in the limit $t \to \infty$

$$\langle E(t) \rangle \to \frac{\gamma \hbar}{m\pi} \int\limits_0^\infty dw \frac{\omega \bar{N}(\omega)}{(\gamma/m)^2 + \omega^2} + \frac{\gamma \hbar}{2m\pi} \int\limits_0^\infty dw \frac{\omega}{(\gamma/m)^2 + \omega^2}. \tag{3.4.6}$$

In this formula, one sees a characteristic of all quantum noise problems; the infinite influence of the zero point fluctuations of the heat bath. The first term is finite, and can in principle be evaluated. In the limit of very weak coupling, $\gamma \to 0$ we note that

$$\lim_{\gamma \to 0} \frac{1}{\pi} \frac{\gamma/m}{(\gamma/m)^2 + \omega^2} = \delta(\omega) \tag{3.4.7}$$

so that

$$\frac{\gamma \hbar}{m\pi} \int\limits_0^\infty dw \frac{\omega \bar{N}(\omega)}{(\gamma/m)^2 + \omega^2} = \tfrac{1}{2}\hbar \lim_{\omega \to 0} \omega \bar{N}(\omega) = \tfrac{1}{2}kT \tag{3.4.8}$$

which is exactly the thermal energy expected from both classical and quantum equipartition formulae for a free particle. However, the final term is actually infinite, but we can get some idea of its nature by using a frequency cutoff, which is equivalent to setting $\bar{\kappa}(\omega)$ equal to zero above the cutoff, and constant below it. Replacing the limit ∞ by λ, the final term becomes

$$\frac{\gamma \hbar}{4m\pi} \ln(1 + m^2\lambda^2/\gamma^2). \tag{3.4.9}$$

The most important thing about this term is its temperature independence. We cannot observe the actual energy of the particle, only energy changes, so that this added term does not affect the thermodynamics. For any reasonable cutoff, it is also small compared with $\tfrac{1}{2}kT$. For example, if $\gamma/m = 10^6\,\mathrm{Hz}$, $\lambda = 10^{18}\,\mathrm{Hz}$, and $T = 1 \deg K$, then

$$\tfrac{1}{2}kT \quad = 7 \times 10^{-24}\,\mathrm{Joule},$$
$$(3.4.9) = 5 \times 10^{-28}\,\mathrm{Joule}.$$

Exercise. Evaluate $\langle E(t) \rangle$ as a function of time, and show that a cutoff is needed for any $t > 0$.

b) Asymptotic Mean Square Displacement: We can now solve the position equation of motion (3.4.2), to get

$$q(t) - q_0 = \frac{p_0(1 - e^{-\gamma t/m})}{\gamma} + \frac{1}{m} \int\limits_0^t dt' \int\limits_0^{t'} dt'' e^{-\gamma(t'-t'')/m} \xi(t''). \tag{3.4.10}$$

As in classical Brownian motion, let us compute the mean square distance travelled in a time interval starting at the time t_1 and ending at the time t_2—the initial time can be chosen equal to t_0, the time at which the system and bath are assumed uncorrelated, but this is not necessary. Thus, we want to compute $\langle [q(t_2) - q(t_1)]^2 \rangle$, and using (3.4.10) and substituting for the resulting correlation functions as previously, we find that

$$
\langle [q(t_2) - q(t_1)]^2 \rangle = \langle p_0^2 \rangle \frac{(e^{-\gamma t_2/m} - e^{-\gamma t_1/m})^2}{\gamma^2}
$$
$$
+ \frac{\gamma \hbar}{2\pi} \int\limits_{-\infty}^{\infty} d\omega \, \frac{\omega \coth(\hbar \omega/2kT)}{\gamma^2 + m^2 \omega^2}
$$
$$
\times \left[\frac{e^{-i\omega t_2} - e^{-i\omega t_1}}{-i\omega} + \frac{e^{-\gamma t_2/m} - e^{-\gamma t_1/m}}{\gamma/m} \right]
$$
$$
\times \left[\frac{e^{i\omega t_2} - e^{i\omega t_1}}{i\omega} + \frac{e^{-\gamma t_2/m} - e^{-\gamma t_1/m}}{\gamma/m} \right]. \quad (3.4.11)
$$

This integral contains transient terms, which depend on the time t_1 and t_2, and not simply their difference. Further, by noting that the hyperbolic cotangent approaches 1 as $\omega \to \infty$, it is easy to see that the integral arising from the product of the last two terms in the brackets is divergent unless we introduce a cutoff. If we do introduce a cutoff, so the integrals are finite, and then consider that t_1 and t_2 are very large, though their difference is not, then all the transient terms vanish, and

$$
\langle (q(t_2) - q(t_1))^2 \rangle = \frac{\gamma \hbar}{2\pi} \int\limits_{-\infty}^{\infty} d\omega \, \frac{\omega \coth(\hbar \omega/2kT)}{\gamma^2 + m^2 \omega^2} \frac{4 \sin^2 \frac{\omega}{2}(t_2 - t_1)}{\omega^2},
$$
$$
(3.4.12)
$$

and the integral is convergent with or without a cutoff. We can now consider the situation where $|t_2 - t_1|$ is large compared with the values of ω^{-1} of significance in the integral, and use the standard formula used in time dependent perturbation theory

$$
4 \frac{\sin^2 \frac{\omega}{2}(t_2 - t_1)}{\omega^2} \to 2\pi |t_2 - t_1| \delta(\omega) \quad (3.4.13)
$$

to get the asymptotic formula

$$
\langle [q(t_2) - q(t_1)]^2 \rangle \approx \frac{2kT}{\gamma} |t_2 - t_1| \quad (3.4.14)
$$

which is exactly the classical result. The interpretation of this result is thus as follows: The model assumes that at time t_0 (here equal to zero) the bath and the Brownian particle are uncorrelated. Immediately there is a large transient evolution, which appears to allow the particle to absorb a large amount of energy from the bath, and to travel a large mean square distance. The meaning of "large" in this context being

that the quantities are dependent on the cutoff. Both of these effects can be traced to the linearly rising nature of $\omega \coth(\hbar\omega/2kT)$ as ω gets large, i.e., they are a result of the very large zero point energies available in the very high frequency oscillators of the heat bath. However, after this transient, for sufficiently large time differences the diffusion of the particle is given by exactly the same formula as in classical stochastics.

This transient behaviour is thus simply the evolution needed to destroy the rather artificial initial state with a density operator consisting of a direct product of a bath term and a system term.

c) **The Case of Zero Temperature:**   In this case the replacement (3.4.13) is not valid, since at zero temperature

$$\omega \coth(\hbar\omega/2kT) \to |\omega| \tag{3.4.15}$$

and thus the argument of the integral is sharply varying at $\omega = 0$. The formula (3.4.12) yields instead

$$\langle [q(t_2) - q(t_1)]^2 \rangle = \frac{4\gamma\hbar}{\pi} \int_0^\infty \frac{d\omega \, \sin^2 \frac{\omega}{2}(t_2 - t_1)}{\omega(\gamma^2 + m^2\omega^2)} \tag{3.4.16}$$

which is still convergent. An asymptotic expression for large $|t_2 - t_1|$ can be developed.

By a setting $x = m\omega/\gamma$, and $a = \gamma(t_2 - t_1)/m$ we can reduce the evaluation of (3.4.16) to that of the real part of

$$\int_0^\infty dx \, \frac{1 - e^{iax}}{x(x^2 + 1)} = ia \int_0^\infty e^{iax} \ln\left(x/\sqrt{x^2 + 1}\right) dx \tag{3.4.17}$$

after a partial integration. This can then be written as

$$= \frac{d}{d\epsilon} \left\{ ia \int_0^\infty e^{iax} \left(x/\sqrt{x^2 + 1}\right)^\epsilon dx \right\} \bigg|_{\epsilon=0} \tag{3.4.18}$$

and for large a, this can be written asymptotically as

$$\approx \frac{d}{d\epsilon} \left\{ ia\Gamma(1 + \epsilon)(ia)^{-(1+\epsilon)} \right\} \bigg|_{\epsilon=0} \tag{3.4.19}$$

by use of Watson's lemma [3.9]. Carrying out the differentiation, the result is

$$\approx \ln(ia) - \Gamma'(1) \tag{3.4.20}$$

and hence

$$\langle [q(t_2) - q(t_1)]^2 \rangle \approx \frac{2\hbar}{\pi\gamma} \ln\left(\frac{\gamma(t_2 - t_1)}{m}\right) \tag{3.4.21}$$

at $T = 0$. This is truly a *quantum* Brownian motion, since it is induced solely by the fluctuations in the zero point energy of the bath oscillators which is made clear by the factor $\hbar$ in (3.4.21). It is of course, much slower than the motion induced by thermal fluctuations, and does *not* follow the classical diffusion law. It also represents a *very* slow diffusion. If we take a proton, $m \approx 10^{-27}$ kg, with $\gamma/m \approx 10^6$ Hz, we find $\langle [q(t_2) - q(t_1)]^2 \rangle \approx 3 \times 10^{-14} \ln[10^6(t_2 - t_1)]$m^2. If we wait perhaps for a day, $t_2 - t_1 \approx 10^5$s, then the mean square distance travelled is about 10^{-12} m^2. Thermal Brownian motion following the law (3.4.14) would mask this unless $T \ll 4 \times 10^{-16}$ K, which is an almost impossibly low temperature.

3.4.3   The Harmonic Oscillator

For definiteness let us consider an electrical implementation of a damped harmonic oscillator, an LRC circuit. The system variables are the canonical variables P and Q, with a Hamiltonian

$$H_{\text{sys}} = \frac{P^2}{2L} + \frac{Q^2}{2C}. \tag{3.4.22}$$

Using this Hamiltonian, and the choice of interaction $X \rightarrow Q$, we deduce the equations of motion from (3.1.12,13)

$$\dot{Q}(t) = P(t)/L \tag{3.4.23}$$

$$\dot{P}(t) = -Q(t)/C - \int_{t_0}^{t} f(t - t')\dot{Q}(t')\,dt' - f(t - t_0)Q(t_0) + \xi(t). \tag{3.4.24}$$

At this stage we shall not assume that $f(t - t')$ is a delta function, since we have already seen that this gives rise to divergences for the case of the free particle.

a) **Fourier Transform Solution:**   The simplest method of solution is to take the Fourier transform, defined by

$$Q(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} d\omega\, e^{i\omega t}\tilde{Q}(\omega)$$

$$P(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} d\omega\, e^{i\omega t}\tilde{P}(\omega) \tag{3.4.25}$$

$$\xi(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} d\omega\, e^{i\omega t}\tilde{\xi}(\omega).$$

We now must Fourier transform the equations of motion (3.4.23,24), and assume the initial condition is in the remote past, i.e., set $t_0 \rightarrow -\infty$. We use the convolution

theorem for Fourier transforms

$$\frac{1}{\sqrt{2\pi}} \int\limits_{-\infty}^{\infty} dt\, e^{-i\omega t} \left\{ \int\limits_{-\infty}^{\infty} f^*(t-t')X(t')dt' \right\} = \sqrt{2\pi}\, \tilde{f}(\omega)^* \tilde{X}(\omega) \qquad (3.4.26)$$

on the second term on the right hand side of (3.4.24). Since, however, the t' integral goes only up to t, we need the Fourier transform of $f(t)$ multiplied by a step function; this means that the Fourier transform of the equations will be

$$i\omega\tilde{Q}(\omega) = \tilde{P}(\omega)/L \qquad (3.4.27)$$

$$i\omega\tilde{P}(\omega) = -\tilde{Q}(\omega)/C - i\omega R(\omega)\tilde{Q}(\omega) + \tilde{\xi}(\omega) \qquad (3.4.28)$$

in which

$$R(\omega) = \int\limits_{0}^{\infty} dt\, e^{-i\omega t} f(t). \qquad (3.4.29)$$

We use a quantity $r(\omega)$, defined by

$$f(t) = \frac{1}{\pi} \int\limits_{-\infty}^{\infty} d\omega\, r(\omega)e^{i\omega t}. \qquad (3.4.30)$$

Notice that the definition of $f(t)$ in (3.1.23) already specifies $r(\omega)$, so we can immediately see that

$$r(\omega) = r(-\omega) = \frac{\pi}{2}G(\omega)\frac{dn}{d\omega}. \qquad (3.4.31)$$

Using the result

$$\int\limits_{0}^{\infty} dt\, e^{i(\omega'-\omega)t} = \pi\delta(\omega'-\omega) + i\frac{P}{\omega'-\omega}, \qquad (3.4.32)$$

where P means that the principal value is to be taken of any integral of the relevant part the RHS of (3.4.32), we find

$$R(\omega) = r(\omega) + \frac{i}{\pi}P\int\frac{d\omega'r(\omega')}{\omega'-\omega} \qquad (3.4.33)$$

$$\equiv r(\omega) + i\,s(\omega). \qquad (3.4.34)$$

The function $R(\omega)$ is known in non-equilibrium statistical mechanics as a *complex impedance*. [3.10] Notice that, by definition $r(\omega) = r(-\omega)$ and hence $s(\omega) = -s(-\omega)$ so that $R(\omega)^* = R(-\omega)$.

The solutions of (3.4.27,28) are then

$$\tilde{Q}(\omega) = (-\omega^2 L + i\omega R(\omega) + 1/C)^{-1}\tilde{\xi}(\omega) \qquad (3.4.35)$$

$$\tilde{P}(\omega) = i\omega L\tilde{Q}(\omega). \qquad (3.4.36)$$

We note also that the commutation relations and mean anticommutator formula for $\xi(t)$ imply

$$[\tilde{\xi}(\omega), \tilde{\xi}(\omega')] \quad = -2\hbar\omega r(\omega)\delta(\omega + \omega') \tag{3.4.37}$$

$$\langle[\tilde{\xi}(\omega), \tilde{\xi}(\omega')]_+\rangle = 4\hbar\omega r(\omega)(\bar{N}(\omega) + \tfrac{1}{2})\delta(\omega + \omega'). \tag{3.4.38}$$

b) Preservation of Commutation Relations: In the special case that $r(\omega) = r = $ constant, we can show explicitly that the canonical commutation relations are preserved. Writing the canonical commutator in the form

$$[Q(t), P(t)] = \frac{1}{2\pi}\int\int d\omega d\omega' e^{i(\omega t + \omega' t)}[\tilde{Q}(\omega), \tilde{P}(\omega')] \tag{3.4.39}$$

and using (3.4.37) this can be simplified to

$$[Q(t), P(t)] = \frac{i\hbar}{\pi}\int d\omega \frac{Lr\omega^2}{(\omega^2 L - 1/C)^2 + \omega^2 r^2} = i\hbar. \tag{3.4.40}$$

c) Mean Energy: Statistical mechanics tells us that the mean energy of the Harmonic oscillator of a temperature T is given by

$$\langle H \rangle = \frac{\sum_n (n + \tfrac{1}{2})\hbar\Omega e^{-n\hbar\Omega/kT}}{\sum_n e^{-n\hbar\Omega/kT}} = \hbar\Omega(\bar{N}(\Omega) + \tfrac{1}{2}), \tag{3.4.41}$$

but, as noted in Sect. 1.4.1, this is an approximation, valid only in the limit of weak coupling. To verify that our methods agree with statistical mechanics, let us work out the mean energy directly from the solutions for $P(t)$ and $Q(t)$ implicit in (3.4.35,36). We find

$$\langle H \rangle = \left\langle \frac{P(t)^2}{2L} + \frac{Q(t)^2}{2C} \right\rangle \tag{3.4.42}$$

$$= \frac{1}{2\pi}\int d\omega d\omega' \left\{ \frac{\langle\tilde{P}(\omega)\tilde{P}(\omega')\rangle}{2L} + \frac{\langle\tilde{Q}(\omega)\tilde{Q}(\omega')\rangle}{2C} \right\}. \tag{3.4.43}$$

We can work out the averages by using the solutions (3.4.35,36). Because we use Fourier transforms, these will be stationary averages if the statistics of $\tilde{\xi}(\omega)$ are assumed stationary. We can then straightforwardly show from solving the Fourier transformed equations, that

$$\langle H_{\text{sys}} \rangle = \frac{1}{4\pi}\int\limits_{-\infty}^{\infty} d\omega \frac{R(\omega)\hbar\omega[2\bar{N}(\omega) + 1]}{(\omega^2 L - 1/C)^2 + \omega^2 |R(\omega)|^2}\left\{ \frac{1}{C} + \omega^2 L \right\}. \tag{3.4.44}$$

This result does approximate the desired result. This can be seen as follows:

i)   Assume $R(\omega) \rightarrow 0$ sufficiently rapidly as $\omega \rightarrow \pm\infty$ that the integral does not diverge; and that $|R(\omega)|^2$ is sufficiently small that the denominator has a

sharp resonance behaviour at $\omega = 1/\sqrt{LC} \equiv \Omega$. Then we can replace most ω dependent factors by their values at $\omega = \Omega$, where the sharp resonance peak occurs, and use the integral in (3.4.40), to show that

$$\langle H \rangle \approx \hbar \Omega \left(\bar{N}(\Omega) + \tfrac{1}{2} \right) \tag{3.4.45}$$

as happens in statistical mechanics.

ii)   However, as in the case of the Brownian particle (Sect. 3.4.2), if $R(\omega)$ is exactly constant, the mean energy actually diverges, for exactly the same reasons—the rising spectrum of the zero point contributions to the spectrum of quantum noise.

3.4.4 The Two Level Atom

We give a treatment here of the model of Sect. 1.5.1. Thus, in this case we assume that in the quantum Langevin equations (3.1.28), we set

$$H_{\text{sys}} \quad = \tfrac{1}{2} \hbar \Omega \sigma_z \tag{3.4.46}$$

$$X \quad = \sqrt{\hbar/\Omega}\, \sigma_x \tag{3.4.47}$$

and in such atomic systems it is conventional to define γ by $\tilde{\kappa}(\omega) = \sqrt{\gamma/2\pi}$, in contrast to (3.2.8). Using the Pauli matrix commutation relations (these are summarised in Sect. 9.1.2), the quantum Langevin equations for each of the system operators, $\sigma_x, \sigma_y, \sigma_z$, are

$$\dot{\sigma}_x \quad = -\Omega \sigma_y \tag{3.4.48}$$

$$\dot{\sigma}_y \quad = \quad \Omega \sigma_x + \frac{1}{\sqrt{\hbar \Omega}} [\xi(t), \sigma_z]_+ \tag{3.4.49}$$

$$\dot{\sigma}_z \quad = -\gamma \; - \frac{1}{\sqrt{\hbar \Omega}} [\xi(t), \sigma_y]_+. \tag{3.4.50}$$

In deriving the second two equations, $\dot{\sigma}_x$ has been replaced using the first equation. Unlike the case of the Brownian particle, or that of the harmonic oscillator, these equations are not exactly soluble. However, an approximate treatment can be readily given, which amounts to that of Sect. 1.5.1 and a more detailed evaluation of some of the non-Markovian effects is presented in Sect. 3.7. The high frequency treatment of this model is also extensively investigated in Chap. 9.

3.4.5 The Rotating Wave Approximation

Let us consider the harmonic oscillator of Sect. 3.4.3 in the situation in which the resonant frequency $\Omega = 1/\sqrt{LC}$ is very large, and the damping, determined by

$f(t)$, is small. We want to show that the behaviour of this oscillator may be approximated by a rather simpler kind of expression as long as our only interest is in the behaviour of the oscillator within a relatively narrow bandwidth around the resonant frequency.

To do this we define the destruction operator

$$a(t) = \frac{1}{\sqrt{2\hbar}} \left\{ \frac{L}{C} \right\}^{\frac{1}{4}} Q(t) + \frac{i}{\sqrt{2\hbar}} \left\{ \frac{L}{C} \right\}^{-\frac{1}{4}} P(t) \qquad (3.4.51)$$

in terms of which the equation of motion becomes (setting $t_0 = -\infty$)

$$\dot{a}(t) = -i\Omega a(t) - \frac{1}{2L} \int\limits_{-\infty}^{t} f(t-t') \left[a(t') - a^{\dagger}(t') \right] dt'$$

$$+ \frac{i}{\sqrt{2\hbar}} \left\{ \frac{L}{C} \right\}^{-\frac{1}{4}} \xi(t). \qquad (3.4.52)$$

We now move to a rotating frame by setting

$$a(t) = e^{-i\Omega t} A(t). \qquad (3.4.53)$$

Since the dominant term in (3.4.52) is the term $-i\Omega a(t)$, we can expect that $A(t)$ changes only slowly with time—more precisely, $A(t)$ varies slowly compared with $f(t)$. We can then make an approximate equation by replacing $A(t')$ and $A^{\dagger}(t')$ with $A(t)$ and $A^{\dagger}(t)$, and using the definition (3.4.29) of $R(\omega)$. We get

$$\dot{A}(t) = \frac{i}{\sqrt{2\hbar}} \left\{ \frac{L}{C} \right\}^{-\frac{1}{4}} e^{i\Omega t} \xi(t) - A(t) \frac{R(\Omega)}{2L} - e^{2i\Omega t} A^{\dagger}(t) \frac{R^{*}(\Omega)}{2L}. \quad (3.4.54)$$

We now make some further approximations.

i)   The last term oscillates very rapidly on the time scale which is of interest, and we can justifiably set it equal to its cycle averaged value, namely zero. This is the core of what is meant by the *rotating wave approximation,* because the solution is only valid when averaged over many cycles of the factor $e^{2i\Omega t}$.

ii) Associated with the first approximation is the realisation that any solutions to these equations will automatically be valid only over a slow time scale for $A(t)$, so only the frequency components of $e^{i\Omega t} \xi(t)$ with frequency near zero will have any significant effect on the solutions.

Let us define noise operators $b(t), b^{\dagger}(t)$ by

$$b(t) \quad = ie^{i\Omega t} \xi(t) / \sqrt{2\hbar\,\Omega r(\Omega)} \qquad (3.4.55)$$

$$b^{\dagger}(t) \quad = -ie^{-i\Omega t} \xi(t) / \sqrt{2\hbar\,\Omega r(\Omega)}. \qquad (3.4.56)$$

We can now compute the commutator of these two operators by inserting the Fourier decomposition of $\xi(t)$ given in (3.4.25); we get

$$[b(t), b^{\dagger}(t')] = \frac{1}{2\pi} \int\limits_{-\infty}^{\infty} d\omega\, e^{i\omega(t-t')} \frac{(\Omega - \omega) r(\Omega - \omega)}{\Omega r(\Omega)} \qquad (3.4.57)$$

and if we are only interested in $\omega \approx 0$ (that is, time scales slow compared to $1/\Omega$), we may replace the coefficient in this Fourier transform by its value at $\omega = 0$. We then obtain

$$[b(t), b^\dagger(t')] \approx \delta(t - t').\tag{3.4.58}$$

Similar reasoning shows that we can also approximate the other commutators as

$$[b(t), b(t')] = [b^\dagger(t), b^\dagger(t')] = 0.\tag{3.4.59}$$

Furthermore, we can also similarly write the approximations for the expectations of anticommutators as

$$\langle [b(t), b(t')]_+ \rangle = \langle [b^\dagger(t), b^\dagger(t')]_+ \rangle = 0\tag{3.4.60}$$

$$\langle [b(t), b^\dagger(t')]_+ \rangle = (2\bar{N}(\Omega) + 1)\delta(t - t').\tag{3.4.61}$$

In terms of these variables (3.4.54) becomes

$$\dot{A}(t) = -\frac{R(\Omega)}{2L}A(t) + \sqrt{\frac{r(\Omega)}{L}}\, b(t).\tag{3.4.62}$$

iii)   Finally let us note that $R(\Omega)$ differs from $r(\Omega)$ by an imaginary term only (defined by (3.4.34)), which effectively changes the oscillator frequency Ω to some slightly different frequency, which can be neglected in all practical applications. The final rotating wave approximation Langevin equation then becomes, after moving out of the rotating frame

$$\dot{a}(t) = -i\Omega a(t) - \frac{\gamma}{2}a(t) + \sqrt{\gamma}\, b(t)\tag{3.4.63}$$

in which $\gamma = R(\Omega)/L$.

In fact the last term should really be $e^{-i\Omega t}b(t)$, but since we make the white noise approximations (3.4.60,61), the exponential factor has no effect. However, it should be remembered that this equation will only give a good representation of the behaviour of $a(t)$ in a rather narrow bandwidth around Ω.

The rotating wave approximation has the major advantage of eliminating divergences. This arises because the counter-rotating terms have been dropped and because we approximate the noise spectrum by a flat spectrum around the frequency of interest. It is not easy to justify it rigorously in this context, but it will become clear when we derive the master equation in Sect.3.6.3.

Exercise. Solve (3.4.63), and show that it predicts that the equilibrium mean value of the energy is $\hbar\Omega[\bar{N}(\Omega) + \frac{1}{2}]$.

3.5   The Adjoint Equation

The nonlinear nature of the quantum Langevin equations in nearly all cases of interest makes them difficult to solve and hence to use. In Sect. 1.5.1 an alternative point

of view was given, based upon a white noise approximation to quantum noise—this yielded the *master equation*, (1.5.31), for the density operator of a two level atomic system. This equation is very powerful in the situations where it is valid, for example in nearly all quantum optical situations. The master equation is an approximate equation for the reduced density operator of the *system*, which is obtained by tracing over the bath variables. The quantum Langevin equation is, in contrast, an equation for system *operators*. In order to connect the two formalisms it is useful to find an equation for the system density operator which is exact, and then to investigate approximations which lead to master equations of various kinds. The equation that makes this connection is the adjoint equation, which will now be derived.

3.5.1   Derivation of the Adjoint Equation

In the theory of classical noise the derivation of a Fokker-Planck equation equivalent to a Langevin equation can be done directly using the theory of stochastic differential equations, as is done in S.M.Sect.4.3.4, and the central fact used is the delta-correlated nature of the noise source. However, we know that physically the noise is *not* delta-correlated, but merely has a short correlation time. This means that in the classical case we can regard Langevin equations like

$$\dot{q} = p/m$$
$$\dot{p} = -V'(q) - \gamma\dot{q} + \xi(t) \tag{3.5.1}$$

as being equivalent to a Liouville equation for a corresponding probability density function $P(q, p, t)$; namely

$$\frac{\partial P}{\partial t} = -\frac{\partial}{\partial q}\left(\frac{p}{m}P\right) + \frac{\partial}{\partial p}\left([V'(q) + \gamma\frac{p}{m} - \xi(t)]P\right) \tag{3.5.2}$$

in which $\xi(t)$ is regarded as a given function of time—albeit a random function. Quantities of interest can be computed in principle by solving this equation for each realisation of $\xi(t)$, and averaging over the various possible realisations. This kind of Liouville equation is known as the *stochastic Liouville equation* and was first introduced by *Kubo*[3.11]. Exact solution is of course rarely possible, and approximation techniques have to be used.

The adjoint equation is the quantum analogue of the stochastic Liouville equation. Let us consider firstly the case where the density operator in the Heisenberg picture factorises into a direct product $\rho_{sys} \otimes \rho_B$. Let $Y(t)$ be an arbitrary operator in the Heisenberg picture, and Y a Schrödinger picture version of the same operator. Then we can consistently define a quantity $\mu(t)$ by

$$\text{Tr}_{sys}\{Y(t)\rho_{sys}\} = \text{Tr}_{sys}\{Y\mu(t)\} \tag{3.5.3}$$

if the equality is true for all system operators Y and $Y(t)$. In fact, this implicit definition of $\mu(t)$ can be made explicit as follows. Suppose e_i are a complete set of Schrödinger picture system operators, in the sense that any system operator can be

expressed as a linear combination of them, and suppose that they are also orthogonal with respect to the trace :

$$\text{Tr}_{\text{sys}}\left\{e_i^\dagger e_j\right\} = \delta_{ij}. \tag{3.5.4}$$

From this and the assumed completeness it follows that any operator A can be written

$$A = \sum_i \text{Tr}_{\text{sys}}\{e_i A\}\, e_i^\dagger. \tag{3.5.5}$$

In particular, if G is any operator, then

$$Ge_i = \sum_j G_{ij} e_j \tag{3.5.6}$$

where

$$G_{ij} = \text{Tr}_{\text{sys}}\left\{e_j^\dagger G e_i\right\}. \tag{3.5.7}$$

Let us denote the corresponding Heisenberg operators by $e_i(t)$. Since the time evolution is unitary, all algebraic relations will be preserved. In particular, if $G(t)$ is the Heisenberg operator corresponding to G, then the Heisenberg version of (3.5.6) is

$$G(t)e_i(t) = \sum_j G_{ij} e_j(t), \tag{3.5.8}$$

where the coefficients G_{ij} are exactly the same as in the Schrödinger version (3.5.6), i.e., are given by (3.5.7), in terms of the Schrödinger picture operators.

Let us now construct $\mu(t)$ explicitly. We can write

$$\mu(t) = \sum_i \text{Tr}_{\text{sys}}\{e_i(t)\rho_{\text{sys}}\}\, e_i^\dagger \tag{3.5.9}$$

since it is clear from (3.5.9) that

$$\text{Tr}_{\text{sys}}\{e_i\mu(t)\} = \text{Tr}_{\text{sys}}\{e_i(t)\rho_{\text{sys}}\} \tag{3.5.10}$$

and the general form (3.5.3) follows by completeness; i.e. if

$$Y = \sum Y_i e_i \qquad \text{then} \qquad Y(t) = \sum_i Y_i e_i(t) \tag{3.5.11}$$

and thus $\mu(t)$ satisfies the implicit definition (3.5.3). Let us now take the quantum Langevin equation in the form (3.1.28). The equation of motion for $\dot{e}_i(t)$ is then

$$\dot{e}_i(t) = \frac{i}{\hbar}\left[H_{\text{sys}}, e_i(t)\right] - \frac{i}{2\hbar}\left[[X(t), e_i(t)], \xi(t) - \gamma\dot{X}(t)\right]_+. \tag{3.5.12}$$

To evaluate $\dot{\mu}(t)$, we need to evaluate $\text{Tr}_{\text{sys}}\{\dot{e}_i(t)\rho_{\text{sys}}\}$, and this will contain terms of the form (as well as similar terms with $\xi(t)$ on the left)

$$\text{Tr}_{\text{sys}}\{A(t)e_i(t)B(t)\xi(t)\rho_{\text{sys}}\} = \text{Tr}_{\text{sys}}\left\{\sum_j G_{ij}e_j(t)\rho_{\text{sys}}\right\}\xi(t) \qquad (3.5.13)$$

where

$$G_{ij} = \text{Tr}_{\text{sys}}\left\{Ae_iBe_j^\dagger\right\} = \text{Tr}_{\text{sys}}\left\{Be_j^\dagger Ae_i\right\}, \qquad (3.5.14)$$

and $A(t), B(t)$ may be $X(t), e_i(t), \dot{X}(t)$, etc. The corresponding term in $\dot{\mu}(t)$ is then

$$\sum_{i,j} \text{Tr}_{\text{sys}}\left\{e_j(t)\rho_{\text{sys}}\right\}\text{Tr}_{\text{sys}}\left\{Be_j^\dagger Ae_i\right\}e_i^\dagger\xi(t) \qquad (3.5.15)$$

$$= \sum_j \text{Tr}_{\text{sys}}\left\{e_j(t)\rho_{\text{sys}}\right\}Be_j^\dagger A\xi(t)$$

$$= B\mu(t)A\xi(t). \qquad (3.5.16)$$

Using this kind of result, it is then straightforward to derive the *adjoint equation* for $\mu(t)$:

$$\boxed{\dot{\mu}(t) = -\frac{i}{\hbar}[H_{\text{sys}}, \mu(t)] + \frac{i}{2\hbar}\left[[\gamma\dot{X} - \xi(t), \mu(t)]_+, X\right].} \qquad (3.5.17)$$

In this equation, the operator $\dot{X}$ is given by

$$\dot{X} = \frac{i}{\hbar}[H_{\text{sys}}, X]. \qquad (3.5.18)$$

3.5.2   Comments on the Adjoint Equation

a) **Equivalence to the Langevin Equation:**   The adjoint equation is an exact consequence of the quantum Langevin equation, but is not really a version of the same thing in the Schrödinger picture. It is rather like the situation in which a quantum system is driven by a classical field, but in that case, $\gamma\dot{X}$ would be missing.

b) **A Commutative Noise Representation:**   Rather miraculously, the operator nature of $\xi(t)$ can be almost completely eliminated. In the adjoint equation, $\xi(t)$ occurs only as an anticommutator. We can define an operator $\alpha(t)$ by

$$\alpha(t)\mu(t') = \tfrac{1}{2}[\xi(t), \mu(t')]_+ \qquad \text{for all } t \text{ and } t'. \qquad (3.5.19)$$

Then in fact

$$\alpha(t)\alpha(t') = \alpha(t')\alpha(t). \qquad (3.5.20)$$

The proof is quite simple. Using straightforward algebra it is easy to show that

$$[\alpha(t), \alpha(t')]\mu = \tfrac{1}{4}\big[[\xi(t), \xi(t')], \mu\big] \tag{3.5.21}$$

However the commutator $[\xi(t), \xi(t')]$ is a c-number, so that the commutator $[\alpha(t), \alpha(t')]$ is zero. By definition the multiplication of $\alpha(t)$ with $\alpha(t')$ is associative, so that $\alpha(t)$ is in fact equivalent to a c-number random function. This has enabled *Parkins* and *Gardiner* [3.12] to simulate the solutions of the adjoint equation by using an appropriate random sequence of numbers to represent $\alpha(t)$. The techniques are demonstrated in Sects. 3.7 and 9.3. It is also possible to use approximation techniques based on classical stochastic processes on the adjoint equation when it is represented in this form, as has been done by *Ritsch* and *Zoller* [3.13].

c) Correlation Functions Using the Adjoint Equation: Since the $e_i(t)$ are a complete set of operators, it is sufficient to compute the correlation functions (using the notation of Sect. 3.5.1)

$$\mathrm{Tr}_{\mathrm{sys}}\big\{e_i(t)e_j(t')\rho_{\mathrm{sys}}\big\} \equiv \mu_{ij}(t, t') \tag{3.5.22}$$

from which

$$\langle e_i(t)e_j(t')\rangle = \mathrm{Tr}_{\mathrm{B}}\big\{\mu_{ij}(t, t')\rho_B\big\}. \tag{3.5.23}$$

What is the equation of motion for $\mu_{ij}(t, t')$? We consider the case $t \geq t'$, so that an initial condition at $t = t'$ can be obtained from the fact that

$$\mu_{ij}(t', t') = \mathrm{Tr}_{\mathrm{sys}}\big\{e_i(t')e_j(t')\rho_{\mathrm{sys}}\big\} = \mathrm{Tr}_{\mathrm{sys}}\big\{e_i e_j \mu(t')\big\}. \tag{3.5.24}$$

We use the quantum Langevin equation (3.5.12), and define

$$\mu_j(t, t') = \sum_i e_i^\dagger \mu_{ij}(t, t') \tag{3.5.25}$$

and proceed as for the derivation of the adjoint equation. However, we meet a term corresponding to (3.5.13)

$$\mathrm{Tr}_{\mathrm{sys}}\big\{A(t)e_i(t)B(t)\xi(t)e_j(t')\rho_{\mathrm{sys}}\big\}. \tag{3.5.26}$$

In this term we can use our procedure to get a term in $\dot{\mu}_j(t, t')$

$$\sum_k \mathrm{Tr}_{\mathrm{sys}}\big\{e_k(t)\xi(t)e_j(t')\rho_s\big\}Be_k^\dagger A \tag{3.5.27}$$

and the operator $\xi(t)$ is still in the middle of the trace. We can bring it fully to the right by using the commutation relation (3.1.33), by means of which (3.5.27) becomes

$$B\mu_j(t, t')A\xi(t) + \sum_k \mathrm{Tr}_{\mathrm{sys}}\left\{e_k(t)\frac{d}{dt}\big[\gamma u(t'-t)[X(t), e_j(t')]\big]\rho_{\mathrm{sys}}\right\}Be_k^\dagger A. \tag{3.5.28}$$

Since we are considering only $t > t'$, we can write

$$
= B\mu_j(t,t')A\xi(t) - \sum_k \mathrm{Tr}_{\mathrm{sys}} \left\{ e_k(t')\gamma\delta(t-t')[X(t'),e_j(t')]\rho_{\mathrm{sys}} \right\} Be_k^\dagger A
$$
(3.5.29)

$$
= B\mu_j(t,t')A\xi(t) - \gamma\delta(t-t') \sum_k \mathrm{Tr}_{\mathrm{sys}} \left\{ e_k[X,e_j]\mu(t') \right\} Be_k^\dagger A. \quad (3.5.30)
$$

The conclusion is that $\mu_j(t,t')$ obeys the adjoint equation in t for $t > t'$, but there is an initial delta function transient. This is equivalent to modifying the initial condition by an additional term

$$
\frac{i\gamma}{2\hbar} \left[X, [X,e_j]\mu(t') \right],
$$
(3.5.31)

which is often rather small, especially in quantum optical situations.

   This transient arises because of the factorised form of density operator chosen, and is similar to those found in the exact solutions for quantum Brownian motion and the harmonic oscillator.

3.5.3   Summary of the Adjoint Equation

Using the quantity $\alpha(t)$, the adjoint equation takes the form

$$
\dot\mu(t) = -\frac{i}{\hbar}[H_{\mathrm{sys}},\mu(t)] + \frac{i\gamma}{2\hbar} \left[[\dot X,\mu(t)]_+, X \right] - \frac{i}{\hbar}\alpha(t)[\mu(t),X] \quad (3.5.32)
$$

in which $\alpha(t)$ can be regarded as a c-number quantity which, in the case of a thermal bath, is Gaussian and has the mean and correlation function given by (3.3.10). The transformation of the operator nature of the input noise into something essentially classical makes for a tremendous simplification in the use of the adjoint equation—in fact it enables us to carry out many calculations using the techniques of ordinary stochastic processes.

a) General Form of the Adjoint Equation: As written in (3.5.32), the adjoint equation corresponds only to the first Markov approximation of Sect. 3.1.2. The more general quantum Langevin equation (3.1.12,13) is of course similar except for the modification of the X term. The general equation contains the term

$$
\int_{t_0}^{t} f(t-t')\dot X(t')dt'.
$$
(3.5.33)

This cannot be expressed directly in terms of system operators at time t, but an approximate expression can be derived *in the case that the noise is small*. In this case H_{sys} is almost time independent and we can write approximately

$$
X(t') = \exp\left(\frac{iH_{\mathrm{sys}}(t'-t)}{\hbar} \right) X(t) \exp\left(\frac{-iH_{\mathrm{sys}}(t'-t)}{\hbar} \right).
$$
(3.5.34)

This does give an expression for $X(t')$ in terms of $X(t)$, so that we can *define* the Schrödinger operator $X(t')$ by

$$X(t') = \exp\left(\frac{iH_{\text{sys}}t'}{\hbar}\right) X \exp\left(\frac{-iH_{\text{sys}}t'}{\hbar}\right), \tag{3.5.35}$$

then the adjoint equation becomes

$$\dot{\mu}(t) = -\frac{i}{\hbar}[H_{\text{sys}}, \mu(t)]$$

$$+ \frac{i}{2\hbar}\left[\int_{-\infty}^{t}\left[\dot{X}(t')f(t')dt', \mu(t)\right]_{+}, X\right] - \frac{i}{\hbar}\alpha(t)[\mu(t), X] \tag{3.5.36}$$

where t_0 has been set to $-\infty$, and it is assumed that $f(t_0) \to 0$ in this limit. This general form of the adjoint equation is not an exact consequence of the initial model, but will be valid in weak noise situations, even if the noise is not Markovian.

b) Applications: It will not be easy to get exact solutions for either of these forms of the adjoint equation, and a number of approximation techniques will be developed, most notably, those of the *Master Equation*. However there is also a possibility of showing that the quasiclassical Langevin equation mentioned in Sect. 1.2 derives its validity from the adjoint equation—this will be done in Sect. 4.5.2 .

3.6   The Master Equation

The adjoint equation gives a way of (in principle) evaluating $\mu(t)$ for any possible realisation of $\alpha(t)$. This is far more than is usually required, since the detailed correlation between the noise, $\alpha(t)$, and the system is not usually of interest. However, average quantities are of interest, that is, averages over the possible values of $\alpha(t)$. Thus, we are interested in averaging $\mu(t)$ over $\alpha(t)$, using the known statistics of $\alpha(t)$. This is in fact a well studied classical problem, since the adjoint equation can be written in the form

$$\frac{d\mu}{dt} = [A_0 + \alpha(t)A_1]\mu \tag{3.6.1}$$

in which A_0 and A_1 are linear operators given by

$$A_0\mu = -\frac{i}{\hbar}[H_{\text{sys}}, \mu] + \frac{i}{2\hbar}\left[\int_{-\infty}^{0}\left[\dot{X}(t')f(t')dt', \mu(t)\right]_{+}, X\right] \tag{3.6.2}$$

$$A_1\mu = -\frac{i}{\hbar}[\mu, X]. \tag{3.6.3}$$

Van Kampen [3.14] has studied this problem in some detail, and has shown how to obtain a reduced equation for $\langle\mu(t)\rangle$, the average of $\mu(t)$ over $\alpha(t)$. If we define

$$\rho(t) = \mathrm{Tr}_B\{\mu(t)\rho_B\} \equiv \langle\mu(t)\rangle \tag{3.6.4}$$

then all the averages of system variables follow from (3.5.3),

$$\left.\begin{aligned}
\langle Y(t)\rangle &= \mathrm{Tr}_{\mathrm{sys}}\{\mathrm{Tr}_B\{Y(t)\rho_{\mathrm{sys}}\otimes\rho_B\}\} = \mathrm{Tr}_B\{\mathrm{Tr}_{\mathrm{sys}}\{Y\mu(t)\}\rho_B\} \\
&= \mathrm{Tr}_{\mathrm{sys}}\{Y\,\mathrm{Tr}_B\{\mu(t)\rho_B\}\} \\
&= \mathrm{Tr}_{\mathrm{sys}}\{Y\rho(t)\}.
\end{aligned}\right\} \tag{3.6.5}$$

Thus, an equation of motion for $\rho(t)$ will enable us to deduce the time development of the averages of system operators. The mathematical result is an expansion in terms of a smallness parameter, which measures how small or how fast the fluctuations are. The result, to second order in this parameter, is

$$\dot{\rho}(t) = A_0\rho(t) + \int\limits_0^\infty d\tau\langle\alpha(t)A_1e^{A_0\tau}\alpha(t-\tau)A_1e^{-A_0\tau}\rangle\rho(t). \tag{3.6.6}$$

The relevant smallness parameter is $\|A_1\|\,\|\alpha\|\,\tau_c$, where

$\quad\|A_1\|$ is a measure of the magnitude of the operator A_1

$\quad\|\alpha\|$ is the root mean square amplitude of $\alpha(t)$

$\quad\tau_c$ is the correlation time of $\alpha(t)$.

The method of derivation of (3.6.6) is known as van Kampen's cumulant expansion method.

a) Outline of van Kampen's Cumulant Expansion: Although it is not appropriate to derive (3.6.6) in full detail, an indication of the major steps in its derivation is appropriate, since the technique is fundamental to very many problems in quantum noise theory, and can be adapted to other situations where the existence of a commutative noise operator $\alpha(t)$ is either undesirable or impossible. The method is basically a perturbation theory method. A new variable $u(t)$ is first defined by

$$\mu(t) = \exp(A_0t)u(t) \tag{3.6.7}$$

so that the equation of motion becomes

$$\frac{du}{dt} = A_1(t)\alpha(t)u(t) \tag{3.6.8}$$

where

$$A_1(t) = \exp(-A_0t)A_1\exp(A_0t). \tag{3.6.9}$$

In most cases A_0 is the most important part of the total evolution operator in (3.6.1)—in terms of the particular case here, for example, the systematic motion

is normally much more significant than the noise. The transformation to this *interaction picture*, defined by $u(t)$, thus removes the rapid time evolution induced by H_{sys}, and leaves us to deal only with the noise. But notice that the *damping* is contained in A_0 —it is the second part of (3.6.2), so that A_0 contains both a large and a small part. We integrate (3.6.8) to get

$$u(t) = u(0) + \int_0^t dt' A_1(t')\alpha(t')u(t') \tag{3.6.10}$$

and then iterate this to get

$$u(t) = u(0) + \int_0^t dt' A_1(t')\alpha(t')u(0)$$
$$+ \int_0^t dt' \int_0^{t'} dt'' A_1(t')\alpha(t') A_1(t'')\alpha(t'')u(t''). \tag{3.6.11}$$

Basic assumptions are now

i) $u(0)$ is independent of $\alpha(t')$ for $t' \geq 0$; $\tag{3.6.12}$

ii) $\langle \alpha(t') \rangle = 0$; $\tag{3.6.13}$

and this will allow us to average $u(t)$ in (3.6.11) and then to differentiate, to get

$$\frac{d\langle u(t) \rangle}{dt} = \int_0^t dt' \langle A_1(t)\alpha(t) A_1(t')\alpha(t')u(t') \rangle. \tag{3.6.14}$$

This equation is of course not in a very useful form, since the mean of $u(t)$ occurs on the left, whereas only $\langle \alpha(t)\alpha(t')u(t') \rangle$ occurs on the right. It is here that some assumptions must be made concerning this triple average. The various possibilities are detailed by *van Kampen* [3.14], who also gives some insight into the history of this problem, including the range of approximation methods employed by a variety of authors. The end result is that an expansion in $\|A_1\|\,\|\alpha\|\,\tau_c$ is possible, and using this expansion enables one to say that to lowest order

$$\langle \alpha(t)\alpha(t')u(t') \rangle \approx \langle \alpha(t)\alpha(t') \rangle \langle u(t') \rangle \tag{3.6.15}$$

and, further, to the same accuracy

$$\langle u(t') \rangle \approx \langle u(t) \rangle \quad \text{provided } |t - t'| \ll \tau_c. \tag{3.6.16}$$

One can make these as substitutions in (3.6.14), to get, on setting $t' = t - \tau$,

$$\frac{d\langle u(t) \rangle}{dt} = \int_0^t d\tau \langle \alpha(t) A_1(t)\alpha(t-\tau) A_1(t-\tau) \rangle \langle u(t) \rangle. \tag{3.6.17}$$

The only remaining step is to let the range of integral over τ be $(0, \infty)$. This is again able to be shown to be as accurate as (3.6.17) to lowest order, and arises basically by noting that $\langle \alpha(t)\alpha(t-\tau) \rangle \sim 0$ as long as $\tau \gg \tau_c$, so that after a few correlation times, after the initial setting up of the system, t is essentially infinite compared to τ_c. Substituting for $\mu(t)$ using the definition (3.6.7) of $u(t)$, we derive the result (3.6.6).

b) Application of the Cumulant Expansion: To apply the cumulant expansion to quantum noise now requires a rather careful consideration of the various parameters involved. The equations derived will always be first order in time, and will involve $\rho(t)$ at only one time. The process of derivation however shows that t is only able to be specified to a precision of about τ_c, which in thermal situations is indeed usually very small, but in principle at very low temperature can become rather long. This means that (3.6.6) will not be valid at very low temperature. This is the penalty one pays for a simple equation, for (3.6.6) is indeed rather simple. It does turn out to be capable of describing a very wide range of phenomena—outside its range of validity, one must turn to the adjoint equation or to other methods.

c) Range of Validity of the Cumulant Expansion: The relevant correlation time is the *thermal correlation time* $\tau_T = \hbar/2\pi kT$ (3.3.20). The value of $\|\alpha\|$ is given by the square root of the coefficient of the exponential in the asymptotic form for the correlation function (3.3.19) i.e.,

$$\|\alpha\| = 2kT\sqrt{\gamma\pi/\hbar} \,. \tag{3.6.18}$$

To estimate the size of the operator A_1 is then all that remains. In the case that we are dealing with something like a particle in a potential well, specified as in Sect. 3.4.1, the operator A_1 is given by

$$A_1 = \frac{i}{2\hbar}[x, \quad] \to -\frac{1}{2}\frac{\partial}{\partial p}. \tag{3.6.19}$$

If we are near thermal equilibrium, $\mu(t)$ has a p dependence which is proportional to $\exp(-p^2/2mkT)$, so we can estimate

$$\left\|\frac{1}{2}\frac{\partial}{\partial p}\right\| \approx \frac{1}{2\sqrt{mkT}}. \tag{3.6.20}$$

Putting these together gives a condition for the validity of (3.6.6)

$$\hbar\gamma/m \ll kT \tag{3.6.21}$$

which is equivalent to

$$\tau_T \ll \tau_D \tag{3.6.22}$$

where

τ_D is the damping constant of the system $= m/\gamma$
τ_T is the thermal correlation time of $E(t)$.

These are admittedly rather crude estimates, but should suffice for our purposes. The condition (3.6.22) is in the end rather reasonable—it simply requires that the correlation time of the noise be much less than the typical time scale of the damped motion of the system. Two cases can now be distinguished. In one of these, that of *quantum Brownian motion*, it is assumed that the correlation time of the noise is much shorter than the typical time scale of the systematic motion and the damping. The equation that arises is interesting, and has strong analogies with classical Brownian motion equations. However, it has not been widely applied for the most obvious of reasons—quantum Brownian motion is experimentally not a widely studied subject. In the second case, the assumption of rather slow systematic motion is not made, but in order to arrive at a reasonably tractable equation it is necessary to assume that the damping is very weak compared to the systematic motion. This is precisely what is needed in the study of quantum optics, where the systematic motion takes place at optical frequencies, but the damping (which is in this case radiation into the electromagnetic field) is very weak—the lifetime of an excited atom is typically many thousands of periods. The equation so derived is very widely applied, and is capable of describing most situations which arise in the field of quantum optics.

3.6.1   The Quantum Brownian Motion Master Equation

We now focus on the situation in which the correlation time of the noise operator $\alpha(t)$ is quite short. The term

$$\int_0^\infty d\tau \langle \alpha(t)\alpha(t-\tau)\rangle A_1 e^{A_0\tau} A_1 e^{-A_0\tau} \tag{3.6.23}$$

which arises in (3.6.6) can be considerably simplified if it is assumed that the factor $\exp(A_0\tau)$ changes very little in a correlation time; i.e., if we assume that

$$\exp(A_0\tau) \approx 1 \qquad \text{for } \tau \approx \tau_c. \tag{3.6.24}$$

From the expression for $\langle \alpha(t)\alpha(t-\tau)\rangle$ given in (3.3.10) we find that if the noise is assumed thermal

$$\int_0^\infty dt \langle \alpha(t)\alpha(t-\tau)\rangle = \frac{\pi kT}{2} G(\omega) \frac{dn}{d\omega}\bigg|_{\omega=0} \tag{3.6.25}$$

and in the case that we take the simpler form of the adjoint equation, arising from the assumption of a constant $G(\omega)dn/d\omega$, this reduces to

$$\int_0^\infty d\tau \langle \alpha(t)\alpha(t-\tau)\rangle = \gamma kT. \tag{3.6.26}$$

If we now substitute for A_1 using the definition (3.6.3), we find

$$\int_0^\infty dt \langle \alpha(t)\alpha(t-\tau)\rangle A_1 e^{A_0\tau} A_1 e^{-A_0\tau} \rho(t) \rightarrow -\frac{\gamma kT}{\hbar^2} \left[X, \left[X, \rho(t) \right] \right] \tag{3.6.27}$$

and from (3.6.6), we derive

$$\frac{d\rho}{dt} = -\frac{i}{\hbar}[H_{\text{sys}}, \rho] - \frac{i\gamma}{2\hbar}\left[X, [\dot{X}, \rho]_+\right] - \frac{\gamma kT}{\hbar^2}[X, [X, \rho]] \qquad (3.6.28)$$

which is known as the quantum Brownian motion master equation. We now treat some particular applications.

3.6.2   Quantum Brownian Motion of a Particle in a Potential

Suppose we take

$$\begin{aligned} H_{\text{sys}} &\rightarrow \frac{p^2}{2m} + V(x) \\ X &\rightarrow x \end{aligned} \qquad (3.6.29)$$

so that the quantum Langevin equation becomes

$$\begin{aligned} \dot{x} &= p/m \\ \dot{p} &= -V'(x) - \gamma p/m + \xi(t) \end{aligned} \qquad (3.6.30)$$

and the master equation (3.6.28) becomes

$$\frac{d\rho}{dt} = -\frac{i}{\hbar}[H_{\text{sys}}, \rho] - \frac{i\gamma}{2m\hbar}[x, [p, \rho]_+] - \frac{\gamma kT}{\hbar^2}[x, [x, \rho]]. \qquad (3.6.31)$$

As an operator equation, this looks rather formidable, but as an equation for the position space matrix elements $\langle x|\rho|y\rangle$, it can be seen to be more tractable. By using the explicit form (3.6.29) for H_{sys}, and the relation

$$p|x\rangle = i\hbar\frac{\partial}{\partial x}|x\rangle \qquad (3.6.32)$$

we can transform (3.6.31) to

$$\begin{aligned} \frac{d}{dt}\langle x|\rho|y\rangle = \Bigg\{ &\frac{i\hbar}{2m}\left(\frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2}\right) - \frac{i}{\hbar}(V(x) - V(y)) \\ &- \frac{\gamma}{2m}(x - y)\left(\frac{\partial}{\partial x} - \frac{\partial}{\partial y}\right) - \frac{\gamma kT}{\hbar^2}(x - y)^2 \Bigg\}\langle x|\rho|y\rangle \end{aligned} \qquad (3.6.33)$$

which is a 2nd order partial differential equation—by no means a difficult thing to understand.

a) Stationary Solution (Approximate): The differential equation (3.6.33) is considerably simplified by a change of variables to

$$\begin{aligned} x &= u + \hbar z \\ y &= u - \hbar z \\ \langle x|\rho|y\rangle &= P(u, z) \end{aligned} \qquad (3.6.34)$$

so that

$$
\begin{aligned}
\frac{\partial}{\partial x} &= \frac{1}{2}\left(\frac{\partial}{\partial u} + \frac{1}{\hbar}\frac{\partial}{\partial z}\right) \\
\frac{\partial}{\partial y} &= \frac{1}{2}\left(\frac{\partial}{\partial u} - \frac{1}{\hbar}\frac{\partial}{\partial z}\right).
\end{aligned}
\tag{3.6.35}
$$

In these variables, (3.6.33) becomes

$$
\frac{\partial P}{\partial t} = \left\{\frac{i}{2m}\frac{\partial^2}{\partial u \partial z} + \frac{(V(u + \hbar z) - V(u - \hbar z))}{i\hbar} - \frac{\gamma}{m}z\frac{\partial}{\partial z} - 4\gamma kTz^2\right\}P.
\tag{3.6.36}
$$

In the limit that $\hbar$ is small, we can make the replacement

$$
\frac{i}{\hbar}\left(V(u + \hbar z) - V(u - \hbar z)\right) \rightarrow 2i z V'(u)
\tag{3.6.37}
$$

and it is notable that Planck's constant no longer appears explicitly anywhere in the equation. By explicit substitution, one sees that the stationary solution is given by

$$
P_s(u, z) = \mathcal{N} \exp\left\{-\frac{V(u)}{kT} - 2mkTz^2\right\}
\tag{3.6.38}
$$

where

$$
\mathcal{N}^{-1} = \sqrt{\frac{\pi}{2mkT}} \int du \exp\left\{-\frac{V(u)}{kT}\right\}.
\tag{3.6.39}
$$

Of course Planck's constant does appear, since z involves $\hbar$: explicitly

$$
\langle x|\rho|y\rangle = \mathcal{N} \exp\left\{-\frac{V\left(\frac{1}{2}(x + y)\right)}{kT} - \frac{mkT(x - y)^2}{2\hbar^2}\right\}.
\tag{3.6.40}
$$

It can be seen that in this basis the density operator ρ is very nearly diagonal, because of the exponential cutoff in $x - y$ over distances of order $\hbar/\sqrt{mkT}$.

Exercise. Show that symmetrised moments of x and p can be computed from (3.6.40), and that these are exactly the same as those given by classical physics.

b) Free Particle: We set $V(x) = 0$, and express P in terms of a Fourier transform with respect to the variable u

$$
P(u, z) = \int dq e^{iqu}\tilde{P}(q, z).
\tag{3.6.41}
$$

This Fourier transform can also be regarded as the characteristic function, i.e.

$$
\tilde{P}(q, z) = \langle\exp(-iq\hat{x} + 2i\hat{p}z)\rangle
\tag{3.6.42}
$$

where $\hat{x}, \hat{p}$ are the position and momentum operators. (Use the fact that $\hat{p}$ is a displacement operator, and factorise the exponential using the Baker-Hausdorff theorem.) The master equation now becomes

$$\frac{\partial \tilde{P}}{\partial t} + \left(\frac{q + 2\gamma z}{2m} \right) \frac{\partial \tilde{P}}{\partial z} = -4\gamma kTz^2 \tilde{P} \tag{3.6.43}$$

and this can be solved by the method of characteristics, as in Sect. 3.8.4 of S.M. The subsidiary equation is

$$\frac{dt}{1} = \frac{2m dz}{q + 2\gamma z} = -\frac{d\tilde{P}}{4\gamma kTz^2 \tilde{P}}. \tag{3.6.44}$$

Two particular integrals are obtained by integrating the first two and the second two of these equations:

$$u(z, t, q, \tilde{P}) = w e^{-\gamma t/m} \tag{3.6.45}$$

$$v(z, t, q, \tilde{P}) = \tilde{P} \exp(2mkTw^2 - 4mqkTw/\gamma) w^{mkTq^2/\gamma^2} \tag{3.6.46}$$

where

$$w = z + q/2\gamma. \tag{3.6.47}$$

The general solution of the partial differential equation (3.6.43) can be written in the form $v = F(u)$, where F is an arbitrary function, and thus, the solution becomes

$$\tilde{P} = \exp(-2mkTw^2 + 4mqkTw/\gamma) w^{-mkTq^2/\gamma^2} F(we^{-\gamma t/m}). \tag{3.6.48}$$

Assuming that at $t = 0$, $\tilde{P} = f(q, w)$, we can deduce that

$$\tilde{P} = f(q, we^{-\gamma t/m}) \exp\left\{ -2mkTw^2 \left(1 - e^{-2\gamma t/m} \right) \right.$$
$$\left. + \frac{4mqkTw}{\gamma} \left(1 - e^{-\gamma t/m} \right) \right\} e^{-(q^2 kT/\gamma)t}. \tag{3.6.49}$$

From this solution, we note that

i)   there are two characteristic time constants

$$\tau_D = m/\gamma, \tag{3.6.50}$$

which is the time constant of frictional damping, and

$$\tau_q = \gamma/kTq^2, \tag{3.6.51}$$

which is a time constant with the same kind of q dependence as turns up in classical diffusion theory.

ii)   There is always a range of q such that $\tau_D < \tau_q$, so we can characterise the time evolution of the $\tilde{P}$ by two periods. In a time of order τ_D, after substituting for w in terms of z

$$\tilde{P} \rightarrow f(q, 0) \exp(-2mkTz^2) \exp\left[-\frac{kTq^2}{\gamma} \left(t + \frac{m}{2\gamma} \right) \right] \tag{3.6.52}$$

and at this stage , only q such that

$$q^2 \ll \frac{\gamma^2}{mkT} \tag{3.6.53}$$

are important.

c) **Example—Damping of Quantum Coherence:** *Savage* and *Walls* [3.15] considered the effect of noise and damping on the interference effects expected quantum mechanically from the superposition of two plane wave states. We consider a system initially in the pure state $|\varphi\rangle$, expressible as a sum of two plane waves, so that

$$\langle x|\varphi\rangle = \sqrt{\tfrac{1}{2}} \left(e^{i k_1 x} + e^{i k_2 x} \right). \tag{3.6.54}$$

Initially

$$\begin{aligned} P(u,z) &= \langle u + \hbar z|\varphi\rangle\langle\varphi|u - \hbar z\rangle \\ &= \tfrac{1}{2}\left\{ e^{2i k_1 \hbar z} + e^{2i k_2 \hbar z} + e^{i\hbar z(k_1+k_2)} \left(e^{iu(k_1-k_2)} + e^{-iu(k_1-k_2)} \right) \right\} \end{aligned} \tag{3.6.55}$$

and

$$\begin{aligned} \tilde{P}(q,z) &= \tfrac{1}{2}\delta(q) \left(e^{2i k_1 \hbar z} + e^{2i k_2 \hbar z} \right) \\ &\quad + \tfrac{1}{2}e^{i\hbar z(k_1+k_2)} \left\{ \delta(q - k_1 + k_2) + \delta(q + k_1 - k_2) \right\} \end{aligned} \tag{3.6.56}$$

Using (3.6.49), we can compute $f(q,w)$, and hence $\tilde{P}(q,w,t)$ which is found to be

$$\begin{aligned} \tilde{P}(q,w,t) &= \tfrac{1}{2}\Bigg[\delta(q) \left\{ \exp\left(2i\, k_1 \hbar\, w e^{-\gamma t/m} \right) + \exp\left(2i\, k_2 \hbar\, w e^{-\gamma t/m} \right) \right\} \\ &\quad + \left\{ \delta(q + k_1 - k_2) + \delta(q - k_1 + k_2) \right\} \\ &\qquad \times \exp\left(i(k_1 + k_2)\hbar\,(w e^{-\gamma t/m} - q/2\gamma) \right) \Bigg] \\ &\quad \times \exp\Bigg\{ -2mkTw^2 \left(1 - e^{-2\gamma t/m} \right) \\ &\qquad + \frac{4mkTqw}{\gamma}\left(1 - e^{-\gamma t/m} \right) - \frac{q^2 kTt}{\gamma} \Bigg\}. \end{aligned} \tag{3.6.57}$$

This can be inverted to yield the density operator, but we shall only examine the diagonal matrix elements of the density operator here, since these refer to the measurable probabilities, which in the case of no damping, exhibit an interference pattern. Thus, from (3.6.55), initially

$$\langle x|\rho|x\rangle = 1 + \cos\left((k_1 - k_2)x \right), \tag{3.6.58}$$

which clearly shows an interference term. To evaluate the diagonal matrix element from (3.6.57) we set $z = 0$, i.e. $w = q/2\gamma$, and carry out the Fourier inversion on the result; we find

$$\langle x|\rho(t)|x\rangle = 1 + e^{-\eta(t)}\cos\left((k_1 - k_2)x - \frac{\hbar}{2\gamma}(1 - e^{-\gamma t/m})(k_1^2 - k_2^2)\right)$$
(3.6.59)

where

$$\eta(t) = (k_1 - k_2)^2\frac{mkT}{\gamma^2}\left\{\frac{\gamma t}{m} - \frac{3}{2} + 2e^{-\gamma t/m} - \tfrac{1}{2}e^{-2\gamma t/m}\right\}.$$
(3.6.60)

Two effects are notable here

i)   The position of the interference fringe moves, as a result of the extra term in the argument of the cosine in (3.6.59).

ii)   The visibility of the fringe is damped out by the factor $e^{-\eta(t)}$, and ultimately, there are no fringes.

3.6.3   The Quantum Optical Case

One of the most powerful instruments in the subject of quantum optics has been the quantum optical master equation. It has a rather long history, originating in the theory of nuclear magnetic resonance in the early fifties [3.16], and moving into quantum optics in the mid-sixties. In this section I will derive a rather general form of this equation using the quantum Langevin equation as a basis. This is not the only derivation, and it is important to realise that the rather particular nature of the heat bath Hamiltonian assumed in the derivation of the quantum Langevin equation is not necessary to obtain the same master equation. The more usual derivations are given in Sects. 5.1.1–3. In this derivation we resolve operators in (3.6.6) into eigenoperators of H_{sys}, namely

$$X = \sum_m (X_m^+ + X_m^-)$$
(3.6.61)

where

$$[H_{\text{sys}}, X_m^\pm] = \pm\hbar\omega_m X_m^\pm.$$
(3.6.62)

This is always possible if the eigenvectors of H_{sys} form a complete set. If we now consider the case when all the ω_m are much larger than γ (as is always the case in quantum optics), then we can omit the γ dependent term in A_0 (as defined in (3.6.2)) in the terms $e^{\pm A_0\tau}$ in (3.6.6), and thus make the replacement

$$e^{\pm A_0\tau}X_m^\pm \rightarrow e^{\pm i\omega_m\tau}X_m^\pm e^{\pm A_0\tau}.$$
(3.6.63)

Using the correlation function (3.3.10), one finds, after some labour, the *quantum optical master equation*:

$$
\begin{aligned}
\dot{\rho}_S(t) = & -\frac{i}{\hbar}[H_{sys}, \rho_S] \\
& -\sum_m \frac{\pi \omega_m}{2\hbar}\left(\bar{N}(\omega_m) + 1\right)\kappa(\omega_m)^2 [\rho_S X_m^+ - X_m^- \rho_S, X] \\
& -\sum_m \frac{\pi \omega_m}{2\hbar}\bar{N}(\omega_m)\kappa(\omega_m)^2 [\rho_S X_m^- - X_m^+ \rho_S, X] \\
& +\frac{i}{2\hbar}\sum_m P \int_{-\infty}^{\infty} \frac{d\omega \kappa(\omega)^2}{\omega_m - \omega}\omega\left(\bar{N}(\omega) + \tfrac{1}{2}\right)[[\rho_S, X_m^+ - X_m^-], X] \\
& +\frac{i}{4\hbar}\sum_m P \int_{-\infty}^{\infty} \frac{d\omega \kappa(\omega)^2}{\omega_m - \omega}\omega_m[[\rho_S, X_m^+ + X_m^-]_+, X]. \qquad (3.6.64)
\end{aligned}
$$

In deriving this equation, it is necessary :

i) To use the adjoint equation in the form implied by (3.6.1–2), where, in particular, the damping is taken with a memory function $f(t)$. We will see in Chap.5 that the second and third lines are the most significant, and account for damping and noise. The last two lines involve principal value integrals, and because of this are rather small. They give rise to rather small frequency shifts, known as the *Stark* shift, which is the term proportional to $\bar{N}(\omega)$ in the fourth line, and the *Lamb* shift, which is the sum of the term in the fifth line and the term proportional to $\tfrac{1}{2}$ in the fourth line. The Stark shift is thus the result of thermal noise, while the Lamb shift, which does not vanish at absolute zero, is the result of vacuum fluctuations. Notice that the integrals are both divergent unless $\kappa(\omega)$ drops off sufficiently rapidly at high frequencies. In the case of an atom, we can say that the actual size of the atom gives a cutoff. In a full quantum electrodynamics treatment (for which the original Hamiltonian would not apply) there is no physically justifiable cutoff, and in that case the full power of renormalisation theory is required. These are dealt with more explicitly in Sect. 3.7.

ii) To use the relationship

$$
\int_{-\infty}^{\infty} d\omega f(\omega) \int_0^{\infty} d\tau\, e^{-i\omega\tau} = \int \left[\pi\delta(\omega) - P\frac{i}{\omega}\right] f(\omega)\, d\omega
$$

$$
= \pi f(0) - iP \int_{-\infty}^{\infty} d\omega f(\omega)/\omega \qquad (3.6.65)
$$

where the symbol $P\int$ means the Cauchy principal value of the integral.

a) The Rotating Wave Approximation: This kind of master equation is well attested to be capable of describing almost all kinds of phenomena which can be experienced in quantum optics, although it is not often written down in precisely the form I have given. As used in quantum optics, it is basically due to *Louisell* [3.17]. Usually the rotating wave approximation is also made, which involves the following.

i)   Define an interaction picture system density operator by

$$\rho_I(t) = \exp\left(\frac{i}{\hbar}H_{sys}t\right)\rho_S(t)\exp\left(-\frac{i}{\hbar}H_{sys}t\right). \tag{3.6.66}$$

Note that the interaction picture master equation no longer has a term corresponding to the first line in (3.6.64), and use the relation (3.6.63) to commute $\exp(\pm\frac{i}{\hbar}H_{sys}t)$ with $X_m^\pm$. This generates certain terms with factors such as $\exp(i(\omega_m + \omega_n)t)$, which is a very rapid time variation on the timescale of atomic decays, thus we may neglect them completely at optical frequencies.

ii)   The Lamb and Stark shift terms are also dropped, since they are very small, leaving the interaction picture master equation in the form :

$$\dot{\rho}_I = \sum_m \frac{\pi\omega_m}{2\hbar}\left(\bar{N}(\omega_m) + 1\right)\kappa(\omega_m)^2(2X_m^-\rho_I X_m^+ - \rho_I X_m^+ X_m^- - X_m^+ X_m^-\rho_I)$$

$$+ \sum_m \frac{\pi\omega_m}{2\hbar}\bar{N}(\omega_m)\kappa(\omega_m)^2(2X_m^+\rho_I X_m^- - \rho_I X_m^- X_m^+ - X_m^- X_m^+\rho_I). \tag{3.6.67}$$

In this form, the master equation describes transitions in an atomic system in a radiation field. Adaptations to include small additional nonlinearities and driving fields are commonly made by adding terms as follows.

b) Driving Fields—Inputs and Outputs: It is very common to consider a situation where a laser beam is incident on an atom. This means that the heat bath (in this case the electromagnetic field) is no longer characterised as having thermal statistics, but has as well some coherent excitation in a small range of modes. It is helpful to go back to the transmission line model of Sect. 3.2 to understand how to include an incident field. From this point of view, the inclusion of a coherent driving field is no problem, since we need only specify the "in" field. One simply makes the requirement

$$\langle A_{in}(t)\rangle = a_{in}(t) \tag{3.6.68}$$

which gives the mean time dependent excitation. The statistics of any fluctuations can be specified by setting

$$\xi(t) = 2\sqrt{\gamma c}\{\dot{A}_{in}(t) - \dot{a}_{in}(t)\} \tag{3.6.69}$$

and specifying the relevant correlation functions of $\xi(t)$. For example, a coherent driving field superimposed on a thermal background is obtained by taking the correlation function (3.3.9) for $\xi(t)$ as defined by (3.6.69). The quantum mechanical Langevin equation becomes

$$\dot{Y} = \frac{i}{\hbar}[H_{sys}, Y] - \frac{2i\sqrt{\gamma c}}{\hbar}\dot{a}_{in}(t)[X, Y] - \frac{i}{2\hbar}\left[\xi(t) - \gamma\dot{X}, [X, Y]\right]_{+}$$
(3.6.70)

and the master equations (3.6.64,67) acquire an extra term

$$\frac{2i\sqrt{\gamma c}}{\hbar}a_{in}(t)[X, \rho],$$
(3.6.71)

and this corresponds to simply adding to the system Hamiltonian a corresponding driving term

$$H_{Driving} = 2\sqrt{\gamma c}\, a_{in}(t)X.$$
(3.6.72)

c) Small Anharmonicity: It is common in quantum optics to consider the system to consist of a single mode of the electromagnetic field inside a cavity, which communicates with the electromagnetic bath and driving modes through an almost perfect mirror. The system would then be a perfect harmonic oscillator—however one also introduces some kind of weak nonlinearity via a nonlinear medium within the cavity. It is thus possible to write

$$H_{sys} = H_0 + H_{nl}$$
(3.6.73)

where H_{nl} is very small compared to H_{sys}. How does this affect the analysis? Typically the effects of H_{nl} are of the same order of magnitude as the damping, so H_{nl} can be neglected in all the procedures leading to (3.6.64,67). In particular this means that

i) ω_m are the transition frequencies of H_0, and *not* of H_{sys}, and since H_0 is harmonic all the ω_m are equal.

ii) X_m are eigenoperators of H_0, *not* of H_{sys} .

Thus the relevant interaction picture is defined in terms of H_0, so that (3.6.64) is modified simply by adding a term

$$-\frac{i}{\hbar}[H_{nl}, \rho_1],$$
(3.6.74)

and we can replace ω_m by a single frequency, Ω, say. Notice that there is an interesting transition region between small nonlinearity, which has this effect, and large nonlinearity, which modifies the whole master equation by modifying the relevant energy levels.

d) Stationary Solution is the Boltzmann Distribution: If we neglect the Stark and Lamb shift terms in the master equation (3.6.64) the stationary solution is obviously the Boltzmann distribution, for the equations (3.6.64) necessarily imply that

$$\exp(-H_{sys}/kT)X_m^{\pm} = \exp(\pm\hbar\,\omega_m/kT)X_m^{\pm}\exp(-H_{sys}/kT)$$
(3.6.75)

from which, using the definition of $\bar{N}$ in (3.6.64), it is obvious that the corresponding terms in the two summations cancel. A general and correct inclusion of the effects of the Lamb and Stark shifts is more tricky, but cannot alter this conclusion as a lowest order result.

Let us now look at the result of Benguria and Kac (see Sect. 1.4.1) that the Boltzmann distribution solution for the stationary state requires quantum Gaussian statistics for $\xi(t)$ in the case (as here) that $[\xi(t_1), \xi(t_2)]$ is a c-number. This admittedly much less rigorous, but certainly far more physically transparent result does not seem to require Gaussian statistics, since only the correlation functions are involved. This is much the same as in the classical case, for there, the proof that there is a white noise limit of a non-white noise stochastic Liouville equation does not require a Gaussian physical noise. Nevertheless, the resultant Fokker-Planck equation is exactly equivalent to a white noise stochastic differential equation with Gaussian noise.

3.7 Adjoint Equation Approach
to Macroscopic Quantum Coherence

In the case that a system moves in a double well potential $V(x)$ one generally obtains closely spaced energy levels, as in Fig.3.3, which in the macroscopic limit are degenerate. The degeneracy is removed by the quantum tunnelling through the potential barrier, a phenomenon which is well attested in atomic and molecular physics.
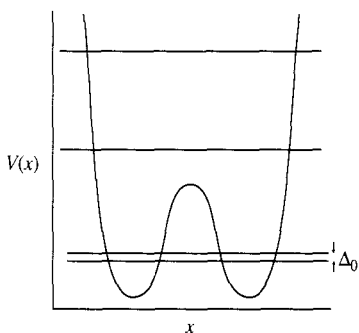
Fig. 3.3   Double well potential, with the lowest energy levels almost degenerate

However, at some macroscopic level, this effect becomes invisible. The "particle", instead of tunnelling back and forth with the characteristic frequency given by the energy difference between the narrowly split levels, stays on one side or the other. The quantum mechanical description appropriate to a microscopic system would predict that the particle would be equally likely to be found on either side of the barrier. However, such an effect is very difficult to see at a macroscopic level.

3.7.1 Dissipation and Tunnelling

In the last ten years it has become clear that coherent quantum tunnelling in a double well system is sensitively dependent on dissipation that results from the coupling of the tunnelling coordinate to the environment. A considerable amount of research has been dedicated to this problem, motivated by its relevance to the possible observation of macroscopic quantum coherence in, for instance, the SQUID (superconducting quantum interference device), where the motion of the trapped magnetic flux may be governed by a symmetric double well potential. Hence, it is possible, in principle, for the flux to exhibit coherent oscillations between two metastable states.

The pioneering work of *Caldeira* and *Leggett* [3.18] showed how in practice, dissipation due to the system-environment coupling plays a vital role in the dynamics of quantum tunnelling. Later work by *Chakravarti* and *Leggett* [3.19] showed how in the problem of quantum tunelling in a double well potential, dissipation will lead to a loss of coherence which may ultimately preclude any observation of coherent oscillations.

The methods introduced by Caldeira and Leggett, and almost universally used by other workers, were based on path integral formulations of exactly the same model of a system coupled to a heat bath of harmonic oscillators as is the basis of this chapter. These treatments were very successful in predicting corrections to the tunnelling rate in the problem of escape from a metastable state, and the predictions appear to have been verified by experiment [3.20]. Path integral methods are beyond the scope of this book, though they do provide a body of methodology which is very useful in quantum noise problems. Therefore, it is of interest to show how the adjoint equation can provide a method for computing dissipative corrections to tunnelling. In this section we present a summary of the recent work of *Parkins* et al. [3.21], in which it is shown that the adjoint equation can provide a treatment of macroscopic quantum tunnelling.

Let us consider a quantum system in which a symmetric double well potential yields two closely spaced low lying states, separated by a tunnel splitting Δ_0, which are well separated in energy from the remaining higher states. We cannot make the Markovian approximation which led to the master equation in Sect. 3.6.3, because at the low temperatures of interest the correlation time of the noise is quite long. On the other hand, there is no way in which the equations of motion can be solved numerically with a driving term arising from the noise with very high frequency components—equations with such a driving term are simply too singular to be successfully integrated numerically.

The solution is to divide the noise into high and low frequency parts, and to perform a stochastic average over only the high frequency portion of the bath, using a cumulant expansion technique, but to treat the low frequency part numerically. This yields an equation with the following structure:

i) Standard master equation terms for the higher states of the system (together with frequency renormalisation terms corresponding to the Lamb shift terms of quantum optics.)

ii) Terms describing the lower frequency transitions which take place between the two lowest levels, driven by the low frequency part of the noise. The magnitude

and nature of this low frequency noise term depends on the particular frequency cutoff used to define the boundary between the high and low frequency parts of the bath. However, the transition frequencies are renormalised also, in a way which we eventually find compensates for this dependence on the arbitrary cutoff.

iii) Finally, there are terms which describe the interaction between the high frequency and low frequency transitions.

With the assumption that the upper levels are not appreciably populated, one can derive a closed set of three equations describing the dynamics of the two lowest levels.

3.7.2   Reduction to a Two Level Problem

We use the adjoint equation in the form (3.5.36), slightly renotated

$$\dot{\mu}(t) = -\frac{i}{\hbar}[H_{\text{sys}}, \mu(t)] + \frac{i}{2\hbar}[\ [\Gamma - \xi(t), \mu(t)]_+, X \ ] \tag{3.7.1}$$

where

$$\Gamma = \frac{1}{4c} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} dx dy \tilde{\kappa}(x) \tilde{\kappa}(y) \dot{X}(-|x-y|/c) \tag{3.7.2}$$

and in which

$$H_{\text{sys}} = \frac{p^2}{2m} + V(x). \tag{3.7.3}$$

We consider a quantum system in which the double well potential, $V(x)$, assumed symmetric for simplicity, yields two discrete low lying states $|1\rangle$ and $|2\rangle$, separated by a tunnel splitting Δ_0, which are together well separated in energy from the remaining higher states. The situation is illustrated in Fig.3.3. If V_0 is the height of the potential barrier separating the two wells, and ω_0 is the typical oscillation frequency in one of the wells, then in the limit $V_0 \gg \hbar \omega_0$, Δ_0 may be calculated using standard WKB techniques which give

$$\Delta_0 \simeq B\omega_0 \exp(-CV_0/\hbar \omega_0) \tag{3.7.4}$$

where B and C are constants which depend upon the shape of the potential. Hence, in this limit, $\Delta_0 \ll \omega_0$.

We expand the operator X into eigenoperators of the system Hamiltonian as follows

$$X = q_{23}(\mathcal{A} + \mathcal{A}^\dagger) + q_{12}\left(|2\rangle\langle1| + |1\rangle\langle2|\right), \tag{3.7.5}$$

where

$$[H_{\text{sys}}, |1\rangle\langle2| \ ] = -\hbar \Delta_0 |1\rangle\langle2|. \tag{3.7.6}$$

The quantities $\pm q_{12}$ give the positions of the minima of the potential, and, for the case of macroscopic quantum coherence, would have to be of such a magnitude as to allow a description in terms of macroscopically distinguishable states. Meanwhile A and $A^{\dagger}$ are operators that mediate transitions between the higher energy states, with

$$A|1\rangle = A^{\dagger}|1\rangle = A|2\rangle = 0. \tag{3.7.7}$$

a) High and Low Frequency Behaviour: If we now consider the bath to which the system is coupled, it is appropriate to consider separately the contributions from the high frequency and low frequency oscillators. For this purpose, we write the noise $\alpha(t)$ as a sum of two *independent* noises β_{hf} and β_{lf}, which correspond respectively to the high and low frequency portion of the bath. That is, we write

$$\alpha(t) \equiv \beta_{hf}(t) + \beta_{lf}(t) \tag{3.7.8}$$

where

$$\langle \beta_{hf}(t)\beta_{hf}(t')\rangle = \frac{\hbar}{2} \int_0^\infty d\omega \kappa_{hf}(\omega)^2 \omega \coth\left(\frac{\hbar\omega}{2kT}\right) \cos[\omega(t-t')], \quad (3.7.9)$$

and

$$\langle \beta_{lf}(t)\beta_{lf}(t')\rangle = \frac{\hbar}{2} \int_0^\infty d\omega \kappa_{lf}(\omega)^2 \omega \coth\left(\frac{\hbar\omega}{2kT}\right) \cos[\omega(t-t')]. \quad (3.7.10)$$

At this stage we shall allow coupling functions $\kappa_{hf}(\omega)$ and $\kappa_{lf}(\omega)$ to remain quite general. However, we note that the following properties will hold,

$$\kappa_{hf}(\Delta_0)^2 \simeq 0 \quad , \quad \kappa_{lf}(\omega_0)^2 \simeq 0. \tag{3.7.11}$$

It is not obvious that such a splitting into two independent noises can always be done. However, the situation being treated here is that of *thermal statistics*. It will be shown in Sect. 4.4.5 that the statistics of thermal quantum noise are quantum Gaussian, which means that $\alpha(t)$ is itself a Gaussian stochastic process. Since Gaussian processes are specified by the first and second order correlations, it is sufficient to show that the correlation function $\langle \alpha(t)\alpha(t')\rangle$ is able to be correctly represented by this procedure.

b) Average over High Frequency Noises: Let us perform an average over the high frequency portion of the bath (i.e., a stochastic average over the noise $\beta_{hf}(t)$) in the adjoint equation. Such an average is facilitated by the cumulant expansion technique of van Kampen, Sect. 3.6.3, which we have employed previously in the derivation of quantum optical and quantum Brownian motion master equations. This procedure gives an equation of motion, correct to second order in a smallness parameter proportional to the effective correlation time of the high frequency part of the noise, which is of the order of magnitude of $1/\omega_c$, and is thus very small compared with the timescale of interest. This equation takes the form

$$\frac{d}{dt}\langle \mu(t)\rangle_{hf} = [A_0 + \beta_{lf}(t)A_1]\langle \mu(t)\rangle_{hf}$$

$$+ \int_0^\infty d\tau \langle \beta_{hf}(t)A_1 \exp(A_0\tau)\beta_{hf}(t-\tau)A_1 \exp(-A_0\tau)\rangle \langle \mu(t)\rangle_{hf}$$

$$\tag{3.7.12}$$

where $\langle \ \rangle_{hf}$ denotes the average over the high frequency portion of the bath. Since A_0 and A_1 involve only operators in the system space, β_{hf} is statistically independent of, and commutes with, A_0 and A_1. To evaluate the integral appearing in (3.7.12), it is standard in quantum optical problems to make the approximation, corresponding to (3.7.12),

$$\exp(A_0\tau)\, X \exp(-A_0\tau) \simeq X(-\tau)$$
$$= \exp(-i H_{sys}\tau/\hbar)\, X \exp(i H_{sys}\tau/\hbar). \tag{3.7.13}$$

since damping is generally weak compared with the systematic motion. For the quantum coherence problem we are considering here, this may not necessarily be so. Nevertheless, we shall make this approximation, since it may also be the case that A_0 is very small, in which case $\exp(A_0 t)$ does not vary significantly over the effective correlation time of $\beta_{hf}(t)$. It is also appropriate to make a rotating wave approximation in the high frequencies.

If the temperature is very low, one can neglect any population in the higher energy levels, and derive a closed equation for the density matrix elements in the lowest two levels—see [3.21] for the detailed derivation. Using the notation

$$S_x = \langle 2|\bar{\mu}|1\rangle + \langle 1|\bar{\mu}|2\rangle$$
$$S_y = i\left(\langle 2|\bar{\mu}|1\rangle - \langle 1|\bar{\mu}|2\rangle\right) \tag{3.7.14}$$
$$S_z = \langle 2|\bar{\mu}|2\rangle - \langle 1|\bar{\mu}|1\rangle,$$

and, for convenience, redefining a rescaled $\beta_{lf}(t)$ by

$$\beta_{lf}(t) \equiv \frac{q_{12}}{\hbar}[\,\xi_{lf}(t),\ ]_+, \tag{3.7.15}$$

we have

$$\dot{S}_x = -\Delta_y S_y$$
$$\dot{S}_y = \Delta_x S_x + \beta_{lf}(t) S_z \tag{3.7.16}$$
$$\dot{S}_z = -\frac{\kappa(\Delta_0)^2 q_{12}^2}{\hbar}\pi\Delta_0 - \beta_{lf}(t) S_y,$$

where Δ_x and Δ_y are given by

$$\Delta_x = \Delta_0 - \frac{q_{12}^2}{\hbar}P\int_0^\infty d\omega\,\kappa_{hf}(\omega)^2\omega\left(\frac{1}{\omega-\Delta_0}-\frac{1}{\omega+\Delta_0}\right)$$
$$+\frac{q_{23}^2}{4\hbar}\left[P\int_0^\infty d\omega\,\kappa_{hf}(\omega)^2\omega\left(\frac{1}{\omega-\omega_0}-\frac{1}{\omega+\omega_0}\right)\right.$$
$$\left.-\int_0^\infty d\omega\,\kappa(\omega)^2\omega_0\left(\frac{1}{\omega-\omega_0}-\frac{1}{\omega+\omega_0}\right)\right] \tag{3.7.17}$$

$$\Delta_y = \Delta_0 + \frac{q_{23}^2}{4\hbar}\left[P\int_0^\infty d\omega\,\kappa_{hf}(\omega)^2\omega\left(\frac{1}{\omega-\omega_0}-\frac{1}{\omega+\omega_0}\right)\right.$$
$$\left.-\int_0^\infty d\omega\,\kappa(\omega)^2\omega_0\left(\frac{1}{\omega-\omega_0}-\frac{1}{\omega+\omega_0}\right)\right]. \tag{3.7.18}$$

These equations are of essentially the same form as would be obtained in the adjoint equation equivalent to the quantum Langevin equation of Sect. 3.4.4, but with the major difference that the frequencies and are renormalised as in (3.7.17,18), and the noise includes only low frequency components.

c) Higher Order Terms in the Cumulant Expansion: In fact, the equations (3.7.16) are not quite in their final form. Parkins and Gardiner show that higher order terms in the cumulant expansion lead to a correction of the inhomogeneous term in the equation for $\dot{S}_z$ in particular. This requires us to make the replacement

$$\frac{\kappa(\Delta_0)^2 q_{12}^2}{\hbar}\pi\Delta_0 \;\rightarrow\; \frac{\kappa(\Delta_0)^2 q_{12}^2}{\hbar}\pi\Delta_0$$
$$\times \left\{ 1 - \frac{q_{12}^2}{\hbar}P\int_0^\infty d\omega\,\kappa_{\mathrm{hf}}(\omega)^2\omega\left[\frac{4\omega\Delta_0}{(\omega^2-\Delta_0^2)^2}\right]\right\}.$$

$$(3.7.19)$$

The inclusion of this correction is consistent with the order of the frequency correction terms already appearing in (3.7.16).

d) Specification of Cutoffs: To proceed further, one must specify the functions $\kappa(\omega)$, $\kappa_{\mathrm{lf}}(\omega)$, and $\kappa_{\mathrm{hf}}(\omega)$ appearing above. We will assume smooth cutoff behaviour of the form

$$\kappa(\omega)^2 = \kappa^2 e^{-\omega/\omega_b}, \tag{3.7.20}$$

$$\kappa_{\mathrm{lf}}(\omega)^2 = \kappa^2 e^{-\omega/\omega_c}, \tag{3.7.21}$$

and hence

$$\kappa_{\mathrm{hf}}(\omega)^2 = \kappa^2\left(e^{-\omega/\omega_b} - e^{-\omega/\omega_c}\right), \tag{3.7.22}$$

with $\omega_b \gg \omega_c$. The frequency ω_c is a parameter characterising the division between the high and low frequency portions of the bath.

Note that if the method is valid, the final results should be largely independent of ω_c, and of the form of cutoff behaviour. While ω_c is an artificial frequency, chosen for our convenience, the upper cutoff frequency ω_b reflects the finite size of the system, and gives a measure of the highest frequency response of the bath. Hence, ω_b represents a real frequency, characteristic of the particular system we are considering.

With the forms (3.7.20–22), the frequencies Δ_x and Δ_y can be evaluated as

$$\Delta_x = \Delta_0 - \Delta_0\frac{q_{12}^2\kappa^2}{\hbar}\ln\left(\frac{\omega_b^2}{\omega_c^2}\right) - \omega_0\frac{q_{23}^2\kappa^2}{2\hbar}e^{\omega_0/\omega_b}\mathrm{Ei}\left(-\omega_0/\omega_b\right) \tag{3.7.23}$$

and

$$\Delta_y = \Delta_0 - \omega_0\frac{q_{23}^2\kappa^2}{2\hbar}e^{\omega_0/\omega_b}\mathrm{Ei}\left(-\omega_0/\omega_b\right). \tag{3.7.24}$$

Here Ei (x) is the exponential integral as defined in [3.22]. The expression (3.7.19) for the inhomogeneous term becomes

$$-\pi \frac{q_{12}^2 \kappa^2}{\hbar} \Delta_0 \left\{ 1 - \frac{q_{12}^2 \kappa^2}{\hbar} \ln \left(\frac{\omega_b^2}{\omega_c^2} \right) \right\} = -\pi \frac{q_{12}^2 \kappa^2}{\hbar} \Delta_z. \tag{3.7.25}$$

The problem is now contained entirely in the closed set of stochastic differential equations (3.7.16), with Δ_x and Δ_y given by (3.7.23–24), where the random noise $\beta_{lf}(t)$ is Gaussian with zero mean and correlation function

$$\langle \beta_{lf}(t)\beta_{lf}(t') \rangle = \frac{2q_{12}^2}{\hbar^2} \langle [\xi_{lf}(t), \xi_{lf}(t')]_+ \rangle$$

$$= \frac{2q_{12}^2 \kappa^2}{\hbar} \int_0^\infty d\omega \, \omega e^{-\omega/\omega_c} \coth \left(\frac{\hbar \omega}{2kT} \right) \cos[\omega(t-t')].$$

$$\tag{3.7.26}$$

The quantity of most interest is, of course, $\langle S_x(t) \rangle$, where now $\langle \ \rangle$ denotes the average over $\beta_{lf}(t)$. In the undamped limit, $\langle S_x(t) \rangle$ displays coherent oscillations between -1 and $+1$, that is, between the left and right wells.

3.7.3   Solution by Stochastic Simulation

We are faced here by a rather complicated noise spectrum, which, from (3.7.26), can be evaluated explicitly as

$$\langle \beta_{lf}(t)\beta_{lf}(0) \rangle = 2\alpha \left\{ \frac{\omega_c^2(1-\omega_c^2 t^2)}{(1+\omega_c^2 t^2)^2} + \frac{1}{t^2} - \left(\frac{\pi kT}{\hbar} \right)^2 \operatorname{cosech}^2 \left(\frac{\pi kTt}{\hbar} \right) \right\}. \tag{3.7.27}$$

As discussed in Sect. 3.3.3, this correlation function possesses a long time tail, whose precise form is temperature dependent. At zero temperature, the long time tail has a t^{-2} dependence, i.e., for large t

$$\langle \beta_{lf}(t)\beta_{lf}(0) \rangle \rightarrow -\frac{2\alpha}{t^2}, \tag{3.7.28}$$

while for finite temperatures, we find an exponential decay, i.e.,

$$\langle \beta_{lf}(t)\beta_{lf}(0) \rangle \rightarrow -2\alpha \left(\frac{2\pi kT}{\hbar} \right)^2 \exp \left(-\frac{2\pi kTt}{\hbar} \right), \tag{3.7.29}$$

with a characteristic thermal correlation time $\hbar/2\pi kT$. For intermediate times and at finite temperatures the correlation function exhibits a more complicated algebraic decay [3.23].

In spite of this complicated behaviour, we are, by noting that (3.7.25) is merely a Fourier transform, able to formulate a means of producing a sequence of random numbers with the appropriate correlation properties. In particular, we may write

$$\beta_{lf}(t) = \int_{-\infty}^\infty d\omega \, g(\omega) \sqrt{S(\omega)} e^{i\omega t}, \tag{3.7.30}$$

where

$$S(\omega) = \alpha\,\omega\exp(-\omega/\omega_c)\coth\left(\frac{\hbar\,\omega}{2\,kT}\right),\tag{3.7.31}$$

and $g(\omega)$ is a random function, with the properties

$$\langle g(\omega)\rangle = 0 \quad,\quad \langle g(\omega)g(\omega')\rangle = \delta(\omega - \omega').\tag{3.7.32}$$

The form (3.7.30) can de discretised, with $g(\omega)$ becoming a sequence of Gaussian random numbers $\{g_i\}$ with zero mean and unit variance ($\langle g_i g_j\rangle = \delta_{ij}$). The resulting discrete summation is then computed using a fast Fourier transform routine, yielding a sequence of random numbers $\{\beta_{\mathrm{lf}}^n\}$ which may be used in simulations of the equations (3.7.16).
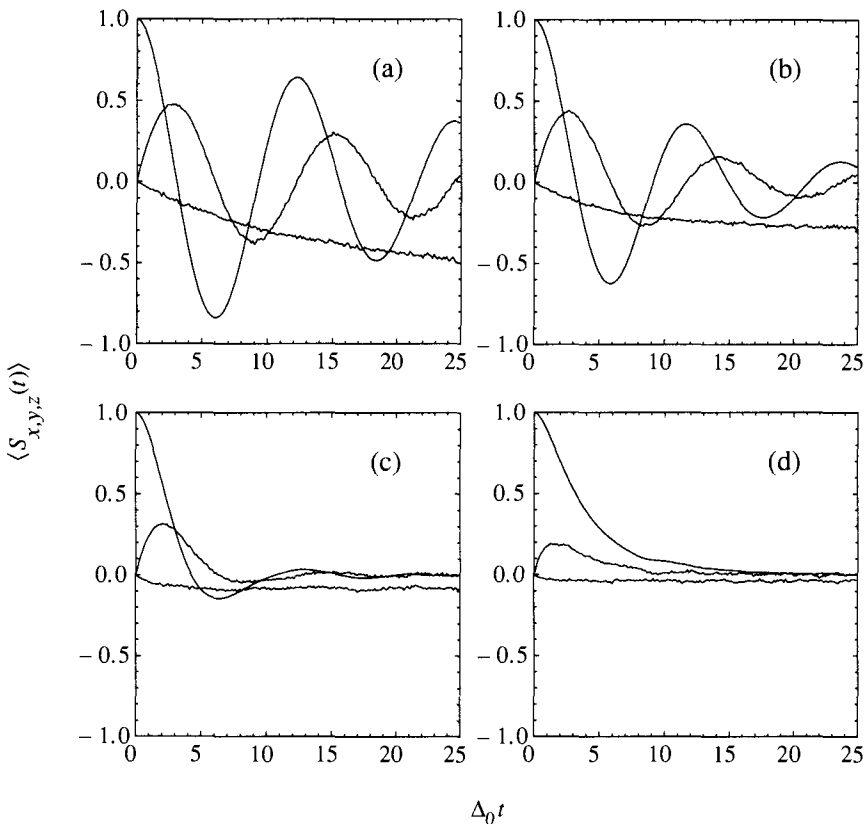
Fig. 3.4(a–d)   Averages of $S_{x,y,z}(t)$ with $\alpha = 0.05$, and values of the absolute temperature given by: (a) $\hbar\,\Delta_0/kT = \infty$, (b) $\hbar\,\Delta_0/kT = 2$, (c) $\hbar\,\Delta_0/kT = 0.5$, (d) $\hbar\,\Delta_0/kT = 0.2$, computed from simulation

a) Numerical Integration: Simulation of the equations (3.7.16) is carried out using the fully implicit numerical integration scheme used by *Smith* and *Gardiner* [3.24]. It is necesary to use a routine with good stability, for otherwise the solutions become unbounded in a surprisingly short time.

b) Numerical Results: In our simulations, we have typically computed the averages from 4000 trials. The results are, within limits, largely independent of the choice of ω_c/Δ_0. For $\alpha = 0.05$, we find only a small shift in frequency as ω_c/Δ_0 is changed by an order of magnitude. The effect becomes smaller with decreasing α. We note, from (3.7.27), that this change in ω_c/Δ_0 corresponds to a change in the variance of the noise, $\langle \beta_{\mathrm{lf}}^2 \rangle/\Delta_0^2$, by a factor of 100.

In Fig.3.4 we display a sequence of results obtained with with $\alpha = 0.05$, for varying temperatures of the bath. We choose $\omega_b/\Delta_0 = 10^5$.

The influence of temperature on coherent oscillations is clearly exhibited in each case. With increasing temperature, the oscillations are gradually destroyed, ultimately giving way to incoherent relaxation at a temperature given approximately by $\hbar \Delta_r/k\pi\alpha$.

4. Phase Space Methods

Ever since the early days of Planck the harmonic oscillator has played a central rôle in quantum mechanics. This rôle arises from the fact that in practice very many forces are nearly harmonic as well as from the all pervading nature of electromagnetic fields, which, like all Bose fields, are exactly equivalent to assemblies of harmonic oscillators.

The harmonic oscillator has an infinite number of equally spaced energy levels, and thus represents one extreme case of systems with discrete energy levels. Systems with a finite number of energy levels do not exist, but in many situations it is possible to consider that interesting processes involve only a few energy levels of some system, and in these situations it is advantageous to consider an idealised Hamiltonian whose full range of energy levels comprises only those of interest. The most idealised extreme of this is known as the *two level atom*, in which we consider processes involving only two energy levels. Such an idealisation is relevant to the theory of atomic decays, which involve only an upper level and a lower level, as well as the electromagnetic field to which the atom is coupled.

From the realm of quantum optics there have been developed a very rich profusion of techniques for dealing with both of these kinds of system—such methods are particularly necessary when dealing with systems composed of assemblies of either of the elementary harmonic oscillator or two level atom systems, for calculations involving such complexity can very rapidly become unwieldy without some clever techniques. Much of the thrust of these techniques lies in their ability to exploit classical analogues—most particularly analogues with classical noise theory. Using these techniques, such as the P-representation of *Glauber* and *Sudarshan* [4.1] and the *Wigner* representation [4.2], purely harmonic systems can be reduced to non-operator systems. However the essentially quantum mechanical nature of the problem is present in terms of the *interpretation* of the apparently classical variables.

This chapter first develops these phase space techniques for harmonic oscillators. The phase space techniques are not actually applicable to their fullest advantage until quantum noise systems are dealt with in the next and subsequent chapters, but even in the case of non-noisy systems they bring a clarity to the problem of the transition from the quantum world to the classical world. The central idea which unifies all the harmonic oscillator techniques is the *coherent state*, first introduced by *Glauber* [4.1], which is the quantum state which most closely approaches the classical description of harmonic physics.

4.1   The Harmonic Oscillator in One Variable

The observation by Galileo that the period of a pendulum is independent of the amplitude of its motion introduced the two most important features of the harmonic oscillator into physics: namely, the amplitude independent period itself, and equally significantly, the fact that many physical systems, (such as a pendulum) are for small amplitudes good approximations to the harmonic oscillator.

