Relative-Entropy Endpoint–Path Gaps as a Diagnostic of RG Coarse-Graining Failure

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Abstract

Endpoint matching is ubiquitous in RG/EFT practice, but it implicitly assumes that *endpoint information is sufficient* to represent what remains distinguishable within a scale window under the declared interface and coarse-graining rules. This work makes that assumption testable by treating the *viewpoint*—the observable interface and protocol, regulator/scheme choices, model class and reconstruction map, divergence, and aggregation rule—as part of the mathematical specification, and by defining a viewpoint-locked *endpoint–path gap*, namely the *strict* endpoint–path gap Γ_{path}, which makes endpoint-compression failure testable; we report Γ_{geo} only as a domain-gated operational signal (Sec. 8.3) rather than a theorem-level substitute.

Unless explicitly stated, the term "gap" refers to the strict quantity Γ_{path}; the computable proxy Γ_{geo} is introduced later (Sec. 8.3).

Fix a viewpoint V. Route I defines a Route-I endpoint divergence Δ_{I} as a relative-entropy-type discrepancy between endpoint predictive objects. Route II induces an information metric G from a local second-order expansion and defines an information-geometric trajectory length $L_{\mathrm{II}} = \int \sqrt{\beta^\top G\,\beta}\,\mathrm{d}\gamma$. At the strict level, for relative entropy along a viewpoint-induced scale chain we establish an exact endpoint–path decomposition $\Delta_{\mathrm{path}} = \Delta_{\mathrm{pair}} + \Gamma_{\mathrm{path}}$ with $\Gamma_{\mathrm{path}} \geq 0$, where $\Gamma_{\mathrm{path}} = 0$ is equivalent to *endpoint sufficiency* for distinguishing the compared scale-chain processes under V. At the computable level, we define the geometric proxy gap $\Gamma_{\mathrm{geo}} := L_{\mathrm{II}} - d_{\mathrm{end}}$, where d_{end} is the endpoint geodesic distance under the same induced structure. We emphasize that $\Gamma_{\mathrm{geo}} = 0$ is a geometric (geodesic) statement and does not, by itself, imply the strict sufficiency criterion $\Gamma_{\mathrm{path}} = 0$ without additional assumptions. On the domain certified by Condition $\mathcal{G}_{\mathrm{geo}}$, Γ_{geo} is invariant under smooth reparameterizations of couplings and monotone reparameterizations of scale, and $\Gamma_{\mathrm{geo}} = 0$ holds if and only if the trajectory segment is a minimizing geodesic between the endpoints.

We provide closed-form calibrations in one-loop, threshold-free QED/EFT windows: a one-dimensional truncation yields the baseline $\Gamma_{\mathrm{geo}} = 0$, while a minimal two-dimensional EFT truncation $\mathcal{M}_2 = \{\alpha, u\}$ generically yields $\Gamma_{\mathrm{geo}}(T) > 0$ for $u_0 \neq 0,\ p > 0,\ T > 0$, together with controlled small-slope approximations. Importantly, $\Gamma_{\mathrm{geo}}(T) > 0$ here is a controlled signal that endpoint matching underestimates full-trajectory accumulation once an interface-visible irrelevant direction is included. These results furnish a reproducible, viewpoint-conditioned criterion for locating windows where intermediate-scale structure remains distinguishable but cannot be compressed into endpoints at the declared resolution. **Contributions and deliverables:** (i) a strict endpoint–path decomposition $\Delta_{\mathrm{path}} = \Delta_{\mathrm{pair}} + \Gamma_{\mathrm{path}}$ with an exact sufficiency criterion $\Gamma_{\mathrm{path}} = 0$; (ii) an invariant, computable proxy gap $\Gamma_{\mathrm{geo}} = L_{\mathrm{II}} - d_{\mathrm{end}}$ on the declared domain of Condition $\mathcal{G}_{\mathrm{geo}}$; and (iii)

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closed-form one-loop QED/EFT calibrations together with a domain-gated reporting protocol for reproducible use.
Reporting semantics. Outputs are domain-gated and status-first (`ADMISSIBLE`/`MISMATCH` /`ABYSS`/`REC-CRITICAL`); numerical proxies are reported only when the corresponding gates certify they are in-domain.

Keywords: viewpoint-relative; relative entropy; information geometry; renormalization group; coarse-graining; QED; EFT; truncation gap.

Contents

1 Introduction

In the standard practice of the renormalization group (RG) and effective field theory (EFT), scale dependence is often accessed through *endpoint objects*[1–4]: one compares renormalized couplings, effective parameters, or predictive/observable outputs at a few fixed momentum points between two energy scales μ_0 and μ_1, and then performs matching, calibration, and constraints accordingly. Endpoint comparison is operationally efficient, but it carries an implicit premise: under the declared interfaces and coarse-graining rules, endpoint information is sufficient to represent the distinguishable structural content of scale evolution *within the window.* Here a *window* refers to a finite scale interval $[\mu_0, \mu_1]$ equipped with an explicitly declared interface, protocol, and resolution clause.

Main message. Fix a viewpoint V (defined below) and consider the comparison task of discriminating two induced scale-chain processes. Endpoint-only *comparison* is sufficient for that task if and only if the strict endpoint–path residual Γ_{path} vanishes. Equivalently, under the locked viewpoint, the endpoints are sufficient statistics for discriminating $\mathbb{P}_Z$ from $\mathbb{Q}_Z$ as defined in Sec. 5. A nonzero residual identifies intermediate-scale structure that remains distinguishable under the declared interface and cannot be compressed into endpoints at the declared resolution; it therefore provides an operational diagnostic of coarse-graining/truncation breakdown along RG flows. At the computable level, the geometric proxy Γ_{geo} inherits a sharp zero criterion only on the declared validity domain of Condition $\mathcal{G}_{\mathrm{geo}}$ (Sec. 8.3).

Contributions and main results. The technical ingredients used in this paper (relative entropy, DPI/chain rule, and Fisher information geometry) are standard. The contribution is to assemble them into a single V-locked diagnostic that separates *endpoint-compressible* windows from windows where intermediate-scale structure remains distinguishable at the declared resolution. Concretely, under a fixed viewpoint V, the paper provides:

- **Viewpoint as part of the mathematical specification.** All divergences and lengths are defined as V-conditioned objects; altering $\mathcal{O}, R, S, \mathcal{M}, \Pi_{\mathrm{rec}}, D$, or Agg changes the diagnostic specification (Sec. 2).
- **Strict endpoint–path bridge with a nonnegative residual.** Introducing path-process objects and applying DPI and the chain rule yields the exact decomposition $\Delta_{\mathrm{path}} = \Delta_{\mathrm{pair}} + \Gamma_{\mathrm{path}}$ with $\Gamma_{\mathrm{path}} \geq 0$, and an explicit equality condition equivalent to endpoint sufficiency for distinguishing the compared scale flows (Sec. 5 and Appendix A).
- **Computable geometric proxy with invariant meaning and a sharp zero criterion.** On the domain where Route II yields a genuine endpoint distance under a positive-definite metric (Condition $\mathcal{G}_{\mathrm{geo}}$), defining $\Gamma_{\mathrm{geo}} := L_{\mathrm{II}} - d_{\mathrm{end}}$ yields a quantity

that is invariant under smooth coordinate reparameterizations and monotone reparameterizations of the scale parameter, and satisfies $\Gamma_{\rm geo} = 0$ if and only if the RG trajectory segment is a minimizing geodesic between the endpoints (Theorems 5.1–5.2).

- **Closed, reproducible calibration examples in standard QED/EFT windows.** In a one-loop, threshold-free QED window, the one-dimensional truncation yields $\Gamma_{\rm geo} = 0$ as a baseline calibration (Sec. 6); extending to the minimal two-dimensional truncation $\mathcal{M}_2 = \{\alpha, u\}$ yields $\Gamma_{\rm geo}(T) > 0$ generically for $u_0 \neq 0,\ p > 0,\ T > 0$, with closed forms and a variance-controlled small-slope approximation (Sec. 7 and Appendix B).

Viewpoint-locking and the information gap. A segment of scale evolution is itself a coarse-graining process [2, 5]. Even if endpoint objects appear consistent under a given interface, intermediate scales may carry additional distinguishable structure. When such structure cannot be absorbed by endpoint compression at the declared resolution, the "endpoint-only" description and the "full-trajectory" description cease to be equivalent. This paper formalizes the resulting incompressible discrepancy as a *viewpoint-conditioned information gap* (denoted by Γ)[1] and adopts the following operational criterion: under a fixed viewpoint V and a declared resolution clause, if the gap transitions from negligible to non-negligible, then the window is identified as a window where endpoint compression fails.

To make this criterion reproducible, we elevate the viewpoint to a first-order structural object. Informally, a viewpoint specifies *what is observed*, *how scale is traversed and coarse-grained*, *what model family and reconstruction are used*, and *how discrepancies are measured and aggregated*. Formally, a viewpoint is specified by

$$\mathsf{V} = (\mathcal{O}, \text{protocol}, R, S, \text{threshold rule}, \mathcal{M}, \Pi_{\rm rec}, D, \text{Agg}),$$

including the observable interface and protocol ($\mathcal{O}$), coarse-graining procedure (R), renormalization scheme (S), threshold rules, model class ($\mathcal{M}$) and reconstruction map ($\Pi_{\rm rec}$), divergence functional (D), and aggregation rule (Agg). All reported quantities—the Route-I endpoint divergence $\Delta_{\rm I}$[2], the induced-geometry length $L_{\rm II}$[3], and the gap terms $\Gamma_{\rm path}, \Gamma_{\rm geo}$—are treated as V-conditioned objects. Accordingly, changes in $\mathcal{O}, R, S, \mathcal{M}, \Pi_{\rm rec}, D$, or Agg represent *changes of specification* (and hence changes of the diagnostic task), rather than alternative expressions of a single viewpoint-independent quantity (Sec. 2).

Notation in this section is anticipatory; formal definitions are given in Secs. 3–5.

1.1 Relation to existing entropic and information-geometric RG approaches

Relative-entropy and information-geometric quantities have a long history in RG and EFT, but they are typically deployed with goals that are distinct from the present diagnostic.

Entropic RG monotones and irreversibility. A major thread in quantum and statistical field theory is the construction of monotone quantities along RG flows ("c-/a-/F-type" theorems and their entropic variants), often phrased in terms of entanglement entropy, relative entropy,

[1] Here Γ denotes the residual history contribution in the endpoint–path decomposition; its strict definition is $\Gamma_{\rm path}$ in Sec. 5 (via the chain rule) and Appendix A, and its computable proxy is $\Gamma_{\rm geo}$ in Sec. 8.3.

[2] A relative-entropy-type discrepancy between endpoint predictive objects; see Sec. 3.

[3] Obtained by integrating the information speed along an RG trajectory; see Sec. 4.

or related convexity/monotonicity structures. These results capture *irreversibility* of coarse-graining in a viewpoint-independent continuum setting and provide powerful constraints on admissible flows. The present work does not propose a new monotone, nor does it target irreversibility as the primary object. Instead, it targets a different question that arises in operational RG/EFT practice: *given a declared interface, protocol, and resolution clause, when is endpoint-only reporting sufficient to represent what remains distinguishable within a window?* This paper's gap quantities are window-local and viewpoint-conditioned, and they are designed to flag *endpoint-compression failure* rather than to certify global irreversibility.

Relative-entropy diagnostics already used in RG/EFT. Relative entropy and closely related distinguishability measures are widely used as diagnostics of truncation quality, model mismatch, and information loss under coarse-graining. In many applications, however, the compared objects are fixed-scale states or distributions, and the diagnostic is expressed directly at endpoints (or at a small number of scales) under a fixed measurement prescription. By contrast, the strict object Γ_{path} here is defined on *path-process* distributions induced by the declared scale protocol and interface, and the endpoint quantity Δ_{pair} appears as the endpoint marginal term in an *exact endpoint–path decomposition*. The strict decomposition is a direct consequence of DPI and the chain rule; the point here is to use it to formulate a *testable compressibility statement*: $\Gamma_{\text{path}} = 0$ if and only if endpoints are sufficient statistics for the comparison task under the locked viewpoint (Sec. 5 and Appendix A).

Information geometry and Fisher metrics in RG. Information geometry (Fisher–Rao type metrics, parameter-manifold geometry, and related constructions)[6–9] is likewise well established, including in contexts where RG trajectories are studied as curves on a coupling manifold. The present work uses standard second-order expansions to induce an information metric, but the diagnostic role is different: (i) the metric is induced *from the same viewpoint-locked divergence* used to quantify endpoint discrepancy, so Route I and Route II are not independent ad hoc choices; (ii) the key computable object is the *length–distance gap* $\Gamma_{\text{geo}} := L_{\text{II}} - d_{\text{end}}$, whose invariant meaning and sharp zero criterion ($\Gamma_{\text{geo}} = 0$ iff the segment is a minimizing geodesic) hold only on a declared validity domain (Condition $\mathcal{G}_{\text{geo}}$); and (iii) the paper treats the induced geometry as an V-conditioned structure, so that changes in interface/protocol/scheme/model class are recorded as specification changes rather than silently absorbed into coordinate choices.

Summary of the distinction. In short, the paper's point of departure from these existing directions is the combination of: *(a) endpoint–path compressibility as the diagnostic target; (b) viewpoint-locking as part of the mathematical object, enabling reproducibility across implementations; and (c) a computable proxy gap with explicit domain gating and a sharp zero criterion.* This triad prevents the construction from reducing to a restatement of standard DPI/chain-rule facts, while remaining compatible with (and informed by) the entropic and geometric structures used elsewhere.

1.2 Construction routes, examples, and paper map

Under a fixed V, this paper provides two complementary construction routes. Route I (the endpoint route) rewrites the difference between endpoint-induced predictive objects under the interface $\mathcal{O}$ as a relative-entropy-type distance[10, 11], yielding the Route-I endpoint divergence Δ_{I} (Sec. 3). Route II (the full-trajectory route) localizes the relative entropy in Route I at neighboring scales, induces an information metric G via a second-order expansion[6–9], and combines it with the RG vector field β to define the information speed

$$\Psi = \sqrt{\beta^{\top} G\, \beta},$$

which is then integrated along the scale variable γ to obtain the path length

$$L_{\mathrm{II}} = \int \Psi \, \mathrm{d}\gamma$$

(Sec. 4).

