

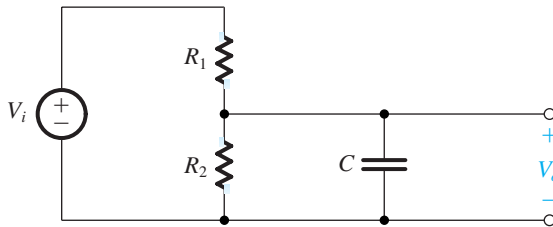
## APPENDIX F

# $s$ -DOMAIN ANALYSIS: POLES, ZEROS, AND BODE PLOTS

In analyzing the frequency response of an amplifier, most of the work involves finding the amplifier voltage gain as a function of the complex frequency  $s$ . In this  $s$ -domain analysis, a capacitance  $C$  is replaced by an admittance  $sC$ , or equivalently an impedance  $1/sC$ , and an inductance  $L$  is replaced by an impedance  $sL$ . Then, using usual circuit-analysis techniques, one derives the voltage transfer function  $T(s) \equiv V_o(s)/V_i(s)$ .

### EXERCISE

**F.1** Find the voltage transfer function  $T(s) \equiv V_o(s)/V_i(s)$  for the STC network shown in Fig. EF.1.



**Figure EF.1**

**Ans.** 
$$T(s) = \frac{1/CR_1}{s + 1/C(R_1 \parallel R_2)}$$

Once the transfer function  $T(s)$  is obtained, it can be evaluated for **physical frequencies** by replacing  $s$  by  $j\omega$ . The resulting transfer function  $T(j\omega)$  is in general a complex quantity whose magnitude gives the magnitude response (or transmission) and whose angle gives the phase response of the amplifier.

In many cases it will not be necessary to substitute  $s = j\omega$  and evaluate  $T(j\omega)$ ; rather, the form of  $T(s)$  will reveal many useful facts about the circuit performance. In general, for all

the circuits dealt with in this book,  $T(s)$  can be expressed in the form

$$T(s) = \frac{a_m s^m + a_{m-1} s^{m-1} + \cdots + a_0}{s^n + b_{n-1} s^{n-1} + \cdots + b_0} \quad (\text{F.1})$$

where the coefficients  $a$  and  $b$  are real numbers, and the order  $m$  of the numerator is smaller than or equal to the order  $n$  of the denominator; the latter is called the **order of the network**. Furthermore, for a **stable circuit**—that is, one that does not generate signals on its own—the denominator coefficients should be such that *the roots of the denominator polynomial all have negative real parts*. The problem of amplifier stability is studied in Chapter 10.

## F.1 Poles and Zeros

An alternate form for expressing  $T(s)$  is

$$T(s) = a_m \frac{(s - Z_1)(s - Z_2) \cdots (s - Z_m)}{(s - P_1)(s - P_2) \cdots (s - P_n)} \quad (\text{F.2})$$

where  $a_m$  is a multiplicative constant (the coefficient of  $s^m$  in the numerator),  $Z_1, Z_2, \dots, Z_m$  are the roots of the numerator polynomial, and  $P_1, P_2, \dots, P_n$  are the roots of the denominator polynomial.  $Z_1, Z_2, \dots, Z_m$  are called the **transfer-function zeros** or **transmission zeros**, and  $P_1, P_2, \dots, P_n$  are the **transfer-function poles** or the **natural modes** of the network. A transfer function is completely specified in terms of its poles and zeros together with the value of the multiplicative constant.

The poles and zeros can be either real or complex numbers. However, since the  $a$  and  $b$  coefficients are real numbers, the complex poles (or zeros) must occur in **conjugate pairs**. That is, if  $5 + j3$  is a zero, then  $5 - j3$  also must be a zero. A zero that is purely imaginary ( $\pm j\omega_Z$ ) causes the transfer function  $T(j\omega)$  to be exactly zero at  $\omega = \omega_Z$ . This is because the numerator will have the factors  $(s + j\omega_Z)(s - j\omega_Z) = (s^2 + \omega_Z^2)$ , which for physical frequencies becomes  $(-\omega^2 + \omega_Z^2)$ , and thus the transfer fraction will be exactly zero at  $\omega = \omega_Z$ . Thus the “trap” one places at the input of a television set is a circuit that has a transmission zero at the particular interfering frequency. Real zeros, on the other hand, do not produce transmission nulls. Finally, note that for values of  $s$  much greater than all the poles and zeros, the transfer function in Eq. (F.1) becomes  $T(s) \approx a_m/s^{n-m}$ . Thus the transfer function has  $(n - m)$  zeros at  $s = \infty$ .