The Hamiltonian for a particle moving in a potential $V(x)$ is given in both classical and quantum mechanics by

$$H = \frac{p^2}{2m} + V(x) \tag{4.1.1}$$

and here

$$
\begin{aligned}
p \quad &= \text{momentum of particle} \\
x \quad &= \text{position of the particle} \\
m \quad &= \text{mass of the particle} \\
V(x) &= \tfrac{1}{2}kx^2 \text{, for the harmonic oscillator.}
\end{aligned}
\tag{4.1.2}
$$

But there are other realisations of the harmonic oscillator; e.g.,the electrical example of the LC circuit, in which

$$
\begin{aligned}
x \quad &= Q = \text{charge on a capacitor } C \\
m \quad &= L = \text{inductance} \\
p \quad &= \text{"magnetic momentum"} = L\dot{Q} \\
V(x) &= Q^2/2C.
\end{aligned}
\tag{4.1.3}
$$

As written, the electric circuit is an exact realisation of a harmonic oscillator, unlike the pendulum, and is the prototype of another large class of representations; namely a single mode of a free electromagnetic field behaves like a harmonic oscillator as will be shown in Sect. 7.1

We will use the language appropriate to the Hamiltonian (4.1.1) in most of our discussion, but it should always be borne in mind that we are talking about a general harmonic oscillator, which may be realised in many other ways.

4.1.1   Equations of Motion—Classical

Hamilton's equations of motion yield the well known equations

$$
\begin{aligned}
m\dot{x} &= p \\
\dot{p} \quad &= -kx
\end{aligned}
\tag{4.1.4}
$$

with the solutions

$$
\begin{aligned}
x(t) &= x(0)\cos\omega t \\
\omega \quad &= \sqrt{k/m}\,.
\end{aligned}
\tag{4.1.5}
$$

The quantity ω is of course the angular frequency of the oscillator. The motion of the particle is bounded as a consequence of the boundedness of the sine and cosine functions, or as a consequence of conservation of energy.

In the limit $\omega \rightarrow 0$ this is no longer the case, since

$$x(t) \rightarrow x(0) + \frac{p(0)}{m}t \tag{4.1.6}$$

unless m is infinite. This qualitative difference is very important in applications, since it is very frequently assumed that the amplitude of the oscillation is bounded and quite small—such approximations cannot be made for $\omega \rightarrow 0$, where (4.1.6) corresponds to the motion of a free particle.

4.1.2   Equations of Motion—Quantum

The canonical commutation relations

$$[x, p] = i\hbar \tag{4.1.7}$$

form the basis of the quantisation of the harmonic oscillator. The oscillator may be treated in either the Schrödinger picture, in which time independent operators act on time dependent quantum states $|\psi, t\rangle$ which obey the Schrödinger equation

$$H|\psi, t\rangle = i\hbar \frac{\partial}{\partial t}|\psi, t\rangle, \tag{4.1.8}$$

or in the Heisenberg picture, in which time dependent operators act on time independent states; the operators $a(t)$ obey the Heisenberg equations of motion

$$[H, a(t)] = -i\hbar \frac{da(t)}{dt}. \tag{4.1.9}$$

The two pictures are related by

Schrödinger   *Heisenberg*

$$|\psi, t\rangle \qquad\qquad\qquad = \exp(-iHt/\hbar)|\psi\rangle \tag{4.1.10}$$

$$\exp(iHt/\hbar)a\exp(-iHt/\hbar) = a(t)$$

In our treatment of quantum noise, the Schrödinger picture will usually be the most convenient, but nevertheless the Heisenberg picture will play an important role, and will often give more easily interpretable equations. As a point of notation, we shall usually indicate which picture is being used by the explicit functional dependence on time t. Thus $|\psi\rangle$ is a Heisenberg state, and a is a Schrödinger operator, while $a(t)$ is a Heisenberg operator. However the notation will not be turned into a formal pedantry; where little confusion can arise, and brevity makes it advantageous, t may not be written explicitly.

4.1.3   The Schrödinger Picture: Energy Eigenvalues and Number States

The harmonic oscillator is best solved in the Schrödinger picture. We introduce the destruction operator a and its Hermitian conjugate $a^\dagger$ by

$$
\begin{aligned}
a &= \frac{\mathrm{i}p}{\sqrt{2\hbar\omega m}} + \sqrt{\frac{k}{2\hbar\omega}}\, x \\
a^\dagger &= \frac{-\mathrm{i}p}{\sqrt{2\hbar\omega m}} + \sqrt{\frac{k}{2\hbar\omega}}\, x.
\end{aligned}
\tag{4.1.11}
$$

In terms of these operators

$$
\begin{aligned}
H &= \hbar\omega(a^\dagger a + \tfrac{1}{2}) \\
[a, a^\dagger] &= 1.
\end{aligned}
\tag{4.1.12}
$$

The solution of the Schrödinger equation is achieved if the eigenvalues and eigenvectors of $a^\dagger a$ can be determined. This is done straightforwardly. One first notes that

$$
\begin{aligned}
[H, a] &= -\hbar\omega a \\
[H, a^\dagger] &= \hbar\omega a^\dagger.
\end{aligned}
\tag{4.1.13}
$$

From these we see that if

$$
H|E\rangle \quad = E|E\rangle
\tag{4.1.14}
$$

then

$$
\begin{aligned}
Ha^\dagger|E\rangle &= (E + \hbar\omega)a^\dagger|E\rangle \\
Ha|E\rangle &= (E - \hbar\omega)a|E\rangle.
\end{aligned}
\tag{4.1.15}
\tag{4.1.16}
$$

If follows that there is a whole sequence of eigenvalues of the form $E + n\hbar\omega$, for any integral n either positive or negative. Because H is a positive operator it cannot have any negative eigenvalues. But if E_0 is the least non-negative eigenvalue, then the only way to terminate the decreasing sequence defined by (4.1.16) is to require

$$
a|E_0\rangle = 0.
\tag{4.1.17}
$$

From this and (4.1.12) it follows that $E_0 = \tfrac{1}{2}\hbar\omega$, and that the eigenvalues of H are

$$
E_n = (n + \tfrac{1}{2})\hbar\omega \qquad (n = 0, 1, 2, \cdots),
\tag{4.1.18}
$$

and we will denote the corresponding eigenstates by $|n\rangle$. The general solution of the Schrödinger equation is thus of the form

$$
|\psi, t\rangle = \sum_n \exp\left[\mathrm{i}(n + \tfrac{1}{2})\omega t\right] A_n |n\rangle
\tag{4.1.19}
$$

where the A_n are certain coefficients and the states $|n\rangle$ are normalised and orthogonal

$$
\langle m|n\rangle = \delta_{mn}.
\tag{4.1.20}
$$

From the definition of H in terms of $a^\dagger a$, (4.1.12), it is clear that

$$a^\dagger a |n\rangle = n|n\rangle, \qquad (4.1.21)$$

from which we note that

$$\langle n|a^\dagger a|n\rangle = n. \qquad (4.1.22)$$

It follows from (4.1.16) that $a|n\rangle$ is proportional to $|n-1\rangle$: hence we deduce that (up to an arbitrary phase), we can set

$$\begin{aligned}
a|n\rangle &= \sqrt{n}\ |n-1\rangle \\
a^\dagger|n\rangle &= \sqrt{n+1}\ |n+1\rangle
\end{aligned} \qquad (4.1.23)$$

with the resulting expression for the number state

$$|n\rangle = \frac{(a^\dagger)^n}{\sqrt{n!}}|0\rangle. \qquad (4.1.24)$$

The number n is conventionally regarded as the number of quanta in the particular state, each with energy $\hbar\omega$, and hence the states $|n\rangle$ are called "number states".

a) Wavefunctions for the Harmonic Oscillator in Position Space: If the eigenstates of the position operator x are written $|x\rangle$, then the wavefunction is

$$\psi(x,t) = \langle x|\psi,t\rangle \qquad (4.1.25)$$

and the corresponding operators become

$$\begin{aligned}
x|\psi,t\rangle &\rightarrow x\psi(x,t) \\
p|\psi,t\rangle &\rightarrow -i\hbar\frac{d}{dx}\psi(x,t).
\end{aligned} \qquad (4.1.26)$$

We define

$$\psi_n(x) = \langle x|n\rangle \qquad (4.1.27)$$

and the lowest energy wavefunction is determined from

$$a|0\rangle = 0 \Rightarrow \frac{d\psi_0(x)}{dx} = -\eta x\psi_0(x) \qquad (4.1.28)$$

where

$$\eta = \sqrt{km}/\hbar. \qquad (4.1.29)$$

The solution of (4.1.28), normalised to $\int\limits_{-\infty}^{\infty} dx|\psi_0(x)|^2 = 1$, is

$$\psi_0(x) = \left(\frac{\eta}{\pi}\right)^{\frac{1}{4}}\exp(-\eta x^2/2). \qquad (4.1.30)$$

The $\psi_n(x)$ are given by using (4.1.24) which from (4.1.26) implies

$$\psi_n(x) = (n!)^{-n/2} \left(\frac{d}{dx} - \eta x \right)^n \psi_0(x) \tag{4.1.31}$$

and by using Roderigues' formula for Hermite polynomials [4.3] we find

$$
\begin{aligned}
\psi_n(x) &= (2^n n!)^{-\frac{1}{2}} \left(\frac{\eta}{\pi} \right)^{\frac{1}{4}} \exp(-\tfrac{1}{2}\eta x^2) H_n(x\sqrt{\eta}) \\
\eta &= \sqrt{km}/\hbar
\end{aligned} \tag{4.1.32}
$$

where $H_n(x)$ is the Hermite polynomial.

b) Wavefunctions for the Harmonic Oscillator in Momentum Space: Writing the eigenfunctions of momentum as $|p\rangle$, the momentum space wavefunction is

$$\psi(p,t) = \langle p|\psi,t\rangle \tag{4.1.33}$$

and the correspondence is

$$
\begin{aligned}
x|\psi,t\rangle &\to i\hbar \frac{d}{dp}\psi(p,t) \\
p|\psi,t\rangle &\to p\psi(p,t).
\end{aligned} \tag{4.1.34}
$$

Similar reasoning to the above leads to

$$\langle p|n\rangle = \psi_n(p) = (2^n n!)^{\frac{1}{2}} \left(\frac{\hbar^2}{\pi\eta} \right)^{\frac{1}{4}} \exp(-\hbar^2 p^2/2\eta) H_n\left(\hbar p/\sqrt{\eta}\right). \tag{4.1.35}$$

4.1.4   The Heisenberg Picture

In the Heisenberg picture, the equations of motion are exactly the same as the classical equations of motion, namely

$$
\begin{aligned}
\dot{x}(t) &= p(t)/m \\
\dot{p}(t) &= -kx(t)
\end{aligned} \tag{4.1.36}
$$

and their solutions are of course

$$x(t) = x(0)\cos\omega t + \frac{p(0)}{\omega m}\sin\omega t \tag{4.1.37}$$

$$p(t) = -\omega m x(0)\sin\omega t + p(0)\cos\omega t.$$

If we know the quantum state of the system, these equations provide us with the solution for all physical quantities. In particular, if the state is $|\psi\rangle$, and we use the notation for any operator A

$$\langle A \rangle = \langle \psi | A | \psi \rangle, \tag{4.1.38}$$

then $\langle x(t) \rangle$ and $\langle p(t) \rangle$ are given in terms of initial values of these quantities by (4.1.37). Simple substitution gives the solutions for the Heisenberg picture creation and destruction operators (4.1.11) as

$$\begin{aligned} a(t) &= e^{-i\omega t} a(0) \\ a^\dagger(t) &= e^{i\omega t} a^\dagger(0). \end{aligned} \tag{4.1.39}$$

4.2   Coherent States and The Classical Limit

It is well known that the limit $\hbar \rightarrow 0$ corresponds to the approach of quantum mechanics to classical mechanics. The energy eigenfunctions have an asymptotic form in this case that is most easily determined by the WKB method of solution of the Schrödinger equation. For the harmonic oscillator, the eigenfunctions are asymptotically

$$\psi_n(x) \simeq \frac{1}{\sqrt{\pi}} \left[\frac{k}{2mE - kx^2} \right]^{\frac{1}{4}} \frac{\cos}{\sin} \left(\int \frac{\sqrt{2mE - kx^2}\, dx}{\hbar} \right) \tag{4.2.1}$$

where

$$E = (n + \tfrac{1}{2})\hbar\omega \tag{4.2.2}$$

and the choice of cos or sin corresponds to n even or odd.

This is valid in the region in which the particle is classically allowed to exist, i.e., in the region where $|x| < \sqrt{2mE/k}$ and the kinetic energy is consequently positive. Outside that region the wavefunction vanishes faster than any power of $\hbar$.

The average probability distribution over the fast sinusoidal oscillations is obtained by squaring $\psi(x)$, and setting the $\cos^2$ or $\sin^2$ terms equal to $\frac{1}{2}$: it is

$$P(x) \simeq \frac{1}{2\pi} \left[\frac{k}{2mE - kx^2} \right]^{\frac{1}{2}}. \tag{4.2.3}$$

This is recognisable as the classical distribution, in that the probability density to be at position x is inversely proportional to the speed at which the particle moves there.

A similar result obviously holds for the momentum space wavefunction. Both these results have the defect that is obvious from (4.2.3), namely momentum and

position are not specified comparably, and neither approximates the classical situation in which momentum and position oscillate—we get simply averages over all phases of the classical motion.

To investigate what kind of states might correspond more closely with classical states, we must consider how one might construct states which specify x, p, and E with as little uncertainty as possible in all of them simultaneously. To this end we must investigate what limits are placed on such uncertainties in general, and this is best done via the Heisenberg picture.

4.2.1   Coherent States as Quasi-Classical States

The eigenfunctions of energy do not give states which approach the classical motion in the limit as $\hbar \rightarrow 0$, as we have just seen in (4.2.1). What one would like is a set of quantum states in which the uncertainties in $x(t)$ and $p(t)$ are related to their initial values in the same way as the classical values are, and in which the uncertainty in the variables is in some way distributed between the position and the momentum to much the same extent.

Suppose we define

$$
\begin{aligned}
A(t) &= a(t) - \langle a(t) \rangle \\
A^\dagger(t) &= a^\dagger(t) - \langle a^\dagger(t) \rangle
\end{aligned}
\tag{4.2.4}
$$

then these operators have the same commutation relations as $a(t)$ and $a^\dagger(t)$. Using (4.1.37), we see that the variances of $x(t)$ and $p(t)$, and the correlation between $x(t)$ and $p(t)$ are given by

$$
\begin{aligned}
&\langle \Delta x(t)^2 \rangle \\
&= \frac{\hbar}{2\sqrt{km}} \{ \langle A(0)^2 \rangle e^{-2i\omega t} + \langle A^\dagger(0)^2 \rangle e^{2i\omega t} + \langle A^\dagger(0)A(0) + A(0)A^\dagger(0) \rangle \}
\end{aligned}
\tag{4.2.5}
$$

$$
\begin{aligned}
&\langle \Delta p(t)^2 \rangle \\
&= -\frac{\hbar\sqrt{km}}{2} \{ \langle A(0)^2 \rangle e^{-2i\omega t} + \langle A^\dagger(0)^2 \rangle e^{2i\omega t} - \langle A^\dagger(0)A(0) + A(0)A^\dagger(0) \rangle \}
\end{aligned}
\tag{4.2.6}
$$

$$
\begin{aligned}
&\langle \tfrac{1}{2} \{ \Delta p(t) \Delta x(t) + \Delta x(t) \Delta p(t) \} \rangle \\
&= -\frac{i\hbar}{2} \{ \langle A(0)^2 \rangle e^{-2i\omega t} - \langle A^\dagger(0)^2 \rangle e^{2i\omega t} \}.
\end{aligned}
\tag{4.2.7}
$$

These variances will be time independent if (and only if) both $\langle A(0)^2 \rangle$ and $\langle A(0)^{\dagger 2} \rangle$ are zero. This corresponds to what we would like to see—the system moving in a wavepacket with well defined *constant* uncertainties in the canonical variables.

The minimum uncertainty arises by minimising $\langle A^\dagger(0)A(0) + A(0)A^\dagger(0) \rangle$. But we have just noted that $A(t)$ and $A^\dagger(t)$ satisfy the same commutation relations as a and $a^\dagger$, so that the eigenvalues of $A^\dagger(0)A(0)$ will be the same as those of $a^\dagger a$,

since the derivation in Sect. 4.1.3 depends only on the commutation relations. Thus the eigenvalues of $A^\dagger(0)A(0) + A(0)A^\dagger(0)$ are of the form $m + \frac{1}{2}$, where m is a non-negative integer, and clearly the minimum mean is obtained in the state $m = 0$. We call this state $|\alpha\rangle$, and in analogy to (4.1.17), $|\alpha\rangle$ satisfies

$$A(0)|\alpha\rangle = 0. \tag{4.2.8}$$

Furthermore, we define the quantity α by

$$\alpha = \langle a\rangle. \tag{4.2.9}$$

Thus, the minimum uncertainty state is $|\alpha\rangle$, which satisfies

$$\boxed{a|\alpha\rangle = \alpha|\alpha\rangle, \tag{4.2.10}}$$

and this is known as a *coherent state*. It was first introduced by *Glauber* [4.1].

This equation can be considered as an alternative definition of a coherent state; namely, the coherent state $|\alpha\rangle$ is the eigenvector of the destruction operator. Since the destruction operator is not Hermitian, the coherent states need not be (and are not) orthogonal; nor is the eigenvalue real.

From (4.2.5–7) we see that

$$\langle\alpha|\Delta x(t)^2|\alpha\rangle = \hbar/2\sqrt{km}$$
$$\langle\alpha|\Delta p(t)^2|\alpha\rangle = \hbar\sqrt{km}/2 \tag{4.2.11}$$
$$\langle\alpha|\{\Delta p(t)\Delta x(t) + \Delta x(t)\Delta p(t)\}|\alpha\rangle = 0.$$

By comparing with (1.1.6), one can see that the minimum uncertainty theoretically possible is indeed achieved in the coherent state.

Exercise. Since a is not Hermitian, an eigenvector need not exist, although in fact it does exist. Show that there is no eigenstate of the creation operator $a^\dagger$.

4.2.2   Coherent State Solution for the Harmonic Oscillator

The solutions (4.1.39) are equivalent to

$$e^{iHt/\hbar}ae^{-iHt/\hbar} = e^{-i\omega t}a. \tag{4.2.12}$$

The solution of the Schrödinger equation with an initial coherent state $|\alpha\rangle$ is

$$|\alpha, t\rangle = e^{-iHt/\hbar}|\alpha\rangle \tag{4.2.13}$$

and using (4.2.12) and (4.2.10) we see that

$$a|\alpha, t\rangle = e^{-iHt/\hbar}\left(e^{iHt/\hbar}ae^{-iHt/\hbar}\right)|\alpha\rangle \tag{4.2.14}$$

$$= e^{-iHt/\hbar}e^{-i\omega t}\alpha|\alpha\rangle \tag{4.2.15}$$

$$= \alpha e^{-i\omega t}|\alpha, t\rangle. \tag{4.2.16}$$

Thus, from (4.2.10) we see that $|\alpha, t\rangle$ must itself be proportional to the coherent state $|\alpha e^{-i\omega t}\rangle$. This means that the coherent states form a set of solutions of the Schrödinger equation which are determined completely by the classical solutions, which are given by the real and imaginary parts of $\alpha e^{-i\omega t}$.

The precise constant of proportionality cannot be determined without a determination of the phase of $|\alpha\rangle$. This is a matter of definition, which is made precise in the next section. The full solution is given in Sect. 4.3.2.

The formulae for the variances (4.2.11) show that they are proportional to Planck's constant, so that in the classical limit they become negligible. Thus the coherent state solution does represent exactly what was wanted—a set of solutions which directly approach the classical solutions.

4.3   Coherent States

This section is largely of the nature of a compendium of properties of coherent states which are often needed. The reader need not study it deeply, but merely use it for reference.

4.3.1   Properties of the Coherent States

a) **Expression in Terms of Number States:**   By using the relations (4.1.23) it is easy to show that the only solution to the defining equation (4.2.10) which also satisfies $\langle\alpha|\alpha\rangle = 1$ is (up to a phase)

$$|\alpha\rangle = \exp\left(-\tfrac{1}{2}|\alpha|^2\right) \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} |n\rangle. \tag{4.3.1}$$

b) **Unitary Transformation of the Vacuum:**   We can also show that

$$|\alpha\rangle = \exp\left(\alpha a^\dagger - \alpha^* a\right) |0\rangle. \tag{4.3.2}$$

This involves the use of the Baker-Hausdorff formula: for any two operators A and B such that $[A, B]$ commutes with both of them one can write

$$\begin{aligned}
\exp(A + B) &= \exp(A)\exp(B)\exp\left(-\tfrac{1}{2}[A, B]\right) \\
&= \exp(B)\exp(A)\exp\left(\tfrac{1}{2}[A, B]\right)
\end{aligned} \tag{4.3.3}$$

and this is proved in Appendix A. Using this identity, we see that (4.3.2) gives

$$|\alpha\rangle = \exp\left(-\tfrac{1}{2}|\alpha|^2\right)\exp\left(\alpha a^\dagger\right)\exp\left(-\alpha^* a\right)|0\rangle \tag{4.3.4}$$

and noting that $a|0\rangle = 0$, we see

$$|\alpha\rangle = \exp\left(-\tfrac{1}{2}|\alpha|^2\right) \sum \frac{\alpha^n (a^\dagger)^n}{n!} |0\rangle, \tag{4.3.5}$$

which yields (4.3.1) when we use the expression (4.1.24) for $|n\rangle$.

c) Scalar Product:

$$\langle\alpha|\beta\rangle = \exp\left(\alpha^*\beta - \tfrac{1}{2}|\alpha|^2 - \tfrac{1}{2}|\beta|^2\right) \tag{4.3.6}$$

$$|\langle\alpha|\beta\rangle|^2 = \exp\left(-|\alpha - \beta|^2\right). \tag{4.3.7}$$

Notice that no two coherent states are actually orthogonal to each other. However, if α and β are significantly different from each other, the two states are almost orthogonal.

d) Completeness Formula:

$$1 = \frac{1}{\pi} \int d^2\alpha \, |\alpha\rangle\langle\alpha|. \tag{4.3.8}$$

Here,

$$\begin{aligned}
\alpha &= \alpha_x + i\,\alpha_y, \\
d^2\alpha &= d\alpha_x d\alpha_y
\end{aligned} \tag{4.3.9}$$

and the integral is over the whole complex plane. We prove this as follows : If $|A\rangle$ is an arbitrary vector, then write

$$|A\rangle = \sum_n A_n |n\rangle \tag{4.3.10}$$

so that

$$\frac{1}{\pi} \int d^2\alpha \, |\alpha\rangle\langle\alpha|A\rangle = \frac{1}{\pi} \sum_n \int A_n |\alpha\rangle\langle\alpha|n\rangle d^2\alpha. \tag{4.3.11}$$

Substitute (4.3.1) and change to polar coordinates by

$$\alpha = re^{i\theta} \tag{4.3.12}$$

$$d^2\alpha = r\, dr\, d\theta. \tag{4.3.13}$$

Hence,

$$(4.3.11) = \frac{1}{\pi} \sum_{m,n} \int A_n e^{-r^2} r^{n+m} e^{i(m-n)\theta} (n!m!)^{-\frac{1}{2}} |m\rangle r\, dr\, d\theta$$

$$= 2 \sum_n \int A_n e^{-r^2} r^{2n+1} (n!)^{-1} |n\rangle \, dr \tag{4.3.14}$$

where we have used

$$2\delta_{n,m} = \frac{1}{\pi} \int d\theta \, e^{i(m-n)\theta}. \tag{4.3.15}$$

Now noting

$$\int_0^\infty dr \, e^{-r^2} r^{2n+1} = n!/2, \tag{4.3.16}$$

we find

$$(4.3.11) = \sum_n A_n |n\rangle = |A\rangle. \tag{4.3.17}$$

Formulae (4.3.6,7) together indicate *that the coherent states are not orthogonal* for different α and β, and that since there is a factor $1/\pi$ in front of the integral (4.3.8) *the coherent states are overcomplete*. In fact, for any r, (4.3.1) shows that we can write

$$|n\rangle = \frac{1}{2\pi} \exp\left(\tfrac{1}{2}r^2\right) r^{-n}\sqrt{n!} \int d\theta \, e^{-in\theta} |\alpha\rangle \tag{4.3.18}$$

which indicates that the states for any fixed $r = |\alpha|$ are complete. This overcompleteness is, however, not a disadvantage because of the very simple connection between coherent states and the physics of classical states, and because of the fact that the *Bargmann states*, defined by

$$||\alpha\rangle = \exp\left(\tfrac{1}{2}|\alpha|^2\right) |\alpha\rangle = \sum_{n=0}^\infty \frac{\alpha^n}{\sqrt{n!}} |n\rangle, \tag{4.3.19}$$

are analytic functions of α. This property is very important in what follows.

e) Expansion of Arbitrary States in Terms of Coherent States: Consider an arbitrary state $|f\rangle$. Then using the completeness relation (4.3.8) we have

$$|f\rangle = \frac{1}{\pi} \int d^2\alpha |\alpha\rangle f(\alpha^*) \exp\left(-\tfrac{1}{2}|\alpha|^2\right), \tag{4.3.20}$$

where

$$f(\alpha^*) = \langle \alpha|f\rangle \exp\left(\tfrac{1}{2}|\alpha|^2\right) = \langle \alpha||f\rangle \tag{4.3.21}$$

is *an analytic function of* α^*. With this proviso, the expansion (4.3.20) is unique. If functions of both α^* and α are permitted, the expansion is no longer unique, as *Glauber* [4.1] shows.

The scalar product of two states $|f\rangle$ and $|g\rangle$ is straightforwardly shown to be

$$\langle g|f\rangle = \frac{1}{\pi} \int d^2\alpha [g(\alpha^*)]^* f(\alpha^*) e^{-|\alpha|^2} \tag{4.3.22}$$

which is obviously a Hilbert space of analytic functions. It provides, in fact, the soundest mathematical starting point for the study of the harmonic oscillator, the creation destruction operators and all the formalism of this chapter.

f) Expansion of an Operator in Coherent States: Consider any operator T in the quantum Hilbert space. Using the identity resolution twice,

$$
\begin{aligned}
T &= 1 \cdot T \cdot 1 = \frac{1}{\pi^2} \int d^2\alpha \, d^2\beta \, |\alpha\rangle\langle\alpha|T|\beta\rangle\langle\beta| \\
&= \frac{1}{\pi^2} \int d^2\alpha \, d^2\beta \, |\alpha\rangle\langle\beta| T(\alpha^*, \beta) \exp\left(-\tfrac{1}{2}|\alpha|^2 - \tfrac{1}{2}|\beta|^2\right)
\end{aligned}
\tag{4.3.23}
$$

where

$$
\begin{aligned}
T(\alpha^*, \beta) &= \exp\left(\tfrac{1}{2}|\alpha|^2 + \tfrac{1}{2}|\beta|^2\right) \langle\alpha|T|\beta\rangle \\
&= \langle\alpha||T||\beta\rangle
\end{aligned}
\tag{4.3.24}
$$

and from the analyticity of the states $||\alpha\rangle$, $||\beta\rangle$, we see that $T(\alpha^*, \beta)$ *is an analytic function of* α^* *and* β, *and, with this proviso, is unique.* Notice, for example, that if

$$
T = (a^\dagger)^m (a^n),
\tag{4.3.25}
$$

then

$$
\begin{aligned}
T(\alpha^*, \beta) &= \langle\alpha|(a^\dagger)^m(a^n)|\beta\rangle \exp\left(\tfrac{1}{2}|\alpha|^2 + \tfrac{1}{2}|\beta|^2\right) \\
&= (\alpha^*)^m (\beta)^n \langle\alpha|\beta\rangle \exp\left(\tfrac{1}{2}|\alpha|^2 + \tfrac{1}{2}|\beta|^2\right) \\
&= (\alpha^*)^m (\beta)^n \exp(\alpha^*\beta).
\end{aligned}
\tag{4.3.26}
$$

g) Any Operator T is Determined by Its Expectation in all Coherent States: For if T is any operator,

$$
\langle\alpha|T|\alpha\rangle = \sum_{n,m} \langle n|T|m\rangle e^{-|\alpha|^2} (\alpha^*)^n (\alpha)^m / \sqrt{n!m!}
\tag{4.3.27}
$$

so that

$$
\langle n|T|m\rangle = \frac{1}{\sqrt{n!m!}} \frac{\partial^n}{\partial\alpha^{*n}} \frac{\partial^m}{\partial\alpha^m} \left(e^{\alpha\alpha^*}\langle\alpha|T|\alpha\rangle\right)\Bigg|_{\alpha=0}.
\tag{4.3.28}
$$

The derivatives are formal derivatives and, as in analytic function theory, are to be interpreted as

$$
\begin{aligned}
\alpha &= x + iy \\
\frac{\partial}{\partial\alpha} &= \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right) \\
\frac{\partial}{\partial\alpha^*} &= \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)
\end{aligned}
\tag{4.3.29}
$$

in which case, $\langle n|T|m\rangle$ are the combinations of coefficients of power series in real variables x and y.

h) Coherent States are Eigenstates of a: Namely,

$$a|\alpha\rangle = \alpha|\alpha\rangle$$

and   (4.3.30)

$$\langle\alpha|a^\dagger = \langle\alpha|\alpha^*$$

which are proved directly from the definition and were the original basis for investigating them.

i) Normal Products: In evaluating matrix elements, *normal products* of operators in which all destruction operators stand to the right of creation operators, are useful. Thus,

$$\begin{aligned}
\langle\alpha|a^\dagger a a^\dagger|\beta\rangle &= \langle\alpha|a^\dagger a^\dagger a + a^\dagger[a, a^\dagger]|\beta\rangle \\
&= \langle\alpha|a^\dagger a^\dagger a + a^\dagger|\beta\rangle \\
&= (\alpha^{*2}\beta + \alpha^*)\langle\alpha|\beta\rangle.
\end{aligned}$$ (4.3.31)

The symbol $::$ around an expression means that it is to be considered a normal product: thus,

$$: (a + a^\dagger)(a + a^\dagger) := a^{\dagger 2} + a^2 + 2a^\dagger a.$$ (4.3.32)

From (4.3.30) it follows that the matrix element between coherent states $\langle\alpha|$ and $|\beta\rangle$ of any normally ordered function $F(a^\dagger, a)$ of creation and destruction operators is given by $F(\alpha^*, \beta)$. Thus, for example,

$$\langle\alpha|:(a + a^\dagger)^3:|\beta\rangle = (\beta + \alpha^*)^3.$$ (4.3.33)

j) Poissonian Number Distribution of Coherent States: The state $|n\rangle$ is known as an n *quantum state* since its energy is $n\hbar\omega$ more than that of the vacuum $|0\rangle$, the zero quantum state. In quantum mechanics, therefore, the probability of observing n quanta in a coherent state $|\alpha\rangle$ is

$$P_\alpha(n) = |\langle n|\alpha\rangle|^2 = \left|\exp(-\tfrac{1}{2}\alpha^2)\frac{\alpha^n}{\sqrt{n!}}\right|^2$$ (4.3.34)

$$= \frac{e^{-|\alpha|^2}|\alpha|^{2n}}{n!}$$ (4.3.35)

which is a Poisson distribution with mean $|\alpha|^2$. Since the number n corresponds to the eigenvalue of the number operator N, we have

$$\left.\begin{aligned}
\langle N\rangle &= \langle\alpha|N|\alpha\rangle = \sum_n nP(n) = |\alpha|^2 \\
\langle N^2\rangle &= \langle\alpha|a^\dagger a a^\dagger a|\alpha\rangle \\
&= \langle\alpha|a^\dagger a^\dagger a a + a^\dagger[a, a^\dagger]a|\alpha\rangle \\
&= |\alpha|^4 + |\alpha|^2.
\end{aligned}\right\}$$ (4.3.36)

Hence,

$$\langle N(N-1)\rangle = |\alpha|^4 = \langle N\rangle^2 \tag{4.3.37}$$

as required for a Poisson.

k) Wavefunction for a Coherent State: Using the definition of a and $a^\dagger$ in terms of p and x (4.1.11), and the usual $p \to -i\hbar\, d/dx$, the coherent state definition (4.2.10)

$$a|\alpha\rangle = \alpha|\alpha\rangle \to \left(\sqrt{\frac{1}{2\eta}}\frac{d}{dx} + \sqrt{\frac{\eta}{2}}\,x\right)\langle x|\alpha\rangle = \alpha\langle x|\alpha\rangle \tag{4.3.38}$$

for which the normalised solution is

$$\langle x|\alpha\rangle = \left(\frac{\eta}{\pi}\right)^{\frac{1}{4}}\exp\left[-(\mathrm{Im}\,\alpha)^2 - \frac{\eta}{2}\left(x - \alpha\sqrt{\frac{2}{\eta}}\right)^2\right]. \tag{4.3.39}$$

Defining

$$\alpha = \frac{i\bar{p}}{\hbar}\frac{1}{\sqrt{2\eta}} + \bar{x}\sqrt{\frac{\eta}{2}} \tag{4.3.40}$$

we can write

$$\langle x|\alpha\rangle = \left(\frac{\eta}{\pi}\right)^{\frac{1}{4}}\exp\left[-\frac{\eta}{2}(x-\bar{x})^2 + \frac{i\bar{p}x}{\hbar}\right]. \tag{4.3.41}$$

This corresponds to a Gaussian wavepacket centred on $\bar{x}$ modulated by a plane wave $\exp(i\bar{p}x/\hbar)$, which corresponds to a mean momentum $\bar{p}$, i.e. a minimum uncertainty Gaussian wavepacket.

4.3.2   Coherent States are Driven Oscillator Wavefunctions

The driven harmonic oscillator is described by adding a term $xE(t)$ to the Hamiltonian for the harmonic oscillator. If $x(t)$ is a position co-ordinate of an oscillator, then $E(t)$ could be proportional to some classical external field of force, for example gravitation, or an electric field. The Schrödinger picture Hamiltonian expressed in terms of a and $a^\dagger$ can be written (with $F(t) = \left(\hbar\sqrt{2\eta}\right)^{-1}E(t)$)

$$H = \hbar\omega(a^\dagger a + \tfrac{1}{2}) + \hbar F(t)(a + a^\dagger). \tag{4.3.42}$$

We can solve the Schrödinger equation by use of the interaction picture as follows. Write

$$\begin{aligned}H_0 &= \hbar\omega\left(a^\dagger a + \tfrac{1}{2}\right),\\ H_I(t) &= \hbar F(t)(a + a^\dagger).\end{aligned} \tag{4.3.43}$$
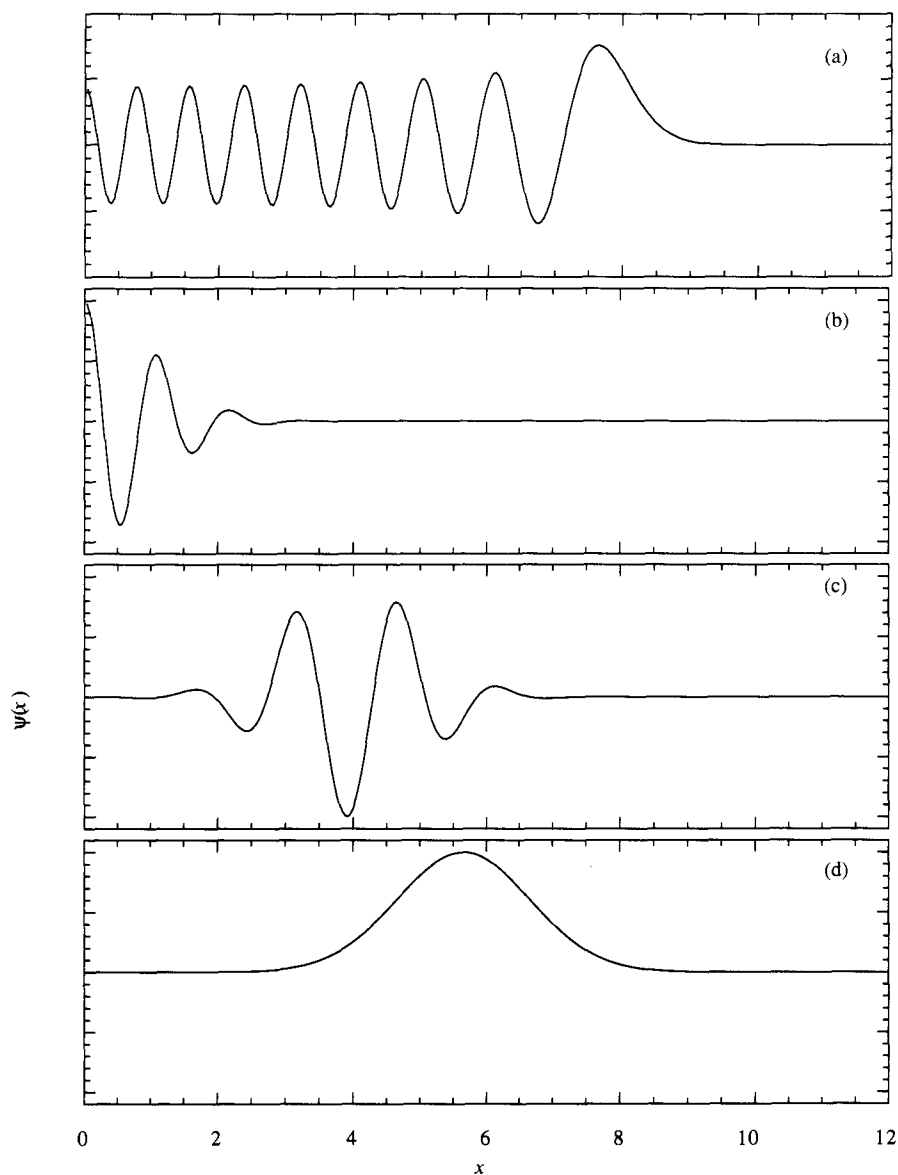
Fig. 4.1 Plot of coherent state wavefunctions, compared with the energy eigenfunction with the same mean energy. (a) Energy eigenfunction from (4.1.32). (b), (c), (d) Coherent state wavefunctions for $\omega t = 0, \pi/4, \pi/2$ from the Gaussian wavepacket (4.3.41)

Then the Schrödinger equation is

$$\frac{d|\psi(t)\rangle}{dt} = -\frac{i}{\hbar}[H_0 + H_I(t)]|\psi(t)\rangle. \tag{4.3.44}$$

If we let

$$|\phi(t)\rangle = \exp\left(\frac{iH_0 t}{\hbar}\right)|\psi(t)\rangle \tag{4.3.45}$$

then

$$\frac{d|\phi(t)\rangle}{dt} = -\frac{i}{\hbar}\exp\left(\frac{iH_0 t}{\hbar}\right)H_I(t)\exp\left(\frac{-iH_0 t}{\hbar}\right)|\phi(t)\rangle. \tag{4.3.46}$$

Notice that the solutions (4.1.39) imply

$$\begin{aligned}
\exp\left(\frac{iH_0 t}{\hbar}\right)a\exp\left(\frac{-iH_0 t}{\hbar}\right) &= e^{-i\omega t}a \\
\exp\left(\frac{iH_0 t}{\hbar}\right)a^\dagger\exp\left(\frac{-iH_0 t}{\hbar}\right) &= e^{i\omega t}a^\dagger
\end{aligned} \tag{4.3.47}$$

so that we find

$$\frac{d|\phi(t)\rangle}{dt} = -i F(t)\left\{a^\dagger e^{i\omega t} + a e^{-i\omega t}\right\}|\phi(t)\rangle. \tag{4.3.48}$$

The two terms satisfy the conditions of the Baker-Hausdorff generalisation as given in Appendix 4A, hence

$$|\phi(t)\rangle = \exp\left\{-i\int_0^t dt' (a^\dagger e^{i\omega t'} + a e^{-i\omega t'})F(t')\right\}|\phi(0)\rangle \tag{4.3.49}$$

and hence

$$|\psi(t)\rangle = \exp\left\{-i(a^\dagger + a)\int_0^t dt' F(t')\right\}\exp\left\{-\frac{iH_0 t}{\hbar}\right\}|\psi(0)\rangle. \tag{4.3.50}$$

a) The Initial State is the Vacuum State: In this case we get a coherent state solution:

$$|\psi(t)\rangle = e^{-i\omega t/2}|\alpha(t)\rangle \tag{4.3.51}$$

where

$$\alpha(t) = -i\int_0^t dt' e^{i\omega(t-t')}F(t'). \tag{4.3.52}$$

Apart from the phase factor arising from the zero point energy, the solution is a coherent state whose parameter $\alpha(t)$ is given by the solution of the classical equation of motion.

b) The Initial State is a Coherent State :   Let the initial state be $|\alpha_0\rangle$. Using the representation of the coherent state (4.3.2), and following a similar procedure, we find that

$$|\psi(t)\rangle = e^{-i\omega t/2} \exp\left\{\tfrac{1}{2}\left(\alpha(t)\alpha_0^* e^{i\omega t} - \alpha^*(t)\alpha_0 e^{-i\omega t}\right)\right\} |\alpha(t) + \alpha_0 e^{-i\omega t}\rangle.$$

$$(4.3.53)$$

We see that if $F(t)$ vanishes the solution is simply given by

$$|\psi(t)\rangle = e^{-i\omega t/2} |\alpha_0 e^{-i\omega t}\rangle.$$

$$(4.3.54)$$

This is the minimum uncertainty state of Sect. 4.2.1.

c) Other Initial Conditions: A wide variety of initial conditions is available, but we have seen that only in the case of initial coherent states (including the vacuum as a special case) do we find that the state vector evolves so as to remain a coherent state. Thus, although we have found a large class of states which behave very similarly to classical states, we have not explained why they should be preferred to any other kind of state. That can only be explained in the context of damping theory, as is discussed in Sect. 7.3.1 where it is shown that the coupling of a quantum system to an environment consisting of a large number of degrees of freedom can select out the coherent state as that which will be observed in practice.

4.4   Phase Space Representations
of the Harmonic Oscillator Density Operator

The classical limit of quantum mechanics is most naturally given by a coherent state description. For example, in Sect. 4.2.1 we saw that the coherent states were the unique states which give a time independent wavefunction in which both $x(t)$ and $p(t)$ have a minimum uncertainty, and in Sect. 4.3.2 it was shown that a classical driving field would automatically drive any coherent state into another coherent state—the parameter $\alpha(t)$ being the solution of the corresponding classical equation.

In all practical situations which arise in noisy systems the description of the system is best carried out in terms of the density operator for the system. In this section we will give a number of methods of expressing the density operator in terms of a c-number function of a coherent state variable α. These methods lead to considerable simplification of the quantum equations of motion, and are therefore of considerable practical use.

4.4.1   The Q-Representation

We have seen already in Sect. 4.3.1f that any operator can be expanded in terms of coherent states, but in fact this does not yield a very useful way of expressing the physics of the density operator in terms of coherent states. A more useful and interesting result is that of Sect. 4.3.1g, that any operator T is determined by its diagonal coherent state matrix elements $\langle\alpha|T|\alpha\rangle$. This result has embodied in it the overcompleteness and non-orthogonality of the coherent states, for such a result is not possible for a complete orthonormal set: the non-diagonal matrix elements are also required.

The connection can be made in a variety of ways, the most common of which are known as the Q- and P-representations, and the Wigner function representation. They arise from the consideration of three different aims. In the Q-representation we attempt to generate a probability density for the system by using the diagonal matrix elements of the density operator for this purpose. The P-representation arises from trying to represent any density operator as an ensemble of coherent states. Finally Wigner invented the function which bears his name as a result of his desire to find a joint distribution for canonically conjugate variables p and q which would be something like a classical probability distribution.

All of these phase space representations have their advantages and disadvantages—nevertheless they are all very useful.

a) Definition: We can define the Q-function corresponding to a density operator ρ by

$$Q(\alpha, \alpha^*) = \frac{1}{\pi}\langle\alpha|\rho|\alpha\rangle. \tag{4.4.1}$$

We note that the normalisation of the density operator gives a normalisation of $Q(\alpha, \alpha^*)$, for

$$1 = \mathrm{Tr}\,\{\rho\} = \mathrm{Tr}\left\{\frac{1}{\pi}\int d^2\alpha|\alpha\rangle\langle\alpha|\rho\right\} \tag{4.4.2}$$

$$= \frac{1}{\pi}\int d^2\alpha\langle\alpha|\rho|\alpha\rangle \tag{4.4.3}$$

so that

$$\int d^2\alpha\, Q(\alpha, \alpha^*) = 1. \tag{4.4.4}$$

A similar procedure shows that the averages of *antinormally* ordered products of creation and destruction operators is

$$\langle a^r(a^\dagger)^s\rangle = \mathrm{Tr}\left\{a^r(a^\dagger)^s\rho\right\} = \int d^2\alpha\,\alpha^r(\alpha^*)^s Q(\alpha, \alpha^*). \tag{4.4.5}$$

Thus, the Q-function can be viewed as a kind of probability: $Q(\alpha, \alpha^*)$ is positive, normalised, and the (antinormally ordered) quantum moments can be determined in terms of simple moments of $Q(\alpha, \alpha^*)$.

b) Examples of Q-functions:

i) Coherent State: The density operator is that of a coherent state

$$\rho = |\alpha_0\rangle\langle\alpha_0|$$ (4.4.6)

and using (4.3.7)

$$Q(\alpha, \alpha^*) = \frac{1}{\pi}|\langle\alpha|\alpha_0\rangle|^2 = \frac{1}{\pi}\exp(-|\alpha - \alpha_0|^2).$$ (4.4.7)

The exponential form of this expression shows that $Q(\alpha, \alpha^*)$ is only significantly different from zero if $\alpha \simeq \alpha_0$.

In terms of the macroscopic momentum and position variables,

$$\begin{aligned}
\alpha &= \bar{x}\sqrt{\frac{\eta}{2}} + \frac{i}{\hbar}\bar{p}\frac{1}{\sqrt{2\eta}} \\
\alpha_0 &= \bar{x}_0\sqrt{\frac{\eta}{2}} + \frac{i}{\hbar}\bar{p}_0\frac{1}{\sqrt{2\eta}},
\end{aligned}$$ (4.4.8)

one finds that the Q-function is only significantly different from zero if

$$\begin{aligned}
\bar{x} - \bar{x}_0 &\le (2\hbar)^{\frac{1}{2}}(km)^{-\frac{1}{4}} \\
\bar{p} - \bar{p}_0 &\le (2\hbar)^{\frac{1}{2}}(km)^{\frac{1}{4}}.
\end{aligned}$$ (4.4.9)

Essentially, the Q-function is in this case a kind of probability distribution concentrated in the minimum uncertainty region around the position $\bar{x}_0, \bar{p}_0$.

ii) A Number State: In this case the density operator is

$$\rho = |n\rangle\langle n|$$ (4.4.10)

and

$$\begin{aligned}
Q_n(\alpha, \alpha^*) &= \frac{1}{\pi}|\langle\alpha|n\rangle|^2 \\
&= \frac{1}{\pi}\exp(-|\alpha|^2)(|\alpha|^2)^n/n!.
\end{aligned}$$ (4.4.11)

The Q-function is independent of the phase of α, and has a maximum at $|\alpha|^2 = n$. The mean and variance of $|\alpha|^2$ are straightforwardly computed as

$$\begin{aligned}
\langle|\alpha|^2\rangle &= n + 1 \\
\text{var}\{|\alpha|^2\} &= n + 1
\end{aligned}$$ (4.4.12)

so that for sufficiently large n the distribution is quite sharply peaked around $|\alpha|^2 = n + 1$

iii) A Thermal State: The density operator for a thermal state is

$$\rho_T = \left(1 - e^{-\hbar\omega/kT}\right)\sum_n |n\rangle\langle n|e^{-n\hbar\omega/kT}$$ (4.4.13)

and the Q-function is

$$
Q(\alpha, \alpha^*) = \frac{\left(1 - e^{-\hbar\omega/kT}\right)}{\pi} \sum_n e^{-n\hbar\omega/kT} \exp(-|\alpha|^2) |\alpha|^{2n}/n! \tag{4.4.14}
$$

so that

$$
Q_T(\alpha, \alpha^*) = \frac{\left(1 - e^{-\hbar\omega/kT}\right)}{\pi} \exp\left\{-|\alpha|^2 \left(1 - e^{-\hbar\omega/kT}\right)\right\}. \tag{4.4.15}
$$

This is a bivariate Gaussian in α and α^*, with

$$
\begin{aligned}
\langle\alpha\rangle &= \langle\alpha^*\rangle = 0 \\
\langle\alpha^2\rangle &= \langle\alpha^{*2}\rangle = 0 \\
\langle|\alpha|^2\rangle &= \left(1 - e^{-\hbar\omega/kT}\right)^{-1}.
\end{aligned} \tag{4.4.16}
$$

Notice that as $kT \to 0$, $\langle|\alpha|^2\rangle \to 1$. This is correct, since $\langle|\alpha|^2\rangle = \langle aa^\dagger\rangle$.

At sufficiently large temperatures, we can approximate the $\exp(-\hbar\omega/kT)$ factor, to get

$$
Q_T(\alpha, \alpha^*) \to \frac{\hbar\omega}{\pi kT} \exp\left(-\frac{\hbar\omega|\alpha|^2}{kT}\right) \qquad \text{as } T \to \infty \tag{4.4.17}
$$

which looks like a Boltzmann distribution.

c) The Q-function as a Quasiprobability: The Q-function reduces the evaluation of means to that of ordinary averages over the two variables α, α^*, and behaves very like a probability distribution. The main problem of the Q-function is that not all positive normalisable Q-functions correspond to positive definite normalisable density operators. For example, we know that

$$
\langle aa^\dagger\rangle = \sum_n (n + 1)\langle n|\rho|n\rangle \geq 1 \tag{4.4.18}
$$

for any density operator. Hence

$$
\langle|\alpha|^2\rangle \geq 1 \tag{4.4.19}
$$

for any Q-function. This means that $\delta^2(\alpha)$ is not a permissible Q-function, since it would give $\langle|\alpha|^2\rangle = 0$. Similarly, no delta function like $\delta^2(\alpha - \alpha_0)$ is permissible. The physical effect is quite expected: from (4.2.5–7) we can see that a delta function Q-function would mean that $\langle\Delta x(t)^2\rangle$ and $\langle\Delta p(t)^2\rangle$ were negative.

d) The Density Operator in Terms of the Q-function: A final question has to be answered: Namely, given the Q-function, how does one construct the density operator? The answer is simple, but the derivation is slightly roundabout. We first show that $Q(\alpha, \alpha^*)$ can be expressed as an absolutely convergent power series in

α, and α^*. To do this we need to prove that for any density operator ρ, the number state matrix elements satisfy

$$|\langle n|\rho|m\rangle| \le 1. \tag{4.4.20}$$

We know that ρ can be expressed as

$$\rho = \sum P_A|A\rangle\langle A| \tag{4.4.21}$$

so that

$$|\langle n|\rho|m\rangle| \le \sum P_A|\langle n|A\rangle|\,|\langle m|A\rangle|. \tag{4.4.22}$$

We note that for any x and y

$$2xy \le x^2 + y^2 \tag{4.4.23}$$

so that

$$|\langle n|\rho|m\rangle| \le \sum \tfrac{1}{2}P_A\left\{\,|\langle n|A\rangle|^2 + |\langle m|A\rangle|^2\,\right\}. \tag{4.4.24}$$

However, since

$$\sum|\langle n|A\rangle|^2 = \langle A|A\rangle = 1 \tag{4.4.25}$$

we can deduce

$$|\langle n|A\rangle|^2 \le 1 \qquad \text{for all } n, \tag{4.4.26}$$

hence

$$|\langle n|\rho|m\rangle| \le \sum P_A = 1. \tag{4.4.27}$$

Using the expression for coherent states (4.3.1), we see that

$$Q(\alpha, \alpha^*) = e^{-\alpha\alpha^*}\sum_{n,m}\frac{\langle n|\rho|m\rangle}{\pi\sqrt{n!m!}}\alpha^m\alpha^{*n}. \tag{4.4.28}$$

The double power series is clearly absolutely convergent, because of the bound (4.4.20), and since $e^{-\alpha\alpha^*}$ has an absolutely convergent power series, the product of these two series is also absolutely convergent.

Suppose this power series is

$$Q(\alpha, \alpha^*) = \sum_{n,m}Q_{n,m}\alpha^m(\alpha^*)^n. \tag{4.4.29}$$

Then, we can straightforwardly compute

$$\boxed{\langle n|\rho|m\rangle = \pi\sqrt{n!m!}\sum_r Q_{n-r,m-r}.} \tag{4.4.30}$$

As an alternative expression we can write ρ as a *normally ordered* power series in a and $a^\dagger$;

$$\boxed{\rho = \pi\sum Q_{n,m}(a^\dagger)^n a^m.} \tag{4.4.31}$$

If such a power series exists, then the definition $a|\alpha\rangle = \alpha|\alpha\rangle$ immediately shows that $\langle\alpha|\rho|\alpha\rangle/\pi$ is identical with (4.3.30): thus the density operator is obtained by replacing $\alpha^m(\alpha^*)^n$ in the power series for $Q(\alpha, \alpha^*)$ by the corresponding *normally ordered* product of creation and destruction operators.

4.4.2 The Quantum Characteristic Function

Since the Q-function determines the density operator, the Fourier transform of the Q-function will also have this property. The (normally ordered) *quantum characteristic function* is defined by

$$\chi(\lambda, \lambda^*) = \mathrm{Tr}\left\{\rho \exp(\lambda a^\dagger) \exp(-\lambda^* a)\right\}. \qquad (4.4.32)$$

The quantum characteristic function has the property, analogous to that of the classical characteristic function, that the *normally ordered* moments are given by its derivatives at $\lambda = 0$; precisely

$$\langle (a^\dagger)^r a^s \rangle = (-)^s \frac{\partial^{r+s}}{\partial^r \lambda \partial^s \lambda^*} \chi(\lambda, \lambda^*)\bigg|_{\lambda=0}. \qquad (4.4.33)$$

Using the Baker-Hausdorff theorem (Sect. 4.3.1b) to reverse the order of the exponentials, and the completeness formula (4.3.8), we find

$$\chi(\lambda, \lambda^*) = \mathrm{Tr}\left\{\frac{1}{\pi}\int d^2\alpha\, \rho \exp(|\lambda|^2) \exp(-\lambda^* a)|\alpha\rangle\langle\alpha| \exp(\lambda a^\dagger)\right\} \qquad (4.4.34)$$

so that

$$\chi(\lambda, \lambda^*) = \exp(|\lambda|^2) \int d^2\alpha\, \exp(\lambda\alpha^* - \alpha\lambda^*) Q(\alpha, \alpha^*). \qquad (4.4.35)$$

This is a Fourier transform in two real variables; $\mathrm{Re}(\alpha)$ and $\mathrm{Im}(\alpha)$. Since $Q(\alpha, \alpha^*)$ is positive and is integrable, the Fourier transform exists, and is bounded as $|\lambda| \to \infty$.

a) Some Examples of Quantum Characteristic Functions:

i) Coherent State:

$$\rho = |\alpha\rangle\langle\alpha|; \qquad \chi(\lambda, \lambda^*) = \exp(\lambda\alpha^* - \alpha\lambda^*). \qquad (4.4.36)$$

ii) Number State:

$$\rho = |n\rangle\langle n|; \qquad \chi(\lambda, \lambda^*) = \sum_{m=0}^{n} \frac{n!}{(m!)^2(n-m)!}(-|\lambda|^2)^m. \qquad (4.4.37)$$

iii) Thermal State: This is computed from the Fourier transform of (4.4.15) using (4.4.35)

$$\chi(\lambda, \lambda^*) = \exp\left\{-\frac{|\lambda|^2}{e^{\hbar\omega/kT} - 1}\right\}. \qquad (4.4.38)$$

4.4.3   The P-Representation

The density operator corresponding to a coherent state $|\alpha\rangle$ is of course given by $|\alpha\rangle\langle\alpha|$. It might happen that the variable α is not well defined, and we may wish to consider an ensemble of coherent states; that is, we may consider a density operator of the form

$$\rho = \int d^2\alpha\, P(\alpha, \alpha^*) |\alpha\rangle\langle\alpha|. \tag{4.4.39}$$

Glauber and *Sudarshan* [4.1] both showed that such a representation of the density operator did exist for a large class of ρ, and indeed *Klauder* et al. [4.4] showed that, provided $P(\alpha, \alpha^*)$ is permitted to be a sufficiently singular generalised function, such a representation exists for any density operator ρ.

Notice that

$$
\begin{aligned}
1 = \mathrm{Tr}\,\{\rho\} &= \int d^2\alpha\, P(\alpha, \alpha^*) \sum_n \langle n|\alpha\rangle\langle\alpha|n\rangle \\
&= \int d^2\alpha\, P(\alpha, \alpha^*) \langle\alpha|\alpha\rangle
\end{aligned}
\tag{4.4.40}
$$

so that

$$\int d^2\alpha P(\alpha, \alpha^*) = 1. \tag{4.4.41}$$

Further, for any normal product (Sect. 4.3.1i) of creation and destruction operators (that is, a product in which all creation operators are to the left of all destruction operators) we can write

$$\langle (a^\dagger)^r a^s \rangle = \mathrm{Tr}\,\{(a^\dagger)^r a^s \rho\} = \mathrm{Tr}\,\{a^s \rho (a^\dagger)^r\} \tag{4.4.42}$$

$$= \int d^2\alpha\, P(\alpha, \alpha^*) \sum_n \langle n|a^s|\alpha\rangle\langle\alpha|(a^\dagger)^r|n\rangle \tag{4.4.43}$$

and using the property of the coherent state (4.3.30), we deduce

$$\langle (a^\dagger)^r a^s \rangle = \int d^2\alpha\, (\alpha^*)^r \alpha^s P(\alpha, \alpha^*). \tag{4.4.44}$$

Thus the quantity $P(\alpha, \alpha^*)$ plays the role of a kind of probability density for the variables α and α^*, in that the means of *normally ordered* products of quantum operators are simple moments of $P(\alpha, \alpha^*)$.

a) Relationship Between the P- and Q-functions: The P- and Q-functions play equivalent roles for normal and antinormal products respectively, and their relationship is best exhibited by use of the quantum characteristic function. The *normally*

ordered quantum characteristic function defined by (4.4.34) is easily shown to be given by

$$\chi(\lambda, \lambda^*) = \int d^2\alpha \, \exp(\lambda\alpha^* - \lambda^*\alpha) P(\alpha, \alpha^*). \qquad (4.4.45)$$

The relationships (4.4.45) and (4.4.35) thus provide the connection between the P- and Q-functions. Looking at the examples of characteristic functions, we immediately discover that the P-function may correspond to a characteristic function which does not have an ordinary Fourier transform. For example, from (4.4.37) we see that $\chi(\lambda, \lambda^*)$ can be a polynomial, which means that $P(\alpha, \alpha^*)$ would be a finite sum of delta functions and finite order derivatives of a delta function. In fact it is easy to see that a quantum characteristic functions could be any convergent infinite series in λ and λ^*, and this would give P-function consisting of an infinite sum of derivatives of delta functions of arbitrarily high order. Formal manipulations can be justified for these [4.4], and in the sense of these rather singular distributions, it can be shown that a P-function always exists. However the usefulness of such a P-function in practice is limited.

The relations (4.4.45) and (4.4.35) can be Fourier transformed to give the more direct connection

$$Q(\alpha, \alpha^*) = \frac{1}{\pi} \int d^2\beta \, \exp(-|\alpha - \beta|^2) P(\beta, \beta^*). \qquad (4.4.46)$$

b) Examples of P-functions: The usefulness of the P-function becomes apparent when one sees that for many common situations, it does exist. For example

i) Coherent State:

$$\rho = |\alpha_0\rangle\langle\alpha_0|, \qquad P(\alpha, \alpha^*) = \delta^2(\alpha - \alpha_0). \qquad (4.4.47)$$

ii) Thermal State: Computed from the Fourier transform of (4.4.38)

$$P(\alpha, \alpha^*) = \frac{1}{\pi} \left(e^{\hbar\omega/kT} - 1 \right) \exp\left\{ -|\alpha|^2 (e^{\hbar\omega/kT} - 1) \right\}. \qquad (4.4.48)$$

In fact the P-function is naturally adapted to the study of thermal or quasithermal situations. For as $T \to 0$, the thermal Gaussian P-function (4.4.48) becomes infinitely sharp—thus we can say that all the broadening of this P-function is associated with the existence of thermal fluctuations. On the other hand at high temperatures,

$$P(\alpha, \alpha^*) \to \frac{\hbar\omega}{\pi kT} \exp\left(-\frac{\hbar\omega|\alpha|^2}{kT} \right) \qquad \text{as } T \to \infty \qquad (4.4.49)$$

which is the same as the high temperature limit of the corresponding Q-function, (4.4.17), and essentially amounts to the classical Boltzmann distribution.

iii) A Number State: No functional form exists for a P-function for a number state, but a solution can be given in terms of derivatives of delta functions. From the definition (4.3.1) of the coherent state it is straightforward that

$$|n\rangle\langle n| = \left[\frac{1}{n!}\left(\frac{\partial^2}{\partial\alpha\partial\alpha^*}\right)^n \left(e^{|\alpha|^2}|\alpha\rangle\langle\alpha|\right)\right]_{\alpha=0} \tag{4.4.50}$$

from which it follows that we may write

$$P_n(\alpha,\alpha^*) = \frac{1}{n!}e^{|\alpha|^2}\left(\frac{\partial^2}{\partial\alpha\partial\alpha^*}\right)^n \delta^2(\alpha). \tag{4.4.51}$$

This result is an example of the kind of generalised function required by the existence proof of Klauder and Sudarshan. Any density operator can be expressed as a series of terms of the kind $|n\rangle\langle m|$, and it is easy to generalise the result to terms of this kind and get a formal expression of the density operator in terms of such expressions as (4.4.51). To prove that the resulting expression converges is the content of the result of Klauder and Sudarshan.

iv) A Gaussian P-Function: Suppose the P-function is a Gaussian with non-zero mean,

$$P(\alpha,\alpha^*) = \frac{1}{\pi\sigma^2}\exp\left\{-\frac{(\alpha-\bar{\alpha})(\alpha-\bar{\alpha})^*}{\sigma^2}\right\}. \tag{4.4.52}$$

Then the number state matrix elements of the density operator are

$$\langle n|\rho|m\rangle = \int d^2\alpha \frac{\alpha^m\alpha^{*n}}{\sqrt{n!m!}}e^{-|\alpha|^2}\frac{1}{\pi\sigma^2}\exp\left\{-\frac{(\alpha-\bar{\alpha})(\alpha-\bar{\alpha})^*}{\sigma^2}\right\} \tag{4.4.53}$$

$$= \exp\left(-\frac{|\bar{\alpha}|^2}{\sigma^2}\right)\frac{(2|\bar{\alpha}|)^{m-n}}{(m-n)!}\frac{\sigma^{2n}}{(1+\sigma^2)^{m+1}}$$

$$\times\sqrt{\frac{m!}{n!}}M\left(m+1,m-n+1,\frac{|\bar{\alpha}|^2}{\sigma^2(1+\sigma^2)}\right) \tag{4.4.54}$$

where $m \geq n$ and $M(a,b,z)$ is the confluent hypergeometric function.[4.3]

c) Power Series for the P-function: If the density operator can be expressed as an antinormally ordered power series

$$\rho = \sum_{nm}P_{n,m}a^n(a^\dagger)^m, \tag{4.4.55}$$

then the P-function has the same power series. We simply insert the resolution of the identity $\int d^2\alpha\,|\alpha\rangle\langle\alpha|/\pi = 1$; thus

$$\rho = \frac{1}{\pi}\sum P_{n,m}\int d^2\alpha\,a^n|\alpha\rangle\langle\alpha|(a^\dagger)^m \tag{4.4.56}$$

$$= \frac{1}{\pi}\int d^2\alpha\sum P_{n,m}\alpha^n(\alpha^*)^m|\alpha\rangle\langle\alpha| \tag{4.4.57}$$

so that

$$\boxed{P(\alpha,\alpha^*) = \frac{1}{\pi}\sum P_{n,m}\alpha^n(\alpha^*)^m.} \tag{4.4.58}$$

4.4.4   The Wigner Function

Historically, this was the first quasiprobability, introduced by *Wigner* as long ago as 1932 [4.2]. In our context, it is most easily defined using its characteristic function, $\chi_W(\lambda, \lambda^*)$, which is defined by

$$\chi_W(\lambda, \lambda^*) = \mathrm{Tr}\left\{ \rho \exp\left(\lambda a^\dagger - \lambda^* a\right) \right\} \tag{4.4.59}$$

and using the Baker-Hausdorff theorem

$$\chi_W(\lambda, \lambda^*) = \exp(\tfrac{1}{2}|\lambda|^2)\chi_A(\lambda, \lambda^*) = \exp(-\tfrac{1}{2}|\lambda|^2)\chi(\lambda, \lambda^*). \tag{4.4.60}$$

The Wigner function itself is then defined as a Fourier transform,

$$W(\alpha, \alpha^*) = \frac{1}{\pi^2} \int d^2\lambda \exp(-\lambda\alpha^* + \lambda^*\alpha)\chi_W(\lambda, \lambda^*). \tag{4.4.61}$$

a) Moments of the Wigner Function: The moments of $W(\alpha, \alpha^*)$ are equal to the averages of *symmetrically ordered products* of creation and destruction operators, which we denote by $\left\{a^r(a^\dagger)^s\right\}_{\mathrm{sym}}$. The symmetrical product is the average of all ways of ordering the operators, for example

$$\left\{a^2(a^\dagger)^2\right\}_{\mathrm{sym}} = \frac{1}{6}\big\{(a^\dagger)^2 a^2 + a^\dagger a a^\dagger a + a^\dagger a^2 a^\dagger \tag{4.4.62}$$
$$+ a(a^\dagger)^2 a + a a^\dagger a a^\dagger + a^2 a^{\dagger 2}\big\}.$$

Note that $\left\{a^r(a^\dagger)^s\right\}_{\mathrm{sym}}$ is the coefficient of $\lambda^s(-\lambda^*)^r$ in $(\lambda a^\dagger - \lambda^* a)^{r+s}$, which means that

$$\exp(\lambda a^\dagger - \lambda^* a) = \sum_{r,s} \frac{(-\lambda^*)^r \lambda^s}{r!s!} \left\{a^r(a^\dagger)^s\right\}_{\mathrm{sym}}. \tag{4.4.63}$$

We then see that, by integrating by parts, in (4.4.61)

$$\int d^2\alpha\, \alpha^r(\alpha^*)^s W(\alpha, \alpha^*) = \left(\frac{\partial}{\partial\lambda}\right)^s \left(-\frac{\partial}{\partial\lambda^*}\right)^r \chi_W(\lambda, \lambda^*) \tag{4.4.64}$$

and using (4.4.59), we find

$$\left\langle\left\{a^r(a^\dagger)^s\right\}_{\mathrm{sym}}\right\rangle = \int d^2\alpha\, \alpha^r(\alpha^*)^s W(\alpha, \alpha^*). \tag{4.4.65}$$

b) Existence of the Wigner Function: The Wigner function always exists for any density matrix. The proof arises by noting that the operator

$$q \equiv -\mathrm{i}(\lambda a^\dagger - \lambda^* a) \tag{4.4.66}$$

is Hermitian, and we may take the trace in (4.4.59) with respect to the eigenstates of q which we call $|q'\rangle$. Thus

$$\chi_W(\lambda, \lambda^*) = \int dq' \langle q'|\rho|q'\rangle e^{iq'} \tag{4.4.67}$$

so that

$$|\chi_W(\lambda, \lambda^*)| \le \int dq' \langle q'|\rho|q'\rangle = 1. \tag{4.4.68}$$

Hence the Fourier transform of $\chi_W(\lambda, \lambda^*)$ always exists, with the worst singularity being that of a delta function. (This also shows that $|\chi_A(\lambda, \lambda^*)| \le \exp(-|\lambda|^2/2)$, from (4.4.60), and hence the Q-function always exists as a very regular function).

c) Wigner's Original Formulation: Wigner's original formulation of the Wigner function was as a quasiprobability for the position and momentum of a particle. He was motivated by the same concerns as are expressed in Sect. 4.2, that in the limit of large quantum numbers, the probability distribution corresponding to an eigenfunction does not approach some kind of appropriate classical distribution around the classical motion. Let us write the Wigner function in terms of position and momentum, through the use of (4.1.11) and (4.3.40). We find that

$$\bar{W}(\bar{x}, \bar{p}) = \frac{1}{(2\pi)^2} \int\int d\sigma\, d\mu \,\mathrm{Tr}\,\{\rho \exp(i\mu(x - \bar{x}) + i\sigma(p - \bar{p}))\} \tag{4.4.69}$$

where now $\bar{W}(\bar{x}, \bar{p})$ is a quasiprobability such that

$$\langle\{p^n x^m\}_{\mathrm{sym}}\rangle = \int\int d\bar{p}\, d\bar{x}\, \bar{W}(\bar{x}, \bar{p})\bar{p}^n \bar{x}^m \tag{4.4.70}$$

and where, using (4.4.61), we can write

$$\bar{W}(\bar{x}, \bar{p}) = \frac{1}{2\hbar} W(\alpha, \alpha^*) \tag{4.4.71}$$

in which, because of the definition (4.1.29) of η,

$$\alpha = \frac{i\bar{p}}{\hbar}\frac{1}{\sqrt{2\eta}} + \bar{x}\sqrt{\frac{\eta}{2}} \tag{4.4.72}$$

Suppose now we
   i) Write the trace in terms of position eigenstates

$$\mathrm{Tr}\,\{A\} = \int dx \langle x|A|x\rangle. \tag{4.4.73}$$

   ii) Use the fact that $\exp(-ipa/\hbar)$ is a displacement operator

$$\exp(-ipa/\hbar)|x\rangle = |x + a\rangle. \tag{4.4.74}$$

   iii) Use the Baker-Hausdorff theorem to write

$$\exp(i\mu x + i\sigma p) = \exp(i\mu x)\exp(i\sigma p)\exp(-i\hbar\sigma\mu/2). \tag{4.4.75}$$

Then, after a little manipulation, we find

$$\bar{W}(\bar{x}, \bar{p}) = \frac{1}{\pi\hbar} \int dy \, \langle \bar{x} + y | \rho | \bar{x} - y \rangle \exp(-2iy\bar{p}/\hbar). \tag{4.4.76}$$

If the density matrix is that corresponding to a pure state $|\psi\rangle$ then we obtain Wigner's original form

$$\bar{W}(\bar{x}, \bar{p}) = \frac{1}{\pi\hbar} \int dy \, \psi^*(\bar{x} + y)\psi(\bar{x} - y) \exp(-2iy\bar{p}/\hbar). \tag{4.4.77}$$

In this form, the Wigner function behaves like a joint probability density in $\bar{x}$ and $\bar{p}$. For example

$$\int d\bar{p} \, \bar{W}(\bar{x}, \bar{p}) = \langle \bar{x} | \rho | \bar{x} \rangle. \tag{4.4.78}$$

This is easily derived by using the representation of the delta function. It is not so easy to prove

$$\int d\bar{x} \, \bar{W}(\bar{x}, \bar{p}) = \langle \bar{p} | \rho | \bar{p} \rangle. \tag{4.4.79}$$

However, we find

$$\int d\bar{x} \, \bar{W}(\bar{x}, \bar{p}) = \frac{1}{\pi\hbar} \int \int d\bar{x} dy \, \langle \bar{x} + y | \rho | \bar{x} - y \rangle \exp(-2iy\bar{p}/\hbar). \tag{4.4.80}$$

Change variables to

$$u = \bar{x} + y, \qquad v = \bar{x} - y \tag{4.4.81}$$

and derive (4.4.79) by using the relation

$$|p\rangle = \frac{1}{\sqrt{2\pi\hbar}} \int dx \, |x\rangle e^{ipx/\hbar} \tag{4.4.82}$$

What one sees is that the probability densities in $\bar{x}$ and $\bar{p}$ respectively are obtained by integrating out the other variable, as for a classical joint probability density.

d) Examples of Wigner Functions:

i) Coherent State:
The density operator is

$$\rho = |\alpha_0\rangle\langle\alpha_0|, \tag{4.4.83}$$

and using (4.4.36) we find that

$$W(\alpha, \alpha^*) = \frac{1}{\pi^2} \int d^2\lambda \, \exp\left[-\lambda(\alpha^* - \alpha_0^*) + \lambda^*(\alpha - \alpha_0) - \tfrac{1}{2}|\lambda|^2\right], \tag{4.4.84}$$

hence

$$W_c(\alpha, \alpha^*) = \frac{2}{\pi} \exp(-2|\alpha - \alpha_0|^2). \tag{4.4.85}$$

ii) Thermal State: We Fourier transform the quantum characteristic function (4.4.38), after multiplying by $\exp(-\frac{1}{2}|\lambda|^2)$. After some minor manipulation, we find

$$W_T(\alpha, \alpha^*) = \frac{2}{\pi}\tanh(\hbar\omega/2kT) \exp\left[-2|\alpha|^2\tanh(\hbar\omega/2kT)\right]. \qquad (4.4.86)$$

iii) A Number State: if we write

$$z = e^{-\hbar\omega/kT} \qquad (4.4.87)$$

then the density operator for a thermal state is, from (4.4.13),

$$\rho_T = (1-z)\sum_n z^n|n\rangle\langle n|. \qquad (4.4.88)$$

We can identify the coefficient of $(1-z)z^n$ in the thermal Wigner function with the Wigner function for $|n\rangle\langle n|$. Substituting z in the expression (4.4.86) we find

$$W_T(\alpha, \alpha^*) = \frac{2}{\pi}(1-z)\exp(-2|\alpha|^2)\left\{\frac{1}{1+z}\exp\left(4|\alpha|^2\frac{z}{1+z}\right)\right\}. \quad (4.4.89)$$

The expression in the large curly brackets is a generating function for Laguerre polynomials [4.3], explicitly,

$$\sum_{n=0}^{\infty}L_n(x)z^n = (1-z)^{-1}\exp\left(\frac{xz}{z-1}\right). \qquad (4.4.90)$$

Hence,

$$W_n(\alpha, \alpha^*) = \frac{2(-)^n}{\pi}\exp(-2|\alpha|^2)L_n(4|\alpha|^2). \qquad (4.4.91)$$

4.4.5   Gaussian Density Operators

The concept of a Gaussian variable is fundamental to the study of classical statistical mechanics, and a quantum generalisation is obviously of interest, and indeed of use. Because of the operator nature of quantum variables the usual classical definitions of the concept "Gaussian" are not able to be exactly generalised to the quantum case. Fortunately, in the case of harmonic oscillator operators there is a unique way of defining a Gaussian density operator, and with this definition, if any one of the P- , Q- and Wigner functions is Gaussian, then all are Gaussian.

a) Definition of a Gaussian Density Operator: Let us consider only the situations in which $\langle a\rangle = \langle a^\dagger\rangle = 0$, since the incorporation of a non-zero mean is simply a matter of changing variables. A Gaussian P-function has already been considered

in Sect. 4.4.3b ii. The logical definition of a Gaussian density operator is that it is of the form

$$\rho = \mathcal{N} \exp(-na^{\dagger}a - \tfrac{1}{2}ma^{\dagger 2} - \tfrac{1}{2}m^{*}a^{2}) \tag{4.4.92}$$

where $\mathcal{N}$ is a suitable normalisation factor, and for simplicity we choose the phase of the operators so that m is real. A particular special case is that of a thermal state

$$\rho_{T} = Z^{-1} \exp\left\{ -\frac{\hbar\omega}{kT} a^{\dagger}a \right\}, \tag{4.4.93}$$

and we now show that the general Gaussian density operator can be expressed in terms of this kind of density operator. We do this by finding a unitary transformation which will diagonalise the general Gaussian density operator.

Define

$$U(\theta) = \exp\left(\frac{\theta}{2}(a^{2} - a^{\dagger 2}) \right) \tag{4.4.94}$$

which is obviously unitary if θ is real. Now let

$$a(\theta) = U(\theta)a U^{-1}(\theta). \tag{4.4.95}$$

Then

$$\begin{aligned}
\frac{\partial a(\theta)}{\partial \theta} &= \tfrac{1}{2}U(\theta)[a^{2} - a^{\dagger 2}, a] U^{-1}(\theta) &&= a^{\dagger}(\theta) \\
\frac{\partial a^{\dagger}(\theta)}{\partial \theta} &= \tfrac{1}{2}U(\theta)[a^{2} - a^{\dagger 2}, a^{\dagger}]U^{-1}(\theta) &&= a(\theta)
\end{aligned} \tag{4.4.96}$$

and the solution of the coupled equations is

$$\begin{aligned}
a(\theta) &= a \cosh\theta + a^{\dagger} \sinh\theta \\
a^{\dagger}(\theta) &= a^{\dagger} \cosh\theta + a \sinh\theta.
\end{aligned} \tag{4.4.97}$$

Now select θ so that $U(\theta)\rho U^{-1}(\theta)$ is a function of $a^{\dagger}a$ only. This occurs if

$$\tanh 2\theta = -m/n \tag{4.4.98}$$

in which case

$$U(\theta)\rho U^{-1}(\theta) = \mathcal{N} \exp(-n\,\mathrm{sech}\,2\theta a^{\dagger}a) \tag{4.4.99}$$

$$= \mathcal{N} \exp(-\sqrt{n^{2} - m^{2}}\, a^{\dagger}a). \tag{4.4.100}$$

b) Computation of the Quantum Characteristic Function: We can now use this result to compute the Wigner characteristic function of the Gaussian density operator. The Wigner characteristic function is, from (4.4.59)

$$\chi_{W}(\lambda, \lambda^{*}) = \mathrm{Tr}\left\{ \rho \exp(\lambda a^{\dagger} - \lambda^{*}a) \right\} \tag{4.4.101}$$

and we can insert the unitary transformation into this to get

$$\chi_W(\lambda, \lambda^*) = \mathrm{Tr}\left\{ U(\theta)\rho U^{-1}(\theta)U(\theta)\exp(\lambda a^\dagger - \lambda^* a)U^{-1}(\theta) \right\} \qquad (4.4.102)$$

$$= \mathrm{Tr}\left\{ \mathcal{N}\exp(-\sqrt{n^2 - m^2}\, a^\dagger a) \right.$$
$$\left. \times \exp\left(-(\lambda^*\cosh\theta - \lambda\sinh\theta)a + (\lambda\cosh\theta - \lambda^*\sinh\theta)a^\dagger \right) \right\}.$$
$$(4.4.103)$$

But the density operator is now of the same form as that of a thermal state, so we can use the result for its Wigner characteristic function to deduce

$$\chi_W(\lambda, \lambda^*) = \exp\left\{ -\tfrac{1}{2}|\lambda^*\cosh\theta - \lambda\sinh\theta|^2 \coth\left(\tfrac{1}{2}\sqrt{n^2 - m^2}\right) \right\} \qquad (4.4.104)$$

and substituting for $\cosh\theta$ and $\sinh\theta$ by (4.4.98), we find the quantum characteristic function is given by

$$\chi(\lambda, \lambda^*) = \exp(\tfrac{1}{2}|\lambda|^2)\chi_W(\lambda, \lambda^*) \qquad (4.4.105)$$

$$= \exp\left\{ \tfrac{1}{2}|\lambda|^2 + \tfrac{1}{2}\left[\frac{m(\lambda^2 + \lambda^{*2})}{2\sqrt{n^2 - m^2}} - \frac{|\lambda|^2 n}{\sqrt{n^2 - m^2}} \right]\coth\left[\tfrac{1}{2}\sqrt{n^2 - m^2}\right] \right\}.$$
$$(4.4.106)$$

c) P-, Q- and Wigner Functions: Let us define quantities

$$N \qquad = -\frac{1}{2} + \frac{n/2}{\sqrt{n^2 - m^2}}\coth\left[\tfrac{1}{2}\sqrt{n^2 - m^2}\right] \qquad (4.4.107)$$

$$M = M^* = \frac{m/2}{\sqrt{n^2 - m^2}}\coth\left[\tfrac{1}{2}\sqrt{n^2 - m^2}\right] \qquad (4.4.108)$$

so that the characteristic function can be written

$$\chi(\lambda, \lambda^*) = \exp\left(-N|\lambda|^2 + \tfrac{1}{2}M^*\lambda^2 + \tfrac{1}{2}M\lambda^{*2} \right). \qquad (4.4.109)$$

From this characteristic function it is straightforward to show that the P-, Q- and

Wigner functions are

$$P(\alpha, \alpha^*) = \frac{1}{\pi} \left(N^2 - |M|^2 \right)^{-\frac{1}{2}}$$
$$\times \exp \left\{ \frac{-N|\alpha|^2 + \frac{1}{2}M^*\alpha^2 + \frac{1}{2}M\alpha^{*2}}{N^2 - |M|^2} \right\} \tag{4.4.110}$$

$$Q(\alpha, \alpha^*) = \frac{1}{\pi} \left((N+1)^2 - |M|^2 \right)^{-\frac{1}{2}}$$
$$\times \exp \left\{ \frac{-(N+1)|\alpha|^2 + \frac{1}{2}M^*\alpha^2 + \frac{1}{2}M\alpha^{*2}}{(N+1)^2 - |M|^2} \right\} \tag{4.4.111}$$

$$W(\alpha, \alpha^*) = \frac{1}{\pi} \left((N+\tfrac{1}{2})^2 - |M|^2 \right)^{-\frac{1}{2}}$$
$$\times \exp \left\{ \frac{-(N+\frac{1}{2})|\alpha|^2 + \frac{1}{2}M^*\alpha^2 + \frac{1}{2}M\alpha^{*2}}{(N+\frac{1}{2})^2 - |M|^2} \right\}. \tag{4.4.112}$$

These three functions are obviously very similar. For all of them we find

$$\langle \alpha^2 \rangle = \langle \alpha^{*2} \rangle = M \tag{4.4.113}$$

but

$$\langle |\alpha|^2 \rangle_Q \quad = N + 1 \tag{4.4.114}$$
$$\langle |\alpha|^2 \rangle_P \quad = N \tag{4.4.115}$$
$$\langle |\alpha|^2 \rangle_W \quad = N + \tfrac{1}{2}. \tag{4.4.116}$$

However, the conditions for these functions to exist as normalisable Gaussians are different from the conditions for them to correspond to physical density operators. There is a physical requirement

$$N(N+1) \geq |M|^2 \tag{4.4.117}$$

which arises from the requirement that

$$\langle (a^\dagger + za)(a + z^*a^\dagger) \rangle \geq 0 \qquad \text{for all } z. \tag{4.4.118}$$

Notice that this means that there are density operators which satisfy this condition, but do not yield normalisable P-functions, since the P-function is normalisable only if $N^2 \geq |M|^2$, which is weaker than (4.4.116). Conversely, the conditions for normalisability of the Q-function and the Wigner function, respectively $(N+1)^2 \geq |M|^2$ and $N(N+1) \geq |M|^2 - 1/4$, are both weaker than the condition for a physical density operator, so that there exist normalisable Q- and Wigner functions which do not correspond to a positive density operator.

d) Higher Moments for Gaussian Density Operators: Since a Gaussian density operator implies a Gaussian P-, Q- and Wigner function, the relations for the

moments of classical Gaussian variables can be used to obtain corresponding relations in the quantum case. Thus, according to S.M.Sect. 2.8.1, if X_i are Gaussian variables with $\langle X_i \rangle = 0$, then

$$\langle X_i X_j X_k \cdots \rangle = \frac{(2N)!}{N!2^N} \{\sigma_{ij}\sigma_{kl}\sigma_{mn} \cdots \}_{\text{sym}} \tag{4.4.119}$$

where $\sigma_{ij} = \langle X_i X_j \rangle$, the order of the moment is $2N$, and the subscript "sym" means the symmetrised form of the product of σ's.

Because a Gaussian density operator gives Gaussian P-, Q-, or Wigner functions, we will be able to deduce from (4.4.119) relations for normal, antinormal and symmetric ordering. Thus, for example

$$
\begin{aligned}
\langle a^{\dagger 4} \rangle &= 3 \langle a^{\dagger 2} \rangle \langle a^{\dagger 2} \rangle \\
\langle a^{\dagger 3} a \rangle &= 3 \langle a^{\dagger 2} \rangle \langle a^{\dagger} a \rangle \\
\langle a^{\dagger 2} a^2 \rangle &= \langle a^{\dagger 2} \rangle \langle a^2 \rangle
\end{aligned}
\tag{4.4.120}
$$

and similar equations will be true for antinormal and symmetric ordering.

In the study of optical correlations and the Hanbury-Brown Twiss effect, (treated in Sect. 6.3.4) we will find formulae involving fourth order moments of two different Gaussian modes. A result which is then applied is the particular case of this general Gaussian result:

$$\langle a_1^{\dagger} a_2^{\dagger} a_2 a_1 \rangle = \langle a_1^{\dagger} a_2^{\dagger} \rangle \langle a_2 a_1 \rangle + \langle a_1^{\dagger} a_2 \rangle \langle a_2^{\dagger} a_1 \rangle + \langle a_1^{\dagger} a_1 \rangle \langle a_2^{\dagger} a_2 \rangle. \tag{4.4.121}$$

In the common case that $\langle a_1^{\dagger} a_2^{\dagger} \rangle = \langle a_2 a_1 \rangle = 0$, this reduces to

$$g^{(2)} = 1 + |g^{(1)}|^2 \tag{4.4.122}$$

where

$$g^{(1)} = \frac{\langle a_1^{\dagger} a_2 \rangle}{\sqrt{\langle a_1^{\dagger} a_1 \rangle \langle a_2^{\dagger} a_2 \rangle}} \tag{4.4.123}$$

$$g^{(2)} = \frac{\langle a_1^{\dagger} a_2^{\dagger} a_2 a_1 \rangle}{\langle a_1^{\dagger} a_1 \rangle \langle a_2^{\dagger} a_2 \rangle}. \tag{4.4.124}$$

These results will be true when a_1, a_2 are any linear combinations of destruction operators, which is what turns up in the theory of photon counting.

4.5   Operator Correspondences and Equations of Motion

The action of an operator on a density operator ρ is mirrored by the action of a corresponding differential operator on the P-, Q- or Wigner function. This can make

the solution of the equations of motion quite simple in certain cases. We already know the basic relations

$$a|\alpha\rangle = \alpha|\alpha\rangle \tag{4.5.1}$$

$$\langle\alpha|a^\dagger = \alpha^*\langle\alpha|. \tag{4.5.2}$$

For the other possible ways of acting with the $a, a^\dagger$, it is convenient to use the Bargmann states, defined in Sect. 4.3.1, $\|\alpha\rangle$, so that

$$a^\dagger \|\alpha\rangle = \sum_n \frac{\alpha^n}{\sqrt{n!}} \sqrt{n+1}\, |n+1\rangle \tag{4.5.3}$$

$$= \frac{\partial}{\partial\alpha} \|\alpha\rangle. \tag{4.5.4}$$

Similarly,

$$\langle\alpha\| a = \frac{\partial}{\partial\alpha^*} \langle\alpha\|. \tag{4.5.5}$$

In terms of the Bargmann states, we can write the P-representation in the form

$$\rho = \int d^2\alpha \, \|\alpha\rangle\langle\alpha\| \, e^{-\alpha\alpha^*} P(\alpha, \alpha^*). \tag{4.5.6}$$

Thus, we have

$$a^\dagger\rho = \int d^2\alpha \frac{\partial}{\partial\alpha}(\|\alpha\rangle)\langle\alpha\| \, e^{-\alpha\alpha^*} P(\alpha, \alpha^*). \tag{4.5.7}$$

and integrating by parts

$$= \int d^2\alpha \, \|\alpha\rangle\langle\alpha\| \, e^{-\alpha\alpha^*} \left(\alpha^* - \frac{\partial}{\partial\alpha}\right) P(\alpha, \alpha^*). \tag{4.5.8}$$

We thus can make an *operator correspondence* between $a^\dagger$ and $\alpha^* - \partial/\partial\alpha$. A similar formula holds for a. Summarising, with the obvious correspondences arising from (4.5.1,2), we have

$$
\begin{aligned}
a\rho &\leftrightarrow \alpha P(\alpha, \alpha^*) \\
a^\dagger\rho &\leftrightarrow \left(\alpha^* - \frac{\partial}{\partial\alpha}\right) P(\alpha, \alpha^*) \\
\rho a &\leftrightarrow \left(\alpha - \frac{\partial}{\partial\alpha^*}\right) P(\alpha, \alpha^*) \\
\rho a^\dagger &\leftrightarrow \alpha^* P(\alpha, \alpha^*).
\end{aligned}
\tag{4.5.9}
$$

For the Q- and Wigner functions, a similar set of relations holds. These are

$$
\begin{aligned}
a\rho &\leftrightarrow \left(\alpha + \frac{\partial}{\partial\alpha^*}\right) Q(\alpha, \alpha^*) \\
a^\dagger\rho &\leftrightarrow \alpha^* Q(\alpha, \alpha^*) \\
\rho a &\leftrightarrow \alpha Q(\alpha, \alpha^*) \\
\rho a^\dagger &\leftrightarrow \left(\alpha^* + \frac{\partial}{\partial\alpha}\right) Q(\alpha, \alpha^*).
\end{aligned}
$$

(4.5.10)

For the Wigner function it is simplest to use (4.4.76) to show that

$$
\begin{aligned}
x\rho &\leftrightarrow \left(x + \frac{i\hbar}{2}\frac{\partial}{\partial p}\right) \bar{W}(x, p) \\
\rho x &\leftrightarrow \left(x - \frac{i\hbar}{2}\frac{\partial}{\partial p}\right) \bar{W}(x, p) \\
p\rho &\leftrightarrow \left(p - \frac{i\hbar}{2}\frac{\partial}{\partial x}\right) \bar{W}(x, p) \\
\rho p &\leftrightarrow \left(p + \frac{i\hbar}{2}\frac{\partial}{\partial x}\right) \bar{W}(x, p).
\end{aligned}
$$

(4.5.11)

Using the relation (4.4.72) between $\alpha, \bar{x},$ and $\bar{p}$, it is straightforward that

$$
\begin{aligned}
a\rho &\leftrightarrow \left(\alpha + \frac{1}{2}\frac{\partial}{\partial\alpha^*}\right) W(\alpha, \alpha^*) \\
a^\dagger\rho &\leftrightarrow \left(\alpha^* - \frac{1}{2}\frac{\partial}{\partial\alpha}\right) W(\alpha, \alpha^*) \\
\rho a &\leftrightarrow \left(\alpha - \frac{1}{2}\frac{\partial}{\partial\alpha^*}\right) W(\alpha, \alpha^*) \\
\rho a^\dagger &\leftrightarrow \left(\alpha^* + \frac{1}{2}\frac{\partial}{\partial\alpha}\right) W(\alpha, \alpha^*).
\end{aligned}
$$

(4.5.12)

4.5.1   Application to the Driven Harmonic Oscillator

We consider the Hamiltonian

$$
H = \hbar\omega(a^\dagger a + \tfrac{1}{2}) + (\lambda a^\dagger + \lambda^* a)
$$

(4.5.13)

for which the von Neumann's equation is

$$
i\hbar\frac{\partial\rho}{\partial t} = \hbar\omega[a^\dagger a, \rho] + \lambda[a^\dagger, \rho] + \lambda^*[a, \rho].
$$

(4.5.14)

We now turn this into an equation for $P(\alpha, \alpha^*)$ by using the operator correspondences (4.5.9). Thus

$$a^\dagger a \rho \rightarrow \left(\alpha^* - \frac{\partial}{\partial \alpha}\right) \alpha P \tag{4.5.15}$$

$$-\rho a^\dagger a \rightarrow -\left(\alpha - \frac{\partial}{\partial \alpha^*}\right) \alpha^* P. \tag{4.5.16}$$

Notice that the order of the operators in (4.5.16) reverses, since acting on ρ they operate from the right, whereas on P, they operate from the left. Also,

$$[a^\dagger, \rho] \rightarrow \left[\left(\alpha^* - \frac{\partial}{\partial \alpha}\right) - \alpha^*\right] P = -\frac{\partial}{\partial \alpha} P \tag{4.5.17}$$

and similarly,

$$[a, \rho] \rightarrow \frac{\partial}{\partial \alpha^*} P \tag{4.5.18}$$

so that we find

$$\frac{\partial P}{\partial t} = \mathrm{i} \left(-\omega \frac{\partial}{\partial \alpha} \alpha + \omega \frac{\partial}{\partial \alpha^*} \alpha^* - \frac{\lambda}{\hbar} \frac{\partial}{\partial \alpha} + \frac{\lambda^*}{\hbar} \frac{\partial}{\partial \alpha^*}\right) P. \tag{4.5.19}$$

This corresponds to a Liouville equation for the P- function. A word of caution. It is tempting to treat α and α^* as independent variables which is not strictly true, and in writing all the above correspondences, one should really write

$$\begin{aligned}
\alpha &= x + \mathrm{i} y \\
\alpha^* &= x - \mathrm{i} y \\
\frac{\partial}{\partial \alpha} &= \frac{1}{2}\left(\frac{\partial}{\partial x} - \mathrm{i} \frac{\partial}{\partial y}\right) \\
\frac{\partial}{\partial \alpha^*} &= \frac{1}{2}\left(\frac{\partial}{\partial x} + \mathrm{i} \frac{\partial}{\partial y}\right)
\end{aligned} \tag{4.5.20}$$

and $\lambda = \mu + \mathrm{i}\nu$. In terms of these real variables, (4.5.19) becomes

$$\frac{\partial P}{\partial t} = \left[\frac{\partial}{\partial x}(\omega y + \nu/\hbar) - \frac{\partial}{\partial y}(\omega x + \mu/\hbar)\right] P \tag{4.5.21}$$

which is a Liouville equation, equivalent to the differential equations

$$\begin{aligned}
\frac{dx}{dt} &= -\omega y - \nu/\hbar \\
\frac{dy}{dt} &= \omega x + \mu/\hbar
\end{aligned} \tag{4.5.22}$$

which are equivalent to one complex differential equation

$$\frac{d\alpha}{dt} = \mathrm{i}(\omega \alpha + \lambda/\hbar) \tag{4.5.23}$$

with the solution

$$\alpha = -\lambda/\hbar\omega + \xi e^{i\omega t}. \tag{4.5.24}$$

The solution for $P(\alpha, \alpha^*)$ is, assuming a deterministic initial condition,

$$P(\alpha, \alpha^*, t) = \delta\left(x - \mathrm{Re}\left\{-\frac{\lambda}{\hbar\omega} + \xi e^{i\omega t}\right\}\right)\delta\left(y - \mathrm{Im}\left\{-\frac{\lambda}{\hbar\omega} + \xi e^{i\omega t}\right\}\right) \tag{4.5.25}$$

$$= \delta^2(\alpha + \lambda/\hbar\omega - \xi e^{i\omega t}) \tag{4.5.26}$$

where the delta function of a complex variable is defined by these equations. Notice that λ may depend on time, in which case (4.5.23) becomes

$$\frac{d\alpha}{dt} = i[\omega\alpha + \lambda(t)/\hbar\,] \tag{4.5.27}$$

whose solution is

$$\alpha(t) = \alpha(0)e^{i\omega t} + i\int\limits_{0}^{t} dt' e^{i\omega(t-t')}\lambda(t')/\hbar \tag{4.5.28}$$

and the corresponding P is

$$P(\alpha, \alpha^*, t) = \delta^2[\alpha - \alpha(t)]. \tag{4.5.29}$$

Exercise. Show that exactly the same equation (4.5.19) arises for the Q-function and the Wigner function.

From the results above, it can be seen that, if one views any one of P, Q, or W as a quasiprobability, one gets an equation of motion which is the same as that of a classical system. This is the basis of the idea of quantum-classical correspondence. It is a formal expression of the fact that in harmonic systems the classical and quantum systems both possess only one frequency.

Exercise. Suppose we have a simplified anharmonic oscillator Hamiltonian

$$H = \tfrac{1}{2}\hbar\omega(a^+a + \tfrac{1}{2}) + g(a^+a)^2. \tag{4.5.30}$$

What is the P-representation equation of motion in this case?

4.5.2 The Wigner Function and the Quasiclassical Langevin Equation

We will now show that it is possible to find a rather simple stochastic differential equation which is in some cases exactly equivalent to the adjoint equation. To do this it is most advantageous to use the Wigner function whch was introduced in Sect. 4.4.4. The equation of motion for the Wigner function, in the case that H_{sys} is the Hamiltonian for a particle moving in a potential, i.e.,

$$H_{\mathrm{sys}} = \frac{p^2}{2m} + V(x), \tag{4.5.31}$$

is derived by the substitution of the operator correspondences for the Wigner function, (4.5.11) into the adjoint equation, (3.5.32). The result is

$$\frac{\partial W}{\partial t} = \left\{ -\frac{\partial}{\partial x}\frac{p}{m} + \frac{\partial}{\partial p}\left(V'(x) + \gamma p/m - \alpha(t)\right) \right\} W$$

$$+ \left\{ \sum_{n=1}^{\infty} \frac{(i\hbar/2)^{2n}}{(2n+1)!} \frac{\partial^{2n+1}}{\partial p^{2n+1}} V^{2n+1}(x) \right\} W. \qquad (4.5.32)$$

There are a number of situations in which the second line vanishes or is negligible:

i)   If $V(x) = Ax + Bx^2$, i.e., we are dealing with a harmonic oscillator, a linear potential, or a free particle.

ii)   If $\hbar \to 0$.

iii)   If $\gamma \to \infty$: the large friction case. For in this case, $\alpha(t)$ also becomes larger, and this represents the dominant noise term. The last line in (4.5.32) is a kind of noise, which is independent of γ, and is then presumably negligible in this limit. (This argument certainly lacks rigour, but can probably be made rigorous.)

In all these cases, we are left with a conventional stochastic Liouville equation for the Wigner function, equivalent to the classical Langevin equation

$$\dot{x} = p/m$$
$$\dot{p} = -V'(x) - \gamma p/m + \alpha(t) \qquad (4.5.33)$$

in which, however, the noise spectrum is given by the right hand side of (3.3.10).

Schmidt [4.5] has called this equation the *quasiclassical Langevin equation*. It is a mixture of classical stochastic theory and quantum mechanics, in which the variables are non-operator quantities, while the spectrum of the noise is given by the expression given by Callen and Welton, as noted in Sect. 1.2. Its validity is limited, however, according to the above derivation as well as that of Schmidt, to situations in which the friction coefficient γ is large. How large γ must be can be determined by the following argument. By using the Wigner function in the adjoint equation we get an exact representation of the quantum mechanical operators. Thus, the terms on the right hand side of (4.5.32) which do not involve $\alpha(t)$ or γ faithfully represent the operator H_{sys}. The characteristic frequencies which turn up will then be the correct quantised frequencies, i.e., they will correspond to the atomic energy levels, and if we apply van Kampen's cumulant expansion to (4.5.32) we will get the master equation, in the same way as was done in Sects. 3.6.1,3. By dropping the terms with higher derivatives, one finds that the characteristic frequencies of the Liouville operator are those of the classical orbits in the potential $V(x)$, which are of course quite different from the quantum frequencies, though they do approach the classical frequencies in the large scale limit. For these to be compatible with each other the friction must be sufficiently large to broaden the quantum energy levels to the extent that the motion is no longer distinguishable from the classical motion. At the very least, the broadening should be so large that the levels overlap. This will happen most easily when levels are very close together

The quasiclassical Langevin equation been used by *Koch* et al. [4.6] to analyse the experiments on quantum noise mentioned in Sect. 1.2 in which the spectrum of

quantum noise was measured. In this analysis, a high friction limit was assumed, which means that the analysis is within the regime of validity of the quasiclassical Langevin equation.