We then introduce path-process objects at the rigorous level and use DPI together with the chain-rule decomposition of relative entropy to establish the endpoint–path bridge

$$\Delta_{\mathrm{path}} = \Delta_{\mathrm{pair}} + \Gamma_{\mathrm{path}}, \qquad \Gamma_{\mathrm{path}} \geq 0,$$

Layering note. The strict bridge objects $\Delta_{\mathrm{pair}}, \Delta_{\mathrm{path}}, \Gamma_{\mathrm{path}}$ are defined for a fixed path-process object under the locked viewpoint and do not involve interface aggregation; no interchange between Agg and the bridge decomposition is assumed unless explicitly stated.

where the equality condition corresponds to the endpoints being sufficient statistics under the viewpoint (Sec. 5 and Appendix A). At the computable level, we propose the geometric gap proxy

$$\Gamma_{\mathrm{geo}} = L_{\mathrm{II}} - d_{\mathrm{end}}, \qquad d_{\mathrm{end}}(\mu_0 \to \mu_1) := \mathrm{dist}_G\big(g(\gamma_0), g(\gamma_1)\big),$$

where d_{end} denotes the endpoint distance induced by the same length structure (it coincides with the geodesic distance when Condition $\mathcal{G}_{\mathrm{geo}}$ holds); within the validity window of the local second-order expansion, the operational approximation $d_{\mathrm{end}} \simeq \sqrt{2\Delta_{\mathrm{I}}}$ is available, thereby yielding a diagnostic that compares "endpoint compression" against "full-trajectory accumulation" (Sec. 5).

To demonstrate operability, we perform two reproducible checks in one-loop QED within a threshold-free window [12–15]. First, under a one-dimensional truncation and a stable interface, we obtain the baseline case $\Gamma_{\mathrm{geo}} = 0$ (Sec. 6); second, within the same window, after extending to a two-dimensional truncation that includes an EFT-irrelevant direction u[1, 3, 4, 16, 17], a minimal nontrivial gap $\Gamma_{\mathrm{geo}}(T) > 0$ generically appears, and we provide closed forms and controlled approximations (Sec. 7). We also provide a reproducible protocol: a required checklist of viewpoint elements to report, pseudocode for the operational procedure, and threshold/window segmentation rules (Sec. 8 and Appendices D–E).

The goal of this paper is to provide an operational, viewpoint-conditioned diagnostic for cross-scale endpoint–path gaps together with invariant meaning (reparameterization invariance and a sharp geodesic zero criterion) and closed-form calibration examples. All statements are made conditional on a locked viewpoint V and a declared resolution clause; under that specification, non-negligibility of the strict gap Γ_{path} (or, operationally, of its proxy Γ_{geo} on the declared domain of Condition $\mathcal{G}_{\mathrm{geo}}$) provides a structural criterion for endpoint-compression failure and a reproducible method for locating where such gaps emerge.

From the next section onward, all quantities are defined and compared conditional on the

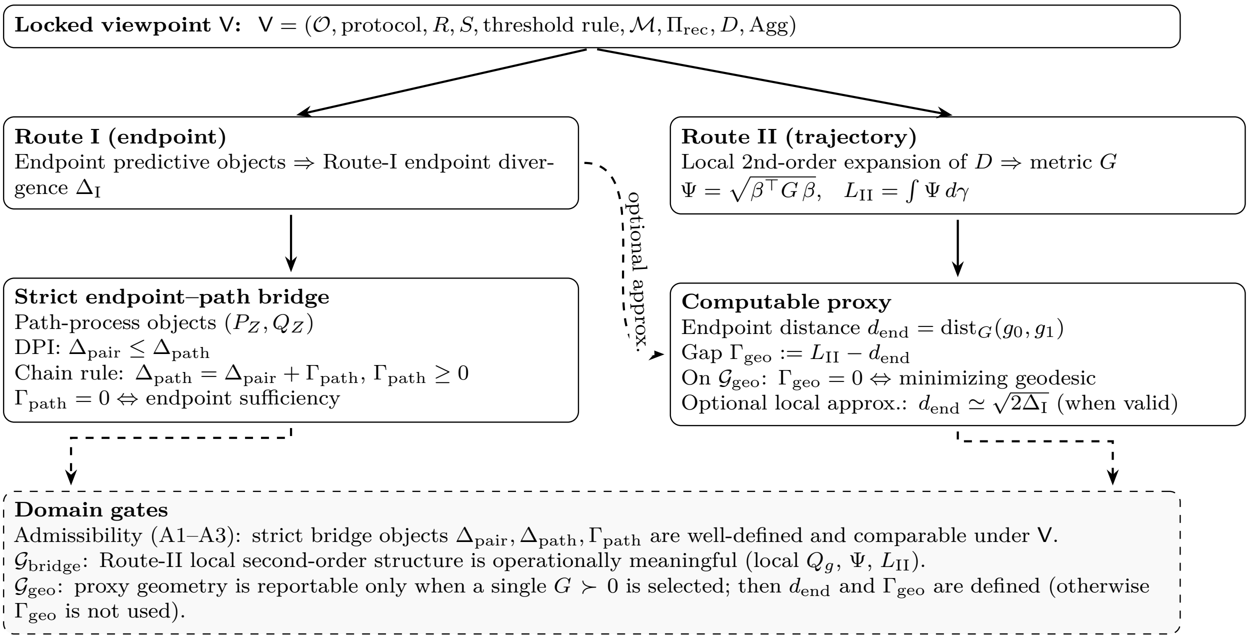

Figure 1: Overview of the viewpoint-locked diagnostic. Viewpoint V specifies the interface and comparison task; Route I yields a Route-I endpoint divergence Δ_{I}. The strict endpoint–path bridge uses the endpoint-pair term Δ_{pair} and gap Γ_{path}, $\Delta_{\text{path}} = \Delta_{\text{pair}} + \Gamma_{\text{path}}$. Route II yields L_{II} and induces the endpoint distance d_{end}, defining the proxy gap $\Gamma_{\text{geo}} = L_{\text{II}} - d_{\text{end}}$ on $\mathcal{G}_{\text{geo}}$. Domain gates $\mathcal{G}_{\text{bridge}}$ and $\mathcal{G}_{\text{geo}}$ delimit where the strict and geometric statements are meaningful. *Notational pitfall:* Δ_{I} (Route I) and Δ_{pair} (the endpoint term in the strict bridge) are distinct objects; they coincide only when the endpoint object is explicitly taken as $E = (Y_0, Y_N)$ under the locked implementation. Zero-gap semantics matches Theorem 5.2: $\Gamma_{\text{geo}} = 0$ holds if and only if the trajectory realizes dist_G (a minimizing geodesic segment between the endpoints).

locked viewpoint V*; statements outside that specification are outside the scope of the paper.*

The paper is organized as follows: Sec. 2 fixes the normative specification of the viewpoint V; Secs. 3–4 define the Route-I endpoint divergence and the Route II information-geometric length, respectively; Sec. 5 establishes the endpoint–path consistency bridge and introduces the gap term; Secs. 6–7 present the baseline and nontrivial one-loop QED examples; Sec. 8 discusses the physical meaning of the gap term, constraint methods, and falsifiable statements; Sec. 9 concludes. Derivation details and closed-form examples are given in Appendices.

Terminology and Structural-Quantity Reference

We adopt a terminology system defined relative to a fixed viewpoint V. The main structural quantities used throughout the paper are summarized in table 1.

Terminology convention (global). In prose, we avoid the bare phrase "endpoint divergence/term"; we always write either *Route-I endpoint divergence* Δ_{I} (Sec. 3) or the *strict-bridge endpoint term* Δ_{pair} (Sec. 5). The phrase *endpoint term* is reserved for Δ_{pair} throughout the strict bridge (remark 5.2). In particular, the Route-I endpoint divergence Δ_{I} (interface-aggregated) is distinct from the strict-bridge endpoint term Δ_{pair}; they coincide only under the explicit implementation choice $E = (Y_0, Y_N)$ under the locked specification.

Symbol	Defined in	Domain / reportability	Brief meaning
Δ_{I}	Sec. 3	Viewpoint-defined; reportable once V and the aggregation rule Agg are declared (implementation level).	The interface-aggregated endpoint discrepancy of predictive objects over $O \in \mathcal{O}$: $\Delta_{\mathrm{I}}(\mu_a \to \mu_b) = \mathrm{Agg}_{O\in\mathcal{O}} D\Big(X^{(O)}_{\mu_a} \| X^{(O)}_{\mu_b}\Big)$ (default Agg $=$ sup); interpreted as the Route-I endpoint structural divergence.
Δ_{pair}	Sec. 5	Strict bridge term on a specified path object; equals Route-I Δ_{I} only under the explicit implementation choice $E = (Y_0, Y_N)$ (see remark 5.2).	The endpoint-pair marginal divergence $D(\mathbb{P}_E \| \mathbb{Q}_E)$ used as the endpoint term in the strict bridge $\Delta_{\mathrm{path}} = \Delta_{\mathrm{pair}} + \Gamma_{\mathrm{path}}$; distinct from Route-I Δ_{I}, which aggregates endpoint discrepancies over the interface family.
L_{II}	Sec. 4	Operational proxy under Condition $\mathcal{G}_{\mathrm{bridge}}$ (local quadratic window) when localized KL elements can be accumulated into an effective length.	The path length accumulated along the RG flow by integrating the information speed Ψ; it measures distinguishability along the entire trajectory.
Ψ	Sec. 4	Operational proxy under Condition $\mathcal{G}_{\mathrm{bridge}}$ (requires a well-defined local quadratic form Q_g / metric G).	Information speed, defined from the RG vector field β and the local Fisher metric G by $\Psi = \sqrt{\beta^\top G \beta}$.
d_{end}	Sec. 5	Reportable only on Condition $\mathcal{G}_{\mathrm{geo}}$ where dist_G (and geodesicity) are well-defined; otherwise treated as out-of-domain for diagnosis (see Sec. 8.3).	The endpoint distance induced by G : $d_{\mathrm{end}} := \mathrm{dist}_G(g_0, g_1)$. In constant-metric/Fisher realizations one has $d_{\mathrm{end}} = \sqrt{2\Delta_{\mathrm{I}}}$, while in general $\sqrt{2\Delta_{\mathrm{I}}}$ is at most a local approximation (see equations (5.11) and (5.12)).
Δ_{path}	Sec. 5	Strict object once the path-process objects $(\mathbb{P}_Z, \mathbb{Q}_Z)$ are specified under the locked viewpoint V.	The joint KL divergence of path objects (scale chains), serving as the strict definition of the “full-trajectory” divergence.
Γ_{path}	Sec. 5	Strict object defined via the chain rule once Δ_{path} and Δ_{pair} are defined.	The strict gap term, representing the incompressible residual information in Δ_{path} beyond the endpoint-pair marginal divergence Δ_{pair}: $\Delta_{\mathrm{path}} = \Delta_{\mathrm{pair}} + \Gamma_{\mathrm{path}}$.
Γ_{geo}	Sec. 5	Reportable only on Condition $\mathcal{G}_{\mathrm{geo}}$; outside that domain it is not formed as a diagnostic quantity (Sec. 8.3).	A computable geometric proxy gap, defined by $\Gamma_{\mathrm{geo}} = L_{\mathrm{II}} - d_{\mathrm{end}}$; used for operational diagnosis on its declared domain and interpreted as a proxy for Γ_{path} via calibration rather than as an unconditional equivalent.
V	Sec. 2	Always declared; all quantities are defined and compared conditional on the locked V.	The viewpoint object, comprising interfaces, protocols, model classes, coarse-graining rules, distance functionals, etc.; it provides the reference frame relative to which all structural quantities are defined.

Table 1: Core structural quantities and their semantics. All quantities are defined relative to a fixed viewpoint V. Decoration: $\hat{X}$ denotes a reported numerical estimate under the locked viewpoint V, and $\tilde{X}$ denotes a reconstruction output.

2 Normative Setup: Scale Axis, Viewpoint Object, Observable Interfaces, Coarse-Graining, and Model Classes

This section fixes the unique comparability convention (normative setup) used throughout the paper. All divergence quantities, length quantities, inequalities, and gap terms appearing later are *defined and compared* only under the viewpoint configuration locked in this section; if any component is changed (observable interface, coarse-graining, scheme, threshold handling, model class, reconstruction rule, or divergence functional), we treat this as a *change of viewpoint*, which yields another set of objects rather than a different writing of the same object.

2.1 Viewpoint object and relativity statement

We treat "viewpoint" as a first-order structural object. A viewpoint V is defined as the following ordered tuple of data:

$$\mathsf{V} := \big(\mathcal{O},\ \text{protocol},\ R,\ S,\ \text{threshold rule},\ \mathcal{M},\ \Pi_{\text{rec}},\ D,\ \text{Agg}\big). \tag{2.1}$$

Here: $\mathcal{O}$ and the observation protocol specify the interfaces for reading/comparing information; R and S specify the theoretical presentation, coarse-graining procedure, and renormalization scheme; the threshold rule specifies the convention for active degrees of freedom as the scale varies; $\mathcal{M}$ and Π_{rec} specify the admissible truncation class and the reconstruction (projection) rule; the divergence functional D specifies how distinguishable information differences are compared *within* a fixed interface; and Agg specifies the fixed cross-interface aggregation rule used when $\mathcal{O}$ is treated as a family of admissible interfaces (e.g. $\sup_{\mathcal{O}}$, an average over $\mathcal{O}$, or another declared aggregation). In the degenerate single-interface case, Agg reduces to the identity and may be absorbed into the choice of D; we nonetheless record Agg explicitly whenever cross-interface comparability is intended. If Agg is non-smooth (e.g. $\sup_{\mathcal{O}}$), then Route-II quantities that rely on a single reportable length structure are meaningful only when Condition $\mathcal{G}_{\text{geo}}$ holds (Sec. 8.3).

Remark 2.1 (Relativity statement)*.* The Route-I endpoint divergence Δ_{I}, the Route-II path length L_{II}, the information speed Ψ, and the gap term (denoted by Γ, with concrete forms $\Gamma_{\text{path}}, \Gamma_{\text{geo}}$) are all viewpoint-relative objects in this paper, and are strictly written as $\Delta_{\text{I}}^{(\mathsf{V})}$, $L_{\text{II}}^{(\mathsf{V})}$, $\Psi^{(\mathsf{V})}$, $\Gamma^{(\mathsf{V})}$. Unless otherwise stated, we work under a fixed viewpoint V and omit the superscripts.

Notation (reserved symbols). Throughout, V (or V) denotes the *viewpoint object*; when a family is considered we write $\mathsf{V} \in \mathfrak{V}$. We denote the finite spatial domain by Ω and its (Lebesgue) volume by $\text{Vol}(\Omega)$.

Notation (decorations). Undecorated symbols denote theoretical (target) objects. A tilde $\tilde{(\cdot)}$ denotes the model-class reconstruction/projection output produced by Π_{rec} (e.g. $\tilde{T}_\mu$). A hat $\hat{(\cdot)}$ denotes a numerical estimate/value reported under the locked viewpoint/workflow V_ε (e.g. $\hat{\Delta}_{\text{I}}, \hat{L_{\text{II}}}, \hat{d_{\text{end}}}, \hat{\Gamma}_{\text{geo}}$; Appendix D).

Notation (local approximation). We use $\simeq$ exclusively for the local second-order (quadratic) approximation induced by the Route-II metricization of D within the admissible window; it is not a global identity unless explicitly stated.