## F.2 First-Order Functions

Many of the transfer functions encountered in this book have real poles and zeros and can therefore be written as the product of first-order transfer functions of the general form

$$T(s) = \frac{a_1 s + a_0}{s + \omega_0} \quad (\text{F.3})$$

where  $-\omega_0$  is the location of the real pole. The quantity  $\omega_0$ , called the **pole frequency**, is equal to the inverse of the time constant of this single-time-constant (STC) network (see Appendix E). The constants  $a_0$  and  $a_1$  determine the type of STC network. Specifically, we

studied in Chapter 1 two types of STC networks, low pass and high pass. For the low-pass first-order network we have

$$T(s) = \frac{a_0}{s + \omega_0} \quad (\text{F.4})$$

In this case the dc gain is  $a_0/\omega_0$ , and  $\omega_0$  is the corner or 3-dB frequency. Note that this transfer function has one zero at  $s = \infty$ . On the other hand, the first-order high-pass transfer function has a zero at dc and can be written as

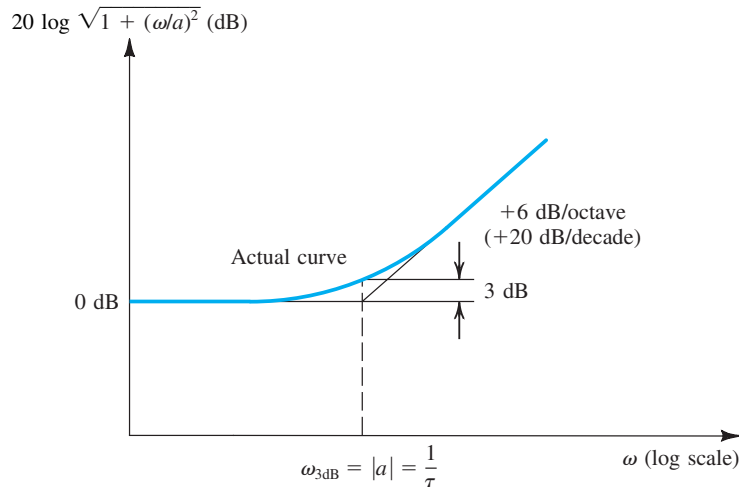
$$T(s) = \frac{a_1 s}{s + \omega_0} \quad (\text{F.5})$$

At this point the reader is strongly urged to review the material on STC networks and their frequency and pulse responses in Appendix E. Of specific interest are the plots of the magnitude and phase responses of the two special kinds of STC networks. Such plots can be employed to generate the magnitude and phase plots of a high-order transfer function, as explained below.

## F.3 Bode Plots

A simple technique exists for obtaining an approximate plot of the magnitude and phase of a transfer function given its poles and zeros. The technique is particularly useful in the case of real poles and zeros. The method was developed by H. Bode, and the resulting diagrams are called **Bode plots**.

A transfer function of the form depicted in Eq. (F.2) consists of a product of factors of the form  $s + a$ , where such a factor appears on top if it corresponds to a zero and on the bottom if it corresponds to a pole. It follows that the magnitude response in decibels of the network can be obtained by summing together terms of the form  $20 \log_{10} \sqrt{a^2 + \omega^2}$ , and the phase response can be obtained by summing terms of the form  $\tan^{-1}(\omega/a)$ . In both cases the terms corresponding to poles are summed with negative signs. For convenience we can extract the constant  $a$  and write the typical magnitude term in the form  $20 \log \sqrt{1 + (\omega/a)^2}$ . On a plot of decibels versus log frequency this term gives rise to the curve and straight-line asymptotes shown in Fig. F.1. Here the low-frequency asymptote is a horizontal straight line



**Figure F.1** Bode plot for the typical magnitude term. The curve shown applies for the case of a zero. For a pole, the high-frequency asymptote should be drawn with a  $-6$ -dB/octave slope.

at 0-dB level and the high-frequency asymptote is a straight line with a slope of 6 dB/octave or, equivalently, 20 dB/decade. The two asymptotes meet at the frequency  $\omega = |a|$ , which is called the **corner frequency**. As indicated, the actual magnitude plot differs slightly from the value given by the asymptotes; the maximum difference is 3 dB and occurs at the corner frequency.

For  $a = 0$ —that is, a pole or a zero at  $s = 0$ —the plot is simply a straight line of 6 dB/octave slope intersecting the 0-dB line at  $\omega = 1$ .