In fact there does exist a body of knowledge called "stochastic electrodynamics" in which various authors [4.7] have built up a theory in which a classical particle interacts with a random electromagnetic field, whose statistics are chosen to give a Planck spectrum. A generalisation to a full three dimensional formulation of inputs and outputs, in which the appropriate wave equation corresponded to electrodynamics, would yield a quasiclassical Langevin equation of exactly this form, which would have the same range of validity. Exact agreement between stochastic electrodynamics and quantum theory is found for assemblies of harmonic oscillators, and for free particles. From this point of view this is not surprising—but the terms in the second line of (4.5.32) will make their presence felt in all other cases, and stochastic electrodynamics cannot be a valid representation of reality for general situations.

Appendix 4A. The Baker-Hausdorff Formula

In the most general form, consider two time dependent operators, $A(t)$, $B(t)$, such that for all t, t'

$$[A(t), A(t')] = [B(t), B(t')] = 0 \tag{4A.1}$$

$$[A(t), B(t')] = f(t, t') \tag{4A.2}$$

and $f(t, t')$ commutes with $A(t'')$ and $B(t'')$ for all t, t', t''.

Suppose that $V(t)$ is an operator such that

$$\frac{dV(t)}{dt} = \{A(t) + B(t)\}V(t), \qquad V(0) = 1. \tag{4A.3}$$

Then write

$$U(t) = \exp\left\{-\int_0^t B(t')\,dt'\right\}V(t) \tag{4A.4}$$

so that, using (4A.1),

$$\frac{dU(t)}{dt} = \exp\left\{-\int_0^t B(t')\,dt'\right\}A(t)U(t) \tag{4A.5}$$

$$= A(t)U(t) - \int_0^t f(t, t')U(t) \tag{4A.6}$$

where, in going from (4A.5) to (4A.6), we have used the formula

$$\exp(-F)G\exp(F) = \sum_n \frac{(-1)^n}{n!}[F, G]_n \tag{4A.7}$$

where the iterated commutator is defined by

$$[F, G]_n = [F, [F, G]_{n-1}] \tag{4A.8}$$

and

$$[F, G]_1 = [F, G], \tag{4A.9}$$

and have used the fact that the first commutator $f(t, t')$ commutes with $B(t'')$ for all t'', so that only the first two terms in the expansion (4A.7) survive. Now using the fact that $f(t, t')$ commutes with all $A(t'')$, we can solve (4A.6), and resubstitute into (4A.4) to get $V(t)$ as

$$V(t) = \exp\left\{\int_0^t B(s)\, ds\right\} \exp\left\{\int_0^t A(s)\, ds\right\} \exp\left\{-\int_0^t ds \int_0^s ds' f(s, s')\right\}. \tag{4A.10}$$

4A1. Corollaries

a) **Baker-Hausdorff Formula:** If $A(t), B(t)$ are constants, A, B, then (4A.3) can be solved by direct integration to give $V(t)$, i.e., after setting $t = 1$

$$V(t) = \exp(A + B) = \exp(A)\exp(B)\exp(-\tfrac{1}{2}[A, B]) \tag{4A.11}$$

which is the Baker-Hausdorff formula.

b) **Generalisation of the Baker-Hausdorff Formula:** If we now substitute in (4A.11)

$$A \to \int_0^t ds\, A(s), \qquad B \to \int_0^t ds\, B(s), \tag{4A.12}$$

which satisfy the conditions required, then (4A.11) becomes

$$V(t) = \exp\left\{\int ds\{A(s) + B(s)\}\right\} \tag{4A.13}$$

which is a slightly more elegant form than (4A.10), but perhaps not so useful.

5. Quantum Markov Processes

The concept of a Markov process is a very powerful and surprisingly general tool in the study of classical irreversible phenomena. Classically, the essence of the Markov idea is that we need only know the probability distribution at time t in order to predict it for all future times. At the deepest level in classical physics this is exactly true—Hamiltons equations of motion are first order in time, so that the knowledge of the canonical co-ordinates and momenta at one time determines them for all future times.

At a coarser level, even classically the Markov property must be regarded as an approximation. For, in both classical and quantum systems, damping of a macroscopic system comes from the transfer of energy to microscopic degrees of freedom, which, in the macroscopic description, give rise to noise terms in the macroscopic equations of motion. If the motion of the noise degrees of freedom is very fast, then on any slower time scale they will appear to be almost delta correlated, and the equation of motion will appear like a white noise stochastic differential equation—that is, a Markov process. It is also very fortunate that it is quite easy to embed a non-Markov process in a larger Markov process, basically by considering no Markov noise to be itself given as the solution of a Markovian equation of motion.

The theory of Markov processes has two main analytic descriptions, by means of stochastic differential equations, or by Fokker-Planck equations, or the corresponding discrete versions of these, and the interplay between one method and the other yields a powerful and flexible way of understanding and computing physically interesting quantities. As well as this there is the dual description of the stochastic differential equation by the Ito and the Stratonovich techniques, a fact which has caused much confusion and even evangelical zeal [5.1]. It is unfortunately the case that both of these techniques have their uses, and it is not possible to avoid the use of either definition. Indeed, there is considerable insight to be gained from knowing and understanding both techniques.

Until quite recently, there has been no description of quantum Markov processes which possesses the completeness of the classical version; that is, a duality between stochastic differential equations and the master equation (the quantum version of the Fokker-Planck equation), nor had the concepts of Ito and Stratonovich quantum stochastic differential equations been formulated. The reason for this lies in the rather rigid requirements of quantum mechanics, the most important of which is to ensure that the canonical commutation relations are preserved.

Nevertheless, quantum Langevin equations and the master equation have quite a long history. Quantum Langevin equations were first actually used in a quantum optical setting by *Senitzky* [5.2]. The first real application was to laser theory, which

was done almost simultaneously by *Lax* [5.3] and *Haken* [5.4]. The master equation approach was developed in a quantum optical context by *Louisell* (see [5.5] for a summary or his book [5.6] for a more detailed presentation.) In all of these approaches it was necessary to develop rather careful methods which did not obscure the quantum mechanics underlying the system under study. An approach based on the use of the Pauli master equation, which does not consider off diagonal density operator matrix elements, would simply not have been adequate.

This chapter will develop all quantum analogues of the classical Markovian techniques. It starts with two further derivations of the master equation, one of them a traditional "physical" derivation, and the other based on the method of adiabatic elimination of fast variables, using projection operator techniques to assist in the algebra. This method gives a rather clear understanding of the limits involved. These derivations, unlike those in Chap.3, do not depend on any particular model of a heat bath, and therefore demonstrate that the concept of a master equation is really much more general than one would have supposed from the work of that chapter.

The next section is devoted to the formulation of the full multitime structure of a quantum Markov process, giving abstract formulae for all multitime correlation functions, and shows how close the structures of classical and quantum Markov processes are to each other.

The next section develops the theory of quantum stochastic differential equations as a Markovian version of the noise input and output theory of Chap.3. The quantum stochastic differential equation (QSDE) is a much more rigidly prescribed object than its classical counterpart, and methods are given for writing down both Ito and Stratonovich versions corresponding to a given coupling to a heat bath. Here the heat bath is of harmonic oscillators. The final section develops the equivalence between the master equation and the QSDE, and shows how we may relate the correlation structure of the outputs to those of the inputs and the operators whose equations of motion are given by the QSDE.

5.1   The Physical Basis of the Master Equation

In the previous chapter a derivation of the master equation was given. This section concentrates on two further ways of deriving the master equation. The first is firmly grounded in physical intuition, and gives the clearest physical idea of the physics behind the master equation. An alternative derivation based on the idea of projection operators is presented next, for this gives a particularly compact way of deriving the master equation at the cost of introducing a certain degree of abstraction.

5.1.1   Derivation of the Quantum Optical Master Equation

We consider a situation described by a Hamiltonian of the form

$$H = H_{\text{sys}} + H_B + H_{\text{Int}} \tag{5.1.1}$$

where the constituent Hamiltonians describe respectively the system, the heat bath, and the interaction. The total density operator for the system and reservoir, $\rho_{tot}(t)$, in the Schrödinger picture, satisfies

$$\dot{\rho}_{tot} = -\frac{i}{\hbar}[H_{sys} + H_B + H_{Int}, \rho_{tot}]. \tag{5.1.2}$$

We want an equation for a quantity which can give us the means of *system* operators, and this is, of course, given by the *reduced density operator*

$$\hat{\rho}(t) = \text{Tr}_B\{\rho_{tot}(t)\}. \tag{5.1.3}$$

a) Transformation to the Interaction Picture: It is most convenient to transform the equation of motion into the interaction picture, defined by setting

$$\rho_I(t) = \exp\left[\frac{i}{\hbar}(H_{sys} + H_B)t\right] \rho_{tot}(t) \exp\left[-\frac{i}{\hbar}(H_{sys} + H_B)t\right]. \tag{5.1.4}$$

It is straightforward to show that $\rho_I(t)$ obeys the interaction picture equation of motion

$$\dot{\rho}_I(t) = -\frac{i}{\hbar}\left[H_{Int}(t), \rho_I(t)\right] \tag{5.1.5}$$

where

$$H_{Int}(t) = \exp\left[\frac{i}{\hbar}(H_{sys} + H_B)t\right] H_{Int} \exp\left[-\frac{i}{\hbar}(H_{sys} + H_B)t\right]. \tag{5.1.6}$$

Notice that, from (5.1.3,4),

$$\hat{\rho}(t) = \text{Tr}_B\left\{\exp\left[-\frac{i}{\hbar}(H_{sys} + H_B)t\right] \rho_I(t) \exp\left[\frac{i}{\hbar}(H_{sys} + H_B)t\right]\right\} \tag{5.1.7}$$

and since H_B is a function only of bath variables, we can use the cyclic property of the trace to cancel the factors involving H_B, to get

$$\hat{\rho}(t) = \exp\left[-\frac{i}{\hbar}H_{sys}t\right] \rho(t) \exp\left[\frac{i}{\hbar}H_{sys}t\right], \tag{5.1.8}$$

where

$$\rho(t) \equiv \text{Tr}_B\{\rho_I(t)\} \tag{5.1.9}$$

is the interaction picture reduced density operator.

b) Initial Conditions: We assume that the system and the bath are initially independent, so that the total density operator factorises into a direct product

$$\rho_{tot}(0) = \hat{\rho}(0) \otimes \rho_B. \tag{5.1.10}$$

In addition, we assume that *the reservoir is so large that its statistical properties are unaffected by the weak coupling to the system*. This assumption is essential for the derivation.

c) Integration of the Equations of Motion: We can return to (5.1.5) and integrate from 0 to t subject to the initial condition (5.1.10). This gives, after two iterations

$$\rho_I(t) = \rho_I(0) - \frac{i}{\hbar} \int\limits_0^t dt' \left[H_{\text{Int}}(t'), \rho_I(0) \right]$$

$$- \frac{1}{\hbar^2} \int\limits_0^t dt' \int\limits_0^{t'} dt'' \left[H_{\text{Int}}(t'), \left[H_{\text{Int}}(t''), \rho_I(t'') \right] \right]. \qquad (5.1.11)$$

One could proceed to iterate this way, and would obtain a power series in the perturbation $H_{\text{Int}}(t)$. However, this would not be very useful—to obtain an exponential decay law, for example, clearly requires an infinite number of iterations. One therefore differentiates (5.1.11) with respect to t, to get an integro-differential equation for $\rho_I(t)$

$$\dot{\rho}_I(t) = -\frac{i}{\hbar} \left[H_{\text{Int}}(t), \rho_I(0) \right] - \frac{1}{\hbar^2} \int\limits_0^t dt' \left[H_{\text{Int}}(t), \left[H_{\text{Int}}(t'), \rho_I(t') \right] \right]. $$

$$(5.1.12)$$

d) Trace over the Bath Variables: If we trace both sides over the bath variables, and use (5.1.9), we obtain

$$\dot{\rho}(t) = -\frac{1}{\hbar^2} \int\limits_0^t dt' \, \text{Tr}_{\text{B}} \left\{ \left[H_{\text{Int}}(t), \left[H_{\text{Int}}(t'), \rho_I(t') \right] \right] \right\} \qquad (5.1.13)$$

where we have assumed

$$\text{Tr}_{\text{B}} \left\{ H_{\text{Int}}(t) \rho_I(0) \right\} = 0 \qquad (5.1.14)$$

and

$$\rho_I(0) = \rho_{\text{tot}}(0) = \hat{\rho}(0) \otimes \rho_B \qquad (5.1.15)$$

in (5.1.10). This means that we assume that the interaction has no diagonal elements in the representation in which H_B is diagonal. In practice this is not a great restriction, and we can always redefine H_{sys} and H_{Int} to include any such diagonal elements in H_{sys}.

e) Weak Coupling Assumption: At this point we assume that H_{Int} is very much less than either H_B or H_{sys}, and that the reservoir density operator is not significantly affected by the interaction. If this is the case, we might be justified in replacing $\rho_I(t')$ in (5.1.13) by a factorised approximation

$$\rho_I(t') \approx \rho(t') \otimes \rho_B \qquad (5.1.16)$$

in which we have made the following assumptions :

i)   The bath density operator is not significantly changed by the interaction.

ii)   However we have allowed the system density operator to change significantly, on the ground that the system is much smaller than the bath, and hence the fractional effect of the interaction is much bigger on the system.

iii)   We have assumed that the density operator may be written approximately as a direct product. In fact, a much weaker assumption is normally sufficient, since in practice the interaction can be written as a sum of terms like

$$X_{sys} A_B \tag{5.1.17}$$

where both of these are reasonably simple operators which act respectively only in the system and the bath spaces. All we really need then, is that approximations like

$$\text{Tr}_B \left\{ \left[A_B(t), [A_B(t'), \rho_I(t')] \right] \right\} \approx \rho(t') \otimes \text{Tr}_B \left\{ \left[A_B(t), [A_B(t'), \rho_B] \right] \right\} \tag{5.1.18}$$

are valid. Essentially, this means that bath correlation functions are not significantly changed by the interaction.

f) Markov Approximation: Using the weak coupling assumption, we find that (5.1.13) can be reduced to an integro-differential equation for $\rho(t)$; namely

$$\dot{\rho}(t) = -\frac{1}{\hbar^2} \int_0^t dt' \, \text{Tr}_B \left\{ \left[H_{\text{Int}}(t), [H_{\text{Int}}(t'), \rho(t') \otimes \rho_B] \right] \right\}. \tag{5.1.19}$$

The final approximation is now to turn this into a differential equation, and this is done by making an assumption about the bath correlation functions, such as (5.1.18), which turn up in this equation. Since the interaction is assumed weak, the rate of change of the interaction picture system density operator will be quite slow compared to that of the bath operators, such as $A_B(t)$, $A_B(t')$, which will vary on a time scale determined from (5.1.6) by H_B. In fact, the bath correlation functions will generally be determined by a *thermal* choice of ρ_B, and in this case the correlation times will be like the thermal correlation time of Sect. 3.3.3, which are much shorter than the typical time constant expected from $\rho(t)$. If this is the case, we can say that the factor $\rho(t')$ changes insignificantly over the time taken for the correlation functions in (5.1.19) to vanish. In this case we can

i)   Set $\rho(t') \rightarrow \rho(t)$.

ii)   For $t \gg$ thermal correlation time, we can let the lower limit of the time integral go to $-\infty$.

In this case, we deduce the master equation :

$$\dot{\rho}(t) = -\frac{1}{\hbar^2} \int_0^\infty dt \, \text{Tr}_B \left\{ \left[H_{\text{Int}}(t), [H_{\text{Int}}(t-\tau), \rho(t) \otimes \rho_B] \right] \right\}. \tag{5.1.20}$$

In this form, by inserting appropriate forms for H_{Int} and H_{sys}, we can derive a number of useful master equations. The approximation, $\rho(t') \to \rho(t)$ is known as the *Markov approximation* since it yields a first order (operator) differential equation for $\rho(t)$—this means that the knowledge of $\rho(t)$ at one point in time $t = t_0$ is sufficient to determine $\rho(t)$ for all $t > t_0$. This is exactly analogous to the Markov property in classical stochastic processes, where a similar property is true. The Markov property is a highly desirable property from a mathematical point of view, because a whole structure of measurement theory can be built around it in a compact and self-contained way. The elegance of this structure leads to the formulation in the abstract of the concept of the quantum Markov process as a branch of mathematics. Nevertheless, it is important to remember that it is an assumption, based on the existence of short correlation times in the heat bath, and the use of perturbation theory.

5.1.2 A Derivation Based on Projection Operators

We start from the equation of motion (5.1.2) in the form

$$\dot{\rho}_{\text{tot}} = (\mathcal{L}_{\text{sys}} + \mathcal{L}_B + \mathcal{L}_{\text{Int}})\rho_{\text{tot}} \qquad (5.1.21)$$

where the *Liouvillians* $\mathcal{L}$ are a shorthand notation for the commutator terms. One is interested in an equation for

$$\hat{\rho}(t) \equiv \text{Tr}_B \{\rho_{\text{tot}}(t)\}, \qquad (5.1.22)$$

and this can be viewed as a projection of the operator $\rho_{\text{tot}}(t)$ into a subspace. One can formalise this by writing a projection operator, P, given by

$$P\rho_{\text{tot}}(t) = \text{Tr}_B \{\rho_{\text{tot}}(t)\} \otimes \rho_B \qquad (5.1.23)$$

where ρ_B is a density operator for the bath. One does not need to specify ρ_B at this stage, but it will turn out to be most convenient to choose

$$\mathcal{L}_B \rho_B = 0 \qquad (5.1.24)$$

which is a condition satisfied by a thermal equilibrium bath density operator; that is by $\exp(-H_B/kT)$. We also define

$$\mathcal{Q} = 1 - P \qquad (5.1.25)$$

and from (5.1.21), can derive two equations. First write

$$\rho_{\text{tot}} = P\rho_{\text{tot}} + \mathcal{Q}\rho_{\text{tot}} \equiv v(t) + w(t). \qquad (5.1.26)$$

We then use the following properties

i) $\qquad P\mathcal{L}_{\text{sys}} = \mathcal{L}_{\text{sys}} P, \qquad (5.1.27)$

which is true because P and $\mathcal{L}_{\text{sys}}$ operate in different spaces and

ii) $\qquad P\mathcal{L}_B = \mathcal{L}_B P = 0. \qquad (5.1.28)$

The first equality is true because the equation of motion given by the bath Hamiltonian must conserve probability; the second is true because of the definition (5.1.23) and the choice (5.1.24).

iii) $P \mathcal{L}_{\text{Int}} P = 0$ (5.1.29)

This is equivalent to the condition (5.1.14), and is normally true

iv) $P^2 = P, \qquad Q^2 = Q$ (5.1.30)

which are true by definition (5.1.23).

The most easily understood analysis comes from using the Laplace transform of the equations of motion. Defining

$$\bar{v}(s) = \int_0^\infty e^{-st} v(t) dt$$

$$\bar{w}(s) = \int_0^\infty e^{-st} w(t) dt$$
(5.1.31)

we can soon derive

$$s\bar{v}(s) - v(0) = \mathcal{L}_{\text{sys}}\bar{v}(s) + P\mathcal{L}_{\text{Int}}\bar{w}(s)$$ (5.1.32)

$$s\bar{w}(s) - w(0) = (\mathcal{L}_{\text{sys}} + \mathcal{L}_B + Q\mathcal{L}_{\text{Int}})\bar{w}(s) + Q\mathcal{L}_{\text{Int}}\bar{v}(s),$$ (5.1.33)

and if we solve for $\bar{w}(s)$, and substitute for $\bar{w}(s)$ in (5.1.32), we get

$$s\bar{v}(s) - \left\{ v(0) + P\mathcal{L}_{\text{Int}}[s - \mathcal{L}_{\text{sys}} - \mathcal{L}_B - Q\mathcal{L}_{\text{Int}}]^{-1} w(0) \right\}$$
$$= \left\{ \mathcal{L}_{\text{sys}} + P\mathcal{L}_{\text{Int}}[s - \mathcal{L}_{\text{sys}} - \mathcal{L}_B - Q\mathcal{L}_{\text{Int}}]^{-1} Q\mathcal{L}_{\text{Int}} \right\} \bar{v}(s).$$ (5.1.34)

Two limits can now be distinguished.

a) The Weak Coupling Limit: Suppose the interaction is proportional to a small parameter introduced by setting

$$\mathcal{L}_{\text{Int}} \to \epsilon \mathcal{L}_{\text{Int}}.$$ (5.1.35)

Then making this substitution, and preserving only lowest order terms in the homogeneous and inhomogeneous terms of (5.1.34) separately

$$s\bar{v}(s) - \left\{ v(0) + \epsilon P\mathcal{L}_{\text{Int}}[s - \mathcal{L}_{\text{sys}} - \mathcal{L}_B]^{-1} w(0) \right\}$$
$$= \left\{ \mathcal{L}_{\text{sys}} + \epsilon^2 P\mathcal{L}_{\text{Int}}[s - \mathcal{L}_{\text{sys}} - \mathcal{L}_B]^{-1} Q\mathcal{L}_{\text{Int}} \right\} \bar{v}(s).$$ (5.1.36)

The term of order ϵ can be neglected as a small change to the initial condition whose influence does not accumulate with time. However, the second order term on the right hand side can be significant if (as is the case) it gives rise to a kind of behaviour totally different from that of $\mathcal{L}_{\text{sys}}$—in any case the solution of (5.1.36) will yield an

accumulation of this term. Neglecting the correction to the initial condition, and inverting the Laplace transform we find

$$\dot{v}(t) = \mathcal{L}_{\text{sys}} v(t) + \epsilon^2 \, \mathcal{P} \mathcal{L}_{\text{Int}} \int_0^\infty d\tau \, \exp\left\{ (\mathcal{L}_{\text{sys}} + \mathcal{L}_B)\tau \right\} \mathcal{Q} \mathcal{L}_{\text{Int}} v(t - \tau),$$

(5.1.37)

and inserting the definitions of the various quantities involved, it is not hard to see that this equation is equivalent to the master equation (5.1.19). The Markov approximation is then easily justified, since the time scale of $\rho(t)$ is of order ϵ^2, which is much slower than all other time scales.

b) The Quantum Brownian Motion Limit: Suppose we have a parametric dependence of the form

$$\gamma^2 \mathcal{L}_B + \gamma \mathcal{L}_{\text{Int}} + \mathcal{L}_{\text{sys}}$$

(5.1.38)

and that γ is very large, corresponding to fast, large fluctuations. This is similar to the adiabatic elimination problem of classical stochastics. A similar analysis shows that the equation of motion is, in the limit of $\gamma \to \infty$,

$$\dot{v}(t) = \mathcal{L}_{\text{sys}} v(t) - \mathcal{P} \mathcal{L}_{\text{Int}} \mathcal{L}_B^{-1} \mathcal{L}_{\text{Int}} v(t)$$

(5.1.39)

and it is not difficult to show that this corresponds to the quantum Brownian motion master equation of Sect. 3.6.1

5.1.3   Relationship to the Quantum Optical Master Equation

The master equation in the form (5.1.20) can eventually be reduced to exactly the same form as that in Sect. 3.6.3. We note that in that case we can write

$$H_B = \frac{1}{2} \sum_n \left\{ p_n^2 + \omega_n^2 q_n^2 \right\}$$

(5.1.40)

$$H_{\text{Int}} = - \sum_n \kappa_n p_n X$$

(5.1.41)

$$H_{\text{sys}} = H_{\text{sys}}(\mathbf{Z}) + \frac{1}{2} \sum_n \kappa_n^2 X^2$$

(5.1.42)

and in the weak coupling situation, the κ_n are implicitly understood to be proportional to the smallness parameter ϵ. Thus, the Liouvillian can be written in the form

$$\mathcal{L} = \mathcal{L}_{\text{sys}} + \mathcal{L}_B + \epsilon^2 \mathcal{L}'_{\text{sys}} + \epsilon \mathcal{L}_{\text{Int}}.$$

(5.1.43)

The terms in this equation are straightforward. It is possible now to show that (5.1.37) (which is itself the same as (5.1.20)) is identical with the master equation

of Sect. 3.6.3 derived by van Kampen's cumulant expansion. The essence of the demonstration relies on the following.

i)   Note that

$$\exp\{(\mathcal{L}_{\text{sys}} + \mathcal{L}_B)\tau\}g = e^{-iH_B\tau/\hbar}e^{-iH_{\text{sys}}\tau/\hbar}ge^{iH_B\tau/\hbar}e^{iH_{\text{sys}}\tau/\hbar}. \tag{5.1.44}$$

ii)   Note also that, since $P\mathcal{L}_{\text{Int}}P = 0$ and $v = P\rho_{\text{tot}}$,

$$\mathcal{Q}\mathcal{L}_{\text{Int}}v = \mathcal{L}_{\text{Int}}v. \tag{5.1.45}$$

iii) Make the Markov approximation, $v(t-\tau) \rightarrow v(t)$.

iv)   Remember that $\mathcal{L}_B P = 0$. $\tag{5.1.46}$

v)   Then the second term in (5.1.37) becomes

$$-\frac{\epsilon^2}{\hbar^2}P\left[\sum \kappa_n p_n(0)X(0), \left[\int_0^\infty d\tau \sum \kappa_n p_n(-\tau)X(-\tau), \rho_B \otimes \hat{\rho}(t)\right]\right] \tag{5.1.47}$$

where

$$\begin{aligned} p_n(t) &= e^{iH_Bt/\hbar}p_n e^{-iH_Bt/\hbar} \\ X(t) &= e^{iH_{\text{sys}}t/\hbar}Xe^{-iH_{\text{sys}}t/\hbar} \end{aligned} \tag{5.1.48}$$

and $\hat{\rho}(t)$ is as in (5.1.3,8). Now let us define

$$\xi(t) = \sum \kappa_n p_n(t) \tag{5.1.49}$$

which is similar to the definition (3.1.14), but differs in the fact that here $p_n(t)$ are, because of the definition (5.1.48), like free field canonical operators. Thus the definition (5.1.49) is essentially of identical form to that in (3.1.14). Using the definition of P, we find that (5.1.47) is

$$\begin{aligned} -\frac{\epsilon^2}{\hbar^2}\rho_B \otimes \int_0^\infty d\tau \, \big\{ &\langle \xi(0)\xi(-\tau)\rangle \, [X(0), X(-\tau)\hat{\rho}(t)] \\ &+ \langle \xi(-\tau)\xi(0)\rangle \, [\hat{\rho}(t)X(-\tau), X(0)] \big\} \end{aligned} \tag{5.1.50}$$

where

$$\langle \xi(0)\xi(-\tau)\rangle = \text{Tr}_B \{\xi(0)\xi(-\tau)\rho_B\}. \tag{5.1.51}$$

vi)   Now note that

$$\tfrac{1}{2}\langle [\xi(0), \xi(-\tau)]_+\rangle = \langle \alpha(\tau)\alpha(0)\rangle \tag{5.1.52}$$

where $\alpha(t)$ is as defined in (3.5.19), and

$$\tfrac{1}{2}\langle [\xi(0), \xi(-\tau)]\rangle = \tfrac{1}{2}i\hbar\frac{d}{dt}f(t). \tag{5.1.53}$$

Here $f(t)$ is defined in (3.1.15) and this latter relation follows from (3.1.29). Then expressing the bath correlation functions in terms of the commutator and anticommutators, we get two kinds of term. Those involving the anticommutator correspond exactly to the integral term in (3.6.6). The other term is integrated by parts, and a boundary term cancels with the term arising from $\epsilon^2 \mathcal{L}_{\text{sys}}'$, to yield the $A_0\rho(t)$ term, evaluated in the situation where $f(t)$ is not necessarily a delta function, as in (3.6.2).

5.1.4   Quantum Optical Master Equation with Arbitrary Bath

The master equation in the form (5.1.20) makes no restriction on the precise kind of bath or indeed on the bath state. We can consider the bath Hamiltonian to be arbitrary, and attempt to derive a master equation for a quite general H_{Int}. Let us suppose that the Schrödinger picture H_{Int} can be written

$$H_{\mathrm{Int}} = \hbar \sum_m (X_m^+ \Gamma_m + X_m^- \Gamma_m^\dagger) \tag{5.1.54}$$

where the $X_m^\pm$ are eigenoperators of the system Hamiltonian satisfying

$$[H_{\mathrm{sys}}, X_m^\pm] = \pm \hbar \, \omega_m X_m^\pm. \tag{5.1.55}$$

This form is quite general, since any system operator can be decomposed into eigenoperators of H_{sys}. If we substitute (5.1.54) into the master equation, there are a number of different terms. To illustrate the method of derivation and demonstrate the approximations, consider the term

$$- \int_0^t dt' \sum_{m,n} X_m^+ \mathrm{e}^{i\omega_m t} X_n^- \mathrm{e}^{-i\omega_n t'} \rho(t') \mathrm{Tr}_{\mathrm{B}} \left\{ \Gamma_m(t) \Gamma_n^\dagger(t') \rho_B \right\}. \tag{5.1.56}$$

i)   We assume that the bath state is stationary, so that $\mathrm{Tr}_{\mathrm{B}} \left\{ \Gamma_m(t) \Gamma_n^\dagger(t') \rho_B \right\}$ is a function of $t - t'$ only. This means that terms with $\omega_m \neq \omega_n$ are rapidly varying functions of t, and will make many oscillations during the typical time scale over which $\rho(t)$ alters significantly. This is the *rotating wave approximation* or RWA, first noted in Sect. 3.6.3a.

ii)   We make the Markov approximation as in the previous section, thus replacing $\rho(t')$ by $\rho(t)$.

iii)   After doing this, the term takes the form

$$- \sum_m X_m^+ X_m^- \rho(t) \int_0^t d\tau \mathrm{e}^{i\omega_m \tau} \mathrm{Tr}_{\mathrm{B}} \left\{ \Gamma_m(\tau) \Gamma_m^\dagger(0) \rho_B \right\}. \tag{5.1.57}$$

The other terms take similar forms. The time t can be considered to be as large as we please compared with the correlation time of the trace, and can be considered infinite.

iv) Thus, everything can be written in terms of

$$\int_0^\infty d\tau \, e^{i\omega_m\tau} \text{Tr}_B \left\{ \Gamma_m(\tau)\Gamma_m^\dagger(0)\rho_B \right\} \equiv \tfrac{1}{2}K_m + i\delta_m$$

$$\int_0^\infty d\tau \, e^{-i\omega_m\tau} \text{Tr}_B \left\{ \Gamma_m(0)\Gamma_m^\dagger(\tau)\rho_B \right\} \equiv \tfrac{1}{2}K_m - i\delta_m$$

$$\int_0^\infty d\tau \, e^{i\omega_m\tau} \text{Tr}_B \left\{ \Gamma_m^\dagger(\tau)\Gamma_m(0)\rho_B \right\} \equiv \tfrac{1}{2}G_m + i\epsilon_m \qquad (5.1.58)$$

$$\int_0^\infty d\tau \, e^{-i\omega_m\tau} \text{Tr}_B \left\{ \Gamma_m^\dagger(0)\Gamma_m(\tau)\rho_B \right\} \equiv \tfrac{1}{2}G_m - i\epsilon_m$$

(The correlations involving terms like $\Gamma\Gamma$ and $\Gamma^\dagger\Gamma^\dagger$ are usually assumed to be zero if the bath is in a thermal state. However, if the bath is *squeezed* — see Chap. 10 — this need not be the case.) The master equation finally takes the form (in the interaction picture)

$$\dot\rho(t) = -i \sum_m [\delta_m X_m^+ X_m^- + \epsilon_m X_m^- X_m^+, \rho]$$

$$+ \tfrac{1}{2} \sum_m K_m \left(2 X_m^- \rho X_m^+ - X_m^+ X_m^- \rho - \rho X_m^+ X_m^- \right) \qquad (5.1.59)$$

$$+ \tfrac{1}{2} \sum_m G_m \left(2 X_m^+ \rho X_m^- - X_m^- X_m^+ \rho - \rho X_m^- X_m^+ \right)$$

The effect of the ϵ_m and δ_m terms is to add a small perturbing Hamiltonian term—these are *Lamb* and *Stark shift* terms (see Sect. 3.6.3), and are usually neglected. The final terms are just those of the RWA master equation previously derived in (3.6.67).

v) Notice from (5.1.58), assuming that the Γ_m are indeed destruction operators of some kind, that G_m should vanish at $T = 0$, while K_m should not. Thus the terms in (5.1.59) which involve K_m are present at absolute zero, and must represent transitions from upper to lower energy levels. The terms involving G_m represent transitions to higher energy levels.

Exercise. By considering the diagonal matrix elements of (5.1.59), show that the identification of upward and downward transitions is indeed correct.

vi) In conclusion, we see that the master equation form is expected to have a wide range of validity. However, the only characteristics of the bath that have any relevance are the correlation functions as in (5.1.58)

Exercise—A Bath of Two Level Atoms. Consider a heat bath which consists of a large number of two level atoms, so that the bath Hamiltonian is

$$H_B = \sum \tfrac{1}{2}\hbar \,\Omega_a \sigma_z^a. \qquad (5.1.60)$$

Suppose the coupling to the bath is given by

$$\Gamma_m = \sum_a g_a^m \sigma_a^-, \qquad \Gamma_m^\dagger = \sum_a g_a^m \sigma_a^+ \tag{5.1.61}$$

where the g_a^m are real numbers. Suppose also that the density operator ρ_B is the direct product of individual density matrices

$$\rho_a = \begin{pmatrix} N_a^+ & 0 \\ 0 & N_a^- \end{pmatrix}, \qquad \text{with} \qquad N_a^+ + N_a^- = 1. \tag{5.1.62}$$

Show that, by making the approximation

$$\begin{aligned} \sum_a &\to \int d\Omega\, n(\Omega) \\ g_a^m &\to g^m(\Omega) \qquad N_a^\pm \to N^\pm(\Omega) \end{aligned} \tag{5.1.63}$$

where $n(\Omega)$ is a density of states with energy $\hbar\Omega$, that in this case

$$\begin{aligned} K_m &= 2\pi \left\{ g_m(\omega_m) \right\}^2 n(\omega_m) N^-(\omega_m) \\ G_m &= 2\pi \left\{ g_m(\omega_m) \right\}^2 n(\omega_m) N^+(\omega_m) \\ \delta_m &= \mathrm{P} \int \frac{d\Omega\, n(\Omega) g(\Omega)^2 N^-(\Omega)}{\omega_m - \Omega} \\ \epsilon_m &= \mathrm{P} \int \frac{d\Omega\, n(\Omega) g(\Omega)^2 N^+(\Omega)}{\omega_m - \Omega}. \end{aligned} \tag{5.1.64}$$

Note that there is no restriction on N_a^+ and N_a^-—this means that $N_a^+ > N_a^-$ is permissible. Physically such a population inversion could only be maintained by some kind of pumping, as indeed happens in a laser (see Chap.9).

5.1.5   Relationship to the Quantum Brownian Motion Master Equation

If we take the quantum Brownian motion limit in the case that the various Hamiltonians are as defined in (5.1.40–42), we find the quantum Brownian motion master equation in exactly the same form (3.6.28) as derived in Sect. 3.6.1.

5.1.6   Notational Matters

Abstractly viewed, the master equation of either kind can be written in the form

$$\frac{d\rho(t)}{dt} = L(t)\rho(t) \tag{5.1.65}$$

where $L(t)$ is a linear operator whose form depends on the particular master equation. The operator $L(t)$ may or may not have an explicit time dependence, depending on a number of factors. Notice that the master equations as displayed in Chap. 3 are originally written in the Schrödinger picture, though to make the rotating wave approximation in (3.6.67) it is necessary to go to the interaction picture.

The derivation of Sect. 5.1.1 gives an interaction picture master equation, while that of Sect. 5.1.2 does not use the interaction picture. If none of the terms in the Hamiltonian $H = H_{\text{sys}} + H_{\text{Int}} + H_B$ has any explicit time dependence, then neither will there be any explicit time dependence in the resulting operator $L(t)$ if this is the operator relevant for evolution in the Schrödinger picture. Thus, we can say that for the quantum Brownian motion case

$$L\rho = -\frac{i}{\hbar}[H_{\text{sys}}, \rho] - \frac{i\gamma}{2\hbar}\left[X, [\dot{X}, \rho]_+\right] - \frac{\gamma kT}{\hbar^2}\left[X, [X, \rho]\right] \tag{5.1.66}$$

and in the quantum optical case

$$\begin{aligned}
L\rho = &-\frac{i}{\hbar}[H_{\text{sys}}, \rho] \\
&- \sum_m \frac{\pi\omega_m}{2\hbar}\left(\bar{N}(\omega_m) + 1\right)\kappa(\omega_m)^2[\rho X_m^+ - X_m^-\rho, X] \\
&- \sum_m \frac{\pi\omega_m}{2\hbar}\bar{N}(\omega_m)\kappa(\omega_m)^2[\rho X_m^- - X_m^+\rho, X] \\
&+ \frac{i}{2\hbar}\sum_m P\int_{-\infty}^{\infty}\frac{d\omega\,\omega\kappa(\omega)^2}{\omega_m - \omega}\left(\bar{N}(\omega) + \tfrac{1}{2}\right)[[\rho, X_m^+ - X_m^-], X] \\
&+ \frac{i}{4\hbar}\sum_m P\int_{-\infty}^{\infty}\frac{d\omega\,\omega_m\kappa(\omega)^2}{\omega_m - \omega}[[\rho, X_m^+ + X_m^-]_+, X].
\end{aligned}$$
$$\tag{5.1.67}$$

In the interaction picture an unavoidable time dependence will arise in the quantum Brownian motion case, since the operator X does not commute with H_{sys}. However, in the quantum optical case the resulting time dependence can be eliminated by the procedure used to derive the rotating-wave master equation, (3.6.64). This is possible because in this case the time evolution arising from H_{sys} is very much faster than that of the damping, which means that all the resulting time dependence can be averaged over many cycles—leaving only the time independent terms. Explicit time dependence of $L(t)$ will arise in practice only from explicit time dependence in H_{sys} or H_{Int} (or even in H_B). In this case the Laplace transform methods of Sect. 5.1.2 do not work, and must be adapted accordingly. It can be seen therefore, that the Schrödinger picture gives the simplest general way of looking at the master equation, since explicit time dependences then only arise from some explicit time dependence in the physics of the situation.

A general notation, that will be used in the remainder of this book will be of the form

$$\dot{\rho}(t) = L(t)\rho(t) \tag{5.1.68}$$

where $L(t)$ will be written L when no explicit time dependence is present. The solution of an equation like (5.1.68) is given in terms of the time-ordered product

$$\rho(t) = T\exp\left\{\int_{t_0}^{t}dt'L(t')\right\}\rho(t_0). \tag{5.1.69}$$

This is a formula of little use for computing answers, but it is of use for analytical purposes. Of particular use is the alternative method of writing the solution as a product limit

$$\rho(t) = \lim_{N \to \infty} \prod_{n=0}^{N} [1 + L(t_n)\Delta t_n]\rho(t_0) \tag{5.1.70}$$

where

$$t_0 < t_1 < t_2 \ldots < t_N < t \tag{5.1.71}$$

and

$$\Delta t_n = t_{n+1} - t_n. \tag{5.1.72}$$

5.2 Multitime Structure of Quantum Markov Processes

The solutions (5.1.69) or (5.1.70) can be formally written

$$\rho(t) = V(t, t_0)\rho(t_0), \tag{5.2.1}$$

where $V(t, t_0)$ is the evolution operator, and satisfies the same equation as $\rho(t)$,

$$\dot{V}(t, t_0) = L(t)V(t, t_0), \tag{5.2.2}$$

but with the initial condition

$$V(t_0, t_0) = 1. \tag{5.2.3}$$

The notation in terms of $V(t, t_0)$ will be of particular use when we formulate the concept of a quantum Markov process. Since $V(t, t_0)$ obeys the differential equation (5.2.2), it follows that

$$V(t, t_1)V(t_1, t_0) = V(t, t_0). \tag{5.2.4}$$

This is known as the *semigroup property* of the evolution operator [5.7].

5.2.1 Computation of Multitime Averages

From the reduced density operator it is possible to compute any average of any product of system operators, provided these are all evaluated at the same time. This is obvious, since the product of any number of system operators is again a system operator. Unfortunately this is not sufficient to satisfy the curiosity of a physicist. It is very often necessary to evaluate time correlation functions of system operators,

since these have a direct connection with measurable quantities, such as spectra, etc. For example, we may wish to evaluate a correlation function such as

$$\langle A(t+\tau)B(t)\rangle = \mathrm{Tr}_{\mathrm{sys}}\{\mathrm{Tr}_{\mathrm{B}}\{A(t+\tau)B(t)\rho_s \otimes \rho_B\}\} \qquad (5.2.5)$$

where $A(t+\tau)$ and $B(t)$ are Heisenberg picture system operators. An exact answer for (5.2.5) is of course given by

$$\langle A(t+\tau)B(t)\rangle = \mathrm{Tr}_{\mathrm{sys}}\left\{\mathrm{Tr}_{\mathrm{B}}\left\{e^{iH(t+\tau)}Ae^{-iH(t+\tau)}e^{iHt}Be^{-iHt}\rho_s \otimes \rho_B\right\}\right\}. \qquad (5.2.6)$$

However, this exact equation is not really what we would like, which would be an equation involving only the system operators, and the reduced dynamics on the system, as given by the master equation. A transformation of (5.2.6) to a form which does involve only the reduced dynamics is quite simple to give. We can rewrite (5.2.6) as

$$\langle A(t+\tau)B(t)\rangle = \mathrm{Tr}_{\mathrm{sys}}\left\{A\,\mathrm{Tr}_{\mathrm{B}}\left\{e^{-iH\tau/\hbar}B\rho_{\mathrm{tot}}(t)e^{iH\tau/\hbar}\right\}\right\}, \qquad (5.2.7)$$

since A is proportional to the identity in the bath space. The equation of motion for the term

$$X(\tau,t) \equiv e^{-iH\tau/\hbar}B\rho_{\mathrm{tot}}(t)e^{iH\tau/\hbar} \qquad (5.2.8)$$

in terms of τ is

$$i\hbar\frac{\partial}{\partial\tau}X(\tau,t) = [H,X(\tau,t)], \qquad (5.2.9)$$

and in exactly the same way as we derived the master equations in Sects. 5.1.1,2, we can show that $\mathrm{Tr}_{\mathrm{B}}\{X(\tau,t)\}$ also obeys the master equation as a function of τ. We now use the evolution operator to write

$$\begin{aligned}\mathrm{Tr}_{\mathrm{B}}\{X(\tau,t)\} &= V(t+\tau,t)\mathrm{Tr}_{\mathrm{B}}\{X(0,t)\}\\ &= V(t+\tau,t)\{B\rho(t)\}\end{aligned} \qquad (5.2.10)$$

where in this case $\rho(t)$ is the Schrödinger picture reduced density operator, as in Sect. 5.1.6. Putting these together, we find that the correlation function is given by

$$\boxed{\langle A(t+\tau)B(t)\rangle = \mathrm{Tr}_{\mathrm{sys}}\{AV(t+\tau,t)\{B\rho(t)\}\}. \qquad (5.2.11)}$$

a) Definition of Operator Ordering: It is important to realise that when we write an equation like (5.2.11) that $L(t)$ and $V(t+\tau,t)$ are *two sided* operators (as is clear from the definitions (5.1.66,67) while the operators A,B and the density operator $\rho(t)$ are ordinary quantum mechanical operators. This means that there is an element of non-associativity. For example

$$L\{A\rho(t)\} \neq \{LA\}\rho(t), \qquad (5.2.12)$$

as is clear from the definitions (5.1.66,67). A similar property pertains to the evolution operator. However, in practice, the explicit specification of the ordering by brackets can be cumbersome, and the convention is that any evolution operator operates on everything to its right. Thus, conventionally, we would write (5.2.11) as

$$\langle A(t+\tau)B(t)\rangle = \mathrm{Tr}_{\mathrm{sys}}\left\{AV(t+\tau)B\rho(t)\right\}. \tag{5.2.13}$$

b) General Formulae for Time-Ordered Correlation Functions: It is quite straightforward to show that time ordered correlation functions can always be written in a form analogous to (5.2.13). Thus, we consider a correlation function of the form

$$\langle A_0(s_0)A_1(s_1)\ldots A_m(s_m)B_n(t_n)B_{n-1}(t_{n-1})\ldots B_0(t_0)\rangle \tag{5.2.14}$$

where the terms are ordered

$$\begin{aligned}
t_n &\geq t_{n-1} \geq \ldots \geq t_0, \\
s_m &\geq s_{m-1} \geq \ldots \geq s_0.
\end{aligned} \tag{5.2.15}$$

The relative order of the s and the t is not specified. This kind of time ordered correlation function looks a little arbitrary at first glance, but, as we have seen in Sect. 2.3.2 this is the class of correlation function which can be obtained by repeated measurements on the system. To develop the formula for (5.2.14), assume that s_m is the latest time and that t_n is the next latest. Then

$$\langle A_0(s_0)A_1(s_1)\ldots A_m(s_m)B_n(t_n)B_{n-1}(t_{n-1})\ldots B_0(t_0)\rangle =$$
$$\mathrm{Tr}_s\left\{\mathrm{Tr}_B\left\{A_m(s_m)B_n(t_n)\ldots B_0(t_0)\rho_s\otimes\rho_B A_0(s_0)\ldots A_{m-1}(s_{m-1})\right\}\right\} \tag{5.2.16}$$

and using the same argument as for (5.2.11),

$$= \mathrm{Tr}_s\Big\{A_m V(s_m,t_n)B_n$$
$$\times\mathrm{Tr}_B\left\{B_{n-1}(t_{n-1})\ldots B_0(t_0)\rho_s\otimes\rho_B A_0(s_0)\ldots A_{m-1}(s_{m-1})\right\}\Big\}. \tag{5.2.17}$$

It will be noted that the particular interlacing of the times t with the times s is significant. The precise procedure can be described as follows:
(1) Order the times t and s in sequence, and rename them τ_i, so that

$$\tau_0 \leq \tau_1 \leq \tau_2 \leq \ldots \leq \tau_{r-1} \leq \tau_r \qquad (r = n+m+1). \tag{5.2.18}$$

(2) Let F_i be the Schrödinger picture operator corresponding to τ_i, and define the operator f_i by

$$\begin{aligned}
f_i\rho &= F_i\rho \qquad \text{if } F_i \text{ is one of the } B's \\
f_i\rho &= \rho F_i \qquad \text{if } F_i \text{ is one of the } A's.
\end{aligned} \tag{5.2.19}$$

(3) Then

$$\langle A_0(s_0)A_1(s_1)\ldots A_m(s_m)B_n(t_n)B_{n-1}(t_{n-1})\ldots B_0(t_0)\rangle$$
$$= \mathrm{Tr}_s\left\{f_r V(\tau_r,\tau_{r-1})f_{r-1}V(\tau_{r-1},\tau_{r-2})\ldots V(\tau_1,\tau_0)f_0\rho(t_0)\right\}. \tag{5.2.20}$$

Exercise. Show that (5.2.20) gives the following

$$\langle A(t_0)B(t_1)C(t_1)D(t_0)\rangle = \text{Tr}_{\text{sys}}\{BCV(t_1,t_0)D\rho(t_0)A\} \qquad (5.2.21)$$

$$\langle A(t)B(t+\tau)\rangle \qquad = \text{Tr}_{\text{sys}}\{BV(t+\tau,t)\rho(t)A\} \qquad (5.2.22)$$

$$\langle B(t+\tau)A(t)\rangle \qquad = \text{Tr}_{\text{sys}}\{BV(t+\tau,t)A\rho(t)\}. \qquad (5.2.23)$$

5.2.2   The Markov Interpretation

The time development equation for the density operator ρ, together with the formula (5.2.20) for the time-ordered correlation functions provide the ingredients for the definition of a quantum Markov process. The essential formula in classical Markov theory is the Chapman-Kolmogorov equation for the conditional probability, $p(x_1, t_1 | x_0, t_0)$; namely,

$$\int dy\, p(x_1,t_1|y,t)p(y,t|x_0,t_0) = p(x_1,t_1|x_0,t_0). \qquad (5.2.24)$$

This has the same formal structure as the equation for the evolution operator

$$V(t_1,t)V(t,t_0) = V(t_1,t_0). \qquad (5.2.25)$$

To show this we simply define an operator $W(t_1,t_0)$ by

$$\{W(t_1,t_0)f\}(x_1) = \int dx_0 p(x_1,t_1|x_0,t_0)f(x_0). \qquad (5.2.26)$$

Similarly, the correlation function equation

$$\begin{aligned}
\langle f_r(x_r,\tau_r)&f_{r-1}(x_{r-1},\tau_{r-1})\dots f_0(x_0,\tau_0)\rangle \\
&= \int dx_r dx_{r-1}\dots dx_0 f_r(x_r)f_{r-1}(x_{r-1})\dots f_0(x_0) \\
&\quad \times p(x_r\tau_r|x_{r-1},\tau_{r-1})p(x_{r-1},\tau_{r-1}|x_{r-2},\tau_{r-2})\dots \\
&\quad \times p(x_1,\tau_1|x_0,\tau_0)p(x_0,\tau_0)
\end{aligned} \qquad (5.2.27)$$

is equivalent to

$$\begin{aligned}
\langle f_r(x_r,\tau_r)&f_{r-1}(x_{r-1},\tau_{r-1})\dots f_0(x_0,\tau_0)\rangle \\
&= \int dx\Big\{f_r(x)\big\{W(\tau_r,\tau_{r-1})f_{r-1}W(\tau_{r-1},\tau_{r-2})f_{r-2}\dots \\
&\quad \times W(\tau_1,\tau_0)f_0 p(\tau_0)\big\}(x)\Big\},
\end{aligned} \qquad (5.2.28)$$

which is in exact analogy to (5.2.20)—the differences being that here there is no time-ordering required since the functions commute, and the quantum mechanical

trace is replaced by the integral over x. The physical essence of the Markov property is the existence of an evolution equation as a first order differential equation in time for the density operator or the conditional probability. However, not every such equation will produce positive probabilities or a positive semidefinite density operator. The relevant conditions in classical stochastics give rise to the differential Chapman-Kolmogorov equation (S.M.Chap. 3), while in the quantum Markov process case, it can be shown [5.7] that L must be of the form

$$L\rho = -\frac{i}{\hbar}[H, \rho] + \sum_J [2 A_J \rho A_J^\dagger - \rho A_J^\dagger A_J - A_J^\dagger A_J \rho]. \tag{5.2.29}$$

Here H is some Hermitian operator, and the J are arbitrary. (The rigorous proofs actually require bounded operators for the A_J.) The quantum Brownian motion and quantum optical master equations are both of this form, so it can be concluded that the positivity of the density operator is guaranteed for both of these.

5.2.3   Quantum Regression Theorem

Linear systems are always of particular interest in physics, because of their simplicity and the fact that they are exactly soluble. It is often the case that the equations of motion for the means of certain operators are linear. In that case, it is possible to show that the corresponding time correlation functions obey exactly the same equations. This result was first derived by *Lax* [5.3] and is known as the *quantum regression theorem*. Suppose for certain set of operators Y_i, the master equation can be shown to yield for any initial $\hat{\rho}$,

$$\partial_t \langle Y_i(t) \rangle \qquad = \sum_j G_{ij}(t) \langle Y_j(t) \rangle, \tag{5.2.30}$$

then we assert that

$$\partial_t \langle Y_i(t+\tau) Y_l(t) \rangle = \sum_j G_{ij}(\tau) \langle Y_j(t+\tau) Y_l(t) \rangle. \tag{5.2.31}$$

For

$$\langle Y_i(t+\tau) Y_l(t) \rangle = \mathrm{Tr}_s \{ Y_i V(t+\tau, t) Y_l \rho(t) \} \tag{5.2.32}$$

and the right hand side is an average of Y_i at time $t + \tau$ with the choice of initial density operator

$$\rho \to Y_l \rho(t). \tag{5.2.33}$$

Since by hypothesis any initial ρ is permitted, and (5.2.30) is linear, we may in fact generate any initial condition whatsoever. Hence choosing ρ as given by (5.2.33), the hypothesis (5.2.30) yields the *quantum regression theorem*, (5.2.31).

5.3   Inputs, Outputs and Quantum Stochastic Differential Equations

The quantum optical master equation forms the definition of a quantum Markov process, but unlike the classical Markov process there is as yet in this book no quantum stochastic differential equation in a corresponding analogy to the classical stochastic differential equation. In this section we will show how appropriate "quantum optical" approximations may be made in the input-output formalism of Sect. 3.2 to yield an idealised *white noise* quantum stochastic differential equation formalism which is in direct analogy to the classical stochastic differential equation. The work in this section is largely adapted from the work of *Collett* and *Gardiner* [5.8]

5.3.1   Idealised Hamiltonian

The simplifications necessary are made by introducing the idealised Hamiltonian

$$H = H_{\text{sys}} + H_B + H_{\text{int}} \tag{5.3.1}$$

$$H_B = \hbar \int\limits_{-\infty}^{\infty} d\omega\, \omega b^{\dagger}(\omega) b(\omega) \tag{5.3.2}$$

$$H_{\text{int}} = i\hbar \int\limits_{-\infty}^{\infty} d\omega \kappa(\omega) \left[b^{\dagger}(\omega) c - c^{\dagger} b(\omega) \right] \tag{5.3.3}$$

where the $b(\omega)$ are boson annihilation operators for the bath, with

$$\left[b(\omega), b^{\dagger}(\omega) \right] = \delta(\omega - \omega') \tag{5.3.4}$$

and c is one of the several possible system operators. We do not specify either H_{sys} or the kind of system operators or their commutation relations. There are two principal idealisations in the form of the Hamiltonian—the rotating wave approximation, and the fact that the range of the ω integration is $(-\infty, \infty)$ rather than $(0, \infty)$. These are intimately connected, as follows.

i)   Without the rotating wave approximation the form of H_{int} would be something like

$$i\hbar \int\limits_{0}^{\infty} d\omega \kappa(\omega) \left[b^{\dagger}(\omega) + b(\omega) \right] \left[c - c^{\dagger} \right] \tag{5.3.5}$$

which has counter-rotating terms like $b^{\dagger}(\omega)c^{\dagger}$. The arguments for dropping these are based on the smallness of H_{Int}, so that the motion of the operators c and $c^{\dagger}$ will be given essentially by the free Hamiltonian H_{sys}. In quantum optical systems, only high frequencies are involved. Thus the time dependence of $c^{\dagger}$ is perhaps like $e^{i\Omega t}$ and terms like $b^{\dagger}(\omega)c^{\dagger}$ have a time dependence like

$e^{i(\omega+\Omega)t}$, which is always rapidly oscillating, while $b^{\dagger}(\omega)c$ has a time dependence like $e^{i(\omega-\Omega)t}$ which is almost constant near $\omega \approx \Omega$, i.e., near resonance.

ii)   Essentially, one argues that only terms which are almost resonant are important. This allows one to extend the lower limit of the ω integrals to $-\infty$. Though this is formally absurd, the added terms are all non-resonant, and contribute very little. Notice that the extension of the lower limit of integration to $-\infty$ only makes sense if the rotating wave approximation has been made, since otherwise the terms like $e^{i(\omega+\Omega)t}$ would be resonant at $\omega = -\Omega$, and would produce spurious effects.

iii)   These simplifications, however, do generate a very simple formalism which is essentially a formulation of *quantum white noise*. As in classical white noise, the required delta function correlations can only be produced if the full range of positive and negative frequencies is available—the extension of the range of integration to $(-\infty, \infty)$ is essential for this.

Thus although the Hamiltonians (5.3.2,3) do look physically absurd, within the rotating wave approximation they are very little different from their more realistic counterparts, but have the merit of producing a mathematical formulation which is very simple and, moreover, will be shown to be *Markovian*.

5.3.2   Derivation of the Langevin Equations

We follow a procedure very similar to that used in Chap. 3. From (5.3.1–3) we derive the Heisenberg equations of motion for $b(\omega)$, and an arbitrary system operator a. They are

$$\dot{b}(\omega) = -i\omega b(\omega) + \kappa(\omega)c, \tag{5.3.6}$$

$$\dot{a} = -\frac{i}{\hbar}\left[a, H_{\text{sys}}\right] + \int d\omega \kappa(\omega) \left\{b^{\dagger}(\omega)\left[a, c\right] - \left[a, c^{\dagger}\right]b(\omega)\right\} \tag{5.3.7}$$

and we solve (5.3.6) to obtain

$$b(\omega) = e^{-i\omega(t-t_0)}b_0(\omega) + \kappa(\omega)\int\limits_{t_0}^{t} e^{-i\omega(t-t')}c(t')dt'. \tag{5.3.8}$$

Here $b_0(\omega)$ is the value of $b(\omega)$ at $t = t_0$ and has the same commutation relations as $b(\omega)$. We substitute in (5.3.7) to obtain

$$\dot{a} = -\frac{i}{\hbar}\left[a, H_{\text{sys}}\right]$$
$$+ \int d\omega \kappa(\omega) \left\{e^{i\omega(t-t_0)}b_0^{\dagger}(\omega)\left[a, c\right] - \left[a, c^{\dagger}\right]e^{-i\omega(t-t_0)}b_0(\omega)\right\}$$
$$+ \int d\omega \left[\kappa(\omega)\right]^2 \int\limits_{t_0}^{t} dt' \left\{e^{i\omega(t-t')}c^{\dagger}(t')\left[a, c\right] - \left[a, c^{\dagger}\right]e^{-i\omega(t-t')}c(t')\right\}. \tag{5.3.9}$$

For notational convenience in (5.3.9) we omit the time argument on the system operators when it is t but write it explicitly otherwise. (Thus $a \equiv a(t)$.)

The equations are exact so far. In the same way as in Chap. 3 we introduce what we shall call the first Markov approximation, that the coupling constant is independent of frequency.

First Markov Approximation: This takes a simpler form from that of Sect. 3.1.1c; namely

$$\kappa(\omega) = \sqrt{\gamma/2\pi}. \tag{5.3.10}$$

This approximation can be used to put (5.3.9) into the form of a damping equation. We use the properties

$$\int_{-\infty}^{\infty} d\omega\, e^{-i\omega(t-t')} = 2\pi\delta(t - t') \tag{5.3.11}$$

and

$$\int_{t_0}^{t} c(t')\delta(t - t')\,dt' = \tfrac{1}{2}c(t). \tag{5.3.12}$$

(The second result has the factor of $\tfrac{1}{2}$ because the peak of the delta function is at the end of the interval of integration.) In the same way as in Chap. 3 we define an "in" field by

$$b_{\text{in}}(t) = \frac{1}{\sqrt{2\pi}} \int d\omega\, e^{-i\omega(t-t_0)} b_0(\omega) \tag{5.3.13}$$

which satisfies in this case the commutation relation

$$[b_{\text{in}}(t), b_{\text{in}}^{\dagger}(t')] = \delta(t - t'). \tag{5.3.14}$$

Using (5.3.11–13) we readily derive the *quantum Langevin equation*

$$\dot{a} = -\frac{i}{\hbar}[a, H_{\text{sys}}] - [a, c^{\dagger}]\left[\frac{\gamma}{2}c + \sqrt{\gamma}\,b_{\text{in}}(t)\right] + \left[\frac{\gamma}{2}c^{\dagger} + \sqrt{\gamma}\,b_{\text{in}}^{\dagger}(t)\right][a, c]. \tag{5.3.15}$$

This equation is clearly analogous to that given in (3.1.28), though a slight difference in definitions of the first Markov approximation means that no time derivative terms turn up. We will see that this is a narrow bandwidth approximation to the more exact quantum Langevin equation. The commutation rule (5.3.14) can be similarly justified as having the same effect as the commutator (3.1.32) as long as only a narrow bandwidth is being considered.

Some points to note are:

a) Noise Terms: The terms depending on $b_{in}(t), b_{in}^{\dagger}(t)$ are to be taken as noise terms. The definition of $b_{in}(t)$ in terms of the values $b_0(\omega)$ of $b(\omega)$ at time $t = t_0$ ensures that these operators may be freely specified on the same basis as initial conditions. Similarly, we may freely specify the state of the system to be such that, at $t = t_0$, the system and bath density operators both factorise. It is *not* necessary (nor indeed is it possible) to make any further "independence" assumption. In fact it is not necessary to make any particular assumption about the initial state of the system at all for the Langevin equation to be valid in the form (5.3.15). However, the terms $b_{in}(t), b_{in}^{\dagger}(t)$ can only be reasonably interpreted as noise when the state of the system is initially factorised and the state of $b_{in}(t)$ is incoherent (e.g., a thermal state). In the case, that, for example $b_{in}(t)$ is in a coherent state, we would have a classical driving field being applied to the system, and other intermediate situations can easily be envisaged.

b) There Are Alternative Forms: These arise from the fact that the terms $\frac{1}{2}\gamma c(t) + \sqrt{\gamma}\, b_{in}(t)$ and $\frac{1}{2}\gamma c^{\dagger}(t) + \sqrt{\gamma}\, b_{in}^{\dagger}$ commute with all system operators. For, from (5.3.4–10)

$$\frac{1}{\sqrt{2\pi}} \int d\omega b(\omega) = b_{in}(t) + \frac{\sqrt{\gamma}}{2} c(t). \tag{5.3.16}$$

Since the $b(\omega)$ are bath operators, and commute with all system operators at the same time, this proves the result.

c) Consistency with Calculus: The basic rule of ordinary calculus for noncommuting operators is that of the product—if a_1 and a_2 are two operators

$$\frac{d}{dt}(a_1 a_2) = \dot{a}_1 a_2 + a_1 \dot{a}_2. \tag{5.3.17}$$

This rule is also valid for any commutator, i.e., for any A

$$[A, a_1 a_2] = [A, a_1] a_2 + a_1 [A, a_2]. \tag{5.3.18}$$

Only the first term on the right of (5.3.15) is in the explicit form of a commutator, but the parts involving γ are of the form of commutators multiplied by terms like (5.3.16) which commute with all system operators. Thus the time derivative of any product will be correctly given by (5.3.17) and by the substitution $a \rightarrow a_1 a_2$ in (5.3.15). We find that there are no Ito-like terms, even if the input is noise. We come back to this in Sect. 5.3.3.

d) "Out" Fields: If we consider $t_1 > t$, we can write, analogously to (5.3.8),

$$b(\omega) = e^{-i\omega(t-t_1)} b_1(\omega) - \kappa(\omega) \int\limits_{t}^{t_1} e^{-i\omega(t-t')} c(t') \tag{5.3.19}$$

and similarly define

$$b_{out}(t) = \frac{1}{\sqrt{2\pi}} \int d\omega e^{-i\omega(t-t')} b_1(\omega). \tag{5.3.20}$$

Carrying out the same procedure, we arrive at a version of the Langevin equation written in terms of the "out" operators;

$$\dot{a} = -\frac{i}{\hbar}\left[a, H_{\text{sys}}\right] + \left[a, c^{\dagger}\right]\left[\frac{\gamma}{2}c - \sqrt{\gamma}\,b_{\text{out}}(t)\right] - \left[\frac{\gamma}{2}c^{\dagger} - \sqrt{\gamma}\,b_{\text{out}}(t)\right][a, c]$$

(5.3.21)

in which we see that

$$\begin{aligned} \sqrt{\gamma}\,b_{\text{in}}(t) &\rightarrow \sqrt{\gamma}\,b_{\text{out}}(t), \\ \tfrac{1}{2}\gamma c &\rightarrow -\tfrac{1}{2}\gamma c, \end{aligned}$$

(5.3.22)

and furthermore

$$\frac{1}{\sqrt{2\pi}}\int d\omega b(\omega) = b_{\text{out}}(t) - \frac{\sqrt{\gamma}}{2}c(t)$$

(5.3.23)

is derived similarly to (5.3.16). From this follows the identity that

$$\boxed{b_{\text{out}}(t) - b_{\text{in}}(t) = \sqrt{\gamma}\,c(t)} \qquad\qquad (5.3.24)$$

which also can be used to transform between the forward Langevin equation (5.3.15) and the time-reversed Langevin equation (5.3.21). This is the analogue of the results of Sect. 3.2.2c.

Exercise—Damped Harmonic Oscillator. Set

$$\begin{aligned} H_{\text{sys}} &\rightarrow \hbar\Omega a^{\dagger}a \\ c &\rightarrow a \\ [a, a^{\dagger}] &= 1. \end{aligned}$$

(5.3.25)

Show that the quantum Langevin equation becomes

$$\dot{a} = -\left(i\Omega + \frac{\gamma}{2}\right)a - \sqrt{\gamma}\,b_{\text{in}}(t).$$

(5.3.26)

Notice that this represents a *damped* motion for $\langle a\rangle$ independent of the statistics of $b_{\text{in}}(t)$, because of the linearity of the equation in the variables a and $b_{\text{in}}(t)$.

Exercise—Two Level Atom. Set

$$\begin{aligned} H_{\text{sys}} &\rightarrow \tfrac{1}{2}\hbar\Omega\sigma_z \\ c &\rightarrow \sigma^{-}. \end{aligned}$$

(5.3.27)

Show that the quantum Langevin equations become

$$\begin{aligned} \dot{\sigma}^{-} &= -(i\Omega + \frac{\gamma}{2})\sigma^{-} + \sqrt{\gamma}\,\sigma_z b_{\text{in}}(t) \\ \dot{\sigma}^{+} &= (i\Omega - \frac{\gamma}{2})\sigma^{+} + \sqrt{\gamma}\,\sigma_z b_{\text{in}}^{\dagger}(t) \\ \dot{\sigma}_z &= -\gamma(1 + \sigma_z) - 2\sqrt{\gamma}\{\sigma^{+}b_{\text{in}}(t) + b_{\text{in}}^{\dagger}(t)\sigma^{-}\}. \end{aligned}$$

(5.3.28)

These equations are non-linear in σ and $b_{\text{in}}(t)$—hence we cannot express, for example, $\langle\sigma^{+}b_{\text{in}}(t)\rangle$ as a function of the $\langle\sigma\rangle$. By making white noise assumptions (as in Sect. 5.3.5) these averages can be computed.

Notice that we cannot conclude that the damping is independent of the statistics of the bath, as in the case of the harmonic oscillator.

5.3.3   Inputs and Outputs, and Causality

The quantities $b_{in}(t)$ and $b_{out}(t)$ will be interpreted as inputs and outputs to the system. The condition (5.3.24) can be viewed as a boundary condition, relating input, output, and internal modes, and is the analogue of the boundary condition of Chap. 3.

   If (5.3.15) is solved to give values of the system operators in terms of their past values and those of $b_{in}(t)$, then it is clear that $a(t)$ is independent of $b_{in}(t')$ for $t' > t$; that is, the system variables do not depend on the values of the input in the future. Hence we deduce

$$\left[a(t), b_{in}(t')\right] = 0, \quad t' > t \tag{5.3.29}$$

and using similar reasoning, we deduce from (5.3.24)

$$\left[a(t), b_{out}(t')\right] = 0, \quad t' < t. \tag{5.3.30}$$

Defining the step function $u(t)$ as

$$u(t) = \begin{cases} 1, & t > 0 \\ \frac{1}{2}, & t = 0 \\ 0, & t < 0, \end{cases} \tag{5.3.31}$$

and using (5.3.24) we obtain the specific results

$$\left[a(t), b_{in}(t')\right] = -u(t - t')\sqrt{\gamma}\left[a(t), c(t')\right] \tag{5.3.32}$$
$$\left[a(t), b_{out}(t')\right] = u(t' - t)\sqrt{\gamma}\left[a(t), c(t')\right] \tag{5.3.33}$$

which are analogous to the result (3.1.33). We now have, in principle, a complete specification of a system with input and output. We *specify* the input $b_{in}(t)$, and solve (5.3.15) for $a(t)$. We then compute the output from the known $a(t)$ and $b_{in}(t)$ by use of the boundary condition, (5.3.24).

   The commutators (5.3.32,33) are an expression of quantum causality—that only the future motion of the system is affected by the present input, and that only the future value of the output is affected by the present values of the system operators.

5.3.4   Several Inputs and Outputs

It is quite straightforward to generalise the formalism to a situation in which the system is coupled to several heat baths, so that

$$H_B = \sum_l \hbar \int_{-\infty}^{\infty} d\omega \, b^\dagger(l, \omega) b(l, \omega) \tag{5.3.34}$$

$$H_{int} = i\hbar \sum_l \int_{-\infty}^{\infty} d\omega \kappa(l, \omega)[b^\dagger(l, \omega)c_l - c_l^\dagger b(l, \omega)] \tag{5.3.35}$$

and in this case the quantum Langevin equation becomes

$$\dot{a} = -\frac{i}{\hbar}[a, H_{\text{sys}}] - \sum_l [a, c_l^\dagger]\{\tfrac{1}{2}\gamma_l c_l + \sqrt{\gamma_l}\, b_{\text{in}}(l, t)\}$$
$$+ \sum_l \{\tfrac{1}{2}\gamma_l c_l^\dagger + \sqrt{\gamma_l}\, b_{\text{in}}^\dagger(l, t)\}[a, c_l]. \tag{5.3.36}$$

The relations between the inputs and outputs are of course then

$$b_{\text{out}}(l, t) = b_{\text{in}}(l, t) + \sqrt{\gamma_l}\, c_l(t). \tag{5.3.37}$$

5.3.5   Formulation of Quantum Stochastic Differential Equations

The results of the previous section, while relevant to the study of a system being driven by a noisy input from a heat bath, are not genuinely stochastic results, since no assumptions have been made concerning the density operator of the bath. We will now develop the formalism by defining *white noise* inputs as the idealised noisy input, similar to classical white noise.

a) The Quantum Wiener Process: The fields $b_{\text{in}}(t)$ defined in Sects. 5.3.2,3 will provide the input to the system described by H_{sys}. The particular quantum state or ensemble of quantum states of the "in" operators determines the nature of the input. There will always be some quantum noise arising from the zero-point fluctuations of the input, and depending on the input ensemble, there may be additional noise, such as thermal noise.

   The input ensemble which corresponds most closely to a classical white noise input is not a thermal ensemble, but one in which the input density operator ρ_{in} is such that

$$\text{Tr}\left\{\rho_{\text{in}} b_{\text{in}}^\dagger(t) b_{\text{in}}(t')\right\} \equiv \langle b_{\text{in}}^\dagger(t) b_{\text{in}}(t')\rangle = \bar{N}\delta(t - t').$$
$$\text{Tr}\left\{\rho_{\text{in}} b_{\text{in}}(t) b_{\text{in}}^\dagger(t')\right\} \equiv \langle b_{\text{in}}(t) b_{\text{in}}^\dagger(t')\rangle = (\bar{N} + 1)\delta(t - t'), \tag{5.3.38}$$

which corresponds to

$$\langle b_0^\dagger(\omega) b_0(\omega')\rangle = \bar{N}\delta(\omega - \omega'),$$
$$\langle b_0(\omega) b_0^\dagger(\omega')\rangle = (\bar{N} + 1)\delta(\omega - \omega'). \tag{5.3.39}$$

This corresponds to an ensemble in which the number of quanta per unit bandwidth is constant, and this is not the case in a thermal ensemble, in which in (5.3.39) $\bar{N}$ would be replaced by $\bar{N}(\omega)$, given by

$$\bar{N}(\omega) = 1/\left[\exp(\hbar\omega/kT) - 1\right]. \tag{5.3.40}$$

Thus, to an even larger extent than in classical stochastics, quantum white noise is an idealisation, not actually attained in any real system.

To define quantum stochastic integration, we define the *quantum Wiener process* by

$$B(t, t_0) = \int_{t_0}^{t} b_{in}(t') dt'$$ (5.3.41)

in which we find that

$$\langle B^\dagger(t, t_0) B(t, t_0) \rangle = \bar{N}(t - t_0),$$ (5.3.42)

$$\langle B(t, t_0) B^\dagger(t, t_0) \rangle = (\bar{N} + 1)(t - t_0),$$ (5.3.43)

$$[B(t, t_0), B^\dagger(t, t_0)] = t - t_0.$$ (5.3.44)

In addition, we specify that the distribution of $B(t, t_0)$, $B^\dagger(t, t_0)$ is quantum Gaussian (see Sect. 4.4.5), by which we mean that the density operator is

$$\rho(t, t_0) = (1 - e^{-\kappa}) \exp \left[-\frac{\kappa B^\dagger(t, t_0) B(t, t_0)}{t - t_0} \right]$$ (5.3.45)

in which

$$\bar{N} = 1/(e^{\kappa} - 1).$$ (5.3.46)

It is clear that any normal-ordered moment of order n in $B(t, t_0)$ and $B^\dagger(t, t_0)$ will be a constant times $(t - t_0)^{n/2}$.

The moments of order n with any ordering of $B(t, t_0), B^\dagger(t, t_0)$ will, as a consequence of the commutation relation (5.3.44), also always be proportional to $(t - t_0)^{n/2}$ —a factor of importance in manipulating stochastic differentials.

5.3.6   Quantum Ito Stochastic Integration

In ordinary stochastic integration, there is a choice of the Ito or the Stratonovich definition. The Ito form has some mathematical advantages, which arise from the increment being independent of the integration variable. However, the rules of calculus are not those of ordinary calculus: only in the Stratonovich form is this the case.

In the quantum situation we have the added complication that variables do not commute. We can define both Ito and Stratonovich quantum stochastic integration, and can show that only the Stratonovich form preserves the rules of (noncommuting) calculus. But this can only be proved via a route involving the quantum Ito calculus, so it will be necessary to define both kinds of integration even to use the apparently simpler Stratonovich form.

If $g(t)$ is any *system operator* we define the *quantum Ito integral* by

$$\mathbf{I} \int_{t_0}^{t} g(t') dB(t') = \lim_{n \to \infty} \sum g(t_i) \left[B(t_{i+1}, t_0) - B(t_i, t_0) \right]$$ (5.3.47)

where $t_0 < t_1 < t_2 < \ldots < t_n = t$, and the limit is a mean-square limit in terms of the density operator (5.3.45). A similar definition can be used for $\int_{t_0}^{t} g(t')dB^+(t')$.

The Ito increments $dB(t)$ and $dB^+(t)$ commute with $g(t)$, which follows from (5.3.29) and the definition (5.3.47), since

$$\left[g(t_i), B(t_{i+1}, t_0) - B(t_i, t_0)\right] = -\sqrt{\gamma} \int_{t_i}^{t_{i+1}} dt' u(t_i - t') \left[g(t_i), c(t')\right] = 0.$$

(5.3.48)

Hence

$$\mathbf{I} \int_{t_0}^{t} g(t')dB(t') = \mathbf{I} \int_{t_0}^{t} dB(t')g(t') \tag{5.3.49}$$

and similarly for integrals with respect to $dB^+(t)$.

5.3.7   Ito Quantum Stochastic Differential Equation

If $\bar{N}$ is defined as in (5.3.46), we define an Ito QSDE in the form

$$\begin{aligned}
(\mathbf{I})\,da = &-\frac{i}{\hbar}\left[a, H_{\text{sys}}\right]dt + \frac{\gamma}{2}(\bar{N} + 1)(2c^+ac - ac^+c - c^+ca)dt \\
&+ \frac{\gamma}{2}\bar{N}(2cac^+ - acc^+ - cc^+a)dt \\
&- \sqrt{\gamma}\left[a, c^+\right]dB(t) + \sqrt{\gamma}\,dB^+(t)\left[a, c\right].
\end{aligned} \tag{5.3.50}$$

We will show that this equation is equivalent to the quantum Langevin equation (5.3.15); but first, show that the second order calculus rule appropriate to Ito integration,

$$d(ab) = a\,db + b\,da + da\,db, \tag{5.3.51}$$

is true. To do this we need the identities

$$\begin{aligned}
&\left[dB(t)\right]^2 = \left[dB^+(t)\right]^2 = 0, \\
&dB(t)dB^+(t) = (\bar{N} + 1)dt, \\
&dB^+(t)dB(t) = \bar{N}\,dt.
\end{aligned} \tag{5.3.52}$$

All other products, including $dt\,dB$, $dt\,dB^+(t)$, and higher orders are set equal to zero. These will not be derived here, but are easy to derive in exactly the same way as in the non-quantum case.

The proof is then straightforward. We compute da and db using (5.3.50), substitute these into (5.3.51) using (5.3.52), and the result is that (after some rearrangement) $d(ab)$ as derived from the substitution $a \rightarrow ab$ in (5.3.50) is the same as the form (5.3.51). From (5.3.51) we can derive *the rules of calculus for any polynomial.* In the case that z is a variable which commutes with dz, we obtain

$$df(z) = f'(z)dz + \tfrac{1}{2}f''(z)dz^2 \tag{5.3.53}$$

which leads to Ito rules. Where z does not commute with dz, (5.3.53) can be interpreted in the sense that all products in a power-series expansion of $f'(z)$ and $f''(z)$ are completely symmetrised in terms of z and dz.

5.3.8   The Quantum Stratonovich Integral

The quantum Stratonovich integral is defined by

$$\mathbf{S}\int_{t_0}^{t} g(t')dB(t') = \lim_{n \to \infty} \sum g\left(\frac{t_i + t_{i+1}}{2}\right) \left[B(t_{i+1}, t_0) - B(t_i, t_0)\right].$$

$$\tag{5.3.54}$$

We notice that the Stratonovich increment does *not* commute with $g(t)$, and in fact, using (5.3.32), it is straightforward to show that we must take

$$\mathbf{S}\int_{t_0}^{t} g(t')dB(t') - \mathbf{S}\int_{t_0}^{t} dB(t')g(t') = \frac{\sqrt{\gamma}}{2}\int_{t_0}^{t} dt' \left[g(t'), c(t')\right]. \tag{5.3.55}$$

We can show this more rigorously by deriving the connection between the two kinds of stochastic integral.

5.3.9   Connection between the Ito and Stratonovich Integral

Let us assume that all operators obey the quantum Ito equation (5.3.50). Let us define $\bar{t}_i = \tfrac{1}{2}(t_i + t_{i+1})$, so that we can rewrite (5.3.54) as

$$\mathbf{S}\int_{t_0}^{t} g(t')dB(t')$$

$$= \lim_{n \to \infty} \left[\sum g(\bar{t}_i)\left[B(t_{i+1}) - B(\bar{t}_i)\right] + \sum g(\bar{t}_i)\left[B(\bar{t}_i) - B(t_i)\right]\right].$$

$$\tag{5.3.56}$$

We then write

$$g(\bar{t}_i) = g(t_i) + dg(t_i), \tag{5.3.57}$$

where $dg(t_i)$ is obtained from (5.3.50), with

$$
\begin{aligned}
dt_i &= \bar{t}_i - t_i, \\
dB(t_i) &= B(\bar{t}_i) - B(t_i),
\end{aligned}
\tag{5.3.58}
$$

which will be valid to lowest order. We then find that

$$
\mathbf{S} \int_{t_0}^{t} g(t') dB(t')
$$

$$
\begin{aligned}
= \lim_{n \to \infty} \Big\{ &\sum g(\bar{t}_i) \big(B(t_{i+1}) - B(\bar{t}_i) \big) + \sum g(t_i) \big(B(\bar{t}_i) - B(t_i) \big) \\
&+ \sqrt{\gamma} \sum [g(t_i), c^\dagger(t_i)] \big(B(\bar{t}_i) - B(t_i) \big) \big(B(\bar{t}_i) - B(t_i) \big) \\
&+ \sqrt{\gamma} \sum [g(t_i), c(t_i)] \big(B^\dagger(\bar{t}_i) - B^\dagger(t_i) \big) \big(B(\bar{t}_i) - B(t_i) \big) \Big\}.
\end{aligned}
\tag{5.3.59}
$$

We now use (5.3.52) in the last part, and combine the first two terms into the Ito integral, to get

$$
\mathbf{S} \int_{t_0}^{t} g(t') dB(t') = \mathbf{I} \int_{t_0}^{t} g(t') dB(t') + \tfrac{1}{2} \sqrt{\gamma} \, \bar{N} \int_{t_0}^{t} [g(t'), c(t')] dt'
\tag{5.3.60}
$$

and similarly

$$
\mathbf{S} \int_{t_0}^{t} dB(t') g(t') = \mathbf{I} \int_{t_0}^{t} g(t') dB(t') + \tfrac{1}{2} \sqrt{\gamma} \, (\bar{N} + 1) \int_{t_0}^{t} [g(t'), c(t')] dt'
\tag{5.3.61}
$$

$$
\mathbf{S} \int_{t_0}^{t} g(t') dB^\dagger(t') = \mathbf{I} \int_{t_0}^{t} g(t') dB^\dagger(t') - \tfrac{1}{2} \sqrt{\gamma} \, (\bar{N} + 1) \int_{t_0}^{t} [g(t'), c^\dagger(t')] dt'
\tag{5.3.62}
$$

$$
\mathbf{S} \int_{t_0}^{t} dB^\dagger(t') g(t') = \mathbf{I} \int_{t_0}^{t} g(t') dB^\dagger(t') - \tfrac{1}{2} \sqrt{\gamma} \, \bar{N} \int_{t_0}^{t} [g(t'), c^\dagger(t')] dt'.
\tag{5.3.63}
$$

5.3.10   Stratonovich Quantum Stochastic Differential Equation

Substituting for the Ito integral implicit in (5.3.50), we find the equivalent quantum Stratonovich equations

$$
\begin{aligned}
(\mathbf{S}) \, da = &-\frac{i}{\hbar}[a, H_{\text{sys}}] dt - \frac{\gamma}{2} \big([a, c^\dagger] c - c^\dagger [a, c] \big) dt \\
&- \sqrt{\gamma}[a, c^\dagger] dB(t) + \sqrt{\gamma} \, dB^\dagger(t)[a, c].
\end{aligned}
\tag{5.3.64}
$$

Thus the quantum Ito equation (5.3.50) is equivalent to the quantum Stratonovich equation (5.3.64). This is exactly of the same form as the quantum Langevin equation(5.3.15), which thus justifies the definition of the Ito equation.Notice also that the commutation relations (5.3.32,33) follow from (5.3.60–63).

Finally, we can compute $(\mathbf{S})d(ab)$ from the corresponding Ito form, and readily verify that

$$(\mathbf{S})d(ab) = a\,db + da\,b \tag{5.3.65}$$

so that ordinary (non-commuting) calculus is valid, as expected for a Stratonovich differential. Thus the stochastic calculus for the quantum Langevin equation is that originally assumed.

5.3.11   Comparison of the Two Forms of QSDE

Let us now summarise the differences between the two forms of QSDE.

a) Stratonovich:

i)   The Stratonovich form is the "natural" physical choice, since it is what arises directly from the physical considerations in Sect. 5.3.2.

ii) However, we note that the increment neither commutes with system operators, nor is it stochastically independent of them.

iii) A direct proof that ordinary calculus is true is difficult to present.

iv) Because the commutator of the increment and a system variable depends on the precise form of the QSDE, it is not possible to define the quantum Stratonovich integrals without a knowledge of the QSDE.

v)   The QSDE in Stratonovich form should also be valid for nonwhite noise and is the same for any $\bar{N}$ if the noise is white. The only assumption necessary to obtain the Stratonovich QSDE is that of a constant $\kappa(\omega)$, (5.3.10).

b) Ito:

i)   Not a natural physical choice.

ii) Increment commutes with and is statistically independent of system operators at the same time.

iii) Ordinary calculus is not true, but the appropriate Ito calculus is easy to derive.

iv) Because $\bar{N}$ appears in the QSDE, it is not possible to define the QSDE without knowledge of $\bar{N}$ and bath statistics, and the Ito QSDE (5.3.50) is exact only in the case of quantum white noise.

v)   The Ito QSDE is thus explicitly a white noise theory, and is essentially as valid as the quantum optical master equation, to which we will shortly show it is exactly equivalent.

c) Small Noise Expansions: In Sect. 8.3.1 in the development of laser theory, we will see how macroscopic QSDEs can arise for macroscopic variables which are obtained as the sum of many microscopic variables, all obeying the same QSDE,

but with independent noises. An example is the atoms in a gas laser—in this case the sum of all the noises does not increase as rapidly with the number of atoms as do the systematic terms, but only in the Ito formulation. This means that the Ito damping terms correspond to the observable macroscopic properties.

Exercise. Compare the Ito and Stratonovich formulations of the quantum stochastic differential equations for a damped two level atom. Note that if the noise terms are dropped, the stationary value of σ_z according to Ito is the expected statistical mechanics value, while Stratonovich gives $\sigma_z = -1$. Of course this does not mean Stratonovich is wrong— rather, it means that some approximations are more robust in the Ito formulation than in the Stratonovich.

5.3.12   Noise Sources of Several Frequencies

Let us go back to (5.3.15), and move to an interaction picture in which all system operators are transformed by the substitution

$$a(t) = e^{iH_{sys}t/\hbar} \hat{a}(t) e^{-iH_{sys}t/\hbar}. \tag{5.3.66}$$

This obviously applies to the operators $c, c^\dagger$. We now write, as in (3.6.61), an expansion of $c, c^\dagger$ in eigenoperators of H_{sys},

$$c(t) = \sum_m X_m^-, \quad c^\dagger(t) = \sum_m X_m^+ \tag{5.3.67}$$

where

$$[H_{sys}, X_m^\pm] = \pm\hbar \omega_m X_m^\pm. \tag{5.3.68}$$

Thus,

$$\hat{c}(t) = \sum_m X_m^- e^{-i\omega_m t} \tag{5.3.69}$$

and if we substitute this in the quantum Langevin equation (5.3.15), we get

$$\frac{d\hat{a}}{dt} = -\left[\hat{a}, \sum_m X_m^+ e^{i\omega_m t}\right] \left(\frac{\gamma}{2} \sum_n X_n^- e^{-i\omega_n t} + \sqrt{\gamma}\, b_{in}(t)\right)$$
$$+ \left(\frac{\gamma}{2} \sum_n X_n^+ e^{i\omega_n t} + \sqrt{\gamma}\, b_{in}^\dagger\right) \left[\hat{a}, \sum_m X_m^- e^{-i\omega_m t}\right]. \tag{5.3.70}$$

We can now make a rotating wave approximation in two forms.

i)   In the terms involving products of terms like $X_m^\pm X_n^\mp$, keep only the terms with $m = n$, since other terms oscillate rapidly, and will have negligible effect.

ii)   This will leave terms involving $e^{i\omega_m t} b_{in}(t)$ or its Hermitian conjugate. Only the non-rapidly rotating parts of this can significantly affect the motion of $\hat{a}(t)$, and we can thus assume that this will behave like a white noise term corresponding to this frequency.

Thus we set

$$e^{i\omega_m t} b_{\text{in}}(t) \rightarrow b_{\text{in}}(m, t) \tag{5.3.71}$$

and

$$\left[b_{\text{in}}^{\dagger}(m, t), b_{\text{in}}(n, t) \right] = \delta_{mn}\delta(t - t') \tag{5.3.72}$$

$$\left\langle b_{\text{in}}^{\dagger}(m, t) b_{\text{in}}(n, t) \right\rangle = \bar{N}(\omega_m)\delta_{mn}\delta(t - t') \tag{5.3.73}$$

where $\bar{N}(\omega_m)$ is the Planck factor,

$$\bar{N}(\omega_m) = \left\{ \exp(\hbar\omega_m/kT) - 1 \right\}^{-1}. \tag{5.3.74}$$

Following the same reasoning as before, we can then identify (5.3.70) with the Ito QSDE in the Heisenberg picture

$$
\begin{aligned}
da = {}& -\frac{i}{\hbar}[a, H_{\text{sys}}] \\
& + \sum_m \frac{\gamma_m}{2} \left(\bar{N}(\omega_m) + 1 \right) \left(2 X_m^+ a X_m^- - a X_m^+ X_m^- - X_m^+ X_m^- a \right) \\
& + \sum_m \frac{\gamma_m}{2} \bar{N}(\omega_m) \left(2 X_m^- a X_m^+ - a X_m^- X_m^+ - X_m^- X_m^+ a \right) \\
& - \sqrt{\gamma_m} \sum_m [a, X_m^+] dB(m, t) + \sqrt{\gamma_m} \sum_m dB^{\dagger}(m, t)[a, X_m^-].
\end{aligned}
\tag{5.3.75}
$$

This form is written for generality—the derivation actually yields $\gamma_m = \gamma$ for all m. The derivation of the master equation just given then carries over, and we find a master equation equivalent to (3.6.67), except that we must set

$$\frac{\pi\omega_m}{\hbar}\kappa(\omega_m)^2 = \frac{\gamma}{2}. \tag{5.3.76}$$

Thus the full generality of the quantum optical master equation can only be obtained by allowing γ to depend on frequency—essentially this means dropping the first Markov approximation. This is in fact not too difficult to do, so that we may actually allow γ_m in (5.3.75) to be different from each other.

The resulting equation will be valid when the coupling to the bath is weak and each possible atomic transition will be driven by a noise source tuned to its frequency, and independent of the sources corresponding to other transitions.

5.4 The Master Equation

In this section we will show that the quantum optical master equation can be derived directly from the QSDE. Unlike the case of the more exact quantum Langevin equations of Chap. 3, the equivalence is (in the white noise case) exact. This means that the short correlation time approximation has already been made by assuming quantum white noise.

5.4.1   Description of the Density Operator

We consider uncorrelated initial conditions, that is, we assume that the density matrix can be written as a direct product

$$\rho = \rho_s(t_0) \otimes \rho_B(t_0). \tag{5.4.1}$$

Here, $\rho_s(t_0)$ specifies the initial state of the system variables, and $\rho_B(t_0)$ specifies the initial state of $b_0(\omega)$, and hence of $b_{\mathrm{in}}(t)$ for all t in the future of t_0. We assume $\rho_B(t_0)$ is such that for any interval $B(t, t')$ has the density operator (5.3.45).

5.4.2   Derivation of the Master Equation

The mean of any operator $a(t)$ is given by

$$\langle a(t) \rangle = \mathrm{Tr}_s \mathrm{Tr}_B \left\{ a(t) \rho_s(t_0) \otimes \rho_B(t_0) \right\} \tag{5.4.2}$$

$$= \mathrm{Tr}_s \mathrm{Tr}_B \left\{ U(t, t_0) a(t_0) U^\dagger(t, t_0) \rho_s(t_0) \otimes \rho_B(t_0) \right\}, \tag{5.4.3}$$

where $U(t, t_0)$ is the time evolution operator $\exp\left[-\mathrm{i} H(t - t_0)/\hbar\right]$, with Hamiltonian H as in (5.3.1–3). Using the cyclic property of the trace, we then find

$$\langle a(t) \rangle = \mathrm{Tr}_s \left\{ a(t_0) \hat{\rho}(t) \right\}, \tag{5.4.4}$$

where $\hat{\rho}(t)$ is the time-dependent reduced density matrix, given by

$$\hat{\rho}(t) = \mathrm{Tr}_B \left\{ U^\dagger(t, t_0) \rho_s(t_0) \otimes \rho_B(t_0) U(t, t_0) \right\}. \tag{5.4.5}$$

All these results are exact for any system. We now use the QSDE in the Ito form (5.3.50) (equivalent to the quantum Langevin equation (5.3.15) provided the input is quantum white noise) to derive the master equation. We note that we can write (5.3.50) in the form

$$da(t) = A[\boldsymbol{a}(t)]dt + G^\dagger[\boldsymbol{a}(t)]dB(t) + G[\boldsymbol{a}(t)]dB^\dagger(t). \tag{5.4.6}$$

(Here $\boldsymbol{a}(t)$ is the set of all system operators.) Because of the construction of $dB(t), dB^\dagger(t)$ as Ito increments, which means that $dB(t) = B(t + dt) - B(t)$, and the fact that $\boldsymbol{a}(t)$ depend only on the past values of $B(t)$, we have

$$\langle da(t) \rangle = \langle A[\boldsymbol{a}(t)] \rangle dt \tag{5.4.7}$$

and from (5.4.4) we can evaluate the mean on the right hand side to get

$$\langle da(t) \rangle = \mathrm{Tr}_s \left\{ A[\boldsymbol{a}(t_0)] \hat{\rho}(t) \right\} dt. \tag{5.4.8}$$

For simplicity and clarity, we now shall use the notation

$$\hat{a} = a(t_0) \tag{5.4.9}$$

for the operators, which are essentially the Schrödinger-picture operators. Using the form of $A[\boldsymbol{a}(t)]$ as given in (5.3.50) and the cyclic property of the trace, we derive

$$
\frac{d\langle a(t)\rangle}{dt} = \mathrm{Tr}_s \left\{ \hat{a} \left(\frac{\mathrm{i}}{\hbar} [\hat{\rho}, H_{\mathrm{sys}}] + \frac{\gamma}{2} (\bar{N}+1)(2\hat{c}\hat{\rho}\hat{c}^\dagger - \hat{c}^\dagger\hat{c}\hat{\rho} - \hat{\rho}\hat{c}^\dagger\hat{c}) \right. \right.
$$
$$
\left. \left. + \frac{\gamma}{2}\bar{N}(2\hat{c}^\dagger\hat{\rho}\hat{c} - \hat{c}\hat{c}^\dagger\hat{\rho} - \hat{\rho}\hat{c}\hat{c}^\dagger) \right) \right\}. \tag{5.4.10}
$$

However, from (5.4.4) we have

$$
\frac{d\langle a(t)\rangle}{dt} = \mathrm{Tr}_s \left\{ a \frac{d\hat{\rho}(t)}{dt} \right\}. \tag{5.4.11}
$$

Both these equations are valid for any system operator; hence we derive the master equation

$$
\frac{d\hat{\rho}}{dt} = \frac{\mathrm{i}}{\hbar} \left[\rho, H_{\mathrm{sys}} \right] + \frac{\gamma}{2}(\bar{N}+1)(2\hat{c}\hat{\rho}\hat{c}^\dagger - \hat{c}^\dagger\hat{c}\hat{\rho} - \hat{\rho}\hat{c}^\dagger\hat{c})
$$
$$
+ \frac{\gamma}{2}\bar{N}(2\hat{c}^\dagger\hat{\rho}\hat{c} - \hat{c}\hat{c}^\dagger\hat{\rho} - \hat{\rho}\hat{c}\hat{c}^\dagger) \equiv \hat{L}\hat{\rho}. \tag{5.4.12}
$$

For future reference we define the operator $\hat{L}$ by means of this equation.

5.4.3 Comparison with Previous Results

Comparison with the quantum optical master equation in the rotating wave approximation, (3.6.67), shows that the master equation (5.4.12) is a special case, in which there is only one operator $X_m^- \to c$, and one frequency ω_m. This is in fact exactly the case of Sect. 3.6.3c where

$$
H_{\mathrm{sys}} = H_0 + H_{\mathrm{nl}} \tag{5.4.13}
$$

and H_{nl} is very small compared with H_0. This appears to be a severe limitation on what can be treated this way, though this is not the case for two reasons:

i) Many very interesting systems are indeed almost harmonic.

ii) In fact, the generalisation of the QSDE to noise sources of several frequencies carried out in Sect. 5.3.12 generates the most general kind of rotating wave approximation master equation, as we show in the next section.

5.4.4 Master Equation with Several Frequencies

Using the same method as in the previous section, the master equation corresponding to (5.3.75) is

$$\frac{d\rho}{dt} = \frac{i}{\hbar}[\rho, H_{\text{sys}}] + \sum_m \frac{\gamma_m}{2}\left(\bar{N}(\omega_m) + 1\right)\left(2X_m^-\rho X_m^+ - \rho X_m^+ X_m^- - X_m^+ X_m^-\rho\right)$$

$$+ \sum_m \frac{\gamma_m}{2}\bar{N}(\omega_m)\left(2X_m^+\rho X_m^- - \rho X_m^- X_m^+ - X_m^- X_m^+\rho\right) \qquad (5.4.14)$$

which is the quantum optical master equation (3.6.67).

Exercise—Relation to the Pauli Master Equation. Consider a system with states $|r\rangle$ with $H_{\text{sys}}|r\rangle = \hbar\omega_r|r\rangle$. The possible frequencies will then be $\omega_{mn} \equiv \omega_m - \omega_n$, and we will have a two index notation X_{mn}, γ_{mn}, etc., in (5.4.14). Suppose that all the ω_{mn} are different from each other. Show that this requires

$$X_{mn}^+|r\rangle = \delta_{rn}x_{mn}|m\rangle$$
$$X_{mn}^-|r\rangle = \delta_{mr}x_{mn}^*|n\rangle \qquad (5.4.15)$$

where x_{mn} are certain complex numbers, and that the equations of motion for the matrix elements of the density operator are

$$\frac{d}{dt}\langle s|\rho|r\rangle = \left\{ i\omega_{rs} - \sum_{n<r}\frac{\gamma_{rn}}{2}[\bar{N}(\omega_{rn}) + 1]|x_{rn}|^2 - \sum_{n<s}\frac{\gamma_{sn}}{2}[\bar{N}(\omega_{sn}) + 1]|x_{sn}|^2 \right.$$

$$\left. - \sum_{m>r}\frac{\gamma_{mr}}{2}\bar{N}(\omega_{mr})|x_{mr}|^2 - \sum_{m>s}\frac{\gamma_{ms}}{2}\bar{N}(\omega_{ms})|x_{ms}|^2 \right\}\langle s|\rho|r\rangle$$

$$+ \delta_{rs}\left\{ \sum_{m>r}\gamma_{mr}[\bar{N}(\omega_{mr}) + 1]|x_{mr}|^2\langle m|\rho|m\rangle + \sum_{n<r}\gamma_{rn}\bar{N}(\omega_{rn})|x_{rn}|^2\langle n|\rho|n\rangle \right\}.$$

$$(5.4.16)$$

Notice that the off-diagonal matrix elements obey damped uncoupled equations of motion. The diagonal matrix elements are occupation probabilities, $p_r = \langle r|\rho|r\rangle$. For these probabilities the equation becomes

$$\dot{p}_r = \sum_{m>r}\gamma_{mr}|x_{mr}|^2\{p_m[\bar{N}(\omega_{mr}) + 1] - p_r\bar{N}(\omega_{mr})\}$$

$$+ \sum_{n<r}\gamma_{nr}|x_{nr}|^2\{p_n\bar{N}(\omega_{rn}) - p_r[\bar{N}(\omega_{mr}) + 1]\}. \qquad (5.4.17)$$

This is an ordinary stochastic master equation—essentially the Pauli master equation of Sect. 1.5—whose solutions approach a stationary state given by

$$p_r(\text{ss}) \propto \exp(-\hbar\omega_r/kT). \qquad (5.4.18)$$

5.4.5   Equivalence of QSDE and Master Equation

For every master equation we can write an equivalent QSDE, and conversely. However the master equation can be derived in situations where the heat bath is *not* composed of harmonic oscillators, as was demonstrated in Sect. 5.1.4. Nevertheless, in

this case the *effect* of the heat bath can be represented via a QSDE, which implicitly defines an equivalent harmonic oscillator heat bath. This equivalent form is very useful when discussing inputs and outputs to a field coupled to a system which is also damped by a coupling to a more general bath. By formulating the heat bath as an equivalent QSDE with appropriate inputs and outputs, all sources of damping and noise acquire an equivalent status—all of this in the Markovian limit, of course.

5.4.6 Correlation Functions of Inputs, System, and Outputs

In the case of an optical system the input fields $b_{in}(t)$ are identified as being proportional to appropriate modes of the electromagnetic field impingent on some reasonably well localised system, such as an atom. Thus it is a very reasonable question in such a situation to ask how specified input statistics are transformed into the statistics of the output field. This is a question which has only recently become important to answer, with the advent of *squeezing*, and it was for this purpose that the input-output formalism was derived.