Operational resolution clause. We treat the viewpoint as an operational specification with an explicit resolution / measurement-noise channel N_ε ($\varepsilon > 0$). When needed, we write

$$V := (\mathcal{O},\ \text{protocol},\ R,\ S,\ \text{threshold rule},\ \mathcal{M},\ \Pi_{\text{rec}},\ D,\ \text{Agg}),$$
$$V_\varepsilon := (\mathcal{O},\ \text{protocol}_\varepsilon,\ R,\ S,\ \text{threshold rule},\ \mathcal{M},\ \Pi_{\text{rec}},\ D,\ \text{Agg}),$$

where $\text{protocol}_\varepsilon$ includes N_ε.

Notation convention (locked resolution). We fix $\varepsilon > 0$ and work under the locked viewpoint V_ε. When no ambiguity arises, we suppress ε and viewpoint superscripts; otherwise they are reinstated.

Resolution channel (locked). N_ε is a fixed, reportable coarse-graining map acting on predictive objects (classical: a Markov kernel with strictly positive density; quantum: a full-rank CPTP mixing channel). Changing N_ε constitutes a change of viewpoint. A viewpoint report must specify: (i) the channel family/type (e.g. Gaussian convolution on readout space, discrete smoothing, or a full-rank quantum mixing channel), (ii) the meaning and units of ε in the chosen readout coordinates, and (iii) whether an ε-scan is performed with the *same* channel family while varying only ε.

Regularized predictive objects. Given the protocol-induced predictive object $X^{(O)}_{T_\mu}$ (a distribution or a state) for an interface $O \in \mathcal{O}$, we define the ε-regularized observable object by

$$X^{(O,\varepsilon)}_\mu \;:=\; \mathsf{N}_\varepsilon\Big(X^{(O)}_{T_\mu}\Big), \qquad O \in \mathcal{O}. \tag{2.2}$$

All divergence quantities in later sections may be read as acting on the regularized objects (we keep the notation $D(\cdot\|\cdot)$ unchanged, since N_ε is part of the protocol). Equivalently, one may define a protocol-relative divergence $D_\varepsilon(X\|Y) := D(\mathsf{N}_\varepsilon X\|\mathsf{N}_\varepsilon Y)$.

Admissibility ("effective observability"). We do *not* assume that an arbitrary system automatically satisfies the analytic premises needed for finite divergences, local expansions, or stable reconstruction. Instead we *define* the class of systems and windows for which the viewpoint is operationally meaningful.

Definition 2.1 (Viewpoint-admissible window)**.** Fix a viewpoint V_ε and a scale window $[\mu_0, \mu_1]$ with a discrete grid $\{\mu_k\}$ if needed. We say the pair $(T_\mu, [\mu_0, \mu_1])$ is V_ε-*admissible* if the following hold:

1. **A1 (Finite-resolution support).** For every $O \in \mathcal{O}$ and every $\mu \in [\mu_0, \mu_1]$, the regularized object $X^{(O,\varepsilon)}_\mu$ has full support (classical) / full rank (quantum) on its protocol readout space.

2. **A2 (Absolute continuity / finite divergence).** For every $O \in \mathcal{O}$ and every adjacent pair μ_k, μ_{k+1} in the grid (or for all μ, μ' in the window in the continuous reading), the objects are mutually absolutely continuous under the protocol, so that

$$D\Big(X^{(O,\varepsilon)}_{\mu_k}\Big\|X^{(O,\varepsilon)}_{\mu_{k+1}}\Big) < \infty$$

(and similarly with endpoints).

3. **A3 (Non-degenerate readability).** The protocol/interface family does not collapse the window into a trivial indistinguishability by construction; i.e. the induced readouts are not constant in μ on the window unless this is the claimed physical behavior under the declared resolution.

Route-II (geometric) admissibility. In addition to Definition 2.1, Route-II constructions (and any quantities involving $\dot{g}(\gamma)$, Ψ, L_{II}, d_{end}, or Γ_{geo}) are only formed on subwindows where the local quadratic surrogate is valid (Condition $\mathcal{G}_{\mathrm{bridge}}$). Moreover, reporting Γ_{geo} requires that the locked aggregation/protocol induces a reportable length structure (Condition $\mathcal{G}_{\mathrm{geo}}$); otherwise we output `FLAG:GEO_OUT_OF_DOMAIN` rather than a numerical proxy.

Remark 2.2 (Observational abyss label). If a window fails the admissibility conditions above under a locked viewpoint V_ε (e.g. absolute continuity fails so the divergence blows up or becomes undefined in the $\varepsilon \to 0$ idealization), then the diagnostic quantities are *not* treated as "incorrect"; rather, the window is labeled as *out of operational domain* for that viewpoint. We refer to such a failure region as an *observational abyss* under V_ε, and we report it as a boundary/flag rather than attempting to force a comparison.

For convenience, we may introduce the family of *admissible viewpoints* $\mathfrak{V}$, which denotes the set of viewpoint configurations satisfying the normative clauses (N1–N6) of this section. The structural conclusions of this paper (the endpoint–path inequality and the gap decomposition) hold in form for any $\mathsf{V} \in \mathfrak{V}$; but the numerical values $\Delta_{\mathrm{I}}^{(\mathsf{V})}, L_{\mathrm{II}}^{(\mathsf{V})}, \Gamma^{(\mathsf{V})}$ vary with the viewpoint.

2.2 Scale parameter and scale stepping

Let $\mu > 0$ denote the renormalization scale (also interpretable as a resolution or a matching scale of an effective theory), and fix a reference scale μ_0. Define the dimensionless scale coordinate

$$\gamma := \ln(\mu/\mu_0). \tag{2.3}$$

To establish the endpoint–path relation, we use both a discrete scale grid and a continuous limit:

- **Discrete grid:** take $\mu_0 < \mu_1 < \cdots < \mu_N$, and write $\gamma_k = \ln(\mu_k/\mu_0)$.
- **Continuous limit:** let $\Delta\gamma_k = \gamma_{k+1} - \gamma_k \to 0$, yielding a continuous parameter $\gamma \in [\gamma_0, \gamma_1]$.

We interpret a single scale step as a Wilsonian scale-advancement operation (realizable as coarse-graining or its inverse, depending on the chosen direction and implementation details): from an effective description at scale μ_k to an effective description at scale μ_{k+1}. To avoid introducing process-mechanism drift in comparisons, when comparing two scale flows we always require that they are generated under the same viewpoint clauses by the same class of scale-stepping maps (see the normative clause N3 of this section). Here "the same class" means: adopting the same set of RG transformations / the same set of integration-elimination and rescaling rules; otherwise, differences will be contaminated by mechanism differences and will no longer correspond to "distinguishable information differences under the same process."

2.3 Observable interface $\mathcal{O}$ and observation protocol

To rewrite "theoretical differences" as "distinguishable information differences," one must fix the observable interface set $\mathcal{O}$ together with its observation protocol. We adopt the following abstract setup:

- $\mathcal{O}$ is a set of observables (or a family of observable functionals). Each $O \in \mathcal{O}$ is bound to a fixed experimental/measurement protocol, including the sampling method, definition of statistics, choice of momentum points/energy-scale points, normalization convention, etc.
- For any theoretical description T_μ at scale μ and any $O \in \mathcal{O}$, "theory + protocol" jointly induce a predictive distribution (or a quantum state), denoted by

$$P^{(O)}_{T_\mu}. \tag{2.4}$$

  This induced mapping is fixed throughout the paper; it must not be replaced in different parts by a "more convenient definition," in order to avoid convention drift.

Interface selection principle (QED context) The calibration target of this paper is a one-loop, threshold-free QED window, and hence $\mathcal{O}$ is taken by default to be gauge-invariant observable interfaces; for example, with defining readings of the "effective charge/effective coupling" as the core (specified by a normalization condition on a two-point function, or by a prescription using some class of scattering cross sections at fixed momentum points). We do not enforce a unique choice of O at the normative level, but require that once a concrete interface implementation is chosen in the main text, it is kept consistent throughout all derivations and numerical examples under that implementation.

Protocol as an explicit map (including noise). To avoid implicit convention drift, we emphasize that the protocol is an explicit mapping. For each interface $O \in \mathcal{O}$ and each scale μ, the pipeline is

$$T_\mu \xrightarrow{\text{interface readout}} X^{(O)}_{T_\mu} \xrightarrow{\mathsf{N}_\varepsilon} X^{(O,\varepsilon)}_\mu,$$

where N_ε is the viewpoint-locked resolution channel. All subsequent quantities (including the Route-I endpoint divergence Δ_{I}, the endpoint-pair marginal divergence Δ_{pair}, path objects, and reconstructed coordinates) are computed from the family $\{X^{(O,\varepsilon)}_\mu\}_{O\in\mathcal{O}}$ under the same locked protocol.

Classical vs. quantum reading (protocol decides the object). The framework allows predictive objects to be either classical distributions or quantum states. However, the divergence D is always applied to the *protocol readout object*. If the protocol includes an explicit measurement/POVM (mapping states to outcome distributions), then D is evaluated on the induced *classical* outcome distributions (e.g. KL divergence). If instead the protocol declares that the readout object remains a quantum state and D is a quantum divergence, then the corresponding finiteness and factorization conditions must be interpreted at the state level (e.g. support inclusion for finiteness). In this paper, unless stated otherwise, all empirical reporting uses protocol-induced classical readouts, so that the chain-rule statements are understood in the classical sense.

2.4 Regulator, renormalization scheme, and threshold rule

Relative entropy / relative-entropy-type divergences in field theory typically come with volume and UV-regulator dependence, so we fix the regulator and scheme explicitly[15]:

- **Regulator R:** one may take a finite spatial domain Ω (with volume $\mathrm{Vol}(\Omega)$) and a UV cutoff Λ, or a lattice spacing a, etc. Any regulator implementation is allowed, but it must be kept unchanged within the same derivation chain.
- **Scheme S:** fix a renormalization scheme (e.g. $\overline{\mathrm{MS}}$), and define parameter coordinates under it (e.g. $\alpha(\mu)$). Once fixed, the scheme is not switched within the paper.
- **Threshold rule:** to avoid non-structural non-uniqueness caused by piecewise changes of degrees of freedom when crossing mass thresholds, the main calibration (one-loop QED) part assumes by default a scale window that does not cross thresholds; if threshold-splicing across windows is discussed, we will explicitly specify consistent rules for the entrance/exit of active degrees of freedom in the relevant subsections.

To make relative-entropy-type quantities comparable, we require that one of the following standardization options be fixed in a concrete implementation:

- **Densitization:** for quantities that grow proportionally to $\mathrm{Vol}(\Omega)$, take $\frac{1}{\mathrm{Vol}(\Omega)}$ as a "relative-entropy density";
- **Differencing:** compare differences or ratios under the same regulator so that regulator dependence cancels explicitly;
- **Explicit retention of regulator dependence:** report $(\mathrm{Vol}(\Omega), \Lambda)$ (or equivalent regulator parameters) as part of the output, avoiding an implicit limit.

2.5 Model class $\mathcal{M}$ and reconstruction map Π_{rec}

Wilsonian scale advancement generally generates an infinite-dimensional tower of operators. To turn "cross-scale differences" into computable objects[1, 2, 16], we introduce a truncated model class and fix its reconstruction rule:

- **Full description:** denote the effective description at scale μ by T_μ (viewable as a Wilsonian effective action or an equivalent object).
- **Model class:** choose a family of finite-dimensional model classes $\mathcal{M}$, whose elements are uniquely parameterized by a finite-dimensional parameter $g \in \mathbb{R}^d$, denoted by $\mathbf{M}(g)$. For example:
  - Baseline calibration: $\mathcal{M}_1 = \{\alpha(\mu)\}$ (one-dimensional parameter);
  - EFT extension: $\mathcal{M}_2 = \{\alpha(\mu), u(\mu)\}$ (two-dimensional parameter, where u denotes a dimensionless coefficient along an irrelevant-operator direction).
- **Reconstruction/projection map:** fix a reconstruction map from the full description to the model class,

$$\Pi_{\mathrm{rec}} : \ T_\mu \ \mapsto \ \tilde{T}_\mu = \Pi_{\mathrm{rec}}(T_\mu) \in \mathcal{M} \quad \text{(when single-valued on the window)}, \tag{2.5}$$

and require that Π_{rec} be uniquely specified by a single criterion (numerical approximations are allowed in implementation, but the criterion must not be changed across different parts).

(P) Minimal-information-discrepancy reconstruction (normalized choice) Under fixed $\mathcal{O}$ and a fixed divergence functional D, define the locked objective under V_ε (we suppress ε when it is fixed) by

$$\mathcal{F}_\mu(g) \;:=\; \operatorname*{Agg}_{O\in\mathcal{O}} \; D\Big(P^{(O,\varepsilon)}_{T_\mu} \,\Big\|\, P^{(O,\varepsilon)}_{\mathbf{M}(g)}\Big)\,,$$

with the same aggregation rule as in the viewpoint. Define the reconstruction minimizer set:

$$\mathcal{A}(\mu) \;:=\; \arg\min_{g\in\mathbb{R}^d} \mathcal{F}_\mu(g). \tag{2.6}$$

Single-valued reconstruction (default in-domain case) If $\mathcal{A}(\mu)$ is a singleton, denote its unique element by $g(\mu)$ and define the model-class reconstruction output

$$\tilde{T}_\mu \;:=\; \mathbf{M}\big(g(\mu)\big) \in \mathcal{M},$$

so that Π_{rec} is single-valued on that window.

Set-valued reconstruction and Π_{rec}-criticality. If $\mathcal{A}(\mu)$ is not a singleton, the reconstruction becomes set-valued. Rather than treating this as a numerical pathology, we record it as an operational structural feature. Unless an explicit tie-breaking/selection rule is declared as part of the locked viewpoint (and hence reported), we do not define $g(\mu)$ (and thus do not form a Route-II trajectory) on that window. Consequently, Route-II geometric quantities that require a single-valued trajectory—Ψ, L_{II}, d_{end}, Γ_{geo}, and Condition $\mathcal{G}_{\text{geo}}$—are not formed on such windows unless an explicit tie-breaking/selection rule is locked and reported.

This definition emphasizes that the model class is not an arbitrarily scale-adjusted "fitting tool," but rather a comparability coordinate system bound to the interface $\mathcal{O}$; changing Π_{rec} is equivalent to changing the coordinate system and the comparison problem itself.

Remark 2.3 (Existence and implementation)*.* The existence/uniqueness of $\mathcal{A}(\mu)$ depends on the concrete implementations of $\mathcal{M}$, $\mathcal{O}$, and D. We fix it here as a normative clause: in practical computations, approximate optimization or computable proxies may be used for implementation, but the criterion must not be switched across different scale segments in pursuit of more "ideal" numerical behavior.

Notation. We reserve the uppercase symbol Π for named viewpoint maps. Unless otherwise stated, we fix the endpoint coarse-graining/projection map $\Pi_{\text{end}}(Z) := (Y_0, Y_N)$ throughout the paper, while the locked reconstruction map is denoted by Π_{rec}.