In summary, to obtain the Bode plot for the magnitude of a transfer function, the asymptotic plot for each pole and zero is first drawn. The slope of the high-frequency asymptote of the curve corresponding to a zero is +20 dB/decade, while that for a pole is -20 dB/decade. The various plots are then added together, and the overall curve is shifted vertically by an amount determined by the multiplicative constant of the transfer function.

### Example F.1

An amplifier has the voltage transfer function

$$T(s) = \frac{10s}{(1 + s/10^2)(1 + s/10^5)}$$

Find the poles and zeros and sketch the magnitude of the gain versus frequency. Find approximate values for the gain at  $\omega = 10$ ,  $10^3$ , and  $10^6$  rad/s.

#### Solution

The zeros are as follows: one at  $s = 0$  and one at  $s = \infty$ . The poles are as follows: one at  $s = -10^2$  rad/s and one at  $s = -10^5$  rad/s.

Figure F.2 shows the asymptotic Bode plots of the different factors of the transfer function. Curve 1, which is a straight line intersecting the  $\omega$ -axis at 1 rad/s and having a +20 dB/decade slope, corresponds to the  $s$  term (that is, the zero at  $s = 0$ ) in the numerator. The pole at  $s = -10^2$  results in curve 2, which consists of two asymptotes intersecting at  $\omega = 10^2$ . Similarly, the pole at  $s = -10^5$  is represented by curve 3, where the intersection of the asymptotes is at  $\omega = 10^5$ . Finally, curve 4 represents the multiplicative constant of value 10.

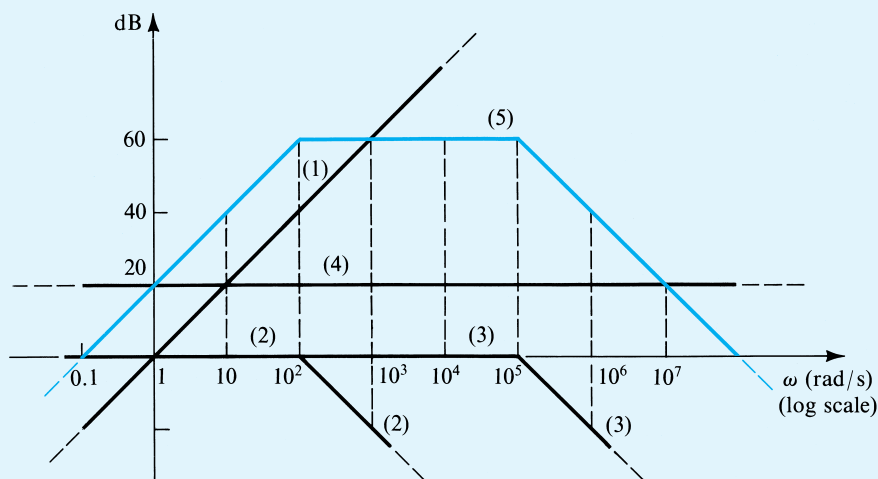
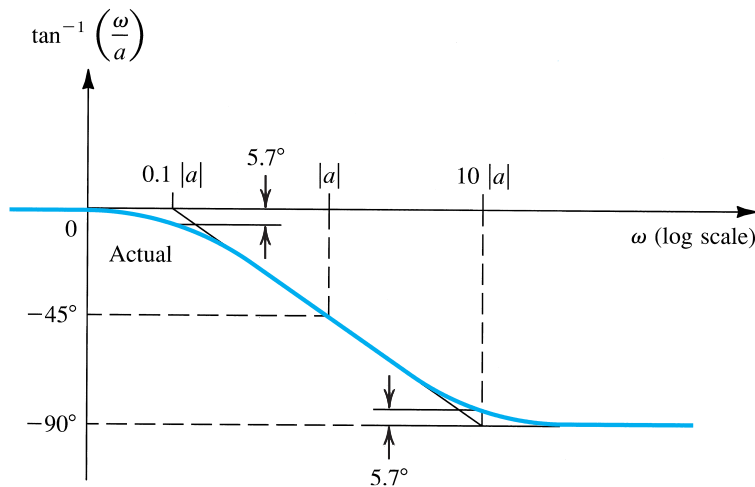


Figure F.2 Bode plots for Example F.1.