We will normally specify the correlation functions of the input, and will wish to compute those of the system and those of the output. From this will arise two principal problems.

First, given the correlation functions of the input, how do we compute those of the system? In certain situations we know the answer. If the QSDE for the system is linear, it can be explicitly solved, and correlations directly calculated. It does not matter for linear systems what kind of statistics the input has, a direct solution for internal modes and output is possible.

In nonlinear situations we can get a master equation exactly equivalent to the internal QSDE, *provided* the input field consists of ordinary white quantum noise, or squeezed white noise. In practice this cannot be exactly true, either because the noise is not white, but has a nonflat spectrum, or because the particular model we have formulated is not exactly valid. However, there *are* many cases where the white-noise approximation is reasonably valid, and the internal modes of these can be treated using the master equation and the numerous techniques available for that. Thus the correlation functions of the internal modes can be calculated in very many cases.

The second consideration is how to compute the correlation functions of the output. We will show that these can be related directly to those of the internal modes, but are not identical in all cases.

a) Output Correlation Functions: The output correlation functions of most interest are the normally ordered correlation functions of the form

$$\langle b_{out}^\dagger(t_1) b_{out}^\dagger(t_2) \dots b_{out}^\dagger(t_n) b_{out}(t_{n+1}) \dots b_{out}(t_m) \rangle. \tag{5.4.19}$$

It does not in fact matter what time ordering is chosen in (5.4.19) since all the $b_{out}(t_r)$ commute with each other. Products which are not normally ordered can be related to normally ordered products by use of the commutation relations for the output field, which we have shown are the same as those for the input field.

Thus we may consider with complete generality that the correlation function is of the form (5.4.19) and that the $b_{\text{out}}^\dagger(t_r)$ are time *antiordered* (i.e., $t_1 < t_2 < \ldots < t_n$) and the $b_{\text{out}}(t_r)$ are *time ordered* (i.e., $t_{n+1} > t_{n+2} > \ldots > t_m$).

We may now substitute for $b_{\text{out}}(t_r)$ by the relation (5.3.24)

$$b_{\text{out}}(t_r) = \sqrt{\gamma}\, c(t_r) + b_{\text{in}}(t_r). \tag{5.4.20}$$

We note that from (5.3.32,33) $c^\dagger(t_r)$ will commute with all $b_{\text{in}}^\dagger(t_s)$ which occur to the right of $c^\dagger(t_r)$, since we have shown $t_r < t_s$. Similarly, $c(t_{r'})$ will commute with all $b_{\text{in}}(t_{s'})$ which occur to the left of $c(t_{r'})$, since we have chosen $t_{r'} > t_{s'}$. Thus we may write the correlation function in a form in which we make the substitution (5.5.4.20) in (5.4.19), and reorder the terms so that

i)   the $c^\dagger(t_r)$ are time *antiordered*, and are to the left of all $c(t_r)$;

ii)   the $c(t_{r'})$ are *time ordered*;

iii) the $b_{\text{in}}^\dagger(t_s)$ all stand to the left of all other operators; and

iv)   the $b_{\text{in}}(t_{s'})$ all stand to the right of all other operators.

We now consider various cases.

b) Input in the Vacuum State: In this case, $b_{\text{in}}(t)\rho_{\text{in}} = \rho_{\text{in}} b_{\text{in}}^\dagger(t) = 0$, and we derive

$$\langle b_{\text{out}}^\dagger(t_1)\, b_{\text{out}}^\dagger(t_2) \ldots b_{\text{out}}^\dagger(t_n) b_{\text{out}}(t_{n+1}) \ldots b_{\text{out}}(t_m) \rangle =$$

$$\gamma^{\frac{m+n}{2}} \left\langle \tilde{T}\left[c^\dagger(t_1)c^\dagger(t_2)\ldots c^\dagger(t_n)\right] T\left[c(t_{n+1})\ldots c(t_m)\right]\right\rangle, \tag{5.4.21}$$

where $\tilde{T}$ is the time-antiordered product and T the time-ordered product.

c) Input is in a Coherent State: In this case

$$\begin{aligned} b_{\text{in}}(t)\rho_B &= \beta(t)\rho_B, \\ \rho_B b_{\text{in}}^\dagger(t) &= \beta^*(t)\rho_B, \end{aligned} \tag{5.4.22}$$

where $\beta(t)$ is the coherent field amplitude; a c-number. In this case we may replace $b, b^\dagger$ by β, β^*, respectively, and since β and β^* are mere numbers, reorder them so that

$$\begin{aligned} &\langle b_{\text{out}}^\dagger(t_1)\, b_{\text{out}}^\dagger(t_2) \ldots b_{\text{out}}^\dagger(t_n) b_{\text{out}}(t_{n+1}) \ldots b_{\text{out}}(t_m) \rangle \\ &= \left\langle \tilde{T}\{[\sqrt{\gamma}\, c^\dagger(t_1) + \beta^*(t_1)]\ldots\} T\{[\sqrt{\gamma}\, c(t_{n+1}) + \beta(t_{n+1})] \right. \\ &\qquad \left. \ldots [\sqrt{\gamma}\, c(t_m) + \beta(t_m)]\}\right\rangle. \end{aligned} \tag{5.4.23}$$

d) The Input is Quantum White Noise: By carrying out this procedure we are left with the problem of evaluating terms like $\langle c^\dagger(t_1)c^\dagger(t_2)b_{\text{in}}(s_1)b_{\text{in}}(s_2)\rangle$ when the density matrix is not coherent or vacuum, but represents a more general state. The solution of this problem for arbitrary statistics of the input is rather difficult, but we can give a way of doing this calculation in the case that the input is quantum white

noise (clearly, this will also allow us to carry out the computation when the input also has an additional coherent part). This requires us, more generally, to calculate terms like $\langle a_1(t)a_2(t')dB(s)dB(s')\rangle$ where $a_1(t)$ and $a_2(t)$ obey a single QSDE of the form (Ito)(5.3.50) which we can write in an abbreviated form

$$d\boldsymbol{a} = A\left(\boldsymbol{a}(t)\right)dt + G^\dagger\left(\boldsymbol{a}(t)\right)dB(t) + G\left(\boldsymbol{a}(t)\right)dB^\dagger(t). \tag{5.4.24}$$

(We shall only treat the case where the simple master equation (5.4.12) is relevant. The more general situation of (5.4.14) is essentially a many-input version of the same procedure; one input and one output being available for each frequency band around the ω_m). For simplicity, let us consider first the evaluation of $\langle a(t')dB(t)\rangle$. If $t > t'$, then $dB(t)$ is independent of $a(t')$, and so we deduce

$$\langle a(t')dB(t)\rangle = 0, \quad t > t'. \tag{5.4.25}$$

For $t < t'$, we argue as follows. We discretise time and define

$$\begin{aligned}
a_0 &= a(t), & a_n &= a(t_n), \\
A_n &= A(\boldsymbol{a}_n), & G_n &= G(\boldsymbol{a}_n),
\end{aligned} \tag{5.4.26}$$

so that (because we use the Ito form of the QSDE)

$$a_n = a_{n-1} + A_{n-1}\Delta t_n + G_{n-1}^\dagger \Delta B_n + G_{n-1}\Delta B_n^\dagger. \tag{5.4.27}$$

To solve the QSDE, we repeatedly use (5.4.27) starting with the initial condition a_0, and we can eventually write

$$a_n = \sum_{r,s} a_n^{r,s}\{\Delta B_1^\dagger\}^r\{\Delta B_1\}^s, \tag{5.4.28}$$

where the coefficients depend on a_0 and $\Delta B_m^\dagger, \Delta B_{m'}$, for $1 \leq m, m' \leq n$. The discretised form of $\langle a(t')dB(t)\rangle$ is (to lowest order in Δt)

$$\langle a_n \Delta B_1\rangle = \langle a_n^{1,0}\rangle\langle\Delta B_1^\dagger \Delta B_1\rangle = \bar{N}\Delta t_1\langle a_n^{1,0}\rangle. \tag{5.4.29}$$

We now need an expression for $\langle a_n^{1,0}\rangle$. Suppose we modify the QSDE (5.4.24) by the substitution

$$\begin{aligned}
dB(t) &\to dB(t) + \epsilon(t)dt, \\
dB^\dagger(t) &\to dB^\dagger(t) + \epsilon^*(t)dt,
\end{aligned} \tag{5.4.30}$$

where $\epsilon(t)$ is a classical c-number driving field. Then discretising again, it is clear that

$$\langle a_n^{1,0}\rangle = \frac{1}{\Delta t_1}\left\langle \frac{\partial}{\partial \epsilon_1^*}a_n^\epsilon\right\rangle_{\epsilon=0}, \tag{5.4.31}$$

and in a continuum notation this becomes

$$\langle a(t')dB(t)\rangle = \left[\bar{N}\frac{\delta}{\delta\epsilon^*(t)}\langle a^\epsilon(t')\rangle\right]_{\epsilon=0}. \tag{5.4.32}$$

We now evaluate this expression. We note that from the master equation (5.4.12)

$$\langle a^\epsilon(t')\rangle = \langle a\rho^\epsilon(t')\rangle$$

$$= \left\langle aT\left[\exp\left(\int\limits_t^{t'}\hat{L}^\epsilon(s)ds\right)\right]\rho(t)\right\rangle \tag{5.4.33}$$

and, discretising, this can be written

$$\langle a_n^\epsilon\rangle = \langle a(1+\hat{L}_n^\epsilon\Delta t_n)(1+\hat{L}_{n-1}^\epsilon\Delta t_{n-1})\ldots(1+\hat{L}_1^\epsilon\Delta t_1)\rho(t)\rangle. \tag{5.4.34}$$

But only the term involving L_1^ϵ depends on ϵ_1^*, and, in fact, the coefficient of ϵ_1^* in $L_1^\epsilon\rho$ is $\sqrt{\gamma}\,[c,\rho]$, so that

$$\frac{\partial}{\partial\epsilon_1^*}\langle a_n^\epsilon\rangle = \langle a(1+L_n^\epsilon\Delta t_n)\ldots(1+L_2^\epsilon\Delta t_2)\times\sqrt{\gamma}\,[c,\rho(t)]\rangle \tag{5.4.35}$$

and we then find

$$\left[\frac{\delta}{\delta\epsilon^*(t)}\langle a^\epsilon(t')\rangle\right]_{\epsilon=0} = \sqrt{\gamma}\,\langle a\exp[\hat{L}(t'-t)][c,\rho(t)]\rangle$$

$$= \sqrt{\gamma}\,\langle[a(t'),c(t)]\rangle. \tag{5.4.36}$$

Thus, we find the general expression

$$\langle a(t')dB(t)\rangle = \sqrt{\gamma}\,\bar{N}\,dt\,u(t'-t)\langle[a(t'),c(t)]\rangle. \tag{5.4.37}$$

We therefore derive in general by similar methods

$$\left.\begin{aligned}\langle a(t')b_{in}(t)\rangle &= \sqrt{\gamma}\,\bar{N}u(t'-t)\langle[a(t'),c(t)]\rangle\\ \langle b_{in}^\dagger(t)a(t')\rangle &= \sqrt{\gamma}\,\bar{N}u(t'-t)\langle[c^\dagger(t),a(t')]\rangle,\\ \langle a(t')b_{in}^\dagger(t)\rangle &= \sqrt{\gamma}\,(\bar{N}+1)u(t'-t)\langle[c^\dagger(t),a(t')]\rangle,\\ \langle b_{in}(t)a(t')\rangle &= \sqrt{\gamma}\,(\bar{N}+1)u(t'-t)\langle[a(t'),c(t)]\rangle\end{aligned}\right\} \tag{5.4.38}$$

and we note that these are consistent with the commutation relations (5.3.32,33). We can now use the results of Sect. 5.3.3 to derive

$$\langle b_{out}(t)b_{out}(t')\rangle = \gamma(\bar{N}+1)\langle T[c(t)c(t')]\rangle - \gamma\bar{N}\langle\bar{T}[c(t)c(t')]\rangle, \tag{5.4.39}$$

$$\langle b_{out}^\dagger(t)b_{out}(t')\rangle = \gamma(\bar{N}+1)\langle c^\dagger(t)c(t')\rangle$$
$$-\gamma\bar{N}\langle c(t')c^\dagger(t)\rangle + \gamma\bar{N}\delta(t-t'), \tag{5.4.40}$$

$$\langle b_{out}^\dagger(t)b_{out}^\dagger(t')\rangle = \gamma(\bar{N}+1)\langle\bar{T}[c^\dagger(t)c^\dagger(t')]\rangle - \gamma\bar{N}\langle T[c^\dagger(t)c^\dagger(t')]\rangle. \tag{5.4.41}$$

Comments

i)   It is very interesting to see that the only correlation functions of system operators which arise in this procedure are the kind of time ordered correlation

functions which arise out of a consideration of the most general kind of measurement as given in (2.3.12). *Barchielli* [5.9] has shown how the process of measuring the correlation function can be viewed as a kind of quantum measurement. Essentially, the quantum field $b(t)$ can be regarded as a kind of physical probe, which gives information about the system.

ii) Indeed, even if we do not use Barchielli's viewpoint, it is certainly true in practice that the usual way of getting information about atoms is to shine light, microwaves, or perhaps sound on it, and to measure a scattered or fluorescent field of some kind. Thus, this *is* a measurement theory, and it is pleasing to see that it is consistent with that of Chap. 2.

6. Applying the Master Equation

The previous chapter can be regarded as the statement of the mathematics of quantum Markov processes, and this chapter will show how the theory can be applied to a number of situations. It starts with an analysis of solutions of the master equation using number state matrix elements, and then moves into the very powerful methods based on the phase space representations introduced in Chap.4. Using these solutions, one is able to show how the quantum theory of damping and noise gives some very reasonable explanations of how classical physics arises as a macroscopic limit of quantum physics.

It finally introduces the ideas of generalised P-representations, introduced by the author and P.D. Drummond, by which many quantum Markov processes may be represented as classical diffusion processes. These have formed a very powerful tool in the practical solution of practical master equations.

6.1 Using the Number State Basis

For some simple systems, solution of the master equation in a number state basis is the most practical method. This section gives a few simple examples.

6.1.1 The Damped Harmonic Oscillator—Quantum Optical Case

We take the master equation in the form (3.6.67). In the case of the harmonic oscillator, we can write (using the notation of (4.1.32))

$$X \rightarrow x = \frac{(a + a^\dagger)}{\sqrt{2\eta}} \tag{6.1.1}$$

and it is clear that there is only one X_m^+ and one X_m^-; thus

$$X_m^- = a/\sqrt{2\eta}$$
$$X_m^+ = a^\dagger/\sqrt{2\eta} \tag{6.1.2}$$

and the master equation takes the form (in the Schrödinger picture)

$$\dot{\rho} = -i\omega[a^\dagger a, \rho]$$
$$+ \tfrac{1}{2}K(\bar{N} + 1)(2a\rho a^\dagger - a^\dagger a\rho - \rho a^\dagger a)$$
$$+ \tfrac{1}{2}K\bar{N}(2a^\dagger \rho a - aa^\dagger \rho - \rho aa^\dagger) \tag{6.1.3}$$

in which

$$\bar{N}(\omega) \rightarrow \bar{N}$$
$$\pi \kappa(\omega)^2 / 2 \hbar \eta \rightarrow K. \tag{6.1.4}$$

Diagonal Matrix Elements:

i) The diagonal matrix element

$$\langle n | \rho | n \rangle \equiv P(n) \tag{6.1.5}$$

represents the probability of there being n quanta in the system. We easily check that (using the properties of $a^\dagger$ and a defined in Sect. 3.1.3)

$$\partial_t P(n) = K(\bar{N} + 1)[(n + 1)P(n + 1) - nP(n)]$$
$$+ K\bar{N}[nP(n - 1) - (n + 1)P(n)]. \tag{6.1.6}$$

Exercise. Show that the equations for the coherences $\langle n | \rho | m \rangle$ decouple, so that elements of the form $\langle n | \rho | n + r \rangle$ for a fixed r have equations independent of those for a different r.

ii) The fact that (6.1.6) is a version of a classical Markov process (a birth-death process) indicates that the connection between the classical and quantum version of a Markov process is more than mere formalism. In fact (6.1.6) is also a version of the Pauli master equation, (1.5.1).

iii) Notice that the transition probabilities of this classical master equation can be written in the form (in the notation of S.M.Chap. 7)

$$t^+(n) = K\bar{N}(n + 1)$$
$$t^-(n) = K(\bar{N} + 1)n \tag{6.1.7}$$

so that the probability of creating a quantum has a part proportional to $n + 1$. A chemical reaction of the form

$$B \rightleftharpoons X$$
$$A + X \rightarrow 2X \tag{6.1.8}$$
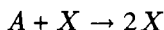
would have a similar master equation.

iv) The solution in the stationary state is

$$P_s(n) = \left(\frac{\bar{N}}{1 + \bar{N}} \right)^n \frac{1}{1 + \bar{N}}. \tag{6.1.9}$$

This is the usual Boltzmann distribution, in which one can identify

$$\frac{\bar{N}}{1 + \bar{N}} = \exp(-\hbar \omega / kT) \tag{6.1.10}$$

which means

$$\bar{N} = \frac{1}{\exp(\hbar \omega / kT) - 1} \tag{6.1.11}$$

which determines $\bar{N}$ in terms of T , or conversely. We note that

$$\langle n \rangle_s = \bar{N}$$
$$\text{var}\{n\}_s = \bar{N}^2. \tag{6.1.12}$$
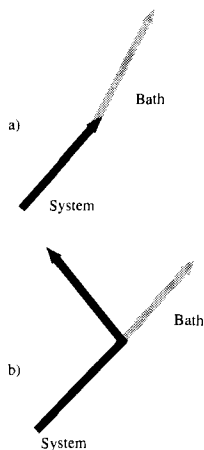
Fig. 6.1   Comparison of standard damping (a), in which a system quantum is destroyed while producing a bath quantum (and conversely), and (b) phase damping, in which a system quantum is scattered when it absorbs or emits a bath quantum

6.1.2   The Phase Damped Oscillator

The damped harmonic oscillator of the previous section has a basic system-bath Hamiltonian of the form

$$H_{\text{Int}} = \sum_i \left\{ \Gamma_i a^\dagger + \Gamma_i^\dagger a \right\}, \tag{6.1.13}$$

which implies that an oscillator quantum can lose energy by creating a bath quantum, and conversely. This kind of damping transfers energy between the bath and the system.

It is possible to have an interaction which can be viewed as a *scattering* process, with an interaction Hamiltonian

$$H_{\text{Int}} = \sum_i \left\{ \Gamma_i a^\dagger a + \Gamma_i^\dagger a^\dagger a \right\} \tag{6.1.14}$$

in which a bath quantum can be absorbed or emitted, but in this case the number of oscillator quanta remains unchanged. The situation is illustrated diagrammatically in Fig. 6.1. Such an interaction commutes with the system Hamiltonian, so no energy is transferred—only the phase of the system state is changed. The master equation is quite simple

$$\dot{\rho} = -i\omega[a^\dagger a, \rho] + \tfrac{1}{2}K(2\bar{N} + 1)\{2a^\dagger a\rho a^\dagger a - (a^\dagger a)^2 \rho - \rho(a^\dagger a)^2\} \tag{6.1.15}$$

and the equations of motion for the matrix elements $\langle n|\rho|m \rangle$ are

$$\frac{d\langle n|\rho|m \rangle}{dt} = \{-i\omega(n - m) - \tfrac{1}{2}K(2\bar{N} + 1)(n - m)^2\}\langle n|\rho|m \rangle \tag{6.1.16}$$

with the solution

$$\langle n|\rho(t)|m \rangle = e^{-i\omega(n-m)t} e^{-(2\bar{N}+1)K(n-m)^2 t/2} \langle n|\rho(0)|m \rangle. \tag{6.1.17}$$

Comments:

i)   Notice that the off-diagonal matrix elements decay away to zero, leaving a stationary diagonal solution, which in fact is

$$\langle n|\rho_s|m \rangle = \delta_{nm} \langle n|\rho(0)|m \rangle. \tag{6.1.18}$$

Thus this interaction produces no thermalisation—its effect is to preserve the occupation probabilities, while erasing all coherences.

ii) The off-diagonal matrix elements decay at a rate proportional to $(n-m)^2$ — that is, the further from the diagonal, the faster the damping. If the system is *macroscopic* $n - m$ will be enormous, so this decay will be very fast. Thus it would be almost impossible to observe the coherences in such a case, because of the very rapid decay.

6.2   Quantum Classical Correspondence

An alternative method of developing the idea of a quantum Markov process is to use the phase space methods of Chap. 4 to derive phase space equations of motion which are very often of the classical Fokker-Planck form. The interpretation of these Fokker-Planck equations is not always straightforward, but the method is a very powerful technique for finding solutions to many quantum Markov problems. Since the phase space equations are classical non-operator equations, a wide variety of standard analytical techniques can be employed for the solution.

This section will concentrate on a number of specific examples, starting with the simplest, the harmonic oscillator.

6.2.1   Use of the P-Representation

We have seen in Chap. 3 that a density operator can be expressed in terms of quasiprobability $P(\alpha)$, which behaves rather like a classical probability, though it need not always be positive or even exist as an ordinary function. If we now express the density matrix under study here as a P-representation

$$\rho = \int d^2\alpha |\alpha \rangle \langle \alpha| P(\alpha), \tag{6.2.1}$$

then using the operator correspondences of (4.5.9) it is easily shown that the quantum master equation (6.1.3) leads to the Fokker-Planck equation (remember that operator products written on the right of ρ are reversed in order when the correspondence is made)

$$\frac{\partial P}{\partial t} = \left\{ \tfrac{1}{2}K \left(\frac{\partial}{\partial\alpha}\alpha + \frac{\partial}{\partial\alpha^*}\alpha^* \right) - i\omega \left(\frac{\partial}{\partial\alpha}\alpha - \frac{\partial}{\partial\alpha^*}\alpha^* \right) + K\bar{N}\frac{\partial^2}{\partial\alpha\partial\alpha^*} \right\} P. \tag{6.2.2}$$

a) Interpretation as a Classical Fokker-Planck Equation: This is a form of complex Ornstein-Uhlenbeck process. To show this we write

$$\alpha = x + \mathrm{i}\,y \tag{6.2.3}$$

and

$$\frac{\partial}{\partial \alpha} = \frac{1}{2}\left(\frac{\partial}{\partial x} - \mathrm{i}\,\frac{\partial}{\partial y}\right) \tag{6.2.4}$$

and get

$$\frac{\partial P}{\partial t} = \tfrac{1}{2}K\left\{\left(\frac{\partial}{\partial x}x + \frac{\partial}{\partial y}y\right) + \omega\left(\frac{\partial}{\partial x}y - \frac{\partial}{\partial y}x\right) + \frac{\bar{N}}{2}\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\right\}P. \tag{6.2.5}$$

b) Stochastic Differential Equations: The classical stochastic differential equations equivalent to (6.2.5) are

$$\begin{aligned}
dx &= -(\tfrac{1}{2}Kx + \omega y)\,dt + \sqrt{\tfrac{1}{2}\bar{N}}\,dW_1(t) \\
dy &= -(\tfrac{1}{2}Ky - \omega x)\,dt + \sqrt{\tfrac{1}{2}\bar{N}}\,dW_2(t).
\end{aligned} \tag{6.2.6}$$

These represent the equations for a damped oscillator, very like those which would occur classically. They are often written as one complex Langevin equation:

$$d\alpha = -(\tfrac{1}{2}K - \mathrm{i}\omega)\alpha\,dt + \sqrt{\bar{N}}\,d\eta(t), \tag{6.2.7}$$

where $d\eta(t)$ is the increment of a complex Wiener process satisfying

$$\begin{aligned}
&\langle d\eta(t)\rangle = \langle d\eta^*(t)\rangle = \langle d\eta(t)d\eta(t')\rangle = \langle d\eta^*(t)d\eta^*(t')\rangle = 0 \\
&\langle d\eta(t)d\eta^*(t)\rangle = dt
\end{aligned} \tag{6.2.8}$$

and explicitly given by

$$d\eta(t) = [dW_1(t) + \mathrm{i}\,dW_2(t)]/\sqrt{2}. \tag{6.2.9}$$

The solutions of the Ornstein-Uhlenbeck process are given in S.M. Sect. 4.4.6. We find

$$\begin{aligned}
&\langle\alpha(t)\rangle = \alpha(0)\exp[-(K/2 - \mathrm{i}\omega)t] \\
&\langle\alpha^*(t)\alpha(t)\rangle = \langle\alpha^*(0)\alpha(0)\rangle e^{-Kt} + \bar{N}(1 - e^{-Kt})
\end{aligned} \tag{6.2.10}$$

and the stationary variances are

$$\begin{aligned}
&\langle\alpha^2\rangle_s = \langle\alpha^{*2}\rangle_s = 0 \\
&\langle\alpha\alpha^*\rangle_s = \langle a^\dagger a\rangle_s = \bar{N}.
\end{aligned} \tag{6.2.11}$$

Only when $\bar{N} = 0$ do we find that $\langle a^\dagger a \rangle_s$ vanishes, i.e., at zero temperature.

c) Initial Coherent State: If the oscillator is initially in a coherent state $|\alpha(0)\rangle$, the P-function is initially of the form $\delta^2 (\alpha - \alpha(0))$. This corresponds to the initial condition for the SDE (6.2.7) having zero variance. Since the equation of motion (6.2.7) has zero noise at zero temperature, $\alpha(t)$ will have zero variance for all time. Thus the time dependent solution for the P-function will be $\delta^2 (\alpha - \alpha(t))$, and hence the time dependent solution for the state will be the time dependent coherent state $|\alpha(t)\rangle$.

This result is generalisable to multivariable harmonic systems with damping which arises from zero temperature absorption of single quanta—in such situations, an initial coherent state remains coherent. For example, it will be shown in Chap. 8 that a coherent light beam passing through a beam splitter remains coherent, and that detection of photons also gives this of damping.

d) Limitations on the Classical Interpretation: At first glance, this appears to yield a complete equivalence between a classical and a quantum system, but this is not so. The most obvious difference is the fact that the probability density which is the solution of the FPE is not required to be positive. Thus, there are initial conditions which cannot be realised classically. The more subtle point is a question of the exact relationship between the conditional probability, for which the Fokker-Planck equation is the evolution equation, and the Markovian nature of the quantum process. Put at its simplest—from the FPE we can calculate $\langle \alpha(t + \tau)\alpha(t) \rangle$ and similar averages—how do these relate to the measurable quantum quantities?

6.2.2   Time Correlation Functions in the P-Representation

As a first example, consider the correlation functions

$$\langle a^\dagger(t + \tau)a(t) \rangle = \mathrm{Tr} \left\{ a^\dagger V(t + \tau, t) \{a\rho(t)\} \right\} \tag{6.2.12}$$

where we have used (5.2.11) to write the correlation function in terms of the evolution operator of the quantum Markov process. Using the P-representation for $\rho(t)$, we can write:

$$\langle a^\dagger(t + \tau)a(t) \rangle = \mathrm{Tr} \left\{ a^\dagger V(t + \tau, t) \int d^2\alpha \, \alpha P(\alpha, t) |\alpha\rangle\langle\alpha| \right\} \tag{6.2.13}$$

where the operator correspondence (4.5.9) has been used. The term

$$V(t + \tau, t) \int d^2\alpha \, \alpha P(\alpha, t) |\alpha\rangle\langle\alpha| \tag{6.2.14}$$

is a solution of the master equation, for which the initial condition at time t is the operator whose P-function is $\alpha P(\alpha, t)$. However, using the FPE (6.2.2), this can be written

$$\int d^2\alpha' \alpha' P(\alpha, t + \tau | \alpha', t) P(\alpha', t) \tag{6.2.15}$$

where $P(\alpha, t+\tau|\alpha', t)$ is the conditional probability which is a solution of the FPE—i.e., the solution of the FPE with the initial condition $\delta^2(\alpha - \alpha')$. Substituting into (6.2.13), and using the cyclic property of the trace, we find

$$\langle a^\dagger(t+\tau)a(t)\rangle = \text{Tr}\left\{a^\dagger \int d^2\alpha \int d^2\alpha'\, \alpha' P(\alpha, t+\tau|\alpha', t) P(\alpha', t)|\alpha\rangle\langle\alpha|\right\} \tag{6.2.16}$$

$$= \int d^2\alpha \int d^2\alpha'\, \alpha^* \alpha' P(\alpha, t+\tau; \alpha', t) \tag{6.2.17}$$

and hence

$$\boxed{\langle a^\dagger(t+\tau)a(t)\rangle = \langle \alpha^*(t+\tau)\alpha(t)\rangle.} \tag{6.2.18}$$

This result is very simple and reinforces the feeling that there is a direct mapping from the quantum process to a classical process. However, on the left hand side of (6.2.18), the operators do not commute, while the c-numbers $\alpha(t+\tau)$ and $\alpha(t)$ on the right hand side do commute. So let us now calculate

$$\langle a(t)a^\dagger(t+\tau)\rangle = \text{Tr}\left\{\{V(t+\tau, t)\rho a\}a^\dagger\right\} \tag{6.2.19}$$

where we have followed the prescription of Sect. 5.2.1b. Using the same procedure as above,

$$(6.2.19) = \text{Tr}\left\{\left(V(t+\tau, t)\int d^2\alpha\left[\left(\alpha - \frac{\partial}{\partial\alpha^*}\right)P(\alpha)\right]|\alpha\rangle\langle\alpha|\right)a^\dagger\right\} \tag{6.2.20}$$

and finally

$$\int d^2\alpha' \int d^2\alpha\left\{\left(\alpha^* - \frac{\partial}{\partial\alpha}\right)P(\alpha, t+\tau|\alpha', t)\right\}\left(\alpha' - \frac{\partial}{\partial\alpha'^*}\right)P(\alpha', t). \tag{6.2.21}$$

In a certain formal sense, this can be written

$$\boxed{\langle a(t)a^\dagger(t+\tau)\rangle = \left\langle\left[\alpha^* - \frac{\partial}{\partial\alpha}\right](t+\tau)\left[\alpha - \frac{\partial}{\partial\alpha^*}\right](t)\right\rangle} \tag{6.2.22}$$

but it must be noted that the notation in (6.2.22) must be interpreted as merely a shorthand for (6.2.21). In particular, the fact that $\alpha' - \partial/\partial\alpha'^*$ acts only on $P(\alpha', t)$ and *not* on $P(\alpha, t+\tau|\alpha', t)$ is not clear from (6.2.22). If we introduce the notation

$$\langle\alpha^*, t+\tau|[\alpha', t]\rangle = \int d^2\alpha\, \alpha^* P(\alpha, t+\tau|\alpha', t), \tag{6.2.23}$$

which is the mean of α at time $t+\tau$ conditional on it having the value α' at time t, then after integrating by parts and discarding surface terms, (6.2.21) can be written

$$\langle a(t)a^\dagger(t+\tau)\rangle = \langle\alpha^*(t+\tau)\alpha(t)\rangle + \int d^2\alpha'\frac{\partial}{\partial\alpha'^*}\langle\alpha^*, t+\tau|[\alpha', t]\rangle P(\alpha', t). \tag{6.2.24}$$

6.2.3   Application to the Damped Harmonic Oscillator

Using standard methods, from the equations (6.2.2) or (6.2.7) we find, in the stationary situation,

$$\langle \alpha^*(t+\tau)\alpha(t)\rangle = \bar{N}e^{-(K/2+i\omega)\tau} \tag{6.2.25}$$

$$\langle \alpha^*, t+\tau | [\alpha', t]\rangle = \alpha'^* e^{-(K/2+i\omega)\tau} \tag{6.2.26}$$

so that

$$\langle a^\dagger(t+\tau)a(t)\rangle = \bar{N}e^{-(K/2+i\omega)\tau}$$
$$\langle a(t)a^\dagger(t+\tau)\rangle = (\bar{N}+1)e^{-(K/2+i\omega)\tau}. \tag{6.2.27}$$

6.2.4   General Form for Time Correlation Functions in the P-Representation

The general result follows from (5.2.20). If $W(t_1, t_2)$ is the evolution operator of the Fokker-Planck equation, then we can make the correspondence:

$$\langle A_0(s_0)A_1(s_1)\dots A_m(s_m)B_n(t_n)B_{n-1}(t_{n-1})\dots B_0(t_0)\rangle$$
$$= \int d^2\alpha f_r(\alpha)W(\tau_r, \tau_{r-1})f_{r-1}(\alpha)W(\tau_{r-1}, \tau_{r-2})\ \dots$$
$$\dots \times W(\tau_1, \tau_0)f_0(\alpha)P(\alpha, \tau_0) \tag{6.2.28}$$

$$\text{where}\quad f_r(\alpha) = \begin{cases} \alpha & \text{if } F_r = a \text{ and } F_r \text{ is one of the } B\text{'s,} \\[2mm] \alpha - \dfrac{\partial}{\partial\alpha^*} & \text{if } F_r = a \text{ and } F_r \text{ is one of the } A\text{'s,} \\[2mm] \alpha^* & \text{if } F_r = a^\dagger \text{ and } F_r \text{ is one of the } A\text{'s,} \\[2mm] \alpha^* - \dfrac{\partial}{\partial\alpha} & \text{if } F_r = a^\dagger \text{ and } F_r \text{ is one of the } B\text{'s.} \end{cases}$$

Here, of course, the operator W has the property for an arbitrary function $f(\alpha)$

$$W(\tau_r, \tau_{r-1})f(\alpha) = \int d^2\alpha' P(\alpha, \tau_r | \alpha', \tau_{r-1})f(\alpha'). \tag{6.2.29}$$

Exercise. Verify the result (6.2.28), and show that it gives the two previous results.

Exercise. Show that

$$\langle a^\dagger(t)a^\dagger(t+\tau)a(t+\tau)a(t)\rangle = \langle |\alpha(t+\tau)|^2 |\alpha(t)|^2\rangle \tag{6.2.30}$$

and that this result can be generalised to any correlation function which can be expressed as a product of two terms, one consisting only of $a^\dagger$'s, ordered with times increasing to the right, and the other of a's, with times decreasing to the right.

6.3   Some Amplifier Models

In this section we consider some amplifiers by using the various possible phase space methods, but not by input-output methods, as in Sect. 7.1. We consider simply the growth of an initial value of a field mode in the amplifying medium from which the signal cannot escape. For these simple models the QSDE methods are much more transparent, but they are also excellent examples of the use of phase space methods.

6.3.1   A Simple Amplifier

If we consider a system consisting of a harmonic oscillator coupled to a bath of two level atoms, we may use the master equation (5.1.59) in the form (omitting frequency shifts, and working in the interaction picture)

$$\dot{\rho}(t) = \tfrac{1}{2}\kappa(2a^\dagger\rho a - aa^\dagger\rho - \rho aa^\dagger) + \tfrac{1}{2}\gamma(2a\rho a^\dagger - a^\dagger a\rho - \rho a^\dagger)a \qquad (6.3.1)$$

and we note that from (5.1.64),

$\kappa > \gamma$ if there is a net inversion,
$\kappa < \gamma$ if there is no net inversion .

In the same way as we derived the Langevin equation (6.2.7) we can derive the equation of motion

$$d\alpha = \tfrac{1}{2}(\kappa - \gamma)\alpha dt + \sqrt{\kappa}\, d\eta(t). \qquad (6.3.2)$$

From this equation it can be seen that if there is a net inversion, the variable $\alpha(t)$ will grow—this arises from energy being fed into the system by the bath. This is the reverse of the damping situation, where energy is dissipated into the bath. However, the amplification is noisy, since the stochastic term is present for all non-zero κ. If the initial density operator corresponds to a pure coherent state $|\alpha_0\rangle$, the time dependent density operator can be deduced from the P-function. This will be a Gaussian, whose means and variances can be found by solving (6.3.2). Explicitly

$$\alpha(t) = \alpha_0 \exp[\tfrac{1}{2}(\kappa - \gamma)t] + \sqrt{\kappa} \int_0^t d\eta(t') \exp\left[\tfrac{1}{2}(\kappa - \gamma)(t - t')\right] \qquad (6.3.3)$$

$$\langle\alpha(t)\rangle = \langle\alpha^*(t)\rangle^* = \alpha_0 \exp\left[\tfrac{1}{2}(\kappa - \gamma)t\right] \qquad (6.3.4)$$

$$\begin{aligned}
\text{var}[\alpha(t)] &= \langle\alpha(t)^2\rangle - \langle\alpha(t)\rangle^2 = 0 \\
\text{var}[\alpha^*(t)] &= \langle\alpha^*(t)^2\rangle - \langle\alpha^*(t)\rangle^2 = 0 \\
\langle\alpha^*(t), \alpha(t)\rangle &= \langle\alpha^*(t)\alpha(t)\rangle - \langle\alpha^*(t)\rangle\langle\alpha(t)\rangle \\
&= \frac{\kappa}{\kappa - \gamma} \{\exp[(\kappa - \gamma)t] - 1\}.
\end{aligned} \qquad (6.3.5)$$

In fact, because of the complex nature of α, α^*, the quantity $\langle \alpha^*(t), \alpha(t) \rangle$ actually expresses the variance. From these, we can say that $P(\alpha, \alpha^*)$ is a Gaussian with these parameters, and that the density operator is a quantum Gaussian, with the appropriate parameters.

i) From (6.3.3) we see that the initial field is continuously amplified with time, but there is an added noise term which also grows at a similar rate. The signal to noise ratio is given by

$$S(t) = \langle \alpha(t) \rangle / \sqrt{\langle \alpha(t), \alpha^*(t) \rangle}$$

$$= \alpha_0 \sqrt{\frac{(\kappa - \gamma)}{\kappa \{ 1 - \exp[-(\kappa - \gamma)t] \}}} \tag{6.3.6}$$

$$\to \alpha_0 \sqrt{(\kappa - \gamma)/\kappa} \quad \text{as } t \to \infty \quad \text{provided } \kappa > \gamma. \tag{6.3.7}$$

The best signal to noise ratio arises from the highest gain situation, $\gamma = 0$. The noise term arises from the constant κ, which shows that if there is gain ($\kappa > \gamma$) then there will be noise.

ii) Suppose the actual operators corresponding to the EM field represented by $a, a^\dagger$ are (in the Schrödinger picture, not the interaction picture)

$$x = (a + a^\dagger)/2$$
$$y = (a - a^\dagger)/2i. \tag{6.3.8}$$

Then

$$\langle x(t) \rangle = \exp[\tfrac{1}{2}(\kappa - \gamma)t]\{x_0 \cos \omega t + y_0 \sin \omega t\}$$
$$\langle y(t) \rangle = \exp[\tfrac{1}{2}(\kappa - \gamma)t]\{-x_0 \sin \omega t + y_0 \cos \omega t\} \tag{6.3.9}$$

and the variances and correlation are

$$\text{var}[x(t)] = \tfrac{1}{4} + \tfrac{1}{2}\langle a^\dagger(t), a(t) \rangle$$

$$= \tfrac{1}{4} + \frac{\kappa}{2(\kappa - \gamma)}\{\exp[(\kappa - \gamma)t] - 1\} \tag{6.3.10}$$

$$\text{var}[y(t)] = \tfrac{1}{4} + \frac{\kappa}{2(\kappa - \gamma)}\{\exp[(\kappa - \gamma)t] - 1\} \tag{6.3.11}$$

and

$$\langle x(t), y(t) \rangle + \langle y(t), x(t) \rangle = 0. \tag{6.3.12}$$

Thus, in the physical operators, x, y, we see three terms:

—The term $1/4$ in the variance: this arises from zero point fluctuations.

—The remainder of the variance: this is noise arising from the amplification process.

—The amplified signal.

iii) A physical picture of what happens is as follows. The bath of atoms is in an optical cavity, and most of the atoms are excited. The process of stimulated emission amplifies any electromagnetic field already present, but this is a noisy process. Further, the process of spontaneous emission adds yet more noise. As the signal grows, the bath of excited atoms will eventually deplete, and this model will break down, unless there is some pumping mechanism which maintains the atoms in their excited states.

6.3.2   Comparison of P-, Q- and Wigner Function Methods

When they were first introduced in Chap. 3, of all the quasiprobabilities the Q-function looked the most attractive. It is positive, and always exists. The Wigner function always exists, but need not be positive, and the P-function need neither exist, nor be positive. In this section, we will compare the various methods on the same example, and show that in fact there are advantages to all of these methods, but that the P-function is most easily interpreted as a quasiprobability because of one major advantage—namely, the P-function for a coherent state $|\alpha_0\rangle\langle\alpha_0|$ is simply $\delta^2(\alpha - \alpha_0)$, while for the Q- and Wigner functions, a delta function is not admissible.

6.3.3   The Degenerate Parametric Amplifier

This can be made optically by putting a crystal with a non-linear dielectric constant in a cavity, and driving it with a light field whose frequency is twice that of the cavity resonance.

This leads to the Hamiltonian (in the Schrödinger picture)

$$H_{\text{sys}} = \hbar\,\omega a^\dagger a + \frac{i\,\hbar\,\epsilon}{2}\left(a^{\dagger\,2}e^{-2i\omega t} - a^2\,e^{2i\omega t}\right). \tag{6.3.13}$$

We assume that ϵ is quite small, so that it is correct to use the same damping term as in the last two sections, and simply add on the nonlinearity (see Sect. 3.6.3c).

a) Use of P-function Methods: The equations of motion for the P-function are then straightforwardly computed, in the interaction picture, to be

$$\frac{\partial P(\alpha)}{\partial t} = \left\{\frac{\kappa}{2}\left(\frac{\partial}{\partial\alpha}\alpha + \frac{\partial}{\partial\alpha^*}\alpha^*\right) - \epsilon\left(\frac{\partial}{\partial\alpha^*}\alpha + \frac{\partial}{\partial\alpha}\alpha^*\right)\right.$$
$$\left. + \kappa\bar{N}\frac{\partial^2}{\partial\alpha\partial\alpha^*} + \frac{\epsilon}{2}\left(\frac{\partial^2}{\partial\alpha^2} + \frac{\partial^2}{\partial\alpha^{*2}}\right)\right\}P. \tag{6.3.14}$$

Notice that this equation is a Fokker-Planck equation of exactly the kind which turns up in classical noise processes, apart from two points:

i)   The noise terms are of two types. The term proportional to $\bar{N}$ represents thermal noise, as it did in the previous examples. The term proportional to ϵ arises

from the non-linearity of the medium, and must be regarded as a purely quantum phenomenon.

ii)   The overall diffusion matrix need not be positive. In fact the condition for positivity is

$$\kappa \bar{N} \geq |\epsilon|. \tag{6.3.15}$$

This means that, provided the temperature (since $\bar{N} = \left[\exp(\hbar\omega/\kappa T) - 1\right]^{-1}$ in the case of a thermal bath) is sufficiently high, this is a genuine Fokker-Planck equation, but otherwise, although (6.3.14) is still valid, the evolution is not the same as that of an ordinary diffusion process. (However, generalised P-representations can deal with this problem—see Sect. 6.4).

The stochastic differential equations are

$$
\begin{aligned}
d\alpha &= \left(-\frac{\kappa}{2}\alpha + \epsilon\alpha^*\right) dt + d\eta(t) \\
d\alpha^* &= \left(-\frac{\kappa}{2}\alpha^* + \epsilon\alpha\right) dt + d\eta^*(t)
\end{aligned}
\tag{6.3.16}
$$

where

$$
\begin{aligned}
d\eta(t)^2 &= \epsilon dt, \\
d\eta^*(t)^2 &= \epsilon dt \\
d\eta^*(t)d\eta(t) &= \kappa\bar{N}dt.
\end{aligned}
\tag{6.3.17}
$$

The equations are best simplified in terms of the *quadrature phases* (see Chap.10 for more about these) :

$$u = \frac{\alpha + \alpha^*}{2}, \quad v = \frac{\alpha - \alpha^*}{2i}, \tag{6.3.18}$$

in terms of which they are

$$
\begin{aligned}
du &= \left(-\frac{\kappa}{2} + \epsilon\right) u \, dt + dW_u(t) \\
dv &= \left(-\frac{\kappa}{2} - \epsilon\right) v \, dt + dW_v(t)
\end{aligned}
\tag{6.3.19}
$$

where

$$
\begin{aligned}
dW_u(t)^2 &= \tfrac{1}{2}(\kappa\bar{N} + \epsilon) dt \\
dW_v(t)^2 &= \tfrac{1}{2}(\kappa\bar{N} - \epsilon) dt \\
dW_u(t)dW_v(t) &= 0.
\end{aligned}
\tag{6.3.20}
$$

The solutions for the quadrature phase equations (6.3.19) are of course quite simple

$$u(t) = \exp\left[(-\tfrac{1}{2}\kappa + \epsilon)t\right] u(0) + \int_0^t dW_u(t') \exp\left[(-\tfrac{1}{2}\kappa + \epsilon)(t - t')\right] \tag{6.3.21}$$

$$v(t) = \exp\left[(-\tfrac{1}{2}\kappa - \epsilon)t\right] v(0) + \int_0^t dW_v(t') \exp\left[-(\tfrac{1}{2}\kappa + \epsilon)(t - t')\right]. \tag{6.3.22}$$

Provided $\epsilon > \frac{1}{2}\kappa$, the variable u displays gain, and provided also $\epsilon < \kappa\bar{N}$, the simple SDE viewpoint is valid. However, at the same time the variable $v(t)$ is attenuated. The variances are also easily computed to be

$$
\begin{aligned}
\text{var}[u(t)] &= e^{(2\epsilon-\kappa)t}\text{var}[u(0)] + \frac{\kappa\bar{N}+\epsilon}{2(2\epsilon-\kappa)}\left(e^{(2\epsilon-\kappa)t} - 1\right) \\
\text{var}[v(t)] &= e^{-(2\epsilon+\kappa)t}\text{var}[v(0)] + \frac{\kappa\bar{N}-\epsilon}{2(2\epsilon+\kappa)}\left(1 - e^{-(2\epsilon+\kappa)t}\right).
\end{aligned} \tag{6.3.23}
$$

Notice that the variances in the actual measurable variables are given by adding 1/4 to each of these, as noted in (6.3.10) .

Comments:

i)   What is most interesting in this system, is the appearance of the terms proportional to ϵ in the diffusion matrix, since these are noise terms which arise from the Hamiltonian (and therefore reversible) motion of the system. However, because of the restriction that $\kappa\bar{N} \geq |\epsilon|$ for positivity of the diffusion matrix, it is in principle not possible to use this method to study the quantum noise in the absence of classical noise.

ii)   However, in the absence of classical noise the interaction picture Heisenberg equations of motion arising from (6.3.13) are

$$
\begin{aligned}
\frac{da}{dt} &= \epsilon a^\dagger \\
\frac{da^\dagger}{dt} &= \epsilon a.
\end{aligned} \tag{6.3.24}
$$

and the solutions are easily found to be computable in terms of

$$
x = \frac{1}{2}(a + a^\dagger), y = \frac{i}{2}(a^\dagger - a) \tag{6.3.25}
$$

as

$$
\begin{aligned}
x(t) &= e^{\epsilon t}x(0) \\
y(t) &= e^{-\epsilon t}y(0),
\end{aligned} \tag{6.3.26}
$$

and hence

$$
\begin{aligned}
\text{var}[x(t)] &= e^{2\epsilon t}\text{var}[x(0)] \\
\text{var}[y(t)] &= e^{-2\epsilon t}\text{var}[y(0)].
\end{aligned} \tag{6.3.27}
$$

Setting $\kappa = 0$ in (6.3.23) and adding 1/4, it can be seen that the solutions (6.3.23) are still valid in the case of no classical noise, even though the method is not. The reason for this will be made clear in Sect. 6.4 on generalised P-representations.

iii)   We can now see however the difference between the classical noise and the quantum noise. In the case of pure quantum noise, we see that

$$
\text{var}[x(t)]\text{var}[y(t)] = \text{var}[x(0)]\text{var}[y(0)] \tag{6.3.28}
$$

and, among other things, this means that the Hamiltonian motion transforms a minimum uncertainty state into another minimum uncertainty state. Essentially, we can say that only the noise required by Heisenberg's uncertainty principle is present.

b) Use of the Q-Function: The Q-function always exists, so it is of interest to see how it might be used in this situation. Using the operator correspondences given in Sect. 4.5 an equation of motion for the Q-function is

$$
\frac{\partial Q(\alpha)}{\partial t} = \left\{ \frac{\kappa}{2} \left(\frac{\partial}{\partial \alpha}\alpha + \frac{\partial}{\partial \alpha^*}\alpha^* \right) - \epsilon \left(\frac{\partial}{\partial \alpha^*}\alpha + \frac{\partial}{\partial \alpha}\alpha^* \right) \right.
$$
$$
\left. + \kappa(\bar{N}+1)\frac{\partial^2}{\partial \alpha \partial \alpha^*} - \frac{\epsilon}{2}\left(\frac{\partial^2}{\partial \alpha^2} + \frac{\partial^2}{\partial \alpha^{*2}} \right) \right\} Q(\alpha). \tag{6.3.29}
$$

Comments:

i) The equation does not look very different from the P-function equation. The only changes are in fact

$$
\bar{N} \to \bar{N} + 1
$$
$$
\text{and} \quad \frac{\epsilon}{2}\left(\frac{\partial^2}{\partial \alpha^2} + \frac{\partial^2}{\partial \alpha^{*2}} \right) \to -\frac{\epsilon}{2}\left(\frac{\partial^2}{\partial \alpha^2} + \frac{\partial^2}{\partial \alpha^{*2}} \right). \tag{6.3.30}
$$

Thus the Hamiltonian noise is changed in sign, and the thermal term is augmented by a term that remains even at zero temperature. The condition for the diffusion matrix to be positive is now

$$
\kappa(\bar{N}+1) \geq |\epsilon|, \tag{6.3.31}
$$

and this is not always satisfied.

ii) At first glance, this is paradoxical—the Q-function is always an acceptable probability function, but the corresponding equation of motion does not have a positive diffusion matrix, and hence does not always have solutions which are positive. The answer to this paradox lies in the restricted range of initial conditions. From (4.4.19), we see that in the case of the Q-function $\langle |\alpha|^2 \rangle \geq 1$, so we can only admit as initial values Q-functions which satisfy this condition. If the condition (6.3.31) is not satisfied, this will mean that an initial state with $\langle |\alpha|^2 \rangle \geq 1$ will evolve in such a way as to reduce the variance in one quadrature phase, but never below a value corresponding to a positive Q-function.

Exercise. Show that the variances of the quadrature phases with respect to the Q-function evolve in time as

$$
\text{var}[u(t)] = e^{(2\epsilon-\kappa)t}\text{var}[u(0)] + \frac{\kappa(\bar{N}+1)-\epsilon}{2(2\epsilon-\kappa)}\left\{ e^{(2\epsilon-\kappa)t} - 1 \right\} \tag{6.3.32}
$$

$$
\text{var}[v(t)] = e^{-(2\epsilon-\kappa)t}\text{var}[v(0)] + \frac{\kappa(\bar{N}+1)+\epsilon}{2(2\epsilon+\kappa)}\left\{ 1 - e^{-(2\epsilon+\kappa)t} \right\} \tag{6.3.33}
$$

and that for no allowable initial conditions can these ever be negative.

c) Use of the Wigner Function: We know that the Wigner function is also guaranteed to exist, and this time a real advantage ensues; the equation for the Wigner function is found to be (by using the correspondences (4.5.12))

$$\frac{\partial W(\alpha)}{\partial t} =$$

$$\left\{ \frac{\kappa}{2} \left(\frac{\partial}{\partial \alpha} \alpha + \frac{\partial}{\partial \alpha^*} \alpha^* \right) - \epsilon \left(\frac{\partial}{\partial \alpha^*} \alpha + \frac{\partial}{\partial \alpha} \alpha^* \right) + \kappa(\bar{N} + \tfrac{1}{2}) \frac{\partial^2}{\partial \alpha \partial \alpha^*} \right\} W$$

(6.3.34)

and in this case there is *no* Hamiltonian noise. A representation in terms of SDEs is possible for all ϵ. Using the same procedures for the P-function shows that the variances (6.3.23) are valid for all $\epsilon, \bar{N}$ and κ.

d) Comments:

i)   The fact that there is no Hamiltonian noise in this case, and that the Hamiltonian noise changes sign between P- and Q-functions indicates that the identification of these terms as noise in any physical sense is subject to some difficulties.

ii)   The other noise term is proportional to $\bar{N} + \tfrac{1}{2}$, which can be interpreted as thermal noise plus vacuum fluctuations. Because the Wigner function corresponds to symmetrically ordered moments, we find that the variances of the operator quadrature phases correspond to those of the Wigner function quadrature phases. These variances satisfy

$$\text{var}[x(t)] \equiv \text{var}[u(t)] = \text{var}[x(0)]e^{(2\epsilon - \kappa)t} + \frac{\kappa(\bar{N} + \tfrac{1}{2})}{2(2\epsilon - \kappa)} \left\{ e^{(2\epsilon - \kappa)t} - 1 \right\}$$

(6.3.35)

$$\text{var}[y(t)] \equiv \text{var}[v(t)] = \text{var}[y(0)]e^{-(2\epsilon + \kappa)t} + \frac{\kappa(\bar{N} + \tfrac{1}{2})}{2(2\epsilon + \kappa)} \left\{ 1 - e^{-(2\epsilon + \kappa)t} \right\}.$$

(6.3.36)

In both cases the original variance is amplified (or attenuated as the case may be) and augmented by an amplified (or attenuated) contribution from the vacuum fluctuations, proportional to $(\bar{N} + \tfrac{1}{2})$.

Exercise—Other Amplifiers. Develop similar treatments of the other kinds of amplifier considered in Sect. 7.1

Exercise—Quantum Brownian Motion Master Equation. Use the Wigner function and the correspondences (4.5.11) to show that the quantum Brownian motion master equation (3.6.28) is equivalent in Wigner function representation to

$$\dot{x} = p/m$$

(6.3.37)

$$\dot{p} = -V'(x) - \gamma p/m + \sqrt{\frac{\gamma kT}{2}}\, \xi(t)$$

(6.3.38)

where

$$\langle \xi(t)\xi(t') \rangle = \delta(t - t')$$

(6.3.39)

provided $V(x)$ is a harmonic or linear potential. This is of course Kramers' equation, which describes classical Brownian motion. If the potential is non-harmonic, extra terms like those in the quasiclassical Langevin equation (4.5.32) will also appear.

6.4   Generalised P-Representations

We have already seen that the analogy between the P-, Q- or Wigner functions, and a genuine probability distribution is imperfect. The P-function does not always exist, the Q-function always exists and is positive, but such a simple Q-function as $\delta^2(\alpha - \alpha_0)$ does not correspond to any physical state. Finally, although the Wigner function always exists, it need not be positive, and like the Q-function, a delta function does not correspond to any physical state. We have seen how these problems arise in the study of the degenerate parametric amplifier in Sect. 6.3.2 where both the P- and Q-functions gave a Fokker-Planck equation whose diffusion matrix was not always positive definite, though the Wigner function did not have this disadvantage. Nevertheless, the solutions involving all of these methods are the same (of course) where valid, and even where the P- and Q-function methods cannot be justified, application of simple Langevin equation methods gives the correct answer. It was this fact that motivated *Drummond* and *Gardiner* [6.1] to introduce a class of generalised P-representations, by expanding in non-diagonal coherent state projection operators. The generalised P-representations are defined as follows. We set

$$\rho = \int\limits_D \Lambda(\alpha, \beta) P(\alpha, \beta) d\mu(\alpha, \beta), \tag{6.4.1}$$

where

$$\Lambda(\alpha, \beta) = \frac{|\alpha\rangle\langle\beta^*|}{\langle\beta^*|\alpha\rangle}, \tag{6.4.2}$$

$d\mu(\alpha, \beta)$ is the integration measure which may be chosen to define different classes of possible representations and D is the domain of integration. The projection operator $\Lambda(\alpha, \beta)$ is analytic in (α, β). Useful choices of the integration measure are:

a) Glauber-Sudarshan P-Representation:

$$d\mu(\alpha, \beta) = \delta^2(\alpha^* - \beta) d^2\alpha \, d^2\beta. \tag{6.4.3}$$

This measure corresponds to the Glauber-Sudarshan P-representation.

b) Complex P-Representation:

$$d\mu(\alpha, \beta) = d\alpha \, d\beta. \tag{6.4.4}$$

Here (α, β) are treated as complex variables which are to be integrated on individual contours C, C'. The existence of this representation under certain circumstances is demonstrated in the next section. In particular, this representation exists for an operator expanded in a finite basis of number states. This is a characteristic situation where nonclassical photon statistics (photon antibunching) may arise, and where the Glauber-Sudarshan P-representation would be singular. This representation is called the complex P-representation since complex values of $P(\alpha, \beta)$ occur. It gives rise to a $P(\alpha, \beta)$ which can be shown to satisfy a FPE obtained by replacing

α, α^* with α, β in the usual P-function FPE. Under certain circumstances, exact solutions to Fokker-Planck equations occur which cannot be normalised as P-functions. These can be handled with the present representation by choosing C, C' (paths of integration) in the complex phase space of (α, β).

c) Positive P-Representation:

$$d\mu(\alpha, \beta) = d^2\alpha \, d^2\beta. \tag{6.4.5}$$

This representation allows (α, β) to vary independently over the whole complex plane. In the next section we will show that $P(\alpha, \beta)$ always exists for a physical density operator and can always be chosen positive, and hence we call it the positive P-representation. This means that $P(\alpha, \beta)$ has all the properties of a genuine probability. It will also be shown that provided any FPE exists for time development in the Glauber-Sudarshan representation, a corresponding FPE exists with a positive semidefinite diffusion coefficient for the positive P-representation. This enables stochastic differential equations, and a correspondence between the quantum Markov process and ordinary diffusion processes to be derived. In all representations, it is, of course, true that observable moments are given by

$$\langle (a^\dagger)^m a^n \rangle = \int_D d\mu(\alpha, \beta) \beta^m \alpha^n P(\alpha, \beta). \tag{6.4.6}$$

6.4.1   The R-Representation

Glauber [6.2] defined an *R-representation* by means of the operator expansion given in Sect. 4.3.1f:

$$\rho = \frac{1}{\pi^2} \int \int d^2\alpha \, d^2\beta \, |\alpha\rangle R(\alpha^*, \beta) \langle\beta| \exp[-\tfrac{1}{2}(|\alpha|^2 + |\beta|^2)] \tag{6.4.7}$$

in which

$$R(\alpha^*, \beta) = \langle\alpha|\rho|\beta\rangle \exp[\tfrac{1}{2}(|\alpha|^2 + |\beta|^2)]. \tag{6.4.8}$$

While the representation is analytic in α^*, β (and therefore nonsingular), it is also by definition non-positive, and has a normalisation that includes a Gaussian weight factor. For this reason, it cannot have a Fokker-Planck equation or any direct interpretation as a quasiprobability. Nevertheless, the existence of this representation does demonstrate that a calculation of normally ordered observables for any ρ is possible with a nonsingular representation.

6.4.2   Existence Theorems

We will show in this section that the generalised P-representations have quite strong existence properties. We do this with a number of theorems. NB: for brevity, we shall use the notation $\boldsymbol{\alpha} = (\alpha, \beta)$.

Theorem 1: A complex P-representation exists for an operator with an expansion in a finite number of number states.

Proof: Let

$$\rho = \sum_{n,m} C_{nm}(a^{\dagger})^m |0\rangle\langle 0| a^n. \tag{6.4.9}$$

Then, by Cauchy's theorem,

$$\rho = \oint_C \oint_{C'} \Lambda(\boldsymbol{\alpha}) P(\boldsymbol{\alpha}) d\mu(\boldsymbol{\alpha}) \tag{6.4.10}$$

with

$$P(\boldsymbol{\alpha}) = \left(-\frac{1}{4\pi^2}\right) e^{\alpha\beta} \sum_{n,m} C_{nm} n! m! \alpha^{-m-1} \beta^{-n-1} \tag{6.4.11}$$

where C, C' are integration paths enclosing the origin.

Theorem 2: A complex P-representation exists for any operator with an expansion on a bounded range of coherent states, i.e., for

$$\rho = \iint_{DD'} \Lambda(\alpha, \beta) C(\alpha, \beta) d^2\alpha \, d^2\beta, \tag{6.4.12}$$

where D, D' are bounded in each complex plane.

Proof: Application of Cauchy's theorem shows that if

$$P(\boldsymbol{\alpha}) = -\frac{1}{4\pi^2} \iint_{DD'} C(\alpha', \beta') / \left[(\alpha - \alpha')(\beta - \beta')\right] d^2\alpha' \, d^2\beta', \tag{6.4.13}$$

then

$$\rho = \iint_{CC'} \Lambda(\boldsymbol{\alpha}) P(\boldsymbol{\alpha}) d\alpha \, d\beta, \tag{6.4.14}$$

where C, C' enclose D, D' respectively. Hence the complex P-representation exists in this case relative to any bounded expansion in coherent state projection operators.

Theorem 3: A positive P-representation exists for any quantum density operator ρ, with

$$P(\boldsymbol{\alpha}) = (1/4\pi^2) \exp(-|\alpha - \beta^*|^2/4) \langle \tfrac{1}{2}(\alpha + \beta^*) | \rho | \tfrac{1}{2}(\alpha + \beta^*) \rangle. \tag{6.4.15}$$

Proof: $P(\boldsymbol{\alpha})$ is positive, since ρ is a density operator, and it is composed of a diagonal matrix element multiplied by a positive function. In order to show that this represents a quantum density operator in the general case, the characteristic function

$$\chi(\lambda, \lambda^*) \equiv \text{Tr} \left\{ \rho \, e^{\lambda a^{\dagger}} e^{-\lambda^* a} \right\} \tag{6.4.16}$$

is used. This has been shown in Sect. 4.4.2 to define the density operator uniquely. In terms of the R-representation for ρ, the characteristic function is

$$\chi(\lambda, \lambda^*) = \frac{1}{\pi} \int R(\alpha^*, \lambda + \alpha) \exp(-\lambda^*\alpha - |\alpha|^2) d^2\alpha. \tag{6.4.17}$$

We now substitute the R-representation for ρ into (6.4.15) which defines $P(\boldsymbol{\alpha})$ in terms of the diagonal matrix elements of ρ. We then define ρ_P to be given by the positive P-representation form (6.4.1) with $P(\alpha)$ as given by the previous process, calculate the corresponding characteristic function $\chi_P(\lambda, \lambda^*)$ using (6.4.16) and show that this is the same as the original characteristic function for ρ. Thus:

$$\chi_P(\lambda, \lambda^*) = \iint P(\boldsymbol{\alpha}) \exp(\lambda\beta - \lambda^*\alpha) d^2\alpha\, d^2\beta$$

$$= \frac{1}{4\pi^4} \iiiint R(\alpha'^*, \beta') \exp\left[\lambda\beta - \lambda^*\alpha - \tfrac{1}{2}|\alpha|^2 - \tfrac{1}{2}|\beta|^2 - |\alpha'|^2 \right.$$
$$\left. - |\beta'|^2 + \tfrac{1}{2}\beta'^*(\alpha + \beta^*) + \tfrac{1}{2}\alpha'^*(\alpha^* + \beta)\right] d^2\alpha\, d^2\beta\, d^2\alpha'\, d^2\beta'. \tag{6.4.18}$$

We now make a variable change by defining

$$\begin{aligned}
\gamma &= (\alpha + \beta^*)/2 & \delta &= (\alpha - \beta^*)/2 \\
\alpha &= (\gamma + \delta) & \beta^* &= (\gamma - \delta) \\
d^2\alpha\, d^2\beta &= 4 d^2\gamma\, d^2\delta.
\end{aligned} \tag{6.4.19}$$

Noting that R is an analytic function, the following identity is useful:

$$R(\alpha^*, \gamma) = \frac{1}{\pi} \int R(\alpha^*, \beta) \exp(\gamma\beta^* - |\beta|^2) d^2\beta. \tag{6.4.20}$$

Hence, the above expression for the characteristic function can be simplified to give

$$\chi_P(\lambda, \lambda^*) = \frac{1}{\pi^3} \iiint R(\alpha'^*, \gamma) \exp\left[\lambda(\gamma - \delta)^* - \lambda^*(\gamma + \delta) \right.$$
$$\left. - |\gamma|^2 - |\delta|^2 - |\alpha'|^2 + \alpha'\gamma^*\right] d^2\gamma\, d^2\delta\, d^2\alpha' \tag{6.4.21}$$

$$= \frac{1}{\pi^2} \iint R(\alpha'^*, \gamma) \exp(|\lambda|^2 + \lambda\gamma^* - \lambda^*\gamma - |\gamma|^2$$
$$- |\alpha'|^2 + \alpha'\gamma^*) d^2\gamma\, d^2\alpha' \tag{6.4.22}$$

$$= \frac{1}{\pi} \int R(\alpha^*, \lambda + \alpha) \exp(-\lambda^*\alpha - |\alpha|^2) d^2\alpha. \tag{6.4.23}$$

Hence,

$$\chi_P(\lambda, \lambda^*) = \text{Tr}\left\{\rho e^{\lambda a^\dagger} e^{-\lambda^* a}\right\} = \chi(\lambda, \lambda^*). \tag{6.4.24}$$

The last step follows from the identity with the characteristic function defined relative to the R-representation in (6.4.16). Thus, we deduce that $\rho_P = \rho$.

6.4.3   Definition of the Positive P-Representation by Means of the Quantum Characteristic Function

The proof in Theorem 3 depends heavily on the use of characteristic functions, and can be used as an implicit way of characterising the positive P-representation. We can say that $P(\boldsymbol{\alpha})$ is a positive P-function corresponding to a density operator ρ provided

$$\chi_P(\lambda, \lambda^*) \equiv \int\int P(\boldsymbol{\alpha}) \exp(\lambda\beta - \lambda^*\alpha) d^2\alpha d^2\beta \tag{6.4.25}$$

is identical with the quantum characteristic function of the density operator,

$$\chi(\lambda, \lambda^*) \equiv \mathrm{Tr} \left\{ \rho e^{\lambda a^\dagger} e^{-\lambda^* a} \right\}. \tag{6.4.26}$$

The result of Theorem 3 is to establish that for any ρ there exists at least one $P(\boldsymbol{\alpha})$, that given by (6.4.15). However there can be others—see Sect. 6.4.7b .

a) Convergence Conditions: It would be nice to establish what kinds of $P(\boldsymbol{\alpha})$ can conceivably exist, and in particular, what conditions on $P(\boldsymbol{\alpha})$ are necessary to ensure the convergence of the integral in (6.4.25). This depends on how we evaluate the integrals.

Let us require that the integrals be carried out in polar coordinates in α and β separately; i.e. we define χ_P by setting

$$\alpha = re^{i\theta}, \qquad \beta = r'e^{i\theta'}, \qquad \lambda = \mu e^{i\phi} \tag{6.4.27}$$

and

$$\chi_P(\mu, \phi) = \int_0^\infty r\, dr \int_0^\infty r'\, dr' \left\{ \int_0^{2\pi} d\theta \int_0^{2\pi} d\theta' \exp\left\{ \mu r' e^{i(\theta'+\phi')} \right\} \right.$$
$$\left. \times \exp\left\{ -\mu r e^{i(\theta-\phi)} \right\} P(r, r', \theta, \theta') \right\}. \tag{6.4.28}$$

Since P must be normalised, we can assume the existence of a Fourier series representation

$$P(r, r', \theta, \theta') = \sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} c_{nm}(r, r') e^{in\theta} e^{im\theta'}. \tag{6.4.29}$$

Notice that if $P(r, r', \theta, \theta')$ is a bounded positive function this expansion will exist, and

$$c_{nm}(r, r') = \frac{1}{4\pi^2} \int\int d\theta d\theta' e^{-in\theta - im\theta'} P(r, r', \theta, \theta'), \tag{6.4.30}$$

and it follows that $c_{nm}(r, r')$ is also uniformly bounded by $c_{00}(r, r')$ for all n, m. This also means that we can substitute the expansion into (6.4.28) and integrate term by term over θ, θ'. Now put

$$z = \mu r' e^{i(\theta' - \phi)}, \qquad y = -\mu r e^{i(\theta + \phi)} \tag{6.4.31}$$

so that the n, m term is

$$-c_{n,m}(r, r') \left(\mu e^{i\phi}\right)^n \left(-\mu e^{i\phi}\right)^m \oint_{C'} \frac{dz}{z} z^{-m} e^z \oint_C \frac{dy}{y} y^{-n} e^y \tag{6.4.32}$$

where C', C, are circles with centre at the origin, and radii respectively $\mu r'$ and μr. The contour integrals are of course evaluated by Cauchy's theorem, to give

$$\chi_P(\mu, \phi) = 4\pi^2 \int_0^\infty r\,dr \int_0^\infty r'dr' \sum_{m,n=0}^\infty c_{-n,-m}(r, r') \frac{[\mu r e^{-i\phi}]^n [-\mu r' e^{i\phi}]^m}{n!\,m!}. \tag{6.4.33}$$

A rather strong condition for the existence of $\chi_P(\mu, \phi)$ is that

$$|c_{-n,-m}(r, r')| \leq c_{0,0}(r, r')(r)^{-n-2-\epsilon}(r')^{-m-2-\epsilon} \tag{6.4.34}$$

for some $\epsilon > 0$. Under these conditions the analytic moments

$$\langle a^{\dagger N} a^M \rangle \equiv \int d^2\alpha\, d^2\beta\, \alpha^M \beta^N P(\boldsymbol{\alpha}) \tag{6.4.35}$$

exist for all M and N, which is a necessary condition for the analyticity of the quantum characteristic function.

b) Comments: This condition, (6.4.35), is not nearly as strong as the condition for all moments of $P(\boldsymbol{\alpha})$ to exist, which would require all $c_{n,m}$ to vanish faster than any powers of r and r'. It is also much milder than one might expect from (6.4.25), which would lead one to expect that $P(\boldsymbol{\alpha})$ would need to drop off exponentially as $|\alpha|, |\beta| \to \infty$ in the direction of positive $\lambda\beta, -\lambda^*\alpha$.

6.4.4   Operator Identities

From the definitions (6.4.28) of the nondiagonal coherent state projection operators, the following identities can be obtained. Again, $\boldsymbol{\alpha}$ is used to denote (α, β):

$$\begin{aligned}
a\Lambda(\boldsymbol{\alpha}) &= \alpha\Lambda(\boldsymbol{\alpha}) \\
a^{\dagger}\Lambda(\boldsymbol{\alpha}) &= (\beta + \partial/\partial\alpha)\Lambda(\boldsymbol{\alpha}) \\
\Lambda(\boldsymbol{\alpha})a^{\dagger} &= \beta\Lambda(\boldsymbol{\alpha}) \\
\Lambda(\boldsymbol{\alpha})a &= (\partial/\partial\beta + \alpha)\Lambda(\boldsymbol{\alpha}).
\end{aligned} \tag{6.4.36}$$

By substituting the above identities into (6.4.2), which defines the generalised P-representation and using partial integration (provided the boundary terms vanish),

these identities can be used to generate operations on the P-function depending on the representation.

a) Glauber-Sudarshan P-Representation:
The same results as (6.4.1).

b) Complex P-Representation:

$$
\begin{aligned}
a\rho &\leftrightarrow \alpha P(\boldsymbol{\alpha}) \\
a^{\dagger}\rho &\leftrightarrow (\beta - \partial/\partial\alpha) P(\boldsymbol{\alpha}) \\
\rho a^{\dagger} &\leftrightarrow \beta P(\boldsymbol{\alpha}) \\
\rho a &\leftrightarrow (\alpha - \partial/\partial\beta) P(\boldsymbol{\alpha}).
\end{aligned}
\tag{6.4.37}
$$

c) Positive P-Representation:
We now use the analyticity of $\Lambda(\boldsymbol{\alpha})$ and note that if

$$
\alpha = \alpha_x + \mathrm{i}\,\alpha_y, \qquad \beta = \beta_x + \mathrm{i}\,\beta_y,
\tag{6.4.38}
$$

then

$$
(\partial/\partial\alpha)\Lambda(\boldsymbol{\alpha}) = (\partial/\partial\alpha_x)\Lambda(\boldsymbol{\alpha}) = (-\mathrm{i}\partial/\partial\alpha_y)\Lambda(\boldsymbol{\alpha})
\tag{6.4.39}
$$

and

$$
(\partial/\partial\beta)\Lambda(\boldsymbol{\alpha}) = (\partial/\partial\beta_x)\Lambda(\boldsymbol{\alpha}) = (-\mathrm{i}\partial/\partial\beta_y)\Lambda(\boldsymbol{\alpha})
\tag{6.4.40}
$$

so that as well as all of (4.5.9) being true in this case, we also have

$$
\begin{aligned}
a^{\dagger}\rho &\leftrightarrow (\beta - \partial/\partial\alpha_x) P(\boldsymbol{\alpha}) \leftrightarrow (\beta + \mathrm{i}\partial/\partial\alpha_y) P(\boldsymbol{\alpha}) \\
\rho a &\leftrightarrow (\alpha - \partial/\partial\beta_x) P(\boldsymbol{\alpha}) \leftrightarrow (\alpha + \mathrm{i}\partial/\partial\beta_y) P(\boldsymbol{\alpha}).
\end{aligned}
\tag{6.4.41}
$$

All these correspondences can now be used to derive Fokker-Planck equations when appropriate.

6.4.5   Time-Development Equations

We firstly want to show that the P-representation will not always yield an acceptable Fokker-Planck equation, and that realistic situations arise in which this is the case. Consider a coherently driven single mode interferometer with a nonlinear absorber, for which we set [6.3]

$$
H_1 = \sum_i \hbar\omega_i(\Gamma_i^{\dagger}\Gamma_i + \tfrac{1}{2})
\tag{6.4.42}
$$

$$
H_2 = \sum_i \left[(a^{\dagger})^2 g_i \Gamma_i + a^2 g_i^* \Gamma_i^{\dagger} \right]
\tag{6.4.43}
$$

$$
H_3 = \hbar\omega(a^{\dagger}a + \tfrac{1}{2}) + \mathrm{i}\hbar\,(\epsilon e^{-\mathrm{i}\omega t}a^{\dagger} - \epsilon^* e^{\mathrm{i}\omega t}a).
\tag{6.4.44}
$$

In an interaction picture, we get the master equation and using the methods of Sect. 6.4.37 (assume the heat bath is at zero temperature)

$$\frac{\partial \rho}{\partial t} = [\epsilon a^\dagger - \epsilon^* a, \rho] + \tfrac{1}{2} K [2 a^2 \rho (a^\dagger)^2 - (a^\dagger)^2 a^2 \rho - \rho (a^\dagger)^2 a^2]. \quad (6.4.45)$$

Using the ordinary P-function operator correspondences (5.1.4) we would obtain

$$\frac{\partial}{\partial t} P(\alpha, \alpha^*) = \left[-\frac{\partial}{\partial \alpha}(\epsilon - K \alpha^2 \alpha^*) - \tfrac{1}{2} \frac{\partial^2}{\partial \alpha^2}(K \alpha^2) + \text{c.c.} \right] P(\alpha, \alpha^*). \quad (6.4.46)$$

In terms of real variables,

$$x = (\alpha + \alpha^*)/\sqrt{2}$$
$$y = (\alpha - \alpha^*)/i\sqrt{2}, \quad (6.4.47)$$

the diffusion matrix is

$$\begin{pmatrix} -K/2 & -K \\ -K & K/2 \end{pmatrix} \quad (6.4.48)$$

which is not positive semidefinite. Hence, a time development equation of the form (4.5.9) may develop singularities. We are led, therefore, to alternative equations in the various P-representations.

Naively following the rules which would apply if there was a positive semi-definite diffusion matrix leads to the stochastic differential equations

$$\frac{d}{dt}\begin{pmatrix} \alpha \\ \alpha^* \end{pmatrix} = \begin{pmatrix} \epsilon - K \alpha^2 \alpha^* \\ \epsilon^* - K (\alpha^*)^2 \alpha \end{pmatrix} + i\sqrt{K}\begin{pmatrix} \alpha \xi_1(t) \\ \alpha^* \xi_2(t) \end{pmatrix}, \quad (6.4.49)$$

where $\xi_1(t)$ and $\xi_2(t)$ are independent white noises. However, a paradox arises. Because ξ_1 and ξ_2 are independent, α and α^* do not remain complex conjugate.

We will show that (6.4.46) is, in fact, correct, provided we make the replacement $\alpha^* \to \beta$ and these are regarded as variables of the positive P-representation.

6.4.6 Complex P-Representation

Here the procedure yields a very similar equation to that for the Glauber-Sudarshan case. We assume that, by appropriate re-ordering of the differential operators, we can reduce the quantum mechanical master equation to the form [where $(\alpha, \beta) = \boldsymbol{\alpha} \equiv (\alpha^{(1)}, \alpha^{(2)}); \mu = 1, 2]$:

$$\frac{\partial \rho}{\partial t} = \int_{C, C'} \Lambda(\boldsymbol{\alpha}) \frac{\partial P(\boldsymbol{\alpha})}{\partial t} d\alpha \, d\beta$$

$$= \int_{C, C'} \left\{ \left[A^\mu(\boldsymbol{\alpha}) \frac{\partial}{\partial \alpha^\mu} + \tfrac{1}{2} D^{\mu\nu}(\boldsymbol{\alpha}) \frac{\partial}{\partial \alpha^\mu} \frac{\partial}{\partial \alpha^\nu} \right] \Lambda(\boldsymbol{\alpha}) \right\} P(\boldsymbol{\alpha}) d\alpha \, d\beta.$$

$$(6.4.50)$$

We now integrate by parts and, if we can neglect boundary terms which may be made possible by an appropriate choice of contours C, C', at least one solution is obtained by equating the coefficients of $\Lambda(\boldsymbol{\alpha})$:

$$\frac{\partial P(\boldsymbol{\alpha})}{\partial t} = \left[-\frac{\partial}{\partial \alpha^{\mu}} A^{\mu}(\boldsymbol{\alpha}) + \tfrac{1}{2} \frac{\partial}{\partial \alpha^{\mu}} \frac{\partial}{\partial \alpha^{\nu}} D^{\mu\nu}(\boldsymbol{\alpha}) \right] P(\boldsymbol{\alpha}). \tag{6.4.51}$$

This equation is sufficient to imply (6.4.49), but is not a unique equation because the $\Lambda(\boldsymbol{\alpha})$ are not linearly independent. It should be noted that for this complex P-representation, $A^{\mu}(\boldsymbol{\alpha})$ and $D^{\mu\nu}(\boldsymbol{\alpha})$ are always analytic in $\boldsymbol{\alpha}$; hence, if $P(\boldsymbol{\alpha})$ is initially analytic, (6.4.50) preserves this analyticity as time develops.

6.4.7 Positive P-Representation

We assume that the same equation (6.4.51) is being considered but with a positive P-representation. The symmetric matrix can always be factorised into the form

$$\mathbf{D}(\boldsymbol{\alpha}) = \mathbf{B}(\boldsymbol{\alpha}) \mathbf{B}^{T}(\boldsymbol{\alpha}). \tag{6.4.52}$$

We now write

$$\mathbf{A}(\boldsymbol{\alpha}) \;= \mathbf{A}_{x}(\boldsymbol{\alpha}) + i\, \mathbf{A}_{y}(\boldsymbol{\alpha}) \tag{6.4.53}$$

$$\mathbf{B}(\boldsymbol{\alpha}) \;= \mathbf{B}_{x}(\boldsymbol{\alpha}) + i\, \mathbf{B}_{y}(\boldsymbol{\alpha}) \tag{6.4.54}$$

where $\mathbf{A}_x, \mathbf{A}_y, \mathbf{B}_x, \mathbf{B}_y$ are real. We then find that the master equation yields

$$\begin{aligned}
\frac{\partial \rho}{\partial t} &= \iint d^2\alpha\, d^2\beta \Lambda(\boldsymbol{\alpha})(\partial P(\boldsymbol{\alpha})/\partial t) \\
&= \iint P(\boldsymbol{\alpha}) \left[A_x^{\mu}(\boldsymbol{\alpha})\partial_{\mu}^{x} + A_y^{\mu}(\boldsymbol{\alpha})\partial_{\mu}^{y} + \tfrac{1}{2}(B_x^{\mu\sigma} B_x^{\nu\sigma} \partial_{\mu}^{x}\partial_{\nu}^{x} \right. \\
&\quad \left. + B_y^{\mu\sigma} B_y^{\nu\sigma} \partial_{\mu}^{y}\partial_{\nu}^{y} + 2 B_x^{\mu\sigma} B_y^{\nu\sigma} \partial_{\mu}^{x}\partial_{\nu}^{y}) \right] \Lambda(\boldsymbol{\alpha}) d^2\alpha\, d^2\beta.
\end{aligned} \tag{6.4.55}$$

Here we have written, for notational simplicity, $\partial/\partial\alpha_x^{\mu} = \partial_{\mu}^{x}$, etc., and have used the analyticity of $\Lambda(\boldsymbol{\alpha})$ to make either of the replacements

$$\partial/\partial\alpha^{\mu} \leftrightarrow \partial_{\mu}^{x} \leftrightarrow -i\,\partial_{\mu}^{y} \tag{6.4.56}$$

in such a way as to yield (6.4.50). Now provided partial integration is permissible, we deduce the Fokker-Planck equation

$$\begin{aligned}
\partial P(\boldsymbol{\alpha})/\partial t = \big\{ &-\partial_{\mu}^{x} A_x^{\mu}(\boldsymbol{\alpha}) - \partial_{\mu}^{y} A_y^{\mu}(\boldsymbol{\alpha}) + \tfrac{1}{2} \big[\partial_{\mu}^{x}\partial_{\nu}^{x} B_x^{\mu\sigma}(\boldsymbol{\alpha}) B_x^{\nu\sigma}(\boldsymbol{\alpha}) \\
&+ 2\,\partial_{\mu}^{x}\partial_{\nu}^{y} B_x^{\mu\sigma}(\boldsymbol{\alpha}) B_y^{\nu\sigma}(\boldsymbol{\alpha}) + \partial_{\mu}^{y}\partial_{\nu}^{y} B_y^{\mu\sigma}(\boldsymbol{\alpha}) B_y^{\nu\sigma}(\boldsymbol{\alpha}) \big] \big\} P(\boldsymbol{\alpha}).
\end{aligned} \tag{6.4.57}$$

Again, this is not a unique time-development equation but (6.4.55) is a consequence of (6.4.50).

However, the Fokker-Planck equation (6.4.57) now possesses a positive semi-definite diffusion matrix in a four-dimensional space whose vectors are

$$(\alpha_x^{(1)}, \alpha_x^{(2)}, \alpha_y^{(1)}, \alpha_y^{(2)}) \equiv (\alpha_x, \beta_x, \alpha_y, \beta_y). \tag{6.4.58}$$

We find the drift vector is

$$A(\alpha) \equiv \left(A_x^{(1)}(\alpha), A_x^{(2)}(\alpha), A_y^{(1)}(\alpha), A_y^{(2)}(\alpha)\right) \tag{6.4.59}$$

and the diffusion matrix is

$$D(\alpha) = \begin{pmatrix} \mathbf{B}_x \mathbf{B}_x^T, & \mathbf{B}_x \mathbf{B}_y^T \\ \mathbf{B}_y \mathbf{B}_x^T, & \mathbf{B}_y \mathbf{B}_y^T \end{pmatrix} (\alpha) \equiv B(\alpha) B^T(\alpha), \tag{6.4.60}$$

where

$$B(\alpha) = \begin{pmatrix} \mathbf{B}_x, & 0 \\ \mathbf{B}_y, & 0 \end{pmatrix} (\alpha) \tag{6.4.61}$$

and D is thus explicitly positive semidefinite (and not positive definite). The corresponding Ito stochastic differential equations can be written as

$$\frac{d}{dt} \begin{pmatrix} \alpha_x \\ \alpha_y \end{pmatrix} = \begin{pmatrix} A_x(\alpha) \\ A_y(\alpha) \end{pmatrix} + \begin{pmatrix} B_x(\alpha)\xi(t) \\ B_y(\alpha)\xi(t) \end{pmatrix}, \tag{6.4.62}$$

or recombining real and imaginary parts

$$d\alpha/dt = A(\alpha) + B(\alpha)\xi(t). \tag{6.4.63}$$

Apart from the substitution $\alpha^* \to \beta$, (6.4.57) is just the stochastic differential equation which would be obtained by using the Glauber-Sudarshan representation and naively converting the Fokker-Planck equation with a non-positive-definite diffusion matrix into an Ito stochastic differential equation.

In our derivation, the two formal variables (α, α^*) have been replaced by variables in the complex plane (α, β) that are allowed to fluctuate independently. The positive P-representation as defined here thus appears as a mathematical justification of this procedure.

a) Neglect of Boundary Terms: In the derivation of the positive P-representation Fokker-Planck equation in the previous section, partial integration is used, and it was assumed that boundary terms at infinity could be neglected. The direct derivation in terms of projection operators $\Lambda(\alpha)$ makes it very difficult to assess the magnitude of these. However, defining the positive P-representation through the characteristic function as in Sect. 6.4.63 makes the problem merely one of calculus. The equation of motion for the characteristic function is obtained by the rules

$$
\begin{array}{rcl}
\rho a^\dagger & \leftrightarrow & \dfrac{\partial \chi}{\partial \lambda} \\[2ex]
a\rho & \leftrightarrow & -\dfrac{\partial \chi}{\partial \lambda^*} \\[2ex]
\rho a & \leftrightarrow & \left(\lambda - \dfrac{\partial}{\partial \lambda^*}\right)\chi \\[2ex]
a^\dagger \rho & \leftrightarrow & \left(-\lambda^* + \dfrac{\partial}{\partial \lambda}\right)\chi
\end{array} \tag{6.4.64}
$$

which easily gives the correspondence (6.4.3) provided the boundary terms can be neglected. We can estimate these boundary terms for any particular case, provided we know the behaviour of $\mathbf{A}(\boldsymbol{\alpha}), \mathbf{B}(\boldsymbol{\alpha})$ as $\alpha, \beta \to \infty$. In most cases of interest, these coefficients are low order polynomials in α, β, and we will find that corresponding to the condition (6.4.37,41) we will require a slightly faster drop off as $r, r' \to \infty$.

No-one has yet analysed any particular case. However *Smith* and *Gardiner* [6.4] have exhibited a situation in which there is a discrepancy between positive P and master equation methods. It appears that the existence of non negligible boundary terms is responsible for this, but no definite conclusion has been reached yet.

Of course the equivalence between the SDEs and the Fokker-Planck equation is only an equivalence in distribution, in the sense that only $P(\boldsymbol{\alpha})$ has any meaning, and it is related to reality only by the characteristic function as shown in Theorem 3, Sect. 6.4.34. Thus the fact that individual trajectories travel in an unphysical region is unimportant—the only things measurable are the moments of physical quantities, like $\langle a^\dagger a \rangle$.

b) Non-Uniqueness: It should be noted that all physical moments involve only α, β, *not* α^*, or $\alpha^{\dagger *}$. It is possible for two different P-functions to have identical values for all moments of α and β, and yet to be different from each other.

Exercise. Show that if $P(\alpha, \beta) = f(|\alpha|, |\beta|)$ which is such that

$$4\pi^2 \int r \, dr \, r' dr' f(r, r') = 1 \tag{6.4.65}$$

then for all such P-functions

$$\langle (a^\dagger)^m a^n \rangle = \delta_{n0} \delta_{m0}. \tag{6.4.66}$$

This result illustrates the non-uniqueness of the positive P-function rather dramatically.

From the example in the exercise, it is clear that $P(\alpha, \beta)$ could be significant in a very large domain, and yet correspond to exactly the same situation as $\delta^2(\alpha)\delta^2(\beta)$ i.e. the variable being located at the origin.

For simulations, this gives the alarming possibility that trajectories may spend a large amount of time in regions which taken together give negligible contributions to the physical results. This could be extremely wasteful of computer time.

6.5 Applications of the Generalised P-Representations

In this section, we will consider a number of examples of how to apply the two major generalised P-representations. It will be seen that there are problems where these methods can provide solutions which cannot otherwise be solved.

6.5.1   Complex P-Representation

We consider the example of Sect. 6.4.2, choosing ϵ to be real for simplicity. Using the appropriate operator correspondence, the complex P-representation Fokker-Planck equation is

$$\partial_t P(\alpha,\beta) = \left[-\frac{\partial}{\partial\alpha}(\epsilon - K\alpha^2\beta) - \tfrac{1}{2}\frac{\partial^2}{\partial\alpha^2}(K\alpha^2) - \frac{\partial}{\partial\beta}(\epsilon - K\alpha\beta^2) \right.$$

$$\left. -\tfrac{1}{2}\frac{\partial^2}{\partial\beta^2}(K\beta^2) \right] P(\alpha,\beta). \tag{6.5.1}$$

Rather miraculously, we see that this Fokker-Planck equation satisfies potential conditions of S.M.Sect. 6.4.5. For, in that notation,

$$\mathbf{A} = \begin{pmatrix} \epsilon - K\alpha^2\beta \\ \epsilon - K\beta^2\alpha \end{pmatrix}, \qquad \mathbf{B} = \begin{pmatrix} -K\alpha^2 & 0 \\ 0 & -K\beta^2 \end{pmatrix} \tag{6.5.2}$$

so, using S.M.Sect. 5.3.3,

$$\mathbf{Z} = -\frac{2}{K}\begin{pmatrix} \epsilon/\alpha^2 - K\beta + K/\alpha \\ \epsilon/\beta^2 - K\alpha + K/\beta \end{pmatrix}, \tag{6.5.3}$$

$$\frac{\partial Z_\alpha}{\partial\beta} = \frac{\partial Z_\beta}{\partial\alpha} = 2 \tag{6.5.4}$$

and

$$\phi(\alpha,\beta) = -\int (Z_\alpha d\alpha + Z_\beta d\beta) \tag{6.5.5}$$

$$= -\frac{2\epsilon}{K}\left(\frac{1}{\alpha} + \frac{1}{\beta}\right) + 2\log(\alpha\beta) - 2\alpha\beta, \tag{6.5.6}$$

so that

$$P_s(\alpha,\beta) = (\alpha\beta)^{-2}\exp\left[2\alpha\beta + \frac{2\epsilon}{K}\left(\frac{1}{\alpha} + \frac{1}{\beta}\right)\right]. \tag{6.5.7}$$

The only acceptable contours for this stationary distribution are C, C' which are independent contours in the α and β planes which encircle the essential singularities at $\alpha = 0$ and $\beta = 0$.

A potential solution of this kind is extremely useful and could not be obtained with the Glauber-Sudarshan P-representation. The moments can be obtained from

$$\iint d\alpha\, d\beta\, \beta^m\alpha^n(\alpha\beta)^{-2}\exp\left[2\alpha\beta + \frac{2\epsilon}{K}\left(\frac{1}{\alpha} + \frac{1}{\beta}\right)\right] \tag{6.5.8}$$

and we can expand $\exp(2\alpha\beta)$ in power series and contour integrate term by term, to obtain

$$\langle (a^\dagger)^m a^n \rangle = \sum_{r=0}^{\infty} \frac{(2\epsilon/K)^{n+m-2r-2}\,2^r}{r!(n-r-1)!(m-r-1)!} \tag{6.5.9}$$

which is an easily computed series.

6.6   Applications of the Positive P-Representation

Because the positive P-representation can yield ordinary stochastic differential equations, two procedures have been developed and used quite extensively.

6.6.1   Linearisation

If the noise terms which arise are small, we can use a small noise expansion, in which the fluctuations are linearised around the deterministic solution. One must be aware of course that any results so derived will be valid only in the small fluctuation regime, but, apart from the problem of transitions from one steady state to another in bistable systems, this is largely what is observable anyway.

As an example, consider the problem of Sect. 5.3.22,23, but now using the positive P-representation. We obtain the stochastic differential equation

$$\begin{pmatrix} d\alpha \\ d\beta \end{pmatrix} = \begin{pmatrix} \epsilon - K\alpha^2\beta \\ \epsilon - K\alpha\beta^2 \end{pmatrix} dt + \mathrm{i}\sqrt{K}\begin{pmatrix} \alpha\, dW_1(t) \\ \beta\, dW_2(t) \end{pmatrix}. \tag{6.6.1}$$

It should be noted that this equation does not contain any very obvious small noise parameter. However, a large driving field limit can be obtained by setting

$$K = \tilde{K}/\epsilon^2, \qquad \alpha = \tilde{\alpha}\epsilon, \qquad \beta = \tilde{\beta}\epsilon \tag{6.6.2}$$

so that

$$\begin{pmatrix} d\tilde{\alpha} \\ d\tilde{\beta} \end{pmatrix} = \begin{pmatrix} 1 - \tilde{K}\tilde{\alpha}^2\tilde{\beta} \\ 1 - \tilde{K}\tilde{\alpha}\tilde{\beta}^2 \end{pmatrix} dt + \mathrm{i}\frac{\sqrt{K}}{\epsilon}\begin{pmatrix} \tilde{\alpha}\, dW_1(t) \\ \tilde{\beta}\, dW_2(t) \end{pmatrix}. \tag{6.6.3}$$

A small noise linearisation process can be carried out in this limit of large driving field and small nonlinearity which is, in fact, a situation of practical utility.

6.6.2   Stochastic Simulation—The Domain of Doubled Dimensions

The stochastic differential equations which arise from the positive P-representation have been found to be very sensitive to the particular algorithm used for their solution. Because there are twice as many variables as in the classical case, the deterministic motion is in a phase space of doubled dimensions. The introduction of the quantum noise introduces fluctuations which are not complex conjugate for the equations in α, β, and thus, in simulations, the values of α and β on a trajectory can be far from complex conjugate to each other. Thus, the underlying dynamics is changed fundamentally.

If an algorithm with insufficient stability is used for the solution of these SDEs, the typical behaviour found is that positive P simulations are well behaved up to a certain time, at which stage they run off to infinity in a direction in which α and β are not complex conjugate. A number of authors have found this problem [6.5],

and have usually discarded such trajectories. Use of a better algorithm always shows that the trajectory does not actually go to infinity, but does make a large excursion. *Drummond* and *Mortimer* [6.6] have developed appropriate algorithms, and shown that reliable results (with error estimates) can be obtained using them.

On the other hand, *Smith* and *Gardiner* [6.4] have exhibited a non-linear damping model which can be reliably simulated but in which the answers are not correct if the non-linear damping is very large. They attribute problem to the existence of non-negligible boundary terms in the derivation of the positive P-function equations of motion in the particular problem studied.

6.6.3   Example—Quantum Noise in the Parametric Oscillator

Wolinsky and *Carmichael* [6.7] have given an example in which certain solutions of positive P-representation equations are non-zero only in a bounded domain—thus there is no question of neglected boundary terms at infinity. One considers a simple model of the degenerate parametric amplifier, in which two quantized electromagnetic field modes, with frequencies $\omega, 2\omega$, interact with a classical pump field in a cavity. This is a slightly more realistic model than that described in Sect. 6.5.1 in that the pump field is assumed to drive a cavity, which in turn drives the non-linear crystal, as is usual in practice, since higher driving field strengths can then be attained.

Both fields have losses out of the cavity; the Hamiltonian takes the form (in the interaction picture)

$$H = \tfrac{1}{2} i \hbar \, \bar{g} \left(a^{\dagger 2} b - a^2 b^{\dagger} \right) + i \hbar \, \mathcal{E} (b^{\dagger} - b) + H_{\text{loss}}, \tag{6.6.4}$$

with H_{loss} consisting of the usual loss terms, assumed to be at zero temperature

$$H_{\text{loss}} = \frac{\kappa_1}{2} \left(2 \, a\rho a^{\dagger} - a^{\dagger} a\rho - \rho a^{\dagger} a \right) + \frac{\kappa_2}{2} \left(2 \, b\rho b^{\dagger} - b^{\dagger} b\rho - \rho b^{\dagger} b \right) \tag{6.6.5}$$

Exercise. Derive Fokker-Planck equations in the positive P-representation corresponding to (6.3.3), and then the corresponding Langevin equations. Assume that the time constant for the b mode, κ_2, is large, so that we can adiabatically eliminate the b variable, (by naive methods, i.e., set the derivatives in the equations for the b positive-P variable equal to zero, and substitute the resulting solution in the other equations) and rescale variables according to

$$\alpha \to \alpha/g, \qquad \text{where } g = \bar{g}/\sqrt{2\kappa_1\kappa_2} \tag{6.6.6}$$

$$t \to \kappa_1 t \tag{6.6.7}$$

$$\lambda = -g\mathcal{E}/\kappa_1\kappa_2 \tag{6.6.8}$$

to obtain SDEs.

$$
\begin{aligned}
d\alpha &= \left[-\alpha + \beta(\lambda - \alpha^2) \right] dt + g\sqrt{\lambda - \alpha^2} \, dW_1(t) \\
d\beta &= \left[-\beta + \alpha(\lambda - \beta^2) \right] dt + g\sqrt{\lambda - \beta^2} \, dW_2(t).
\end{aligned}
\tag{6.6.9}
$$

where $dW_1(t)$ and $dW_2(t)$ are Wiener increments.

a) **Stationary Deterministic States:**   The equations (6.6.4) describe trajectories in a four-dimensional phase space. The subspace satisfying the conjugacy condition $\beta = \alpha^*$ is designated the *classical* subspace, since it is within this subspace that the equations of semiclassical nonlinear optics evolve.

The deterministic stationary solutions of (6.6.9), given by neglecting dW_1 and dW_2 are:

i)   In the classical subspace: a stationary solution $\alpha = \beta = 0$. Below a threshold defined by $\lambda < 1$ this solution is stable.

ii)   In the classical subspace: two stationary solutions $\alpha = \beta = \pm\sqrt{\lambda - 1}$. Above the threshold, i.e., for $\lambda > 1$, this solution is stable.

iii)   Below threshold i.e., for $\lambda < 1$, two unstable stationary solutions:

$$\alpha = \beta = \pm i \sqrt{1 - \lambda}. \tag{6.6.10}$$

Above threshold, this solution is transformed into the stable classical solution, ii).

iv)   Both above and below the threshold

$$\alpha = -\beta = \pm\sqrt{\lambda + 1}. \tag{6.6.11}$$

It is this kind of reorganisation of the deterministic nonlinear dynamics in the expanded phase space which can lead to anomalous behaviour (unstable trajectories) in numerical simulations of the stochastic differential equations for large quantum noise (here $g \geq 1$).

b) **The Real Submanifold:**   Consider the bounded manifold $\alpha = x, \beta = y$, with x and y both real and $|x|, |y| \leq \sqrt{\lambda}$. We will denote this manifold by $\Omega(x, y)$. If any point within this manifold is taken as an initial condition in (6.6.9) the resulting stochastic trajectory must at first preserve α and β as real quantities, since λ, g, dW_1 and dW_2 are all real, and the arguments of the square roots are both positive. Thus, α and β remain real, and within $\Omega(x, y)$ for all time. To escape this manifold one of the square roots must obtain a negative argument.