Definition 2.2 (Π_{rec}-critical point / window)**.** A scale point μ is called Π_{rec}-*critical* (under the locked viewpoint V_ε) if $|\mathcal{A}(\mu)| > 1$ in (2.6). A window is called Π_{rec}-*critical* if it contains at least one Π_{rec}-critical scale point.

Remark 2.4 (Interpretation and reporting rule)*.* Π_{rec}-criticality is interpreted as a reconstruction degeneracy / bifurcation under the fixed resolution: multiple distinct parameter points in the chosen model class are operationally indistinguishable under the locked interface family and divergence criterion. In such a window, Route-II geometric objects that require a single-valued C^1 trajectory (e.g. $\dot{g}(\gamma)$, geodesicity, or the strict Riemannian Condition $\mathcal{G}_{\rm geo}$) should be treated as `REC-CRITICAL` for reporting purposes (with Route-II geometric fields withheld) unless a viewpoint-locked tie-breaking rule is explicitly added. The default reporting rule in this paper is: *do not force uniqueness by switching criteria*; instead, report the $\Pi_{\rm rec}$-critical label and (optionally) the diameter of $\mathcal{A}(\mu)$ in the induced metric as a measure of degeneracy under the chosen resolution.

2.6 Divergence functional D and normative clauses (viewpoint stability)

We use relative-entropy-type divergences to characterize distinguishable information differences. The divergence functional $D(\cdot\|\cdot)$ is fixed throughout the paper. Typical choices include the KL divergence (for classical distributions) or its quantum counterpart (for quantum states). In what follows we only use its structural properties to derive the endpoint–path inequality and the gap decomposition, for example:

- Non-negativity: $D(X\|Y) \geq 0$;
- Uniqueness of the zero point: if two objects are indistinguishable under the given interface, then $D = 0$;
- Monotonicity under coarse-graining (data processing inequality, DPI): for an appropriate coarse-graining map Φ (a stochastic map in the classical case, a CPTP map in the quantum case), one has $D(X\|Y) \geq D(\Phi(X)\|\Phi(Y))$.

To ensure that subsequent conclusions do not fail due to viewpoint drift, we organize the viewpoint-stability requirements into six normative clauses:

- **N1 (Interface fixed):** once $\mathcal{O}$ and the observation protocol are chosen, they are not changed throughout the paper; any change is treated as a change of viewpoint.
- **N2 (Regulator and scheme fixed):** the regulator R and scheme S are fixed; the threshold rule is fixed; the chosen handling method among densitization/differencing/explicit retention of regulator dependence is fixed.
- **N3 (Same scale-advancement mechanism):** when comparing two scale flows, they must share the same class of scale-stepping maps (the same scale-advancement mechanism); otherwise, differences are contaminated by mechanism differences and no longer correspond to distinguishable information differences under the same process.
- **N4 (Model class and reconstruction fixed):** the model class $\mathcal{M}$ and reconstruction map $\Pi_{\rm rec}$ are fixed; one must not switch truncation classes or switch optimization criteria across different scale segments to obtain more "ideal" numerical behavior.
- **N5 (Viewpoint equivalence):** if two viewpoints V, V' are equivalent on the predictive objects defined by the interface $\mathcal{O}$, i.e. for the same set of $O \in \mathcal{O}$ and the same set of scale points (or windows) the induced predictive distributions/states agree (or are equivalent under an explicitly specified invertible reparameterization), then we write $\mathsf{V} \sim_{\rm vp} \mathsf{V}'$. Coordinate changes are allowed within an equivalence class, but the transformation rule and its effect on $\Delta_{\rm I}, L_{\rm II}, \Gamma$ must be given explicitly.

- **N6 (Cross-viewpoint comparison):** if $\mathsf{V} \not\sim \mathsf{V}'$, then $\Delta_{\mathrm{I}}^{(\mathsf{V})}, L_{\mathrm{II}}^{(\mathsf{V})}, \Gamma^{(\mathsf{V})}$ should not be directly compared as the same scalar. Cross-viewpoint comparison must first specify a mapping from V to V' (e.g. interface projection, regulator transformation, or reconstruction-rule transformation), and make explicit the transformation law of the objects under the mapping.

After this section, the Route-I endpoint divergence, the Route-II path length, and the gap term appearing in later sections will all be treated as structural objects relative to the fixed viewpoint V; their numerical values and negligibility (e.g. whether $\Gamma^{(\mathsf{V})}$ is negligible under a given resolution) constitute testable statements about whether "intermediate-scale structure can be compressed into endpoints" under that viewpoint.

Standing domain convention (and the meaning of divergence). Unless explicitly stated otherwise, all Route-II constructions and geometric-proxy statements in the sequel (e.g. definitions and claims involving $\dot{g}(\gamma)$, Ψ, L_{II}, d_{end}, and Γ_{geo}) are understood to be made strictly on V_ε-admissible windows (A1–A3) with non-Π_{rec}-critical reconstruction. Outside this domain the framework does not force numerical surrogates based on the current-layer geometry; instead it reports viewpoint-locked status labels (`ABYSS` / `REC-CRITICAL`) plus `FLAGS` (e.g. `FLAG:GEO_OUT_OF_DOMAIN`) together with trigger metadata.

Importantly, a divergence label (e.g. $D = \infty$ or `ABYSS`) is not treated as "meaningless" output. In this framework, non-finiteness is itself a structural statement: under the locked viewpoint, the relevant objects cease to be comparable in the divergence-based sense (classically, absolute continuity fails; quantum mechanically, support inclusion fails), or the local geometric surrogate required by Route II ceases to be well-defined. Accordingly, we do not assign values by ad hoc fixes. Any finite assignment to a divergent quantity should be understood as a *protocol extension* (e.g. an explicitly declared normalization/regulator completion, a finite-resolution parameter ε, or an enlarged model class), and its meaning is inseparable from that declared extension. Thus, "∞" functions here as an informative diagnostic boundary datum: it marks the limit of what the current viewpoint can express, and it guides how the operational specification would need to be extended if one wishes to proceed beyond that boundary.

Viewpoint summary (locked for all reported diagnostics)

Viewpoint Card (main-text, used in Secs. 6–7)

Locked viewpoint. $\mathsf{V}_\varepsilon := (O, \mathrm{protocol}_\varepsilon, R, S, \text{threshold rule}, \mathcal{M}, \Pi_{\mathrm{rec}}, D, \mathrm{Agg})$. All reported diagnostics $(\Delta_I, L_{\mathrm{II}}, d_{\mathrm{end}}, \Gamma_{\mathrm{geo}}, \Gamma_{\mathrm{path}})$ are defined and interpreted *relative to this locked tuple*; changing any component changes the problem.

O **(interfaces)**	Gauge-invariant interface(s) for a one-loop, threshold-free QED window; concretely, readouts centered on an effective charge/coupling definition (e.g. normalization of a two-point function or a prescribed family of cross sections at fixed momentum points). In Sec. 6 we use the *minimal single-interface implementation* $O = \{O\}$ to avoid multi-interface aggregation artifacts.
$\mathrm{protocol}_\varepsilon$	Explicit readout map $T_\mu \mapsto X^{(O)}_{T_\mu} \mapsto X^{(O,\varepsilon)}_\mu$ including the locked resolution channel family N_ε (measurement/estimation noise, discretization/binning, etc.). Unless stated otherwise, reporting uses protocol-induced *classical* readouts, so D is applied to outcome distributions. For Sec. 7, the interface is chosen so that both θ and u are readable, with declared resolution parameters (e.g. σ_θ, σ_u).

(R, S)	Regulator R and renormalization scheme S are fixed throughout the window and not switched mid-chain. Regulator-handling is standardized by one declared convention (densitization / differencing / explicit retention of regulator parameters). Scheme examples include MS-like schemes with fixed parameter coordinates (e.g. $\alpha(\mu)$, $\theta = \alpha^{-1}$).
threshold rule	*Threshold-free window* in Secs. 6–7: the active set of degrees of freedom is unchanged on $[\mu_0, \mu_1]$ (single segment, no threshold crossing). If threshold splicing is discussed elsewhere, entrance/exit rules must be explicitly declared and kept consistent.
$\mathcal{M}$ **(model class)**	*Sec. 6 baseline:* $\mathcal{M}_1 = \{\alpha(\mu)\}$ (equivalently $\theta(\mu) = \alpha^{-1}(\mu)$). *Sec. 7 extension:* $\mathcal{M}_2 = \{\alpha(\mu), u(\mu)\}$, where u is a dimensionless coefficient along an irrelevant direction (with the locked definition used in Sec. 7).
Π_{rec} **(reconstruction)**	Reconstruction is defined by the viewpoint objective (endpoint/along-chain fitting) of the form $F_\mu(g) = \mathrm{Agg}_{O \in O}\, D\big(P^{(O,\varepsilon)}_{T_\mu} \,\|\, P^{(O,\varepsilon)}_{\mathbf{M}(g)}\big)$, together with the declared optimization procedure (e.g. multi-start + local flatness tests) and any tie-breaking rule (default: none).
D **(divergence)**	Relative-entropy-type divergence applied to the protocol readouts (default: classical KL/relative entropy). *Ordering (locked):* we use $D(P\|Q)$ with the target/observed object on the left and the surrogate/reconstructed object on the right. Quantum-state divergences are mentioned only for context and are out of scope for the strict bridge statements.
Agg **(aggregation)**	Aggregation over interfaces as fixed by the viewpoint. In the Sec. 6 minimal implementation $O = \{O\}$, Agg reduces to the identity on that single interface; if multiple interfaces are used, Agg must be declared (e.g. mean/weighted/sup) and used consistently, noting that non-smooth sup aggregation can affect Route-II reportability.

Domain-gated reporting (status first). All outputs are accompanied by STATUS $\in$ $\{\mathrm{ADMISSIBLE}, \mathrm{MISMATCH}, \mathrm{ABYSS}, \mathrm{REC\text{-}CRITICAL}\}$. Here MISMATCH denotes a *proxy-level* inconsistency triggered by the reporting rule for numerical surrogates, and does not, by itself, assert a theorem-level statement about Γ_{path} absent an explicit calibration step. Numeric surrogates (e.g. $d_{\mathrm{end}}, \Gamma_{\mathrm{geo}}$) are reported only when the corresponding domain gates (e.g. Condition $\mathcal{G}_{\mathrm{bridge}}$, $\mathcal{G}_{\mathrm{geo}}$) permit them; otherwise the output is a label plus trigger metadata (status + flags are recorded rather than introducing additional labels).

3 Route I: Definition and Basic Properties of the Route-I Endpoint Divergence

Under the locked viewpoint V defined in Sec. 2, this section gives the definition of the Route I (endpoint-type) divergence quantity $\Delta_{\mathrm{I}}^{(\mathsf{V})}$ and summarizes its basic structural properties. Unless otherwise stated, we omit the superscript (V) in this section.

3.1 Endpoint objects: observable distributions/states at scale μ

Within the viewpoint V, the full effective description at scale μ is denoted by T_μ. Since we allow a truncated model class $\mathcal{M}$ and fix a reconstruction map Π_{rec}, we define the "comparable object" at scale μ as the truncated description

$$\tilde{T}_\mu := \Pi_{\mathrm{rec}}(T_\mu) \in \mathcal{M}. \tag{3.1}$$

The meaning of this convention is that we compare scale objects expressed under the same interface $\mathcal{O}$ and within the same model class $\mathcal{M}$, thereby avoiding mixing incomparable operator-tower details with comparable coordinate differences into a single scalar quantification (normative clause N4).

For any $O \in \mathcal{O}$, "truncated description + fixed observation protocol" induces a predictive object:

- If the interface corresponds to a classical observational distribution, we write

$$P_\mu^{(O)} \;\equiv\; P_{\tilde{T}_\mu}^{(O)}. \tag{3.2}$$

- If the interface corresponds to a comparison of quantum states (or reduced density matrices), we write

$$\rho_\mu^{(O)} \;\equiv\; \rho_{\tilde{T}_\mu}^{(O)}. \tag{3.3}$$

In what follows we uniformly denote by $X_\mu^{(O)}$ the "predictive object at scale μ under interface O" (a distribution or a state), and by $D(\cdot\|\cdot)$ the relative-entropy-type divergence fixed in Sec. 2 (e.g. classical KL divergence or quantum relative entropy).

3.2 Definition of Route I: Route-I endpoint divergence Δ_{I}

Given two scales μ_a, μ_b (typically $\mu_a = \mu_0, \mu_b = \mu_1$), the Route-I endpoint divergence is defined as the viewpoint-locked interface aggregation of endpoint discrepancies over the interface set $\mathcal{O}$:

$$\boxed{\Delta_{\mathrm{I}}(\mu_a \to \mu_b) := \operatorname*{Agg}_{O\in\mathcal{O}} D\Big(X_{\mu_a}^{(O)} \;\|\; X_{\mu_b}^{(O)}\Big)} \tag{3.4}$$

Terminology. We refer to Δ_{I} in (3.4) as the *Route-I endpoint divergence*. The strict endpoint–path bridge later uses the distinct endpoint term $\Delta_{\mathrm{pair}}(0 \to N)$ (the *endpoint-pair marginal divergence*; see Sec. 5 and (5.6)).

Aggregation rule. The aggregation operator Agg is treated as a fixed component of the viewpoint V and must remain unchanged across comparable runs (normative clauses N1 and

N4). Unless stated otherwise, we use the default choice $\mathrm{Agg} = \sup$, so that Δ_{I} is a worst-case (over interfaces) endpoint discrepancy.

Interpretation Δ_{I} compares predictive objects only at the endpoint scales and does not include historical information at intermediate scales, hence it is an endpoint-type divergence. In Sec. 5, Δ_{I} will be related structurally to path-type objects $\Delta_{\mathrm{path}}, \Gamma_{\mathrm{path}}$ and their computable proxies $L_{\mathrm{II}}, \Gamma_{\mathrm{geo}}$.

In Route I we have written $\Delta_{\mathrm{I}}(\mu_a \to \mu_b)$ as a divergence between endpoint objects along a single scale-indexed family. In the strict bridge of Sec. 5 the diagnostic task is stated for a comparison between two scale-chain distributions $\mathbb{P}_Z$ and $\mathbb{Q}_Z$; the corresponding endpoint quantity is the endpoint-pair marginal divergence $\Delta_{\mathrm{pair}}(0 \to N)$ defined in (5.6). When $\mathbb{Q}_Z$ is instantiated as an endpoint-only (or endpoint-matched) surrogate for $\mathbb{P}_Z$ under the same locked viewpoint, the Route-I endpoint divergence can be read as the specialization of that endpoint-pair divergence to the chosen construction.