Adding the four curves results in the asymptotic Bode diagram of the amplifier gain (curve 5). Note that since the two poles are widely separated, the gain will be very close to  $10^3$  (60 dB) over the frequency range  $10^2$  to  $10^5$  rad/s. At the two corner frequencies ( $10^2$  and  $10^5$  rad/s) the gain will be approximately 3 dB below the maximum of 60 dB. At the three specific frequencies, the values of the gain as obtained from the Bode plot and from exact evaluation of the transfer function are as follows:

$\omega$	Approximate Gain	Exact Gain
10	40 dB	39.96 dB
$10^3$	60 dB	59.96 dB
$10^6$	40 dB	39.96 dB

We next consider the Bode phase plot. Figure F.3 shows a plot of the typical phase term  $\tan^{-1}(\omega/a)$ , assuming that  $a$  is negative. Also shown is an asymptotic straight-line approximation of the arctan function. The asymptotic plot consists of three straight lines. The first is horizontal at  $\phi = 0$  and extends up to  $\omega = 0.1|a|$ . The second line has a slope of  $-45^\circ/\text{decade}$  and extends from  $\omega = 0.1|a|$  to  $\omega = 10|a|$ . The third line has a zero slope and a level of  $\phi = -90^\circ$ . The complete phase response can be obtained by summing the asymptotic Bode plots of the phase of all poles and zeros.



**Figure F.3** Bode plot of the typical phase term  $\tan^{-1}(\omega/a)$  when  $a$  is negative.

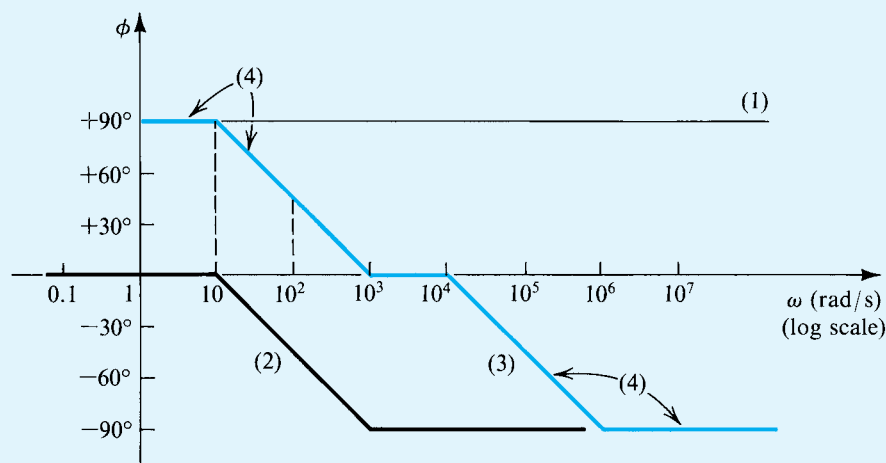
## Example F.2

Find the Bode plot for the phase of the transfer function of the amplifier considered in Example F.1.

### Solution

The zero at  $s = 0$  gives rise to a constant  $+90^\circ$  phase function represented by curve 1 in Fig. F.4. The pole at  $s = -10^2$  gives rise to the phase function

$$\phi_1 = -\tan^{-1} \frac{\omega}{10^2}$$



**Figure F.4** Phase plots for Example F.2.

(the leading minus sign is due to the fact that this singularity is a pole). The asymptotic plot for this function is given by curve 2 in Fig. F.4. Similarly, the pole at  $s = -10^5$  gives rise to the phase function

$$\phi_2 = -\tan^{-1} \frac{\omega}{10^5}$$

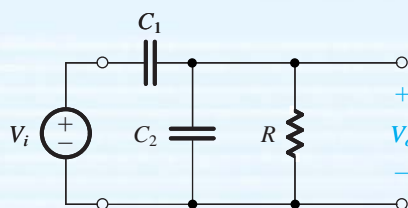
whose asymptotic plot is given by curve 3. The overall phase response (curve 4) is obtained by direct summation of the three plots. We see that at 100 rad/s, the amplifier phase leads by 45° and at 10<sup>5</sup> rad/s the phase lags by 45°.

## F.4 An Important Remark

For constructing Bode plots, it is most convenient to express the transfer-function factors in the form  $(1 + s/a)$ . The material of Figs. F.1 and F.2 and of the preceding two examples is then directly applicable.

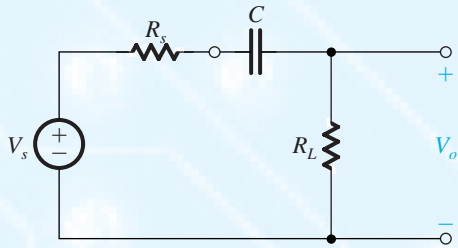
## PROBLEMS

**F.1** Find the transfer function  $T(s) = V_o(s)/V_i(s)$  of the circuit in Fig. PF.1. Is this an STC network? If so, of what type? For  $C_1 = C_2 = 0.5 \mu\text{F}$  and  $R = 100 \text{ k}\Omega$ , find the location of the pole(s) and zero(s), and sketch Bode plots for the magnitude response and the phase response.