The boundary of $\Omega(x, y)$ is defined by the condition that one or other of the square roots which are the coefficients of the noise terms vanishes. Since the stochastic trajectories are strictly continuous functions of time, if the point is to escape $\Omega(x, y)$, this boundary must be crossed; it cannot be jumped. But the noise transverse to the boundary vanishes, while the deterministic flow transverse to the boundary is directed inwards, as illustrated in Fig.6.2. A trajectory reaching the boundary can only move along the boundary under the influence of the other, nonvanishing noise, or return to the interior of $\Omega(x, y)$ under the deterministic flow. Thus, trajectories entering $\Omega(x, y)$ are contained within it by a reflecting boundary condition. The vacuum state, if represented by $\alpha = \beta = 0$, (the natural initial condition), as well as all stable steady states of the deterministic equations are contained within $\Omega(x, y)$. The unstable steady states and those regions of phase space which can lead to unstable numerical trajectories lie outside. Of course, unless directed not to do so, trajectories generated by numerical algorithms, taking finite steps, can jump

Fig. 6.2   Deterministic flow on the real submanifold
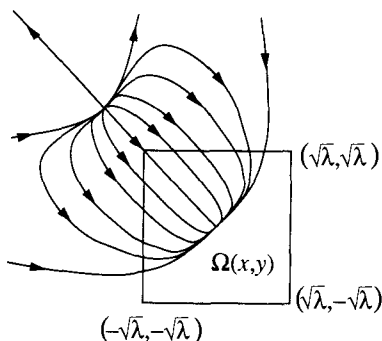
the boundary; this opens the possibility for the instabilities. (But, unfortunately this is not a method of eliminating instabilities except in this special case and perhaps others).

The manifold $\Omega(x, y)$ is alternatively denoted $\Omega(u, v)$, with $u = \frac{1}{2}(x + y)$, and $v = \frac{1}{2}(x - y)$. The line $v = 0$ is a one-dimensional *classical* subspace; the subspace preserving $\alpha = \beta$. The variable v defines a transverse *nonclassical* dimension used by the noise to construct manifestly nonclassical states.

c) Stationary Solution of the Fokker-Planck Equation: With $\alpha = x$ and $\beta = y$ both real, the Fokker-Planck equation corresponding to (6.6.9) can be solved in the steady state to obtain

$$P_{\mathrm{ss}}(x, y) = \mathcal{N} \left[(\lambda - x^2)(\lambda - y^2) \right]^{1/g^2 - 1} e^{2xy/g^2} \tag{6.6.12}$$

with $\mathcal{N}$ determined by normalisation, and $|x|, |y| \leq \sqrt{\lambda}$.

For weak noise $g \ll 1, P_{ss}(x, y)$ is illustrated in Fig. 6.3a. It strictly vanishes on the boundary of $\Omega(x, y)$, but is essentially zero well before the boundary is reached. $P_{ss}(x, y)$ is a slightly broadened version of the classical steady states. The picture is precisely that derived from a linearised treatment of the quantum fluctuations. Explicitly, with $g \ll 1$ and $\lambda < 1$,

$$P_{\mathrm{ss}}(u, v) = \frac{\sqrt{1 - \lambda^2}}{\pi \lambda g/2} \exp \left[-\frac{(1 - \lambda) u^2 + (1 + \lambda) v^2}{\lambda g^2/2} \right]. \tag{6.6.13}$$

This is the regime of experiments generating squeezed states. For example, in [6.8], g was about $\approx 10^{-4}$.

d) Character of the Stationary Distribution: The distribution (6.6.9) is essentially the same as that already investigated in Sect. 6.6.13—one neglects the nonlinearity in the equation of motion by assuming the positive P-function variable corresponding to b is constant, rather than adiabatically eliminating as in the derivation of (6.3.3).

For stronger pumping, this is no longer possible. A numerical investigation demonstrates the varied character of the distribution as g and λ are altered. In Fig. 6.3a,b the distribution is plotted below and above threshold, but still in what

one might call the classical regime. The single peak below threshold and the double peak above threshold occur at the position of the classical stationary states. As long as $g^2 < 1$, the stationary distribution vanishes on the boundary of $\Omega(x, y)$, but when $g^2 > 1$, a singular distribution results.

Before going any further, the question arises: If $P_{ss}(x, y)$ is singular, is the discarding of surface terms justified? The answer is yes, if

i)   A reflecting boundary condition is imposed on the boundary

ii) $(\lambda - x^2)P(x, y) \to 0$ as $x^2 \to \lambda$, and similarly for y.

Under these conditions it is easy to verify that the tranformation from the characteristic function equation to the Fokker-Planck equation equivalent to (6.6.9) is possible with no surface terms. Note that the stationary distribution obeys both conditions for any finite g.
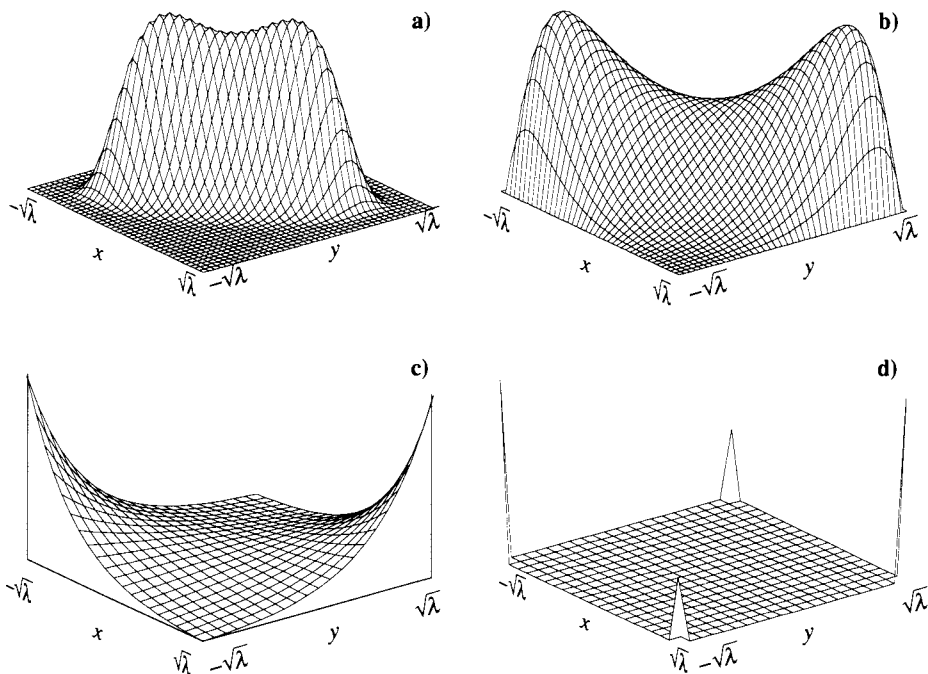
Fig. 6.3a–d Plots of the stationary distribution(6.6.12) (**a**) For $g = 0.2$, in the regime of classical behaviour; (**b**) For $g = 0.7$, clearly showing bistable behaviour; (**c**) $g = 1$ Above threshold, but still in the classical regime; (**d**) For $g = 1.2$, approaching the non-classical regime

So the plots for $\lambda^2 > 1$ are valid, and reveal that the two peaks have migrated to the corners of the $\Omega(x, y)$. Notice the trend as λ gets larger in Fig.2c for $g = 1.2$; we see there distinct, sharp peaks at all four corners for large g, but these peaks become concentrated at the diagonal corners as g grows.

e) Bistable Behaviour :   *Kinsler* and *Drummond* [6.9] have used this method to analyse the bistable behaviour which is apparent in the stationary solutions of the deterministic equations, $\alpha = \beta = \pm\sqrt{\gamma - 1}$, which are stable for $\gamma > 1$. (See a) i) above.) The exact stationary stochastic solution is plotted in Fig.6.3b, and we see the two well defined peaks.

This bistable behaviour was noted by *Graham* [6.10] before the invention of the positive P-representation. Graham used the Wigner function to analyse this model, but no exact treatment could be found, since the Wigner function equations of motion gave third order derivatives.

Exercise. Derive the Wigner function equations of motion corresponding to those derived for the positive P-function in the first exercise of this section. Show that by neglecting third order derivatives in the equations obtained *before any adiabatic elimination*, that a stochastic equation with positive definite diffusion matrix is obtained.

However *Drummond* [6.11] has shown that the neglect of third order derivatives can be very dangerous if one wishes to compute the switching time between one maximum and another in bistable problems. He noted that the switching time must be the inverse of the smallest non-zero eigenvalue of the Fokker-Planck operator, and gave an asymptotic method of computing this in bistable systems. His method involves an expansion in a parameter which measures the distance of the physical situation from a critical point, i.e., that configuration of parameters at which bistability emerges.

Drummond and Kinsler investigated the switching time in this parametric amplifier problem by several methods.

i)   Numerical simulation of the stochastic equations (6.6.9).

ii)   Standard stochastic analytic approximations (as in S.M.Ch.9) applied to the positive-P Fokker-Planck equations.

iii) Numerical solution of the eigenvalue problem for the master equation. This means that one computes what the master equation is equivalent to in a number state basis, as in Sect. 6.6.6–8. However in this case we get infinite matrix equations, which must be truncated at some maximum photon number. With sufficient computing power, this maximum can be made large enough to get good results.

iv)   The methods i) and ii) can be applied to the Wigner function equations.

The result of this investigation shows that the positive-P methods are accurate and correct. They are good for large photon number, while the number state methods are good for small photon number. The Wigner function methods are good in situations of small non-linearity, i.e., small quantum noise, but fail when the quantum noise becomes large.

f) Conclusion: This system represents a reasonably rigorously demonstrable example of a non-trivial positive P-representation in which it can be shown that all boundary terms can be accounted for. It should be also be noted that, with care, the same conclusions can be drawn from a complex P-representation [6.12].

7. Amplifiers and Measurement

This chapter consists of two parts, not closely related, but conceptually close. It is largely a chapter on the applications of input-output theory. The first part, on the theory of linear amplifiers, is a direct application of input-output theory, since amplifiers do of necessity have inputs and outputs, and the relation of one to the other is the central problem under study.

The second part is a partial answer to the problems raised in Chap. 2 , on the foundations of the von Neumann theory of measurement. By means of a few specific examples, it is possible to show how in practice the existence of macroscopic quantum superposition states is ruled out in almost any reasonable situation. Finally, it is shown how the mechanism which destroys the coherences which would characterise any macroscopic quantum superposition state can also give a very plausible explanation of the von Neumann measurement postulate. If a small system is measured by a macroscopic measuring, apparatus, which is of necessity in contact with an environment (i.e., a heat bath), then the indirect effect of the heat bath is to feed noise back into the system in such a way as to destroy any coherences—thus the system density operator is diagonalised. One can then say that, after the measurement, the system is left in a particular quantum state.

7.1 Input-Output Theory of Amplifiers and Attenuators

An attenuator for a signal is a rather simple concept, implemented optically by a piece of grey glass. It is a passive device, and can easily be modelled by the following system. We have a harmonic oscillator, represented by the destruction operator a, coupled to two heat baths. One of the heat baths, with operators $b(\omega)$, is viewed as a device which introduces the input signal, and carries away the output signal. The other, with operators $h(\omega)$, represents the bath which absorbs the power from the signal. Thus the loss mechanism and the mechanism for carrying the signal in and out are of the same kind. The loss part of the Hamiltonian is

$$H_{\text{Loss}} = i\hbar \int d\omega\, \gamma(\omega) \{h(\omega)a^\dagger - ah(\omega)\} \tag{7.1.1}$$

while the input-output part of the Hamiltonian is

$$H_{\text{Sig}} = i\hbar \int d\omega\, \kappa(\omega) \{b(\omega)a^\dagger - ab(\omega)\}. \tag{7.1.2}$$

The quantum Langevin equation for the oscillator is (in the interaction picture)

$$\dot{a} = -\tfrac{1}{2}\gamma a - \sqrt{\gamma}\, h_{in}(t) - \tfrac{1}{2}\kappa a - \sqrt{\kappa}\, b_{in}(t) \tag{7.1.3}$$

which has the solution

$$a(t) = a(0)\, e^{-(\kappa+\gamma)t/2} - \int_0^t dt'\, e^{-(\kappa+\gamma)(t-t')/2}\{\sqrt{\gamma}\, h_{in}(t') + \sqrt{\kappa}\, b_{in}(t')\}. \tag{7.1.4}$$

Let us assume that $(\kappa + \gamma)^{-1}$ is much smaller than the time scales of interest in $b_{in}(t)$; then we can approximate the stationary solution by dropping the initial value term and neglecting the time dependence of the signal and loss bath operators in the integral, to get

$$a(t) = -\frac{2}{\gamma+\kappa}\{\sqrt{\gamma}\, h_{in}(t) + \sqrt{\kappa}\, b_{in}(t)\}. \tag{7.1.5}$$

The boundary condition of the signal is

$$b_{out}(t) = b_{in}(t) + \sqrt{\kappa}\, a(t) \tag{7.1.6}$$

so that

$$b_{out}(t) = \frac{\gamma-\kappa}{\gamma+\kappa} b_{in}(t) - \frac{2\sqrt{\gamma\kappa}}{\gamma+\kappa} h_{in}(t). \tag{7.1.7}$$

The output term then consists of an attenuated version of the input signal, as well as an additional term which represents the noise which arises from the loss medium.

7.2   Amplifiers

To obtain an amplifier we can follow the same procedure, but will need a medium with gain, rather than loss. We have already seen that a master equation can be easily set up which gives gain, namely in Sect. 5.1.4, the master equation for an assembly of two level atoms in which $N_a^+ > N_a^-$. We would like to find a QSDE which corresponds to this master equation, but comparison with the master equation (5.4.12) shows that this would require $\bar{N} > \bar{N} + 1$ so this cannot be done straightforwardly.

7.2.1   The Inverted Oscillator Heat Bath

To obtain gain rather than damping from a medium requires an inversion, and this cannot be achieved with a harmonic oscillator heat bath. To see this, one need
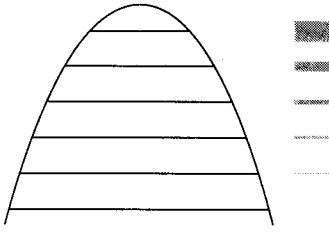
only note the damped harmonic oscillator equations of motion (5.3.26), which give damping for the mean independent of the state of the heat bath.

A device introduced by Glauber approximates an inverted two level system by a harmonic oscillator with energy levels

$$E_n = -(n + \tfrac{1}{2})\hbar\omega \tag{7.2.1}$$

maintained at a *negative* temperature $-T$. The populations of the energy levels will be given by

$$\frac{e^{-n\hbar\omega/kT}}{1 - e^{-\hbar\omega/kT}} \tag{7.2.2}$$

and if T is not too large, nearly all the population is in levels 1 and 2, with most in level 1, thus giving an approximation to an inverted two level atom. Since the raising operator for the energy decreases the number of quanta in the inverted oscillator, the role of creation and destruction operators is reversed.

Thus, the Hamiltonian for an inverted bath may be written in terms of operators $h(\omega), h^\dagger(\omega)$, which have commutation relations

$$[h(\omega), h^\dagger(\omega')] = -\delta(\omega - \omega'), \tag{7.2.3}$$

that is, the roles of h and $h^\dagger$ are reversed. However the bath Hamiltonian is

$$H_H = -\int d\omega\, \hbar\omega \left(h(\omega)h^\dagger(w') + \tfrac{1}{2}\right) \tag{7.2.4}$$

and we find, in the negative temperature state

$$\begin{aligned}
\langle h^\dagger(\omega)h(\omega')\rangle &= \left(\bar{N}(\omega) + 1\right)\delta(\omega - \omega') \\
\langle h(\omega)h^\dagger(\omega')\rangle &= \bar{N}(\omega)\delta(\omega - \omega')
\end{aligned} \tag{7.2.5}$$

with $\bar{N}(\omega)$ as in (5.3.40), in which $-T$ is the value of the negative temperature.

In terms of the inputs and outputs, this means that we can write

$$\begin{aligned}
\langle h_{\text{in}}^\dagger(t)h_{\text{in}}(t')\rangle &= (\bar{N} + 1)\delta(t - t') \\
\langle h_{\text{in}}(t)h_{\text{in}}^\dagger(t')\rangle &= \bar{N}\delta(t - t')
\end{aligned} \tag{7.2.6}$$

to achieve the quantum white noise limit. The effect on the QSDEs and master equations is thus simply to interchange the coefficients of $\bar{N}$ and $\bar{N} + 1$. We then have a simple model of a medium which gives linear gain.

7.2.2   The Amplifier Model

The QSDE for an amplifier can be made from the following model. A gain medium (perhaps composed of an assembly of inverted atoms as in Sect. 5.1.4) is enclosed in an optical cavity. The mode inside the cavity thus experiences the effect of the gain medium, the input optical signal, and the losses via the output optical signal. However, we model the inverted atomic medium by an inverted oscillator heat bath.

The gain part of the Hamiltonian is thus obtained by writing an interaction Hamiltonian

$$H_{\text{gain}} = i\hbar \int d\omega \gamma(\omega) \left[h(\omega)a^\dagger - ah^\dagger(\omega) \right]. \tag{7.2.7}$$

The input-output electromagnetic field is represented by

$$H_{\text{loss}} = i\hbar \int d\omega \, \kappa(\omega) \left[b(\omega)a^\dagger - ab^\dagger(\omega) \right] \tag{7.2.8}$$

as in (5.3.3).

Putting these together the quantum Langevin equation for a is (setting $H_{\text{sys}} = 0$ for simplicity)

$$\dot{a} = \frac{\gamma}{2}a + \sqrt{\gamma}\, h_{\text{in}}(t) - \frac{\kappa}{2}a - \sqrt{\kappa}\, b_{\text{in}}(t) \tag{7.2.9}$$

which has the solution

$$a(t) = a(0)e^{-(\kappa-\gamma)t/2} + \int_0^t dt' e^{-(\kappa-\gamma)(t-t')/2} \left[\sqrt{\gamma}\, h_{\text{in}}(t') - \sqrt{\kappa}\, b_{\text{in}}(t') \right]. \tag{7.2.10}$$

If the loss, κ, exceeds the gain, then this eventually comes to a steady state in which the initial value term vanishes.

These equations are valid for any statistics of $b_{\text{in}}(t)$ and $h_{\text{in}}(t)$ so we can consider situations in which the damping and gain are sufficiently strong that $(\kappa - \gamma)^{-1}$ is a much smaller time than the time scale of the fluctuations in $b_{\text{in}}(t)$ or $h_{\text{in}}(t)$. In that case we can set $b_{\text{in}}^\dagger(t') \rightarrow b_{\text{in}}^\dagger(t), h_{\text{in}}(t') \rightarrow h_{\text{in}}(t)$ in (7.2.10) and obtain approximately

$$a(t) = \frac{2}{\kappa - \gamma} \left\{ \sqrt{\gamma}\, h_{\text{in}}(t) - \sqrt{\kappa}\, b_{\text{in}}(t) \right\}. \tag{7.2.11}$$

The input-output boundary condition on the b operators, is from (5.3.24),

$$b_{\text{out}}(t) = b_{\text{in}}(t) + \sqrt{\kappa}\, a(t) \tag{7.2.12}$$

so that

$$b_{\text{out}}(t) = \frac{\gamma + \kappa}{\gamma - \kappa} b_{\text{in}}(t) - \frac{2\sqrt{\gamma\kappa}}{\gamma - \kappa} h_{\text{in}}(t). \tag{7.2.13}$$

The output then consists of an amplified version of the input, as well as an additional term. If the statistics of the gain medium are effectively random, this is an added noise term.

7.2.3   Added Noise

It is easy to see that (7.2.13) guarantees that the "out" fields have the right commutation relations. But the presence of the term $h_{in}(t)$ says that there will be added noise to the amplified signal. To investigate the noise, however, the equations are best solved exactly by Fourier transform. If $\bar{a}(\omega), \bar{h}_{in}(\omega), \bar{b}_{in}(\omega)$, are the Fourier components of the various quantities involved , then it is not difficult to show that

$$\bar{a}(\omega) = \frac{\sqrt{\gamma}\,\bar{h}_{in}(\omega) - \sqrt{\kappa}\,\bar{b}_{in}(\omega)}{i\omega + (\kappa - \gamma)/2} \tag{7.2.14}$$

$$\bar{b}_{out}(\omega) = \bar{b}_{in}(\omega)\frac{\gamma + \kappa - 2i\omega}{\gamma - \kappa - 2i\omega} - \bar{h}_{in}(\omega)\frac{2\sqrt{\gamma\kappa}}{\gamma - \kappa - 2i\omega}. \tag{7.2.15}$$

We now suppose that there is an input *signal* given by $\langle\bar{b}_{in}(\omega)\rangle$, but that the amplifier bath has simply thermal statistics, so that $\langle\bar{h}_{in}(\omega)\rangle = 0$. Then

$$\langle\bar{b}_{out}(\omega)\rangle = \langle\bar{b}_{in}(\omega)\rangle\frac{\gamma + \kappa - 2i\omega}{\gamma - \kappa - 2i\omega}. \tag{7.2.16}$$

The amplifier has a frequency dependent gain $G(\omega)$ given by

$$G(\omega) = \frac{\gamma + \kappa - 2i\omega}{\gamma - \kappa - 2i\omega} \tag{7.2.17}$$

which is a maximum at zero frequency. If $\gamma < \kappa$, there is no stationary situation, and the amplifier gives a signal which increases without limit. Essentially, the power being fed into the cavity cannot escape fast enough. (Of course the idea of an inverted medium which maintains its inversion independent of power output is not exactly valid, and depletion effects will then need to be considered. The system is then essentially a laser).

The noise is given by the mean square fluctuation power per unit bandwidth. We use the notation for the noise power, $\mathbf{N}\{\bar{b}_{in}(\omega)\}$, defined by

$$\mathbf{N}\{\bar{b}_{in}(\omega)\}\delta(\omega - \omega')$$
$$= \hbar\omega\left\langle\left(\bar{b}_{in}^{+}(\omega) - \langle b_{in}^{+}(\omega)\rangle\right)\left(\bar{b}_{in}(\omega') - \langle\bar{b}_{in}(\omega')\rangle\right)\right\rangle \tag{7.2.18}$$

with a similar notation for $\bar{b}_{out}(\omega)$ and $\bar{h}_{in}(\omega)$. From (7.2.15,17) we can see that

$$\mathbf{N}\{\bar{b}_{out}(\omega)\} = \mathbf{N}\{\bar{b}_{in}(\omega)\}|G(\omega)|^2 + \mathbf{N}\{\bar{h}_{in}(\omega)\}\{|G(\omega)|^2 - 1\}. \tag{7.2.19}$$

The *added noise*, $A(\omega)$, is that amount of noise in the input which would produce the last term on the right hand side, thus

$$A(\omega) = \mathbf{N}\{\bar{h}_{in}(\omega)\}\{1 - |G(\omega)|^{-2}\}. \tag{7.2.20}$$

If the amplifier bath is thermal at temperature $-T$, then

$$A(\omega) = \left\{\frac{\hbar\omega}{e^{\hbar\omega/kT} - 1} + \hbar\omega\right\}\{1 - |G(\omega)|^{-2}\} \tag{7.2.21}$$

and the minimum added noise, which occurs at $T = 0$, is

$$A(\omega)_{T=0} = \hbar\,\omega\{1 - |G(\omega)|^{-2}\}. \tag{7.2.22}$$

Thus

$$0 \le A(\omega)_{T=0} \le \hbar\,\omega \tag{7.2.23}$$

with zero being attained only at unit gain.

Thus, even at absolute zero, quantum noise will be added to the signal unless there is no gain at all.

7.2.4   Signal to Noise Ratio

Using a similar notation for the signal power,

$$\mathbf{S}\{\bar{b}_{\text{in}}(\omega)\} = \hbar\,\omega|\langle\bar{b}_{\text{in}}(\omega)\rangle|^2 \tag{7.2.24}$$

and the corresponding definition of the "out" signal, we are now in a position to consider the signal to noise ratios $(\mathbf{S}/\mathbf{N})$. Since from (7.2.16,24) we know

$$\mathbf{S}\{\bar{b}_{\text{out}}(\omega)\} = |G(\omega)|^2 \mathbf{S}\{\bar{b}_{\text{in}}(\omega)\}, \tag{7.2.25}$$

the signal to noise ratios are related by

$$(\mathbf{S}/\mathbf{N})_{\text{out}}^{-1} = (\mathbf{S}/\mathbf{N})_{\text{in}}^{-1} + \frac{\mathbf{N}\{\bar{h}_{\text{in}}(\omega)\}}{\mathbf{S}_{\text{in}}}\left\{1 - |G(\omega)|^{-2}\right\}. \tag{7.2.26}$$

If we assume that the gain medium is thermal and has temperature $-T$ so that $\mathbf{N}\{\bar{h}_{\text{in}}(\omega)\} = 1 + \bar{N}(\omega)$, and that $|G(\omega)| \to \infty$, we find that

$$(\mathbf{S}/\mathbf{N})_{\text{out}}^{-1} \approx (\mathbf{S}/\mathbf{N})_{\text{in}}^{-1} + \frac{\hbar\,\omega\{1 + \bar{N}(\omega)\}}{\mathbf{S}_{\text{in}}}. \tag{7.2.27}$$

As expected, the signal to noise ratio is degraded even at absolute zero by the added vacuum fluctuations from the oscillator heat bath.

7.2.5   "Noise Temperature" of an Amplifier

It is traditional to express the noise produced in an amplifier in terms of the concept of "noise temperature". The idea behind the concept is to express the amplifier noise as the amount of extra noise which would be produced by increasing the temperature of the input by an amount called the noise temperature.

In fact the noise temperature is not really a very convenient measure, and is only included here for completeness. In (7.2.20) the added noise term in the input is $\mathbf{N}\{\bar{h}_{\text{in}}(\omega)\}\{1 - |G(\omega)|^{-2}\}$. The *noise temperature* will be defined as T_n, where

$$\frac{\hbar\,\omega}{e^{\hbar\omega/kT_n} - 1} = \mathbf{N}\{\bar{h}_{\text{in}}(\omega)\}\{1 - |G(\omega)|^{-2}\} \tag{7.2.28}$$

so that

$$T_n = \frac{\hbar\omega/k}{\log\left[1 + \hbar\omega\{1 - |G(\omega)|^2\}/\mathrm{N}\{\bar{h}_{\mathrm{in}}(\omega)\}\right]}. \tag{7.2.29}$$

Notice that

i)   As $|G| \to \infty$, $T_n \to 0$; there is no extra noise in the case of unit gain.

ii) If the gain medium is thermal at temperature $-T$, then

$$T_n = \frac{\hbar\omega}{k}\left[\log\left(1 + \frac{1 - e^{-\hbar\omega/kT}}{1 - |G(\omega)|^{-2}}\right).\right]^{-1} \tag{7.2.30}$$

iii) If $T \to 0$, giving a perfect inversion, then

$$T_n = \frac{\hbar\omega}{k}\left[\log\left(\frac{2|G(\omega)|^2 - 1}{|G(\omega)|^2 - 1}\right)\right]^{-1} \tag{7.2.31}$$

so that in this case we deduce that

$$0 \le T_n \le \frac{\hbar\omega}{k\log 2}. \tag{7.2.32}$$

iv) An alternative definition can be obtained if, instead of the definition (7.2.18) of noise power in terms of the normally ordered average, we use the symmetrically ordered average, and define the noise temperature as that temperature at which the added noise equals the increase in noise compared to absolute zero. In this case, assuming $T = 0$, we find (using T'_n for this alternative definition)

$$T'_n = \frac{\hbar\omega}{k}\left[\log\left(\frac{3|G(\omega)|^2 - 1}{|G(\omega)|^2 - 1}\right)\right]^{-1} \tag{7.2.33}$$

and

$$0 \le T'_n \le \frac{\hbar\omega}{k\log 3}. \tag{7.2.34}$$

v)   The expressions for noise temperature give the impression of having deep significance, but their form is really only a reflection of an inappropriate way of characterising noise, which I feel should now be replaced by the concept of added noise.

7.2.6   QSDEs in the Case of a Negative Temperature Bath

Exercise. Work through the derivation of the quantum Langevin equation in Sect. 5.3.2 and show that, in the case of a negative temperature bath, as in Sect. 7.2.1, the quantum Langevin equation (5.3.15) becomes

$$\dot{a} = -\frac{i}{\hbar}[a, H_{\text{sys}}] + [a, c^\dagger]\left[\frac{\gamma}{2}c - \sqrt{\gamma}\,h_{\text{in}}(t)\right] - \left[\frac{\gamma}{2}c^\dagger - \sqrt{\gamma}\,h_{\text{in}}^\dagger(t)\right][a, c]. \quad (7.2.35)$$

Show that the corresponding Ito QSDE takes the form

$$
\begin{aligned}
(\mathbf{I})\,da = &-\frac{i}{\hbar}[a, H_{\text{sys}}]dt + \frac{\gamma}{2}\bar{N}(2c^\dagger ac - ac^\dagger c - c^\dagger ca)dt \\
&+\frac{\gamma}{2}(\bar{N}+1)(2cac^\dagger - acc^\dagger - cc^\dagger a)dt \\
&-\sqrt{\gamma}[a, c^\dagger]dH(t) + \sqrt{\gamma}\,dH^\dagger(t)[a, c]
\end{aligned}
\quad (7.2.36)
$$

in which

$$
\begin{aligned}
dH(t)^2 &= dH^\dagger(t)^2 = 0 \\
dH^\dagger(t)dH(t) &= (\bar{N}+1)\,dt \\
dH(t)dH^\dagger(t) &= \bar{N}\,dt.
\end{aligned}
\quad (7.2.37)
$$

7.2.7   Ito QSDEs for Positive and Negative Temperature

Notice that the Ito equation can be written in a form which is valid for both positive and negative temperature baths as follows:
Define for each case

$$
\begin{array}{lll}
T > 0 & ; & T < 0 \\
dW(t) = \sqrt{\gamma}\,dB^\dagger(t) & ; & = \sqrt{\gamma}\,dH(t) \\
dW^\dagger(t) = \sqrt{\gamma}\,dB(t) & ; & = \sqrt{\gamma}\,dH^\dagger(t) \\
\lambda = \gamma(\bar{N}+1) & ; & = \gamma\bar{N} \\
\mu = \gamma\bar{N} & ; & = \gamma(\bar{N}+1).
\end{array}
\quad (7.2.38)
$$

Then

$$
\begin{aligned}
dW(t)^2 &= dW^\dagger(t)^2 = 0 \\
dW^\dagger(t)dW(t) &= \lambda\,dt \\
dW(t)dW^\dagger(t) &= \mu\,dt
\end{aligned}
\quad (7.2.39)
$$

and

$$
\begin{aligned}
(\mathbf{I})\,da = &-\frac{i}{\hbar}[a, H_{\text{sys}}] + \frac{\mu}{2}(2c^\dagger ac - ac^\dagger c - c^\dagger ca)dt \\
&+\frac{\lambda}{2}(2cac^\dagger - acc^\dagger - cc^\dagger a)dt \\
&-[a, c^\dagger]dW(t) + dW^\dagger(t)[a, c].
\end{aligned}
\quad (7.2.40)
$$

The different cases of negative temperature and positive temperature depend on the relative sizes of μ and λ; for positive temperature $\lambda > \mu$, for negative temperature $\lambda < \mu$.

7.2.8 Phase Conjugating Amplifier

Exercise. Consider a modification of the amplifier model of Sect. 7.2.2, made by using the couplings

$$H = i\hbar \int d\omega\, \gamma(\omega)[h(\omega)a - a^\dagger h^\dagger(\omega)] + i\hbar \int d\omega\, \kappa(\omega)[b(\omega)a^\dagger - ab^\dagger(\omega)].$$
(7.2.41)

The second term is quite normal, but the first term can be seen to create a bath quantum and a system quantum at the same time. Both $h(\omega)$ and $b(\omega)$ represent ordinary non-inverted heat baths. The first term could in practice be approximated by a pumped Hamiltonian

$$H = i\hbar \int d\omega\, \mu(\omega)[E(2\omega)h(\omega)a - a^\dagger h^\dagger(\omega)E(2\omega)],$$
(7.2.42)

in which $E(2\omega)$ is in a coherent state created by a strong pump field, and can thus be approximated by a classical field $\mathcal{E}(2\omega)$, and $\gamma(\omega) = \mu(\omega)\mathcal{E}(2\omega)$.

Using the same general notation as for the ordinary amplifier show that the equations of motion are

$$\dot{a} = \frac{\gamma - \kappa}{2}a + \sqrt{\gamma}\, h_{\text{in}}^\dagger(t) - \sqrt{\kappa}\, b_{\text{in}}(t).$$
(7.2.43)

Taking $b_{\text{in}}(t)$ to be the input field, and $h_{\text{out}}(t)$ to be the output, show that this represents an amplifier such that

$$h_{\text{out}}(t) = Gb_{\text{in}}^\dagger(t) + \sqrt{G^2 + 1}\, h_{\text{in}}(t)$$
(7.2.44)

with

$$G = \frac{2\sqrt{\kappa\gamma}}{\kappa - \gamma}.$$
(7.2.45)

Analyse the noise performance of this amplifier.

The amplifier is called "phase conjugating" since a change in the phase $e^{i\phi}$ of the input $b_{\text{in}}(t)$ results in a phase change $e^{-i\phi}$ in the output $h_{\text{out}}(t)$.

Notice that (7.2.44) is very similar to the inverted bath amplifier (7.2.13)—each represents essentially a mixture of creation and destruction operators. This occurs in (7.2.13) because h and $h^\dagger$ have roles which are reversed, as a result of the inverted bath; in (7.2.44) the creation operator term arises because of the abnormal coupling in (7.2.41). However the output signal in the phase conjugating amplifier is in a different channel from the input.

7.2.9 The Degenerate Parametric Amplifier

In the previous example the signal to noise ratio of the amplified input was degraded by the added term arising from the amplifier heat bath. It is possible to have amplification which does not depend on a bath, as was noted by *von Neumann* [7.1], and in this case no added noise occurs.

In this amplifier the gain comes from a *Hamiltonian* term

$$H_{\text{gain}} = \frac{i\hbar\epsilon}{4}(a^{+2} - a^2) \tag{7.2.46}$$

but the arrangement is otherwise the same as in the previous example. At this stage we will not be concerned with how to create such a Hamiltonian in practice—see Sect. 10.2 for an explanation. The quantum Langevin equation now becomes

$$\dot{a} = \frac{\epsilon}{2}a^{\dagger} - \frac{\kappa}{2}a - \sqrt{\kappa}\,b_{\text{in}}(t). \tag{7.2.47}$$

We can solve this equation by introducing the *quadrature phases*

$$X = \frac{1}{2}(a + a^{\dagger}), \qquad Y = \frac{1}{2i}(a - a^{\dagger}) \tag{7.2.48}$$

$$B_{\text{in}}^{X}(t) = \frac{1}{2}[b_{\text{in}}(t) + b_{\text{in}}^{\dagger}(t)], \qquad B_{\text{in}}^{Y}(t) = \frac{1}{2i}[b_{\text{in}}(t) - b_{\text{in}}^{\dagger}(t)], \tag{7.2.49}$$

and these have the commutation relations

$$[B_{\text{in}}^{X}(t), B_{\text{in}}^{Y}(t')] = \frac{i}{2}\delta(t - t'). \tag{7.2.50}$$

The equations of motion are then

$$\dot{X} = \frac{\epsilon - \kappa}{2}X - \sqrt{\kappa}\,B_{\text{in}}^{X}(t), \qquad \dot{Y} = -\frac{\epsilon + \kappa}{2}Y - \sqrt{\kappa}\,B_{\text{in}}^{Y}(t), \tag{7.2.51}$$

a) Approximate Solution: Using the same techniques as in the previous example, one finds the stationary solutions are

$$X = \frac{2\sqrt{\kappa}}{\epsilon - \kappa}B_{\text{in}}^{X}(t), \qquad Y = -\frac{2\sqrt{\kappa}}{\epsilon + \kappa}B_{\text{in}}^{Y}(t) \tag{7.2.52}$$

so that

$$\begin{aligned} B_{\text{out}}^{X}(t) &= \frac{\epsilon + \kappa}{\epsilon - \kappa}B_{\text{in}}^{X}(t) \\ B_{\text{out}}^{Y}(t) &= \frac{\epsilon - \kappa}{\epsilon + \kappa}B_{\text{in}}^{Y}(t). \end{aligned} \tag{7.2.53}$$

Thus $B_{\text{in}}^{X}(t)$ is amplified with a gain $G = (\epsilon + \kappa)/(\epsilon - \kappa) > 1$, whereas $B_{\text{in}}^{Y}(t)$ is attenuated by a factor G^{-1}. In terms of the $b, b^{\dagger}$ variables, we can write

$$b_{\text{out}}(t) = b_{\text{in}}(t)\frac{G + G^{-1}}{2} + b_{\text{in}}^{\dagger}(t)\frac{G - G^{-1}}{2}. \tag{7.2.54}$$

b) Commutation Relations: It is easy to check that the commutation relations are preserved—no added noise term is necessary since the amplification mixes $b_{\text{in}}(t)$ and $b_{\text{in}}^{\dagger}(t)$. This means that the gain is only in half of the signal, the X quadrature $B_{\text{in}}^{X}(t)$, and the Y quadrature $B_{\text{in}}^{Y}(t)$ is attenuated.

c) No Added Noise: We define a Fourier transform variable

$$X(t) = \frac{1}{\sqrt{2\pi}} \int d\omega\, e^{i\omega t} \tilde{X}(\omega) \tag{7.2.55}$$

with a similar definition for the other variables. Then exact solutions are

$$\tilde{B}^X_{out}(\omega) = \tilde{B}^X_{in}(\omega)\frac{i\omega - (\epsilon + \kappa)/2}{i\omega - (\epsilon - \kappa)/2}$$

$$\tilde{B}^Y_{out}(\omega) = \tilde{B}^Y_{in}(\omega)\frac{i\omega + (\epsilon - \kappa)/2}{i\omega + (\epsilon + \kappa)/2}. \tag{7.2.56}$$

Since there is no added term in either of these, the signal to noise ratio for the output will be exactly the same as that of the input. Nevertheless, even though one quadrature is attenuated and the other amplified, the power put into the amplified quadrature more than compensates for that lost in the attenuated signal.

Exercise. Show that the power gain is $\frac{1}{2}\{|G(\omega)|^2 + |G(\omega)|^{-2}\} > 1$, where $G(\omega) = (2i\omega - \epsilon - \kappa)/(2i\omega - \epsilon + \kappa)$.

d) Squeezing: The time integrals of the field operators

$$\left.\begin{array}{l} I^X_{in}(t) = \displaystyle\int\limits_{t}^{t+\Delta t} B^X_{in}(t')dt' \\[4mm] I^Y_{in}(t) = \displaystyle\int\limits_{t}^{t+\Delta t} B^Y_{in}(t')dt' \end{array}\right\} \tag{7.2.57}$$

have canonical commutation relations

$$\left[I^X_{in}(t), I^Y_{in}(t)\right] = \frac{i}{2}\Delta t. \tag{7.2.58}$$

Hence, if there is no mean field,

$$\langle I^X_{in}(t)^2\rangle\langle I^Y_{in}(t)^2\rangle \geq \Delta t^2/16. \tag{7.2.59}$$

If the input field is the vacuum, then

$$\langle I^X_{in}(t)^2\rangle = \langle I^Y_{in}(t)^2\rangle = \Delta t/4 \tag{7.2.60}$$

and we can see that the vacuum is a minimum uncertainty state.

From the approximate form of solution (7.2.53), we see that (7.2.60) is still true for the outputs, but

$$\langle I^X_{out}(t)^2\rangle = G^2\Delta t/4$$

$$\langle I^Y_{out}(t)^2\rangle = G^{-2}\Delta t/4. \tag{7.2.61}$$

The uncertainty in the Y quadrature can be made as small as one pleases by letting $\epsilon \to \kappa$, with of course a corresponding increase in the uncertainty of the X quadrature. The output field is said to be *squeezed*.

Exercise. The canonical variables X and Y describe the field inside the cavity. The exact stationary solutions of (7.2.51) are

$$
X(t) = -\sqrt{\kappa} \int_{-\infty}^{t} dt' e^{-(\kappa-\epsilon)(t-t')/2} B_{\text{in}}^{X}(t')
$$

$$
Y(t) = -\sqrt{\kappa} \int_{-\infty}^{t} dt' e^{-(\kappa+\epsilon)(t-t')/2} B_{\text{in}}^{Y}(t').
$$

$$(7.2.62)$$

Show explicitly that $[X, Y] = i/2$, and that

$$
\langle X^2 \rangle \langle Y^2 \rangle = \kappa^2 / 16(\kappa^2 - \epsilon^2)
$$

$$(7.2.63)$$

so that the internal field is *not* a minimum uncertainty state. In fact the minimum value of $\langle Y^2 \rangle$ is $\frac{1}{8}$, not zero as is the case for the "out" field; see [7.2].

7.3   The Macroscopic Limit in Open Quantum Systems

In various examples throughout this book we have met situations in which the connection of a system to a heat bath will tend to destroy superposition states. One example is found in Sect. 3.6.2c where it was shown that the interference between plane waves was destroyed by damping; another occurs in the phase damped oscillator, in Sect. 6.1.2, where we noted that off-diagonal elements are rapidly damped, and only probabilities play a significant role in the further time evolution, and finally, all of Sect. 3.7 was devoted to a numerical investigation of the possibility of macroscopic quantum coherence between the two states of a particle in a double well potential. Although this damping of coherence seems to happen very frequently, it is not an automatic consequence of damping, but depends on the basis we choose to measure with. An example can perhaps clarify this.

7.3.1   Example—Quantum Brownian Motion

Consider the quantum Brownian motion master equation (3.6.28)

$$
\frac{d\rho}{dt} = -\frac{i}{\hbar} \left[H_{\text{sys}}, \rho \right] - \frac{i\gamma}{2\hbar} \left[X, [\dot{X}, \rho]_+ \right] - \frac{\gamma kT}{\hbar^2} \left[X, [X, \rho] \right].
$$

$$(7.3.1)$$

We shall not, however, specify the coupling operator X, or H_{sys} at this stage. Suppose, however, that X commutes with H_{sys}, so that it is a constant of the motion of

the undamped system. Then the equation of motion for the matrix elements of ρ in a basis of the eigenstate $|x\rangle$ of X, is

$$\frac{d}{dt}\langle x|\rho|y\rangle = \left\{-i\omega_{xy} - \frac{\gamma kT}{\hbar^2}(x - y)^2\right\}\langle x|\rho|y\rangle \tag{7.3.2}$$

where

$$\hbar\omega_{xy} = \langle x|H_{sys}|x\rangle - \langle y|H_{sys}|y\rangle. \tag{7.3.3}$$

It is very clear that the solutions of (7.3.2) are damped exponentially, with a time constant proportional to the square of the distance from the diagonal, $x = y$. Further, this constant has $\hbar^2$ in the denominator, so if $x - y$ has a macroscopic scale, the damping will be exceedingly fast. Only the matrix elements of ρ close to the diagonal will remain, and these do not change with time.

Exercise. It has been assumed that $|x\rangle$ characterise the system—consider the situation in which the states are degenerate, $|x, i\rangle$, and show that coherence remains within the same eigenbasis $|x, i\rangle$, for different i, but the same x.

a) **Destruction of Macroscopic Coherence:**   We see from this example that the density operator rapidly becomes a diagonal density operator, represented therefore only by the probabilities $\langle x|\rho|x\rangle$, and of course the basis in which the diagonalisation occurs is determined by the coupling operator X. That operator which is coupled to the bath, or environment, is diagonalised rapidly—it therefore behaves as a classical variable. A superposition state, represented by a density operator $\frac{1}{2}\{|x_1\rangle + |x_2\rangle\}\{\langle x_1| + \langle x_2|\}$ would rapidly become a mixture, $\frac{1}{2}|x_1\rangle\langle x_1| + \frac{1}{2}|x_2\rangle\langle x_2|$. Notice that this diagonalisation is indeed macroscopic, since it happens rapidly when $\gamma kT(x - y)^2/\hbar^2$ is large. If the difference between x and y is sufficiently small, the decay of these off-diagonal elements will not happen very rapidly.

b) **Competing Diagonalisations:**   Suppose there are two couplings, for example, to two operators with canonical commuation relations, $[X, P] = i\hbar$. Then there is no simultaneous diagonal basis, and exact diagonalisation in both bases simultaneously is impossible. However, the damping term

$$-\frac{\gamma_1 kT}{\hbar^2}[X, [X, \rho]] - \frac{\gamma_2 kT}{\hbar^2}[P, [P, \rho]] \tag{7.3.4}$$

becomes for the x matrix elements

$$\left(-\frac{\gamma_1 kT}{\hbar^2}z^2 + \gamma_2 kT\frac{\partial^2}{\partial z^2}\right)\langle u + \tfrac{1}{2}z|\rho(t)|u - \tfrac{1}{2}z\rangle \tag{7.3.5}$$

with $z = x - y$ and $u = (x + y)/2$. The operator in (7.3.5) is the same as that for the harmonic oscillator, so that the time development will be given by exponentials of the form $\exp(-\lambda_n t)$, with

$$\lambda_n = (2n + 1)\sqrt{\gamma_1\gamma_2}/\hbar \tag{7.3.6}$$

with the eigenfunctions the same as those of the harmonic oscillator, with a characteristic distance scale $\left(\gamma_2 \hbar^2 / \gamma_1\right)^{1/4} \equiv l_c$. Thus, the density operator will diagonalise to a width l_c, and will be a Gaussian in the variable z. A complementary result will hold for the variable P; in essence, we reach a minimum uncertainty wavefunction in the conjugate variables. The degree to which this wavefunction is diagonal in P or X depends on the relative strengths of the damping constants γ_1 and γ_2. The density operator becomes as diagonal as it can in both variables simultaneously, and if the system is macroscopic, this amounts essentially to well defined values of P and X. The situation is illustrated in Fig. 7.2.
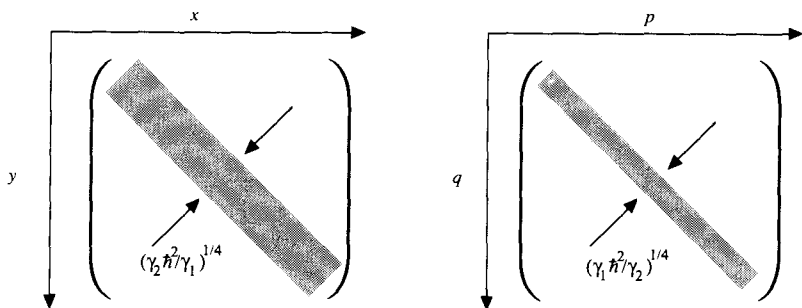
Fig. 7.2   Competing diagonalisations of the density operator by X and P damping

c) Meaning of the Diagonalisation: Once the damping has "diagonalised" the density operator, all we are left with is a probability distribution. If all coupling operators, like X and P, are conserved, the resulting probability distribution is the same as that implied by the original density operator—but no superposition effects will be observable on a macroscopic scale in this basis.

d) Suppose X is not Conserved: The example of Sect. 3.6.2 in fact is such a case; even for a free particle, the position is not conserved, and the degree to which it is not conserved depends on the mass of the particle, and on the value of the momentum, by Newton's law $\dot{x} = p/m$. If m is very large, then we can regard the position as almost constant on the time scale of damping, and all the above conclusions will still be true.

Exercise— Scaling with Mass. Take the formula (3.6.60), and consider the limit $m \to$ large, with fixed damping time, γ/m, and for a superposition state consisting of fixed *velocities* $v_i = \hbar k_i/m$. Show that $\eta(t) \sim m$ for large m. Thus the classical limit corresponds to the situation of very weak H_{sys}. However, notice that if we fix the *momenta*, $p_i = \hbar k_i$, the damping is weaker at large m.

7.3.2   Example—The Quantum Optical Situation

Consider a damped harmonic oscillator at absolute zero, for which the master equation is

$$\dot{\rho} = \gamma \left(2 a\rho a^\dagger - a^\dagger a\rho - \rho a^\dagger a\right). \tag{7.3.7}$$

Using the positive P-representation, we can deduce the equivalent SDEs

$$
\begin{aligned}
d\alpha &= -\gamma\alpha\,dt \\
d\beta &= -\gamma\beta\,dt
\end{aligned}
\tag{7.3.8}
$$

there being no fluctuating term because this is a zero temperature situation. For an initial superposition of coherent states $(|\alpha_1\rangle + |\alpha_2\rangle)/\sqrt{2}$, we find that

$$
\begin{aligned}
P(\alpha,\beta,0) &= \tfrac{1}{2}\delta^2(\alpha-\alpha_1)\delta^2(\beta-\alpha_1^*) + \tfrac{1}{2}\delta^2(\alpha-\alpha_2)\delta^2(\beta-\alpha_2^*) \\
&\quad + \tfrac{1}{2}\langle\alpha_2|\alpha_1\rangle\delta^2(\alpha-\alpha_1)\delta^2(\beta-\alpha_2^*) \\
&\quad + \tfrac{1}{2}\langle\alpha_1|\alpha_2\rangle\delta^2(\alpha-\alpha_2)\delta^2(\beta-\alpha_1^*).
\end{aligned}
\tag{7.3.9}
$$

Notice the normalisation factors in the second two terms which occur because the positive P-function is an expansion in $\Lambda(\alpha,\beta) = |\alpha\rangle\langle\beta^*|/\langle\beta^*|\alpha\rangle$. If α and β are very different, this makes $\Lambda(\alpha,\beta)$ very large, since $\langle\alpha|\beta\rangle$ is very small. Correspondingly, the coefficient of the delta function in (7.3.9) is very small. With time evolution, (7.3.9) will, according to (7.3.8) become

$$
\begin{aligned}
P(\alpha,\beta,t) &= \tfrac{1}{2}\delta^2(\alpha-\alpha_1 e^{-\gamma t})\delta^2(\beta-\alpha_1^* e^{-\gamma t}) \\
&\quad + \tfrac{1}{2}\delta^2(\alpha-\alpha_2 e^{-\gamma t})\delta^2(\beta-\alpha_2^* e^{-\gamma t}) \\
&\quad + \tfrac{1}{2}\langle\alpha_2|\alpha_1\rangle\delta^2(\alpha-\alpha_1 e^{-\gamma t})\delta^2(\beta-\alpha_2^* e^{-\gamma t}) \\
&\quad + \tfrac{1}{2}\langle\alpha_1|\alpha_2\rangle\delta^2(\alpha-\alpha_2 e^{-\gamma t})\delta^2(\beta-\alpha_1^* e^{-\gamma t})
\end{aligned}
\tag{7.3.10}
$$

and hence the density operator is

$$
\begin{aligned}
\rho &= \tfrac{1}{2}\left\{|\alpha_1 e^{-\lambda t}\rangle\langle\alpha_1 e^{-\lambda t}| + |\alpha_2 e^{-\lambda t}\rangle\langle\alpha_2 e^{-\lambda t}|\right\} \\
&\quad + \tfrac{1}{2}\frac{\langle\alpha_1|\alpha_2\rangle}{\langle\alpha_1 e^{-\gamma t}|\alpha_2 e^{-\gamma t}\rangle}|\alpha_2 e^{-\gamma t}\rangle\langle\alpha_1 e^{\gamma t}| \\
&\quad + \tfrac{1}{2}\frac{\langle\alpha_2|\alpha_1\rangle}{\langle\alpha_2 e^{-\gamma t}|\alpha_1 e^{-\gamma t}\rangle}|\alpha_1 e^{-\gamma t}\rangle\langle\alpha_2 e^{-\gamma t}|.
\end{aligned}
\tag{7.3.11}
$$

Notice the extra coefficient in the off diagonal terms. The numerator is in both cases smaller than the denominator, because with the exponential decay, the two states approach each other. Using (4.3.6), we find the modulus of the coefficient of the diagonal terms is

$$
\exp\left\{-\tfrac{1}{2}|\alpha_1-\alpha_2|^2\left(1-e^{-2\gamma t}\right)\right\}.
\tag{7.3.12}
$$

If α_1 and α_2 are macroscopic, then $|\alpha_1|^2$ and $|\alpha_1|^2$ are approximately the number of quanta, so that $|\alpha_1-\alpha_2|^2$ is enormous; thus, for γt the slightest bit different from zero, (7.3.12) is essentially zero—the macroscopic superposition has been destroyed.

However, since they are coherent states, these states are only almost diagonal. This is because of the rotating wave interaction which gives rise to the master equation is essentially a damping which is coupled to *two* operators, $\tfrac{1}{2}(a+a^\dagger)$, and $\tfrac{1}{2}i(a-a^\dagger)$, and the situation is thus similar to that in Sect. 7.3.1b.

7.3.3   Application to a Model of Quantum Measurement

A simple model of a quantum measurement was introduced by *Walls* et al. [7.3]. The basic idea is based on an exposition by *Zurek* [7.4].

A naive view of measurement is given by considering a system which is to be measured, coupled by some Hamiltonian to a meter, which is some other quantum system. With the evolution of time, the states of the meter and the system become correlated, thus knowledge of the state of the meter will give information about the system.

An ideal measurement should not disturb the system being measured—this is of course impossible. The von Neumann hypothesis suggests that the degree of disturbance should be such as not to disturb the variable being measured; in other words, we would expect any reasonable measurement to be such that the variable being measured would commute with the total Hamiltonian.

a) An Optical Example: Let us consider a system and meter, described by harmonic oscillator variables, a, b, with

$$
\begin{aligned}
H &\equiv H_{\text{sys}} + H_{\text{M}} + H_{\text{Int}} \\
&\equiv \hbar \omega a^\dagger a + \hbar \omega b^\dagger b - i \hbar \, a^\dagger a (b \epsilon^* - b^\dagger \epsilon).
\end{aligned}
\tag{7.3.13}
$$

This could describe two modes of the electromagnetic field, coupled by a classical pump field ϵ, which induces a four wave mixing in some intracavity non-linear medium.

Notice that H commutes with $a^\dagger a$, the system photon number, so that $a^\dagger a$ is a candidate for the operator being measured. We also assume that the b field, which carries out the measurement, is itself damped.

This is done by coupling meter variable b to an environment, represented by a heat bath. We can then write, in an interaction picture, QSDE's for a and b, including the damping of the b field,

$$
\begin{aligned}
da &= a(b\epsilon^* - b^\dagger \epsilon)\,dt \\
db &= \left(\epsilon a^\dagger a - \gamma b\right) dt + \sqrt{2\gamma}\, dB(t).
\end{aligned}
\tag{7.3.14}
$$

These equations can be considered in either Ito or Stratonovich form, since the noise term is independent of a and b. Notice that the equation implies that $a^\dagger a$ is a constant of the motion, and to emphasise this we define $N \equiv a^\dagger a$.

b) Solution of the Equations: The equations can be solved exactly, firstly the b equation

$$
b(t) = \frac{\epsilon N}{\gamma} \left(1 - e^{-\gamma t}\right) + b(0) e^{-\gamma t} + \sqrt{2\gamma} \int_0^t e^{-\gamma(t-t')} dB(t').
\tag{7.3.15}
$$

This solution represents the measurement process. One can see that if ϵ is sufficiently large, the first term can have a macroscopic scale, and is proportional to N, the operator to be measured. Thus the measurement process has gain—this will be seen to be an essential feature of the theory.

We now substitute this solution into the first equation;

$$\frac{da}{dt} = aX(t) \tag{7.3.16}$$

with

$$X(t) = \left[\epsilon^* b(0) - \epsilon b^\dagger(0)\right] e^{-\gamma t} + \sqrt{2\gamma} \int_0^t e^{-\gamma(t-t')} \left[\epsilon^* dB(t') - \epsilon dB^\dagger(t')\right]. \tag{7.3.17}$$

Notice that $\left[X(t), X(t')\right] = 0$, so that

$$\begin{aligned}
a(t) &= a(0) \exp\left[\int_0^t dt' X(t')\right] \\
&= \exp\left[\int_0^t dt' X(t')\right] a(0).
\end{aligned} \tag{7.3.18}$$

Notice that $X(t)$ is in fact completely independent of N; the dependence given by the solution (7.3.15) for $b(t)$ cancels. By construction $X(t)$ is quantum Gaussian, so that

$$\left\langle \exp\left[\int_0^t dt' X(t')\right] \right\rangle = \exp\left\{ \frac{1}{2} \left\langle \left[\int_0^t dt' X(t')\right]^2 \right\rangle \right\} \tag{7.3.19}$$

$$= \exp\left\{ \frac{1}{2} \left\langle \int_0^t dt' \int_0^t ds\, X(t') X(s) \right\rangle \right\}. \tag{7.3.20}$$

Exercise. Assume that the bath is at zero temperature, and compute that, in the steady state,

$$\langle X(t') X(t) \rangle = -|\epsilon|^2 e^{-\gamma|t-t'|}. \tag{7.3.21}$$

c) **Phase Destruction:**   We can now see that the phase of $a(t)$ is randomised through the solution (7.3.18), where $X(t)$ behaves like a phase noise, which arises as a result of the meter field $b(t)$. This can be called the "back action" of the meter.

d) **Quantum Characteristic Function:**   Let us compute the joint quantum characteristic function of the system, assuming that it was initially in a state

$$\rho(0) = \sum p_{nm} \left(|n\rangle\langle m|\right)_{\text{sys}} \otimes \left(|0\rangle\langle 0|\right)_{\text{meter}}. \tag{7.3.22}$$

This will involve terms like

$$\langle 0|\langle n| \exp\left[\lambda b^\dagger(t)\right] \exp\left[\mu a^\dagger(t)\right] \exp\left[-\mu^* a(t)\right] \exp\left[-\lambda^* b(t)\right] |m\rangle|0\rangle. \tag{7.3.23}$$

We now use the solution (7.3.15) to replace N in $b, b^\dagger$, with m or n as appropriate, and bring the remainder outside the $\langle n| \quad |m\rangle$ mean; thus (7.3.23) becomes

$$\langle 0| \exp\left[\lambda b_n^\dagger(t)\right] \langle n| \exp\left[\mu a^\dagger(t)\right] \exp\left[-\mu^* a(t)\right] |m\rangle \exp\left[-\lambda^* b_m(t)\right] |0\rangle. \tag{7.3.24}$$

Now :

i)   Write

$$|n,t\rangle = \frac{a^\dagger(t)^n}{\sqrt{n!}} |0\rangle = \exp\left[-n \int_0^t dt' X(t')\right] |n\rangle \tag{7.3.25}$$

and note that $X(t)$ is an operator independent of a.

ii)   Note

$$\exp\left[-\lambda b_m(t)\right] |0\rangle = \exp\left[-\frac{\lambda\epsilon m}{\gamma}\left(1 - e^{-\gamma t}\right)\right] |0\rangle \tag{7.3.26}$$

and similarly for n, and write

$$(7.3.24) = \langle 0| \exp\left[(n-m) \int_0^t dt' X(t')\right] |0\rangle$$

$$\times \exp\left\{-\frac{\lambda^*\epsilon m}{\gamma}\left(1 - e^{-\gamma t}\right)\right\} \times \exp\left\{-\frac{\lambda\epsilon^* n}{\gamma}\left(1 - e^{-\gamma t}\right)\right\}$$

$$\times \chi_{n,m}(\mu, \mu^*). \tag{7.3.27}$$

Here, $\chi_{nm}(\mu, \mu^*)$ is the quantum characteristic function for $|m,t\rangle\langle n,t|$, in terms of the operators $a(t), a^\dagger(t)$, which is thus exactly the same as that for $|m\rangle\langle n|$, since $a(t)$ and $a(0)$ are unitarily related.

Finally, the average can be evaluated using (7.3.21). We find

$$\chi(\lambda, \lambda^*, \mu, \mu^*) = \chi_{n,m}(\mu, \mu^*) \exp\left\{\frac{|\epsilon|^2 (n-m)^2}{\gamma^2}\left(1 - \gamma t - e^{-\gamma t}\right)\right\}$$

$$\times \exp\left\{-\frac{\lambda^*\epsilon m}{\gamma}(1 - e^{-\gamma t}) - \frac{\lambda\epsilon^* n}{\gamma}(1 - e^{-\gamma t})\right\}. \tag{7.3.28}$$

e) Density Operator Solution: This characteristic function corresponds to a state of the form

$$|\alpha_m(t)\rangle\langle \alpha_n(t)| \otimes |n\rangle\langle m| \times R(n,m,t) \tag{7.3.29}$$

where we can compute that

$$|R(n,m,t)| = \exp\left\{-\frac{|\epsilon|^2 (n-m)^2}{2\gamma^2}\left[\gamma t - (1 - e^{-\gamma t}) - (1 - e^{-\gamma t})^2\right]\right\}. \tag{7.3.30}$$

Here the $(1 - e^{-\gamma t})^2$ term comes from the fact that the normalisation of $|\alpha\rangle\langle\beta|$ is not 1, and

$$\alpha_m(t) = \frac{\epsilon m}{\gamma}(1 - e^{-\gamma t}). \tag{7.3.31}$$

f) Interpretation—The Projection Postulate: We see that the measurement process drives the meter into a coherent state $|\alpha_m(t)\rangle$, whose mean is proportional to m. However, if this value is macroscopic, this will also mean that $|\epsilon|^2(n-m)^2/2\gamma^2$ is also macroscopic, and thus for any γt significantly different from zero,

$$|R(n, m, t)| \to \delta_{n,m}. \tag{7.3.32}$$

Thus, the state after the measurement is essentially fully diagonal,

$$\rho \to \sum_{n,n} |n\rangle\langle n| \otimes |\alpha_n(t)\rangle\langle\alpha_n(t)|. \tag{7.3.33}$$

This is the von Neumann projection postulate; the collapse of the wavefunction. Let us suppose the values of n are such that we can easily distinguish the macroscopic states $|\alpha_n(t)\rangle$ and $|\alpha_m(t)\rangle$ for $m \neq n$. Then if we see that the meter is in a state corresponding to $\alpha_n(t)$, we *know* that the meter-system state is

$$|n\rangle\langle n| \otimes |\alpha_n(t)\rangle\langle\alpha_n(t)|. \tag{7.3.34}$$

The calculation we have done shows that the coherences will be destroyed by the action of the macroscopic meter; the problem is then purely probabilistic. No prediction is given for what value is measured, but the diagonalisation of the density operator ensures that if we measure, then we know that the system must be in the state $|n\rangle\langle n|$.

g) Concluding Remarks: Notice that the reason that the density operator takes on this "classical" form is ultimately the commutation relations of $dB(t)$ and $dB^\dagger(t)$, and the quantum noise driving them. The phase destruction factor comes from (7.3.21), and the fact that this is non-zero even though the state of $dB(t)$ and $dB^\dagger(t)$ is the vacuum. (If additional thermal noise is included, the diagonalisation will be faster; but the result a thermal state).

The phase destruction arises from the fact that the coupling to the meter is a Hamiltonian interaction—the same term that allows the system to influence the meter makes the meter influence the system. Because of the coupling chosen, the variable being measured is not itself affected, but variables corresponding to phases are, and the result is the collapse of the wavefunction.

Notice also that it is important for a fast collapse that $\epsilon n/\gamma$ be extremely large, i.e., macroscopic. We thus conclude that quantum mechanics has within it a mechanism which ensures indeterminacy at a microscopic level, but this very mechanism also acts to conceal the effects of quantum mechanical superposition in macroscopic systems.

8. Photon Counting

The most systematically studied field of quantum noise occurs in the field of optics, where the quantum aspect of light is an essential but not always dominant component. In spite of the fact that many optical phenomena like interference and diffraction are well explained by classical physics, it was not until the invention of the laser that it was possible to produce an optical electromagnetic field with anything like the pure wave properties of radio waves. Typical optical sources were discharge lamps, where the rapid thermal agitation of the radiating atoms produced a randomised field which concealed many features—both classical and quantum. For most purposes such an electromagnetic field could be thought of merely as a thermally disturbed classical field.

The laser changed this situation dramatically. Initially, it was not realised that laser light was anything other than a very intense version of the kind of light one could obtain from a thermal source. However it soon became clear that the classical features of laser light were closer to those of radio waves—that is, it was rather like a pure wave field. Pioneering work in the 1960s [8.1] showed that laser light also had a pure quantum aspect which had never before been considered.

To exhibit these new aspects required a new look at quantum electrodynamics, which had until Glauber's seminal articles of 1963 [8.2] developed in the direction of a relativistic theory formulated in a form adapted more to the study of high energy particle physics than the description of optical fields. Glauber showed that in the study of optical fields the many particle aspect of quantum electrodynamics, and the concept of coherence as an aspect of photon counting and electromagnetic field correlations had a rich and varied structure. Indeed it is possible that there are still quantum aspects of optical fields that have not been formulated or studied.

This chapter aims to give a modern description of the quantised electromagnetic field in the optical regime. It starts with a brief resumé of the quantisation of Maxwell's equations, and then moves into aspects of photon counting and optical correlation functions. It is then completed with a formulation of the concept of the photon, and the formulae used to describe photon counting.

8.1 Quantisation of the Electromagnetic Field

The electromagnetic field represents the original conception of a quantised system, dating from *Planck* [8.3] and *Einstein* [8.4]. The formal relationship between field quantisation and harmonic oscillators is well known, and this section will serve mainly to establish notation and provide a reference.

8.1.1 Maxwell's Equations

A convenient starting point for the quantisation of the electromagnetic field is the classical field equations. The free electromagnetic field obeys the source free Maxwell equations (in SI units),

$$
\begin{aligned}
\nabla \cdot \mathbf{B} &= 0 & (\,a\,) \\
\nabla \times \mathbf{E} &= -\frac{\partial \mathbf{B}}{\partial t} & (\,b\,) \\
\nabla \cdot \mathbf{D} &= 0 & (\,c\,) \\
\nabla \times \mathbf{H} &= \frac{\partial \mathbf{D}}{\partial t} & (\,d\,)
\end{aligned}
\tag{8.1.1}
$$

where $\mathbf{B} = \mu_0 \mathbf{H}$, $\mathbf{D} = \epsilon_0 \mathbf{E}$, where μ_0, ϵ_0 are the magnetic permeability and electric permittivity of free space and $\mu_0 \epsilon_0 = c^{-2}$. The electromagnetic field is best represented in terms of the vector and scalar potentials $\mathbf{A}$ and ϕ, in terms of which

$$
\begin{aligned}
\mathbf{B} &= \nabla \times \mathbf{A} \\
\mathbf{E} &= -\nabla \phi - \frac{\partial \mathbf{A}}{\partial t}.
\end{aligned}
\tag{8.1.2}
$$

However there is no unique set of $\mathbf{A}$ and ϕ which specify a given $\mathbf{B}$ and $\mathbf{E}$ since a *gauge transformation*

$$
\begin{aligned}
\mathbf{A}' &= \mathbf{A} + \nabla \chi \\
\phi' &= \phi - \frac{\partial \chi}{\partial t}
\end{aligned}
\tag{8.1.3}
$$

obviously does not change the measurable fields $\mathbf{B}$ and $\mathbf{E}$. For the purposes of optics the *Coulomb gauge* is convenient—this is defined by the choice $\phi = 0$, $\nabla \cdot \mathbf{A} = 0$. Thus, we find that we can write

$$
\begin{aligned}
\mathbf{B} &= \nabla \times \mathbf{A}, & (a) \\
\mathbf{E} &= -\frac{\partial \mathbf{A}}{\partial t} & (b)
\end{aligned}
\tag{8.1.4}
$$

with the transversality or Coulomb gauge condition

$$
\nabla \cdot \mathbf{A} = 0.
\tag{8.1.5}
$$

Substituting (8.1.4) in (8.1.1b) we find that $\mathbf{A}(\mathbf{r}, t)$ satisfies the wave equation

$$
\nabla^2 \mathbf{A}(\mathbf{r}, t) = \frac{1}{c^2} \frac{\partial^2 \mathbf{A}(\mathbf{r}, t)}{\partial t^2}.
\tag{8.1.6}
$$

8.1.2   Expansion in Mode Functions

We separate the vector potential into two complex terms

$$\mathbf{A}(\mathbf{r},t) = \mathbf{A}^{(+)}(\mathbf{r},t) + \mathbf{A}^{(-)}(\mathbf{r},t). \tag{8.1.7}$$

Here $\mathbf{A}^{(+)}(\mathbf{r},t)$ contains only Fourier components with positive frequency, i.e., only terms which vary as $e^{-i\omega t}$ for $\omega > 0$, and $\mathbf{A}^{(-)}(\mathbf{r},t)$ contains amplitudes which vary as $e^{i\omega t}$. We take $\mathbf{A}$ to be real (or quantum mechanically, Hermitian) so $\mathbf{A}^{(-)}(\mathbf{r},t) = \{\mathbf{A}^{(+)}(\mathbf{r},t)\}^*$. It is more convenient to deal with a discrete set of variables rather than the whole continuum. We shall therefore describe the field restricted to a certain volume of space and expand the vector potential in terms of a discrete set of orthogonal mode functions

$$\mathbf{A}^{(+)}(\mathbf{r},t) = \sum_k c_k \mathbf{u}_k(\mathbf{r}) e^{-i\omega_k t} \tag{8.1.8}$$

where the Fourier coefficients c_k are constant for a free field. The set of vector mode functions $\mathbf{u}_k(\mathbf{r})$ which correspond to frequency ω_k will satisfy the wave equation

$$\left(\nabla^2 + \frac{\omega_k^2}{c^2} \right) \mathbf{u}_k(\mathbf{r}) = 0. \tag{8.1.9}$$

The mode functions are also required to satisfy the transversality condition which arises from (8.1.5)

$$\nabla \cdot \mathbf{u}_k(\mathbf{r}) = 0. \tag{8.1.10}$$

They also form an orthonormal set

$$\int_V \mathbf{u}_k^*(\mathbf{r}) \cdot \mathbf{u}_{k'}(\mathbf{r}) d^3\mathbf{r} = \delta_{kk'} \tag{8.1.11}$$

which is complete within the chosen volume. The mode functions depend on the boundary conditions of the physical volume under consideration, e.g., periodic boundary conditions corresponding to travelling wave modes or conditions appropriate to reflecting walls which lead to standing waves. Plane wave mode functions appropriate to a cubical volume of side L with periodic boundary conditions may be written as

$$\mathbf{u}_k(\mathbf{r}) = L^{-\frac{3}{2}} \hat{\mathbf{e}}^{(\lambda)} \exp(i\mathbf{k} \cdot \mathbf{r}) \tag{8.1.12}$$

where $\hat{\mathbf{e}}^{(\lambda)}$ are the unit polarisation vectors satisfying

$$\mathbf{k} \cdot \hat{\mathbf{e}}^{(\lambda)} = 0, \qquad \hat{\mathbf{e}}^{(\lambda)} \cdot \hat{\mathbf{e}}^{(\lambda')} = \delta_{\lambda\lambda'}. \tag{8.1.13}$$

The mode index k describes several discrete variables, the polarisation index ($\lambda = 1, 2$) and the three Cartesian components of the propagation vector $\mathbf{k}$. Each component of the wave vector $\mathbf{k}$ takes the values

$$
\begin{aligned}
k_x &= \frac{2\pi n_x}{L} \\[4pt]
k_y &= \frac{2\pi n_y}{L} \\[4pt]
k_z &= \frac{2\pi n_z}{L}
\end{aligned}
$$

$$n_x, n_y, n_z = 0, \pm 1, \pm 2 \ldots$$

(8.1.14)

and as a consequence of the equation (8.1.9)

$$\omega_k = c|\mathbf{k}|.$$

(8.1.15)

The polarisation vector $\hat{\mathbf{e}}^{(\lambda)}$ is required to be perpendicular to $\mathbf{k}$ by the transversality condition (8.1.10). The vector potential may now be written in the form

$$\mathbf{A}(\mathbf{r}, t) = \sum_k \left(\frac{\hbar}{2\omega_k \epsilon_0} \right)^{\frac{1}{2}} \left(a_k \mathbf{u}_k(\mathbf{r}) e^{-i\omega_k t} + a_k^\dagger \mathbf{u}_k^*(\mathbf{r}) e^{i\omega_k t} \right).$$

(8.1.16)

The corresponding form for the electric field is

$$\mathbf{E}(\mathbf{r}, t) = i \sum_k \left(\frac{\hbar \omega_k}{2\epsilon_0} \right)^{\frac{1}{2}} \left(a_k \mathbf{u}_k(\mathbf{r}) e^{-i\omega_k t} - a_k^\dagger \mathbf{u}_k^*(\mathbf{r}) e^{i\omega_k t} \right).$$

(8.1.17)

The normalisation factors have been chosen such that the amplitudes a_k and $a_k^\dagger$ are dimensionless.

8.1.3   Quantisation by Commutation Relations

The Hamiltonian for the electromagnetic field is given by

$$H = \int \left(\frac{\epsilon_0 \mathbf{E}^2}{2} + \frac{\mathbf{B}^2}{2\mu_0} \right) d^3\mathbf{r}.$$

(8.1.18)

By substituting for $\mathbf{E}$ and $\mathbf{B}$ using the expression (8.1.17) for $\mathbf{E}$ and a similar one for $\mathbf{B}$, and by making use of the conditions (8.1.9,11), this Hamiltonian can be reduced to the form

$$H = \tfrac{1}{2} \sum_k \hbar \omega_k \left(a_k^\dagger a_k + a_k a_k^\dagger \right).$$

(8.1.19)

This is of course exactly the same as the Hamiltonian for an assembly of independent harmonic oscillators. Quantisation is accomplished by assuming the same commutation relations for $a_k, a_k^\dagger$ as would be found from quantising such an assembly, namely

$$[a_k, a_{k'}] = [a_k^\dagger, a_{k'}^\dagger] = 0, \qquad [a_k, a_{k'}^\dagger] = \delta_{kk'}.$$

(8.1.20)

The dynamical states of the electromagnetic field may then be described by assigning an appropriate quantum state to each of the modes—these modes may be assigned and described independently, and all the apparatus developed for harmonic oscillators applied to them.

8.1.4   Quantisation in an Infinite Volume

While quantisation in a finite box is perfectly legitimate, provided the box is very much larger than the dimensions under consideration, its use is nevertheless slightly deceptive, for two reasons.

i)   In quantum optics the quantisation volume is sometimes chosen finite, for physical reasons, since many optical experiments take place either in a cavity, or between highly reflecting mirrors. In such a situation, only a discrete number of field modes can occur, and the description by quantisation of the discrete cavity modes is appropriate. Of course if we want to describe the rest of the universe, we must choose a box which is very large compared to the cavity. It is conceptually simpler to think of the outside world as being actually infinite, than to have two quantisation volumes.

ii)   In a box irreversibility does not occur, since quanta cannot be lost forever— they bounce back from the walls of the box, or return from the other side if periodic boundary conditions are imposed. Of course for a large enough box this may take a long time, but the pure vastness of infinite space into which light can disappear without trace yields the concept of irreversibility directly.

The transition to an infinite volume decomposition is achieved by letting the quantisation volume become infinite. We note that as n_x, n_y, n_z increase by 1,

$$k_x \to k_x + \frac{2\pi}{L}, \text{etc.,} \tag{8.1.21}$$

so that

$$\frac{1}{L^3} \sum_k \to \frac{1}{(2\pi)^3} \sum_\lambda \int d^3\mathbf{k}. \tag{8.1.22}$$

If we also let

$$a_k \to \left(\frac{2\pi}{L}\right)^{\frac{3}{2}} a^\lambda(\mathbf{k}) \tag{8.1.23}$$

we find

$$\left[a^\lambda(\mathbf{k}), a^{\lambda'}(\mathbf{k}')^\dagger\right] = \delta^3(\mathbf{k} - \mathbf{k}')\delta_{\lambda\lambda'} \tag{8.1.24}$$

and

$$\mathbf{A}^{(+)}(\mathbf{r}, t) = \frac{1}{(2\pi)^{\frac{3}{2}}} \int d^3\mathbf{k} \left(\frac{\hbar}{2\omega_k \epsilon_0}\right)^{\frac{1}{2}} \sum_\lambda \hat{\mathbf{e}}^\lambda \exp(i\mathbf{k} \cdot \mathbf{r} - i\omega_k t)a^\lambda(\mathbf{k}) \tag{8.1.25}$$

with corresponding expressions for $\mathbf{A}(\mathbf{r}, t), \mathbf{E}(\mathbf{r}, t)$, etc.

8.1.5 Optical Electromagnetic Fields

It is very rare in optical situations to have an electromagnetic field whose bandwidth is anything more than an infinitesimal fraction of the frequency being used. For example, a typical optical frequency is 6×10^{14} Hz , whereas the light from an atomic transition never has a bandwidth of more than one thousandth of this. This means that there will not be much difference between quantities calculated using $\mathbf{A}(\mathbf{r}, t)$ or $\mathbf{E}(\mathbf{r}, t)$, and quantities calculated using *approximate* expressions in which the $\sqrt{\omega_{\mathbf{k}}}$ factors which occur in the mode representations (8.1.16,17) are replaced by $\sqrt{\Omega}$, where Ω is the frequency being used. This means that we could write, approximately,

$$
\begin{aligned}
\mathbf{E}^{(+)}(\mathbf{r}, t) &\approx i\Omega \mathbf{A}^{(+)}(\mathbf{r}, t) \\
&= \frac{i}{(2\pi)^{\frac{3}{2}}} \left(\frac{\hbar\Omega}{2\epsilon_0}\right)^{\frac{1}{2}} \int d^3\mathbf{k} \sum_\lambda \hat{\mathbf{e}}^\lambda e^{i(\mathbf{k}\cdot\mathbf{r}-\omega_k t)} a^\lambda(\mathbf{k}).
\end{aligned}
$$

$$(8.1.26)$$

The independent variables in this approximation are now the positive and negative frequency parts of $\mathbf{E}(\mathbf{r}, t)$, and these satisfy the equal times commutation relations,

$$
\left[E_i^{(+)}(\mathbf{r}, t), E_j^{(-)}(\mathbf{r}', t)\right] = \frac{\hbar\Omega}{2\epsilon_0} \delta_{ij}^T(\mathbf{r} - \mathbf{r}'). \tag{8.1.27}
$$

The quantity $\delta_{ij}^T(\mathbf{r} - \mathbf{r}')$ is called the *transverse delta function* and is defined by

$$
\delta_{ij}^T(\mathbf{r} - \mathbf{r}') = \left(\delta_{ij} - \frac{\partial_i \partial_j}{\nabla^2}\right) \delta(\mathbf{r} - \mathbf{r}'). \tag{8.1.28}
$$

It arises in the commutator because of the identity

$$
\sum_\lambda \hat{e}_i^{(\lambda)}(\mathbf{k}) \hat{e}_j^{(\lambda)}(\mathbf{k}) = \delta_{ij} - k_i k_j / \mathbf{k}^2, \tag{8.1.29}
$$

which follows from the transversality of the electromagnetic field. By construction, it follows that

$$
\partial_i \delta_{ij}^T(\mathbf{r} - \mathbf{r}') = 0 \tag{8.1.30}
$$

which means that the commutation relation is then consistent with $\nabla \cdot \mathbf{E} = 0$.

The Hamiltonian is now written as in (8.1.19), and yields the equation of motion

$$
\dot{\mathbf{E}}^{(+)}(\mathbf{r}, t) = \frac{i}{\hbar} \left[H, \mathbf{E}^{(+)}(\mathbf{r}, t)\right] = -ic\sqrt{-\nabla^2} \, \mathbf{E}^{(+)}(\mathbf{r}, t). \tag{8.1.31}
$$

The $\sqrt{-\nabla^2}$ is to be interpreted as a factor $|\mathbf{k}|$ in the Fourier transform. This means that the equation (8.1.31) has only positive frequency solutions, as should be the case.

8.1.6   The Photon

From almost the very beginning of quantum theory, the idea of quantisation has been completely bound up with the idea that light consists of discrete "bundles" of energy $h\nu$, called photons. Nevertheless, in the days of the "old quantum theory" almost all physicists had great difficulty in reconciling the idea of the photon with Maxwell's electromagnetic field equations. As an exact concept, the idea of a photon with energy $h\nu$ does present difficulties, since an exact frequency ν can only be specified in a wave field which lasts for an infinite time—thus practical photons, which appear to be moderately well localised in space and time, cannot have a well defined frequency ν.

Furthermore, looking at the Hamiltonian formula (8.1.19), it is obvious that the energy eigenvalues of the system have the form

$$\sum_k \hbar \omega_k (n_k + \tfrac{1}{2}) \tag{8.1.32}$$

which says that the only observable energies are equivalent to those of a number of photons of energies $\hbar \omega_k = h\nu_k$. However, when we consider a genuinely infinite system this simple interpretation is no longer possible, and one must talk about ranges of energies etc., which, while quite straightforward mathematically, are not very enlightening. And even with a finite system, the quantisation volume must be so large that the frequency of any practical photon is only defined up to an accuracy which would encompass an enormous number of very finely spaced possible values of $\hbar \omega_k$.

The basic conceptual stumbling block lies in the formula $E = h\nu$, the foundation stone of quantum theory. It is perfectly straightforward to define a photon which does not have a definite energy, and we can show that all states of the electromagnetic field are linear combinations of states with a finite number of photons.

Let us take a complete set of complex solutions of Maxwell's equations; the vector potential $\pmb{A}^\alpha(\mathbf{r}, t)$, and the electric field $\pmb{\mathcal{E}}^\alpha(\mathbf{r}, t)$, which are assumed to have only positive frequency. Noting that these functions must be transverse, we assume the normalisation and completeness properties

$$\sum_\alpha \pmb{A}_i^\alpha(\mathbf{r}, t) \pmb{\mathcal{E}}_j^{\alpha *}(\mathbf{r}, t) \quad = \frac{i\hbar}{2\epsilon_0} \delta_{ij}^T(\mathbf{r} - \mathbf{r}') \tag{8.1.33}$$

$$\int d^3\mathbf{r}\, \pmb{A}^\alpha(\mathbf{r}, t) \cdot \pmb{\mathcal{E}}^{\beta *}(\mathbf{r}, t) = \frac{2\epsilon_0}{i\hbar} \delta_{\alpha\beta}. \tag{8.1.34}$$

Because $\pmb{\mathcal{E}}^\alpha(\mathbf{r}, t)$, $\pmb{A}^\alpha(\mathbf{r}, t)$ are solutions of Maxwell's equations, we can show that these conditions are preserved as time evolves; that is, the left hand sides are guaranteed to be time independent.

The electromagnetic field operators are then able to be expanded as

$$\mathbf{A}^{(+)}(\mathbf{r}, t) \quad = \sum_\alpha a_\alpha \pmb{A}^{\alpha *}(\mathbf{r}, t) \tag{8.1.35}$$

$$\mathbf{E}^{(+)}(\mathbf{r}, t) \quad = \sum_\alpha a_\alpha \pmb{\mathcal{E}}^{\alpha *}(\mathbf{r}, t) \tag{8.1.36}$$

and the commutation relations between $\mathbf{E}$ and $\mathbf{A}$, as required by the expansions (8.1.16,17) are

$$[A_i(\mathbf{r}, t), E_j(\mathbf{r}', t)] = -\frac{i\hbar}{\epsilon_0}\delta(\mathbf{r} - \mathbf{r}'). \tag{8.1.37}$$

From these commutation relations and the orthogonality and completeness relations, (8.1.33,34), we can deduce the canonical commutation relation

$$[a_\alpha, a_\beta^\dagger] = \delta_{\alpha\beta}. \tag{8.1.38}$$

These operators a_α, $a_\alpha^\dagger$ are thus destruction and creation operators for the *mode* whose mode functions are $\mathbf{A}^\alpha$, $\mathbf{\mathcal{E}}^\alpha$. The states of the electromagnetic field can thus be written in terms of n photon states such as $a_{\alpha_1}^\dagger a_{\alpha_2}^\dagger \dots a_{\alpha_n}^\dagger |0\rangle$. This will be a Heisenberg picture state, with one photon in each of the modes $\alpha_1, \alpha_2, \dots, \alpha_n$, with electromagnetic fields $\mathbf{\mathcal{E}}^{\alpha_i *}(\mathbf{r}, t)$, $\mathbf{\mathcal{E}}^{\alpha_i *}(\mathbf{r}, t)$ etc., which can describe a wide variety of possible situations. In particular, it is possible to speak of a one photon wavepacket when the mode functions correspond to those of a wavepacket.

Thus the concept of a photon is not limited to that of a quantum with energy $h\nu$. Any solution of Maxwell's equations can be chosen as a mode function, after being appropriately normalised as in (8.1.33). The modes orthogonal to this mode can then be constructed if required. The modes corresponding to definite frequency are of course those which diagonalise the Hamiltonian, in which the energy eigenvalues take on the form $\sum \hbar \omega_k [n(k) + \frac{1}{2}]$, but this is not possible in a general kind of mode.

8.1.7   Beams of Light

It is also of interest to consider a beam propagating in one dimension, say along the z axis. If the beam is quite wide compared to the wavelengths of interest, we can consider

$$\bar{E}^{(+)}(z, t) \equiv \frac{2\pi}{A} \int_A dx \, dy \, E^{(+)}(\mathbf{r}, t). \tag{8.1.39}$$

We can show that, if A is very large

$$\bar{E}^{(+)}(z, t) = \frac{i}{\sqrt{2\pi}} \left(\frac{\hbar\Omega}{2\epsilon_0}\right)^{\frac{1}{2}} \int_{-\infty}^{\infty} dq \sum_\lambda \hat{e}^\lambda \, e^{i(qz - \omega_q t)} \, \bar{a}^\lambda(q) \tag{8.1.40}$$

where

$$\bar{a}^\lambda(q) \approx \frac{(2\pi)^2}{A} a^\lambda(\{0, 0, q\})$$

and

$$[\bar{a}^\lambda(q), \bar{a}^{\lambda'}(q')^\dagger] = \delta_{\lambda\lambda'}\delta(q - q'). \tag{8.1.41}$$

In this derivation we have used

$$\delta\left(\{0,0,q\}\right) \approx \frac{1}{(2\pi)^3} \int\limits_A dx dy \int\limits_{-\infty}^{\infty} dz e^{iqz} = \delta(q)\, A/(2\pi)^2. \tag{8.1.42}$$

In this situation, $\mathbf{E}(z,t)$ has only x and y components, and

$$\left[\bar{E}_i^{(+)}(z,t), \bar{E}_j^{(-)}(z',t')\right] = -\frac{\delta_{ij}}{2\pi} \left(\frac{\hbar\Omega}{2\epsilon_0}\right) \int\limits_{-\infty}^{\infty} dq e^{iq(z-z')-i|qc|(t-t')}. \tag{8.1.43}$$

If we are dealing with an optical field in which the frequency is well defined, as is the direction of propagation, we can assume that $|\mathbf{q}| \approx \Omega/c$, and thus there will be little error in setting $|q| = \pm q$, which leads to the commutation relation being approximately

$$\left[\bar{E}_i^{(+)}(z,t), \bar{E}_j^{(-)}(z',t')\right] = -\frac{\hbar\Omega\delta_{ij}}{2\epsilon_0}\delta\left(z - z' \pm c(t - t')\right), \tag{8.1.44}$$

where the choice of sign in the delta function depends on the direction of propagation.

8.2   Photodetection and Photon Counting

The process of measuring a light field is an example of a measurement of the second kind, as discussed in Sect. 2.2.4. The normal measurement process is to absorb the light as it impinges on the detector, whether this is a sophisticated electronic detector or the human eye. The end result is that in an ideal measurement, all of the light is absorbed, and thus the measurement destroys the state being measured.

The general properties of ideal photodetection were first stated by *Glauber* [8.2]. It is not necessary to discuss the details of photodetection to get a fairly accurate idea of what is measured. If we use an absorption process, the basic measurement process is the absorption of one photon from the electromagnetic field, with the consequent production of an electron, which is detected with a certain efficiency ϵ, which in practice can have any value up to 100%. The important point is that in practical photodetection each photon gives rise to no more than one electron, and each electron is produced by only one photon. The measurement process is one that can only be satisfactorily studied by the density operator formalism of Sect. 2.2.2. The event being considered is that of detection of a photon in a time dt and an element of volume $d^3\mathbf{r}$. We take, for an initial radiation field density operator $\rho(i)$ the probability of detection in a volume $d^3\mathbf{r}$ in a time interval dt, to be

$$P(\mathbf{r},t)d^3\mathbf{r}\, dt = \text{Tr}\left\{\epsilon\, E^{(+)}(\mathbf{r},t)\rho(i)E^{(-)}(\mathbf{r},t)\right\} d^3\mathbf{r}\, dt. \tag{8.2.1}$$

This is a shorthand notation for

$$P(\mathbf{r},t)d^3\mathbf{r}\,dt = \mathrm{Tr}\left\{\epsilon\sum_j E_j^{(+)}(\mathbf{r},t)\rho(i)E_j^{(-)}(\mathbf{r},t)\right\}d^3\mathbf{r}\,dt \qquad (8.2.2)$$

which is introduced because the writing out of all the explicit $\sum_j$ terms for pairs of positive and negative frequencies would otherwise become very tedious. It will be considered in all correlation functions that such a summation over the appropriate pairs of positive and negative frequency operators is understood.

The density operator after measurement is

$$\rho(f) = \frac{\epsilon E^+(\mathbf{r},t)\rho(i)E^{(-)}(\mathbf{r},t)}{P(\mathbf{r},t)}. \qquad (8.2.3)$$

It is straightforward to follow the reasoning of Sect. 2.3.1 to show that the probability density of detecting photons at times and places $t_1, \mathbf{r}_1, t_2, \mathbf{r}_2, t_3, \mathbf{r}_3, \ldots$ is

$$P(\mathbf{r}_1,t_1; \mathbf{r}_2,t_2; \ldots; \mathbf{r}_n,t_n)$$
$$= \epsilon^n \mathrm{Tr}\left\{E^{(+)}(\mathbf{r}_n,t_n)\ldots E^{(+)}(\mathbf{r}_1,t_1)\rho(i)E^{(-)}(\mathbf{r}_1,t_1)\ldots E^{(-)}(\mathbf{r}_n,t_n)\right\}$$
$$\times d^3\mathbf{r}_n dt_n \ldots d^3\mathbf{r}_1 dt_1. \qquad (8.2.4)$$

8.2.1 The Physical Basis of the Detection Formulae

The forms (8.2.1,3) are based ultimately on time dependent perturbation theory and some idealisations. Using time dependent perturbation theory we would not obtain (8.2.1) exactly, but rather something like

$$P(\mathbf{r},t) = \sum_{l,m} \epsilon(m)|\langle m|E^{(+)}(\mathbf{r},t)|l\rangle|^2 p(l) \qquad (8.2.5)$$

where $p(l)$ is the probability of being in the state $|l\rangle$ at the beginning. The quantity $\epsilon(m)$ represents that part of the probability which arises from the detector; for example, if the detector is a single atom, it will be proportional to the squared modulus of a dipole moment matrix element. If $\epsilon(m)$ is a constant, ϵ, over the range of the optical frequencies of interest (which is usually a very narrow range) then we can use the completeness of the states $|m\rangle$ and, identifying $\rho(i)$ with $\sum_l |l\rangle p(l)\langle l|$, derive (8.2.1).

Similarly, application of time dependent perturbation theory will yield (8.2.3), as a result of a measurement of the first kind on the time evolved joint density operator. Both these formulae are thus perturbative in nature, and apply only in the case of weak fields. If we have strong fields, higher order terms appear, and the detector becomes non-linear. None of the theory in this chapter will then apply.

One might imagine that a restriction to a perturbative formulation would limit the application of this kind of theory to situations in which the photons were detected

only with low efficiency, since high efficiency would seem to require a strong interaction between the field and the detector. But this is not so, for we can make a thick detector, which only detects a fraction of the photons per unit length, but which ultimately detects all. This is in practice how high efficiency detectors are made.

The formula (8.2.5) when evaluated in time dependent perturbation theory is an average over times much longer than the period of an optical cycle, and this is the origin of the occurrence of $E^{(+)}(\mathbf{r}, t)$ only. There are considerable technical problems with any treatment which does not make this approximation. Such a treatment is necessary in situations in which the detection tekes place for only very short times. The fundamental probem is the same as that noted in the treatment of the harmonic oscillator in Sect. 3.4.3. There we assumed that the system was initially in a state which was factorised into a bath and an oscillator term. As soon as the time evolves, a very large transient occurs. In this case, we would find an infinite counting rate would initially occur, unless the rotating wave approximation is made in the initial coupling Hamiltonian. But this initial infinite counting rate is caused by our assumption that the field and the atom are initially independent, which is not the case, because of vacuum fluctuations. If we assume the atom-field system is initially in the true ground state (which may not be easy to determine) no infinite counting rates occur, and in the long time limit the rotating wave approximation result is obtained. A discussion of this problem, and a solution by means of a canonical transformation, is given by *Drummond* [8.5].

8.2.2   Coherence and Correlation Functions

It is convenient to define the nth order correlation functions by

$$
\begin{aligned}
&G^n\left(x_1, \ldots, x_n, x_{n+1}, \ldots, x_{2n}\right) \\
&= \operatorname{Tr}\left\{\rho E^{(-)}\left(x_1\right) \ldots E^{(-)}\left(x_n\right) E^{(+)}\left(x_{n+1}\right) \ldots E^{(+)}\left(x_{2n}\right)\right\}
\end{aligned}
\tag{8.2.6}
$$

where

$$
x = (\mathbf{r}, t).
\tag{8.2.7}
$$

This is more general than that measurable by a strictly local detection process, for which $x_i = x_{2n+1-i}$. The more general kind of correlation function will arise in case the measurement process is non-local, i.e., when we have $\int d^3 r \, \epsilon(\mathbf{r} - \mathbf{r}') E^{(+)}(\mathbf{r}', t)$ instead of $E^{(+)}(\mathbf{r}, t)$.

A link to the classical correlation functions can be provided by using a P-representation for the various modes of the electromagnetic field. We consider a situation in which the radiation field is described by a coherent state in every mode, $|\{\alpha\}\rangle$ so that, for all a_k,

$$
a_k|\{\alpha\}\rangle = \alpha_k|\{\alpha\}\rangle.
\tag{8.2.8}
$$

Since $E^{(+)}(x)$ is a function only of destruction operators, it is also true that

$$
E^{(+)}(x)|\{\alpha\}\rangle = \mathcal{E}^{(+)}(x)|\{\alpha\}\rangle
\tag{8.2.9}
$$

where

$$\mathcal{E}^{(+)}(x) = i \sum_k \left(\frac{\hbar \omega_k}{2\epsilon_0}\right)^{\frac{1}{2}} \alpha_k \mathbf{u}_k(\mathbf{r}) e^{-i\omega_k t} \tag{8.2.10}$$

is the positive frequency part of a classical electromagnetic field $\mathcal{E}(x)$. The density operator corresponding to $|\{\alpha\}\rangle$ is of course $|\{\alpha\}\rangle\langle\{\alpha\}|$, and substituting into (8.2.6), we see that

$$\begin{aligned}
G^{(n)}&(x_1, \ldots, x_n, x_{n+1}, \ldots, x_{2n+1}) \\
&= \mathcal{E}^{(-)}(x_1) \ldots \mathcal{E}^{(-)}(x_n) \mathcal{E}^{(+)}(x_{n+1}) \ldots \mathcal{E}^{(+)}(x_{2n+1}), \tag{8.2.11}
\end{aligned}$$

that is, the correlation functions all factorise. This property of factorisation is intimately connected with the visibility of the interference fringes.
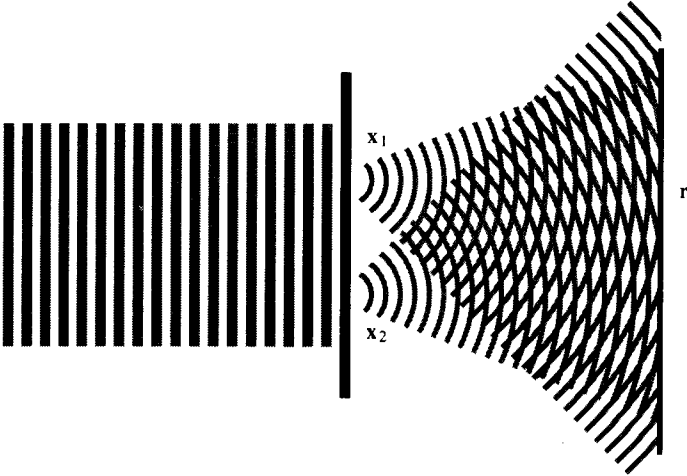
Fig. 8.1   Interference in a double slit experiment

We can imagine combining together the fields at points $\mathbf{x}_1$ and $\mathbf{x}_2$ by the classic two slit experiment, as in Fig.8.1. The field at point $\mathbf{r}$ is given (up to geometrical factors, in the case that $|\mathbf{x}_1 - \mathbf{r}| \simeq |\mathbf{x}_2 - \mathbf{r}|$) by

$$E^{(+)}(\mathbf{r}, t) = E^{(+)}(\mathbf{x}_1, t) e^{ik|\mathbf{r}-\mathbf{x}_1|} + E^{(+)}(\mathbf{x}_2, t) e^{ik|\mathbf{r}-\mathbf{x}_2|} \tag{8.2.12}$$

so that the average rate of photodetection at $\mathbf{r}$, where the beams are combined, is

$$P(\mathbf{r}, t) = \epsilon \langle E^{(-)}(\mathbf{r}, t) E^{(+)}(\mathbf{r}, t) \rangle. \tag{8.2.13}$$

If we assume the distance between the slits is very small compared to the distance to the point of detection $\mathbf{r}$, we can then write approximately

$$\begin{aligned}
|\mathbf{r} - \mathbf{x}_1| &\approx r - \hat{\mathbf{r}} \cdot \mathbf{x}_1 \\
|\mathbf{r} - \mathbf{x}_2| &\approx r - \hat{\mathbf{r}} \cdot \mathbf{x}_2
\end{aligned} \tag{8.2.14}$$

where $\hat{\mathbf{r}} = \mathbf{r}/r$, i.e,. is a unit vector in the direction $\mathbf{r}$. Thus

$$P(\mathbf{r}, t) \approx \epsilon \left[\langle E^{(-)}(\mathbf{x}_1, t) E^{(+)}(\mathbf{x}_1, t) \rangle + \langle E^{(-)}(\mathbf{x}_2, t) E^{(+)}(\mathbf{x}_2, t) \rangle \right.$$

$$\left. + 2\,\mathrm{Re}\left\{ \langle E^{(-)}(\mathbf{x}_2, t) E^{(+)}(\mathbf{x}_1, t) \rangle e^{-ik\hat{\mathbf{r}}\cdot(\mathbf{x}_1 - \mathbf{x}_2)} \right\} \right]. \qquad (8.2.15)$$

We can now consider various cases.

a) Electromagnetic Field in a Coherent State: In this case, as noted above, the correlation functions factorise, and

$$P(\mathbf{r}, t) = \epsilon |\mathcal{E}(\mathbf{x}_1, t) + \mathcal{E}(\mathbf{x}_2, t) e^{-ik\hat{\mathbf{r}}\cdot(\mathbf{x}_1 - \mathbf{x}_2)}|^2. \qquad (8.2.16)$$

If we now assume that $|\mathcal{E}(\mathbf{x}_1, t)|^2 = |\mathcal{E}_2(\mathbf{x}_2, t)|^2 = \mathcal{E}^2$, a constant, so that

$$\mathcal{E}(\mathbf{x}_1, t) = \mathcal{E}\, e^{i\phi(\mathbf{x}_1, t)}, \qquad \mathcal{E}(\mathbf{x}_2, t) = \mathcal{E}\, e^{i\phi(\mathbf{x}_2, t)}, \qquad (8.2.17)$$

then

$$P(\mathbf{r}, t) = 4\epsilon |\mathcal{E}|^2 \cos^2\left(\tfrac{1}{2}\left[k\hat{\mathbf{r}} \cdot (\mathbf{x}_1 - \mathbf{x}_2) + \phi(\mathbf{x}_1, t) - \phi(\mathbf{x}_2, t) \right] \right) \qquad (8.2.18)$$

and it can be seen that as $\hat{\mathbf{r}}$ varies, the $\cos^2$ function will vary between 1 and 0— this gives rise to interference fringes, which can thus be said to be of maximum visibility. This corresponds to the classical concept of *coherence*—the fields at $\mathbf{x}_1$ and $\mathbf{x}_2$ are said to be maximally coherent.

b) Cross Correlation Function Zero: By this we mean that the correlation function $\langle E^{(-)}(\mathbf{x}_2, t) E^{(+)}(\mathbf{x}_1, t) \rangle = 0$, and thus

$$P(\mathbf{r}, t) = \langle E^{(-)}(\mathbf{x}_1, t) E^{(+)}(\mathbf{x}_1, t) \rangle + \langle E^{(-)}(\mathbf{x}_2, t) E^{(+)}(\mathbf{x}_2, t) \rangle, \qquad (8.2.19)$$

that is, the intensity at $\mathbf{r}$ is simply given by the sum of the two intensities, and no fringes are detectable in the photocount distribution.