3.3 Basic properties: non-negativity, the zero point, and viewpoint dependence

Proposition 3.1 (Non-negativity)**.** *For any pair of scales (μ_a, μ_b), one has*

$$\Delta_{\mathrm{I}}(\mu_a \to \mu_b) \geq 0. \tag{3.5}$$

Proof. By the non-negativity of the relative-entropy-type divergence $D(\cdot\|\cdot)$, for each $O \in \mathcal{O}$ we have $D\Big(X^{(O)}_{\mu_a} \| X^{(O)}_{\mu_b}\Big) \geq 0$; applying the aggregation rule Agg (assumed to map nonnegative inputs to a nonnegative scalar) preserves non-negativity. □

Proposition 3.2 (Viewpoint meaning of the zero point (faithful aggregation))**.** *Assume that the aggregation rule* Agg *is* faithful on nonnegative inputs*, in the sense that for any family $\{a_O\}_{O\in\mathcal{O}}$ with $a_O \geq 0$,*

$$\operatorname*{Agg}_{O\in\mathcal{O}} a_O = 0 \quad \Longrightarrow \quad a_O = 0 \ \ \textit{for all } O \in \mathcal{O}.$$

(This holds, for example, for the default choice $\mathrm{Agg} = \sup$ *and for weighted means with strictly positive weights.) If $\Delta_{\mathrm{I}}(\mu_a \to \mu_b) = 0$, then under the viewpoint V, for all $O \in \mathcal{O}$ one has*

$$D\Big(X^{(O)}_{\mu_a} \| X^{(O)}_{\mu_b}\Big) = 0, \tag{3.6}$$

i.e. the endpoints are indistinguishable in the sense of the interface $\mathcal{O}$.

Remark 3.1*.* This conclusion is viewpoint-relative: it only states indistinguishability under the fixed interface $\mathcal{O}$, fixed model class $\mathcal{M}$, and fixed divergence D, and does not imply indistinguishability under a finer interface or a richer model class. If D is taken to be the standard KL divergence or quantum relative entropy[10, 18–20], the zero point typically corresponds to (almost-everywhere) identical distributions or the same quantum state; however, the subsequent arguments in this paper only rely on the structural property of "uniqueness of the zero point."

Proposition 3.3 (Viewpoint dependence)**.** Δ_{I} *depends on all components of the viewpoint* V*. In particular:*

- *Changing the interface set $\mathcal{O}$ or the observation protocol changes the definition of $X_\mu^{(O)}$, and hence changes Δ_{I};*
- *Changing the regulator R, scheme S, or threshold rule changes the parameterization and predictive objects, and may change the densitization/differencing treatment;*
- *Changing the model class $\mathcal{M}$ or the reconstruction map Π_{rec} changes the endpoint objects $\tilde{T}_\mu$, and hence changes the "comparable part" of the divergence;*
- *Changing the divergence functional D changes the numerical value of the divergence and may change the range of applicability of structural properties such as monotonicity.*

Therefore, numerical comparisons of Δ_{I} are meaningful only within the same viewpoint V; cross-viewpoint comparisons must first specify a viewpoint mapping and transformation law (see Sec. 2, normative clause N6).

3.4 Route-I endpoint divergence and coarse-graining monotonicity (preparation for Sec. 5)

We record here a structural fact that will be used in Sec. 5. Within an admissible viewpoint, if scale advancement can be represented at the interface level as a class of coarse-graining maps (channels) Φ_k acting on predictive objects, i.e. for each $O \in \mathcal{O}$ there exists a map such that

$$X_{\mu_{k+1}}^{(O)} = \Phi_k\left(X_{\mu_k}^{(O)}\right), \tag{3.7}$$

and the divergence functional D satisfies monotonicity under coarse-graining (data processing inequality, DPI)[11, 21, 22],

$$D(X\|Y) \;\geq\; D(\Phi(X)\|\Phi(Y)), \tag{3.8}$$

then under the constraint of the "same scale-advancement mechanism" (Sec. 2, normative clause N3), the Route-I endpoint divergence has a compression property consistent with the scale direction.

We do not expand the concrete form here; Sec. 5 will use equation (3.8) to establish the upper-bound relation "the endpoint-pair marginal divergence does not exceed the full-trajectory divergence Δ_{path},"[4] and will introduce the gap term to quantify the historical distinguishable information lost under endpoint compression.

Summary Under a fixed viewpoint V, Route I rewrites theoretical differences between two endpoint scales as the worst-case interface-readable information difference Δ_{I}. The next section will induce an information metric via a local expansion of Δ_{I}, thereby obtaining Route II's information speed Ψ and path length L_{II}, and laying the foundation for the endpoint–path structural relations in Sec. 5.

[4]The path divergence Δ_{path} mentioned here is cited in advance as terminology; its strict definition will be given in Sec. 5.

4 Route I $\to$ Route II: Locally Induced Metric, Information Speed, and Path Length

Under the viewpoint V fixed in Sec. 2, this section localizes the Route-I endpoint divergence from Sec. 3 in the "small-step / neighboring-scale" limit, thereby inducing a local quadratic form on parameter space (a local version of an information metric, or more generally a local structure). On this basis we define the Route II information speed Ψ and the path length L_{II}. Unless otherwise stated, we continue to omit the superscript (V).

4.1 Parameterization and the local comparability window (Condition $\mathcal{G}_{\mathrm{bridge}}$)

Within the model class $\mathcal{M}$, we parameterize the truncated description $\tilde{T}_\mu = \mathbf{M}(g(\mu))$ by a finite-dimensional parameter vector $g = (g^1, \ldots, g^d) \in \mathbb{R}^d$. Accordingly, evolution along the scale axis γ can be written as a parameter trajectory

$$g(\gamma): \ [\gamma_0, \gamma_1] \to \mathbb{R}^d. \tag{4.1}$$

To ensure that the relative-entropy-type divergence in Route I admits a second-order expansion and induces a local quadratic form, we work in a *local comparability window.* For later reference, we label this window condition as "Condition $\mathcal{G}_{\mathrm{bridge}}$":

Domain gate for Route II (geometry-enabled regime). All Route-II objects (e.g. $\dot{g}(\gamma)$, the information speed Ψ, the length proxy L_{II}, and any subsequent geodesic diagnostics) are understood as *conditional* on the operational domain of the locked viewpoint. Concretely, for a fixed V_ε, we only attempt to construct Route II on windows that are V_ε*-admissible* in the sense of Sec. 2 (A1–A3), and that are *not* Π_{rec}*-critical* (i.e. the reconstruction map is single-valued on the window). If either admissibility fails or Π_{rec}-criticality occurs, we do not force a geometric surrogate; instead we report a label:

- **Abyss** (endpoint/path/geometry), when divergence finiteness or local expansion prerequisites fail under the locked protocol;
- Π_{rec}**-Critical**, when the minimizer set $\mathcal{A}(\mu)$ is non-singleton and the reconstructed trajectory is not single-valued.

These labels are treated as diagnostic outputs, not as numerical errors.

Condition $\mathcal{G}_{\mathrm{bridge}}$ as a domain certificate. The bridge condition introduced below (Condition $\mathcal{G}_{\mathrm{bridge}}$) is used as a *certificate* that, on a given window, Route-II local quadratic structure is operationally meaningful under V_ε. When Condition $\mathcal{G}_{\mathrm{bridge}}$ does not hold on a window (and no segmentation is declared), the report returns `STATUS=ABYSS` with `ABYSS_KIND=GEOMETRY` (gate: Condition $\mathcal{G}_{\mathrm{bridge}}$) for Route-II quantities, unless the window is explicitly segmented into subwindows on which the certificate holds.

Condition $\mathcal{G}_{\mathrm{bridge}}$ (local second-order window) Within a sufficiently small scale step $\gamma \to \gamma + \mathrm{d}\gamma$, the following hold:

1. Viewpoint elements are locked: the interface and protocol $\mathcal{O}$ are fixed, and the regulator R and scheme S are fixed (normative clauses N1–N2);

2. The threshold rule is not triggered within the step (or a fixed segmentation rule has been adopted so that the "active degrees of freedom" remain consistent within the step);

3. Local regularity: for each $O \in \mathcal{O}$, the predictive object $X^{(O)}(g)$ varies sufficiently smoothly with g within the window, so that the relative-entropy-type divergence admits a second-order approximation under parameter perturbations;

4. Aggregation regularity (if Agg aggregates over multiple interfaces): the second-order remainder in the local expansion is controlled uniformly over the interfaces being aggregated (in particular, when $\mathcal{O}$ is finite or reduced to a finite representative set), so that aggregation preserves the $o(\|\mathrm{d}g\|^2)$ order;

5. Truncation consistency: the model class $\mathcal{M}$ and reconstruction map Π_{rec} remain unchanged within the window (normative clause N4).

Condition $\mathcal{G}_{\mathrm{bridge}}$ is the technical premise under which the Route I $\to$ Route II bridge holds; its failure typically corresponds to threshold crossing, strongly nonlinear segments, or situations where the model-class truncation is insufficient to stably capture interface-sensitive directions.

4.2 Local expansion of Route I and the induced quadratic form (local form of an information metric)

For any fixed interface $O \in \mathcal{O}$, consider the predictive object $X^{(O)}(g)$ and a parameter perturbation $g \mapsto g + \mathrm{d}g$. Within the window of Condition $\mathcal{G}_{\mathrm{bridge}}$, the relative-entropy-type divergence admits the second-order expansion

$$D\Big(X^{(O)}(g) \,\|\, X^{(O)}(g+\mathrm{d}g)\Big) = \frac{1}{2}\, G^{(O)}_{ij}(g)\,\mathrm{d}g^i\,\mathrm{d}g^j \;+\; o\big(\|\mathrm{d}g\|^2\big)\,, \tag{4.2}$$

where the coefficient tensor $G^{(O)}_{ij}(g)$ is the local quadratic form induced by the second-order term of the relative-entropy expansion under that interface (the classical case corresponds to the Fisher information matrix; the quantum case corresponds to a local information quadratic form induced by relative entropy) [6–8, 11].

Interface-aggregated induced local quadratic form (default: worst-case) Fixing the same aggregation rule Agg as in the definition of Δ_{I} (Sec. 3), we define, under the viewpoint, the local quadratic form

$$Q_g(\mathrm{d}g) := \operatorname*{Agg}_{O\in\mathcal{O}}\; \mathrm{d}g^{\top} G^{(O)}(g)\,\mathrm{d}g. \tag{4.3}$$

It characterizes the interface-aggregated local sensitivity over the admissible interface set $\mathcal{O}$; under the default choice Agg $=$ sup, it reduces to the worst-case local sensitivity.

Remark 4.1 (On Agg $=$ sup and smoothness). Taking Agg $=$ sup over an interface family may cause $Q_g(\cdot)$ to exhibit a piecewise structure in $\mathrm{d}g$, and hence it need not be a smooth quadratic form in general. At the v0 level we treat Q_g as a local structure induced by interface aggregation: in concrete applications one may (i) choose finitely many representative interfaces and take the maximum, or (ii) first fix a set of sufficient-statistic interfaces so that Q_g becomes piecewise smooth. These implementations do not change the definition chain of Route II; their differences belong to the viewpoint-implementation level and should be reported for reproducibility (normative clauses N1 and N4).

From equations (4.2) and (4.3), within the window of Condition $\mathcal{G}_{\mathrm{bridge}}$, the small-step Route I divergence satisfies

$$\Delta_{\mathrm{I}}(\gamma \to \gamma + \mathrm{d}\gamma) = \operatorname*{Agg}_{O\in\mathcal{O}} D\Big(X^{(O)}(\gamma) \| X^{(O)}(\gamma + \mathrm{d}\gamma)\Big) = \frac{1}{2}\, Q_{g(\gamma)}(\mathrm{d}g) + \ o\big(\|\mathrm{d}g\|^2\big)\,, \tag{4.4}$$

where $\mathrm{d}g = g(\gamma + \mathrm{d}\gamma) - g(\gamma)$.

4.3 RG flow and the information speed Ψ

Under a fixed scheme and threshold rule, the parameter trajectory satisfies an RG-type evolution equation

$$\frac{\mathrm{d}g^i}{\mathrm{d}\gamma} = \beta^i(g), \tag{4.5}$$

where $\beta(g)$ is the beta function (or more generally the scale-flow vector field) in the coordinates of the model class. Thus within a small step one has $\mathrm{d}g = \beta(g(\gamma))\,\mathrm{d}\gamma$.

Information line element and speed Using equation (4.3), define the local line element induced by Q_g (more precisely: the local “line element squared”)

$$\mathrm{d}s^2 := Q_{g(\gamma)}(\mathrm{d}g), \qquad \mathrm{d}s := \sqrt{Q_{g(\gamma)}(\mathrm{d}g)}. \tag{4.6}$$

Substituting $\mathrm{d}g = \beta\,\mathrm{d}\gamma$ into equation (4.6) yields the information speed

$$\boxed{\Psi(\gamma) := \frac{\mathrm{d}s}{\mathrm{d}\gamma} = \sqrt{Q_{g(\gamma)}\big(\beta(g(\gamma))\big)} = \sqrt{\operatorname*{Agg}_{O\in\mathcal{O}}\ \beta^i(g(\gamma))\, G^{(O)}_{ij}(g(\gamma))\, \beta^j(g(\gamma))}.} \tag{4.7}$$

Remark 4.2 (Small-step relation to Δ_{I}). From equations (4.4) and (4.6), within the window of Condition $\mathcal{G}_{\mathrm{bridge}}$,

$$\Delta_{\mathrm{I}}(\gamma \to \gamma + \mathrm{d}\gamma) = \tfrac{1}{2}\,\mathrm{d}s^2 + o(\mathrm{d}\gamma^2),$$

i.e. the Route-I endpoint divergence provides the second-order coefficient of the local line element squared in the small-step limit; Route II then defines the path length by a first-order accumulation of $\mathrm{d}s$.

4.4 Route II path length L_{II}, distance dist_G, and calculus relations

In the continuous-scale limit, define the Route II path length (information-geometric path length) along the RG trajectory by

$$\boxed{L_{\mathrm{II}}(\gamma_0 \to \gamma_1) := \int_{\gamma_0}^{\gamma_1} \Psi(\gamma)\,\mathrm{d}\gamma = \int_{\gamma_0}^{\gamma_1} \sqrt{Q_{g(\gamma)}\big(\beta(g(\gamma))\big)}\ \mathrm{d}\gamma.} \tag{4.8}$$

Interpretation. L_{II} is an *information-geometric path length* induced by the local quadratic structure (via Ψ); it is not a KL-type divergence. The strict KL-type *full-trajectory divergence* is Δ_{path}.

Equation (4.8) immediately yields the "calculus relation" of Route II:

$$\frac{\mathrm{d}}{\mathrm{d}\gamma} L_{\mathrm{II}}(\gamma_0 \to \gamma) = \Psi(\gamma), \tag{4.9}$$

where $L_{\mathrm{II}}(\gamma_0 \to \gamma)$ is viewed as a cumulative length function with upper limit γ.