**Figure PF.1**

**D•F.2** (a) Find the voltage transfer function  $T(s) = V_o(s)/V_i(s)$ , for the STC network shown in Fig. PF.2.



**Figure PF.2**

(b) In this circuit, capacitor  $C$  is used to couple the signal source  $V_s$  having a resistance  $R_s$  to a load  $R_L$ . For  $R_s = 10\text{ k}\Omega$ , design the circuit, specifying the values of  $R_L$  and  $C$  to only one significant digit to meet the following requirements:

- The load resistance should be as small as possible.
- The output signal should be at least 70% of the input at high frequencies.
- The output should be at least 10% of the input at 10 Hz.

**F.3** Two STC RC circuits, each with a pole at 100 rad/s and a maximum gain of unity, are connected in cascade with an intervening unity-gain buffer that ensures that they function separately. Characterize the possible combinations (of low-pass and high-pass circuits) by providing (i) the relevant transfer functions, (ii) the voltage gain at 10 rad/s, (iii) the voltage gain at 100 rad/s, and (iv) the voltage gain at 1000 rad/s.

**F.4** Design the transfer function in Eq. (F.5) by specifying  $a_1$  and  $\omega_0$  so that the gain is 10 V/V at high frequencies and 1 V/V at 10 Hz.

**F.5** An amplifier has a low-pass STC frequency response. The magnitude of the gain is 20 dB at dc and 0 dB at 100 kHz. What is the corner frequency? At what frequency is the gain 19 dB? At what frequency is the phase  $-6^\circ$ ?

**F.6** A transfer function has poles at  $(-5)$ ,  $(-7 + j10)$ , and  $(-20)$ , and a zero at  $(-1 - j20)$ . Since this function represents

an actual physical circuit, where must other poles and zeros be found?

**F.7** An amplifier has a voltage transfer function  $T(s) = 10^6 s / (s + 10)(s + 10^3)$ . Convert this to the form convenient for constructing Bode plots [that is, place the denominator factors in the form  $(1 + s/a)$ ]. Provide a Bode plot for the magnitude response, and use it to find approximate values for the amplifier gain at 1, 10,  $10^2$ ,  $10^3$ ,  $10^4$ , and  $10^5$  rad/s. What would the actual gain be at 10 rad/s? At  $10^3$  rad/s?

**F.8** Find the Bode phase plot of the transfer function of the amplifier considered in Problem F.7. Estimate the phase angle at 1, 10,  $10^2$ ,  $10^3$ ,  $10^4$ , and  $10^5$  rad/s. For comparison, calculate the actual phase at 1, 10, and 100 rad/s.

**F.9** A transfer function has the following zeros and poles: one zero at  $s = 0$  and one zero at  $s = \infty$ ; one pole at  $s = -100$  and one pole at  $s = -10^6$ . The magnitude of the transfer function at  $\omega = 10^4$  rad/s is 100. Find the transfer function  $T(s)$  and sketch a Bode plot for its magnitude.

**F.10** Sketch Bode plots for the magnitude and phase of the transfer function

$$T(s) = \frac{10^4(1 + s/10^5)}{(1 + s/10^3)(1 + s/10^4)}$$

From your sketches, determine approximate values for the magnitude and phase at  $\omega = 10^6$  rad/s. What are the exact values determined from the transfer function?

**F.11** A particular amplifier has a voltage transfer function  $T(s) = 10s^2 / (1 + s/10)(1 + s/100)(1 + s/10^6)$ . Find the poles and zeros. Sketch the magnitude of the gain in dB versus frequency on a logarithmic scale. Estimate the gain at  $10^0$ ,  $10^3$ ,  $10^5$ , and  $10^7$  rad/s.

**F.12** A direct-coupled differential amplifier has a differential gain of 100 V/V with poles at  $10^6$  and  $10^8$  rad/s, and a common-mode gain of  $10^{-3}$  V/V with a zero at  $10^4$  rad/s and a pole at  $10^8$  rad/s. Sketch the Bode magnitude plots for the differential gain, the common-mode gain, and the CMRR. What is the CMRR at  $10^7$  rad/s? (Hint: Division of magnitudes corresponds to subtraction of logarithms.)