8.2.3   Normalised Correlation Functions

It is convenient to define normalised correlation functions by

$$g^{(1)}(x_1, x_2) = \frac{G^{(1)}(x_1, x_2)}{\sqrt{G^{(1)}(x_1, x_1) G^{(1)}(x_2, x_2)}} \qquad (8.2.20)$$

$$g^{(2)}(x_1, x_2) = \frac{G^{(2)}(x_1, x_2, x_2, x_1)}{G^{(1)}(x_1, x_1) G^{(1)}(x_2, x_2)}. \qquad (8.2.21)$$

(Here the notation $x \equiv (\mathbf{x}, t)$, introduced in (8.2.7) is again utilised for compactness.) The first order normalised correlation function determines the visibility of the interference fringes seen in the experiment of the previous section. There are inequalities for correlation functions which arise from the Schwartz inequality

$$|\langle AB^\dagger \rangle|^2 \leq \langle A^\dagger A \rangle \langle B^\dagger B \rangle \qquad (8.2.22)$$

which imply that

$$g^{(1)}(x_1, x_2) \leq 1 \tag{8.2.23}$$

$$g^{(2)}(x_1, x_2) \leq 1. \tag{8.2.24}$$

In a coherent state, these become equalities, and we see that if $g^{(1)}(x_1, x_2) = 1$, the interference fringes of (8.2.15) are maximally visible. We define the fringe visibility, $\mathbf{V}$, by

$$\mathbf{V} = \frac{P(\mathbf{r}, t)_{\max} - P(\mathbf{r}, t)_{\min}}{P(\mathbf{r}, t)_{\max} + P(\mathbf{r}, t)_{\min}} \tag{8.2.25}$$

$$= \frac{2 \, |g^{(1)}(\mathbf{x}_2, t; \mathbf{x}_1, t)|}{\sqrt{I_1/I_2} + \sqrt{I_2/I_1}} \tag{8.2.26}$$

where $I_1 = \langle E^{(-)}(\mathbf{x}_1, t) E^{(+)}(\mathbf{x}_1, t) \rangle$ is the intensity at $\mathbf{x}_1$, and I_2 is similarly defined. When $|g^{(1)}| = 1$, the visibility is maximal for given I_1 and I_2, and when $g^{(1)} = 0$, there are no fringes.

8.3 Photon Counting Formulae

Although the formula (8.2.4) for the probability of detecting photons at various times and places is completely general, it does not give directly a practical formula for analysing photodetection. The major problem lies in the effect that the detection process has on the light field being measured. The detection formula requires the knowledge of the field at the places where the photodetection takes place, and in a thick detector, or a highly efficient detector, this field is attenuated by an amount which must be determined.

It is possible to formulate a theory of weak detection, in which the field is not considered to be appreciably attenuated by the detection process, but this is hardly appropriate in practical situations, where the efficiency can be quite high. *Srinivas* and *Davies* [8.6] have developed a method of formulating photodetection which uses measurement theory as its basis, and uses a Markovian view of the processes involved. Using this formulation, it is possible to compute the absorption induced by photodetection, and to develop quite simple formulae, in a way which makes their generality clear.

8.3.1 Development of the Formulae

We want to consider a counting process with a detector taking up a finite volume V, so that the probability of detecting a single photon in the infinitesimal time interval dt is, (in the Schrödinger picture in which $\mathbf{E}$ does not depend on time)

$$P(1, t) dt = \epsilon \, dt \int_V d^3\mathbf{r} \, \mathrm{Tr} \left\{ E^{(-)}(\mathbf{r}) E^{(+)}(\mathbf{r}) \rho(t) \right\}. \tag{8.3.1}$$

We want to derive a formula for the probability of counting a finite number of photons in a *finite* time interval $(t, t + \tau)$. We again use the Schrödinger picture.

The processes under consideration are as follows. We can consider detecting m photons in $(t, t + \tau)$, and introduce the operation $N_{t+\tau, t}(m)$, which represents this process by:

i)   After measuring m photons in this interval, the density operator is

$$\frac{N_{t+\tau, t}(m)\rho}{\text{Tr}\,\{N_{t+\tau, t}(m)\rho\}}.\tag{8.3.2}$$

ii) The probability that m photons are measured is

$$P(t + \tau, t, m) = \text{Tr}\,\{N_{t+\tau, t}(m)\rho\}.\tag{8.3.3}$$

This formula and (8.2.3) tell us that, for example, we may choose

$$N_{t+dt, t}(1)\rho = \epsilon\, dt \int d^3\mathbf{r}\, E^{(+)}(\mathbf{r})\rho(t)E^{(-)}(\mathbf{r})\tag{8.3.4}$$

and this formula will be fundamental to the final result. The exact nature of $N_{t+dt, t}(1)\rho$ however, is not a fundamental assumption, but a description of the kind of detector being used.

a) Markov Assumption: From the very definition of counting, we can write

$$N_{t_1, t_0}(m) = \sum_{m_1 + m_2 = m} N_{t_1, s}(m_1) N_{s, t_0}(m_2) \qquad (t_1 > s > t_0)\tag{8.3.5}$$

which simply says that we can partition the interval (t_1, t_0) anywhere, and count some of the photons in the first interval, and some in the second. The processes of counting in the two intervals are *independent*, as is shown by the fact that the operations in each of the two intervals do not involve the number counted in the other interval. Because of this assumption, the resulting equations will have a Markov property. If the medium which causes the absorption has any kind of "memory" of how many photons it has absorbed, then this need not be true.

b) Semigroup Equations: From this equation we can derive two evolution equations. We define

$$S_{t+\tau, t} = N_{t+\tau, t}(0)\tag{8.3.6}$$

$$T_{t+\tau, t} = \sum_{m=0}^{\infty} N_{t+\tau, t}(m).\tag{8.3.7}$$

Then from (8.3.5) we find that both form semigroups, i.e.,

$$S_{t_1, t_0} = S_{t_1, s} S_{s, t_0}\tag{8.3.8}$$

$$T_{t_1, t_0} = T_{t_1, s} T_{s, t_0}.\tag{8.3.9}$$

The significance of these two operations lies in the fact that S_{t_1, t_0} describes the process of counting no photons, while T_{t_1, t_0} describes the process of counting an

unknown number of photons. More precisely, we see from (8.3.2) that $S_{t_1,t_0}\rho(t_0)$ is the density operator at time t, under the condition that the detector was active during the time interval (t_0, t_1) and *no* photons were counted.

From (8.3.2,3) we can also see that

$$T_{t+\tau,t}\rho(t) = \sum_m \frac{N_{t+\tau,t}(m)\rho(t)}{\text{Tr}\{N_{t+\tau,t}(m)\rho(t)\}} P(t+\tau, t, m) \qquad (8.3.10)$$

which is the average of the density operators for all possible numbers of photons counted. Thus, this gives the field density operator as time evolves and photons are counted, in the situation when we do not know how many photons were counted.

c) Infinitesimal Forms: For sufficiently small τ we know that the probability of counting m photons is proportional to τ^m; this follows from (8.2.5). This means that we can write (since the operation of counting no photons in an infinitesimal time interval must approach the identity)

$$S_{t+\tau,t} = N_{t+\tau,t}(0) = 1 - \tau A(t) \qquad (8.3.11)$$

when τ is infinitesimal, and where $A(t)$ is an operation to be determined. We can also write

$$N_{t+\tau,t}(1) = \tau B(t) \qquad (8.3.12)$$

where in this case we shall assume that

$$B(t)\rho(t) = \epsilon \int d^3\mathbf{r}\, E^{(+)}(\mathbf{r})\rho(t)E^{(-)}(\mathbf{r}), \qquad (8.3.13)$$

and we see that $B(t)$ is actually independent of t—thus we shall omit the t dependence. All other $N_{t+\tau,t}(m)$ vanish faster than τ, and can be determined as follows.

Using the limits (8.3.11,12) we find, from (8.3.5) that

$$\dot{S}_{t_1,t_0} = A(t_1)S_{t_1,t_0} \qquad (8.3.14)$$

$$\dot{T}_{t_1,t_0} = \{A(t_1) + B\}T_{t_1,t_0}. \qquad (8.3.15)$$

We can now determine $A(t_1)$ by consistency arguments.

i)   Equation (8.3.10) requires that $\text{Tr}\{\dot{T}_{t_1,t_0}\rho\} = 0$, so that the trace of ρ will be preserved equal to 1. Thus from (8.3.15) we can see that

$$\text{Tr}\{A(t)\rho\} = -\text{Tr}\{B\rho\} = -\epsilon \int d^3\mathbf{r}\,\text{Tr}\{E^{(+)}(\mathbf{r})\rho E^{(-)}(\mathbf{r})\}. \qquad (8.3.16)$$

This means that we can write

$$A(t)\rho = -\tfrac{1}{2}\epsilon \int d^3\mathbf{r}\,\{E^{(-)}(\mathbf{r})E^{(+)}(\mathbf{r})\rho + \rho E^{(-)}(\mathbf{r})E^{(+)}(\mathbf{r})\}$$
$$- \frac{\mathrm{i}}{\hbar}[H(t), \rho] + \mathcal{L}(t)\rho. \qquad (8.3.17)$$

ii)   Here $H(t)$ is any Hermitian operator, and $\mathcal{L}(t)\rho$ is any operation which preserves ρ positive definite and with trace 1, and does not contain any Hamiltonian term. Because of the arbitrariness of $H(t)$ and $\mathcal{L}(t)$, this is in fact the most general operation whose trace satisfies to (8.3.16). From the results of Sects. 5.1,2, this means that $\mathcal{L}(t)$ is some kind of master equation operator. Thus $\rho(t) \equiv T(t,t_0)\rho(t_0)$ obeys a master equation which can be written

$$\dot{\rho}(t) = \tfrac{1}{2}\epsilon \int d^3\mathbf{r}\,\left\{2E^{(+)}(\mathbf{r})\rho E^{(-)}(\mathbf{r}) - E^{(-)}(\mathbf{r})E^{(+)}(\mathbf{r})\rho\right.$$

$$\left. - \rho E^{(-)}(\mathbf{r})E^{(+)}(\mathbf{r})\right\} - \frac{i}{\hbar}\left[H(t),\rho\right] + \mathcal{L}(t)\rho. \tag{8.3.18}$$

iii) The quantities $H(t)$ and $\mathcal{L}(t)$ are now determined by the physics of the situation. Obviously $H(t)$ determines the free evolution of the field, and must be the electromagnetic Hamiltonian,

$$H_{EM}(t) = \tfrac{1}{2}\int d^3\mathbf{r}\,\{\epsilon_0\mathbf{E}^2 + \mathbf{B}^2/\mu_0\} \tag{8.3.19}$$

unless we want to modify it to take account of a refractive index.

The master equation operator in the first line of (8.3.18) can fairly be described as the evolution operator associated with the detection process, and $\mathcal{L}(t)$ describes any other Markovian evolution that may be necessary. For example, it could contain terms describing such absorption of electromagnetic radiation as does not lead to photoelectrons, or it could contain the terms necessary to take account of the fact that the detector may not be at zero temperature.

If we want to consider an ideal Markovian photodetector, we set $\mathcal{L}(t) = 0$, as we shall mostly from now on.

8.3.2   Master Equation and Quantum Stochastic Differential Equations

Since the field density operator obeys the master equation (8.3.18) we can write a corresponding QSDE for the field operators. Since there is a continuum of damping terms, one at each point in space, there will be a continuum of quantum noise input terms. The required generalisation is quite straightforward, and the QSDE (in the Stratonovich form) takes the form, for any field operator $F(\mathbf{r},t)$,

$$dF(\mathbf{r},t) = -\frac{i}{\hbar}\left[F(\mathbf{r},t),H_{EM}\right]dt$$

$$- \int d^3\mathbf{r}'\left\{\sum_i\left[F(\mathbf{r},t),E_i^{(-)}(\mathbf{r}',t)\right]\left(\frac{\epsilon}{2}E_i^{(+)}(\mathbf{r}',t)dt + \sqrt{\epsilon}\,dB_i(\mathbf{r}',t)\right)\right.$$

$$\left. - \sum_i\left(\frac{\epsilon}{2}E_i^{(-)}(\mathbf{r}',t)dt + \sqrt{\epsilon}\,dB_i^\dagger(\mathbf{r}',t)\right)\left[F(\mathbf{r},t),E_i^{(+)}(\mathbf{r}',t)\right]\right\} \tag{8.3.20}$$

where the spatially distributed quantum white noises are at zero temperature,commute at *different* times, and satisfy

$$\left[dB_i(\mathbf{r}, t), dB_j^\dagger(\mathbf{r}', t) \right] = \delta_{ij}\delta(\mathbf{r} - \mathbf{r}')dt \tag{8.3.21}$$

$$\langle dB_i^\dagger(\mathbf{r}, t)dB_j(\mathbf{r}', t') \rangle = 0, \tag{8.3.22}$$

which corresponds to zero temperature of the bath. Notice that the full form of the photodetector interaction as in (8.2.2) has been used explicitly.

The QSDE (8.3.20) is most easily interpreted when the approximations of Sect. 8.1.7 are made—using the commutation relations in the form (8.1.27), the equation of motion analogous to (8.1.31) is

$$\dot{\mathbf{E}}^{(+)}(\mathbf{r}, t) = -ic\sqrt{-\nabla^2}\,\mathbf{E}^{(+)}(\mathbf{r}, t) - \frac{\hbar\Omega\epsilon}{4\epsilon_0}\mathbf{E}^{(+)}(\mathbf{r}, t) - \frac{\hbar\Omega\sqrt{\epsilon}}{2\epsilon_0}\mathbf{b}(\mathbf{r}, t), \tag{8.3.23}$$

where $\mathbf{b}(\mathbf{r}, t)$ is a zero temperature *transverse* vector quantum white noise which satisfies

$$\left[b_i(\mathbf{r}, t), b_j^\dagger(\mathbf{r}', t') \right] = \delta_{ij}^T(\mathbf{r} - \mathbf{r}')\delta(t - t') \tag{8.3.24}$$

$$\langle b_i^\dagger(\mathbf{r}, t)b_j(\mathbf{r}', t') \rangle = 0 \tag{8.3.25}$$

and is defined by

$$b_i(\mathbf{r}, t)dt = \sum_j \int d^3\mathbf{r}'\delta_{ij}^T(\mathbf{r} - \mathbf{r}')dB_j(\mathbf{r}', t). \tag{8.3.26}$$

The equation (8.3.23) for $\mathbf{E}^{(+)}$ is a wave equation, written in a form that allows positive frequency solutions only, supplemented by a damping term and a quantum noise term, both of which are the direct result of the detection process.

a) Solutions of the Propagation Equation: The propagation equation (8.3.23) is not straightforwardly amenable to intuitive comprehension because of the $\sqrt{-\nabla^2}$. However, it can be understood in two ways. First, notice that because the photodetection medium is contained within the volume V, the equation is only valid within V; outside V we should set ϵ equal to zero. To take this into account, ϵ should be written explicitly as a function of $\mathbf{r}, \epsilon(\mathbf{r})$. For compactness, let us also define

$$\mu = \frac{\hbar\Omega}{4\epsilon_0} \tag{8.3.27}$$

and an operator D by

$$D = -ic\sqrt{-\nabla^2}, \tag{8.3.28}$$

so that the propagation equation can be written formally as

$$\dot{\mathbf{E}}^{(+)}(\mathbf{r}, t) = \left[D - \mu\epsilon(\mathbf{r}) \right]\mathbf{E}^{(+)}(\mathbf{r}, t) - 2\mu\sqrt{\epsilon(\mathbf{r})}\,\mathbf{b}(\mathbf{r}, t). \tag{8.3.29}$$

This has a formal solution in the form

$$E^{(+)}(\mathbf{r}, t) = E_{SIG}^{(+)}(\mathbf{r}, t) + E_{NOISE}^{(+)}(\mathbf{r}, t) \tag{8.3.30}$$

since it is a first order (operator) differential equation in the time variable. The "SIG" term depends on $E^{(+)}(\mathbf{r}, t)$ for various $\mathbf{r}$ and some initial value t_0, and the "NOISE" term depends on $\mathbf{b}(\mathbf{r}, t')$ for $t_0 < t' < t$. This division into dependency on initial condition at t_0 and noise at later times is important in using the photodetection formulae.

As a method of actually constructing a solution (8.3.29) and (8.3.30) are almost useless. However, if we differentiate (8.3.23) again we can derive a simpler equation

$$\ddot{E}^{(+)}(\mathbf{r}, t) - c^2 \nabla^2 E^{(+)}(\mathbf{r}, t) + 2\mu\epsilon \dot{E}^{(+)}(\mathbf{r}, t) + (\mu\epsilon)^2 E^{(+)}(\mathbf{r}, t)$$
$$= -2\mu\epsilon^{\frac{3}{2}} \mathbf{b}(\mathbf{r}, t) - 2(\partial_t - ic\sqrt{-\nabla^2} + \mu\epsilon)\mu\sqrt{\epsilon}\, \mathbf{b}(\mathbf{r}, t), \tag{8.3.31}$$

where ϵ has been assumed constant. To match between the vacuum and the detector, one must match $\mathbf{B}$ and $\mathbf{E}$ as usual—see the next exercise.

b) Absorption as a Result of Detection: The homogeneous part of (8.3.31) clearly exhibits the absorption which arises as a consequence of the detection. Provided we neglect the last term (which should be very small if absorption is weak) in the first line of this equation, it has a form would result classically from an electrical conductivity $2\epsilon_0\mu\epsilon = \hbar\Omega\epsilon/2$. The noise term is very complicated, but can be simplified by assuming that only those frequency and wavelength components of the noise of magnitude Ω and Ω/c respectively will contribute, so that we can reduce equation (8.3.31) approximately to

$$\ddot{E}^{(+)}(\mathbf{r}, t) - c^2 \nabla^2 E^{(+)}(\mathbf{r}, t) + 2\mu\epsilon \dot{E}^{(+)}(\mathbf{r}, t)$$
$$= -2\mu\sqrt{\epsilon}\,(\mu\epsilon + 2i\Omega)\mathbf{b}(\mathbf{r}, t). \tag{8.3.32}$$

This equation is only approximate, and its solution will not exactly satisfy the dependences on signal and noise just mentioned.

Exercise. Consider a one-dimensional version of (8.3.32) and find the solutions ignoring the quantum noise term, which as noted above does not contribute to the counting formulae. Assume a counting medium in the range $(-\infty, 0)$, and specify the incoming field as

$$A^{(+)}(t, x) = a_{in}^{(+)}(t + x/c) + a_{out}^{(+)}(t - x/c) \tag{8.3.33}$$

with

$$B^{(+)}(t, x) = \partial_x A^{(+)}(t, x) \tag{8.3.34}$$

$$E^{(+)}(t, x) = -\partial_t A^{(+)}(t, x) \tag{8.3.35}$$

being required to be continuous at $x = 0$.

Introduce Fourier transforms by

$$E^{(+)}(t, x) = \frac{1}{\sqrt{2\pi}} \int\limits_{-\infty}^{\infty} d\omega \tilde{E}(\omega, x)\, e^{-i\omega t} \tag{8.3.36}$$

with similar definitions for other variables. If

$$\nu = \sqrt{\omega^2 + 2i\mu\epsilon\omega} \tag{8.3.37}$$

is the root with *negative* imaginary part, show that

$$\tilde{E}_{\text{out}}(\omega, 0) = \left(\frac{\omega - \nu}{\omega + \nu}\right) \tilde{E}_{\text{in}}(\omega, 0) \tag{8.3.38}$$

$$\tilde{E}(\omega, x) = \frac{\omega/2}{\omega + \nu} \tilde{E}_{\text{in}}(\omega, 0) e^{i\nu x/c} \tag{8.3.39}$$

$$\approx \tilde{E}_{\text{in}}(\omega, 0) e^{i\omega x/c + \mu\epsilon x/c} \tag{8.3.40}$$

when $\mu\epsilon \ll \omega$.

8.3.3   Photon Counting Probabilities

There are two counting probability densities which are of interest:

a) Coincidence Probability Density: This is the probability density that one count is observed at each of the times $t_1, t_2, \ldots, t_n$ together with other possible counts in the time interval $(0, t)$, such that

$$0 < t_1 < t_2 < \ldots < t_n < t, \tag{8.3.41}$$

and it is given by

$$P(t_1, t_2, \ldots, t_n) = \text{Tr}\left\{T_{t,t_n} B T_{t_n,t_{n-1}} B \ldots B T_{t_1,0} \rho\right\}. \tag{8.3.42}$$

This can be seen as follows; at the times $t_1, t_2 \ldots, t_n$ the operator B acts to carry out the measurement, but between times the evolution operator $T_{t_1,t_{r-1}}$ acts, which corresponds to evolution with an unknown number of possible counts in between.

This is a coincidence measurement—or more accurately, a probability density for delayed coincidences. The quantity $P(t_1, t_2, \ldots, t_n) dt_1 dt_2 \ldots dt_n$ is the probability that one count occurs in each of the pre-assigned intervals $dt_1, dt_2, \ldots, dt_n$, without specifying what happens in any other time intervals. This kind of measurement can be quite practicably carried out as long as not too many times are being considered.

b) Elementary Probability Density: If we set

$$Q(t_1, t_2, \ldots, t_n) = \text{Tr}\left\{S_{t,t_n} B S_{t_n,t_{n-1}} B \ldots B S_{t_1,0} \rho\right\} \tag{8.3.43}$$

This represents the probability density that counts occur as before at $t_1, t_2, \ldots, t_n$; but *no counts* occur between these intervals. It can give the distribution of time intervals between counts. For example, the probability of a count between t_2 and $t_2 + dt_2$ given that the last count was between t_1 and $t_1 + dt_1$ is

$$G(t_2) dt_2 = Q(t_2, t_1) dt_2 / Q(t_1). \tag{8.3.44}$$

This is also a measurement that can be practicably carried out.

c) Relation Between Elementary and Coincidence Probability Densities: From the definition, the coincidence probability density can be expressed as the sums of probabilities for n counts and a further $0, 1, 2, 3, \ldots$ more counts at some other points in the interval. It is helpful to extend the definitions of $P(t_1, t_2, \ldots t_n)$ and $Q(t_1, t_2, \ldots t_n)$ as follows. As defined, in both cases it is required that $t_1 < t_2 < \ldots < t_n$. Let us consider extending their definitions to allow all of the t_i to extend over $(0, t)$, but require them to be totally symmetric functions of $t_1, t_2, \ldots, t_n$. Using this meaning, we are led to

$$P(t_1) = \sum_{s=1}^{\infty} \frac{1}{(s-1)!} \int_0^t d\tau_2 \ldots \int_0^t d\tau_s Q(t_1, \tau_2, \ldots, \tau_s) \tag{8.3.45}$$

$$P(t_1, t_2) = \sum_{s=2}^{\infty} \frac{1}{(s-2)!} \int_0^t d\tau_3 \ldots \int_0^t d\tau_s Q(t_1, t_2, \tau_3, \ldots, \tau_s) \tag{8.3.46}$$

$$\vdots \qquad\qquad\qquad \vdots$$

$$P(t_1, t_2, \ldots, t_n) = \sum_{s=n}^{\infty} \frac{1}{(s-n)!} \int_0^t d\tau_{n+1} \ldots \int_0^t d\tau_s Q(t_1, t_2, \ldots, t_n, \tau_{n+1}, \ldots, \tau_s)$$

$$\tag{8.3.47}$$

These formulae simply follow from adding up the probabilities of all events involving exactly the number s of counts, where s is at least the number of counts measured in the coincidence probability.

It is also useful to have an inverse relation. We define generating functionals

$$G(f) = \sum_{n=0}^{\infty} \frac{1}{n!} \int_0^t dt_1 \ldots \int_0^t dt_n f(t_1) f(t_2) \ldots f(t_n) Q(t_1, t_2, \ldots, t_n)$$

$$\tag{8.3.48}$$

and

$$H(f) = \sum_{n=0}^{\infty} \frac{1}{n!} \int_0^t dt_1 \ldots \int_0^t dt_n f(t_1) f(t_2) \ldots f(t_n) P(t_1, t_2, \ldots, t_n). \tag{8.3.49}$$

Then using the symmetry of $P(t_1, \ldots, t_n)$ and $Q(t_1, \ldots, t_n)$ the definition (8.3.47) shows

$$H(f) = G(f + 1) \tag{8.3.50}$$

where by $f + 1$, we mean the function $f(t) + 1$. The inversion is then simply a matter of writing $G(f) = H(f - 1)$, which, when expanded explicitly in (8.3.48),

gives the inverse formula

$$Q(t_1, t_2 \ldots t_n) = \sum_{s=n}^{\infty} \frac{(-1)^{n-s}}{(s-n)!} \int_0^t d\tau_{n+1} \ldots \int_0^t d\tau_s P(t_1, t_2, \ldots, t_n, \tau_{n+1}, \ldots \tau_s).$$

(8.3.51)

Exercise. By considering all possible ways in which s counts can occur either before or after t_1, show

$$P(t_1) = \sum_{s=1}^{\infty} \sum_{r=1}^{s-1} \int_{t_1}^t d\tau_s \int_{t_1}^{\tau_s} d\tau_{s-1} \ldots \int_{t_1}^{\tau_{r+2}} d\tau_{r+1} \int_0^{t_1} d\tau_{r-1} \int_0^{\tau_{r-1}} d\tau_{r-2} \ldots \int_0^{\tau_2} d\tau_1$$

$$\times Q(\tau_1, \tau_2, \ldots, \tau_{r-1}, t_1, \tau_{r+1}, \ldots, \tau_s)$$

(8.3.52)

and then using the symmetry of Q in all its arguments, show the truth of (8.3.45), and the other formulae.

d) Coincidence Probability Density in Terms of QSDE Solutions: The definition of the evolution operator T_{t_1, t_2} corresponds exactly to that used for the evolution operator in Sect. 5.2, and by using the formula (5.2.20) for multitime correlation functions, we can see that equation (8.3.42) can be interpreted as

$$P(t_1, t_2, \ldots, t_n) = \text{Tr} \left\{ \epsilon^n \int d^3 \mathbf{r}_n E^{(+)}(\mathbf{r}_n, t_n) \int d^3 \mathbf{r}_{n-1} E^{(+)}(\mathbf{r}_{n-1}, t_{n-1}) \right.$$

$$\ldots \int d^3 \mathbf{r}_1 E^{(+)}(\mathbf{r}_1, t_1) \rho$$

$$\left. \times \int d^3 \mathbf{r}_1 E^{(-)}(\mathbf{r}_1, t_1) \ldots \int d^3 \mathbf{r}_n E^{(-)}(\mathbf{r}_n, t_n) \right\}$$

(8.3.53)

where the $E^{(+)}(\mathbf{r}, t)$ are *solutions of the QSDE* as given approximately by the solutions of (8.3.32). This corresponds exactly to the formula (8.2.4), but we now know the solutions for $E^{(+)}(\mathbf{r}_n, t_n)$. The correlation function

$$\text{Tr} \left\{ E^{(+)}(\mathbf{r}_1, t_1) E^{(+)}(\mathbf{r}_2, t_2) \ldots E^{(+)}(\mathbf{r}_n, t_n) \rho E^{(-)}(\mathbf{r}_n, t_n) \ldots E^{(-)}(\mathbf{r}_1, t_1) \right\}$$

(8.3.54)

is therefore the central object of study.

As noted in the previous section, we can write each $E^{(\pm)}(\mathbf{r}, t)$ in the form

$$E^{(\pm)}(\mathbf{r}, t) = E_{\text{SIG}}^{(\pm)}(\mathbf{r}, t) + E_{\text{NOISE}}^{(\pm)}(\mathbf{r}, t)$$

(8.3.55)

where the "NOISE" term depends on the $E^{(\pm)}(\mathbf{r}', t)$ for various $\mathbf{r}'$ and for $t > 0$, and the "SIG" term depends on $E^{(\pm)}(\mathbf{r}', 0)$ for various $\mathbf{r}'$. Now all the $\mathbf{b}(\mathbf{r}, t)$ will commute with all the $E^{(-)}(\mathbf{r}', 0)$, and therefore can be commuted with all the $E^{(-)}$ to stand against ρ, where $\mathbf{b}(\mathbf{r}, t)\rho = 0$, since the bath is at zero temperature. Thus, the correlation function can be written as

$$\langle E_{\text{SIG}}^{(-)}(\mathbf{r}_n, t_n) E_{\text{SIG}}^{(-)}(\mathbf{r}_{n-1}, t_{n-1}) \ldots E_{\text{SIG}}^{(+)}(\mathbf{r}_1, t_1) E_{\text{SIG}}^{(+)}(\mathbf{r}_1, t_1) \ldots E_{\text{SIG}}^{(+)}(\mathbf{r}_n, t_n) \rangle.$$

(8.3.56)

Here the $E_{\text{SIG}}^{(-)}$ quantities are computed by the first part of (8.3.55). But this simply means that, to compute the correlation function one should calculate the quantum field at each location by using the boundary conditions and values of $\epsilon(\mathbf{r})$ and the propagation equation (8.3.32), but *omitting the noise* term on the right.

e) Elementary Probability Density: If we define

$$I(t) = \int_V d^3\mathbf{r}\, \mathbf{E}_{\text{SIG}}^{(-)}(\mathbf{r},t) \cdot \mathbf{E}_{\text{SIG}}^{(+)}(\mathbf{r},t) \tag{8.3.57}$$

(where the full form is now used) then the coincidence probability density for counting photons in the volume V is

$$P(t_1, t_2 \ldots t_n) = \epsilon^n \langle :I(t_1)I(t_2)\ldots I(t_n): \rangle \tag{8.3.58}$$

and using (8.3.51), the elementary probability density can be written

$$Q(t_1, t_2, \ldots t_n) = \epsilon^n \left\langle :I(t_1)I(t_2)\ldots I(t_n)\exp\left\{-\epsilon \int_0^t dt\, I(t)\right\}: \right\rangle. \tag{8.3.59}$$
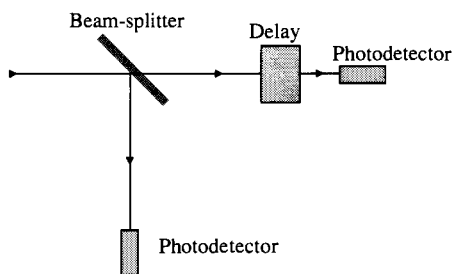
Fig. 8.2 Apparatus used in the Hanbury-Brown Twiss experiment. The beam is divided by the beam splitter, and delayed in one arm

8.3.4 Intensity Correlations and the Hanbury-Brown Twiss Experiment

This experiment provides a measure of the two-time coincidence probability density, $P(t_1, t_2) = \epsilon^2 \langle :I(t_1)I(t_2): \rangle$. The experiment is put into effect by the apparatus of Fig.8.2. The detectors are arranged so that there is a time delay τ between them. Thus the coincidence probability density is $\epsilon^2 \langle :I(t_1)I(t_2): \rangle$.

a) Thermal Light: In the case of thermal light we can expect the statistics to be quantum Gaussian. Assuming that the light is excited in only one mode $a(t)$, the coincidence probability is proportional to

$$G^{(2)}(t, t+\tau) = \langle a^\dagger(t)a^\dagger(t+\tau)a(t+\tau)a(t) \rangle. \tag{8.3.60}$$

We can use the result of Sect. 4.4.5d to note that if we define

$$g^{(2)}(t, t+\tau) = \frac{\langle a^\dagger(t) a^\dagger(t+\tau) a(t+\tau) a(t) \rangle}{\langle a^\dagger(t) a(t) \rangle \langle a^\dagger(t+\tau) a(t+\tau) \rangle} \tag{8.3.61}$$

$$g^{(1)}(t, t+\tau) = \frac{\langle a^\dagger(t) a(t+\tau) \rangle}{\sqrt{\langle a^\dagger(t) a(t) \rangle \langle a^\dagger(t+\tau) a(t+\tau) \rangle}} \tag{8.3.62}$$

then

$$g^{(2)}(t, t+\tau) = 1 + |g^{(1)}(t, t+\tau)|^2 \tag{8.3.63}$$

and obviously

$$g^{(1)}(t, t) = 1. \tag{8.3.64}$$

In all real situations, for sufficiently large τ, $g^{(1)}(t, t+\tau)$ factorises. Thus

$$g^{(1)}(t, t+\tau) \rightarrow \frac{\langle a^\dagger(t) \rangle \langle a^\dagger(t+\tau) \rangle}{\sqrt{\langle a^\dagger(t) a(t) \rangle \langle a^\dagger(t+\tau) a(t+\tau) \rangle}} \rightarrow 0 \tag{8.3.65}$$

in the case of thermal light, for which the mean amplitude must be zero—only the intensity is non-zero.

Thus we find that $g^{(2)}(t, t) = 2$, and $g^{(2)}(t, \infty) = 1$. If, for example, the first order correlation function has an exponential behaviour,

$$g^{(1)}(t, t+\tau) = e^{-\gamma\tau}, \tag{8.3.66}$$

then the second order correlation function is given by

$$g^{(2)}(t, t+\tau) = 1 + e^{-2\gamma\tau}. \tag{8.3.67}$$
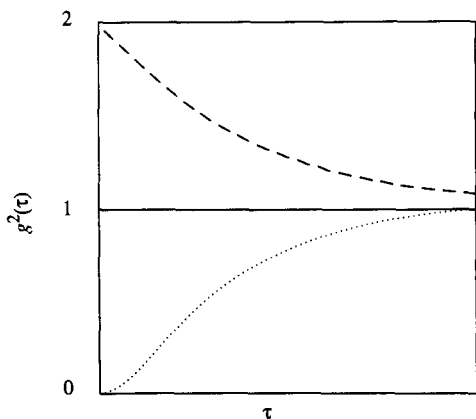
Fig. 8.3 Plots of the intensity correlation functions for thermal light (dashed line), laser light (solid line), and antibunched light (dotted line)

b) Laser Light: On the other hand, if we have an ideal coherent source (which is well approximated by laser light) then the factorisation property of the correlation functions means that

$$g^{(2)}(t, t+\tau) = 1. \tag{8.3.68}$$

Thus we find the rather paradoxical result, that maximum coherence yields no correlation at $\tau = 0$, while what one would expect is the most random situation yields a correlation at $\tau = 0$.

c) General Situation: There do exist optical fields with $g^{(2)}(t, t) = 0$, in marked distinction to the Gaussian situation. For example, in Sect. 9.2.3e, it is shown that light produced by fluorescence from a two level atom driven by an optical field always has this property. Such light is said to be *antibunched*, while Gaussian light can be said to be *bunched*. In the case of bunched light, the fact that $g^{(2)}(t, t) > 1$ indicates that there is a high probability of detecting another photon immediately after one has been detected; thus the photons have a tendency to arrive together, i.e., in bunches.

d) Antibunching: In contrast, in the case of antibunching, the probability of detecting a photon immediately after one has been detected is zero; the photons arrive with some time interval between them, in a rather regular way—hence the name antibunching. Such a situation cannot occur classically, as can be seen from the following argument. Let us suppose that there is a joint P-function $P(\alpha_1, t_1, \alpha_2, t_2)$, whereby we can write

$$G^{(1)}(t_1, t_2) = \int \int d^2\alpha_1 d^2\alpha_2 \, \alpha_1^* \alpha_2 P(\alpha_1, t_1, \alpha_2, t_2) \tag{8.3.69}$$

$$G^{(2)}(t_1, t_2) = \int \int d^2\alpha_1 d^2\alpha_2 \, |\alpha_1|^2 |\alpha_2|^2 P(\alpha_1, t_1, \alpha_2, t_2). \tag{8.3.70}$$

The simplest situation is one in which $P(\alpha_1, t_1, \alpha_2, t_2) = \delta[\alpha_1 - \alpha(t_1)]\delta[\alpha_2 - \alpha(t_2)]$ so that

$$\begin{aligned} G^{(1)}_{\text{coherent}}(t_1, t_2) &= \alpha(t_1)^* \alpha(t_2) \\ G^{(2)}_{\text{coherent}}(t_1, t_2) &= |\alpha(t_1)|^2 |\alpha(t_2)|^2. \end{aligned} \tag{8.3.71}$$

By a classical situation, we mean a situation in which the field at the two times can be described as a random field, distributed over possible values with a probability distribution like $P(\alpha_1, t_1, \alpha_2, t_2)$, which must however be *non-negative*. But from (8.3.69,70), we see that

$$G^{(2)}(t_1, t_2) = \langle |\alpha_1^* \alpha_2|^2 \rangle \geq |\langle \alpha_1^* \alpha_2 \rangle|^2 = |G^{(1)}(t_1, t_2)|^2 \tag{8.3.72}$$

which means that for the classical situation, $g^{(2)}(t, t) \geq 1$.

The antibunched situation can therefore only arise because the P-function is negative or distribution valued, as is permitted in the quantum mechanical situation.

8.3.5   Mandel's Counting Formula

The probability of detecting exactly n counts in the time interval $(0, t)$ is

$$
p(n, [0, t]) = \int_0^t dt_1 \int_0^{t_1} dt_2 \dots \int_0^{t_{n-1}} dt_n Q(t_1, t_2, \dots, t_n) \tag{8.3.73}
$$

and using the fact that $Q(t_1, \dots, t_n)$ is defined to be a symmetric function of its arguments,

$$
p(n, [0, t]) = \frac{1}{n!} \int_0^t dt_1 \int_0^t dt_2 \dots \int_0^t dt_n Q(t_1, t_2, \dots, t_n) \tag{8.3.74}
$$

and now using (8.3.59), we are led to *Mandel's counting formula*

$$
p(n, [0, t]) = \frac{\epsilon^n}{n!} \left\langle : \left\{ \int_0^t I(t') dt' \right\}^n \exp\left\{ -\epsilon \int_0^t dt' I(t') \right\} : \right\rangle. \tag{8.3.75}
$$

a) Photon Number Operator: The quantity $I(t)$ is proportional to what might be called the photon number operator, $N(V, t)$, for the number of photons in the volume V. In fact, if we define

$$
N(V, t) = \frac{2\epsilon_0}{\hbar \Omega} \int_V d^3 \mathbf{r} : \mathbf{E}^{(-)}(\mathbf{r}, t) \cdot \mathbf{E}^{(+)}(\mathbf{r}, t) : \tag{8.3.76}
$$

which is proportional to $I(t)$, then $N(V, t)$ has eigenvalues $0, 1, 2, \dots$ for photon number states which are localised within V, i.e., states which can be generated using operators (for various $\mathbf{f}(\mathbf{r})$)

$$
a^\dagger(\mathbf{f}, t) = \int_V d^3 \mathbf{r} \, \mathbf{f}(\mathbf{r}) \cdot \mathbf{E}^{(-)}(\mathbf{r}, t) \tag{8.3.77}
$$

acting on the vacuum. Using the commutation relations (8.1.27), one finds

$$
\left[a(\mathbf{f}, t), a^\dagger(\mathbf{f}, t) \right] = 1 \tag{8.3.78}
$$

provided $\mathbf{f}(\mathbf{r})$ is normalised appropriately;

$$
\int_V d^3 \mathbf{r} \, \mathbf{f}^*(\mathbf{r}) \cdot \mathbf{f}(\mathbf{r}) = 2\epsilon_0 / \hbar \Omega. \tag{8.3.79}
$$

Further, for *any* normalised $\mathbf{f}(\mathbf{r})$

$$
\left[N(V, t), a^\dagger(\mathbf{f}, t) \right] = a^\dagger(\mathbf{f}, t). \tag{8.3.80}
$$

Together with the commutation relations (8.3.78), this is sufficient to prove the eigenvalues of $N(V,t)$ are non-negative integers. If $\mathbf{f}_i(\mathbf{r})$ are a complete set of functions normalised like (8.3.79), then

$$N(V,t) = \sum_i a^\dagger(\mathbf{f}_i, t) a(\mathbf{f}_i, t). \tag{8.3.81}$$

Thus, the photon counting formula can be written as

$$p(n,[0,t]) = \frac{\bar{\epsilon}^n}{n!} \left\langle : \left\{ \int_0^t N(V,t')dt' \right\}^n \exp\left\{ -\bar{\epsilon} \int_0^t N(V,t')dt' \right\} : \right\rangle \tag{8.3.82}$$

where

$$\bar{\epsilon} = \frac{\hbar\Omega\epsilon}{2\epsilon_0}. \tag{8.3.83}$$

The photon number operator represents the number of photons *inside* the detector, and it is very reasonable that this should determine the counting rate.

b) Inefficient Detection: In the formalism developed so far the only way in which a photon is absorbed is by being detected. We can include the possibility of absorption without detection simply by choosing $\mathcal{L}(t)$ in (8.3.18) to be

$$\mathcal{L}(t) = \tfrac{1}{2}\kappa \int d^3\mathbf{r} \left\{ 2E^{(+)}(\mathbf{r})\rho E^{(-)}(\mathbf{r}) \right.$$
$$\left. -E^{(-)}(\mathbf{r})E^{(+)}(\mathbf{r})\rho - \rho E^{(-)}(\mathbf{r})E^{(+)}(\mathbf{r}) \right\}, \tag{8.3.84}$$

that is, of exactly the same form as the detection term, but with a different constant κ. The efficiency of detection of photons inside the detector (i.e., not including the possibility of escape from the detector), will be $\epsilon/(\epsilon+\kappa)$, and all counting formulae will be simply obtained by computing the fields using $\epsilon+\kappa$ instead of ϵ in the QSDEs.

Exercise. Using the solution (8.3.38–40) show that the total counting rate in a detector like that described matches with the power not reflected off the surface if the detector efficiency is 100%, i.e., there is no extra absorption in the detector.

c) Mandel's Formula as a Photon Counting Formula: Under the condition that the detector is thin and very efficient, the number of photocounts corresponds to the measurement of the photon number operator. We assume that the detector is located at $z < 0$, and the absorption length γ^{-1} of the detector material is so small that we can approximate the time dependence by $e^{-i\omega t}$ for times less than $(\gamma c)^{-1}$. If $\mathcal{T}$ is the transmission coefficient of the light into the detector medium, then we can say

$$E^{(-)}(\mathbf{r},t) = \mathcal{T}E^{(-)}(x,y,0)e^{\gamma z - i\omega z/c}. \tag{8.3.85}$$

Here $E^{(-)}$ on the RHS represents the value of the incoming field to the immediate right of the detector surface. The situation is illustrated in Fig.8.4. Using the one dimensional model solution of (8.3.40), we can see that

$$\gamma = \mu\epsilon/c = \hbar\Omega\epsilon/4\epsilon_0 c = \bar{\epsilon}/2c. \tag{8.3.86}$$

Assuming also that the light is travelling along the z axis, we find that

$$\bar{\epsilon} \int_0^t N(V, t')_{\text{SIG}} \, dt' = |T|^2 \int_A dx \, dy \int_0^{ct} dz \, E^{(-)}(\mathbf{r}, 0) E^{(+)}(\mathbf{r}, 0) \tag{8.3.87}$$

$$= |T|^2 N_{\text{IN}}(V, 0) \tag{8.3.88}$$

where V is the volume of a tube of cross sectional area A and length ct, where t is the time for which one counts. (It has been assumed that the length of the detector along the z direction is much greater than the attenuation length, allowing the integration of a factor $e^{2\gamma z}$.)
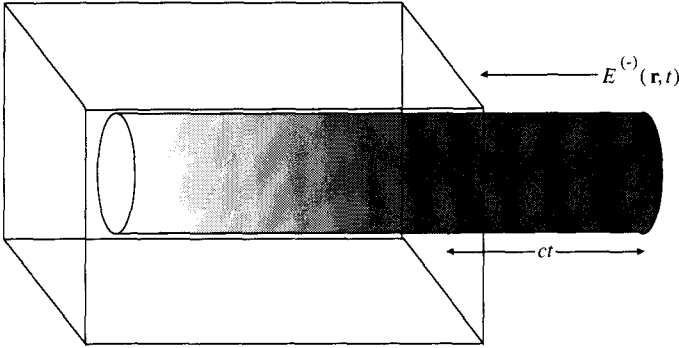
Fig. 8.4 Photodetection in a spatially distributed detector

If the counting is not perfectly efficient, due to other absorption as in b) above, then $|T|^2$ is replaced by e, which is the product of $|T|^2$ and the intrinsic efficiency of the detector. Thus, Mandel's formula may be written

$$p(n, [0, t]) = \frac{e^n}{n!} \langle :N_{\text{IN}}(V, 0)^n \exp\{-eN_{\text{IN}}(V, 0)\} : \rangle. \tag{8.3.89}$$

Since the photon number operator is formed out of creation and destruction operators, we can consider eigenvalues of $N_{\text{IN}}(V, 0)$ with a definite number of photons; these can be written in general as $|\mathbf{n}\rangle$, where n_i are the eigenstates of the operator $a^\dagger(\mathbf{f}_i, t)a(\mathbf{f}_i, t)$. Using the creation and destruction operators of Chap. 4, it is easy to show that

$$\frac{e^n}{n!} :(a^\dagger a)^n \exp(-ea^\dagger a): |r\rangle = \frac{r!}{n!(r-n)!} e^n (1-e)^{r-n} |r\rangle \tag{8.3.90}$$

or

$$\frac{e^n :(a^\dagger a)^n \exp(-ea^\dagger a):}{n!} = \sum_{r=0}^n e^n (1-e)^{r-n} \frac{r!}{n!(r-n)!} |r\rangle\langle r|. \tag{8.3.91}$$

The special case of $e = 1$, perfect efficiency, gives

$$\frac{:(a^\dagger a)^n \exp(-ea^\dagger a):}{n!} = |n\rangle\langle n|. \tag{8.3.92}$$

Exercise. Generalise (8.3.92) to the case of N modes, to show that

$$\frac{:\left\{\sum_{i=1}^{N} a_i^\dagger a_i\right\}^n \exp\left\{-\sum_{i=1}^{N} a_i^\dagger a_i\right\}:}{n!} = P_n \tag{8.3.93}$$

where P_n is the projector into states with total photon number equal to n.

Mandel's formula can thus be written, in the case of 100% efficiency

$$p(n, [0, t]) = \langle P_n \rangle = \sum_{\mathbf{n}} \delta(n, \sum n_i) \langle \mathbf{n}|\rho|\mathbf{n}\rangle. \tag{8.3.94}$$

Thus, the probability of n photocounts in time t is the probability that there are n photons in the incoming volume V of light counted. *This means that a photodetector in fact measures the photon number operator for the number of photons in the volume V. We can thus work out the moments of the photocount distribution simply by computing the moments of $N_{\text{in}}(V, 0)$.*

8.3.6   Applications to Particular States

a) Number State: Assuming only one mode is excited in the volume V, and the state is a number state $|r\rangle$, then

$$\begin{aligned}
p(n, [0, t]) &= \frac{r!}{n!(r-n)!} e^n (1 - e)^{r-n} & n \leq r \\
&= 0 & n > r.
\end{aligned} \tag{8.3.95}$$

This is a Bernoulli distribution. When the counting efficiency is perfect, we find

$$p(n, [0, t]) = \delta_{n,r} \tag{8.3.96}$$

which means that we can only get the result that the number of photons counted is equal to the number of photons in the volume.

b) Coherent State $|\alpha\rangle$:

$$p(n, [0, t]) = \frac{(e|\alpha|^2)^n}{n!} \exp\left(-e|\alpha|^2\right). \tag{8.3.97}$$

This is a Poisson distribution for any value of e, the efficiency. In fact, since the system is essentially a harmonic oscillator damped by a zero temperature bath, this corresponds to the optical state remaining a coherent state as it is attenuated or partially transmitted—this is essentially the result of Sect. 6.2.1c, that a coherent state remains coherent as it is linearly attenuated.

c) Thermal State: Here ρ_T is given by (4.4.13), and we find that

$$p(n,[0,t]) = \frac{\left[e\bar{N}(T)\right]^n}{\left[1 + e\bar{N}(T)\right]^{n+1}} \tag{8.3.98}$$

where, as usual

$$\bar{N}(T) = \left(e^{\hbar\omega/kT} - 1\right)^{-1}. \tag{8.3.99}$$

We can work out the moments of the photocurrent distribution by simply computing the moments of $N_{\mathrm{in}}(V,0)$.
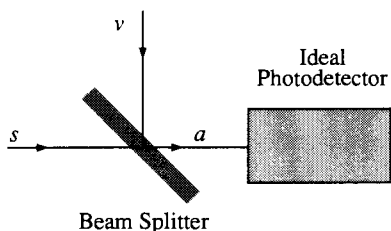
Fig. 8.5   Model of an inefficient photodetector formed by a perfect photodetector preceded by a beam splitter

8.3.7   Model for Efficiency Less Than 100%

An inefficient detector may be modeled by a perfect detector with a beam splitter, of amplitude transmissivity e in front of it as in Fig.8.5 The detected beam can be written

$$a = s\sqrt{e} + v\sqrt{1-e} \tag{8.3.100}$$

Since v is in the vacuum state, all of its normally ordered moments are zero, which means that the normally ordered moments of $a^\dagger a$ are

$$\langle: (a^\dagger a)^n :\rangle = e^n \langle: (s^\dagger s)^n :\rangle \tag{8.3.101}$$

which gives the counting distribution (8.3.89) directly for a; namely that of an inefficient detector. This result is very useful when one needs to model an inefficient detector.

8.4   Homodyne and Heterodyne Detection

In practical detection of light beams considerable advantages can ensue by mixing the signal to be detected with a strong coherent signal, called a *local oscillator*, before detection. The mixing is linear in the fields being mixed, whereas detection
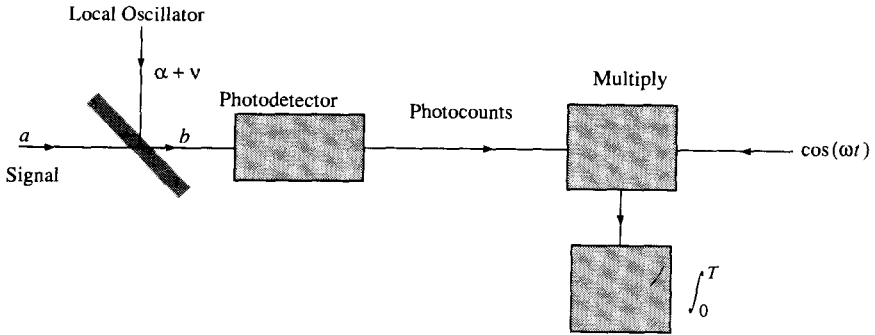
Fig. 8.6   Arrangements for homodyne or heterodyne detection

is proportional to intensity, so this procedure mixes the signals. Two kinds of detection are in use; *homodyne detection*, in which the local oscillator has the same frequency as that of the detected signal, and *heterodyne detection*, where the local oscillator has a different frequency.

8.4.1   Schematic Setup of Homodyne and Heterodyne Detection

The basic concept is illustrated in Fig.8.6. The local oscillator and signal are mixed in a beam splitter, and then detected. The output photocurrent is then multiplied by a $\cos(\omega t)$ function, and the result integrated for a time T, very much larger than ω^{-1}. The photodetector will not normally be 100% efficient, but we will model it as an ideal photodetector preceded by a beam splitter.

8.4.2   General Formulae

The signal at the photodetector is

$$b(t) = r\alpha(t) + rv(t) + ta(t). \tag{8.4.1}$$

Here $\alpha(t)$ is the *mean value* of the local oscillator field, thus $\langle v(t) \rangle = 0$. The quantity being studied is thus

$$N(\omega, T) = \int\limits_{0}^{T} dt\, b^{\dagger}(t) b(t) \cos(\omega t). \tag{8.4.2}$$

The assumption is that $\cos(\omega t)$ is rather slowly varying, and (8.4.2) is effectively got by measuring the photon number operator repeatedly. These values are then multiplied by $\cos(\omega t)$, (or $\sin(\omega t)$ as appropriate), and added. We shall only carry out the calculations explicitly for $\cos(\omega t)$.

We now assume that $r\alpha(t)$ is very large, and expanding (8.4.2) to the first two orders only

$$N(\omega, T) = \int_0^T dt \left\{ |r|^2 |\alpha(t)|^2 + r^*\alpha^*(t) \left[ta(t) + rv(t) \right] \right.$$

$$\left. + r\alpha(t) \left[t^*a^\dagger(t) + r^*v^\dagger(t) \right] \right\} \cos(\omega t). \tag{8.4.3}$$

Here we assume a single frequency local oscillator, i.e.

$$\alpha(t) = Ae^{-i\omega_L t - i\phi}, \tag{8.4.4}$$

and let us choose, for convenience, r and t to be *real*. We define a Fourier variable

$$\tilde{a}(\omega) = \frac{1}{T} \int_0^T e^{i\omega t} a(t) \, dt \tag{8.4.5}$$

with a similar notation for $\tilde{v}(\omega)$. We then find

$$N(\omega, T) = \int_0^T r^2 A^2 \cos \omega t \, dt$$

$$+ \frac{rAT}{2} e^{i\phi} \left\{ t \left[\tilde{a}(\omega_L - \omega) + \tilde{a}(\omega_L + \omega) \right] + r \left[\tilde{v}(\omega_L - \omega) + \tilde{v}(\omega_L + \omega) \right] \right\}$$

$$+ \frac{rAT}{2} e^{-i\phi} \left\{ t \left[\tilde{a}(\omega_L - \omega)^\dagger + \tilde{a}(\omega_L + \omega)^\dagger \right] + r \left[\tilde{v}(\omega_L - \omega)^\dagger + \tilde{v}(\omega_L + \omega)^\dagger \right] \right\}. \tag{8.4.6}$$

a) **Mean Detected Signal:** We now define the mean field,

$$\tilde{\mathcal{E}}(\omega) = \langle \tilde{a}(\omega) \rangle \tag{8.4.7}$$

so that the mean signal detected is, on the assumption that $T \gg 1/\omega$, and terms of order ω^{-1} are neglected in comparison with terms of order T,

$$\langle N(\omega, T) \rangle = r^2 A^2 T \delta_{\omega,0} + rtAT \, \mathrm{Re} \left\{ e^{i\phi} \left[\tilde{\mathcal{E}}(\omega_L - \omega) + \tilde{\mathcal{E}}(\omega_L + \omega) \right] \right\}. \tag{8.4.8}$$

b) **Variance of Detected Signal:** We are also interested in the variance of $N(\omega, T)$. We need to assume a variance for $v(t)$, and we shall assume that of the vacuum, i.e.

$$\langle v^\dagger(t) v(t') \rangle = 0, \qquad \langle v(t) v^\dagger(t') \rangle = \delta(t - t'), \tag{8.4.9}$$

so that the variance can be computed to be

$$
\begin{aligned}
\left\langle \left[N(\omega, T) - \langle N(\omega, T) \rangle \right]^2 \right\rangle = {} & \left(\frac{rAT}{2} \right)^2 \left\{ \frac{2r^2}{T}(1 + \delta_{\omega,0}) \right. \\
& + t^2 \left\langle \left[e^{i\phi} \{ \delta\tilde{a}(\omega_L - \omega) + \delta\tilde{a}(\omega_L + \omega) \} \right. \right. \\
& \left. \left. \left. + e^{-i\phi} \{ \delta\tilde{a}(\omega_L - \omega)^\dagger + \delta\tilde{a}(\omega_L + \omega)^\dagger \} \right]^2 \right\rangle \right\}
\end{aligned}
\tag{8.4.10}
$$

(where δ means the difference from the mean value). There are two terms in this expression; that proportional to r^2 is *local oscillator noise*, and arises from the fluctuations in the local oscillator part; the other is the signal noise.

8.4.3   Coherent Signal Detection

If the signal is itself coherent, the signal noise takes on exactly the same form as the vacuum, i.e.

$$
\langle a(t) a^\dagger(t') \rangle = \delta(t - t'), \quad \langle a^\dagger(t) a(t') \rangle = 0
\tag{8.4.11}
$$

and

$$
\langle \left[N(\omega, T) - \langle N(\omega, T) \rangle \right]^2 \rangle = \frac{T(rA)^2}{2} \{ 1 + \delta_{\omega,0} \}
\tag{8.4.12}
$$

(where use has been made of $r^2 + t^2 = 1$). We can compute the signal to noise ratio in some practical cases.

a) **Homodyne Detection:** Here we set $\omega = 0$. We take the *signal* to be that part by which the $\langle N(\omega, T) \rangle$ differs from the "background", $r^2 A^2 T$.

$$
\text{Sig (Ho)} = 2rtAT \, \text{Re} \left\{ e^{i\phi} \tilde{\mathcal{E}}(\omega_L) \right\}
\tag{8.4.13}
$$

$$
\text{Noise (Ho)} = T(rA)^2
\tag{8.4.14}
$$

$$
\text{S/N (Ho)} = 4t^2 T \left[\text{Re} \left\{ e^{i\phi} \tilde{\mathcal{E}}(\omega_L) \right\} \right]^2 .
\tag{8.4.15}
$$

b) **Heterodyne Detection:** Here we set $\omega \neq 0$, and we assume that $\omega_L - \omega = \omega_{\text{Sig}}$, where ω_{Sig} is the frequency around which the signal has significant power. The background term now vanishes, which is an advantage for this method. It is in practice not common to have significant power in both $\omega_L + \omega$ and $\omega_L - \omega$, but it is of course possible. We then find

$$
\text{Sig (Het)} = rtAT \, \text{Re} \left\{ e^{i\phi} \tilde{\mathcal{E}}(\omega_{\text{Sig}}) \right\}
\tag{8.4.16}
$$

Noise (Het) $= \frac{1}{2}T(rA)^2$ \hfill (8.4.17)

$$\mathbf{S/N}\,(\text{Het}) = 2t^2 T \left[\text{Re}\left\{e^{i\phi}\bar{\mathcal{E}}(\omega_{\text{Sig}})\right\}\right]^2 . \hspace{2cm} (8.4.18)$$

c) Comment on Signal to Noise Ratios: Note that the signal to noise ratio in heterodyne detection is only half of that for homodyne detection. On the other hand, there is no constant level, as in homodyne detection.

d) Imperfect Photodedectors: If the photodetector has an efficiency of less than 100%, then there is an additional noise source. This can be seen by using the model of the inefficient detector as a perfect detector behind a beam splitter, see Sect. 8.3.7. Thus the light that reaches the photodetector is

$$c(t) = \sqrt{e}\, b(t) + \sqrt{1-e}\, w(t), \hspace{2cm} (8.4.19)$$

where $w(t)$ is a vacuum white noise. In the limit of $\alpha(t)$ large, we find that

$$N(\omega,T) = \int_0^T dt\, \cos\omega t \left\{ er^2 |\alpha(t)|^2 \right.$$
$$+ \sqrt{e}\, r\alpha^*(t) \left[\sqrt{e}\, ta(t) + \sqrt{e}\, rv(t) + \sqrt{1-e}\, w(t)\right]$$
$$\left. + \sqrt{e}\, r\alpha(t) \left[\sqrt{e}\, ta(t) + \sqrt{e}\, rv(t) + \sqrt{1-e}\, w(t)\right]^\dagger \right\}. \;(8.4.20)$$

By making $\alpha(t)$ arbitrarily large, and r correspondingly small (to the degree that this may be technically achievable), the term with $v(t)$ can be made as small as desired, but the $w(t)$ term cannot—it can only be eliminated by making the efficiency e as close to 100% as possible.

An arbitrarily large signal to noise ratio is achievable, however, by counting for an arbitrarily long time. But if one is interested in the *noise* itself, which is the case in the study of squeezed light, then great efforts must be made to increase e. Efficiencies of about 90% are in practice achievable. [8.7]

The results for signal to noise ratios are similar to (8.4.15,18), and are obtained by the replacement $t \rightarrow t\sqrt{e}$ in these formulae—as might be expected.

8.4.4   Balanced Homodyne/Heterodyne Detection

In balanced detection, the beam splitter is chosen to have equal reflectivity and transmissivity, i.e., $|r| = |t| = 1/\sqrt{2}$, and two detectors are used, as in Fig.8.7. The photocurrents are then subtracted, and processed as before. In the case that the detectors are 100% efficient, the signal to be measured is

$$N_B(\omega,T) = \int_0^T dt\, \left\{b_1^\dagger(t)b_1(t) - b_2^\dagger(t)b_2(t)\right\} \cos\omega t \hspace{1cm} (8.4.21)$$
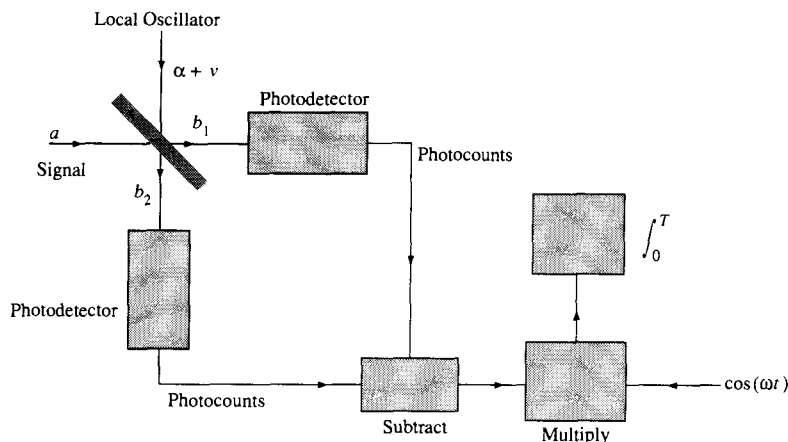
Fig. 8.7 Arrangements for balanced detection

with

$$b_1(t) = \frac{1}{\sqrt{2}} [\alpha(t) + v(t) + a(t)]$$

$$b_2(t) = \frac{1}{\sqrt{2}} [-\alpha(t) - v(t) + a(t)].$$

(8.4.22)

Many terms cancel, and we find

$$N_B(\omega, T) = \int\limits_0^T dt \left\{ \alpha(t)^* a(t) + \alpha(t) a^\dagger(t) + v^\dagger(t) a(t) + v(t) a^\dagger(t) \right\} \cos \omega t.$$

(8.4.23)

Again, one assumes that $\alpha(t)$ is very large, so that the final two terms can be neglected. However, $\alpha(t)$ need not be nearly so strong as in unbalanced detection since it does not appear multiplied by $r \ll 1$. Thus balanced detection provides a more satisfactory technique, particularly as the constant term is no longer present. The formulae are essentially the same as those for ordinary detection, with the replacement $r \to 1, t \to 1$ (*not* $r \to 1/\sqrt{2}, t \to 1/\sqrt{2}$, since the differencing in (8.4.21) provides another factor of 2.)

Of course to get the full advantage of the technique, one must have a perfectly balanced beam splitter, which can never be exactly so.

a) Imperfect Detectors: If the photodetectors are not perfect, we have two additional independent noise sources at each detector. Thus, the inputs measured will be

$$c_1(t) = \sqrt{e_1}\, b_1(t) + \sqrt{1 - e_1}\, w_1(t)$$

$$c_2(t) = \sqrt{e_2}\, b_2(t) + \sqrt{1 - e_2}\, w_2(t).$$

(8.4.24)

Here $w_1(t)$ and $w_2(t)$ may be taken as vacuum quantum white noise. In the limit that $\alpha(t)$ is large (and with $e_1 = e_2 \equiv e$ for simplicity) we find

$$N_B(\omega, T) \rightarrow \int_0^T dt \left\{ e\left[\alpha(t)^* a(t) + \alpha(t) a^\dagger(t)\right] + \right.$$

$$+ \sqrt{\frac{e(1-e)}{2}}\, \alpha(t)^* \left[w_1(t) + w_2(t)\right]$$

$$\left. + \sqrt{\frac{e(1-e)}{2}}\, \alpha(t) \left[w_1(t) + w_2(t)\right]^\dagger \right\} \cos \omega t. \quad (8.4.25)$$

If e is very close to 1, the final terms are negligible, but otherwise they can be significant, as in the case of unbalanced detection.

Exercise. In fact, because there are two *independent* noise sources in balanced detection, the signal to noise ratio is half that of unbalanced detection. Show this.

8.5   Input-Output Formulations of Photodetection

The formulation of photodetection given in Sect. 8.3 is based on what really must be regarded as a phenomenological theory, for that is all the Srinivas-Davies theory can claim to be. The formulation is manifestly only an approximation, obtained by assuming a narrow bandwidth detector, and by averaging over many cycles of the relevant optical frequency. Of course both of these are in almost all current situations very good approximations, but their introduction is necessary not only because they are reasonable, but also because ambiguities arise when they are not made.

I want to introduce here a different view of the problem, based on the input-output formalism. The major new idea is to treat the photodetection process as the counting of electrons, represented by an appropriate fully quantised field, rather than merely the counting of photoionisations, which lies at the heart of the Srinivas-Davies formalism. The QSDEs developed in Sect. 8.3.2 have noise terms, which were not interpreted there. But the whole formalism developed there suggests that they must be related to the individual absorptions of photons with the consequent emissions of electrons. Thus, it is tempting to interpret the "out" field corresponding to $\mathbf{b}(\mathbf{r}, t)$ as the electron field operator. However this must be justified in more detail before being put into application.

In this section I start with a very simple model, essentially of one atom and a one dimensional electromagnetic field. The electrons are treated as fully quantised *bosons*, because this allows for great simplicity and clarity. It is able to be shown that, under the conditions which normally apply for a practical photodetector, the "out" electron field has the same statistics as the "in" photon field—and as a natural

consequence we can derive Mandel's formula. The next section shows that the results still hold, in a degree of approximation, even with Fermi electrons. This involves the adaptation of the input-output formalism to handle fermions.

The final section reverts to Bose electrons, but treats the fully quantised propagation absorption and photoproduction problem in some detail, without making the rotating wave approximation. In this final model, we see a physical interpretation of the formalism of Sect. 8.3.2 in which the "out" noises are indeed the electron field operators.

8.5.1   A One Atom Model

In order to see why the photoelectron statistics are those of the incident light, let us consider the simplest possible model of a photodetector—a single atom (idealised as a harmonic oscillator), localised at some point in a one-dimensional space, and tuned to the frequency of the incoming light. In photodetection the electrons are never produced with any significant occupation of the available states so the distinction between bosons and fermions is immaterial; let us therefore consider Bose electrons.

We consider a simple Hamiltonian, in which a single atom interacts with the operators of both the electromagnetic field and of the electrons.

$$
H = \hbar \Omega a^\dagger a + \int_{-\infty}^{\infty} d\omega \hbar \omega b^\dagger(\omega) b(\omega) + \int_{-\infty}^{\infty} d\omega \hbar \omega d^\dagger(\omega) d(\omega)
$$

$$
+ i\hbar \int_{-\infty}^{\infty} \bar\gamma(\omega) \left[b^\dagger(\omega) a - b(\omega) a^\dagger \right] d\omega
$$

$$
+ i\hbar \int_{-\infty}^{\infty} \bar\kappa(\omega) \left[d^\dagger(\omega) a - d(\omega) a^\dagger \right] d\omega. \tag{8.5.1}
$$

Here a is a destruction operator for the atom, $b(\omega)$ is a destruction operator for an electromagnetic field mode, $d(\omega)$ is a destruction operator for an electron field mode. The coupling between electrons, photons and the atom is of the kind that allows an atom to be excited by the electromagnetic field, and to de-excite by emission of either an electron or a photon—the processes are all completely reversible.

The quantum Langevin equation corresponding to (8.5.1) is

$$
\dot a(t) = -i\Omega a(t) - \tfrac{1}{2}(\kappa + \gamma) a(t) - \sqrt{\gamma}\, b_{\text{in}}(t) - \sqrt{\kappa}\, d_{\text{in}}(t). \tag{8.5.2}
$$

Her the "first Markov approximation of Sect. 5.3.2 has been made, so that $\bar\gamma(\omega) = \sqrt{\gamma/2\pi}$, $\bar\kappa(\omega) = \sqrt{\kappa/2\pi}$. The "in" fields are essentially the inputs into the oscillator from the electromagnetic and electron fields; thus $b_{\text{in}}(t)$ represents the incoming electromagnetic field, whose statistics we wish to measure. The process of measurement is to count emitted electrons, so we wish to know not the "in" field d_{in}, but a corresponding "out" field,

$$
d_{\text{out}}(t) = d_{\text{in}}(t) + \sqrt{\kappa}\, a(t). \tag{8.5.3}
$$

To get the relationship between $d_{out}(t)$ and $b_{in}(t)$ now requires us to solve the equations. We go to a frame rotating at angular frequency Ω by writing $a(t) = e^{-i\Omega t}A(t)$ with a similar notation for $B_{in}(t), D_{in}(t), D_{out}(t)$, etc. Now assume that $\kappa + \gamma$ is sufficiently large that we can solve the equation resulting from (8.5.2) by adiabatic elimination, thus

$$A(t) = -\frac{2\sqrt{\gamma}\,B_{in}(t) + 2\sqrt{\kappa}\,D_{in}(t)}{\kappa + \gamma} \tag{8.5.4}$$

and using (8.5.3), we get

$$D_{out}(t) = \frac{\gamma - \kappa}{\gamma + \kappa}D_{in}(t) - \frac{2\sqrt{\gamma\kappa}}{\gamma + \kappa}B_{in}(t). \tag{8.5.5}$$

It is clear that the normally ordered moments of $D_{out}(t)$ are equal to those of $2\sqrt{\gamma\kappa}\,A_{in}(t)/(\gamma + \kappa)$ provided that the state of $D_{in}(t)$ corresponds to the vacuum, that is, provided we are not directing electrons at the atom, which is of course the case. If $\gamma = \kappa$, we get the ideal result that $D_{out}(t) = A_{in}(t)$, so that the measurement of $D_{out}(t)$ is equivalent to the measurement of $A_{in}(t)$. As was shown by *Ghielmetti* [8.8], this is exactly equivalent to Mandel's formula. In the case of lower efficiency, when $\gamma \neq \kappa$, the effect is similar to that of a beam splitter; that is, the output electron field is a mixture of the input photon field and the vacuum fluctuations arising from the input electron field. This is similar to a view introduced by *Yuen* and *Shapiro* [8.9], that such a photodetector is equivalent to an ideal photodetector preceded by a beam splitter.

Mandel's formula is not exactly equivalent to (8.5.5), which would suggest that phase information is also supplied to the electron field. Mandel's formula only requires that the normally ordered moments of the *electron number operator*

$$N(t_a, t_b) = \int_{t_a}^{t_b} D_{out}^\dagger(t)\,D_{out}(t)\,dt \tag{8.5.6}$$

are given by substituting for $D_{out}(t)$, as mentioned, thus we can say that measuring $N(t_a, t_b)$ counts the number of electrons arriving in the interval (t_a, t_b), and this is equal to

$$\frac{4\gamma\kappa}{(\gamma + \kappa)^2}\int_{t_a}^{t_b} A_{in}^\dagger(t)\,A_{in}(t)\,dt. \tag{8.5.7}$$

The equality of *normally ordered* moments, with the factor $\epsilon = 4\gamma\kappa/(\gamma + \kappa)^2 \leq 1$ as the detector efficiency is equivalent to Mandel's formula. Notice also that the detector efficiency is strictly no greater than 100%, and reaches 100% when $\gamma = \kappa$— a fact which can be regarded as impedance matching. This is not a relevant result to practical photodetection, which is a distributed process, in which attenuation takes place over a finite distance. A more appropriate model is presented in Sect. 8.5.3.

8.5.2   A Model Using Fermi Electrons

The use of Fermi electrons is much more difficult because the harmonic oscillator can no longer be used as a model of an atom. It is simply not possible to set up a physics in which a photon raises the atom to an excited state, which emits an electron and returns to the *same* state. Instead, a three level model is the bare minimum, as in Fig.8.8. The emitted electron leaves the atom in an intermediate state, which then makes a transition to the ground state by absorption of another electron from a bath of electrons. This is of course the essence of what happens in practice.
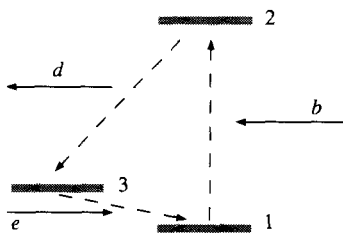
Fig.8.8   Three level atom model of a photodetector

a) Input-Output Method with Fermions: To use fermions, an appropriate formulation of the input-output methods is essential. Suppose we consider ionisation of an atom, in which a fermion can be attached or detached. The Hamiltonian is

$$H = H_{sys} + H_{int} + H_B \tag{8.5.8}$$

where H_{sys} is the free atom Hamiltonian, and

$$H_B = \int d\omega\, \hbar\,\omega\, d^\dagger(\omega)d(\omega) \tag{8.5.9}$$

$$H_{int} = i\hbar \int d\omega\, \bar{\kappa}(\omega)\left\{ d^\dagger(\omega)c - c^\dagger d(\omega)\right\}, \tag{8.5.10}$$

which is exactly the same as in the boson formulation. However, in this case, $d(\omega), d^\dagger(\omega)$, are fermion operators so that

$$[d(\omega), d^\dagger(\omega')]_+ = \delta(\omega - \omega'). \tag{8.5.11}$$

A complication is now introduced by the concept of fermionic and bosonic system operators. At equal times $d(\omega), d^\dagger(\omega)$ commute with all bosonic system operators and anticommute with all fermionic operators. Clearly the operator c which occurs in the interaction Hamiltonian must be Fermionic.

b) Quantum Langevin Equation: The same procedure as for the boson bath case can now be applied, to yield the Fermionic quantum Langevin equation,

$$\dot{a} = -\frac{i}{\hbar}[a, H_{sys}] - \left\{ [a, c^\dagger]_\mp \left(\tfrac{1}{2}\kappa c + \sqrt{\kappa}\,d_{in}(t)\right) \right.$$
$$\left. \mp \left(\tfrac{1}{2}\kappa c^\dagger + \sqrt{\kappa}\,d_{in}^\dagger(t)\right)[a, c]_\mp \right\} \tag{8.5.12}$$

where + corresponds to a fermionic a and − corresponds to a bosonic a. The bosonic equation is the same as in the usual formalism.

c) Inputs, Outputs and Causality: As in the usual formalism, we find

$$d_{\text{out}}(t) = d_{\text{in}}(t) + \sqrt{\kappa}\,c(t) \tag{8.5.13}$$

and

$$
\begin{aligned}
{[a(t), d_{\text{in}}(t')]}_{\pm} &= -u(t - t')\sqrt{\kappa}\,[a(t), c(t')]_{\pm} \\
{[a(t), d_{\text{out}}(t')]}_{\pm} &= u(t' - t)\sqrt{\kappa}\,[a(t), c(t')]_{\pm}.
\end{aligned} \tag{8.5.14}
$$

d) Fermionic Quantum White Noise: In this case we define

$$\langle d_0^{\dagger}(\omega) d_0(\omega') \rangle = \tilde{N}\delta(\omega - \omega'), \qquad \langle d_0(\omega) d_0^{\dagger}(\omega') \rangle = (1 - \tilde{N})\delta(\omega - \omega') \tag{8.5.15}$$

where $\tilde{N} = \{\exp(\hbar\omega/kT) + 1\}^{-1} \leq 1$.

e) The Detector Model: Define

$$
\left.
\begin{aligned}
|1\rangle\langle 1| &= n \\
|2\rangle\langle 2| &= m \\
|3\rangle\langle 3| &= l = 1 - m - n \\
|1\rangle\langle 2| &= p
\end{aligned}
\right\} \quad \text{bosonic,} \tag{8.5.16}
$$

$$
\left.
\begin{aligned}
|3\rangle\langle 2| &= q \\
|1\rangle\langle 3| &= r
\end{aligned}
\right\} \quad \text{fermionic.}
$$

There is a field b for the photons, and two separate electron fields d and e. The first represents ejected electrons, and the second the returning electrons. The interaction Hamiltonian is

$$
\begin{aligned}
H_{\text{int}} = {}& i\hbar \int d\omega\, \tilde{\kappa}(\omega) \left\{ d^{\dagger}(\omega)q - q^{\dagger}d(\omega) \right\} \\
&+ i\hbar \int d\omega\, \tilde{\lambda}(\omega) \left\{ e^{\dagger}(\omega)r^{\dagger} - re(\omega) \right\} \\
&+ i\hbar \int d\omega\, \tilde{\gamma}(\omega) \left\{ b^{\dagger}(\omega)p - p^{\dagger}b(\omega) \right\},
\end{aligned} \tag{8.5.17}
$$

and this leads to the equations of motion (in the interaction picture)

$$
\begin{aligned}
\dot{n} &= \gamma m + \lambda l + \sqrt{\gamma}\left\{ p^{\dagger}b_{\text{in}} + b_{\text{in}}^{\dagger}p \right\} + \sqrt{\lambda}\left\{ r^{\dagger}e_{\text{in}}^{\dagger} + e_{\text{in}}r \right\} \\
\dot{m} &= -(\kappa + \gamma)m - \sqrt{\gamma}\left\{ p^{\dagger}b_{\text{in}} + b_{\text{in}}^{\dagger}p \right\} - \sqrt{\kappa}\left\{ q^{\dagger}d_{\text{in}} + d_{\text{in}}^{\dagger}q \right\} \\
\dot{l} &= -\dot{m} - \dot{n} \\
\dot{p} &= -\frac{\kappa}{2}p - \sqrt{\kappa}\,r d_{\text{in}} + \sqrt{\lambda}\,q e_{\text{in}}^{\dagger} - \frac{\gamma}{2}p - \sqrt{\gamma}\,(n - m)b_{\text{in}} \\
\dot{q} &= -\tfrac{1}{2}(\gamma + \kappa + \lambda)q - \sqrt{\gamma}\,r^{\dagger}b_{\text{in}} - \sqrt{\kappa}\,(1 - n)d_{\text{in}} - \sqrt{\lambda}\,e_{\text{in}}p. \\
\dot{r} &= -\frac{\lambda}{2}r - \sqrt{\gamma}\,q^{\dagger}b_{\text{in}} - \sqrt{\kappa}\,d_{\text{in}}^{\dagger}p - \sqrt{\lambda}\,(1 - m)e_{\text{in}}^{\dagger}.
\end{aligned} \tag{8.5.18}
$$

f) Input-Output Boundary Conditions: We want to specify b_{in}, the incoming photon field, and to compute d_{out}, the ejected electron field. We also need to specify that :

d_{in} corresponds to a vacuum—no input electrons are present in the channel to which the ejected electron couples.

e_{in} corresponds to a fully filled bath—there are numerous electrons available in the channels required to return the atom to the ground state.

We know that

$$b_{out} = b_{in} + \sqrt{\gamma}\,p$$
$$d_{out} = d_{in} + \sqrt{\kappa}\,q \qquad\qquad (8.5.19)$$
$$e_{out}^\dagger = e_{in}^\dagger + \sqrt{\lambda}\,r.$$

g) Adiabatic Elimination Limit: As before, assume that γ and κ are large and we can solve the equations for q and p by adiabatic elimination.

$$p = \frac{-\sqrt{\gamma}\,(n-m)b_{in} - \sqrt{\kappa}\,r d_{in} + \sqrt{\lambda}\,q e_{in}^\dagger}{\frac{1}{2}(\gamma + \kappa)} \qquad (8.5.20)$$

$$q = \frac{-\sqrt{\gamma}\,r^\dagger b_{in} - \sqrt{\kappa}\,(1-n)d_{in} - \sqrt{\lambda}\,e_{in}p}{\frac{1}{2}(\gamma + \kappa + \lambda)}. \qquad (8.5.21)$$

The r equation cannot be solved by adiabatic elimination, since we do not put any fast requirement on λ. We can approximate the r equation by noting that the excitation amplitude will be small in practice, for linear operation of the detector, so that we can say $n \sim 1, m \sim l \sim r \sim p \sim q \sim 0$, so that

$$r^\dagger \approx -\sqrt{\lambda} \int\limits_{-\infty}^{t} dt'e^{-\frac{1}{2}\lambda(t-t')}e_{in}(t') \qquad (8.5.22)$$

and

$$q \approx \frac{2\sqrt{\gamma\lambda}}{\gamma + \kappa + \lambda} \int\limits_{-\infty}^{t} dt'e^{-\frac{1}{2}\lambda(t-t')}e_{in}(t')b_{in}(t) - \frac{2\sqrt{\kappa}}{\gamma + \kappa + \lambda}d_{in}(t)$$
$$+ \frac{4\sqrt{\lambda\gamma}\,e_{in}(t)b_{in}(t)}{(\gamma + \kappa + \lambda)(\gamma + \kappa)}. \qquad (8.5.23)$$

We will only be interested in normally ordered products, and since d_{in} is the vacuum, it will contribute nothing. It is not difficult to see that although the other two terms are similar, the first one is much bigger than the other if $\gamma, \kappa \gg \lambda$.

Thus we find

$$d_{out} \simeq \frac{\gamma - \kappa + \lambda}{\gamma + \kappa + \lambda}d_{in} + \frac{2\sqrt{\gamma\kappa\lambda}}{\gamma + \kappa + \lambda} \int\limits_{-\infty}^{t} dt'e^{-\frac{1}{2}\lambda(t-t')}e_{in}(t')b_{in}(t). \qquad (8.5.24)$$

The normally ordered correlation functions of of d_{out} are then given by those of the second term. We take, for a filled $e_{\text{in}}(t)$ that

$$\langle e_{\text{in}}^{\dagger}(t)e_{\text{in}}(t')\rangle = \delta(t-t') \qquad \langle e_{\text{in}}(t)e_{\text{in}}^{\dagger}(t')\rangle = 0 \tag{8.5.25}$$

so that, for example

$$\langle d_{\text{out}}^{\dagger}(t)d_{\text{out}}(t')\rangle = \left(\frac{4\gamma\kappa}{\gamma+\kappa+\lambda}\right)e^{-\frac{1}{2}\lambda|t-t'|}\langle b_{\text{in}}^{\dagger}(t)b_{\text{in}}(t')\rangle. \tag{8.5.26}$$

Provided λ is much *smaller* than the time constant of the correlation function $\langle b_{\text{in}}^{\dagger}(t)b_{\text{in}}(t')\rangle$, this says that the electron correlation function is that of the input photons, up to an efficiency factor of $4\gamma\kappa/(\gamma+\kappa)$, since $\gamma \ll \gamma+\kappa$. This is the same result that we found in the Bose model. The higher order normally ordered correlation functions follow similarly for *even* order correlations, but the odd order correlations will vanish, as should indeed be the case.

Thus the inclusion of Fermi electrons does yield exactly the kind of result predicted by Mandel's formula, without the spurious equality of odd order moments which happens with a Bose electron model. Nevertheless the result for even order moments in the degree of approximation used is exactly the same as that of the Bose model. The reason for this is very simple. The Bose electron model is completely linear, in that the output electron fields are linear functions of the input photon fields. The Fermi electron model cannot be linearised in this sense, since Fermi fields can never be equal to Bose fields, but it is linearised in the only sense that is allowable; namely there is a linear relation between the d_{out} and a bilinear function of the input Bose field and another Fermi field, which arises from the returning electrons. The equivalence of correlation functions is true only if λ, the relaxation time constant of the re-absorption of the electrons, is small compared to those of the photon absorption and electron production. This is appropriate, since it means that the virtual level to which the atom returns after emitting the photoelectron is almost stable, and correlations between photoelectrons and reabsorbed electrons cannot arise.

8.5.3   A Spatially Distributed Detector Model

The fact that we can get essentially the same results from both Bose and Fermi electrons is encouraging, but both of these models are far too simplified. For example, the result of perfect photodetection if $\gamma = \kappa$ is not what one expects from a single atom—rather, it is the property of a lumped oscillator model of a detector. Most treatments of photodetection assume that the medium acts locally as a very weak inefficient detector, with high efficiency obtained by the eventual absorption in a sufficiently thick medium. Let us therefore model a detector of this kind, but simplify matters by assuming again Bose electrons. The model is one dimensional, as

in Fig.8.9. An appropriate Lagrangian is

$$L = \tfrac{1}{2} \int\limits_{-\infty}^{\infty} dx \left\{ \dot{A}(x,t)^2 - c^2 [\partial_x A(x,t)]^2 \right\}$$

$$+ \int\limits_{-\infty}^{0} dx \left\{ \tfrac{1}{2}\dot{u}(x,t)^2 - \tfrac{1}{2}\omega^2 u(x,t)^2 + u(x,t) \left[2\sqrt{\kappa v}\, \dot{D}(x,t,0) + \gamma \dot{A}(x,t) \right] \right\}$$

$$+ \tfrac{1}{2} \int\limits_{-\infty}^{0} dx \int\limits_{0}^{\infty} ds \left\{ \dot{D}(x,t,s)^2 - v^2 [\partial_s D(x,t,s)]^2 \right\}. \qquad (8.5.27)$$
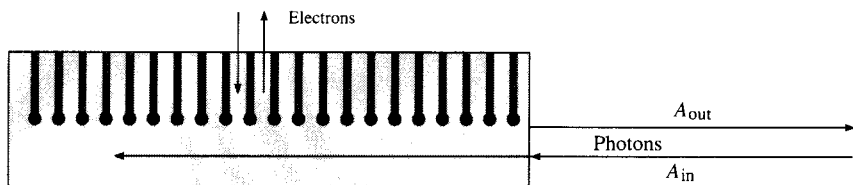
Fig. 8.9   Spatially distributed model of a photodetector

This Lagrangian corresponds to a one dimensional electrodynamics on the range $(-\infty, \infty)$, coupled in the left half of space to a distribution of atoms represented by the operators $u(x,t)$. These atoms are coupled *locally* to a distribution of electron baths, represented by the field $D(x,t,s)$. For the electron baths and the atoms the co-ordinate x does not represent any dynamics, but merely acts as a label to distinguish the different electron baths and atoms at each point—the dynamics takes place in another variable s. Writing the bath Lagrangian in the form given in (8.5.27) has two main advantages over the more conventional Hamiltonian method.
—The input-output formalism is easily developed.
—The identification of outputs corresponding to electrons produced is straightforward.
The physics represented is something like that of a solid state photodetector, in which the electrons produced are immediately localised by thermal and other interactions in the solid, so that the photoelectric processes at two points depend on each other only to the extent enforced by the coherence of the electromagnetic field. The basic concept is thus the same as in the previous part, and it is merely a matter of solving for a D_{out} field in terms of the electromagnetic field being input from the right.

a) Equations of Motion: For the three fields, we find, for $x < 0$

$$\ddot{A} - c^2 \partial_x^2 A = -\gamma \dot{u}$$
$$\ddot{u} + \omega^2 u = \gamma \dot{A} + 2\sqrt{\kappa v}\, \dot{D}(x,t,0) \qquad (8.5.28)$$
$$\ddot{D} - v^2 \partial_s^2 D = -2\sqrt{\kappa v}\, \delta(s)\dot{u}.$$

Since $D(x, t, s)$ represents the electron bath, we have to solve for it in terms of the electron inputs and outputs. The solution of the equation can be written in two equivalent forms;

$$
\begin{aligned}
D(x, t, s) &= D_{\text{in}}(x, t + s/v) + D_{\text{in}}(x, t - s/v) \\
&\quad - \sqrt{\frac{\kappa}{v}}\, u(x, t - s/v) \\
&= D_{\text{out}}(x, t - s/v) + D_{\text{out}}(x, t + s/v) \\
&\quad + \sqrt{\frac{\kappa}{v}}\, u(x, t + s/v).
\end{aligned}
\tag{8.5.29}
$$

We can substitute the first of equations (8.5.29) into the equation for u given in (8.5.28) to get as our basic pair of equations, for $x < 0$, (that is, inside the photodetection medium).

$$
\begin{aligned}
\ddot{A} - c^2 \partial_x^2 A &= -\gamma \dot{u} \\
\ddot{u} + \omega^2 u &= \gamma \dot{A} - \kappa \dot{u} + 2\sqrt{\kappa v}\, \dot{D}_{\text{in}}(x, t).
\end{aligned}
\tag{8.5.30}
$$

To the right, $x > 0$, where there is no medium, we simply have the free wave equation $\ddot{A} - c^2 \partial_x^2 A = 0$. The other variables do not exist for $x > 0$.

b) Solution of the Equations of Motion: If we define a Fourier transform variable by

$$
u(t, x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} d\Omega\, e^{-i\Omega t} \tilde{u}(\Omega, x)
\tag{8.5.31}
$$

with a similar definition for $\tilde{A}(\Omega, x)$ and $\tilde{D}(\Omega, x, s)$, we find that the u equation can be reduced to

$$
\tilde{u} = \frac{(-i\Omega)\left[2\sqrt{\kappa v}\, \tilde{D}_{\text{in}}(\Omega, x) + \gamma \tilde{A}\right]}{\omega^2 - \Omega^2 - i\kappa\Omega}
\tag{8.5.32}
$$

which, when substituted into the equation for $\tilde{A}$, yields

$$
-\tilde{A}\left\{\Omega^2 + \frac{\gamma^2 \Omega^2}{\omega^2 - \Omega^2 - i\Omega\kappa}\right\} - c^2 \partial_x^2 \tilde{A} = \frac{2\gamma\Omega^2 \sqrt{\kappa v}}{\omega^2 - \Omega^2 - i\Omega\kappa}\tilde{D}_{\text{in}}(\Omega, x).
\tag{8.5.33}
$$

This is a wave equation with both dispersion and absorption, as expected for such a medium. Notice, however, the appearance of a "quantum noise" term on the right-hand side, which arises from the vacuum fluctuations in the electron baths. This acts to ensure the preservation of canonical commutation relations.

Let us define a quantity ν by

$$
\nu^2 = \Omega^2 + \frac{\gamma^2 \Omega^2}{\omega^2 - \Omega^2 - i\Omega\kappa}.
\tag{8.5.34}
$$

In particular, we define ν so that it is the root of (8.5.34) with a *negative imaginary part*. (Notice that this means for large Ω that $\nu \sim -\Omega$). In terms of ν, (8.5.33) can be written

$$-\nu^2 \tilde{A} - c^2 \partial_x^2 \tilde{A} = (\nu^2 - \Omega^2) \frac{2\sqrt{\kappa v}}{\gamma} \tilde{D}_{\text{in}}(\Omega, x). \tag{8.5.35}$$

A general solution of this equation which satisfies the boundary condition that $\tilde{A}(\Omega, x)$ be finite as $x \to -\infty$ is (noting that $\text{Im}(\nu) < 0$ by definition)

$$\tilde{A}(\Omega, x) = Q e^{i\nu x/c} + \frac{2\sqrt{\kappa v}}{\gamma} \int\limits_{-\infty}^{\infty} dx' (\nu^2 - \Omega^2) \tilde{D}_{\text{in}}(\Omega, x') \frac{e^{-i\nu|x-x'|/c}}{2i\nu c} \tag{8.5.36}$$

where Q is to be determined from boundary conditions.

c) Boundary Conditions of the Interface: The conditions of continuity of electric and magnetic fields must be met at $x = 0$, the edge of the medium. The solution of the wave equation to the right of $x = 0$ is of the form

$$A(x, t) = a_{\text{in}}(t + x/c) + a_{\text{out}}(t - x/c), \tag{8.5.37}$$

so that the boundary conditions are

$$\dot{A}(x, t)|_{x \to 0-} = \dot{a}_{\text{in}}(t + x/c) + \dot{a}_{\text{out}}(t - x/c) \tag{8.5.38}$$

and

$$\partial_x A(x, t)|_{x \to 0-} = \frac{1}{c}\left\{ \dot{a}_{\text{in}}(t + x/c) - \dot{a}_{\text{out}}(t - x/c) \right\}. \tag{8.5.39}$$

Taking the Ω frequency component of these boundary conditions and combining with (8.5.36) gives

$$Q = \frac{2\Omega \tilde{a}_{\text{in}}}{\Omega - \nu} + \frac{2\sqrt{\kappa v}}{\gamma}(\Omega + \nu)^2 \int\limits_{-\infty}^{\infty} dx \tilde{D}_{\text{in}}(\Omega, x) \frac{e^{i\nu x/c}}{2i\nu c} \tag{8.5.40}$$

$$\tilde{a}_{\text{out}} = \left(\frac{\Omega + \nu}{\Omega - \nu}\right) \tilde{a}_{\text{in}} + \frac{4\sqrt{\kappa v}}{\gamma}(\Omega + \nu)\nu \int\limits_{-\infty}^{\infty} dx \tilde{D}_{\text{in}}(\Omega, x) \frac{e^{i\nu x/c}}{2i\nu c}. \tag{8.5.41}$$

The equation for $\tilde{a}_{\text{out}}$ shows a reflected term, with reflection coefficient $\left|\frac{\Omega+\nu}{\Omega-\nu}\right|^2$, and a term which arises from the quantum fluctuations of the electron bath.

d) Attenuation Length: Since ν has an imaginary part, the factor $e^{i\nu x/c}$ in (8.5.36) represents an attenuation of the input beam with depth x. The characteristic absorption depth is $l_a = \frac{1}{2}\{|\text{Im}(\nu/c)|\}^{-1}$. An exact expression is very complicated, but if we assume that the second term in (8.5.34) is always smaller than Ω^2, then

$$l_a \approx \frac{(\omega^2 - \Omega^2)^2 + \Omega^2 \kappa^2}{\Omega^2 \kappa \gamma^2}. \tag{8.5.42}$$

This shows a strong resonant behaviour—the maximum absorption obviously takes place when $\omega \simeq \Omega$. The ratio of the wavelength to the minimum attenuation length is $\lambda/l_a \propto \gamma^2/\Omega\kappa$, and if this ratio is small, the approximations required to derive (8.5.42) are valid. This is also the regime in which one would expect to use such a detector.

e) Preservation of Commutation Relations: Although it is quite tricky, it can be shown that these solutions do preserve all canonical commutation relations.

f) Detector Efficiency and Statistics: We now want to solve for the D_{out} field, which corresponds to the ejected electrons which are counted. Notice that (8.5.29) implies that

$$D_{\text{out}}(x,t) = D_{\text{in}}(x,t) - \sqrt{\frac{\kappa}{v}}\, u(x,t). \tag{8.5.43}$$

We use the solution for $u(x,t)$ implicit in (8.5.32), into which we substitute the solution (8.5.36) for $A(t,x)$. The solution for $\tilde{D}_{\text{out}}$ can be written:

$$\begin{aligned}
\tilde{D}_{\text{out}}(\Omega,x) = {}& -\frac{2i}{\gamma}\sqrt{\frac{\kappa}{v}}(\Omega+\nu)e^{i\nu x/c}\,\tilde{a}_{\text{in}}(\Omega) \\
& + \left(\frac{\omega^2 - \Omega^2 + i\kappa\Omega}{\omega^2 - \Omega^2 - i\kappa\Omega}\right)\tilde{D}_{\text{in}}(x,\Omega) \\
& + \frac{2i\kappa}{\gamma^2\Omega}(\nu^2 - \Omega^2)(\Omega+\nu)^2 \int\limits_{-\infty}^{\infty} dx'\,\tilde{D}_{\text{in}}(\Omega,x')\frac{e^{i\nu(x'+x)/c}}{2i\nu c} \\
& + \frac{2i\kappa}{\gamma^2\Omega}(\nu^2 - \Omega^2)^2 \int\limits_{-\infty}^{\infty} dx'\,\tilde{D}_{\text{in}}(\Omega,x')\frac{e^{-i\nu|x-x'|/c}}{2i\nu c}.
\end{aligned} \tag{8.5.44}$$

While this appears rather complicated, it does in fact represent the physics of photodetection rather naturally. There are two basic components, one arising from the input electromagnetic field, and the rest from the input electron field. The input from the latter will be assumed to consist of vacuum fluctuations only, and serves mainly to ensure the preservation of commutation relations of $\tilde{D}_{\text{out}}(\Omega,x)$. As in the initial simple one atom model, the normally ordered moments of $\tilde{D}_{\text{out}}(\Omega,x)$ will be given by those of the $\tilde{a}_{\text{in}}(\Omega)$ term, thus ensuring that a formula like Mandel's formula is valid *locally*.

g) Output Electron Flux: If we were dealing with real electrons, rather than our present Bose electrons, we would now simply calculate the electric current operator from the output field. However, with this kind of Bose electron, the electric current is not easy to define, since a charged particle can only be represented by a non-Hermitian field, which in turn would require us to use a rather more sophisticated model of the atom than a harmonic oscillator.

The only electron flux available in this formalism is the energy flux per unit length, given by

$$N(s,t,x) = -v^2 : \frac{\partial D}{\partial t}\frac{\partial D}{\partial s} : \tag{8.5.45}$$

corresponding to a kind of Poynting vector. Thus, in a time interval (t_a, t_b), the energy flowing into the electron bath per unit length of the detector is, at the point s,

$$-v^2 \int_{t_a}^{t_b} : \frac{\partial}{\partial t} D_{\text{out}}(t - s/v, x) \frac{\partial}{\partial s} D_{\text{out}}(t - s/v, x) : dt \qquad (8.5.46)$$

$$= v \int_{t_a}^{t_b} : \dot{D}_{\text{out}}(t - s/v, x)^2 : dt. \qquad (8.5.47)$$

This operator quantity is normally ordered to eliminate vacuum contributions. Using (8.5.44) we can see that there will be terms arising from $\tilde{D}_{\text{in}}(\Omega, x)$, and $\tilde{a}_{\text{in}}(\Omega)$; but if there is no input electron field, the $\tilde{D}_{\text{in}}$ terms will give no contribution to the expectations of normally ordered products; so that the normally ordered moments of (8.5.45) which are what is measured, will be given by substituting into (8.5.47) the terms in (8.5.44) which involve $\tilde{a}_{\text{in}}(\Omega)$.