Endpoint distance induced by Q_g (for later use of d_{end}) *Notation.* We denote this endpoint distance by d_{end} ("end" for endpoint). To compare endpoints and full trajectories under the same induced structure, we also introduce the endpoint distance dist_G induced by Q_g: for any piecewise C^1 curve $c : [0,1] \to \mathbb{R}^d$, define its length functional

$$L_G(c) := \int_0^1 \sqrt{Q_{c(t)}\big(\dot{c}(t)\big)}\,\mathrm{d}t. \tag{4.10}$$

For two endpoints g_a, g_b, define the distance

$$\boxed{\mathrm{dist}_G(g_a, g_b) := \inf_{c(0)=g_a,\, c(1)=g_b} L_G(c).} \tag{4.11}$$

Interpretation note. dist_G is the global geodesic distance (infimum over curves), not the length of an observed discrete trajectory segment.
Gate note. The Riemannian/geodesic interpretation of dist_G (and the formation/reporting of Γ_{geo}) is invoked only when Condition $\mathcal{G}_{\mathrm{geo}}$ holds (Sec. 8.3); otherwise we treat dist_G purely via the infimum definition (4.11) and do not report Γ_{geo}. When $Q_g(\mathrm{d}g)$ is given in $\mathrm{d}g$ by a smooth positive-definite bilinear form, dist_G reduces to the geodesic distance under the corresponding Riemannian metric; in general dist_G is defined as the endpoint distance induced by Q_g [8, 9].

In Sec. 5 we will use

$$d_{\mathrm{end}} := \mathrm{dist}_G\big(g(\gamma_0), g(\gamma_1)\big)$$

as an endpoint-equivalent length, and within the local window of Condition $\mathcal{G}_{\mathrm{bridge}}$ provide an operational approximation relating it to $\sqrt{2\Delta_{\mathrm{I}}}$.

Discrete-scale approximation If a discrete scale grid $\{\gamma_k\}$ is used, one may use the Riemann-sum approximation

$$L_{\mathrm{II}}(\gamma_0 \to \gamma_N) \approx \sum_{k=0}^{N-1} \Psi(\gamma_k)\,\Delta\gamma_k, \qquad \Psi(\gamma_k) = \sqrt{Q_{g(\gamma_k)}\big(\beta(g(\gamma_k))\big)}. \tag{4.12}$$

4.5 Scope and failure modes (preparation for interpreting the gap term Γ)

The bridge in this section relies on Condition $\mathcal{G}_{\mathrm{bridge}}$ (local second-order window) and viewpoint locking (Sec. 2, clauses N1–N4), and hence is mainly applicable to:

- threshold-free windows, or windows where threshold handling has been fixed and segmentation is explicit;

- truncations whose model class is sufficient to stably capture the main sensitivity directions of the interface $\mathcal{O}$;
- local variations that do not invalidate the second-order approximation.

In contrast, when the following situations occur, they should be treated as outside the window of Condition $\mathcal{G}_{\text{bridge}}$ or reorganized according to segmentation rules:

1. the model class is too narrow, so that directions generated at intermediate scales are strongly projected out, making the local structure unstable within the window;
2. the effective sensitivity directions of the interface family $\mathcal{O}$ switch across different scale segments, making Q_g induced by sup strongly piecewise;
3. threshold crossing changes the content of active degrees of freedom, so that β and the local quadratic form must be defined piecewise under the prescribed threshold rule.

These situations will be incorporated in Sec. 5 into the structural meaning of the gap term (Γ_{path} and its computable proxy Γ_{geo}): the gap term quantifies the difference between "endpoint compression" and "full-trajectory accumulation," i.e. the distinguishable information in intermediate-scale history discarded by an endpoint description.

Summary This section induces a local quadratic form Q_g from a local relative-entropy expansion of Route I, then defines the information speed Ψ via the RG vector field β, and obtains the Route II path length L_{II} by integrating along the scale axis. We also introduce the endpoint distance dist_G induced by Q_g, providing a unified "length scale" for later endpoint–path matching and for defining the geometric gap Γ_{geo}. The next section will build on this to give the endpoint–path structural inequality and gap decomposition, and interpret the testable meaning of the gap term.

5 Consistency Bridge: Endpoint–Path Inequalities and the Gap Term Γ

Under a fixed viewpoint V, this section places the Route-I endpoint divergence Δ_{I} from Sec. 3 and the path length L_{II} from Sec. 4 within a single structural framework, derives controlled endpoint–path inequalities (rather than presupposing equality), and introduces gap terms (collectively denoted by Γ) to quantify the cross-scale historical differences compressed away by an "endpoint-only" description relative to a "full scale-evolution" description. Unless otherwise stated, we continue to omit the superscript (V).

5.1 Path objects: scale chains and path distributions (or process states)

To capture the "full trajectory" rather than only endpoints, we adopt a discrete scale grid

$$\mu_0 < \mu_1 < \cdots < \mu_N,$$

and view scale evolution as a path process. Under a fixed viewpoint V, for each scale μ_k and for the comparable object induced by a chosen interface implementation (including observation protocol, regulator/scheme, threshold rule, model class, and reconstruction), Sec. 3 yields a predictive object at that scale point (a distribution in the classical case, a state in the quantum case). For the strict path construction, let Y_k denote the corresponding protocol readout random variable (or its sufficient statistic) at scale μ_k, whose law is induced by that predictive object.

To place the endpoint-pair marginal divergence and the full-trajectory divergence into the same relative-entropy framework, we introduce the path variable

$$Z := (Y_0, Y_1, \dots, Y_N). \tag{5.1}$$

Note: this section gives the formal definition of the path divergence Δ_{path} anticipated at the end of Sec. 3, and establishes its structural correspondence with the endpoint-pair marginal term Δ_{pair} *(with implementation-level alignment to the Route-I endpoint divergence Δ_{I} only under the explicit endpoint choice $E = (Y_0, Y_N)$; see remark 5.2).*

For two scale flows being compared (e.g. two sets of initial conditions, two RG trajectories, or two reconstruction sequences), under the same scale-advancement mechanism (Sec. 2, N3) and the same interface implementation they induce two path-process objects

$$\mathbb{P}_Z, \qquad \mathbb{Q}_Z. \tag{5.2}$$

In the diagnostic use case emphasized in this paper, $\mathbb{P}_Z$ may represent the scale-chain induced by a less-truncated (or higher-dimensional) description, while $\mathbb{Q}_Z$ represents an endpoint-matched but more aggressively coarse-grained/truncated surrogate under the same viewpoint. We do not require writing all details of $\mathbb{P}_Z, \mathbb{Q}_Z$ explicitly; we only require that, under the fixed viewpoint, the joint object over the scale chain can be treated as a joint distribution (classical case) or a process state (quantum case) for which relative entropy is defined. For multiple interfaces, there are two admissible implementation choices: either take Y_k to be the joint object of the interface-family readouts (convenient for a strict "mother object"; the two choices coincide only when the joint construction is explicitly matched to the declared aggregation rule), or construct a separate Z for each representative interface and report the interface choice for reproducibility (see N1, N4).

5.2 The strict mother object for Route II: path divergence Δ_{path}

On the above path objects, define the strict version of the "full-trajectory divergence" by

$$\boxed{\Delta_{\text{path}}(0 \to N) := D(\mathbb{P}_Z \parallel \mathbb{Q}_Z)\,.} \tag{5.3}$$

Whenever we invoke the endpoint–path chain decomposition in the main text (see Appendix A), we take D to be the classical KL divergence or quantum relative entropy, or any divergence admitting the same chain rule. It measures the distinguishable information difference, under the viewpoint V, between two scale flows viewed as "full process objects." This quantity is the strict mother object for the endpoint–path structural relations; its subsequent decomposition follows directly from the chain rule for relative entropy [10, 11].

Remark 5.1 (Strict object and computable proxy). The L_{II} in Sec. 4 is an information-geometric length obtained by integrating along the scale axis. Within Condition $\mathcal{G}_{\text{bridge}}$ (local second-order window), it can be used as a computable proxy of Δ_{path} insofar as the scale-chain process admits a stepwise localization compatible with the second-order expansion (so that local KL elements can be accumulated into an effective length). In general, this paper does *not* claim an identity $L_{\text{II}} = \Delta_{\text{path}}$; the strict semantics remain anchored by the chain-rule object Γ_{path} in equation (5.10), while L_{II} and Γ_{geo} serve as operational proxies for reproducible diagnosis. This section provides both the strict endpoint–path inequality (based on Δ_{path}) and the operational matching with L_{II} (for practical calibration).

5.3 Endpoint-pair marginal divergence as a coarse-graining of path divergence: an endpoint upper bound (DPI)

Let the endpoint map (coarse-graining) Π_{end} retain only the two ends of the path variable:

$$\Pi_{\text{end}}(Z) = (Y_0, Y_N). \tag{5.4}$$

Then the endpoint marginals satisfy

$$\Pi_{\text{end}}(\mathbb{P}_Z) = \mathbb{P}_{Y_0,Y_N}, \qquad \Pi_{\text{end}}(\mathbb{Q}_Z) = \mathbb{Q}_{Y_0,Y_N}.$$

If the divergence D satisfies monotonicity under coarse-graining (data processing inequality, DPI), then

$$D(\mathbb{P}_Z \| \mathbb{Q}_Z) \;\geq\; D(\mathbb{P}_{Y_0,Y_N} \| \mathbb{Q}_{Y_0,Y_N}). \tag{5.5}$$

Alignment with the Route-I endpoint divergence (implementation level) Under a fixed interface implementation/aggregation rule, the strict endpoint quantity induced by the path objects is the *endpoint-pair marginal divergence* $\Delta_{\text{pair}}(0 \to N)$, defined by

$$\boxed{\Delta_{\text{pair}}(0 \to N) \;:=\; D(\mathbb{P}_{Y_0,Y_N} \| \mathbb{Q}_{Y_0,Y_N}).} \tag{5.6}$$

In the single-interface instantiation, if (as an explicit implementation choice) Route I takes the endpoint object to be the endpoint pair $E := (Y_0, Y_N)$, then the Route-I endpoint divergence coincides with Δ_{pair} as defined in equation (5.6). Otherwise, Δ_{I} remains a distinct Route-I quantity (see Remark efrem:endpoint-term-lock for the strict-bridge endpoint terminology). If Route I aggregates over multiple interfaces (e.g. via a sup over $\mathcal{O}$), then Δ_{I} is obtained by applying the same aggregation rule to the corresponding endpoint-pair divergences; in that case, equation (5.5) applies to each interface projection separately, yielding the corresponding endpoint upper bounds. This belongs to the viewpoint-implementation level and should be reported for reproducibility (see N1, N4).

Therefore, from equations (5.5) and (5.6) we obtain the endpoint–path upper-bound relation:

$$\boxed{\Delta_{\mathrm{pair}}(0 \to N) \;\le\; \Delta_{\mathrm{path}}(0 \to N).} \tag{5.7}$$

Remark 5.2 (Terminology lock (endpoint term in the strict bridge)). Throughout Sec. 5 (and in all inequalities/identities below derived from DPI and the chain rule, such as (5.7) and (5.10)), the endpoint term always means the *endpoint-pair marginal divergence* $\Delta_{\mathrm{pair}}(0 \to N)$ defined in (5.6). It is *not* the Route-I endpoint divergence $\Delta_{\mathrm{I}}(\mu_a \to \mu_b)$ defined in (3.4) in Sec. 3.

Equation (5.7) expresses the basic endpoint–path inequality: retaining only endpoints does not increase distinguishable information difference; the endpoint-pair marginal divergence is at most the full-trajectory divergence [11, 21, 22].

5.4 Gap decomposition: $\Delta_{\mathrm{path}} = \Delta_{\mathrm{pair}} + \Gamma_{\mathrm{path}}$

Decompose the path variable into endpoints

$$E := (Y_0, Y_N)$$

and history

$$\mathcal{H} := (Y_1, \dots, Y_{N-1}).$$

The chain-rule decomposition of relative entropy gives

$$D(\mathbb{P}_{E,\mathcal{H}} \| \mathbb{Q}_{E,\mathcal{H}}) = D(\mathbb{P}_E \| \mathbb{Q}_E) + \mathbb{E}\big[\, D(\mathbb{P}_{\mathcal{H}|E} \| \mathbb{Q}_{\mathcal{H}|E}) \,\big]_{\mathbb{P}_E}. \tag{5.8}$$

The endpoint term is exactly the endpoint-pair marginal divergence in equation (5.6),

$$\Delta_{\mathrm{pair}}(0 \to N) := D(\mathbb{P}_E \| \mathbb{Q}_E).$$

Accordingly, define the path gap term

$$\boxed{\Gamma_{\mathrm{path}}(0 \to N) := \mathbb{E}\big[\, D(\mathbb{P}_{\mathcal{H}|E} \| \mathbb{Q}_{\mathcal{H}|E}) \,\big]_{\mathbb{P}_E} \;\ge\; 0,} \tag{5.9}$$

and obtain the exact identity

$$\boxed{\Delta_{\mathrm{path}}(0 \to N) = \Delta_{\mathrm{pair}}(0 \to N) + \Gamma_{\mathrm{path}}(0 \to N).} \tag{5.10}$$

Type note. $\Gamma_{\mathrm{path}}(0 \to N) \ge 0$ is the strict chain-rule residual; $\Gamma_{\mathrm{path}}(0 \to N) = 0$ is the sharp endpoint-sufficiency criterion within the locked viewpoint.

Equation (5.10) gives the structural core of the endpoint–path relation: the difference between endpoint and path divergences is not an "error," but a strictly nonnegative structural term Γ_{path} with a clear conditional meaning; it measures the "historical distinguishable information" compressed away by the coarse-graining that retains only the endpoints $E = (Y_0, Y_N)$.

Remark 5.3 (Notation (alignment of gap terminology)). We use Γ as a collective name for the *endpoint–trajectory discrepancy* that remains after compressing a scale chain to its endpoints. At the strict level, this discrepancy is quantified by the chain-rule residual Γ_{path} in (5.10), where $\Gamma_{\text{path}} \geq 0$ and $\Gamma_{\text{path}} = 0$ is the sharp criterion for endpoint sufficiency under the locked viewpoint. At the computable level, we use the geometric proxy residual $\Gamma_{\text{geo}} := L_{\text{II}} - d_{\text{end}}$ (Sec. 8.3), which is meaningful only on the Route-II domain gates and is used for operational diagnosis. The paper does *not* claim an unconditional equivalence between Γ_{geo} and Γ_{path}; any correspondence requires additional assumptions and should be treated as calibration rather than a theorem.

Equality condition (structural statement) From equation (5.9), $\Gamma_{\text{path}} = 0$ holds if and only if $D(\mathbb{P}_{\mathcal{H}|E} \| \mathbb{Q}_{\mathcal{H}|E}) = 0$ holds $\mathbb{P}_E$-almost surely, i.e. the two path processes have indistinguishable conditional history distributions given the endpoints $E = (Y_0, Y_N)$. Equivalently, under the viewpoint V, the endpoints E are sufficient statistics for distinguishing the two scale flows: the history provides no additional distinguishing information [10, 22].