Suppose we assume that the input field is stationary, so that

$$\langle \tilde{a}_{\text{in}}(\Omega)\tilde{a}_{\text{in}}(\Omega') \rangle = \delta(\Omega + \Omega')S(\Omega). \qquad (8.5.48)$$

Then, we can substitute (8.5.44) into (8.5.47), and take the average, to get

$$v \int_{t_a}^{t_b} \langle : \dot{D}_{\text{out}}(t - s/v, x)^2 : \rangle dt$$

$$= \frac{2\kappa}{\pi\gamma^2}(t_b - t_a) \int_{-\infty}^{\infty} d\Omega |\Omega|^2 |\Omega + \nu|^2 e^{-2\,\text{Im}(\nu x/c)} S(\Omega). \qquad (8.5.49)$$

The total power absorbed is given by integrating this over all negative x; namely

$$\text{total absorbed power} = \int_{-\infty}^{\infty} d\Omega \frac{\kappa(t_b - t_a)}{\pi\gamma^2 \text{Im}(\nu/c)} S(\Omega)|\Omega|^2 |\Omega + \nu|^2. \qquad (8.5.50)$$

It is not too difficult to show that input power = total power absorbed + power reflected, i.e. from (8.5.41), using the coefficient of $\tilde{a}_{\text{in}}(\Omega)$, we can show

$$\frac{(8.5.50)}{t_b - t_a} + \frac{c}{2\pi} \int_{-\infty}^{\infty} d\Omega \left|\frac{\Omega + \nu}{\Omega - \nu}\right|^2 S(\Omega)|\Omega|^2 = \frac{c}{2\pi} \int_{-\infty}^{\infty} d\Omega S(\Omega)|\Omega|^2. \qquad (8.5.51)$$

h) Detector Bandwidth: It is paradoxical that in fact the maximum efficiency is achieved by the weakest detection. If the attenuation length is very large, i.e., $\gamma^2 \ll \Omega/\kappa$, the efficiency approaches 100% at all frequencies. However this is only a crude counting efficiency. To deduce the detector bandwidth one must work out the time correlation function $\langle : N(s, t, x)N(s', t', x') : \rangle$, and appropriately integrate this over the unobserved parameters.

9. Interaction of Light with Atoms

Systems composed of a number of free atoms interacting with a few modes of the electromagnetic field are particularly well studied both theoretically and experimentally. Such systems are much simpler than those consisting of solids or liquids, where a much wider variety of kinds of behaviour is possible. Theoretical treatments often simplify matters even more by assuming that the atoms only have two levels, which experimentally can only be achieved by rather complicated optical pumping techniques. However, such experiments as have been done with two level atoms show impressive agreement with theory [9.1].

This chapter consists of an introductory part, describing the algebra of two level systems, and then treats the behaviour of a single two level atom in thermal and coherent light fields. These results are then used in two major applications, the theory of a gas laser, and the theory of optical bistability induced by an atomic vapour inside an optical cavity.

9.1 Two Level Systems

A fundamental building block in quantum theory is the two level system. This is an idealisation of an atom, or molecule, which has only two energy levels. Although it is in principle possible for an atomic system to have only two *bound* states, there would also be a continuum of unbound or scattering states. In the ideal two level atom, these continuum states do not exist. The basic idea is that we can consider situations in which transitions to the continuum are negligible, and similarly, any transitions to other energy levels are negligible. If the energy differences between the energy levels are all very different this makes sense, since radiation which might induce a transition between a given pair of levels will have an energy which will be completely detuned from any other transition. The only situation for which this does not make sense is the harmonic oscillator, since the energy levels are evenly spaced, so that it is impossible to consider incident radiation as only exciting one transition.

9.1.1   Pauli Matrix Description

The two level atom is described by a wavefunction which is a two component matrix

$$u = \begin{pmatrix} u^{(+)} \\ u^{(-)} \end{pmatrix} \tag{9.1.1}$$

and all operators can be expressed in terms of the Pauli matrices

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \qquad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{9.1.2}$$

and the identity matrix, $\mathbf{1}$. The Hamiltonian for such a two level system then can always be written

$$H = \tfrac{1}{2}\hbar\,\Omega\boldsymbol{\sigma}\cdot\hat{\mathbf{n}} \tag{9.1.3}$$

where $\hat{\mathbf{n}}$ is a unit vector—if this vector is along the z axis, then

$$H = \tfrac{1}{2}\hbar\,\Omega\sigma_z = \begin{pmatrix} \tfrac{1}{2}\hbar\,\Omega & 0 \\ 0 & -\tfrac{1}{2}\hbar\,\Omega \end{pmatrix}. \tag{9.1.4}$$

9.1.2   Pauli Matrix Properties

The proofs are left as an exercise :

i) $\sigma_i\sigma_j = \delta_{ij} + i\sum_k \epsilon_{ijk}\sigma_k.$ $\hspace{2cm}$ (9.1.5)

ii) $(\boldsymbol{\sigma}\cdot\mathbf{A})(\boldsymbol{\sigma}\cdot\mathbf{B}) = \mathbf{A}\cdot\mathbf{B} + i\boldsymbol{\sigma}\cdot\mathbf{A}\times\mathbf{B};$ $\hspace{1.5cm}$ (9.1.6)

iii) $\mathbf{A}$ and $\mathbf{B}$ are c-number or operator vectors.

iii) $[\sigma_i,\sigma_j] = 2i\sum_k \epsilon_{ijk}\sigma_k.$ $\hspace{2.5cm}$ (9.1.7)

iv) $[\sigma_i,\sigma_j]_+ = 2\delta_{ij}.$ $\hspace{3.5cm}$ (9.1.8)

v)   If G is any two dimensional matrix

$$G = \tfrac{1}{2}\left\{\mathbf{1}\,\mathrm{Tr}\,\{G\} + \sum_i \sigma_i\,\mathrm{Tr}\,\{\sigma_i G\}\right\}. \tag{9.1.9}$$

vi) $\exp\{i\theta\boldsymbol{\sigma}\cdot\mathbf{n}\} = \cos\theta + i\boldsymbol{\sigma}\cdot\mathbf{n}\sin\theta.$ $\hspace{1.5cm}$ (9.1.10)

vii)   If

$$\sigma_+ = \tfrac{1}{2}(\sigma_x + i\sigma_y) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \qquad (9.1.11)$$

$$\sigma_- = \tfrac{1}{2}(\sigma_x - i\sigma_y) = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \qquad (9.1.12)$$

then

$$\sigma_+^2 = \sigma_-^2 = 0 \qquad (9.1.13)$$

$$\sigma_+\sigma_- = \tfrac{1}{2}(1 + \sigma_z) \qquad (9.1.14)$$

$$\sigma_-\sigma_+ = \tfrac{1}{2}(1 - \sigma_z) \qquad (9.1.15)$$

$$\sigma_z\sigma_+ = -\sigma_+\sigma_z = \sigma_+ \qquad (9.1.16)$$

$$\sigma_z\sigma_- = -\sigma_-\sigma_z = -\sigma_-. \qquad (9.1.17)$$

viii)   For any matrix G

$$G = \tfrac{1}{2}1\,\mathrm{Tr}\,\{G\} + \tfrac{1}{2}\sigma_z\mathrm{Tr}\,\{\sigma_z G\} + \sigma_+\mathrm{Tr}\,\{\sigma_- G\} + \sigma_-\mathrm{Tr}\,\{\sigma_+ G\}. \qquad (9.1.18)$$

xi) $$\exp(a + \mathbf{b}\cdot\boldsymbol{\sigma}) = e^a\left\{\cosh|\mathbf{b}| + \frac{\boldsymbol{\sigma}\cdot\mathbf{b}}{|\mathbf{b}|}\sinh|\mathbf{b}|\right\}. \qquad (9.1.19)$$

9.1.3   Atoms with More Than Two Levels

Notice that atoms with more than two levels can be handled by similar methods. If the atom has eigenstates $|i\rangle$, one defines matrices by

$$\lambda_{ij} = |i\rangle\langle j|. \qquad (9.1.20)$$

These matrices are of course linearly complete, and also have a closed algebra corresponding to (9.1.6)

$$\lambda_{ij}\lambda_{kl} = \delta_{jk}\lambda_{il}. \qquad (9.1.21)$$

With care, almost all two level problems in this chapter can be generalised.

9.2   Two Level Atom in the Electromagnetic Field

The quantum Langevin equation for the two level atom was derived in Sect. 3.4.4, where it was noted that exact solutions were not possible. Using the same notation as in that section, it is straightforward to substitute into the master equation, and to show that one obtains essentially the same results as in Sect. 1.5.1

$$X \to \sqrt{\frac{\hbar}{\Omega}}\,\sigma_z = \sqrt{\frac{\hbar}{\Omega}}\,(\sigma^+ + \sigma^-) \qquad (9.2.1)$$

$$H_{\text{sys}} \to \tfrac{1}{2}\hbar\,\Omega\sigma_z. \tag{9.2.2}$$

We see that the process of splitting the operator X into eigenoperators as in (3.6.61) corresponds exactly to that in (9.2.1) so that m takes on only one value, and

$$X^{\pm} \to \hbar\,\sigma^{\pm}. \tag{9.2.3}$$

Using the Pauli matrix algebra, the master equation can be reduced to

$$
\begin{aligned}
\dot{\rho} = &-\tfrac{1}{2}\mathrm{i}\Omega[\sigma_z,\rho] \\
&+\frac{\pi}{2}\left(\bar{N}(\Omega)+1\right)\kappa(\Omega)^2\left\{2\sigma^-\rho\sigma^+-\rho\sigma^+\sigma^--\sigma^+\sigma^-\rho+\sigma^-\rho\sigma^-+\sigma^+\rho\sigma^+\right\} \\
&+\frac{\pi}{2}\bar{N}(\Omega)\kappa(\Omega)^2\left\{2\sigma^+\rho\sigma^--\rho\sigma^-\sigma^+-\sigma^+\sigma^-\rho+\sigma^-\rho\sigma^-+\sigma^+\rho\sigma^+\right\} \\
&+\frac{\mathrm{i}}{2\Omega}\mathrm{P}\int_{-\infty}^{\infty}\frac{\omega\kappa(\omega)^2 d\omega}{\Omega-\omega}\left(\bar{N}(\omega)+\tfrac{1}{2}\right)\left\{-[\sigma_z,\rho]-2\sigma^+\rho\sigma^++2\sigma^-\rho\sigma^-\right\}.
\end{aligned}
\tag{9.2.4}
$$

Exercise. Using the algebra of Pauli matrices, including in particular the formulae in (9.1.12–18) derive the Master equation in the form (9.2.4).

Exercise. Using the cyclic property of the trace the Pauli matrix algebra, and

$$\frac{d}{dt}\langle\sigma^+\rangle = \mathrm{Tr}\{\dot{\rho}\sigma^+\}, \quad \text{etc.,} \tag{9.2.5}$$

show that the equations of motion for the means of $\sigma^{\pm}, \sigma_z$ are

$$\frac{d}{dt}\langle\boldsymbol{\sigma}\rangle = \boldsymbol{\Omega}\langle\boldsymbol{\sigma}\rangle - \boldsymbol{\gamma} \tag{9.2.6}$$

in which

$$
\boldsymbol{\sigma} = \begin{pmatrix}\sigma^+ \\ \sigma^- \\ \sigma_z\end{pmatrix}, \qquad \boldsymbol{\gamma} = \begin{pmatrix}0 \\ 0 \\ \gamma\end{pmatrix} \tag{9.2.7}
$$

and

$$
\boldsymbol{\Omega} = \begin{pmatrix}
\mathrm{i}(\Omega+\Delta\Omega)-\gamma(\bar{N}+\tfrac{1}{2}) & \gamma(\bar{N}+\tfrac{1}{2})+2\mathrm{i}\,\Delta\Omega & 0 \\
\gamma(\bar{N}+\tfrac{1}{2})-2\mathrm{i}\,\Delta\Omega & -\mathrm{i}(\Omega+\Delta\Omega)-\gamma(\bar{N}+\tfrac{1}{2}) & 0 \\
0 & 0 & -\gamma(2\bar{N}+1)
\end{pmatrix}.
\tag{9.2.8}
$$

Quantities in (9.2.7,8) are defined by

$$\bar{N} = \bar{N}(\Omega) \tag{9.2.9}$$

$$\gamma = \pi\kappa(\Omega)^2 \tag{9.2.10}$$

$$\Delta\Omega = \frac{1}{\Omega}\mathrm{P}\int_{-\infty}^{\infty}\frac{\omega\kappa(\omega)^2 d\omega}{\Omega-\omega}\left(\bar{N}(\omega)+\tfrac{1}{2}\right). \tag{9.2.11}$$

9.2.1 Lamb and Stark Shifts

From the structure of (9.2.8), $\Delta\Omega$ represents a frequency shift, in the sense that it occurs in (9.2.4) as the coefficient of a term $[\sigma_z, \rho]$ which is of the same kind as the original system Hamiltonian. Thus, in interaction with the bath (or radiation field in this case) this frequency observed will be $\Omega + \Delta\Omega$.

i) If $\kappa(\omega)$ is constant, $\Delta\Omega$ is divergent—this means that for a point atom this kind of formulation does not make sense. The original model is unrealistic, in that there is no kinetic term in the atomic Hamiltonian, and thus recoil is impossible. In fact, it is only in the case of a fully relativistic Hamiltonian that the Lamb shift can be made finite, and even then after QED renormalisation, which involves infinities.

ii) For the purposes of this book it is best to regard the model of an atom fixed in space as having a small but finite extent, so that $\kappa(\omega)$ drops off sufficiently rapidly at high ω to make the integral (9.2.11) convergent.

iii) Of course the $\bar{N}(\omega)$ term is finite at all $T > 0$, and it vanishes when $T = 0$. This part is a shift caused by the thermal radiation which the atom experiences. This is called the Stark shift; the Lamb shift is that part proportional to the $\frac{1}{2}$, and has its source in the vacuum fluctutations.

iv) Notice that one can modify the density of states, perhaps by use of a cavity, so that $\kappa(\omega)^2$ varies strongly around $\omega \approx \Omega$. If $\kappa(\omega)^2$ is smooth, the principal value integral is almost zero, but if it changes rapidly in this region the cancellation will not occur, and the Lamb and Stark shifts can both be very large.

v) One should keep in mind that these frequency shifts are small in comparison to Ω, but are nevertheless of the same order of magnitude as the damping γ— indeed they have a common origin. The damping is more noticeable only because it gives a decay which is otherwise absent. The frequency shifts can normally be neglected; or at least absorbed into Ω.

9.2.2 Rotating Wave Approximation

But it is not quite fair to say this, since there are a larger number of other terms. However these do not have a large effect. The simplest way to argue is via the method of the rotating wave approximation of Sect. 3.6.3a. Applied directly to (9.2.6), this says that $\Omega + \Delta\Omega$ is much larger than all other coefficients, so to a first approximation the time dependence of $\langle \sigma^\pm \rangle$ is $e^{\pm i(\Omega + \Delta\Omega)t}$. If we substitute

$$\langle \sigma^\pm(t) \rangle = S^\pm(t) e^{\pm(i\Omega + \Delta\Omega)t}$$
$$\langle \sigma_z(t) \rangle = S_z(t), \tag{9.2.12}$$

the equations of motion become

$$\dot{S}^\pm = -\tfrac{1}{2}\gamma(2\bar{N} + 1)S^\pm + \left\{\tfrac{1}{2}\gamma(2\bar{N} + 1) \pm 2i\,\Delta\Omega\right\} S^\pm e^{\pm 2i(\Omega + \Delta\Omega)t}$$
$$\dot{S}_z = -\gamma(2\bar{N} + 1)S_z - \gamma. \tag{9.2.13}$$

The very rapidly varying terms $e^{\pm 2i(\Omega + \Delta\Omega)t}$ average out to zero on the time scale of the other time constants, $\gamma(2\bar{N} + 1)$ and are thus neglected. This means the solutions can be approximated by

$$\langle \sigma^{\pm}(t) \rangle = \langle \sigma^{\pm}(0) \rangle \exp \left\{ \pm (i\Omega + \Delta\Omega)t - \tfrac{1}{2}\gamma(2\bar{N} + 1)t \right\} \tag{9.2.14}$$

$$\langle \sigma_z(t) \rangle = \langle \sigma_z(0) \rangle \exp \left\{ -\gamma(2\bar{N} + 1)t \right\} - \frac{1}{2\bar{N} + 1} \left\{ 1 - \exp(2\bar{N} + 1)t \right\} \tag{9.2.15}$$

Compared to the equations of motion for a free spin system, we see exponential damping factors, which cause $\langle \sigma^{\pm} \rangle$ to approach zero, and $\langle \sigma_z \rangle$ to approach the value $-1/(2\bar{N} + 1)$. Using $\bar{N} = \left(\exp(\hbar\Omega/\kappa T) - 1 \right)^{-1}$ gives the correct result for the Boltzmann distribution over the energy levels.

Exercise. Instead of making the rotating wave approximation solve (9.2.6) by computing the eigenvalues and eigenvectors of the coefficient matrix. In particular, show that the two complex eigenvalues are, to first order in Ω^{-1}

$$\pm i(\Omega + \Delta\Omega) - \tfrac{1}{2}\gamma(2\bar{N} + 1) \pm \frac{i}{\Omega} \left\{ \frac{3(\Delta\Omega)^2}{2} + \frac{\gamma^2(2\bar{N} + 1)^2}{4} \right\} \tag{9.2.16}$$

which agree with (9.2.14) to the order Ω^0.

9.2.3   Master Equation and QSDE

The practical investigation of atomic spectra is carried out by shining light onto atoms, and observing light which is emitted. Because the world is three dimensional, the mode which represents the incoming light is not usually the same as that which represents the outgoing light. As a simplified representation of the situation, let us consider an atom coupled to two inputs, and write down the Ito QSDEs using terms (5.3.36) and (5.3.50).
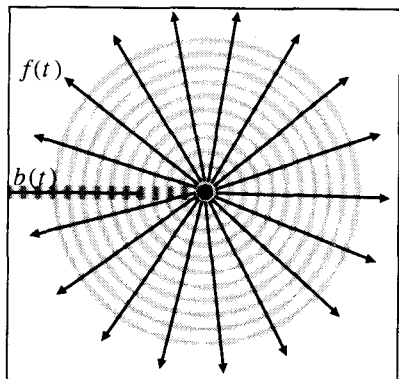
Fig. 9.1   Atom coupled to an almost plane wave mode and to a spherical wave mode

We use the interaction picture, and thus write

$$H_{\text{sys}} = 0$$
$$b_{\text{in}}(1,t) = b_{\text{in}}(t); \quad b_{\text{in}}(2,t) = f_{\text{in}}(t); \quad c_1 = c_2 = \sigma^- \tag{9.2.17}$$

and

$$\langle b_{\text{in}}^\dagger(t) b_{\text{in}}(t) \rangle = \bar{N}_0 \delta(t - t'); \quad \langle f_{\text{in}}^\dagger(t) f_{\text{in}}(t') \rangle = 0. \tag{9.2.18}$$

Using the Pauli matrix formulae of Sect. 9.1.2, we find the three QSDE's

$$\dot{\sigma}^+ = -\left\{ \frac{\gamma_1}{2}(2\bar{N}_0 + 1) + \frac{\gamma_2}{2} \right\} \sigma^+ + \sigma_z \left\{ \sqrt{\gamma_1}\, b_{\text{in}}^\dagger(t) + \sqrt{\gamma_2}\, f_{\text{in}}^\dagger(t) \right\} \tag{9.2.19}$$

$$\dot{\sigma}^- = -\left\{ \frac{\gamma_1}{2}(2\bar{N}_0 + 1) + \frac{\gamma_2}{2} \right\} \sigma^- + \sigma_z \left\{ \sqrt{\gamma_1}\, b_{\text{in}}(t) + \sqrt{\gamma_2}\, f_{\text{in}}(t) \right\} \tag{9.2.20}$$

$$\dot{\sigma}_z = -\left\{ \gamma_1(2\bar{N}_0 + 1) + \gamma_2 \right\} \sigma_z - (\gamma_1 + \gamma_2)$$
$$- 2\sigma^+ \left(\sqrt{\gamma_1}\, b_{\text{in}}(t) + \sqrt{\gamma_2}\, f_{\text{in}}(t) \right) - 2\sigma^- \left(\sqrt{\gamma_1}\, b_{\text{in}}^\dagger(t) + \sqrt{\gamma_2}\, f_{\text{in}}^\dagger(t) \right). \tag{9.2.21}$$

a) Input and Output fields: We assume that the atom is illuminated with the field $b_{\text{in}}(t)$, and observed with the field $f_{\text{out}}(t)$, where, of course

$$f_{\text{out}}(t) = f_{\text{in}}(t) + \sqrt{\gamma_2}\, \sigma^-(t). \tag{9.2.22}$$

We will want to compute the 2nd and 4th order correlation functions, which correspond to the amplitude and intensity correlation functions, and are experimentally accessible. These are best done using the master equation, and the formulae of Sect. 5.2.1.

b) Master Equation Evolution Operator: For simplicity, we define

$$\gamma = \gamma_1 + \gamma_2$$
$$\bar{N} = \frac{\gamma_1 \bar{N}_0}{\gamma_1 + \gamma_2} \tag{9.2.23}$$
$$\kappa = \tfrac{1}{2}\left[\gamma_1(2\bar{N}_0 + 1) + \gamma_2 \right] \equiv \tfrac{1}{2}\gamma(2\bar{N} + 1).$$

The equations of motion for the atomic averages are thus

$$\langle \dot{\sigma}^+ \rangle = -\kappa \langle \sigma^+ \rangle$$
$$\langle \dot{\sigma}^- \rangle = -\kappa \langle \sigma^- \rangle \tag{9.2.24}$$
$$\langle \dot{\sigma}_z \rangle = -2\kappa \langle \sigma_z \rangle - \gamma$$

with the solutions

$$\langle \sigma^+(t) \rangle = e^{-\kappa t} \langle \sigma^+(0) \rangle$$
$$\langle \sigma^-(t) \rangle = e^{-\kappa t} \langle \sigma^-(0) \rangle \tag{9.2.25}$$
$$\langle \sigma_z(t) \rangle = e^{-2\kappa t} \langle \sigma_z(0) \rangle - \frac{1}{2\bar{N} + 1} \left(1 - e^{-2\kappa t} \right).$$

Using the Pauli matrix properties, we can write the density matrix $\rho(t)$ as

$$\rho(t) = \tfrac{1}{2}\left\{1 + \langle\sigma_z(t)\rangle\sigma_z\right\} + \langle\sigma^+(t)\rangle\sigma^- + \langle\sigma^-(t)\rangle\sigma^+ \tag{9.2.26}$$

which gives us the general solution to the master equation. Indeed, it gives a description of the evolution operator $V(t)$ acting on any matrix *with unit trace*.

In general if $\mathrm{Tr}\{\rho(t)\} \neq 1$ one writes

$$\rho(t) = \tfrac{1}{2}\mathrm{Tr}\{\rho(0)\} + \tfrac{1}{2}\langle\sigma_z(t)\rangle\sigma_z + \langle\sigma^+(t)\rangle\sigma^- + \langle\sigma^-(t)\rangle\sigma^+ \tag{9.2.27}$$

in which

$$\langle\sigma_z(t)\rangle = \mathrm{Tr}\{\sigma_z\rho(t)\} \tag{9.2.28}$$

i.e. we ignore the fact that $\rho(t)$ does not have unit trace in this definition.

c) Stationary Solutions—Boltzmann Formula:

As $t \to \infty$, we find

$$\langle\sigma^+(t)\rangle, \quad \langle\sigma^-(t)\rangle \to 0 \tag{9.2.29}$$

$$\langle\sigma_z(t)\rangle \to -1/(2\bar{N}+1) \tag{9.2.30}$$

$$\rho(t) \to \rho_s = \tfrac{1}{2}\left[1 - \sigma_z/(2\bar{N}+1)\right] = \begin{pmatrix} \frac{N}{2N+1} & 0 \\ 0 & \frac{N+1}{2N+1} \end{pmatrix}. \tag{9.2.31}$$

The relations (9.2.23) show that the equations of motion are the same as if we had only one noise, characterised by $\bar{N}$. If this is a thermal field, approximated by white noise, then $\bar{N} = 1/\left(e^{\hbar\Omega/kT} - 1\right)$, and the stationary density operator shows that the ratio of the population in the upper level to that in the lower level is $e^{-\hbar\Omega/kT}$, the Boltzmann formula.

d) Amplitude Correlation Function:

Using the formulae of Sect. 9.2.22 and (5.4.6), we note that the input $f_{\mathrm{in}}(t)$ is in the vacuum state, so that (5.4.21) may be used to get

$$\langle f_{\mathrm{out}}^\dagger(t)f_{\mathrm{out}}(t')\rangle = \gamma_2\langle\sigma^+(t)\sigma^-(t')\rangle \tag{9.2.32}$$

and using (5.2.23), the stationary output amplitude correlation function can be written in terms of

$$\langle\sigma^+(t)\sigma^-(0)\rangle_s = \mathrm{Tr}\{\sigma^+ V(t,0)\sigma^-\rho_s\}. \tag{9.2.33}$$

Notice now that

i)   Using the Pauli matrix formulae and (9.2.31)

$$\sigma^-\rho_s = \frac{\bar{N}}{2\bar{N}+1}\sigma^-. \tag{9.2.34}$$

ii) σ^- does not have unit trace: hence we can use the solutions (9.2.25) where, for example, we now have

$$\langle\sigma^+(0)\rangle = \text{Tr}\left\{\sigma^+\sigma^-\right\} = 1$$
$$\langle\sigma^-(0)\rangle = \text{Tr}\left\{\sigma^-\sigma^-\right\} = 0 \qquad (9.2.35)$$
$$\langle\sigma_z(0)\rangle = \text{Tr}\left\{\sigma_z\sigma^-\right\} = 0.$$

Thus

$$V(t)\{\sigma^-\rho_s\} = \frac{\bar{N}}{2\bar{N}+1}\sigma^-e^{-\kappa t}. \qquad (9.2.36)$$

iii) Hence

$$\langle\sigma^+(t)\sigma^-(0)\rangle_s = \text{Tr}\left\{\sigma^+V(t)\{\sigma^-\rho_s\}\right\} = \frac{\bar{N}}{2\bar{N}+1}e^{-\kappa t}. \qquad (9.2.37)$$

Similarly,

$$\langle\sigma^-(t)\sigma^+(0)\rangle_s = \frac{\bar{N}+1}{2\bar{N}+1}e^{-\kappa t}. \qquad (9.2.38)$$

and thus

$$\langle f_{\text{out}}^\dagger(t)f_{\text{out}}(0)\rangle_s = \frac{\gamma_2\bar{N}}{2\bar{N}+1}e^{-\gamma(\bar{N}+\frac{1}{2})|t|}. \qquad (9.2.39)$$

e) **Intensity Correlation Function:** In this case, the correlation function related to photon counting, as explained in Sect. 8.3.4, is

$$\langle f_{\text{out}}^\dagger(0)f_{\text{out}}^\dagger(t)\ f_{\text{out}}(t)f_{\text{out}}(0)\rangle_s = \gamma_2^2\langle\sigma^+(0)\sigma^+(t)\sigma^-(t)\sigma^-(0)\rangle_s \qquad (9.2.40)$$

$$= \text{Tr}\left\{\sigma^+\sigma^-V(t)\{\sigma^-\rho_s\sigma^+\}\right\}. \qquad (9.2.41)$$

Here we note that

$$\sigma^-\rho_s\sigma^+ = \frac{\bar{N}}{2\bar{N}+1}\{\tfrac{1}{2}(1-\sigma_z)\} \qquad (9.2.42)$$

and the term inside the curly brackets has unit trace. Thus, proceeding as before, we find

$$V(t)(\sigma^-\rho_s\sigma^+) = \frac{\bar{N}}{2\bar{N}+1}\left\{\tfrac{1}{2}-\tfrac{1}{2}\sigma_z\left[e^{-2\kappa t}+\frac{1-e^{-2\kappa t}}{2\bar{N}+1}\right]\right\} \qquad (9.2.43)$$

so that (9.2.40) gives

$$\langle f_{\text{out}}^{\dagger}(0) f_{\text{out}}^{\dagger}(t) f_{\text{out}}(t) f_{\text{out}}(0) \rangle = \gamma_2^2 \left(\frac{\bar{N}}{2\bar{N}+1} \right)^2 \left(1 - e^{-2\kappa t} \right). \qquad (9.2.44)$$

Comments:

i)   In terms of the notation of Sect. 8.3.4, we find that

$$g^{(2)}(t)_s = (1 - e^{-2\kappa t}) \qquad (9.2.45)$$

$$g^{(1)}(t)_s = e^{-\kappa t}. \qquad (9.2.46)$$

Thus, $g^{(2)}(0) = 0$, which is completely non-classical behaviour, called anti-bunching. From the formula (9.2.40), we see that this vanishing at $t = 0$ is not really a property of the solutions of the equations, but rather of the fact that $\sigma^+(0)\sigma^+(0) = \sigma^-(0)\sigma^-(0) = 0$.

ii)   The antibunching says that there is no probability of counting a photon immediately after one has just been counted. This can be viewed as the requirement that the atom will be in its ground state after emitting a photon, and it will take a time of order $(2\kappa)^{-1}$ before it is again significantly excited. The property of antibunching was first pointed out by *Carmichael* and *Walls* [9.2] in the related problem of *coherent* resonance fluorescence, in which it is assumed that $b_{\text{in}}(t)$ is a coherent field, with mean $\beta(t)$, and thus $\bar{N}_0 = 0$. The resonance fluorescence problem can be solved by exactly the same methods as given here, but is more complicated because the equations for the averages do not decouple, as they do in (9.2.24).

9.2.4   Two Level Atom in a Coherent Driving Field

For further application to the laser, and for its own intrinsic interest, we are led to the situation in which we drive the atom with a coherent driving field. To do this we follow the methods of Chap. 5, and modify the QSDEs for the atom in a thermal field by adding a constant term to the incoming noise, thus

$$\sqrt{\gamma}\, b_{\text{in}}(t) \rightarrow \sqrt{\gamma}\, b_{\text{in}}(t) + \mathcal{E} e^{i\phi} \qquad (9.2.47)$$

where $\mathcal{E} e^{i\phi}$ represents the coherent field by the amplitude $\mathcal{E} > 0$ and the phase ϕ. (Actually, $\mathcal{E}$ is the product of the field amplitude and an appropriate coupling constant, which has been absorbed into one constant for simplicity). If we now define

$$\langle \sigma^+ \rangle = -\frac{i}{2} e^{-i\phi} (S_x + i S_y) = \langle \sigma^- \rangle^*, \qquad \langle \sigma_z \rangle = S_z, \qquad (9.2.48)$$

then the mean value equations corresponding to (9.2.24) become

$$\dot{S}_x = -\kappa S_x$$
$$\dot{S}_y = -\kappa S_y + 2\,\mathcal{E} S_z \qquad\qquad (9.2.49)$$
$$\dot{S}_z = -2\kappa S_z - 2\,\mathcal{E} S_y - \gamma.$$

a) Stationary Solutions: Setting the time derivatives to zero, we get the stationary solutions, $\bar{\mathbf{S}}$,

$$\bar{S}_x = 0$$
$$\bar{S}_y = -\frac{2\gamma\mathcal{E}}{2\kappa^2 + 4\mathcal{E}^2} \qquad\qquad (9.2.50)$$
$$\bar{S}_z = -\frac{\gamma\kappa}{2\kappa^2 + 4\mathcal{E}^2}.$$

The stationary density matrix is then

$$\rho_s = \tfrac{1}{2}\left(1 + \sigma_z \bar{S}_z\right) + \sigma^+ \langle\sigma^-\rangle + \sigma^- \langle\sigma^+\rangle \qquad\qquad (9.2.51)$$
$$= \tfrac{1}{2}\left(1 + \bar{\mathbf{S}}\cdot\boldsymbol{\sigma}'\right) \qquad\qquad (9.2.52)$$

where

$$\boldsymbol{\sigma}' = \left\{ i\left(e^{i\phi}\sigma^+ - e^{-i\phi}\sigma^-\right), \left(e^{i\phi}\sigma^+ + e^{-i\phi}\sigma^-\right), \sigma_z \right\} \qquad\qquad (9.2.53)$$

and

$$(\boldsymbol{\sigma})'^{\pm} = \pm i e^{\pm i\phi}\sigma^{\pm}. \qquad\qquad (9.2.54)$$

From the above, we can see that we may simply make the replacement $\boldsymbol{\sigma}' \to \boldsymbol{\sigma}$, since the algebra is the same. In what follows we will not explicitly write $\boldsymbol{\sigma}'$, though it should be understood.

b) Time Dependent Solutions: The equations of motion (9.2.49) can be written symbolically as

$$\dot{\mathbf{S}}(t) = -A(\mathbf{S}(t) - \bar{\mathbf{S}}) \qquad\qquad (9.2.55)$$

with

$$A = \begin{pmatrix} -\kappa & 0 & 0 \\ 0 & -\kappa & 2\,\mathcal{E} \\ 0 & -2\,\mathcal{E} & -2\kappa \end{pmatrix} \qquad\qquad (9.2.56)$$

and solutions given by

$$\mathbf{S}(t) = e^{-At}\left(\mathbf{S}(0) - \bar{\mathbf{S}}\right) + \bar{\mathbf{S}}. \qquad\qquad (9.2.57)$$

Using the property (9.1.19) for exponentiating the 2×2 submatrix, one can write

$$e^{-At} = \begin{pmatrix} e^{-\kappa t} & 0 & 0 \\ 0 & f_1(t) & f_3(t) \\ 0 & -f_3(t) & f_2(t) \end{pmatrix} \qquad\qquad (9.2.58)$$

where

$$f_1(t) = e^{-\frac{3}{2}\kappa t}\left(\cosh \Lambda t + \frac{\kappa}{2\Lambda}\sinh \Lambda t\right)$$

$$f_2(t) = e^{-\frac{3}{2}\kappa t}\left(\cosh \Lambda t - \frac{\kappa}{2\Lambda}\sinh \Lambda t\right) \qquad (9.2.59)$$

$$f_3(t) = e^{-\frac{3}{2}\kappa t}\frac{2\mathcal{E}}{\Lambda}\sinh \Lambda t$$

and

$$\Lambda = \sqrt{\frac{\kappa^2}{4} - 4\mathcal{E}^2}\,. \qquad (9.2.60)$$

The solutions change from hyperbolic to circular functions when $\mathcal{E} > \kappa/4$; for larger $\mathcal{E}$ one makes the replacement

$$\Lambda \to i\Omega = i\sqrt{4\mathcal{E}^2 - \frac{\kappa^2}{4}} \qquad (9.2.61)$$

where the frequency Ω is known as the *Rabi frequency*.

c) The Time Dependent Density Matrix: This is of course given (using (9.2.27)) by

$$\rho(t) = \tfrac{1}{2}\mathrm{Tr}\,\{\rho(0)\}\{\mathbf{1} + \sigma \cdot \mathbf{S}(t)\}, \qquad (9.2.62)$$

where for a genuine density matrix, $\mathrm{Tr}\,\{\rho(0)\}$ should be equal to 1, but for generality, we will see that it is best to not make this restriction.

d) Correlation Functions: All the two-time correlation functions can be written in terms of the expression

$$\mathrm{Cor}(t) = \mathrm{Tr}\,\{(a + \mathbf{b}\cdot\sigma)V(t,0)\tfrac{1}{2}(1 + \mathbf{S}(0)\cdot\sigma)\} \qquad (9.2.63)$$

where the evolution operator $V(t,0)$ is the notation of Sect. 5.2. Using the time dependent solutions,

$$\mathrm{Cor}(t) = a + \mathbf{b}\cdot\mathbf{S}(t), \qquad (9.2.64)$$

and this can be written, using the solution (9.2.57), as

$$\mathrm{Cor}(t) = a + \mathbf{b}\cdot\tilde{\mathbf{S}} + \mathbf{b}\cdot e^{-At}\left(\mathbf{S}(0) - \tilde{\mathbf{S}}\right). \qquad (9.2.65)$$

e) Amplitude Correlation Function: Using the formulae of (5.2.21–23), we want to evaluate the stationary correlation function

$$\langle \sigma^+(t)\sigma^-(0)\rangle_s = \begin{cases} \mathrm{Tr}\,\{\sigma^+ V(t,0)\sigma^-\rho_s\} & t > 0 \\ \mathrm{Tr}\,\{\sigma^- V(-t,0)\rho_s\sigma^+\} & t < 0. \end{cases} \qquad (9.2.66)$$

By comparing with (9.2.63), one sees that here $a = 0$ and

$$\mathbf{b} = \begin{cases} \left(\tfrac{1}{2},\ \tfrac{1}{2}i,\ 0\right) & t > 0 \\ \left(\tfrac{1}{2},\ -\tfrac{1}{2}i,\ 0\right) & t < 0. \end{cases} \qquad (9.2.67)$$

Using the formula (9.1.6), one finds

$$\sigma^- \rho_s = -\frac{i\bar{S}_y}{4}\left\{1 + \boldsymbol{\sigma} \cdot (ig, g, -1)\right\} \tag{9.2.68}$$

where

$$g = \frac{1 + \bar{S}_z}{2i\bar{S}^-} = \frac{1 + \bar{S}_z}{\bar{S}_y}. \tag{9.2.69}$$

Thus using (9.2.65) (multiplied by a factor $-i\bar{S}_y/2$)

$$\langle \sigma^+(t)\sigma^-(0)\rangle_s = \frac{\bar{S}_y^2}{4} + \frac{(1+\bar{S}_z)}{4}e^{-\kappa|t|} + \frac{1+\bar{S}_z - \bar{S}_y^2}{4}f_1(|t|)$$
$$- \frac{\bar{S}_y(1+\bar{S}_z)}{4}f_3(|t|) \tag{9.2.70}$$

where we have implicitly included the similarly derivable result for $t < 0$, by writing $|t|$ instead of t. Again, we can derive in the same way the reverse correlation function

$$\langle \sigma^-(t)\sigma^+(0)\rangle_s = \frac{\bar{S}_y^2}{4} + \frac{(1-\bar{S}_z)}{4}e^{-\kappa|t|} + \frac{1-\bar{S}_z - \bar{S}_y^2}{4}f_1(|t|)$$
$$+ \frac{\bar{S}_y(1-\bar{S}_z)}{4}f_3(|t|). \tag{9.2.71}$$

Reverting to the transformation $\sigma \to \sigma'$, we see that indeed this does not change the result.

f) Intensity Correlation Function: We follow much the same procedure to evaluate

$$\langle \sigma^+(0)\sigma^+(t)\sigma^-(t)\sigma^-(0)\rangle = \mathrm{Tr}\left\{\sigma^+\sigma^- V(t,0)\sigma^-\rho_s\sigma^+\right\}$$

$$= \frac{(1+\bar{S}_z)^2}{4}\left(1 - f_2(t)\right) + \frac{\bar{S}_y(1+\bar{S}_z)}{4}f_3(t) \qquad t > 0. \tag{9.2.72}$$

g) Spectrum of Fluorescent Light: The spectrum of the fluorescent light is given by the Fourier transform of $\langle b^\dagger_{\mathrm{out}}(t)b_{\mathrm{out}}(0)\rangle$, which, by using the formula (5.4.40), relating the output correlation function to the internal correlation function, is given by the Fourier transform of $\langle \sigma^+(t)\sigma^-(0)\rangle$, i.e.

$$S(\omega) = \frac{\gamma_2}{2\pi}\int_{-\infty}^{\infty} e^{i\omega t}\langle \sigma^+(t)\sigma^-(0)\rangle_s\, dt. \tag{9.2.73}$$

Noting that $\langle \sigma^+(-t)\sigma^-(0)\rangle = \langle \sigma^+(0)\sigma^-(t)\rangle = \langle \sigma^+(t)\sigma^-(0)\rangle^*$, we see that

$$S(\omega) = \frac{\gamma_2}{\pi}\mathrm{Re}\left\{\int_0^{\infty}\langle \sigma^+(t)\sigma^-(0)\rangle e^{-i\omega t}\, dt\right\}. \tag{9.2.74}$$

The Fourier transforms are straightforward to evaluate, but the spectrum which results is an unsightly formula. In the very strong field limit, $\Lambda \rightarrow i\Omega, \bar{S}_y = \bar{S}_z = 0$, and

$$S(\omega) = \frac{\gamma_2}{4\pi} \frac{\kappa}{\kappa^2 + \omega^2}$$
$$+ \frac{\gamma_2}{4\pi} \left\{ \frac{\frac{3}{4}\kappa}{\left(\frac{3}{2}\kappa\right)^2 + (\omega + \Omega)^2} + \frac{\frac{3}{4}\kappa}{\left(\frac{3}{2}\kappa\right)^2 - (\omega + \Omega)^2} \right\}. \qquad (9.2.75)$$

We see three peaks; a central peak, and two sidebands at the Rabi frequency, Ω, as shown in Fig.9.2.
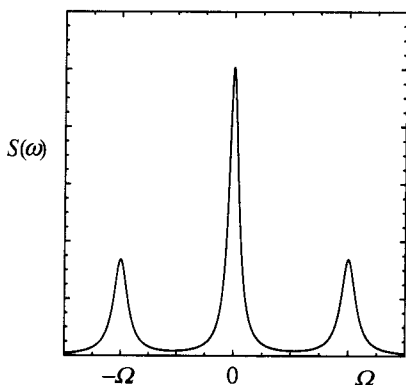
Fig. 9.2 The spectrum of resonance fluorescence as given by (9.2.75)

9.3   The Laser

The impetus to develop a useful form of quantum noise theory as a computational tool came from the invention of the laser, and the consequent need to understand the properties of laser light. The basis of laser action is stimulated emission of radiation. As we have seen in Sect. 7.5.1, if a bath of atoms is coupled to a mode of the electromagnetic field and can be maintained in a situation in which there is a net inversion, the electromagnetic field will be amplified.

A practical laser must include two more features. The first is the fact that the mode of the electromagnetic field will most often be a standing wave mode between two mirrors, which are not perfect—indeed for practical use, some light must get through the mirrors to be available to the user, so that one mirror will be deliberately made partially reflecting.

The second feature is that of depletion of the bath of inverted atoms. As lasing takes place, atoms lose their inversion. Some mechanism must be provided to pump them back up again, and this cannot be absolutely perfect.

A simple model of a laser is found by considering a system composed of:

i) A single mode of the light field, confined between mirrors with some small losses through them. This is basically a damped harmonic oscillator, represented by a system Hamiltonian

$$H_F = \hbar \omega a^\dagger a \tag{9.3.1}$$

coupled to a heat bath via a coupling

$$H_{FB} = a^\dagger \Gamma_F + a\Gamma_F^\dagger. \tag{9.3.2}$$

The precise nature of the heat bath Hamiltonian is not very important unless we are concerned with the output field—in that case it consists of the appropriate modes of the free electromagnetic field, and the output field can be determined via the input-output methods of Sect. 5.3.2.
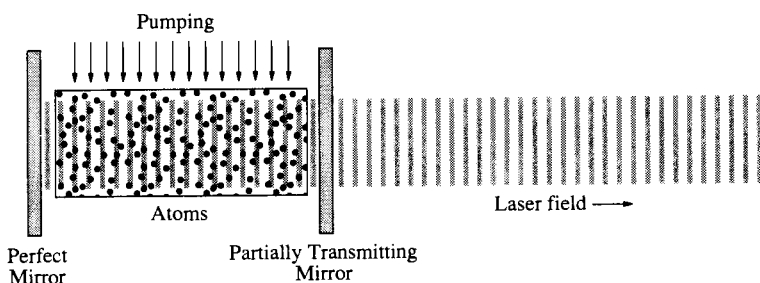
Fig. 9.3 Configuration of the essential elements of a gas laser

ii) A large number, N, of two level atoms, which are held in an inverted state by some appropriate pumping. These are represented as in Sect. 5.1.4, but are not treated as a heat bath. Instead, they are coupled to another bath, which is the source of the pumping. The Hamiltonian for the atoms is

$$H_A = \sum_{\mu=1}^{N} \tfrac{1}{2}\hbar \omega \sigma_z^\mu \tag{9.3.3}$$

where the σ^μ are different representations of the Pauli matrices, one for each atom, and the transition frequency ω is (for simplicity) chosen exactly equal to that of the mode of the light field. The coupling to the pumping bath is given by the Hamiltonian

$$H_{AB} = \sum_{\mu=1}^{N} \left(\Gamma_{A,\mu}\sigma_\mu^+ + \Gamma_{A,\mu}^\dagger \sigma_\mu^- \right). \tag{9.3.4}$$

iii) The atoms and the field are now coupled by a RWA coupling

$$H_{AF} = ig\hbar \sum_\mu \left(a^\dagger \sigma_\mu^- - a\sigma_\mu^+ \right). \tag{9.3.5}$$

This model is in the form given by *Haken* [9.3], and is also very similar to those of *Lax*, *Louisell* and *Gordon* [9.4].

The laser is most naturally treated with master equation techniques, which allow the application of techniques of adiabatic elimination of fast variables, like those used in Sect. 5.1.2. There are two natural time scales; the lifetime of a photon in the laser cavity and the natural lifetime of an excited atom. By assuming that the lifetime of the atoms is very small, their motion can be assumed to follow that of the much slower laser field, and thus a quite simple equation can be derived.

9.3.1   Quantum Langevin Equations for the Laser

We can use the techniques used in Sect. 5.3 to write quantum stochastic differential equations for the variables $a, \sigma_\mu^\pm, \sigma_z^\mu$. The heat bath coupling to the atomic variables has a negative temperature, so the damping parts of equations take the form given in (7.2.36). We use the notation

$$c \to \sigma_\mu^- \tag{9.3.6}$$

$$\gamma(\bar{N} + 1) \to W_{12} \tag{9.3.7}$$

$$\gamma\bar{N} \to W_{21} \tag{9.3.8}$$

in this equation. Similarly, the damping part field mode a is taken from the Ito QSDE form (5.3.50), with

$$\bar{N} \to n \tag{9.3.9}$$

$$\frac{\gamma}{2} \to \kappa. \tag{9.3.10}$$

The equations of motion are then straightforwardly worked out as (in the interaction picture)

$$(\mathbf{I})\,da = \left(-\kappa a + g \sum_\mu \sigma_\mu^-\right) dt - \sqrt{2\kappa}\, dB(t) \tag{9.3.11}$$

$$(\mathbf{I})\,d\sigma_\mu^+ = \left(ga^\dagger \sigma_\mu^z - \tfrac{1}{2}(W_{12} + W_{21})\sigma_\mu^+\right) dt + \sigma_\mu^z dH_\mu^\dagger(t) \tag{9.3.12}$$

$$(\mathbf{I})\,d\sigma_\mu^z = \left\{-2g\left(a^\dagger \sigma_\mu^- + a\sigma_\mu^+\right) - (W_{12} + W_{21})\sigma_\mu^z + (W_{12} - W_{21})\right\} dt$$
$$-2\sigma_\mu^+ dH_\mu(t) - 2\sigma_\mu^- dH_\mu^\dagger(t). \tag{9.3.13}$$

Here, the Ito noise terms satisfy the relations

$$dB^\dagger(t)dB(t) = \bar{n}dt, \quad dB(t)dB^\dagger(t) = (\bar{n} + 1)dt \tag{9.3.14}$$

$$dH_\mu^\dagger(t)dH_\nu(t) = \delta_{\mu\nu}W_{12}dt, \quad dH_\mu(t)dH_\nu^\dagger(t) = \delta_{\mu\nu}W_{21}dt \tag{9.3.15}$$

and all other products vanish.

Since the equations are linear, we can simply add them up, and derive equations of motion for the "total spin operator" $\mathbf{S} = \sum_\mu \boldsymbol{\sigma}_\mu$; namely

$$(\mathbf{I})da\ = (-\kappa a + gS^-)dt - \sqrt{2\kappa}\,dB(t) \qquad (9.3.16)$$

$$(\mathbf{I})dS^+ = \left(ga^+ S_z - \tfrac{1}{2}(W_{12} + W_{21})S^+\right)dt + dH^\dagger(t) \qquad (9.3.17)$$

$$(\mathbf{I})dS_z = \left\{-2g(a^+ S^- + aS^+) - (W_{12} + W_{21})S_z + N(W_{12} - W_{21})\right\}dt$$
$$- 2dH_z(t). \qquad (9.3.18)$$

Here, $dH(t)$, $dH^\dagger(t)$, and $dH_z(t)$ are defined as the sums of the noises occurring in (9.3.12,13), and satisfy rather simple product formulae, which are easily evaluated using the fact that the Ito increment commutes with and is independent of the variables σ_μ. Thus, for example

$$dH^\dagger(t)dH(t) = \sum_{\mu\nu} \sigma_\mu^z dH_\mu^\dagger(t)\sigma_\nu^z dH_\nu(t) \qquad (9.3.19)$$

$$= \sum_{\mu\nu} W_{12}\delta_{\mu\nu}dt = NW_{12}dt. \qquad (9.3.20)$$

We can devise all the rules similarly; the complete set of nonvanishing products is

$$dH^\dagger(t)dH(t)\ = NW_{12}dt$$
$$dH(t)dH^\dagger(t)\ = NW_{21}dt$$
$$dH^\dagger(t)dH_z(t) = W_{12}S^+ dt$$
$$dH_z(t)dH^\dagger(t) = -W_{21}S^+ dt \qquad (9.3.21)$$
$$dH_z(t)dH(t)\ = W_{12}S^- dt$$
$$dH(t)dH_z(t)\ = -W_{21}S^- dt$$
$$dH_z(t)^2\ = \tfrac{1}{2}\left\{(W_{21} - W_{12})S_z + (W_{12} + W_{21})N\right\}dt.$$

a) Asymptotic Expansion in the Number of Atoms: In any normal laser the number, N, of atoms is very large. Noting that $\mathbf{S}$ is the sum of N parts, we can define appropriate scaled variables by

$$\bar{a}\ = a/\sqrt{N}$$
$$\bar{g}\ = g\sqrt{N}$$
$$\bar{S}\ = \mathbf{S}/N \qquad (9.3.22)$$
$$d\bar{H}(t)\ = dH(t)/\sqrt{N}$$
$$d\bar{H}_z(t) = dH_z(t)/\sqrt{N}.$$

This can be viewed as a kind of system size expansion in N, the number of atoms, and is a quantum analogue of the system size expansion introduced by *van Kampen* [9.5] in classical stochastic systems.

The scaled equations now are of the same appearance as the original equations, (9.3.16–18), except that the deterministic terms are of order N^0, while the noises are all of order $N^{-\frac{1}{2}}$: explicitly

$$d\tilde{a} = \left(-\kappa\tilde{a} + \tilde{g}\tilde{S}^-\right) dt - \sqrt{\frac{2\kappa}{N}}\, dB(t)$$

$$d\tilde{S}^+ = \left(\tilde{g}\tilde{a}^\dagger \tilde{S}_z - \tfrac{1}{2}(W_{12} + W_{21})\tilde{S}^+\right) dt + \sqrt{\frac{1}{N}}\, d\tilde{H}^\dagger(t)$$

$$d\tilde{S}_z = \left\{ -2\tilde{g}\left(\tilde{a}^\dagger \tilde{S}^- + \tilde{a}\tilde{S}^+\right) - (W_{12} + W_{21})\tilde{S}_z \right.$$
$$\left. + (W_{12} - W_{21}) \right\} dt - \frac{2}{\sqrt{N}} d\tilde{H}_z(t) \tag{9.3.23}$$

with

$$d\tilde{H}^\dagger(t)d\tilde{H}(t) = W_{12}\, dt; \qquad d\tilde{H}(t)d\tilde{H}^\dagger(t) = W_{21}\, dt$$
$$d\tilde{H}^\dagger(t)d\tilde{H}_z(t) = W_{12}\tilde{S}^+ dt; \quad d\tilde{H}_z(t)d\tilde{H}^\dagger(t) = -W_{21}\tilde{S}^+ dt \tag{9.3.24}$$
$$d\tilde{H}_z(t)^2 = \tfrac{1}{2}\left\{(W_{21} - W_{12})\tilde{S}_z + W_{12} + W_{21}\right\} dt$$

and the corresponding Hermitian conjugate equations.

b) Ito Form of QSDE is Appropriate for Asymptotic Expansions: It is implicit in the formulation (9.3.23), that the noise terms are small compared with the deterministic terms. In the Ito form, the mean values of the noise terms are zero, and their mean squares in (9.3.24) are small. If we use Stratonovich equations, the mean values of the noises are proportional to the mean values of the W_{12} and W_{21} terms in (9.3.23)—this is how the damping terms arise in the averages of the Stratonovich equations, even though they are not in the equations themselves. Thus the averages of the Stratonovich noise terms are proportional to N^0, and thus no analogous expansion in $1/\sqrt{N}$ is possible.

c) Lowest Order—Macroscopic Laser Equations: To zeroth order in N we can neglect the quantum nature of the variables since the commutators are all of order $1/N$, and hence arise from higher order contributions. We are then led to the deterministic laser equations for the variables,

$$\bar{\alpha} = \tilde{a}$$
$$\bar{v} = \tilde{S}^- \tag{9.3.25}$$
$$\bar{D} = \tilde{S}_z$$

in the form

$$\boxed{\begin{aligned}
\dot{\bar{\alpha}} &= -\kappa\bar{\alpha} + \tilde{g}\bar{v} \\
\dot{\bar{v}} &= \tilde{g}\bar{\alpha}\bar{D} - \bar{\gamma}\bar{v} \\
\dot{\bar{D}} &= -2\tilde{g}\left(\bar{\alpha}^*\bar{v} + \bar{\alpha}\bar{v}^*\right) - 2\bar{\gamma}(\bar{D} - d)
\end{aligned}} \tag{9.3.26}$$

Fig. 9.4   Macroscopic stationary solutions of the laser equations
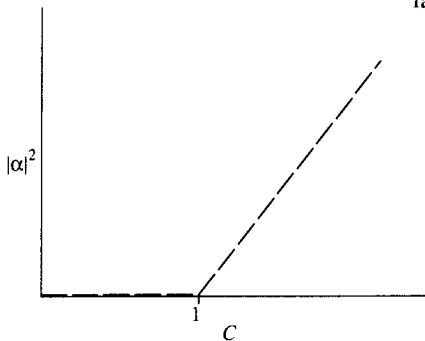
with

$$\bar{d} = \frac{W_{12} - W_{21}}{W_{12} + W_{21}}, \qquad \bar{\gamma} = \tfrac{1}{2}(W_{12} + W_{21}). \tag{9.3.27}$$

d) Stationary Solutions of the Macroscopic Laser Equations: The three equations (9.3.26) have within them a rather rich range of behaviours. It is simplest to start first with the stationary solution, obtained by setting $\dot{\tilde{\alpha}} = \dot{\tilde{v}} = \dot{\tilde{D}} = 0$. Elementary algebra shows that the stationary solution for the field variable $\tilde{\alpha}$ is given by

$$\tilde{\alpha} \left[1 - \frac{C}{1 + |\tilde{\alpha}|^2/\tilde{n}_0} \right] = 0 \tag{9.3.28}$$

where we define

$$C \equiv \frac{\bar{d}\tilde{g}^2}{\bar{\gamma}\kappa} \tag{9.3.29}$$

which is known as the *pump* or *co-operativity parameter*, and

$$n_0 \equiv \tilde{n}_0 N = \frac{\bar{\gamma}^2}{2g^2} \tag{9.3.30}$$

which is called the *saturation photon number*. There are two solutions for (9.3.28),

$$\tilde{\alpha} = 0 \tag{9.3.31}$$

and

$$|\tilde{\alpha}|^2 = \tilde{n}_0(C - 1). \tag{9.3.32}$$

Clearly, the second is only possible if $C > 1$. We can see that C may be varied by changing $W_{12} - W_{21}$, that is, by changing the pumping of the atoms. The behaviour of the system is completely different, depending on whether C is greater than or less than 1, and there is a threshold at $C = 1$. Above threshold, when $C > 1$, a mean coherent field arises—this is the lasing action. The intensity, $|\alpha|^2$, is proportional to the pumping, and to the number of atoms, if we consider that $g \approx 1/\sqrt{N}$. For fixed g, the lasing intensity is proportional to the square of the number of atoms, which is the hallmark of a coherent process.

Below threshold, when $C < 1$, there is no mean coherent field—in fact in this macroscopic field, $\bar{\alpha} = 0$. We cannot discuss what field is actually present without considering the fluctuations.

e) Time Dependent Solutions: There are three time constants in this problem, γ^{-1}, κ^{-1}, and $\bar{g}^{-1}$, and the full analysis of behaviour depends on their relative sizes. In a practical laser the largest of these is γ, which determines the rate at which the atoms radiate. We can then use the procedure known as adiabatic elimination, which means that the variables whose equations have a time development determined by γ rapidly approach quasi-stationary values obtained by setting equal to zero the left-hand side of the appropriate equations. Thus we transform the equations (9.3.26) by setting $\dot{\bar{v}} = \dot{D} = 0$, and substitute into the equation for α—the resulting equation is

$$\dot{\bar{\alpha}} = -\kappa\bar{\alpha}\left(1 - \frac{C}{1 + |\bar{\alpha}|^2/\tilde{n}_0}\right). \tag{9.3.33}$$

If we use real and imaginary parts of $\bar{\alpha} = x + iy$, the equation (9.3.33) can be written in terms of the gradient of a potential

$$\dot{x} = -\frac{\partial}{\partial x}V\left(\sqrt{x^2 + y^2}\right) \tag{9.3.34}$$

$$\dot{y} = -\frac{\partial}{\partial y}V\left(\sqrt{x^2 + y^2}\right) \tag{9.3.35}$$

where

$$V(r) = \tfrac{1}{2}\kappa r^2 - \tfrac{1}{2}\tilde{n}_0\kappa C \ln(1 + r^2/\tilde{n}_0). \tag{9.3.36}$$

This potential (see Fig.9.5) clearly shows that for $C < 1, x = y = 0$ is a stable stationary solution, while for $C > 1$ the stable solution is given by $x^2 + y^2 = \tilde{n}_0(C - 1)$, which means that there is neutral stability for all solutions of this equation. This will mean that above the threshold at $C = 1$ the field variable develops an amplitude which is determined by C, but the phase of the field is arbitrary.

f) Solution to Next Order: We can expand all the variables to first order in $1/\sqrt{N}$, to derive equations to the next order. Writing

$$\begin{aligned}
\tilde{a} &= \bar{\alpha} + N^{-\frac{1}{2}}\alpha_1 \\
\tilde{S}^- &= \bar{v} + N^{-\frac{1}{2}}v_1 \\
\tilde{S}_z &= \bar{D} + N^{-\frac{1}{2}}D_1,
\end{aligned} \tag{9.3.37}$$

we can derive equations for the operators α_1, v_1, D_1, which are linear, with coefficients which are functions of $\bar{\alpha}, \bar{v}, \bar{D}$, the deterministic solutions, which are c-numbers. Thus, the equations can be written as
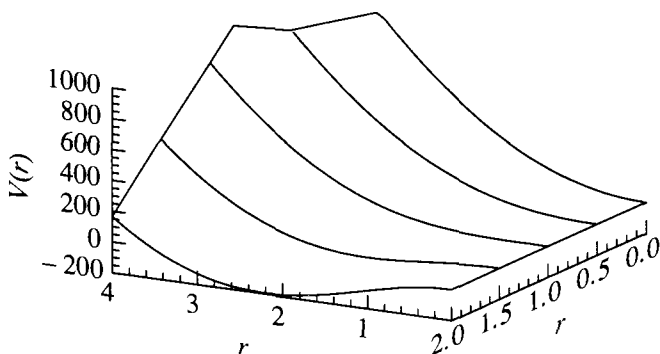
Fig. 9.5 The laser potential $V(r)$ for different values of the pump parameter C

$$\begin{bmatrix} d\alpha_1 \\ d\alpha_1^\dagger \\ dv_1 \\ dv_1^\dagger \\ dD_1 \end{bmatrix} = \begin{bmatrix} -\kappa & 0 & g & 0 & 0 \\ 0 & -\kappa & 0 & g & 0 \\ \bar{D} & 0 & -\bar{\gamma} & 0 & \bar{\alpha} \\ 0 & \bar{D} & 0 & -\bar{\gamma} & \bar{\alpha}^* \\ -2\bar{g}\bar{v}^* & -2\bar{g}\bar{v} & -2\bar{g}\bar{\alpha}^* & -2\bar{g}\bar{\alpha} & -2\bar{\gamma} \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_1^\dagger \\ v_1 \\ v_1^\dagger \\ D_1 \end{bmatrix} dt + \begin{bmatrix} -\sqrt{2\kappa}\,dB(t) \\ -\sqrt{2\kappa}\,dB^\dagger(t) \\ dH(t) \\ dH^\dagger(t) \\ -2\,dH_z(t) \end{bmatrix}.$$

$$(9.3.38)$$

It must be remembered that these are operator equations, though this does not make them difficult to solve, especially if the zeroth order solution is a stationary state, so that the 5×5 matrix has constant matrix elements.

Exercise. Write (9.3.38) as

$$d\mathbf{v} = -A\mathbf{v}\,dt + d\mathbf{B}(t). \tag{9.3.39}$$

Show that, in a similar manner to that used for classical SDEs, the stationary solution is

$$\mathbf{v}(t) = \int_{-\infty}^{t} e^{-A(t-t')}\,d\mathbf{B}(t'). \tag{9.3.40}$$

The matrix of products (9.3.24) can be written as

$$d\mathbf{B}(t)d\mathbf{B}^T(t) = G\,dt. \tag{9.3.41}$$

Using this, follow the reasoning in S.M. Sect. 4.4.6 to show that the stationary variance matrix can be written

$$\langle \mathbf{v}(t)\mathbf{v}^T(t)\rangle = \sigma \tag{9.3.42}$$

and that it satisfies the equation

$$A\sigma + \sigma A^T = G. \tag{9.3.43}$$

The autocorrelation function is

$$\langle \mathbf{v}(t)\mathbf{v}^T(s)\rangle_s = e^{-A(t-s)}\sigma, \tag{9.3.44}$$

and the spectrum matrix is

$$S(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-i\omega\tau}\langle \mathbf{v}(t+\tau)\mathbf{v}^T(t)\rangle_s \tag{9.3.45}$$

$$= \frac{1}{2\pi}(A + i\omega^{-1})G(A^T - i\omega)^{-1}. \tag{9.3.46}$$

Hence work out the laser spectrum, i.e. the spectrum of flucuations of $a(t)$ to first order in $1/\sqrt{N}$.

Exercise. Of course the previous exercise assumes that A is nonsingular. Show that above threshold A is singular. After reading Sect. 9.3.3 you will see that this problem is related to the existence of phase diffusion.

9.3.2   Derivation of Laser Equations

a) Master Equation in the Interaction Picture: The QSDE approach to the laser has the advantage of possessing simple equations, and of dealing directly with the quantities of interest, the operators $\mathbf{S}$ and a. But it does have the disadvantage that any practical calculations must be carried out in the perturbation expansion in $N^{-\frac{1}{2}}$, which assumes that N is large.

The QSDEs (9.3.11–13) are equivalent to a master equation. From these, the standard techniques give a master equation which can be written (in the interaction picture of the atom and field Hamiltonians)

$$\frac{\partial\rho}{\partial t} = -\frac{i}{\hbar}[H_{AF},\rho] + \frac{\partial\rho}{\partial t}\bigg|_A + \frac{\partial\rho}{\partial t}\bigg|_F. \tag{9.3.47}$$

The last two terms are the damping terms arising from the two bath-system couplings given in (9.3.2,3), namely

$$\frac{\partial\rho}{\partial t}\bigg|_A = \frac{W_{21}}{2}\sum_{\mu=1}^N \left(2\sigma_\mu^-\rho\sigma_\mu^+ - \sigma_\mu^+\sigma_\mu^-\rho - \rho\sigma_\mu^+\sigma_\mu^-\right)$$

$$+ \frac{W_{12}}{2}\sum_{\mu=1}^N \left(2\sigma_\mu^+\rho\sigma_\mu^- - \sigma_\mu^-\sigma_\mu^+\rho - \rho\sigma_\mu^-\sigma_\mu^+\right) \tag{9.3.48}$$

and

$$\frac{\partial\rho}{\partial t}\bigg|_F = \kappa n\left(2a^\dagger\rho a - aa^\dagger\rho - \rho aa^\dagger\right) + \kappa(n+1)\left(2a\rho a^\dagger - a^\dagger a\rho - \rho a^\dagger a\right). \tag{9.3.49}$$

We can use the techniques of adiabatic elimination directly on this master equation, and derive in the end some laser equations which do *not* depend on the large N limit.

b) Evolution Operators in the P-representation: We represent the joint density operator of the atoms and the light field by a P-representation thus

$$\rho = \int d^2\alpha |\alpha\rangle\langle\alpha|\rho(\alpha), \tag{9.3.50}$$

where $\rho(\alpha)$ is an operator in the many atom space. The methods of Sects. 4.5.1, 6.2.1 can be used in the interaction picture to give the equation of motion for $\rho(\alpha)$:

$$\frac{\partial\rho}{\partial t} = \left\{\kappa\left(\frac{\partial}{\partial\alpha}\alpha + \frac{\partial}{\partial\alpha^*}\alpha^*\right) + 2\kappa n\frac{\partial^2}{\partial\alpha\partial\alpha^*}\right\}\rho + \frac{\partial\rho}{\partial t}\bigg|_A$$
$$+ g\sum_\mu\left\{\alpha^*[\sigma_\mu^-,\rho] - \alpha[\sigma_\mu^+,\rho] - \frac{\partial}{\partial\alpha}\sigma_\mu^-\rho - \frac{\partial}{\partial\alpha^*}\rho\sigma_\mu^+\right\}. \tag{9.3.51}$$

The operator on the RHS of (9.3.51) divides into three parts, in a way that is not immediately obvious. Since we are considering the situation in which the atoms relax much faster than the field, we can assume that they are in a quasi-stationary state, which depends however on the electric field α. The relevant equation of motion for the atoms is thus given by

$$\frac{\partial\rho}{\partial t}\bigg|_A = L_1\rho$$

$$\equiv \quad \tfrac{1}{2}W_{12}\sum_{\mu=1}^N\left(2\sigma_\mu^-\rho\sigma_\mu^+ - \sigma_\mu^+\sigma_\mu^-\rho - \rho\sigma_\mu^-\sigma_\mu^+\right)$$

$$+ \tfrac{1}{2}W_{21}\sum_{\mu=1}^N\left(2\sigma_\mu^+\rho\sigma_\mu^- - \sigma_\mu^-\sigma_\mu^+\rho - \rho\sigma_\mu^-\sigma_\mu^+\right)$$

$$+ g\sum_{\mu=1}^N\left(\alpha^*[\sigma_\mu^-,\rho] - \alpha[\sigma_\mu^+,\rho]\right) \tag{9.3.52}$$

which is obtained by including the first part of the summation in (9.3.51) together with the purely atomic part of the master equation.

c) Stationary Means of Atomic Operators: Equation (9.3.52) is simply the equation of motion for N driven two level atoms, as in Sects. 9.2.3,4. The only connection between the atoms is the common external field, α. As in the previous treatment, we define some parameters

$$\bar{d} = \frac{W_{12} - W_{21}}{W_{12} + W_{21}} \tag{9.3.53}$$

$$\bar{\gamma} = \tfrac{1}{2}(W_{12} + W_{21}) \tag{9.3.54}$$

so that the parameters of Sects. 9.2.3,4 are given by (remembering that the bath temperature is now negative, so that γ is negative)

$$
\begin{aligned}
\kappa &\to \bar{\gamma} \\
\gamma &\to -2\bar{d}\bar{\gamma} \ , \\
\mathcal{E} &\to g\alpha
\end{aligned}
\tag{9.3.55}
$$

and thus the stationary means are (from (9.2.48,50))

$$
\langle \sigma_\mu^- \rangle = \langle \sigma_\mu^+ \rangle^* = \frac{\bar{d}g\alpha/\bar{\gamma}}{1 + |\alpha|^2/n_0}
\tag{9.3.56}
$$

where

$$
n_0 = \bar{\gamma}^2/2g^2
\tag{9.3.57}
$$

is a parameter known as the *saturation photon number*, which will be explained soon.

d) Adiabatic Elimination: The master equation can now be written as

$$
\frac{\partial \rho}{\partial t} = L_1\rho + L_2\rho + L_3\rho
\tag{9.3.58}
$$

with L_1 defined in (9.3.52), and

$$
L_3\rho = \left\{ \frac{\partial}{\partial \alpha}\left(\kappa\alpha - g\sum_\mu\langle\sigma_\mu^-\rangle\right) + \frac{\partial}{\partial\alpha^*}\left(\kappa\alpha^* - g\sum\langle\sigma_\mu^+\rangle\right) + 2\kappa n\frac{\partial^2}{\partial\alpha\partial\alpha^*} \right\}\rho
\tag{9.3.59}
$$

$$
L_2\rho = -g\sum_\mu\left\{ \frac{\partial}{\partial\alpha}\left(\sigma_\mu^- - \langle\sigma_\mu^-\rangle\right)\rho + \frac{\partial}{\partial\alpha^*}\rho\left(\sigma_\mu^+ - \langle\sigma_\mu^+\rangle\right) \right\}.
\tag{9.3.60}
$$

The adiabatic elimination is done by the projector techniques of Sect. 5.1.2 and S.M. Chap. 6. The projector onto the stationary atomic state, ρ_A^s, is defined by

$$
\begin{aligned}
PL_1 &= L_1P = 0 \\
PL_2P &= 0 \\
PL_3 &= L_3P.
\end{aligned}
\tag{9.3.61}
$$

There is no parametric dependence explicitly written for L_1, L_2, L_3, but it is clear that

$$
L_1 \sim \gamma, \quad L_3 \sim \kappa, \quad L_2 \sim g
\tag{9.3.62}
$$

so that if γ is much larger than κ and g, the adiabatic elimination will be justified and we will be able to derive the equation of motion

$$
\frac{\partial P(\alpha)}{\partial t} = L_3P(\alpha) - \mathrm{Tr}_A\left\{ L_2L_1^{-1}L_2\rho_A^s \right\}P(\alpha)
\tag{9.3.63}
$$

in which

$$P(\alpha) = \text{Tr}_A \{\boldsymbol{\rho}(\alpha)\}. \tag{9.3.64}$$

e) Evaluation of Noise Terms: The second term on the RHS of (9.3.63) can, as explained in S.M.(6.5.33), be put in the form

$$\text{Tr}_A \left\{ L_2 \int_0^\infty e^{L_1 t} (L_2 \rho_A^s) \, dt \right\} \tag{9.3.65}$$

of which a typical term is

$$g^2 \frac{\partial}{\partial \alpha} \int_0^\infty \text{Tr}_A \left\{ (\sigma_\mu^- - \langle\sigma_\mu^-\rangle) \, e^{L_1 t} \frac{\partial}{\partial \alpha^*} \rho_A^s \, (\sigma_\mu^+ - \langle\sigma_\mu^+\rangle) \right\} dt. \tag{9.3.66}$$

We can show that this term is small, so we will be able to move the $\partial/\partial\alpha^*$ to the left, with the addition of a small term to L_3, which will be neglected. In this case

$$(9.3.66) \rightarrow \quad g^2 \frac{\partial^2}{\partial\alpha\partial\alpha^*} \int_0^\infty \left\{ \langle\sigma_\mu^+(0)\sigma_\mu^-(t)\rangle - \langle\sigma_\mu^+\rangle\langle\sigma_\mu^-\rangle \right\} dt. \tag{9.3.67}$$

One can now look up the correlation functions for the driven atom and substitute them. In the case of not too large a field, we can in fact simply use the correlation function for the atom without a driving field, for which $\langle\sigma_\mu^+\rangle = \langle\sigma_\mu^-\rangle = 0$. Thus from (9.2.37)

$$\langle\sigma_\mu^+(t)\sigma_\mu^-(0)\rangle = \langle\sigma_\mu^+(0)\sigma_\mu^-(t)\rangle = \tfrac{1}{2}(1 + \bar{d})e^{-\bar{\gamma}t} \tag{9.3.68}$$

and substituting into (9.3.67), and similar terms

$$(9.3.65) \rightarrow \quad (1 + \bar{d})\frac{g^2}{\bar{\gamma}} \frac{\partial^2}{\partial\alpha\partial\alpha^*}. \tag{9.3.69}$$

f) Laser Equation: Substituting (9.3.69) and the values for $\langle\sigma_\mu^\pm\rangle$, (9.3.56) into (9.3.63), we finally derive

$$\frac{\partial P(\alpha)}{\partial t} = \left\{ \kappa \frac{\partial}{\partial\alpha} \left[\alpha \left(1 - \frac{C}{1 + |\alpha|^2/n_0} \right) \right] + \kappa \frac{\partial}{\partial\alpha^*} \left[\alpha^* \left(1 - \frac{C}{1 + |\alpha|^2/n_0} \right) \right] \right.$$
$$\left. + \left[2\kappa n + N(1 + \bar{d})\frac{g^2}{\bar{\gamma}} \right] \frac{\partial^2}{\partial\alpha\partial\alpha^*} \right\} P(\alpha) \tag{9.3.70}$$

where it should be noted that there are N terms exactly the same as (9.3.69), and the pump parameter C is defined by

$$C = \frac{N\bar{d}g^2}{\bar{\gamma}\kappa}. \tag{9.3.71}$$

g) Behaviour of Parameters: The situation of most interest occurs when the noise terms are small, but the terms proportional to C, which depend on the pumping, are significant. In order to derive the adiabatically eliminated equation (9.3.70), it was necessary to assume large $\bar{\gamma}$, which does seem to indicate that the noise term must be small, since it is divided by $\bar{\gamma}$. However, one must not forget that N, the number of atoms, is normally very large. Bearing in mind that $1 + \bar{d}$ is of order of magnitude 1, we can see that in order to be confident of a finite numerator, we must have g^2 of the same order of magnitude as $1/N$, which can be formalised by defining a scaled quantity $\bar{g}$, by

$$g = \bar{g}/\sqrt{N} \tag{9.3.72}$$

where it is understood that $\bar{g}$ is a quantity that is of order of magnitude 1.

This means that we can write a reduced saturation photon number by use of the equation

$$n_0 = \tilde{n}_0 N = \frac{\bar{\gamma}^2}{2\bar{g}^2} N \tag{9.3.73}$$

and this shows that n_0 is expected to scale with N. The cooperativity parameter C, can be written

$$C = \frac{\bar{d}\bar{g}^2}{\bar{\gamma}\kappa} \tag{9.3.74}$$

and does not scale with N—indeed it can have a variety of values, and is the parameter which determines the behaviour of the laser.

We can make the behaviour of the solutions with respect to N appear simplest by also defining a reduced variable $\bar{\alpha}$ by

$$\alpha = \bar{\alpha}\sqrt{N}, \tag{9.3.75}$$

in terms of which the Fokker-Planck equation (9.3.70) takes the form

$$\frac{\partial P}{\partial t} = \left\{ \frac{\partial}{\partial \bar{\alpha}} \left[\kappa\bar{\alpha} \left(1 - \frac{C}{1 + |\bar{\alpha}|^2/\tilde{n}_0} \right) \right] + \frac{\partial}{\partial \bar{\alpha}^*} \left[\kappa\bar{\alpha}^* \left(1 - \frac{C}{1 + |\bar{\alpha}|^2/\tilde{n}_0} \right) \right] \right.$$
$$\left. + \frac{2}{N} \frac{\partial^2}{\partial \bar{\alpha} \partial \bar{\alpha}^*} Q(\bar{\alpha}) \right\} P \tag{9.3.76}$$

where in this case $Q(\bar{\alpha})$ is actually independent of $\bar{\alpha}$, and is given by

$$Q(\bar{\alpha}) = \kappa n + \frac{\bar{g}^2(1 + \bar{d})}{2\bar{\gamma}}. \tag{9.3.77}$$

The form (9.3.77) for $Q(\bar{\alpha})$ arose because we assumed that $|\bar{\alpha}|^2 \ll \tilde{n}_0$. If this is not so, the correlation function expressions (9.3.68) have to be replaced by appropriate expressions derived from (9.2.70,71), which can be done with sufficient labour. For complete generality, we can adopt the form (9.3.76).
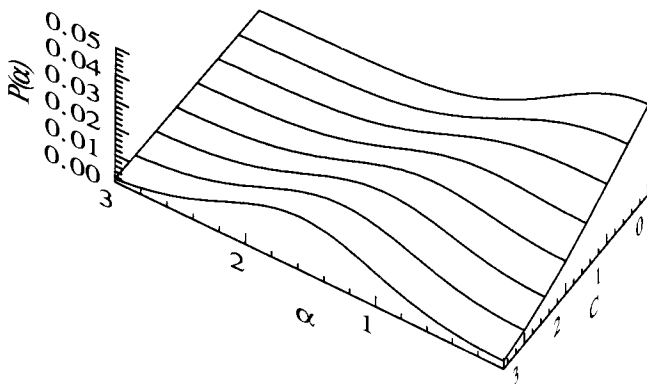
Fig. 9.6   Plot of the P-function $P(\alpha)$ for the laser

9.3.3   Solutions of the Laser Equations

a) Stochastic Differential Equations: The Fokker-Planck equation (9.3.76) is equivalent to the stochastic differential equation

$$\frac{d\tilde{\alpha}}{dt} = -\kappa\tilde{\alpha}\left(1 - \frac{C}{1 + |\tilde{\alpha}|^2/\tilde{n}_0}\right) + \frac{1}{\sqrt{N}}\tilde{\Gamma}(t) \tag{9.3.78}$$

and its complex conjugate. The correlation function of the noise term $\tilde{\Gamma}(t)$ is

$$\langle\tilde{\Gamma}^*(t)\tilde{\Gamma}(t')\rangle = 2Q(\tilde{\alpha})\delta(t - t') \tag{9.3.79}$$

with other correlation functions equal to zero.

b) Macroscopic Laser Equations: If we take the limit $N \to \infty$ we obtain deterministic equations in which the noise on the RHS of (9.3.78) is simply dropped. One obtains the same equations as in Sect. 9.3.1c–e, and the analysis is exactly the same.

c) Stationary Solution of the Fokker Planck Equation: Although it is possible to solve for the stationary solution of Fokker-Planck equation (9.3.76) exactly (in terms of integrals), the result is complicated and uninformative. It is simpler to consider the situation in which the $|\tilde{\alpha}^2|$ dependence is neglected, and take Q to be a constant. In that case, the solution is quite straightforward, and we find

$$P(\alpha, \alpha^*) = \mathcal{N}\exp\left(-\frac{N}{Q}V(|\tilde{\alpha}|)\right) \tag{9.3.80}$$

where $V(|\alpha|)$ is the classical potential given in (9.3.36). Using the explicit form of this potential, the solution is

$$\boxed{P(\alpha, \alpha^*) = \mathcal{N}(1 + |\tilde{\alpha}|^2/\tilde{n}_0)^{\tilde{n}_0\kappa CN/2Q}\exp\left(-\frac{N\kappa|\tilde{\alpha}|^2}{2Q}\right). \tag{9.3.81}}$$

This P-function is plotted in Fig. 9.6.

This distribution takes on two characteristic shapes, depending on whether the laser is above or below threshold. Below threshold, the distribution has the form of a single peak centred on the origin. Above threshold, the distribution develops a maximum at the classical stationary value, given by $|\tilde{\alpha}| = \sqrt{\bar{n}_0(C-1)}$. This is therefore an annular distribution in the variables $x = \text{Re}(\tilde{\alpha})$, $y = \text{Im}(\tilde{\alpha})$, showing that the amplitude $|\tilde{\alpha}|$ has a well defined most probable value, while the phase of $\tilde{\alpha}$ is indeterminate. The lasing action determines the amplitude quite precisely, while not determining the phase.

d) Linearised solution of the Langevin Equation: Let us write the Langevin equation in the form

$$\frac{d\tilde{\alpha}}{dt} = -\kappa\tilde{\alpha}f(|\tilde{\alpha}|^2) + \frac{1}{\sqrt{N}}\tilde{\Gamma}(t) \qquad (9.3.82)$$

and assume that N is large, so that we may set $\tilde{\alpha} = \tilde{\alpha}_0 + \epsilon$, where $\tilde{\alpha}_0$ is a stationary solution of the equation without the fluctuating term. Thus, we can approximate (9.3.82) by

$$\frac{d\tilde{\alpha}_0}{dt} + \frac{d\epsilon}{dt} = -\kappa(\tilde{\alpha}_0 + \epsilon)\left\{f(|\tilde{\alpha}|^2) + \epsilon\frac{\partial f}{\partial\tilde{\alpha}_0} + \epsilon^*\frac{\partial f}{\partial\tilde{\alpha}_0^*}\right\} + \frac{1}{\sqrt{N}}\tilde{\Gamma}(t). \quad (9.3.83)$$

e) Below Threshold: $\tilde{\alpha}_0 = 0$, so that to lowest order in ϵ, the equation is

$$\dot{\epsilon} = -\kappa\epsilon + \frac{1}{\sqrt{N}}\tilde{\Gamma}(t) \qquad (9.3.84)$$

(where we note that $f(0) = 1$.) This is a simple Ornstein-Uhlenbeck equation (S.M. Sects. 3.8.4, 4.4.4. In the stationary state the time correlation function is

$$\langle\epsilon^*(t)\epsilon(t')\rangle = \frac{Q}{\kappa N}e^{-\kappa|t-t'|}. \qquad (9.3.85)$$

In this case, the correlation function (and in fact the total dynamical behaviour) of the light field is determined by the properties of the cavity, whose decay constant is κ, and the number of atoms N.

f) Above Threshold: The annular symmetry complicates matters here, and it is best to transform to polar co-ordinates by setting

$$\tilde{\alpha} = Re^{i\phi}. \qquad (9.3.86)$$

We follow the procedure of S.M. Sect. 4.4.5. Firstly, set $R = e^{\mu}$ so that

$$\mu + i\phi = \ln(\tilde{\alpha}). \qquad (9.3.87)$$

We write the Langevin equations in SDE form as

$$d\tilde{\alpha} = -\kappa\tilde{\alpha}f(|\tilde{\alpha}|^2)dt + \frac{1}{\sqrt{N}}d\tilde{\Gamma}(t) \qquad (9.3.88)$$

where

$$
\begin{aligned}
d\tilde{\Gamma}(t)^2 &= d\tilde{\Gamma}^*(t)^2 = 0 \\
d\tilde{\Gamma}(t)d\tilde{\Gamma}^*(t) &= 2Qdt
\end{aligned}
\tag{9.3.89}
$$

and interpret them as Ito equations, since the Fokker-Planck equation always comes out in this form. Then, in changing variables, we have to expand the infinitesimals to second order, so that

$$
d\mu + i\,d\phi = \frac{d\tilde{\alpha}}{\bar{\alpha}} - \frac{(d\tilde{\alpha})^2}{\bar{\alpha}^2}
\tag{9.3.90}
$$

but in this case, using (9.3.89), we see that the second order term is zero. Hence

$$
d\mu + i\,d\phi = -\kappa f(e^{2\mu})dt + \frac{e^{-\mu-i\phi}}{\sqrt{N}}d\tilde{\Gamma}(t)
\tag{9.3.91}
$$

so that

$$
\begin{aligned}
d\mu &= -\kappa f(e^{2\mu})dt + \frac{e^{-\mu}}{\sqrt{N}}\,d\Gamma_R(t) \\
d\phi &= \frac{e^{-\mu}}{\sqrt{N}}\,d\Gamma_\phi(t).
\end{aligned}
\tag{9.3.92}
$$

As discussed in S.M.Sect. 4.4.5, $d\Gamma_R(t)$ and $d\Gamma_\phi(t)$ are given by

$$
\begin{aligned}
d\Gamma_R(t) &= \mathrm{Re}\left(e^{-i\phi}d\tilde{\Gamma}(t)\right) \\
d\Gamma_\phi(t) &= \mathrm{Im}\left(e^{-i\phi}d\tilde{\Gamma}(t)\right)
\end{aligned}
\tag{9.3.93}
$$

and are independent stochastic sources, i.e.

$$
\begin{aligned}
d\Gamma_R(t)^2 &= d\Gamma_\phi(t)^2 = Qdt \\
d\Gamma_R(t)d\Gamma_\phi(t) &= 0.
\end{aligned}
\tag{9.3.94}
$$

We now let $R = \exp(\mu)$, so that

$$
dR = \exp(\mu)d\mu + \tfrac{1}{2}\exp(\mu)d\mu^2
\tag{9.3.95}
$$

and find the equivalent SDEs :

$$
\begin{aligned}
dR &= \left\{-\kappa R f(R^2) + \frac{Q}{2NR}\right\}dt + \frac{1}{\sqrt{N}}\,d\Gamma_R(t) \\
d\phi &= \frac{1}{R\sqrt{N}}\,d\Gamma_\phi(t).
\end{aligned}
\tag{9.3.96}
$$

The correlation function of interest is $\langle \bar{\alpha}^*(t)\bar{\alpha}(t')\rangle$, since this is proportional to $\langle a^\dagger(t)a(t')\rangle$, which is the quantity whose Fourier transform is the optical spectrum. Using the polar variables, we find

$$
\langle \bar{\alpha}^*(t)\bar{\alpha}(t')\rangle = \langle R(t)R(t')\exp[i\phi(t') - i\phi(t)]\rangle.
\tag{9.3.97}
$$

From (9.3.96), it is clear that $R(t)$ fluctuates about

$$R = |\tilde{\alpha}_0| = \sqrt{\tilde{n}_0(C-1)} \tag{9.3.98}$$

with fluctuations of order $1/N$. Thus, to order $1/N$, we can write

$$\langle \tilde{\alpha}^*(t)\tilde{\alpha}(t')\rangle = \tilde{n}_0(C-1)\langle \exp i \left(\phi(t') - \phi(t)\right)\rangle. \tag{9.3.99}$$

The equation for ϕ is also accurately represented to order $1/N$ by substituting for R by (9.3.98). We then find

$$\phi(t) - \phi(t') = \frac{1}{\sqrt{N\tilde{n}_0(C-1)}} \int_{t'}^{t} d\Gamma_\phi(t'') \tag{9.3.100}$$

so that

$$\langle \left(\phi(t) - \phi(t')\right)^2 \rangle = \frac{Q|t-t'|}{N\tilde{n}_0(C-1)}. \tag{9.3.101}$$

Since Γ_ϕ is Gaussian, $\phi(t) - \phi(t')$ is also, so we can use

$$\langle \exp x \rangle = \exp \tfrac{1}{2}\langle x^2 \rangle \tag{9.3.102}$$

which is valid for Gaussian x with zero mean, to get

$$\langle \tilde{\alpha}^*(t)\tilde{\alpha}(t')\rangle = \tilde{n}_0(C-1) \exp \left(-\frac{Q|t-t'|}{2N\tilde{n}_0(C-1)}\right) \tag{9.3.103}$$

or

$$\boxed{\langle \alpha^*(t)\alpha(t')\rangle = \langle a^\dagger(t)a(t')\rangle = n_0(C-1) \exp \left(-\frac{Q|t-t'|}{2n_0(C-1)}\right). \tag{9.3.104}}$$

9.3.4 The Nature of Laser Light

a) Below Threshold: Below and above threshold we find two different kinds of behaviour. To the extent that the linearisation (9.3.84) is valid, it is clear that $\alpha(t) = \sqrt{N}\,\epsilon(t)$ is a Gaussian function with zero mean, and intensity

$$\langle a^\dagger a \rangle_s = \frac{Q}{\kappa} \tag{9.3.105}$$

and

$$\langle a^\dagger(t)a(0)\rangle_s = \frac{Q}{\kappa} e^{-\kappa|t|}. \tag{9.3.106}$$

This light is thus essentially the same as thermal light, as described in Sect. 8.3.4. Thus, below threshold, the intensity of the light is essentially independent of the number of atoms, N, though careful inspection of the definition (9.3.77) of Q shows that, there is a dependence through $\tilde{g} = g\sqrt{N}$.

Since the light is represented by a Gaussian function, we can use the results of Sect. 8.3.4, to find that

$$g^{(1)}(t) = e^{-\kappa|t|} \tag{9.3.107}$$

$$g^{(2)}(t) = 1 + e^{-2\kappa|t|}. \tag{9.3.108}$$

Thus, below threshold, the laser behaves like an ordinary lamp; in no way is the light distinguishable from that emitted by a number of atoms.

b) At Threshold: We can get an estimate of $g^{(1)}(0)$ and $g^{(2)}(0)$ at threshold by using the method of steepest descents to evaluate the moments using the P-function (9.3.81).

Exercise. Write $1 + |\tilde{\alpha}^2|/n_0 = \exp\left[\log(1 + |\tilde{\alpha}|^2/n_0)\right]$, and expand the logarithm to the second order. Set $C = 1$, and show that at threshold, this approximation yields

$$\langle :(a^\dagger a)^n : \rangle = \frac{2^{n/2}}{\sqrt{\pi}} n_0 \Gamma\left(\frac{n+1}{2}\right) \tag{9.3.109}$$

and hence

$$g^{(2)}(0) = \pi/2 \approx 1.57 \tag{9.3.110}$$

which is intermediate between the Gaussian value of 2, and the Poissonian value of 1.

c) Above Threshold: To a first approximation, above threshold one can ignore the fluctuations, and $\alpha(t)$ is simply a constant, say α_0. This means that all correlations of $a^\dagger(t)$ will, to this accuracy, factorise—the laser produces essentially coherent light above threshold.

A more careful interpretation of the fluctuations follows by still ignoring the amplitude fluctuations, but taking note of the phase fluctuations, which gives the correlation formula (9.3.104). The phase correlation time is

$$\tau_c = N\frac{2\tilde{n}_0(C-1)}{Q} \tag{9.3.111}$$

which grows with N, and can be very large. Thus if the phase is measured, and then remeasured within a time much less than τ_c, almost the same result will be found. The laser field can be represented by a slowly drifting $\alpha(t)$. Taking the Fourier transform of (9.3.104) will give a spectrum with bandwidth τ_c^{-1}; i.e., a very narrow bandwidth.

The intensity correlation functions can be deduced from the Langevin equations (9.3.96).

Exercise. Linearise the R equation (9.3.96) about $\sqrt{\tilde{n}_0(C-1)}$, to obtain

$$\langle R(t)R(0)\rangle \approx \tilde{n}_0^2(C-1) + \frac{Q^2}{2\kappa n_0(C-1)} \exp\left[-2\kappa\tilde{n}_0(C-1)|t|\right] \tag{9.3.112}$$

and thus

$$g^{(2)}(t) \approx 1 + \frac{1}{N} \frac{Q^2}{2\kappa\tilde{n}_0^3(C-1)^2} \exp\left[-2\kappa\tilde{n}_0(C-1)|t|\right]. \tag{9.3.113}$$

For large numbers of atoms N, this is barely distinguishable from the Poissonian value, $g^{(2)}(t) = 1$. Notice however that the time constant is quite different from that of the amplitude correlation function. Thus, the amplitude fluctuations are small ($\sim 1/\sqrt{N}$ times the amplitude), and relatively fast (i.e., the time constant does not increase with N). The phase fluctuations are very slow, and large, and give the principal contribution to the observed laser linewidth.

9.4   Optical Bistability

Optical bistability can be generated by a number of mechanisms, all of which involve a nonlinearity in optical response. A system which does exhibit optical bistability, and has been well studied experimentally, is modelled very similarly to the laser model of the previous section. The major differences are

i)   The atomic bath is not pumped to a negative temperature—indeed its temperature may be zero.

ii)   An external driving field is added; thus the mode of the light field is coupled to an external field $F(t)$.

iii)   The atomic transition frequency, the resonant frequency of the cavity, and the frequency of the driving field are not all exactly the same. This makes it necessary to introduce detuning parameters to describe the situation.

Optical bistability was first correctly treated theoretically by *Bonifacio* and *Lugiato* [9.6] using a different method from that presented here.

a) Detunings: The driving field frequency is taken to be ω, the resonant frequency of the cavity is $\omega + \Delta$, and the resonant frequency of the atoms is $\omega + \theta$. This means that we now work in a frame rotating at frequency ω, in which we find

$$H_F = \hbar\Delta a^\dagger a + i\hbar F a^\dagger - i\hbar F^* a \tag{9.4.1}$$

$$H_A = \sum_{\mu=1}^{N} \tfrac{1}{2}\hbar\,\theta\sigma_\mu^z. \tag{9.4.2}$$

In this frame, the master equation can be written as in (9.3.47), in which however,

$$\left.\frac{\partial\rho}{\partial t}\right|_A = \tfrac{1}{2}W_{21}\sum_{\mu=1}^{N}\left(2\sigma_\mu^-\rho\sigma_\mu^+ - \sigma_\mu^+\sigma_\mu^-\rho - \rho\sigma_\mu^+\sigma_\mu^-\right)$$

$$+ \tfrac{1}{2}W_{12}\sum_{\mu=1}^{N}\left(2\sigma_\mu^+\rho\sigma_\mu^- - \sigma_\mu^-\sigma_\mu^+\rho - \rho\sigma_\mu^-\sigma_\mu^+\right) - \tfrac{1}{2}i\theta\sum_{\mu=1}^{N}\left[\sigma_\mu^z,\rho\right] \tag{9.4.3}$$

$$\frac{\partial\rho}{\partial t}\bigg|_F = \kappa n \left(2 a^\dagger \rho a - a a^\dagger \rho - \rho a a^\dagger\right) + \kappa(n+1)\left(2 a \rho a^\dagger - a^\dagger a \rho - \rho a^\dagger a\right)$$

$$-\mathrm{i}\,\Delta\,[a^\dagger a, \rho] + F\,[a^\dagger, \rho] - F^*\,[a, \rho]. \tag{9.4.4}$$

Otherwise, the equations are the same.

b) Evolution Equations in the P-Representation: We find that, analogously to Sect. 9.3.2d

$$\frac{\partial\boldsymbol{\rho}}{\partial t} = (L_1 + L_2 + L_3)\boldsymbol{\rho}, \tag{9.4.5}$$

in which

$$L_1\rho = \tfrac{1}{2}W_{21}\sum_{\mu=1}^{N}\left(2\sigma_\mu^-\boldsymbol{\rho}\sigma_\mu^+ - \sigma_\mu^+\sigma_\mu^-\boldsymbol{\rho} - \boldsymbol{\rho}\sigma_\mu^+\sigma_\mu^-\right)$$

$$+\tfrac{1}{2}W_{12}\sum_{\mu=1}^{N}\left(2\sigma_\mu^+\boldsymbol{\rho}\sigma_\mu^- - \sigma_\mu^-\sigma_\mu^+\boldsymbol{\rho} - \boldsymbol{\rho}\sigma_\mu^-\sigma_\mu^+\right)$$

$$-\tfrac{1}{2}\mathrm{i}\,\theta\sum_{\mu=1}^{N}[\sigma_\mu^z, \boldsymbol{\rho}] + g\sum_{\mu=1}^{N}\left(\alpha^*\,[\sigma_\mu^-, \boldsymbol{\rho}] - \alpha\,[\sigma_\mu^+, \boldsymbol{\rho}]\right) \tag{9.4.6}$$

$$L_3\rho = \left\{\frac{\partial}{\partial\alpha}\left[-F + (\kappa + \mathrm{i}\,\Delta)\alpha + g\sum_{\mu=1}^{N}\langle\sigma_\mu^-\rangle\right]\right.$$

$$\left.+\frac{\partial}{\partial\alpha^*}\left[-F^* + (\kappa - \mathrm{i}\,\Delta)\alpha^* - g\sum_{\mu=1}^{N}\langle\sigma_\mu^+\rangle\right] + 2\kappa n\frac{\partial^2}{\partial\alpha\partial\alpha^*}\right\}\rho \tag{9.4.7}$$

$$L_2\rho = -g\sum_{\mu=1}^{N}\left\{\frac{\partial}{\partial\alpha}\left(\sigma_\mu^- - \langle\sigma_\mu^-\rangle\right)\boldsymbol{\rho} + \frac{\partial}{\partial\alpha^*}\boldsymbol{\rho}\left(\sigma_\mu^+ - \langle\sigma_\mu^+\rangle\right)\right\}. \tag{9.4.8}$$

Exercise. Show that the equations of motion for the averages of σ_μ are

$$\left.\begin{aligned}
\langle\dot\sigma_\mu^+\rangle &= -(\bar\gamma - \mathrm{i}\,\theta)\langle\sigma_\mu^+\rangle + g\alpha^*\langle\sigma_\mu^z\rangle\\
\langle\dot\sigma_\mu^-\rangle &= -(\bar\gamma + \mathrm{i}\,\theta)\langle\sigma_\mu^-\rangle + g\alpha\langle\sigma_\mu^z\rangle\\
\langle\dot\sigma_\mu^z\rangle &= -2\bar\gamma\langle\sigma_\mu^z\rangle - 2g\alpha\langle\sigma_\mu^+\rangle - 2g\alpha^*\langle\sigma_\mu^-\rangle - 2d\bar\gamma
\end{aligned}\right\} \tag{9.4.9}$$

and that the stationary solutions are given by

$$\langle\sigma_\mu^+\rangle_s = \langle\sigma_\mu^-\rangle_s^* = -\frac{g(\bar\gamma + \mathrm{i}\,\theta)\alpha^*}{\bar\gamma^2 + \theta^2 + 2g^2|\alpha|^2}$$

$$\langle\sigma_\mu^z\rangle = -\frac{\bar\gamma^2 + \theta^2}{\bar\gamma^2 + \theta^2 + 2g^2|\alpha|^2}. \tag{9.4.10}$$

These solutions have been computed with $\bar d = 1$, which, according to (9.3.53), corresponds to $W_{21} = 0$, implying that the atoms are not pumped, and are treated as being at zero temperature.

c) Adiabatic Elimination of Atoms: This can be carried out in the same way as for the laser; the Fokker-Planck equation is

$$\frac{\partial P}{\partial t} = \left\{ \frac{\partial}{\partial \alpha} \left[-F + \alpha \left(\kappa + i\Delta + \frac{Ng^2(\bar{\gamma} - i\theta)}{\bar{\gamma}^2 + \theta^2 + 2g^2|\alpha|^2} \right) \right] \right.$$

$$+ \frac{\partial}{\partial \alpha^*} \left[-F^* + \alpha^* \left(\kappa - i\Delta + \frac{Ng^2(\bar{\gamma} + i\theta)}{\bar{\gamma}^2 + \theta^2 + 2g^2|\alpha|^2} \right) \right]$$

$$\left. + 2\kappa n \frac{\partial^2}{\partial\alpha\partial\alpha^*} + N \left[\frac{\partial^2}{\partial\alpha\partial\alpha^*}Q + \frac{\partial^2}{\partial\alpha^2}R + \frac{\partial^2}{\partial\alpha^{*2}}R^* \right] \right\} P. \qquad (9.4.11)$$

The coefficients Q and R are given by the same reasoning as for the laser, and are

$$Q = g^2 \int_0^\infty dt \left\{ \langle \sigma_\mu^+(0)\sigma_\mu^-(t)\rangle_s + \langle \sigma_\mu^-(t)\sigma_\mu^+(0)\rangle_s - 2\langle \sigma_\mu^+\rangle_s\langle \sigma_\mu^-\rangle_s \right\} \qquad (9.4.12)$$

$$R = g^2 \int_0^\infty \left\{ \langle \sigma_\mu^-(t)\sigma_\mu^-(0)\rangle_s - \langle \sigma_\mu^-\rangle_s^2 \right\} dt. \qquad (9.4.13)$$

To evaluate these is not particularly pleasant in the general case, though the procedure which must be used is quite clear.

i)   Write the equation of motion for the averages, (9.4.9), as

$$\langle \dot{\boldsymbol{\sigma}} \rangle = -A\langle \boldsymbol{\sigma} \rangle + \mathbf{b}. \qquad (9.4.14)$$

ii)   The quantum regression theorem can then be used to say

$$\int_0^\infty dt \left\{ \langle \boldsymbol{\sigma}(t)\sigma^\alpha(0)\rangle_s - \langle \boldsymbol{\sigma}\rangle_s\langle \sigma^\alpha\rangle_s \right\} = \int_0^\infty dt\, e^{-At} \left\{ \langle \boldsymbol{\sigma}\sigma^\alpha\rangle_s - \langle \boldsymbol{\sigma}\rangle_s\langle \sigma^\alpha\rangle_s \right\}$$

$$= A^{-1} \left\{ \langle \boldsymbol{\sigma}\sigma^\alpha\rangle_s - \langle \boldsymbol{\sigma}\rangle_s\langle \sigma^\alpha\rangle_s \right\}. \qquad (9.4.15)$$

iii)   In these last two equations, $\boldsymbol{\sigma} = \begin{pmatrix} \sigma^+ \\ \sigma^- \\ \sigma_z \end{pmatrix}$ and σ^α is one of the components of $\boldsymbol{\sigma}$.

iv)   The terms $\langle \boldsymbol{\sigma}\sigma^\alpha\rangle_s$ can all be evaluated by using the algebra of Pauli matrices, e.g., $\langle \sigma^+\sigma^-\rangle_s = \frac{1}{2}\langle \sigma_z + 1\rangle_s$, etc.

v)   Terms like $\langle \sigma^-(0)\sigma^+(t)\rangle_s$ can be evaluated by using

$$\langle \sigma^-(0)\sigma^+(t)\rangle_s = \langle \sigma^-(t)\sigma^+(0)\rangle_s^*. \qquad (9.4.16)$$

Exercise. Show that if

$$\begin{pmatrix} \int_0^\infty dt\langle \sigma^+(t), \sigma^+(0)\rangle_s & \int_0^\infty dt\langle \sigma^+(t), \sigma^-(0)\rangle_s \\ \int_0^\infty dt\langle \sigma^-(t), \sigma^+(0)\rangle_s & \int_0^\infty dt\langle \sigma^-(t), \sigma^-(0)\rangle_s \\ \int_0^\infty dt\langle \sigma_z(t), \sigma^+(0)\rangle_s & \int_0^\infty dt\langle \sigma_z(t), \sigma^-(0)\rangle_s \end{pmatrix} \equiv \mathbf{S} \qquad (9.4.17)$$

where $\langle A(t), B(0) \rangle_s \equiv \langle A(t) B(0) \rangle_s - \langle A(t) \rangle_s \langle B(0) \rangle_s$, then

$$
\mathbf{S} = A^{-1} \begin{pmatrix} -\langle \sigma^+ \rangle_s^2 & \frac{1}{2}\langle 1 + \sigma_z \rangle_s - |\langle \sigma^+ \rangle_s|^2 \\ \frac{1}{2}\langle 1 - \sigma_z \rangle_s - |\langle \sigma^- \rangle_s|^2 & -\langle \sigma^- \rangle_s^2 \\ \langle \sigma^+ \rangle_s \langle 1 - \sigma_z \rangle_s & -\langle \sigma^- \rangle_s \langle 1 + \sigma_z \rangle_s \end{pmatrix}. \tag{9.4.18}
$$

From the formula the noise coefficients Q and R can be computed. The explicit results are a little complicated.

d) Absorptive Limit: Existence of Bistability: In the case that θ and Δ are zero, i.e. there is no atomic or cavity detuning, the stationary solution of the deterministic equation is given by

$$
F = \alpha \left(\kappa + \frac{N g^2 \bar{\gamma}}{\bar{\gamma}^2 + 2 g^2 |\alpha|^2} \right). \tag{9.4.19}
$$

We can use the fact that the solutions have F an α in phase to write

$$
|F| = |\alpha| \left(\kappa + \frac{N g^2 \bar{\gamma}}{\bar{\gamma}^2 + 2 g^2 |\alpha|^2} \right). \tag{9.4.20}
$$

Using the same notations n_s and C as in the laser, we can write the equation more simply as

$$
|F| = \kappa |\alpha| \left(1 + \frac{C}{1 + |\alpha|^2 / n_0} \right). \tag{9.4.21}
$$

If $C > 4$, there are the solutions for this equation for any n_0, κ, and a range of $|F|$. Graphing the function as in Fig.9.7 demonstrates this clearly.
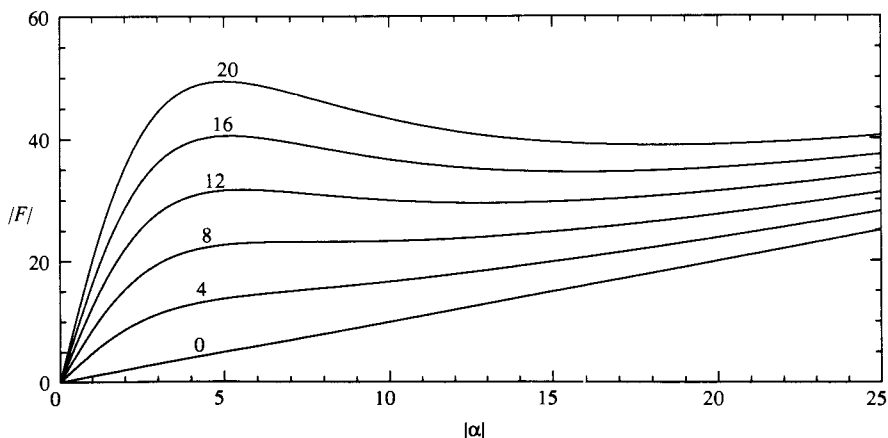
Fig. 9.7 Plots of bistability curves for the equation (9.4.21)

Exercise. Show that (9.4.21) exhibits bistability if $C > 4$. The curve has two branches;

i)   Lower branch; $|\alpha|^2/n_0 \ll 1$,

$$\Rightarrow |F| \approx \kappa |\alpha| (1 + C). \tag{9.4.22}$$

In this branch we see the extra effect of atomic absorption in the factor $(1 + C)$.

ii)   Upper branch: $|\alpha|^2/n_0 \gg 1$,

$$\Rightarrow |F| \approx \kappa |\alpha|. \tag{9.4.23}$$

Here the saturation effect is seen; there is no longer any additional absorption. When the field is sufficiently strong, the atoms are no longer able to absorb light.

e) Dispersive Limit: At very large atomic detuning θ, the deterministic equation becomes, on expanding the denominators of the fraction

$$F \approx \alpha \left(\kappa + i\Delta - iNg^2/\theta + 2iNg^4 |\alpha|^2/\theta^3 \right) \tag{9.4.24}$$

thus

$$|F|^2 \approx |\alpha|^2 \left(\kappa^2 + (\Delta - Ng^2/\theta + 2Ng^4 |\alpha|^2/\theta^3)^2 \right). \tag{9.4.25}$$

There will be a range of values of F for which there are three solutions for $|\alpha|$ provided

$$|\Delta - Ng^2/\theta| > \kappa^2 \sqrt{3}. \tag{9.4.26}$$

The full Fokker-Planck equation is, in this limit,

$$\begin{aligned}
\frac{\partial P}{\partial t} = \Bigg\{ &\frac{\partial}{\partial \alpha} \left[-F + \alpha \left(\kappa + i\Delta - iNg^2/\theta + 2iNg^4 |\alpha|^2/\theta^3 \right) \right] \\
&+ \frac{\partial}{\partial \alpha^*} \left[-F^* + \alpha^* \left(\kappa - i\Delta + iNg^2/\theta - 2iNg^4 |\alpha|^2/\theta^3 \right) \right] \\
&+ 2\kappa \bar{n} \frac{\partial^2}{\partial \alpha \partial \alpha^*} \\
&- iNg^4/\theta^3 \left[\frac{\partial^2}{\partial \alpha^2} \alpha^2 - \frac{\partial^2}{\partial \alpha^{*2}} \alpha^{*2} \right] \Bigg\} P.
\end{aligned} \tag{9.4.27}$$

Notice that this Fokker-Planck equation can be obtained by coupling an anharmonic oscillator, with Hamiltonian

$$H = \hbar \left(\omega + \Delta - \frac{Ng^2}{\theta} \right) a^\dagger a + \frac{\hbar Ng^4}{\theta^3} (a^\dagger)^2 a^2 \tag{9.4.28}$$

to a heat bath. Thus, in this limit, the atoms modify only the Hamiltonian part of the motion, and give no damping, nor any of the noise associated with damping. Optically, one would say that the basic effect of the atoms in the dispersive limit is to provide a non-linear refractive index, and bistability comes about because the wavelength of light is changed, and thus different resonant conditions exist for the same incident field. [9.7]

9.5   Other Phase Space Methods

The methods given in this chapter so far are those which I have felt most comfortable with, and which seem to me to present the theory with the greatest clarity. Thus the quantum Langevin equations for the laser in the scaled form, (9.3.22,23), while being fully quantum mechanical, very straightforwardly show that there is a classical limit when $N \to \infty$, and thus generate the macroscopic laser equations.