5.5 Matching with Route II: the geometric gap Γ_{geo}

The Γ_{path} defined above is a strict object, but in generic field-theoretic problems it is often difficult to compute directly.
Proxy note. Γ_{geo} is an operational proxy (not the strict residual); any inference from Γ_{geo} to Γ_{path} is calibration-dependent, not a theorem.

Sec. 4 provides a computable proxy for Route II: the information speed $\Psi(\gamma)$ and the path length L_{II}.

To place the Route-I endpoint divergence Δ_{I} and the path length L_{II} on the same "length scale," we adopt the endpoint information distance induced by Route II (Sec. 4, equation (4.11)) to define an endpoint-equivalent length:

$$\boxed{d_{\text{end}}(\mu_0 \to \mu_1) := \text{dist}_G\big(g(\gamma_0),\, g(\gamma_1)\big).} \tag{5.11}$$

Here dist_G is the endpoint distance induced by Q_g; when Q_g is given by a smooth positive-definite bilinear form, it reduces to the corresponding geodesic distance.
Notation. We denote this endpoint distance by d_{end} ("end" for endpoint).

Local equivalence (relation to $\sqrt{2\Delta_{\text{I}}}$) Within the local second-order window of Condition $\mathcal{G}_{\text{bridge}}$, if the divergence satisfies the local relation

$$D \simeq \frac{1}{2}\,\mathrm{d}s^2,$$

then for sufficiently close endpoints one may use the operational approximation

$$d_{\text{end}}(\mu_0 \to \mu_1) \;\simeq\; \sqrt{2\,\Delta_{\text{I}}(\mu_0 \to \mu_1)}. \tag{5.12}$$

When an explicit computable expression is needed, we use equation (5.12) as an approximation within the local window of Condition $\mathcal{G}_{\text{bridge}}$. In special constant-metric implementations where a closed-form identity is available under the declared workflow (see Sec. 8.3 for the QED/EFT instantiation), one may treat equation (5.12) as an identity on that declared in-domain setting. Outside such declared special cases, we default to dist_G in equation (5.11) to avoid misreading the local equivalence as a global identity.

Accordingly, define the geometric gap

$$\boxed{\Gamma_{\text{geo}}(\mu_0 \to \mu_1) := L_{\text{II}}(\mu_0 \to \mu_1) - d_{\text{end}}(\mu_0 \to \mu_1).} \tag{5.13}$$

Interpretation L_{II} is the accumulated length along the scale axis (full trajectory), while d_{end} is the endpoint distance under the same induced structure (endpoints); thus Γ_{geo} measures the "extra length of the trajectory relative to an endpoint-equivalent connection," and is used to diagnose whether intermediate-scale structure is compressible into endpoints at a given resolution.

Remark 5.4 (On nonnegativity and proxy character). If within the window of Condition $\mathcal{G}_{\text{bridge}}$, L_{II} is defined as the curve length induced by Q_g, and d_{end} is taken as the corresponding endpoint distance dist_G, then by "curve length is no smaller than endpoint distance" one has

$$\Gamma_{\text{geo}} \geq 0.$$

Outside the window of Condition $\mathcal{G}_{\text{bridge}}$, or when using the local approximation equation (5.12), Γ_{geo} is treated in this paper as a computable diagnostic proxy: its vanishing typically corresponds to a one-dimensional sufficient-statistic case or a geodesic case, while a significant deviation from zero indicates that endpoint compression discards non-negligible intermediate-scale structure. The strict structure remains anchored by the chain-rule object Γ_{path} in equation (5.10).

Theorem 5.1 (Reparameterization invariance of Γ_{geo})**.** *Fix a viewpoint* V *and assume Condition* $\mathcal{G}_{\text{geo}}$ *holds (Sec. 8.3), namely: Route II yields a* C^1 *trajectory* $\gamma \mapsto g(\gamma)$ *on a parameter manifold equipped with a single positive-definite metric field* $G(g)$*, so that* L_{II} *is the* G*-arc length and* $d_{\text{end}} = \text{dist}_G$ *is the induced endpoint distance. Then the geometric proxy gap* $\Gamma_{\text{geo}} := L_{\text{II}} - d_{\text{end}}$ *is invariant under:*

1. *any smooth change of coupling coordinates* $\tilde{g} = \phi(g)$ *with the covariant metric transformation* $\tilde{G}(\tilde{g}) = \left(J^{-1}\right)^{\top} G(g)\left(J^{-1}\right)$*, where* $J = \partial\tilde{g}/\partial g$*;*
2. *any monotone reparameterization of the scale variable* $\tilde{\gamma} = \tilde{\gamma}(\gamma)$*.*

Proof. Both L_{II} (arc length) and $d_{\text{end}} = \text{dist}_G$ (the induced geodesic distance) are invariants of the underlying Riemannian structure $(\mathcal{M}, G)$, independent of coordinate charts and curve parameterizations. Therefore their difference Γ_{geo} is invariant as well. □

Theorem 5.2 (Zero-gap characterization)**.** *Under the hypotheses of Theorem 5.1, one has* $\Gamma_{\text{geo}} \geq 0$*. Moreover,* $\Gamma_{\text{geo}} = 0$ *holds if and only if the trajectory segment* $g([\gamma_0, \gamma_1])$ *realizes the endpoint distance* d_{end}*, i.e. it is a minimizing geodesic between the endpoints (up to monotone reparameterization).*

Proof. For any rectifiable curve, its length is at least the induced distance between its endpoints; this gives $\Gamma_{\rm geo} \geq 0$. Equality holds exactly when the curve is distance-realizing, i.e. when it is a minimizing geodesic segment (up to reparameterization). $\square$

5.6 Testable statements: criteria for negligible/non-negligible Γ

We summarize the meaning of the gap term as testable statements (all stated under a fixed viewpoint V and a given resolution):

- If under the given resolution and model-class truncation the gap term can be treated as negligible (e.g. $\Gamma_{\rm path} \approx_\tau 0$; or, on the declared $\mathcal{G}_{\rm geo}$ domain and only as a calibrated proxy, $\Gamma_{\rm geo} \approx_\tau 0$), then the endpoint-pair marginal divergence $\Delta_{\rm pair}$ is sufficient to represent the full-trajectory divergence $\Delta_{\rm path}$: intermediate-scale history provides no additional distinguishable information, and the system is endpoint-compressible under the viewpoint.

- If the gap term is significantly greater than zero (e.g. $\Gamma_{\rm path} > 0$ or $\Gamma_{\rm geo} > 0$ with a non-negligible deviation), then the endpoint-pair marginal divergence $\Delta_{\rm pair}$ systematically underestimates the full-trajectory divergence $\Delta_{\rm path}$: intermediate scales contribute non-negligible structural content. Common sources include (i) insufficient model-class truncation (directions are projected out), (ii) multi-direction interface sensitivity that cannot be stably carried by the given parameter dimension, or (iii) threshold crossing / piecewise changes of active degrees of freedom that modify the process structure in segments.

In Sec. 6 (one-loop, threshold-free QED window), we will exhibit a baseline case with $\Gamma_{\rm geo} = 0$; in Sec. 7 (EFT-extended two-dimensional truncation), we will present a nontrivial example where $\Gamma_{\rm geo} > 0$ typically appears after an irrelevant direction enters, thereby advancing the endpoint–path gap from a concept to a reproducible structural output.

Summary This section provides two layers of endpoint–path relations: at the strict level, $\Delta_{\rm pair} \leq \Delta_{\rm path}$ (DPI), and $\Delta_{\rm path} = \Delta_{\rm pair} + \Gamma_{\rm path}$ (chain-rule decomposition, $\Gamma_{\rm path} \geq 0$); at the computable level, when the local bridge under Condition $\mathcal{G}_{\rm bridge}$ is valid, the endpoint quantity can be converted into an equivalent length $d_{\rm end}$, and the geometric gap $\Gamma_{\rm geo} = L_{\rm II} - d_{\rm end}$ is defined to diagnose whether "intermediate-scale structure is compressible into endpoints." These structural relations serve as the common theoretical basis for the subsequent one-loop QED calibration and the EFT examples.

6 One-Loop QED Matching Test: Baseline Behavior of $\Gamma_{\rm geo}$ in a Threshold-Free Window

This section selects a threshold-free window (the active set of degrees of freedom is unchanged; regulator/scheme are fixed), and in the one-loop QED approximation instantiates the objects from Secs. 3–5 into reproducible calculations: the Route-I endpoint divergence $\Delta_{\rm I}$, the information speed Ψ, the path length $L_{\rm II}$, and the computable proxy of the gap term $\Gamma_{\rm geo}$ (with the relation to the strict gap $\Gamma_{\rm path}$ stated when needed). The goal is to provide a baseline case: under a one-dimensional truncation and a stable interface, the gap can be treated as negligible at the given resolution, so that endpoints and the full trajectory are consistent within this window.

6.1 Physical window and truncation: single-coupling, threshold-free QED effective description

Take the scale coordinate

$$\gamma = \ln(\mu/\mu_\star), \tag{6.1}$$

where $\mu_\star$ is the reference scale for this section (it may be chosen equal to the reference scale in Sec. 2; endpoint differences do not depend on this choice).

Choose an energy window

$$m_{\rm low} \ll \mu \ll m_{\rm high}, \tag{6.2}$$

so that within this window the set of charged species counted as active remains unchanged (no new mass threshold is crossed), and the RG equation does not switch piecewise on this interval. This is equivalent to fixing the "threshold/active set" as a clause of the viewpoint V: this section only considers a single-segment window under that clause.

Within this window, adopt the minimal truncated model class

$$\mathcal{M}_1 = \{\text{keep only one effective parameter } \alpha(\mu) \text{ or an equivalent coordinate } \theta(\mu)\}, \qquad \theta := \alpha^{-1}. \tag{6.3}$$

The purpose of introducing θ is to write the one-loop QED running as a linear flow (see the next subsection), thereby constraining the relation between "path geometry" and "endpoint difference" to the minimal number of degrees of freedom.

6.2 One-loop QED running: $\beta(\alpha)$ and the endpoint linear law

The one-loop QED β-function can be written as (with $\alpha = e^2/(4\pi)$) [12–15]

$$\boxed{\frac{\mathrm{d}\alpha}{\mathrm{d}\ln\mu} = \beta(\alpha) = b\,\alpha^2, \qquad b = \frac{2}{3\pi}\sum_{f\in\text{active}} n_f\,Q_f^2 \ > 0,} \tag{6.4}$$

where the sum runs over the charged Dirac fermions treated as active in this window; Q_f is the charge number, and n_f is a degeneracy factor (e.g. a color factor, if applicable, absorbed into n_f). This section uses only two structural facts: b is constant within a threshold-free window, and $\beta(\alpha) > 0$.

Changing coordinates to $\theta = \alpha^{-1}$, we have

$$\frac{\mathrm{d}\theta}{\mathrm{d}\ln\mu} = \frac{\mathrm{d}(\alpha^{-1})}{\mathrm{d}\ln\mu} = -b. \tag{6.5}$$

Hence in a threshold-free window we obtain the endpoint linear law

$$\boxed{\theta(\mu_1) - \theta(\mu_0) = -b\ln\frac{\mu_1}{\mu_0}.} \tag{6.6}$$

This shows that in the θ coordinate, the RG flow is a constant-velocity straight line. (The formal Landau pole occurs at scales far above the window considered here and does not enter the present threshold-free one-loop test.)

6.3 A minimal implementation of the viewpoint V: single interface, statistical object, and induced structure

To instantiate the objects in Secs. 3–5 into closed, computable form, this section fixes a "minimal interface implementation" by taking the interface set to be a single interface,

$$\mathcal{O} = \{O\}, \tag{6.7}$$

so as to avoid the non-smooth technical burden from multi-interface aggregation (e.g. $\sup_{O\in\mathcal{O}}$).

Let the interface output an estimator Y for θ. At a fixed resolution, we model its conditional distribution by a local Gaussian model

$$P_\theta(y) = \mathcal{N}(\theta, \sigma^2). \tag{6.8}$$

Here σ makes the "given protocol and resolution" explicit within the viewpoint V: σ is fixed by the protocol (statistics, sample size, systematic error budget, etc.), and represents the "given resolution" in this section.

The endpoint KL divergence for (6.8) is[11]

$$D(P_{\theta_0}\|P_{\theta_1}) = \frac{(\theta_0-\theta_1)^2}{2\sigma^2}. \tag{6.9}$$

Its local second-order term yields the Fisher information (local quadratic form/metric) [6, 7]

$$G(\theta) = \frac{1}{\sigma^2} \qquad \text{(constant)}. \tag{6.10}$$

Therefore this section satisfies Condition $\mathcal{G}_{\mathrm{bridge}}$ (local second-order window) from Sec. 4, and in this model it may be treated as holding everywhere within the window.

Remark 6.1. If one instead uses a concrete physical process as the interface (e.g. a scattering cross section or angular distribution at fixed momentum transfer), then σ and G are determined by the shape of the cross section and statistical errors, and may vary with scale. This section uses (6.8) as a normalized baseline interface; its role is to realize the meaning of the proxy gap Γ_{geo} in a reproducible closed calculation without introducing additional degrees of freedom.

6.4 Geometric proxy endpoint–path consistency ($\Gamma_{\rm geo}$): explicit calculations of $\Delta_{\rm I}, \Psi, L_{\rm II}, d_{\rm end}, \Gamma_{\rm geo}$

(i) Route I: Route-I endpoint divergence $\Delta_{\rm I}$ From the definition in Sec. 3 (no sup is needed in the single-interface case) and (6.9), we obtain

$$\boxed{\Delta_{\rm I}(\mu_0 \to \mu_1) = \frac{(\theta_0 - \theta_1)^2}{2\sigma^2}.} \tag{6.11}$$

(ii) Route II: information speed Ψ and path length $L_{\rm II}$ By the definition chain in Sec. 4, in the one-dimensional, constant-quadratic-form case,

$$\Psi(\gamma) = \sqrt{G(\theta)}\left|\frac{\mathrm{d}\theta}{\mathrm{d}\gamma}\right| = \frac{1}{\sigma}|-b| = \frac{b}{\sigma} \qquad \text{(constant)}. \tag{6.12}$$

Hence

$$\boxed{L_{\rm II}(\mu_0 \to \mu_1) = \int_{\gamma_0}^{\gamma_1} \Psi(\gamma)\,\mathrm{d}\gamma = \frac{b}{\sigma}|\gamma_1 - \gamma_0| = \frac{|\theta_0 - \theta_1|}{\sigma}.} \tag{6.13}$$

The last equality uses the endpoint linear law (6.6), and notes that $|\gamma_1 - \gamma_0| = \left|\ln \frac{\mu_1}{\mu_0}\right|$.