On the other hand, the solution of the quantum Langevin equations is only possible by perturbation methods, valid only when N is large. The adiabatic elimination method of Sects. 9.3.2,3 does not depend on this limit, and thus could be valid for only a few atoms. The resulting equations of this method were not worked out in fullest detail, but have been by *Haake* and *Lewenstein* [9.8] in a slightly different presentation of this method. There are details of difference in the noise terms when this evaluation is carried out.

9.5.1   The Method of Haken, Risken and Weidlich

Historically, other methods have been used to derive these equations. One that is very common was introduced by *Haken*, *Risken* and *Weidlich* [9.9]. One introduces (in the notation of Sect. 9.3.2) a characteristic function, which is a function of variables $\xi, \xi^*, \varsigma, \beta, \beta^*$, by the following method.

i)   Define

$$O_F = e^{i\beta^* a^\dagger} e^{i\beta a}. \tag{9.5.1}$$

$$O_{A\mu} = e^{i\xi^* \sigma_\mu^+} e^{\varsigma \sigma_\mu^z/2} e^{i\xi \sigma_\mu^-}. \tag{9.5.2}$$

ii)   Define the characteristic function by

$$\chi(\beta, \beta^*, \xi, \xi^*, \varsigma) = \text{Tr} \left\{ \rho O_F \prod_{\mu=1}^{N} O_{A\mu} \right\}. \tag{9.5.3}$$

iii)   Define a distribution function f by the Fourier transform

$$f(\alpha, \alpha^*, v, v^*, D) =$$
$$\int d^2\beta \int d^2\xi \int d\varsigma \, e^{-i\left\{ v\xi + v^*\xi^* + D\varsigma/2 + \alpha\beta + \alpha^*\beta^* \right\}} \cdot \chi(\beta, \beta^*, \xi, \xi^*, \varsigma). \tag{9.5.4}$$

It is easy to see that, as far as the field variables α, α^* are concerned, this is exactly the same as the P-function. The other variables can also be related to operators in v, v^*, D, though the results are rather complicated. The resulting Fokker-Planck equation for f is of infinite order, and takes the form

$$\frac{\partial f}{\partial t} = Lf \tag{9.5.5}$$

where the operator L consists of the atomic part, the interaction and field part

$$L = L_A + L_{AL} + L_L. \tag{9.5.6}$$

The different contributions are defined as follows:

$$
\begin{aligned}
L_A = \frac{W_{12}}{2} &\left\{ N \left(e^{-2\frac{\partial}{\partial D}} - 1 \right) + N\, e^{2\frac{\partial}{\partial D}} \frac{\partial^4}{\partial v^2 \partial v^{*2}} + 2N \frac{\partial^2}{\partial v \partial v^*} \right.\\
&+ \frac{\partial}{\partial v} \left[2 \frac{\partial^2}{\partial v \partial v^*} + 2 e^{-2\frac{\partial}{\partial D}} - 1 \right] v \\
&+ \frac{\partial}{\partial v^*} \left[2 \frac{\partial^2}{\partial v \partial v^*} + 2 e^{-2\frac{\partial}{\partial D}} - 1 \right] v^* \\
&\left. - 2 \left[\left(e^{-2\frac{\partial}{\partial D}} - 1 \right) - e^{2\frac{\partial}{\partial D}} \frac{\partial^4}{\partial^2 v \partial^2 v^*} \right] \frac{D}{2} \right\} \\
+ \frac{W_{21}}{2} &\left\{ N \left(e^{2\frac{\partial}{\partial D}} - 1 \right) + \frac{\partial}{\partial v} v + \frac{\partial}{\partial v^*} v^* + 2 \left(e^{2\frac{\partial}{\partial D}} - 1 \right) \frac{D}{2} \right\}
\end{aligned} \tag{9.5.7}
$$

$$
\begin{aligned}
L_{AL} = -g &\left\{ \left[e^{-2\frac{\partial}{\partial D}} v^* - \frac{\partial^2}{\partial v^2} v + \frac{\partial}{\partial v} D \right] \alpha \right. \\
&- \left[-\frac{\partial^2}{\partial v^{*2}} v^* + e^{-2\frac{\partial}{\partial D}} v + \frac{\partial}{\partial v^*} D \right] \alpha^* \\
&\left. + \left[-\frac{\partial}{\partial \alpha} + \alpha^* \right] v - \left[-\frac{\partial}{\partial \alpha^*} + \alpha \right] v^* \right\}
\end{aligned} \tag{9.5.8}
$$

$$
L_L = \kappa \left[\frac{\partial}{\partial \alpha} \alpha + \frac{\partial}{\partial \alpha^*} \alpha^* \right] + 2\kappa\, n_{\text{th}} \frac{\partial^2}{\partial \alpha \partial \alpha^*}. \tag{9.5.9}
$$

a) Scaling Arguments: The equation as presented is almost impossible to use—recourse must be made to scaling arguments. As before, one assumes $g \approx 1/\sqrt{N}$, and one considers scaling with

$$
\begin{aligned}
D_0 &\equiv N \frac{W_{12} - W_{21}}{W_{12} + W_{21}} \\
D &\sim N \\
|\alpha|^2 &\sim N^x \\
v &\sim N^{(x+1)/2}.
\end{aligned} \tag{9.5.10}
$$

With this scaling, all derivatives of order higher than the second are negligible. One can see that the adiabatic elimination results, and the exercise of Sect. 9.3.4b, show that $x = 1/2$ is appropriate at threshold, while $x = 1$ is appropriate above threshold.

In the threshold region, one finds the Fokker-Planck equation simplifies to

$$
\begin{aligned}
\frac{\partial P}{\partial t} = \Bigg\{ & -\frac{\partial}{\partial \alpha}(-\kappa\alpha + gv) - \frac{\partial}{\partial \alpha^*}(-\kappa\alpha^* + gv^*) \\
& -\frac{\partial}{\partial v}(-\bar\gamma\, v + g\alpha D) - \frac{\partial}{\partial v^*}(-\bar\gamma\, v^* + g\alpha^* D) \\
& -\frac{\partial}{\partial D}\Big[-2\bar\gamma\,(D - D_0) - 2g(v^*\alpha + v\alpha^*)\Big] \\
& +\frac{\partial^2}{\partial v \partial v^*}(NW_{12}) \Bigg\} P.
\end{aligned}
\tag{9.5.11}
$$

Here $\bar\gamma = \frac{1}{2}(W_{12} + W_{21})$ as in (9.3.54), and the zero temperature case n_{th} has been assumed. Adiabatic elimination can then be carried out, to yield essentially the same results as have already been derived. The same methods have been used for optical bistability by *Reid*, *Walls*, *Drummond* and their co-workers [9.10].

In the limit of large N, this method is quite simple and straightforward to apply, unlike its derivation. Of course it is only applicable in the limit that N is large, and since this is exactly the same limit in which one can linearise the quantum Langevin equations (9.3.16–18), the results obtained by both must be in agreement.

One can use the method of adiabatic elimination of the atoms on the Fokker-Planck equation (9.5.11), and rather surprisingly, one finds essentially the same equations as obtained without any small noise assumption. See Haken's book [9.3] for details.

9.5.2   A More Direct Phase Space Method

The equations of motion in the previous section are of infinite order in D because the characteristic function chosen involves exponentials. Although this characteristic function has an ordering which readily allows the finding of averages when the atomic operators are normally ordered, it is still quite singular. In fact, O^A can be written explicitly as

$$
O^A = \prod_{\mu=1}^{N}(1 + \mathrm{i}\,\xi^*\sigma_\mu^+)\left[\cos(\varsigma\sigma_{\mu,z}) + \mathrm{i}\,\sin(\varsigma\sigma_{\mu,z})\right](1 + \mathrm{i}\,\xi\sigma_\mu^-)
\tag{9.5.12}
$$

whose Fourier transform (to be defined in (9.5.34)) will contain terms of the form

$$
\delta^m(v)\delta(D - m')\delta^n(v^*)
\tag{9.5.13}
$$

for all m, n such that $0 \le m, n \le N$ and for all m' such that $-N \le m' \le N$.

The P-function conventionally obtained by the truncation procedure of the previous section amounts to a smoothed version of this quite singular distribution. It is an approximation in the sense that physical averages evaluated with a smooth distribution can be quite close to the exact result obtained with the exact, nonsmooth distribution, although the smoothed distribution is not a good pointwise approximation to the exact distribution.

It is clear, however, that O^A is, in fact, simply a linear combination of the Pauli spin matrices and the identity matrix, so a simpler way of looking at the characteristic function is to change to a new set of variables and define

$$\chi = \text{Tr} \{O\rho\} \tag{9.5.14}$$

where

$$O = O^A O^F \tag{9.5.15}$$

and, as before,

$$O^F = e^{i\beta^* a^\dagger} e^{i\beta a} \tag{9.5.16}$$

but

$$
\begin{aligned}
O^A &= \prod_{\mu=1}^{N} \left(b + c^+ \sigma_\mu^+ + c^- \sigma_\mu^- + c\sigma_{\mu,z} \right) \\
&= \prod_{\mu=1}^{N} \left(b + \mathbf{c}.\boldsymbol{\sigma}_\mu \right).
\end{aligned}
\tag{9.5.17}
$$

This characteristic function has six variables, though one of them is redundant, since, by construction, O^A is a homogeneous polynomial in b and $\mathbf{c}$ of degree N, and the physics is thus contained only in the *ratios* of b and $\mathbf{c}$.

We can derive an equation of motion for the characteristic function as follows.

a) Field Contribution: For the terms involving $a, a^\dagger$, we find

$$\text{Tr} \left\{ O \frac{\partial \rho}{\partial t} \bigg|_F \right\} = \kappa \left[-\beta \frac{\partial}{\partial \beta} - \beta^* \frac{\partial}{\partial \beta^*} - 2n\beta\beta^* \right] \chi. \tag{9.5.18}$$

b) Atomic Terms: For the atomic contribution, write O as

$$O = Q^{A,\mu} O^{A,\mu} \tag{9.5.19}$$

where

$$Q^{A,\mu} = O^F \prod_{i \neq \mu} O^{A,i} \tag{9.5.20}$$

and

$$O^{A,\mu} = b + \mathbf{c}.\boldsymbol{\sigma}_\mu. \tag{9.5.21}$$

Hence

$$
\begin{aligned}
\text{Tr} \left\{ O \frac{\partial \rho}{\partial t} \bigg|_A \right\} = \text{Tr} \Bigg\{ &\sum_\mu Q^{A,\mu} O^{A,\mu} \left[\tfrac{1}{2} W_{21} \left(2\sigma_\mu^- \rho \sigma_\mu^+ - \sigma_\mu^+ \sigma_\mu^- \rho - \rho \sigma_\mu^+ \sigma_\mu^- \right) \right. \\
&\left. + \tfrac{1}{2} W_{12} \left(2\sigma_\mu^+ \rho \sigma_\mu^- - \sigma_\mu^- \sigma_\mu^+ \rho - \rho \sigma_\mu^- \sigma_\mu^+ \right) \right] \Bigg\}.
\end{aligned}
\tag{9.5.22}
$$

Using the properties of the trace and of the Pauli matrices we deal with each term individually e.g.,

$$\mathrm{Tr}\left\{\sum_\mu Q^{A,\mu}\left(b+\boldsymbol{c}.\boldsymbol{\sigma}_\mu\right)W_{21}\sigma_\mu^-\rho\sigma_\mu^+\right\}$$

$$=\mathrm{Tr}\left\{\sum_\mu Q^{A,\mu}W_{21}\sigma_\mu^+\left(b+\boldsymbol{c}.\boldsymbol{\sigma}_\mu\right)\sigma_\mu^-\rho\right\} \tag{9.5.23}$$

$$=\mathrm{Tr}\left\{\sum_\mu Q^{A,\mu}\tfrac{1}{2}W_{21}(b-c)\left(\sigma_{z,\mu}+1\right)\rho\right\} \tag{9.5.24}$$

$$=\mathrm{Tr}\left\{\sum_\mu Q^{A,\mu}\tfrac{1}{2}W_{21}(b-c)\left(\frac{\partial}{\partial c}+\frac{\partial}{\partial b}\right)O^{A,\mu}\rho\right\}. \tag{9.5.25}$$

Following a similar method for all the other terms we then substitute into (9.5.22) to obtain

$$\mathrm{Tr}\left\{O\frac{\partial\rho}{\partial t}\Big|_A\right\}=\mathrm{Tr}\left\{\sum_\mu Q^{A,\mu}\left[-\tfrac{1}{2}(W_{21}+W_{12})\left(c^-\frac{\partial}{\partial c^-}+c^+\frac{\partial}{\partial c^+}+2c\frac{\partial}{\partial c}\right)\right.\right.$$

$$\left.\left.+(W_{12}-W_{21})c\frac{\partial}{\partial b}\right]O^{A,\mu}\rho\right\}. \tag{9.5.26}$$

Now using the product rule for differentiation we can turn the sum of derivatives of $O^{A,\mu}$ into one derivative of the product O, so that

$$\mathrm{Tr}\left\{O\frac{\partial\rho}{\partial t}\Big|_A\right\}$$

$$=\left[-\tfrac{1}{2}(W_{21}+W_{12})\left(c^-\frac{\partial}{\partial c^-}+c^+\frac{\partial}{\partial c^+}+2c\frac{\partial}{\partial c}\right)+(W_{12}-W_{21})c\frac{\partial}{\partial b}\right]\chi. \tag{9.5.27}$$

c) **Interaction Terms:** Expanding out the commutator, the interaction term can be evaluated as

$$g\,\mathrm{Tr}\left\{O\left[a^\dagger S^-\rho-\rho a^\dagger S^-+\rho aS^+-aS^+\rho\right]\right\} \tag{9.5.28}$$

where

$$S^-=\sum_\mu\sigma_\mu^- \qquad S^+=\sum_\mu\sigma_\mu^+, \tag{9.5.29}$$

and inserting the definition (9.5.15), and following a procedure similar to (9.5.23–25) for the atomic operators (9.5.28) becomes

$$=g\,\mathrm{Tr}\left\{(b-c)\left[i\beta\frac{\partial}{\partial c^-}+i\beta^*\frac{\partial}{\partial c^+}\right]+\tfrac{1}{2}(i\beta c^++i\beta^*c)\left[\frac{\partial}{\partial c}+\frac{\partial}{\partial b}\right]\right.$$

$$\left.+\frac{\partial}{\partial i\beta^*}\left[-2c\frac{\partial}{\partial c^-}+c^+\frac{\partial}{\partial c}\right]+\frac{\partial}{\partial i\beta}\left[-2c\frac{\partial}{\partial c^+}+c^-\frac{\partial}{\partial c}\right]\right\}\chi. \tag{9.5.30}$$

d) Characteristic Function Equation: substituting (9.5.18,27,30) into the characteristic function equation we obtain the equation

$$
\begin{aligned}
\frac{\partial \chi}{\partial t} = &\left\{ \kappa \left(-\beta \frac{\partial}{\partial \beta} - \beta^* \frac{\partial}{\partial \beta^*} - 2n\beta^*\beta \right) \right. \\
&- \frac{(W_{12} + W_{21})}{2} \left(c^+ \frac{\partial}{\partial c^+} + c^- \frac{\partial}{\partial c^-} + 2c \frac{\partial}{\partial c} \right) + (W_{12} - W_{21})c \frac{\partial}{\partial b} \\
&+ g \left\{ -2c \frac{\partial^2}{\partial c^+ \partial i\beta} + c^- \frac{\partial^2}{\partial c \partial i\beta} \right. \\
&+ i\beta \left[(b - c) \frac{\partial}{\partial c^-} + \tfrac{1}{2}c^+ \left(\frac{\partial}{\partial b} + \frac{\partial}{\partial c} \right) \right] \\
&- 2c \frac{\partial^2}{\partial c^- \partial i\beta^*} + c^+ \frac{\partial^2}{\partial c \partial i\beta^*} \\
&+ \left. \left. i\beta^* \left[(b - c) \frac{\partial}{\partial c^+} + \tfrac{1}{2}c^- \left(\frac{\partial}{\partial b} + \frac{\partial}{\partial c} \right) \right] \right\} \right\} \chi.
\end{aligned}
\tag{9.5.31}
$$

e) Generalised Fokker-Planck Equation: For any given number, N, of atoms the characteristic function is a homogeneous polynomial on b and c of degree N. A Fourier transform does not exist in the ordinary sense, but can exist in the sense of a distribution. If, however, there is a distribution of numbers of atoms, (e.g., a Gaussian distribution) the characteristic function can be well behaved and possess a well behaved Fourier transform. This is sufficient for our needs, and this Fourier transform will be a quasiprobability. Because the means of the variables are defined by

$$
\begin{aligned}
\left\langle \sum_\mu \sigma_\mu \right\rangle &= \left. \frac{\partial \chi}{\partial c} \right|_{b=1, c=0} \\
N &= \left. \frac{\partial \chi}{\partial b} \right|_{b=1, c=0}
\end{aligned}
\tag{9.5.32}
$$

we make the change of variables

$$
\begin{aligned}
b &\to 1 + b' \\
\frac{\partial}{\partial b} &\to \frac{\partial}{\partial b'},
\end{aligned}
\tag{9.5.33}
$$

which means that the expectation values are now evaluated at $b' = 0$. In analogy to the method of the previous section, we could expect to define a quasiprobability function $P(x) = P(\alpha^*, \alpha, v^*, D, v)$, where $x = (\alpha^*, \alpha, v^*, D, v)$, via

$$
\chi(\beta, \beta^*, \xi^*, \varsigma, \xi) = \int d\alpha^* \int d\alpha \int dv^* \int dv \int dD
$$
$$
e^{i\beta^*\alpha^*} e^{i\beta\alpha} e^{i\xi^*v^*} e^{i\varsigma D/2} e^{i\xi v} P(x).
\tag{9.5.34}
$$

In this case we have two independent complex variables and one real variable in both χ and P, so that (9.5.34) defines $P(x)$ to be the five-dimensional Fourier transform of the characteristic function. We may then define $P(x)$ via the inverse Fourier transform

$$P(x) = \int d\beta^* \int d\beta \int d\xi^* \int d\xi \int d\varsigma$$
$$e^{-i\beta^*\alpha^*} e^{i\beta\alpha} e^{-i\xi^*v^*} e^{-i\varsigma D/2} e^{-i\xi v} \chi(\beta^*, \beta, \xi^*, \varsigma, \xi). \tag{9.5.35}$$

However, a major problem arises with the P-representation defined in this way for the laser theory, in that the diffusion matrix of the Fokker-Planck equation obtained is not positive definite in the five-dimensional physical space. To overcome this problem we carry out a procedure analogous to that followed for the positive P-representation and consider all the variables in both the characteristic function and the P-function to be independent complex variables. Thus we no longer have $\alpha^* = (\alpha)^*$ but must treat α and α^* as independent complex variables. To make this clear in the following we shall adopt the notation $\alpha^* = \alpha^+$ to explicitly display the lack of conjugacy. So making all the notational changes to the variables, we define the characteristic function χ in a ten dimension phase space by

$$\chi = \int d^2\alpha^+ \int d^2\alpha \int d^2v^+ \int d^2D \int d^2v \exp\left[i(\beta^+, \beta, \xi^+, \varsigma/2, \xi) \cdot x\right] P(x). \tag{9.5.36}$$

It is now no longer possible to uniquely define the quasiprobability function $P(x)$ by the inverse Fourier transform, as was done between (9.5.34) and (9.5.35). However it is this nonuniqueness which makes the guarantee of positive-definite diffusion possible. We arrive at a generalised Fokker-Planck equation via the correspondences :

$$\left.\begin{array}{ll} i\beta \to -\dfrac{\partial}{\partial\alpha} & \dfrac{\partial}{\partial i\beta} \to \alpha \\[2ex] i\beta^* \to -\dfrac{\partial}{\partial\alpha^*} & \dfrac{\partial}{\partial i\beta^*} \to \alpha^* \end{array}\right\} \tag{9.5.37}$$

$$\left.\begin{array}{ll} c^- \to -\dfrac{\partial}{\partial v} & \dfrac{\partial}{\partial c^-} \to v \\[2ex] c^+ \to -\dfrac{\partial}{\partial v^*} & \dfrac{\partial}{\partial c^+} \to v^* \\[2ex] c \to -\dfrac{\partial}{\partial D} & \dfrac{\partial}{\partial c} \to D \end{array}\right\} \tag{9.5.38}$$

$$\left.\begin{array}{ll} b' \to -\dfrac{\partial}{\partial B} & \dfrac{\partial}{\partial b'} \to B. \end{array}\right\} \tag{9.5.39}$$

We thus obtain the generalised Fokker-Planck equation

$$\frac{\partial}{\partial t}P(\boldsymbol{x}) = \left\{ \kappa \left(\frac{\partial}{\partial \alpha}\alpha + \frac{\partial}{\partial \alpha^+}\alpha^+ + 2n\frac{\partial^2}{\partial\alpha\partial\alpha^+} \right) \right.$$

$$+ \frac{(W_{12}+W_{21})}{2} \left(\frac{\partial}{\partial v}v + \frac{\partial}{\partial v^+}v^+ + 2\frac{\partial}{\partial D}D \right) - (W_{12}-W_{21})\frac{\partial}{\partial D}B$$

$$+ g\left[2\frac{\partial}{\partial D}\alpha v^+ - \frac{\partial}{\partial v}\alpha D - \frac{\partial}{\partial \alpha}v + \frac{\partial^2}{\partial B\partial\alpha}v \right.$$

$$- \frac{\partial^2}{\partial D\partial\alpha}v + \tfrac{1}{2}\frac{\partial^2}{\partial v^+\partial\alpha}(B+D)$$

$$+ 2\frac{\partial}{\partial D}\alpha^+ v - \frac{\partial}{\partial v^+}\alpha^+ D - \frac{\partial}{\partial \alpha^+}v^+ + \frac{\partial^2}{\partial B\partial\alpha^+}v^+$$

$$\left. \left. - \frac{\partial^2}{\partial D\partial\alpha^+}v^+ + \tfrac{1}{2}\frac{\partial^2}{\partial v\partial\alpha^+}(B+D) \right] \right\} P(\boldsymbol{x}). \qquad (9.5.40)$$

This Fokker-Planck equation contains only second derivatives, although it does have a non-positive definite diffusion matrix. Following the similar methods to those used for the positive P-representation it is straightforward to show that we can write SDEs (using complex variables) which take the form

$$\begin{aligned}
\dot{\alpha} &= -\kappa\alpha + gv + \Gamma_\alpha \\
\dot{v} &= -\gamma v + g\alpha D + \Gamma_v \\
\dot{D} &= -2\gamma D + (W_{12}-W_{21})B - 2g(\alpha^+ v + \alpha v^+) + \Gamma_D \\
\dot{B} &= \Gamma_B
\end{aligned} \qquad (9.5.41)$$

where

$$\begin{aligned}
\gamma &= \tfrac{1}{2}(W_{21}+W_{12}) \\
\langle \Gamma_x(t)\Gamma_y(t')\rangle &= \mathbf{B}_{xy}\delta(t-t')
\end{aligned} \qquad (9.5.42)$$

$$\mathbf{B} = \begin{pmatrix}
0 & 2\kappa n & 0 & \tfrac{1}{2}g(B+D) & -gv & gv \\
2\kappa n & 0 & \tfrac{1}{2}g(B+D) & 0 & -gv^+ & gv^+ \\
0 & \tfrac{1}{2}g(B+D) & 0 & 0 & 0 & 0 \\
\tfrac{1}{2}g(B+D) & 0 & 0 & 0 & 0 & 0 \\
-gv & -gv^+ & 0 & 0 & 0 & 0 \\
gv & gv^+ & 0 & 0 & 0 & 0
\end{pmatrix}.$$

$$(9.5.43)$$

It is important to notice that since none of the second order coefficients depend on their respective derivatives, they can be moved inside or outside the derivatives and hence have both an Ito and Stratonovich noise interpretation. Thus the equations (9.5.41) can be interpreted as Ito or Stratonovich SDEs to suit different circumstances.

f) Comments:

i) The field part of the Fokker-Planck equation (9.5.40) is given by the first line, and is simply what would arise from the use of a P-representation, as shown in Sect. 9.3.2, as is expected, since the P-function arises as the Fourier-transform of the normally ordered quantum characteristic function (Sect. 4.4.2).

ii) Notice that N, the number of atoms, does not appear anywhere in this equation—that is, the equation is valid for *any number* of atoms. However, since $\langle B \rangle = N$, the number of atoms can be easily introduced as the initial value for B.

iii) The equation of motion for B has the intriguing property that the time derivative of any moment of B is zero, since $\langle \Gamma_B(t) \Gamma_B(t') \rangle = 0$. Thus the mean number of atoms is indeed constant, even though its quasi-variable B does have a non-trivial equation of motion.

g) Scaling with the Number of Atoms: We can make the scaling

$$
\begin{aligned}
B &= N\tilde{B} \\
D &= N\tilde{D} \\
D_0 &= N\tilde{D}_0 \\
v &= N\tilde{v} \\
\alpha &= N^{\frac{1}{2}}\tilde{\alpha} \\
g &= N^{-\frac{1}{2}}\tilde{g}
\end{aligned}
\tag{9.5.44}
$$

in the generalised Fokker-Planck equation (9.5.40). With this substitution, all the drift terms are independent of N, while all diffusion terms are proportional to $1/N$. We can write explicit stochastic differential equations for the scaled variables as

$$
\begin{aligned}
\dot{\tilde{\alpha}} &= -\kappa\tilde{\alpha} + \tilde{g}\tilde{v} + \frac{1}{\sqrt{N}}\tilde{\Gamma}_\alpha \\
\dot{\tilde{v}} &= -\gamma\tilde{v} + \tilde{g}\tilde{\alpha}\tilde{D} + \frac{1}{\sqrt{N}}\tilde{\Gamma}_v \\
\dot{\tilde{D}} &= -2\gamma(\tilde{D} - \tilde{D}_0\tilde{B}) - 2\tilde{g}(\tilde{\alpha}^+\tilde{v} + \tilde{\alpha}\tilde{v}^+) + \frac{1}{\sqrt{N}}\tilde{\Gamma}_D \\
\dot{\tilde{B}} &= \frac{1}{\sqrt{N}}\tilde{\Gamma}_B
\end{aligned}
\tag{9.5.45}
$$

and the noise terms can be written

$$\tilde{\Gamma}_\alpha = \sqrt{\kappa n}\,(\xi_1 + i\xi_2) + \tfrac{1}{4}\sqrt{\tilde{g}}\,(\tilde{B} + \tilde{D})(\xi_3 + i\xi_4)$$
$$- \tfrac{1}{2}\sqrt{\tilde{g}}\,\tilde{v}\,\big[(\xi_7 + i\xi_8) - (\xi_{11} + i\xi_{12})\big]$$

$$\tilde{\Gamma}_{\alpha^+} = \sqrt{\kappa n}\,(\xi_1 - i\xi_2) + \tfrac{1}{4}\sqrt{\tilde{g}}\,(\tilde{B} + \tilde{D})(\xi_5 + i\xi_6)$$
$$- \tfrac{1}{2}\sqrt{\tilde{g}}\,\tilde{v}^+\big[(\xi_9 + i\xi_{10}) - (\xi_{13} + i\xi_{14})\big] \qquad (9.5.46)$$

$$\tilde{\Gamma}_v = \sqrt{\tilde{g}}\,(\xi_5 - i\xi_6)$$
$$\tilde{\Gamma}_{v^+} = \sqrt{\tilde{g}}\,(\xi_3 - i\xi_4)$$
$$\tilde{\Gamma}_D = \sqrt{\tilde{g}}\,\big[(\xi_7 - i\xi_8) + (\xi_9 - i\xi_{10})\big]$$
$$\tilde{\Gamma}_B = \sqrt{\tilde{g}}\,\big[(\xi_{11} - i\xi_{12}) + (\xi_{13} - i\xi_{14})\big]$$

where the $\xi_i(t)$ are real white Langevin sources, with the correlation properties

$$\langle \xi_i(t)\xi_j(t')\rangle = \delta_{ij}\delta(t - t'). \qquad (9.5.47)$$

This means that the equation for $\tilde{B}$ is the complex Wiener process

$$\dot{\tilde{B}} = \frac{1}{\sqrt{N}}\tilde{\Gamma}_B = \sqrt{\frac{\tilde{g}}{N}}\,\big[(\xi_{11} + \xi_{13}) - i(\xi_{12} + \xi_{14})\big] \qquad (9.5.48)$$

which can be immediately integrated, since none of the variables appears on the right-hand side, to give

$$\tilde{B} = 1 + \frac{1}{\sqrt{N}}\int_0^t \tilde{\Gamma}_B(t')\,dt', \qquad (9.5.49)$$

where the initial condition $B(0) = N$ yields the initial condition $\tilde{B}(0) = 1$.

h) Naive Adiabatic Elimination of Atomic Variables: Although it is not strictly correct, we can eliminate the fast atomic variables by setting $\dot{\tilde{v}} = \dot{\tilde{D}} = 0$ in (9.5.45). Nevertheless, in the case that the noise is small, which must be the case here if N is large, as it always is, the same method as used in the deterministic equation is applicable.

Thus, we set $\dot{\tilde{v}}$ and $\dot{\tilde{D}}$ equal to zero, so that

$$\tilde{v} = \frac{\tilde{g}\tilde{\alpha}\tilde{D}}{\gamma} + \frac{\tilde{\Gamma}_v(t)}{\gamma\sqrt{N}}$$

$$\tilde{D} = \tilde{D}_0 - \frac{\tilde{g}}{\gamma}(\tilde{\alpha}^+\tilde{v} + \tilde{\alpha}\tilde{v}^+) + \frac{\tilde{D}_0}{\sqrt{N}}\int_0^t \tilde{\Gamma}_B(t')\,dt' + \frac{\tilde{\Gamma}_D(t)}{2\gamma\sqrt{N}}$$

$$(9.5.50)$$

and substituting into the equation for $\bar{\alpha}$ we obtain

$$\dot{\bar{\alpha}} = -\kappa\bar{\alpha}\left(1 - \frac{C}{1 + |\bar{\alpha}|^2/\tilde{n}_0}\right) + \frac{1}{\sqrt{N}}\bar{\Gamma}(t) \tag{9.5.51}$$

where for brevity, we have written $\alpha\alpha^+ \rightarrow |\alpha|^2$ and, as before,

$$\tilde{n}_0 = \tfrac{1}{2}\gamma^2/\tilde{g}^2 \tag{9.5.52}$$

and

$$\begin{aligned}
\bar{\Gamma}(t) &= \bar{\Gamma}_\alpha(t) + \frac{1}{\sqrt{2\tilde{n}_0}}\frac{(1 + \tfrac{1}{2}|\bar{\alpha}|^2/\tilde{n}_0)}{(1 + |\bar{\alpha}^2|/\tilde{n}_0)}\bar{\Gamma}_v(t) \\
&\quad + \frac{1}{2\tilde{n}_0}\frac{\bar{\alpha}}{1 + |\bar{\alpha}|^2/\tilde{n}_0}\left\{\tfrac{1}{2}\bar{\Gamma}_D(t) + \gamma\bar{D}_0\int_0^t \bar{\Gamma}_B(t')dt'\right\} \\
&\quad - \frac{1}{\sqrt{8\tilde{n}_0^3}}\frac{\bar{\alpha}^2}{1 + |\bar{\alpha}|^2/\tilde{n}_0}\Gamma_v(t).
\end{aligned} \tag{9.5.53}$$

i) Small Reduced Saturation Photon Number $\bar{n}_0$: To lowest order in $\bar{n}_0$, we can drop all but the first line, and in this case

$$\langle\Gamma(t)\Gamma(t')\rangle = \langle\Gamma^+(t)\Gamma^+(t')\rangle = 0 \tag{9.5.54}$$

$$\begin{aligned}
\langle\Gamma^+(t)\Gamma(t')\rangle &= \left\{2\kappa n + \frac{1}{\sqrt{2\tilde{n}_0}}\frac{(1 + \tfrac{1}{2}|\bar{\alpha}|^2/\tilde{n}_0)}{(1 + |\bar{\alpha}|^2/\tilde{n}_0)}\tilde{g}\left(1 + \frac{\bar{D}_0}{1 + |\bar{\alpha}|^2/\tilde{n}_0}\right)\right\} \\
&\quad \times \delta(t - t').
\end{aligned} \tag{9.5.55}$$

If the noise is being considered near threshold, we can set $|\alpha|^2 \approx 0$, and hence

$$\langle\Gamma^+(t)\Gamma(t')\rangle \approx \delta(t - t')\left\{2\kappa n + \frac{\tilde{g}^2}{\gamma}(1 + \bar{D}_0)\right\}. \tag{9.5.56}$$

In the limit implied by (9.5.56), the equation (9.5.51) is exactly the same as that previously derived in Sect. 9.3.2, apart from the fact that α and α^+ are not explicitly complex conjugate. However the equations of motion are such that all averages of analytic moments are the same as those of the usual laser equations, as is the case with the usual positive P-representation. There will be differences at higher orders, because of the invalidity of the simple adiabatic elimination technique, and also because of the simplifications made in Sect. 9.3.2e, when the adiabatic elimination noise terms were evaluated.

10. Squeezing

Although quantum noise cannot be eliminated, it is not immune to technical manipulation, a result which was first noted by *Caves* et al. [10.1] in a paper on the possibility of detecting gravitational radiation. The *standard quantum limit* is a term introduced to describe the limitations on the precision of specification of two canonically conjugate variables as given by the following naive argument. If $a, a^{\dagger}$ are destruction and creation operators for a harmonic oscillator, and we define the quadrature phase operators, X, Y, by

$$a = \frac{X + \mathrm{i}Y}{\sqrt{2\hbar}}, \qquad a^{\dagger} = \frac{X - \mathrm{i}Y}{\sqrt{2\hbar}},$$

then Heisenberg's uncertainty principle says that

$$\Delta X \, \Delta Y \geq \tfrac{1}{2}\hbar \, .$$

The standard quantum limit assumes that in the minimum uncertainty states of interest, $\Delta X = \Delta Y$, and hence $\Delta X^2 = \Delta Y^2 = \hbar/2$. Such is certainly the case in the lowest energy state of the harmonic oscillator, which corresponds to a zero temperature thermal state. Intuitively one would believe that the uncertainty, i.e., the fluctuations, at absolute zero would surely represent the irreducible minimum.

However, the uncertainty principle itself does not require this; either ΔX or ΔY can be as small as one likes, as long as the product satisfies the uncertainty principle. This chapter is devoted to developing that theme. It will establish the basic method of generating squeezing, with the degenerate parametric amplifier, and show how the input-output formalism, developed by *Collett* and *Gardiner* [10.2] must be used to study the production of squeezed beams of light.

A simple theory of squeezed white noise is then developed, and the corresponding master equations, etc., developed and applied. Finally, the concept of multibeam squeezing, in which different beams of light are quantum mechanically correlated, is developed.

10.1 Squeezed States of the Harmonic Oscillator

Squeezing is defined precisely only in the case of the operators of a harmonic oscillator. We have already met squeezed states in the study of Gaussian density operators, undertaken in Sect. 4.4.5. The *ideal* squeezed state of a harmonic oscillator is taken to be a quantum Gaussian state with minimum uncertainty product. This is most straightforwardly defined by means of the linear transformation which will transform a vacuum state into one with prescribed means and covariance matrix. Because of the Gaussian requirement, this is sufficient to specify the squeezed state.
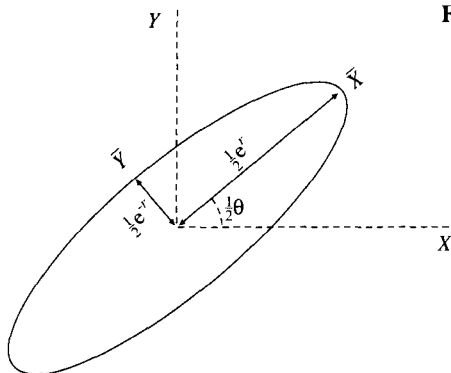
Fig. 10.1 Error ellipse for a squeezed state

10.1.1 Definition of an Ideal Squeezed State

An ideal squeezed state, $|\alpha, \xi\rangle$ is defined as

$$|\alpha, \xi\rangle = D(\alpha)S(\xi)|0\rangle, \tag{10.1.1}$$

where $D(\alpha)$ is the displacement operator

$$D(\alpha) = \exp(\alpha a^\dagger - \alpha^* a) \tag{10.1.2}$$

used in the definition of coherent states in Sect. 4.3.1b, $S(\xi)$ is the operator

$$S(\xi) = \exp\left(\tfrac{1}{2}\xi^* a^2 - \tfrac{1}{2}\xi a^{\dagger 2}\right) \tag{10.1.3}$$

and

$$\xi = r e^{i\theta}. \tag{10.1.4}$$

a) **Means and Variances of a Squeezed State:** For a squeezed state, $|\alpha, \xi\rangle$, the means are given by

$$\langle a \rangle = \alpha, \qquad \langle a^\dagger \rangle = \alpha^*. \tag{10.1.5}$$

The variances have to be defined in a rotated frame; if variables $\bar{X}$ and $\bar{Y}$ are defined by

$$\sqrt{2\hbar}\, a = X + iY = (\bar{X} + i\bar{Y})e^{i\theta/2} \tag{10.1.6}$$

where θ is as defined in (10.1.4), then the variables $\bar{X}, \bar{Y}$ are the independent squeezed variables, i.e.

$$\begin{aligned}
\langle \Delta \bar{X}^2 \rangle &= \tfrac{1}{4}e^{-2r} \\
\langle \Delta \bar{Y}^2 \rangle &= \tfrac{1}{4}e^{2r} \\
\langle \Delta \bar{X} \Delta \bar{Y} \rangle &= 0.
\end{aligned} \tag{10.1.7}$$

Graphically, this can be represented as in Fig. 10.1.

The principal axes of the error ellipse are rotated an angle $\theta/2$ with respect to those of X and Y.

Exercise. Show that the squeezed state *never* has a Glauber-Sudarshan P-function for any non-zero ξ. (It does have of course, a positive P-function.)

Exercise—Complex P-function for a Squeezed State. From the definition (10.1.1) show that the density operator for a squeezed vacuum state (i.e., $\alpha = 0$) with real ξ satisfies the equation

$$\frac{\partial \rho}{\partial \xi} = -2\mathrm{i}[\hat{H}_2, \rho] \tag{10.1.8}$$

with

$$H_2 = \frac{\mathrm{i}}{4}(a^{\dagger 2} - a^2). \tag{10.1.9}$$

Using the operator correspondences for the complex P-function, show that the complex P-function obeys the equation

$$2\frac{\partial P(\alpha, \beta; \xi)}{\partial \xi} = \left(2\beta\frac{\partial}{\partial \alpha} + 2\alpha\frac{\partial}{\partial \beta} - \frac{\partial^2}{\partial \alpha^2} - \frac{\partial^2}{\partial \beta^2}\right)P(\alpha, \beta; \xi). \tag{10.1.10}$$

This corresponds to an Ornstein-Uhlenbeck process in the variables $\mathrm{i}\alpha$ and $\mathrm{i}\beta$. Hence the solution of this equation is

$$P(\alpha, \beta; \xi) = \exp\left\{\alpha\beta + \frac{\coth \xi}{2}(\alpha^2 + \beta^2)\right\}. \tag{10.1.11}$$

The contours of integration for this complex P-function are the imaginary axes of both variables.

Using an appropriate transformation of variables, find the complex P-function for an arbitrary squeezed state.

10.2   The Degenerate Parametric Amplifier

From the results of Sects. 7.2.9 and 6.3.3 and, the equation (10.1.8) is seen to correspond exactly to that of the degenerate parametric amplifier (DPA) in which the coupling to the heat bath (or modes outside the cavity) is zero. Thus t and ξ are proportional, and the DPA inside a perfect cavity produces an increasingly squeezed state as time increases. In the case that the damping produced by the heat bath is not negligible, a stationary state occurs when losses into the bath balance the gain of the amplifier. Using (7.2.63) of the exercise of Sect. 7.2.9c, it can be seen that the internal field in this stationary state does not become perfectly squeezed. In fact the minimum uncertainty achievable is $\langle \Delta Y^2 \rangle = \hbar/8$, not zero. How, therefore, to achieve arbitrary squeezing is not a problem that will be solved this way. However, in Sect. 7.2.9c we showed that the *output* from a degenerate parametric amplifier is arbitrarily squeezed, even though the internal field is not. Since, in fact, squeezed light is used in beams, rather than in internal cavity modes, this is just the result that is needed.

10.2.1 Squeezing in the Degenerate Parametric Amplifier

Although this has already been treated in Sect. 7.2.9, from the point of view of its properties as an amplifier, a treatment will be given here which is more appropriate for the study of squeezing.
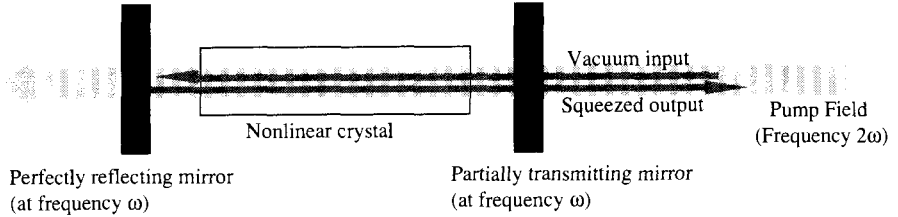
Fig. 10.2 Optical arrangement for a degenerate parametric amplifier

The systematic Hamiltonian for degenerate parametric amplification with a classical pump can be written as

$$H_{\text{sys}} = \hbar\,\omega_0 a^\dagger a + \tfrac{1}{2}\mathrm{i}\,\hbar\,\left[\epsilon e^{-\mathrm{i}\omega_p t}(a^\dagger)^2 - \epsilon^* e^{\mathrm{i}\omega_p t}a^2\right],\tag{10.2.1}$$

where ω_p is the frequency of the pump beam and ϵ a measure of the effective pump intensity. For now the pump and cavity will be considered to be tuned so that $\omega_p = 2\omega_0$. In practice the amplifying medium will be a crystal inside an optical cavity, which of necessity has two ends. The mirrors can be chosen to be almost non-reflecting at ω_p, and quite highly reflecting at ω_0. We can use the formalism of quantum Langevin equations as developed in Sect. 5.3.2 to write an equation of motion for the field mode a inside the cavity in terms of inputs $a_{\text{in}}(t), b_{\text{in}}(t)$, coming from either side of the cavity; the resulting equation of motion is

$$\frac{da}{dt} = -\mathrm{i}\,\omega_0 a + \epsilon e^{-\mathrm{i}\omega_p t}a^\dagger - \tfrac{1}{2}(\gamma_1 + \gamma_2)a - \sqrt{\gamma_1}\,a_{\text{in}}(t) - \sqrt{\gamma_2}\,b_{\text{in}}(t).\tag{10.2.2}$$

We now transform to a rotating frame with $a \to e^{\mathrm{i}\omega_p t/2}a$ and similarly for the input operators—for simplicity, we now use only these operators in the rotating frame, without any distinctive notation. In matrix notation, the equations become

$$\frac{d\mathbf{a}}{dt} = \left[\mathbf{A} - \tfrac{1}{2}(\gamma_1 + \gamma_2)\right]\mathbf{a} - \sqrt{\gamma_1}\,\mathbf{a}_{\text{in}} - \sqrt{\gamma_2}\,\mathbf{b}_{\text{in}}\tag{10.2.3}$$

where

$$\mathbf{A} = \begin{pmatrix} 0 & \epsilon \\ \epsilon^* & 0 \end{pmatrix} = \begin{pmatrix} 0 & |\epsilon|e^{\mathrm{i}\theta} \\ |\epsilon|e^{-\mathrm{i}\theta} & 0 \end{pmatrix},\tag{10.2.4}$$

and the notation

$$\mathbf{a}_{\text{in}} = \begin{pmatrix} a_{\text{in}} \\ a_{\text{in}}^\dagger \end{pmatrix}\tag{10.2.5}$$

is used for $\mathbf{a}, \mathbf{a}_{\text{in}}, \mathbf{b}_{\text{in}}$, etc.

a) Fourier Transform Solution: Define a Fourier transform variable by

$$\tilde{a}(\omega) = \frac{1}{\sqrt{2\pi}} \int dt e^{i\omega t} a(t), \tag{10.2.6}$$

with similar notation for other Fourier transforms. In frequency space (10.2.3) transforms to

$$-i\omega \tilde{\mathbf{a}}(\omega) = \left[\mathbf{A} - \tfrac{1}{2}(\gamma_1 + \gamma_2)\right] \tilde{\mathbf{a}}(\omega) - \sqrt{\gamma_1}\, \tilde{\mathbf{a}}_{\text{in}}(\omega) - \sqrt{\gamma_2}\, \tilde{\mathbf{b}}_{\text{in}}(\omega) \tag{10.2.7}$$

where now, to allow for the rotating frame,

$$\tilde{\mathbf{a}}(\omega) = \begin{pmatrix} \tilde{a}(\omega_s + \omega) \\ \tilde{a}^\dagger(\omega_s - \omega) \end{pmatrix}, \tag{10.2.8}$$

with $\omega_s = \omega_p/2$, and similarly for the input operators.

After performing the matrix inversion we find

$$\tilde{a}(\omega_s + \omega) = -\frac{(\tfrac{1}{2}\gamma_1 + \tfrac{1}{2}\gamma_2 - i\omega)\left[\sqrt{\gamma_1}\, \tilde{a}_{\text{in}}(\omega_s + \omega) + \sqrt{\gamma_2}\, \tilde{b}_{\text{in}}(\omega_s + \omega)\right]}{(\tfrac{1}{2}\gamma_1 + \tfrac{1}{2}\gamma_2 - i\omega)^2 - |\epsilon|^2}$$
$$-\frac{\epsilon\left[\sqrt{\gamma_1}\, \tilde{a}^\dagger_{\text{in}}(\omega_s - \omega) + \sqrt{\gamma_2}\, \tilde{b}^\dagger_{\text{in}}(\omega_s - \omega)\right]}{(\tfrac{1}{2}\gamma_1 + \tfrac{1}{2}\gamma_2 - i\omega)^2 - |\epsilon|^2}. \tag{10.2.9}$$

If both the input fields are vacuum or coherent, they will have zero normally ordered variance, that is,

$$\mathbf{C}_N(a_{\text{in}}, a^\dagger_{\text{in}}) = \mathbf{C}_N(b_{\text{in}}, b^\dagger_{\text{in}}) = \mathbf{0} \tag{10.2.10}$$

where we use the notation, for any operator x

$$\mathbf{C}_N(x, x^\dagger) = \begin{pmatrix} \langle x, x \rangle & \langle x^\dagger, x \rangle \\ \langle x^\dagger, x \rangle & \langle x^\dagger, x^\dagger \rangle \end{pmatrix} \tag{10.2.11}$$

and have used the notation

$$\langle a, b \rangle = \langle ab \rangle - \langle a \rangle \langle b \rangle. \tag{10.2.12}$$

b) Internal Mode Properties: We can compute the normally ordered second moments of the Fourier transform variables directly from the solution (10.2.9). Because the input fields are vacuum fields, the only contribution to the normally ordered variance of the internal field will be from the terms involving antinormally ordered products of the input field operators. The result is easily computed to be

$$\langle \tilde{a}(\omega_s + \omega), \tilde{a}(\omega_s + \omega') \rangle = \frac{(\tfrac{1}{2}\gamma_1 + \tfrac{1}{2}\gamma_2 - i\omega)e^{i\theta}}{2}$$
$$\times \left[\frac{1}{\left(\tfrac{1}{2}\gamma_1 + \tfrac{1}{2}\gamma_2 - |\epsilon|\right)^2 + \omega^2} - \frac{1}{\left(\tfrac{1}{2}\gamma_1 + \tfrac{1}{2}\gamma_2 + |\epsilon|\right)^2 + \omega^2}\right] \delta(\omega + \omega') \tag{10.2.13}$$

$$\langle \tilde{a}^{\dagger}(\omega_s + \omega), \tilde{a}(\omega_s + \omega') \rangle =$$

$$\frac{|\epsilon|}{2} \left[\frac{1}{\left(\frac{1}{2}\gamma_1 + \frac{1}{2}\gamma_2 - |\epsilon|\right)^2 + \omega^2} - \frac{1}{\left(\frac{1}{2}\gamma_1 + \frac{1}{2}\gamma_2 + |\epsilon|\right)^2 + \omega^2} \right] \delta(\omega - \omega'). \tag{10.2.14}$$

Inverting the Fourier transform, and noting the presence of delta functions in (10.2.13,14), the variances of the internal mode are

$$\langle a^{\dagger}(t), a(t) \rangle = \frac{1}{2\pi} \int d\omega d\omega' \langle \tilde{a}^{\dagger}(\omega_s + \omega), \tilde{a}(\omega_s + \omega') \rangle$$

$$= \frac{1}{2} \frac{|\epsilon|^2}{\left(\frac{1}{2}\gamma_1 + \frac{1}{2}\gamma_2\right)^2 - |\epsilon|^2}, \tag{10.2.15}$$

$$\langle a(t), a(t) \rangle = \frac{1}{2\pi} \int d\omega d\omega' \langle \tilde{a}(\omega_s + \omega), \tilde{a}(\omega_s + \omega') \rangle$$

$$= \frac{1}{2} \frac{\epsilon \left(\frac{1}{2}\gamma_1 + \frac{1}{2}\gamma_2\right)}{\left(\frac{1}{2}\gamma_1 + \frac{1}{2}\gamma_2\right)^2 - |\epsilon|^2}. \tag{10.2.16}$$

c) Internal Mode Quadrature Phases: The internal mode quadrature phase operators will be defined by

$$a = e^{i\theta/2}(X_1 + iX_2). \tag{10.2.17}$$

It is convenient to use the *normally ordered variances* to characterise the squeezing, since, in the case of vacuum input, these tend to give simpler formulae. The normally ordered variances of these operators are

$$\langle : X_1, X_1 : \rangle = \frac{\frac{1}{4}|\epsilon|}{\frac{1}{2}\gamma_1 + \frac{1}{2}\gamma_2 - |\epsilon|}$$

$$\langle : X_2, X_2 : \rangle = -\frac{\frac{1}{4}|\epsilon|}{\frac{1}{2}\gamma_1 + \frac{1}{2}\gamma_2 + |\epsilon|}$$

$$\langle : X_1, X_2 : \rangle = 0. \tag{10.2.18}$$

Perfect squeezing in one quadrature, corresponding to an eigenstate of the quadrature phase operator, is achieved with a normally ordered variance of $-1/4$. The best that can be achieved in the case of the parametric amplifier in a cavity is when on oscillation threshold, $|\epsilon| = (\gamma_1 + \gamma_2)/2$, giving

$$\langle : X_2, X_2 : \rangle = -1/8. \tag{10.2.19}$$

Squeezing by a factor of one-half can thus be obtained in the X_2 quadrature of the generating cavity with the X_1 quadrature infinitely unsqueezed. Note that the properties of the internal mode considered in this section depend only on the total damping, not on the damping through each mirror separately.

d) The Output Field: The output fields can be calculated as the boundary condition (5.3.24) (in Fourier transform form)

$$\tilde{a}_{\text{out}}(\omega) = \tilde{a}_{\text{in}}(\omega) + \sqrt{\gamma_1}\, \tilde{a}(\omega). \tag{10.2.20}$$

It is a matter of straightforward substitution to arrive at the result

$$
\begin{aligned}
\tilde{a}_{\text{out}}\,&(\omega_s + \omega) \\
&= -\frac{\left[(\tfrac{1}{2}\gamma_1)^2 - (\tfrac{1}{2}\gamma_2 - i\omega)^2 + |\epsilon|^2\right]\tilde{a}_{\text{in}}(\omega_s + \omega) + \epsilon\gamma_1 \tilde{a}_{\text{in}}^{\dagger}(\omega_s - \omega)}{(\tfrac{1}{2}\gamma_1 + \tfrac{1}{2}\gamma_2 - i\omega)^2 - |\epsilon|^2} \\
&\quad - \frac{\sqrt{\gamma_1\gamma_2}\,(\tfrac{1}{2}\gamma_1 + \tfrac{1}{2}\gamma_2 - i\omega)\tilde{b}_{\text{in}}(\omega_s + \omega) + \epsilon\sqrt{\gamma_1\gamma_2}\,b_{\text{in}}^{\dagger}(\omega_s - \omega)}{(\tfrac{1}{2}\gamma_1 + \tfrac{1}{2}\gamma_2 - i\omega)^2 - |\epsilon|^2}.
\end{aligned}
$$

$$(10.2.21)$$

As with the internal field, only antinormally ordered terms contribute to the variances so that

$$
\begin{aligned}
\langle \tilde{a}_{\text{out}}^{\dagger}&(\omega_s + \omega), \tilde{a}_{\text{out}}(\omega_s + \omega')\rangle \\
&= \frac{|\epsilon|\gamma_1}{2}\left[\frac{1}{\left(\tfrac{1}{2}\gamma_1 + \tfrac{1}{2}\gamma_2 - |\epsilon|\right)^2 + \omega^2} - \frac{1}{\left(\tfrac{1}{2}\gamma_1 + \tfrac{1}{2}\gamma_2 + |\epsilon|\right)^2 + \omega^2}\right]\delta(\omega - \omega'),
\end{aligned}
$$

$$(10.2.22)$$

$$
\begin{aligned}
\langle \tilde{a}_{\text{out}}&(\omega_s + \omega), \tilde{a}_{\text{out}}(\omega_s + \omega')\rangle \\
&= \frac{|\epsilon|\gamma_1}{2}\left[\frac{1}{\left(\tfrac{1}{2}\gamma_1 + \tfrac{1}{2}\gamma_2 - |\epsilon|\right)^2 + \omega^2} + \frac{1}{\left(\tfrac{1}{2}\gamma_1 + \tfrac{1}{2}\gamma_2 + |\epsilon|\right)^2 + \omega^2}\right]\delta(\omega + \omega').
\end{aligned}
$$

$$(10.2.23)$$

Once again it is the variances in the quadrature phases which are of most interest. Defining the output quadrature phases in the same fashion as the internal ones, from (10.2.22)

$$
\begin{aligned}
\langle: \tilde{X}_{1,\text{out}}&(\omega_s + \omega), \tilde{X}_{1,\text{out}}(\omega_s + \omega') :\rangle \\
&= \frac{|\epsilon|(\gamma_1/2)}{\left(\tfrac{1}{2}\gamma_1 + \tfrac{1}{2}\gamma_2 - |\epsilon|\right)^2 + \omega^2}\delta(\omega + \omega')
\end{aligned}
$$

$$(10.2.24)$$

$$
\begin{aligned}
\langle: \tilde{X}_{2,\text{out}}&(\omega_s + \omega), \tilde{X}_{2,\text{out}}(\omega_s + \omega') :\rangle \\
&= -\frac{|\epsilon|(\gamma_1/2)}{\left(\tfrac{1}{2}\gamma_1 + \tfrac{1}{2}\gamma_2 + |\epsilon|\right)^2 + \omega^2}\delta(\omega + \omega').
\end{aligned}
$$

$$(10.2.25)$$

e) **Maximum Squeezing:**   The maximum squeezing is still attained at threshold, $|\epsilon| = \tfrac{1}{2}(\gamma_1 + \gamma_2)$, giving

$$
\begin{aligned}
\langle: \tilde{X}_{2,\text{out}}&(\omega_s + \omega), \tilde{X}_{2,\text{out}}(\omega_s + \omega') :\rangle_{\text{max}} \\
&= -\frac{\gamma_1}{4}\frac{\gamma_1 + \gamma_2}{(\gamma_1 + \gamma_2)^2 + \omega^2}\delta(\omega + \omega').
\end{aligned}
$$

$$(10.2.26)$$

The most interesting thing about these results is that while the squeezing in the total field is independent of γ_2, this is not the case for the individual frequency

components. The δ function in (10.2.26) can be removed by integrating over ω' to give the normally ordered spectrum of the operator $X_{2,\text{out}}$:

$$: S_{2,\text{out}}(\omega_s + \omega) :_{\max} = -\frac{\gamma_1}{4}\frac{\gamma_1 + \gamma_2}{(\gamma_1 + \gamma_2)^2 + \omega^2}, \tag{10.2.27}$$

which is a convenient way of describing the squeezing in the output field. It may be thought of loosely as the squeezing at a particular frequency, although in this case it results from the coupling of pairs of frequencies on either side of resonance: This spectrum is, ignoring the sign, a Lorentzian with peak height $\frac{1}{4}\left[\gamma_1/(\gamma_1 + \gamma_2)\right]$ and width $\gamma_1 + \gamma_2$. Thus for a symmetric double-ended cavity with $\gamma = \gamma_1 = \gamma_2$

$$: S_{2,\text{out}}(0) :_{\max} = -1/8. \tag{10.2.28}$$

The resonant mode of the output field is squeezed by a factor of one-half, the same as the internal field. If, however, the cavity is single-ended, with $\gamma_2 = 0$,

$$: S_{2,\text{out}}(0) :_{\max} = -1/4 \tag{10.2.29}$$

and this is the maximum squeezing available.

We get degradation of squeezing with a two-sided cavity because there are two inputs. With a one-sided cavity, the radiated field from the cavity (the term $\sqrt{\gamma}\,\tilde{a}(\omega)$ in (10.2.20)) and the reflected field $\tilde{a}_{\text{in}}(\omega)$ interfere in one quadrature to cancel out all fluctuations at $\omega = 0$. With a two-sided cavity, there is an additional component from $b_{\text{in}}(\omega)$ which is transmitted, and, being independent of $a_{\text{in}}(\omega)$, cannot interfere with the reflection.

f) Correlation Functions of the Output Field: By Fourier transforming (10.2.22) we can easily deduce the correlation functions. Using the definition (10.2.6), we find that the correlation functions can be compactly represented as

$$\langle a_{\text{out}}^{\dagger}(t)\,a_{\text{out}}(t')\rangle = \frac{\gamma_1}{\gamma_1 + \gamma_2}\left\{\frac{\lambda^2 - \mu^2}{4}\left[\frac{e^{-\mu(t-t')}}{2\mu} - \frac{e^{-\lambda(t-t')}}{2\lambda}\right]\right\} \tag{10.2.30}$$

$$\langle a_{\text{out}}(t)\,a_{\text{out}}(t')\rangle = \frac{\gamma_1}{\gamma_1 + \gamma_2}\left\{\frac{\lambda^2 - \mu^2}{4}\left[\frac{e^{-\mu(t-t')}}{2\mu} + \frac{e^{-\lambda(t-t')}}{2\lambda}\right]\right\} \tag{10.2.31}$$

where

$$\begin{aligned}\lambda &= \tfrac{1}{2}(\gamma_1 + \gamma_2) + |\epsilon| \\ \mu &= \tfrac{1}{2}(\gamma_1 + \gamma_2) - |\epsilon|.\end{aligned} \tag{10.2.32}$$

g) Squeezed White Noise Limit: In the case that the cavity is single-ended, $\gamma_2 \to 0$, and γ_1 and $|\epsilon|$ are both very large, so that $\mu, \lambda \to \infty$, we can set

$$e^{-\mu|t-t|} \to \frac{2}{\mu}\delta(t - t') \tag{10.2.33}$$

and so on, to get

$$\langle a_{\text{out}}^{\dagger}(t)a_{\text{out}}(t')\rangle = N\delta(t - t') \tag{10.2.34}$$

$$\langle a_{\text{out}}(t) a_{\text{out}}(t') \rangle = M\delta(t - t') \tag{10.2.35}$$

with

$$N = \frac{(\lambda^2 - \mu^2)^2}{4\mu^2\lambda^2}, \qquad M = \frac{\lambda^4 - \mu^4}{4\mu^2\lambda^2}. \tag{10.2.36}$$

Notice that in fact here

$$|M|^2 = N(N + 1). \tag{10.2.37}$$

Exercise. Use the squeezed beam $a_{\text{out}}(t)$ as produced by the DPA, as the input to a single-ended passive cavity, and show that the internal mode is an ideal squeezed state. Thus ideal squeezing can be produced inside a cavity, but not by an active medium inside that cavity.

10.2.2   Squeezed White Noise

We can define ideal squeezed white noise on the basis of the previous section; abstractly, we can say that $b(t)$ represents squeezed white noise if

$$\left. \begin{aligned} dB(t)^2 &= M\,dt, \quad dB^\dagger(t)^2 = M^*dt, \\ dB(t)dB^\dagger(t) &= (N+1)\,dt, \quad dB^\dagger(t)dB(t) = N\,dt. \end{aligned} \right\} \tag{10.2.38}$$

Quantum stochastic calculus can be derived in the same way as for ordinary white noise, and the resulting rules are

a) Ito QSDE: This takes the form

$$\begin{aligned} da = &-\frac{i}{\hbar}[a, H_{\text{sys}}] + \frac{\gamma}{2}(N+1)(2c^\dagger ac - ac^\dagger c - c^\dagger ca)\,dt \\ &+ \frac{\gamma}{2}N(2cac^\dagger - acc^\dagger - cc^\dagger a)\,dt - \frac{\gamma}{2}M(2c^\dagger ac^\dagger - ac^\dagger c^\dagger - c^\dagger c^\dagger a)\,dt \\ &- \frac{\gamma}{2}M^*(2cac - acc - cca)\,dt - \sqrt{\gamma}[a, c^\dagger]dB(t) + \sqrt{\gamma}[a, c]dB^\dagger(t). \end{aligned} \tag{10.2.39}$$

b) Relation between Ito and Stratonovich Integrals: This now takes the form

$$(\mathbf{S}) \int\limits_{t_0}^{t} dB^{\dagger}(t')g(t') = (\mathbf{I}) \int\limits_{t_0}^{t} g(t')dB^{\dagger}(t')$$

$$-\frac{1}{2}\sqrt{\gamma}\,N \int\limits_{t_0}^{t} [g(t'),c^{\dagger}(t')]dt' + \frac{1}{2}\sqrt{\gamma}\,M^{*} \int\limits_{t_0}^{t} [g(t'),c(t')]dt'$$

$$(10.2.40)$$

$$(\mathbf{S}) \int\limits_{t_0}^{t} g(t')dB(t') = (\mathbf{I}) \int\limits_{t_0}^{t} g(t')dB(t')$$

$$+\frac{1}{2}\sqrt{\gamma}\,N \int\limits_{t_0}^{t} [g(t'),c(t')]dt' - \frac{1}{2}\sqrt{\gamma}\,M \int\limits_{t_0}^{t} [g(t'),c^{\dagger}(t')]dt.$$

$$(10.2.41)$$

(The Stratonovich integrals with $g(t')$ and increment permuted differ from these by the replacement $\bar{N} \rightarrow \bar{N}+1$, corresponding to the commutation relation (5.3.55) still being true.)   The Stratonovich QSDE corresponding to (10.2.39) is exactly (5.3.64) as is expected. As previously noted, the Stratonovich QSDE is independent of the statistics of the incoming field, and in this sense is more general than the Ito QSDE.

c) Master Equation: The master equation can be derived as in Sect. 5.4.2; it is

$$\frac{\partial\rho}{\partial t} = \frac{i}{\hbar}[\rho,H_{\text{sys}}] + \frac{\gamma}{2}(N+1)(2\,c\rho c^{\dagger} - c^{\dagger}c\rho - \rho c^{\dagger}c)$$
$$+ \frac{\gamma}{2}N(2\,c^{\dagger}\rho c - cc^{\dagger}\rho - \rho cc^{\dagger}) - \frac{\gamma}{2}M(2\,c^{\dagger}\rho c^{\dagger} - c^{\dagger}c^{\dagger}\rho - \rho c^{\dagger}c^{\dagger})$$
$$- \frac{\gamma}{2}M^{*}(2\,c\rho c - cc\rho - \rho cc).$$

$$(10.2.42)$$

This equation can be expected to give the correct description of a system driven by a squeezed noise (which arises from a squeezed vacuum) provided the squeezing is reasonably constant over a bandwidth significantly larger than that expected from the system.

d) Squeezed White Noise in a Non-Rotating Frame: When we have ordinary white noise, there is no difference between $dB(t)$, and $dB(t)e^{-i\Omega t} \equiv d\bar{B}(t)$; both $dB(t)$ and $d\bar{B}(t)$ have the same stochastic and commutator properties. However, with squeezing,

$$d\bar{B}(t)^2 = M e^{-2i\Omega t}dt, \quad d\bar{B}^{\dagger}(t)^2 = M^{*}e^{2i\Omega t}dt. \qquad (10.2.43)$$

Looking at the equation (10.2.39), one can see that if H_{sys} is the full system Hamiltonian, the operator c will have, to lowest order, a time dependence of the form $e^{i\Omega t}$

for some Ω, and this would produce rapidly changing coefficients of M, M^*, which would be negligible when time averaged. If however $d\bar{B}(t)$ is understood, then $M \to M e^{-2i\Omega t}$, and cancels this time dependence out. What this means is that in practice, squeezed white noise does have a characteristic frequency Ω, and the form (10.2.38), which may be regarded as the fundamental form, has $\Omega = 0$. For the master equation and QSDE to be true, the squeezing frequency must be matched to the transition frequency of interest.

For an experimental laser physicist this is not a major point—in practice one has real squeezed noise, which has an obvious central frequency, and is squeezed only over a rather narrow bandwidth, which nevertheless may be sufficiently wide to enable the white noise limit to be used sometimes.

The equations (10.2.40–42) are best interpreted as being valid in an interaction picture rotating at frequency Ω, which makes the squeezed white noise representation in the form (10.2.38) valid—H_{sys} is then only that part of the Hamiltonian not removed by this process.

e) Quadrature Phases of Squeezed White Noise:

Writing in terms of input fields, and not in the interaction picture, the correlation functions of the squeezed white noise are

$$
\begin{aligned}
\langle b(t)b(t')\rangle &= M\,e^{-i\Omega(t+t')}\delta(t-t') \\
\langle b^\dagger(t)b^\dagger(t')\rangle &= M^*\,e^{\Omega(t-t')}\delta(t-t') \\
\langle b^\dagger(t)b(t')\rangle &= N\,\delta(t-t') \\
\langle b(t)b^\dagger(t')\rangle &= (N+1)\delta(t-t').
\end{aligned}
\tag{10.2.44}
$$

We can define quadrature phases of squeezed white noise by setting $M = |M|e^{-2i\theta}$, and

$$
\begin{aligned}
X(t) &= \tfrac{1}{2}\left[b^\dagger(t)e^{i(\theta-\Omega t)} + b(t)e^{-i(\theta-\Omega t)}\right] \\
Y(t) &= \tfrac{1}{2}i\left[b^\dagger(t)e^{i(\theta-\Omega t)} - b(t)e^{-i(\theta-\Omega t)}\right]
\end{aligned}
\tag{10.2.45}
$$

and we then find

$$
\begin{aligned}
\langle X(t)X(t')\rangle &= \tfrac{1}{2}(N+|M|+\tfrac{1}{2})\delta(t-t') \\
&\sim (N+\tfrac{1}{2})\delta(t-t') \qquad \text{if } N\to\infty \text{ and } |M| = \sqrt{N(N+1)}
\end{aligned}
\tag{10.2.46}
$$

$$
\begin{aligned}
\langle Y(t)Y(t')\rangle &= \tfrac{1}{2}(N+\tfrac{1}{2}-|M|)\delta(t-t') \\
&\sim \delta(t-t')/16N \qquad \text{if } N\to\infty \text{ and } |M| = \sqrt{N(N+1)}.
\end{aligned}
\tag{10.2.47}
$$

Thus, in this notation, the Y quadrature is squeezed, and the X quadrature unsqueezed.

10.3   Squeezed Light on a Single Atom

As a simple application, consider a two-level atom interacting with squeezed light, whose frequency is tuned to that of the atomic transition. We use the QSDE (10.2.39) with

$$c \to \sigma^-, \qquad c^\dagger \to \sigma^+ \tag{10.3.1}$$

and choose M, M^* real, since this is merely a phase choice. In the interaction picture $H_{\text{sys}} \to 0$, and we get the Ito QSDEs for the variables σ_x, σ_y, σ_z

$$
\begin{aligned}
\dot\sigma_x &= -\gamma(N + M + \tfrac{1}{2})\sigma_x + \tfrac{1}{2}\sqrt{\gamma}\,\sigma_z\left\{b_{\text{in}}^\dagger(t) + b_{\text{in}}(t)\right\} \\
\dot\sigma_y &= -\gamma(N - M + \tfrac{1}{2})\sigma_y - \tfrac{1}{2}\mathrm{i}\,\sqrt{\gamma}\,\sigma_z\left\{b_{\text{in}}^\dagger(t) - b_{\text{in}}(t)\right\} \\
\dot\sigma_z &= -\gamma(2N + 1)\sigma_z - \gamma \\
&\quad - \sqrt{\gamma}\,\sigma_x\left\{b_{\text{in}}^\dagger(t) + b_{\text{in}}(t)\right\} + \mathrm{i}\,\sqrt{\gamma}\,\sigma_y\left\{b_{\text{in}}^\dagger(t) - b_{\text{in}}(t)\right\}.
\end{aligned}
\tag{10.3.2}
$$

a) Mean Value Solutions: The equations of motion for the mean values take the form

$$
\begin{aligned}
\langle \dot\sigma_x \rangle &= -\gamma(N + M + \tfrac{1}{2})\langle \sigma_x \rangle \equiv -\gamma_x \langle \sigma_x \rangle \\
\langle \dot\sigma_y \rangle &= -\gamma(N - M + \tfrac{1}{2})\langle \sigma_y \rangle \equiv -\gamma_y \langle \sigma_y \rangle \\
\langle \dot\sigma_z \rangle &= -\gamma(2N + 1)\langle \sigma_z \rangle - \gamma = -\gamma_z \langle \sigma_z \rangle - \gamma.
\end{aligned}
\tag{10.3.3}
$$

If we have pure squeezed state, in which $M = \sqrt{N(N+1)}$, for large N,

$$
\left.
\begin{aligned}
N + M + \tfrac{1}{2} &\quad\to\quad 2N + 1 \\
N - M + \tfrac{1}{2} &\quad\to\quad 1/8N
\end{aligned}
\right\} \text{ as } N \to \infty.
\tag{10.3.4}
$$

We can see therefore that for large squeezing both γ_z and γ_x become large. In fact γ_x and γ_y are proportional to the coefficients of intensity of the respective quadrature phases of the input, as defined in (10.2.45,46).

Since this noise can approach zero in the Y quadrature phase, it is clear that the decay constant γ_y can be as small as we please, while at the same time γ_x becomes extremely large. Thus on a time scale short compared with γ_y^{-1}, but substantially larger than γ_x^{-1} and γ_z^{-1}, we find

$$\langle \sigma_x \rangle \to 0, \qquad \langle \sigma_z \rangle \to -1/(2N + 1), \tag{10.3.5}$$

while $\langle \sigma_y \rangle$ is essentially unchanged. Thus the projection of the original orientation of the Bloch vector on the direction of the low-noise quadrature phase is preserved—nevertheless the inversion $\langle \sigma_z \rangle$ decays rapidly to its stationary value.

b) Correlation Functions of the Output: To detect this effect one can look at the spectrum of the fluorescent light. According to the quantum regression theorem

and standard methods connected with it (see Sect. 5.2.3), the stationary correlation functions of the atom are

$$\langle \sigma^+(t)\sigma^-(0)\rangle = \tfrac{1}{2}[N/(2N+1)][\exp(-\gamma_x t) + \exp(-\gamma_y t)]$$
$$\langle \sigma^-(t)\sigma^+(0)\rangle = \tfrac{1}{2}[(N+1)/(2N+1)][\exp(-\gamma_x t) + \exp(-\gamma_y t)]$$
$$\langle \sigma^+(t)\sigma^+(0)\rangle = \tfrac{1}{2}[(N+1)/(2N+1)][\exp(-\gamma_x t) - \exp(-\gamma_y t)]$$
$$\langle \sigma^-(t)\sigma^-(0)\rangle = \tfrac{1}{2}[N/(2N+1)][\exp(-\gamma_x t) - \exp(-\gamma_y t)].$$

$$(10.3.6)$$

The above has been formulated in the case that there is one input and one output channel. In the case of interaction of light with an atom, this is realistic if we view the channels as corresponding to the various partial waves, since only the electric dipole partial wave (corresponding to the electric dipole approximation) interacts appreciably with the atom. If the experiment can be carried out with a squeezed electric dipole wave, then the output would be of the same kind.

In this case the amplitude correlation function of the fluorescent light (whose Fourier transform is the spectrum) is computed from the boundary condition

$$b_{\mathrm{out}}(t) = \sqrt{\gamma}\,\sigma^-(t) + b_{\mathrm{in}}(t). \tag{10.3.7}$$

We find

$$\langle b_{\mathrm{out}}^\dagger(t)b_{\mathrm{out}}(0)\rangle = N\delta(t) + \gamma u(t)\langle [\sigma^+(t), N\sigma^-(0) - M\sigma^+(0)]\rangle$$
$$+ \gamma u(-t)\langle [N\sigma^+(t) - M\sigma^-(t), \sigma^-(0)]\rangle + \gamma\langle \sigma^+(t)\sigma^-(0)\rangle$$

$$(10.3.8)$$

(where $u(t) = 0$ for $t < 0$; $u(t) = 1$ for $t > 0$). This correlation function contains elements arising from the input correlation function $N\delta(t)$ and elements from the atom itself—arising because the fluorescent light consists of a radiated part and a reflected part. The correlation function can be evaluated as

$$\langle b_{\mathrm{out}}^\dagger(t)b_{\mathrm{out}}(0)\rangle = N\delta(t) + \tfrac{1}{2}[\gamma M/(2N+1)][\exp(-\gamma_y t) - \exp(-\gamma_x t)] \tag{10.3.9}$$

which displays the characteristic decay constants.

$S(\omega)$
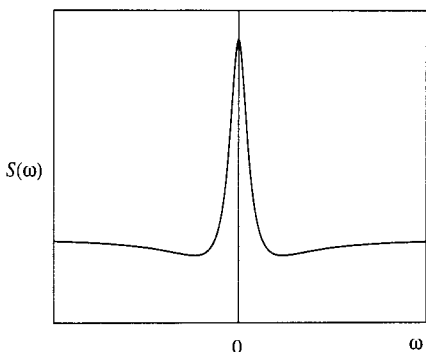
Fig. 10.3   Spectrum corresponding to the correlation function (10.3.9)

The spectrum obtained by the Fourier transform of (10.3.9) consists of a flat background, from $\delta(t)$, plus a negative peak of width γ_x (which will be very broad), and finally a positive peak of width γ_y, which can be as narrow as we please. Such narrowing is an effect which can only be produced by some reduction of noise, i.e., squeezing. The narrowed width is in fact directly proportional to the squeezing, and thus provides a measurement of squeezing.

10.4   Simulation Methods Based on the Adjoint Equation

The squeezed white noise results are intriguing, and elegant, but one must ask to what extent these results will be preserved when real squeezed light is used, since it will almost certainly not be exactly white. In fact, the correlation functions (10.2.30,31) show two time constants λ^{-1}, and μ^{-1}. According to (10.2.24), the maximum squeezing is attained at threshold, when $\mu \to 0$, and this means that one of the time constants becomes *infinite*, which of course means that the squeezed white noise approximation cannot be used. Consequently, neither the master equation nor QSDEs can be used. However, the adjoint equation introduced in Sect. 3.5 can be used here, and has been fruitfully applied to computations of these effects by *Parkins* and *Gardiner*, using simulation methods [10.3] and also by *Ritsch* and *Zoller* [10.4], using eigenfunction methods.

10.4.1   Adjoint Equation for Squeezed Light

We can take the quantum Langevin equations (4.3.51–53), with the resulting equations

$$
\begin{aligned}
\dot{\sigma}_x &= -\Omega\sigma_y \\
\dot{\sigma}_y &= \Omega\sigma_x + (\hbar\Omega)^{-\frac{1}{2}}\left[\xi(t),\sigma_z\right]_+ \\
\dot{\sigma}_z &= -\gamma - (\hbar\Omega)^{-\frac{1}{2}}\left[\xi(t),\sigma_y\right]_+
\end{aligned}
\tag{10.4.1}
$$

where $\xi(t) = \sqrt{2\gamma c}\,\dot{A}_{\text{in}}(0,t)$ is the incoming electric field evaluated at the atom. This light beam is assumed to have all excitation in a rather narrow bandwidth around Ω, and we can represent this by defining the quadrature phase operators $X_{\text{in}}(t), Y_{\text{in}}(t)$, by

$$
\xi(t) = -\sqrt{2\gamma c}\left[Y_{\text{in}}(t)\cos(\Omega t) + X_{\text{in}}(t)\sin(\Omega t)\right].
\tag{10.4.2}
$$

One now moves to an interaction picture, defined by

$$
\begin{aligned}
S_x(t) \pm \mathrm{i}\,S_y(t) &= \mathrm{e}^{\mp\mathrm{i}\Omega t}(\sigma_x(t) \pm \mathrm{i}\sigma_y(t)) \\
S_z(t) &= \sigma_z(t),
\end{aligned}
\tag{10.4.3}
$$

and dropping rapidly rotating terms like $e^{\pm i\Omega t}$, one obtains

$$\dot{S}_x = -\tfrac{1}{2}c_0 \left[X_{\mathrm{in}}(t), S_z\right]_+$$
$$\dot{S}_y = -\tfrac{1}{2}c_0 \left[Y_{\mathrm{in}}(t), S_z\right]_+ \qquad (10.4.4)$$
$$\dot{S}_z = -\gamma + \tfrac{1}{2}c_0 \left\{ \left[X_{\mathrm{in}}(t), S_x\right]_+ + \left[Y_{\mathrm{in}}(t), S_y\right]_+ \right\}$$

where $c_0 = \sqrt{(2\gamma c/\hbar\,\Omega)}$

a) Adjoint Equation: Using the same techniques as in Sect. 3.5 the adjoint equation can be derived in this interaction picture. If $\mu(t)$ is the adjoint equation density operator, then it is defined by

$$\mathrm{Tr}_s \left\{ \rho S_i(t) \right\} = \mathrm{Tr}_s \left\{ \mu(t)\sigma_i \right\} \rho_B \qquad (10.4.5)$$

where σ_i are the Pauli matrices. For ease of working, it is simplest to work in terms of $\bar{\mathbf{S}} \equiv \mathrm{Tr}_s \left\{ \mathbf{S}\mu(t) \right\}$, for which the equations of motion are

$$\dot{\bar{S}}_x = -\alpha_X(t)\bar{S}_z$$
$$\dot{\bar{S}}_y = -\alpha_Y(t)\bar{S}_z \qquad (10.4.6)$$
$$\dot{\bar{S}}_z = -\gamma + \left[\alpha_X(t)\bar{S}_x + \alpha_Y(t)\bar{S}_y\right].$$

where

$$\alpha_X(t)\mu \equiv \tfrac{1}{2}c_0 \left[X_{\mathrm{in}}(t), \mu\right]_+ \qquad \alpha_Y(t)\mu \equiv \tfrac{1}{2}c_0 \left[Y_{\mathrm{in}}(t), \mu\right]_+ . \qquad (10.4.7)$$

With these definitions, $\alpha_X(t)$ and $\alpha_Y(t)$ can be shown to commute with each other and with everything else. This implies that the equations can be treated as classical c-number equations, and so we need only specify the statistics of the operators α_X and α_Y, which behave simply like classical random quantities.

b) Correlation Functions of Noises: The correlation functions (10.2.30,31), are appropriate to the degenerate parametric amplifier, the most successful source of squeezed light so far found. Using these to characterise our input, we find the α correlation functions are

$$\langle\alpha_X(t)\alpha_X(t')\rangle = (\lambda^2 - \mu^2)\frac{e^{-\mu|t-t'|}}{2\mu} + \delta(t - t')$$

$$\langle\alpha_Y(t)\alpha_Y(t')\rangle = -(\lambda^2 - \mu^2)\frac{e^{-\lambda|t-t'|}}{2\lambda} + \delta(t - t'). \qquad (10.4.8)$$

The first term in each expression gives the effect of squeezing, while the second δ-correlated term represents vacuum fluctuations. That this is, in fact, a valid description of the output has been confirmed by the experiments of *Kimble* and co-workers [10.5]. The parameters λ and μ are related to experiment by

$$\lambda = \tfrac{1}{2}\gamma_c + \epsilon, \qquad \mu = \tfrac{1}{2}\gamma_c - \epsilon, \qquad (10.4.9)$$

where γ_c is the cavity damping of the parametric amplifier and ϵ is the amplifier driving strength. The practical limitations of the broadband theory are highlighted by the fact that typically $\gamma_c \approx 10^8\,\mathrm{s}^{-1}$ and $\epsilon \approx 0.3\gamma_c$ (at best), which are of the order of the atomic decay rate γ^{-1}. The problem becomes more extreme if we consider the limiting case $\epsilon \to \frac{1}{2}\gamma_c$. In this limit one achieves arbitrarily large squeezing in the Y quadrature, and its correlation time becomes extremely large, and varies on a very slow time scale; in fact, its correlation time μ^{-1} approaches infinity. This kind of behaviour will always be present in light from a system which depends on squeezing produced very close to an instability.

10.4.2   Solution of the Equations of Motion

The equations (10.4.6) can be compactly written as

$$\dot{u} = \sqrt{\gamma}\,\big[A\alpha_x(t) + B\alpha_y(t)\big]\,u + c, \qquad (10.4.10)$$

where

$$u = \begin{pmatrix} \bar{S}_x \\ \bar{S}_y \\ \bar{S}_z \end{pmatrix}, \qquad A = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix},$$

$$B = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}, \qquad c = \begin{pmatrix} 0 \\ 0 \\ -\gamma \end{pmatrix}. \qquad (10.4.11)$$

A number of approximation techniques can be applied to these equations. These techniques are all standard techniques for classical stochastic equations, so we will not develop them too fully. Briefly one may do the following:

i)   Treat $\alpha_x(t)$ and $\alpha_y(t)$ as white noise—this leads to the results of Sect. 10.3.

ii) Omit the squeezed term altogether, but keep the other terms. The equations can then be solved exactly. This gives the essential features of more precise methods.

iii) Make a white noise approximation to the squeezed quadrature, whose correlation time remains finite as the squeezing increases.

iv) If squeezing is large, one may assume that the unsqueezed quadrature (whose correlation time can become very large) varies so slowly that it may be treated as almost constant. This enables a kind of rotating wave approximation to be used.

For details of these techniques, the reader is referred to the work of *Parkins* and *Gardiner* [10.6].

10.4.3   Numerical Methods

The incident noise contains the wide bandwidth vacuum noise, and a finite band-width squeezed part. We can write

$$\alpha_X(t) = \alpha_X^w(t) + \alpha_X^c(t), \tag{10.4.12}$$

where

$$\langle \alpha_X^w(t) \alpha_X^w(t') \rangle = \delta(t - t') \tag{10.4.13}$$

$$\langle \alpha_X^c(t) \alpha_X^c(t') \rangle = (\lambda^2 - \mu^2) \frac{e^{-\mu|t-t'|}}{2\mu}, \tag{10.4.14}$$

and similarly for $\alpha_Y(t)$. The equations of motion become

$$
\begin{aligned}
\dot{\bar{S}}_x &= -\gamma \bar{S}_x - \sqrt{\gamma}\, \alpha_X^c(t) \bar{S}_z, \\
\dot{\bar{S}}_y &= \gamma \bar{S}_y - \sqrt{\gamma}\, \alpha_Y^c(t) \bar{S}_z \\
\dot{\bar{S}}_z &= \gamma - \gamma \bar{S}_z + \sqrt{\gamma}\, \alpha_X^c \bar{S}_x + \sqrt{\gamma}\, \alpha_Y^c(t) \bar{S}_y.
\end{aligned}
\tag{10.4.15}
$$

These equations can be solved numerically by two methods.

a) Expansion in Eigenfunctions: We know that $\alpha_x^c(t)$ and $\alpha_y^c(t)$ are Gaussian, with correlation functions like (10.4.14). Of course if $\lambda \neq \mu$, one of them must have a negative variance, but this can be simulated by choosing an imaginary quantity. Assuming $\mu > \lambda$, we can write

$$\alpha_x^c(t) = i \bar{\alpha}_x(t) \tag{10.4.16}$$

where now $\bar{\alpha}_x(t)$ and $\alpha_y(t)$ are independent Ornstein-Uhlenbeck processes; i.e., we can supplement (10.4.15) with

$$
\begin{aligned}
d\alpha_x(t) &= -\mu \alpha_x(t) dt + i g_x dW_x(t) \\
d\alpha_y(t) &= -\lambda \alpha_y(t) dt + g_y dW_y(t)
\end{aligned}
\tag{10.4.17}
$$

where $dW_x(t), dW_y(t)$ are independent white noise increments.

A procedure used by *Ritsch* and *Zoller* [10.7] uses the following logic to develop a numerical scheme to solve these :

i)   The five equations (10.4.15,17) form a Markovian system, whose Fokker-Planck equation may be written

$$\frac{\partial P}{\partial t} = (\kappa^2 L_1 + \kappa L_2 + L_3) P \tag{10.4.18}$$

where μ and λ are proportional to κ; L_1 describes the motion of $\alpha_x(t), \alpha_y(t)$; L_3 describes that part of the motion of $\bar{S}$ which is independent of $\alpha_x(t), \alpha_y(t)$, and L_2 describes the coupling of $\bar{S}$ to $\alpha_x(t), \alpha_y(t)$.

ii)   The limit $\kappa \to \infty$ is the white noise limit. One seeks a procedure which implements this limit as an expansion in κ^{-1}. This can be achieved using an expansion in eigenfunctions of L_1. Since L_1 represents an Ornstein-Uhlenbeck

process in two variables, the eigenfunctions are products of Gaussians and Hermite polynomials, as explained in S.M.Sect. 5.2.6c. The eigenvalues are of the form

$$\lambda_{n_1,n_2} = -(n_1\mu + n_2\lambda) \equiv -\kappa^2 \lambda_{n_1,n_2}. \tag{10.4.19}$$

iii) For simplicity, consider a system in which L_1 represents a one variable Ornstein-Uhlenbeck process. Take the Laplace transform of (10.4.18), and expand

$$\tilde{P}(s) = \sum \tilde{w}_n(s) E_n \tag{10.4.20}$$

where

$$L_1 E_n = -\lambda_n E_n. \tag{10.4.21}$$

We also require a relation of the form

$$L_2 E_n = a_n E_{n-1} + b_n E_{n+1} \tag{10.4.22}$$

where a_n, b_n, are not operators in the space of L_1. This is the case here, since L_2 is proportional to α_x, and if E_n is the Hermite polynomial $H_n(\alpha_x)$, then (10.4.22) will follow from the recursion relation for Hermite polynomials

$$xH_n(x) = \tfrac{1}{2}H_{n+1}(x) + nH_{n-1}(x). \tag{10.4.23}$$

iv) Now use the expansion (10.4.20), and the recursion relation (10.4.22) to derive the difference equation

$$(s + \kappa^2 \lambda_n - L_3)\tilde{w}_n(s) = \kappa \{a_{n+1}\tilde{w}_{n+1}(s) + b_{n-1}\tilde{w}_{n-1}(s)\}. \tag{10.4.24}$$

v) If κ is large, there are two independent solutions to the difference equation (10.4.24),

$$\tilde{w}_n(s) \to \kappa^{-n}, \qquad \text{or} \quad \to \kappa^n. \tag{10.4.25}$$

Only the first kind of solution is relevant to any reasonable initial condition. By truncating the difference equation at $n = N$, one can express all the $\tilde{w}_n(s)$ in terms of $\tilde{w}_N(s)$ (formally), and this solution shows that in this case the first kind of solution then eventuates from an initial condition in which

$$w_n(0) = 0, \qquad n > 1, \tag{10.4.26}$$

which would arise from an initial condition in which the probability distribution was initially factorised, and in which the noise term was stationary.

vi) This means that we can carry out a numerical procedure by using the difference-differential equations corresponding to the Laplace inverse of the difference equations (10.4.24), truncated at some order N. Thus we have a closed set of equations which can be solved numerically.

vii) The generalisation of the two variable situation is straightforward. But the method depends strongly on the two variable analogue of relation (10.4.22).

This is not exclusively a property of the Ornstein-Uhlenbeck process, but neither is it universally true.

viii) The procedure then, is to choose a truncation level N_1, N_2 in the two variable case, and solve the difference-differential equations for appropriate initial conditions. N_1, N_2 can then be altered, and the convergence checked. The usual initial condition will be that the atom and $\alpha_x(t), \alpha_y(t)$ are initially independent of the atomic $\mu(t)$, and are stationary. This means that $w_{n_1 n_2} = 0$ unless $n_1 = n_2 = 0$. Ritsch and Zoller have found that in practice convergence is quite rapid. However, the method is essentially an extension of the white noise approximation, and cannot be expected to work well in situations where $\alpha_y^c(t)$ develops a *very* long correlation time.
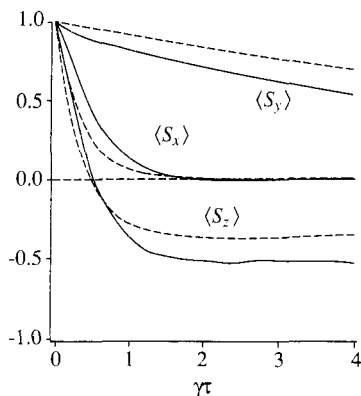
Fig. 10.4 Comparison of the spin averages for simulations (solid line); white noise theory (dashed line)

b) Simulation Methods: It should be remembered that the operators $\alpha_x^c(t)$ and $\alpha_y^c(t)$ are able to be represented by anything which will reproduce their correct mean values and correlation functions. Since they are quantum Gaussian, yet abelian operators, we can represent $\alpha_x^c(t)$ by Gaussian random numbers with an appropriate correlation function. However, $\alpha_y^c(t)$ has a negative variance, so instead we can use pure imaginary Gaussian random numbers. On a computer it is possible to use satndard algorithms to generate a sequence of independent Gaussian random numbers, g_i. A sequence of Gaussian random numbers, α_i, with an exponentially decaying correlation function can be generated from these by the simple algorithm.

$$\alpha_{i+1} = (1 - \delta)\alpha_i + \delta g_i \tag{10.4.27}$$

and, by appropriate scaling, a discretised version of the correlation functions for $\alpha_x^c(t)$ and $\alpha_y^c(t)$ can be developed. The differential equations can then be numerically solved by any appropriate method.

Since there are both real and imaginary terms in the equations, the values of $\bar{\mathbf{S}}(t)$ obtained will have an imaginary part. However, on average, provided the initial conditions are real, the solutions will be real—that is, in practical situations, one takes simply the real parts of the computed averages over $\alpha_x^c(t)$ and $\alpha_y^c(t)$.

Parkins and *Gardiner* carried out simulations using this kind of algorithm, and found that convergence was surprisingly rapid—for details see [10.6], and also

Sect. 3.7, where similar techniques are used. The major effect is a destruction of the inhibition effect when the squeezing becomes too strong, caused largely by the very slow fluctuations of the unsqueezed quadrature, as seen in Fig.10.2.

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