(iii) Endpoint-equivalent length $d_{\rm end}$ and the baseline gap $\Gamma_{\rm geo}$ By the strict definition in Sec. 5, the endpoint-equivalent length is taken as the endpoint distance induced by the Route II structure,

$$d_{\rm end} := \mathrm{dist}_G\big(\theta(\mu_0), \theta(\mu_1)\big).$$

In the present one-dimensional constant-metric model, dist_G reduces to a straight-line length, so

$$\boxed{d_{\rm end}(\mu_0 \to \mu_1) = \frac{|\theta_0 - \theta_1|}{\sigma}.} \tag{6.14}$$

Moreover, in this model the local equivalence relation not only holds but becomes an exact identity:

$$d_{\rm end} = \sqrt{2\Delta_{\rm I}}.$$

Implementation note. This identity holds in the baseline Gaussian constant-metric realization (equal-covariance); it is not assumed elsewhere.

Therefore, by the definition in Sec. 5, the baseline value of the geometric gap is

$$\boxed{\Gamma_{\rm geo} = L_{\rm II} - d_{\rm end} = \frac{|\theta_0 - \theta_1|}{\sigma} - \frac{|\theta_0 - \theta_1|}{\sigma} = 0.} \tag{6.15}$$

Thus, under the baseline conditions "threshold-free window + one-dimensional truncation $\mathcal{M}_1$ + stable interface resolution,"

$$\boxed{\Gamma_{\rm geo}(\mu_0 \to \mu_1) = 0 \quad \text{(treatable as endpoint–path consistency at this resolution).}} \tag{6.16}$$

This conclusion corresponds to a clear structural condition: when the viewpoint compresses the system to a single effective coordinate and the induced structure does not switch piecewise within the window, the RG trajectory coincides with the endpoint geodesic connection, and hence “endpoint-only” and “full scale process” are consistent within the window.

6.5 When a nonzero Γ_{geo} appears: trigger conditions from baseline to nontrivial cases

The reason that $\Gamma_{\mathrm{geo}} = 0$ here is a geometric degeneration from “one-dimensional flow + constant induced structure.” In this subsection we focus on the computable proxy gap Γ_{geo} and write it explicitly; the strict chain-rule residual Γ_{path} is defined in Sec. 5 and is referenced only when needed. To obtain an observable $\Gamma_{\mathrm{geo}} > 0$ in the same physical context, it is necessary to break at least one of the following conditions:

1. **Threshold crossing:** the active-species set changes, making (6.4)–(6.6) piecewise; endpoint compression discards segment information, providing a nonzero source for the strict gap Γ_{path}. If the interface/induced structure also switches piecewise at thresholds (or threshold-segment labels are explicitly included in the comparable coordinate system), then Γ_{geo} will also respond nontrivially to the segmented structure.

2. **Model-class extension:** extend $\mathcal{M}_1$ to $\mathcal{M}_2$ (e.g. by adding the first EFT irrelevant-operator coefficient u), so that the flow enters a two- (or higher-) dimensional parameter space; in the generic case the RG trajectory does not coincide with the endpoint geodesic, producing a systematic difference between L_{II} (full-trajectory accumulation) and $d_{\mathrm{end}} := \mathrm{dist}_G$ (endpoint-equivalent length), hence a natural source of $\Gamma_{\mathrm{geo}} > 0$.

3. **Switching of interface-sensitive directions:** even if the model class is unchanged, if the observation protocol makes the induced structure strongly scale-dependent within the window (e.g. the dominant statistical error source changes with μ, so that σ or the equivalent G exhibits strong scale dependence), then the difference between the “full-trajectory accumulation” L_{II} and the “endpoint length” d_{end} can also become nonzero.

The next section advances to a nontrivial case with minimal additional cost: still within the QED/EFT context, we take $\mathcal{M}_1 \to \mathcal{M}_2$ (introducing one irrelevant direction or matching parameter), and provide an externally reproducible example with $\Gamma_{\mathrm{geo}} > 0$, thereby advancing “gap diagnosis” from a baseline test to a structural output.

7 EFT Extension Example: A Nonzero Gap Γ_{geo} Under a Two-Dimensional Truncation

Under the same threshold-free window setup as in Sec. 6, this section extends the truncated model class from one dimension $\mathcal{M}_1 = \{\alpha\}$ to two dimensions $\mathcal{M}_2 = \{\alpha, u\}$, where u denotes a dimensionless coefficient of an EFT irrelevant direction. The purpose of this extension is not to introduce new physical assumptions, but to explicitly include, within the comparable coordinate system, an additional structural direction that may be generated/carried by intermediate scales under Wilsonian scale advancement [1, 2, 16]. This yields a reproducible structural conclusion: at a fixed viewpoint resolution, one generally finds

$$\Gamma_{\text{geo}} > 0, \tag{7.1}$$

i.e. endpoint compression is insufficient to represent full-trajectory accumulation.

To remain consistent with the scale convention in Sec. 2, we continue to use $\gamma = \ln(\mu/\mu_0)$ as the scale coordinate.

To represent the coarse-graining direction from UV to IR with an increasing parameter, we use the monotone reparameterization ℓ defined in (7.2). Thus ℓ increases monotonically as μ decreases. All path-related expressions in this section will be written with ℓ as the parameter; this is only a change of variables and does not alter the earlier definition chain.

At the same time, to make the parameter along the coarse-graining direction (decreasing resolution) monotonically increasing, we introduce

$$\ell := -\gamma = \ln(\mu_0/\mu), \qquad \ell \in [0, T],\ T > 0, \tag{7.2}$$

to describe the coarse-graining step from UV toward IR. Throughout, we treat Ψ as a non-negative speed and L_{II} as an arclength; monotone reparameterizations of the scale coordinate (including $\ell = -\gamma$) do not change the reported L_{II} and hence Γ_{geo} (Theorem 5.1).

7.1 Choosing a physical irrelevant direction u

In the EFT description of QED (within a fixed window), one may introduce a higher-dimensional operator direction of dimension $d > 4$ as a representative irrelevant direction [3, 4]. To give a minimal and general example, this section does not bind to a unique operator form, and only requires:

1. This direction is irrelevant in the Wilsonian sense, with canonical scaling exponent

$$p := d - 4 > 0, \tag{7.3}$$

   and it is suppressed in the IR under coarse-graining (increasing ℓ);

2. Under the chosen observation interface (including protocol and resolution), it has observable sensitivity to some statistical direction (otherwise it would be projected to an indistinguishable direction at the interface level, and the gap could be treated as negligible under that viewpoint).

Let the Wilson coefficient along this direction be $C_d(\mu)$, and define the dimensionless coordinate

$$u(\mu) := \mu^p\, C_d(\mu), \qquad p > 0. \tag{7.4}$$

In the minimal computable example of this section, we only use the structural facts $p > 0$ and the monotone decay behavior of u, and do not rely on finer operator details.

7.2 Two-dimensional model class and scale flow (minimal RG approximation)

Take the two-dimensional truncated model class

$$\mathcal{M}_2 = \{\alpha(\mu), u(\mu)\}, \qquad \theta := \alpha^{-1}. \tag{7.5}$$

In a threshold-free window, one-loop QED gives (consistent with Sec. 6)

$$\frac{\mathrm{d}\theta}{\mathrm{d}\ell} = \beta_\theta, \qquad \beta_\theta > 0 \text{ is a constant within the window.} \tag{7.6}$$

(With $\ell = -\gamma$, this constant is $\beta_\theta = b$ in the notation of Sec. 6.) For the irrelevant direction u, under the minimal approximation "canonical scaling dominates and higher-order mixing terms with α are neglected," take

$$\frac{\mathrm{d}u}{\mathrm{d}\ell} = -p\, u, \qquad p > 0, \tag{7.7}$$

so that

$$\theta(\ell) = \theta_0 + b\ell, \qquad u(\ell) = u_0\, e^{-p\ell}. \tag{7.8}$$

The geometric meaning of (7.8) is that, in the (θ, u) plane, the scale trajectory is generally a curved curve (unless $u_0 = 0$ or in degenerate cases).

Remark 7.1. If a more specific EFT is needed, one may add mixing terms such as $\mathrm{O}(\alpha\, u)$ in (7.7). The mechanism for "$\Gamma_{\mathrm{geo}} > 0$" in this section does not depend on these higher-order details; the decisive factors are $\dim(\mathcal{M}) \geq 2$ and that the trajectory is generically not collinear/affine with respect to the endpoint geodesic connection.

7.3 A two-dimensional interface implementation of the viewpoint V: making both θ and u readable

To make the gap a reproducible quantity, this section fixes a two-dimensional interface implementation (or a single interface containing a two-dimensional sufficient statistic). Abstractly: at each scale point, the observation protocol outputs two types of statistical readouts (Y_θ, Y_u), which are sensitive to θ and u, respectively, and at a given resolution can be modeled by local Gaussian models:

$$Y_\theta \mid \theta \sim \mathcal{N}(\theta, \sigma_\theta^2), \qquad Y_u \mid u \sim \mathcal{N}(u, \sigma_u^2), \tag{7.9}$$

where $(\sigma_\theta, \sigma_u)$ are fixed by the protocol (sample size, systematic errors, fitting strategy, etc.) and form part of the viewpoint V. To minimize degrees of freedom, we take the two readouts to be approximately independent under this interface implementation; if correlations exist, the induced structure will acquire off-diagonal terms, but the gap mechanism itself is unchanged.

The local quadratic form induced by (7.9) is taken in this section to be a constant diagonal form

$$\mathrm{d}s^2 = \frac{\mathrm{d}\theta^2}{\sigma_\theta^2} + \frac{\mathrm{d}u^2}{\sigma_u^2}, \qquad G = \mathrm{diag}\big(\sigma_\theta^{-2}, \sigma_u^{-2}\big)\,. \tag{7.10}$$

7.4 Explicit calculations: Δ_{I}, d_{end}, L_{II}, and Γ_{geo}

Let the coarse-graining window length be $T := \ell_1 - \ell_0 > 0$, and take $\ell_0 = 0,\ \ell_1 = T$. From (7.8) the endpoint differences are

$$\Delta\theta = \theta(T) - \theta(0) = bT, \qquad \Delta u = u(T) - u(0) = u_0\left(e^{-pT} - 1\right). \tag{7.11}$$

(i) Route I: Route-I endpoint divergence Δ_{I} For Gaussian families with the same covariance, the closed form of KL divergence gives the Route-I endpoint divergence

$$\boxed{\Delta_{\mathrm{I}} = \frac{1}{2}\left(\frac{(\Delta\theta)^2}{\sigma_\theta^2} + \frac{(\Delta u)^2}{\sigma_u^2}\right).} \tag{7.12}$$

(ii) Endpoint-equivalent length $d_{\mathrm{end}} := \mathrm{dist}_G$ By the definition in Sec. 5, the endpoint-equivalent length is the endpoint distance induced by the present structure, $d_{\mathrm{end}} := \mathrm{dist}_G\big((\theta_0, u_0), (\theta_T, u_T)\big)$. Under the constant diagonal metric (7.10), dist_G reduces to a scaled Euclidean distance, hence

$$\boxed{d_{\mathrm{end}} = \sqrt{\frac{(bT)^2}{\sigma_\theta^2} + \frac{u_0^2(1 - e^{-pT})^2}{\sigma_u^2}}\,.} \tag{7.13}$$

Note that in this "constant metric + equal-covariance Gaussian family" implementation, $d_{\mathrm{end}} = \sqrt{2\Delta_{\mathrm{I}}}$ holds exactly (an identity under the declared V_ε conventions in this section). Outside such constant-metric Gaussian/Fisher implementations, one should default to $d_{\mathrm{end}} = \mathrm{dist}_G$ and treat $\sqrt{2\Delta_{\mathrm{I}}}$ only as an optional, viewpoint-declared approximation audited on-domain. In this single-interface instantiation, the endpoint object is the endpoint pair, hence Δ_{pair} coincides with the Route-I endpoint divergence Δ_{I} for the locked interface implementation.

(iii) Route II: information speed and path length L_{II} From (7.6), (7.7), and (7.10), the information speed is

$$\Psi(\ell) = \sqrt{\frac{1}{\sigma_\theta^2}\left(\frac{\mathrm{d}\theta}{\mathrm{d}\ell}\right)^2 + \frac{1}{\sigma_u^2}\left(\frac{\mathrm{d}u}{\mathrm{d}\ell}\right)^2} = \sqrt{\frac{b^2}{\sigma_\theta^2} + \frac{p^2u_0^2e^{-2p\ell}}{\sigma_u^2}}. \tag{7.14}$$

The path length is

$$\boxed{L_{\mathrm{II}} = \int_0^T \Psi(\ell)\,\mathrm{d}\ell = \int_0^T \sqrt{\frac{b^2}{\sigma_\theta^2} + \frac{p^2u_0^2e^{-2p\ell}}{\sigma_u^2}}\,\mathrm{d}\ell.} \tag{7.15}$$

This integral can be computed numerically in a stable way; if a closed form is needed, it can be written as a combination of logarithms and radicals via an elementary substitution (recommended to place in an appendix; not expanded in the main text).

(iv) Geometric gap: $\Gamma_{\rm geo} \geq 0$, and generically strictly positive By the definition in Sec. 5,

$$\boxed{\Gamma_{\rm geo} := L_{\rm II} - d_{\rm end}.} \tag{7.16}$$

Under the constant-metric setting (7.10), rescale coordinates as

$$x(\ell) := \theta(\ell)/\sigma_\theta, \qquad y(\ell) := u(\ell)/\sigma_u,$$

then $L_{\rm II}$ is the Euclidean arc length of the curve $(x(\ell), y(\ell))$, while $d_{\rm end}$ is the chord length between endpoints under the same Euclidean metric. Therefore one must have

$$\boxed{\Gamma_{\rm geo} \geq 0.} \tag{7.17}$$

Moreover, the necessary and sufficient condition for $\Gamma_{\rm geo} = 0$ is that the image of the curve lies on the endpoint line segment and does not turn back (the tangent direction does not reverse). In this section $x(\ell) = x_0 + (b/\sigma_\theta)\ell$ is monotonically increasing, so there is no turning back; thus $\Gamma_{\rm geo} = 0$ is equivalent to the curve being a straight segment in the (x, y) plane, i.e.

$$y(\ell) \text{ as a function of } x(\ell) \text{ is affine (a straight line).} \tag{7.18}$$

For the exponentially decaying trajectory in (7.8), $u(\ell) = u_0 e^{-p\ell}$, this condition fails unless $u_0 = 0$ (degenerating back to one dimension) or $p = 0$ (no longer an irrelevant direction). Hence we obtain the minimal nontriviality statement of this section:

$$\boxed{u_0 \neq 0,\ p > 0,\ T > 0 \implies \Gamma_{\rm geo} > 0.} \tag{7.19}$$

Theorem 7.1 (Strict positivity of $\Gamma_{\rm geo}$ in the minimal two-dimensional truncation)**.** *In the two-dimensional implementation of Sec. 7 with constant diagonal metric* (7.10) *and minimal flow* (7.8) *(equivalently* (7.6)*–*(7.7)*), the geometric proxy gap satisfies, for any window length $T > 0$,*

$$u_0 \neq 0,\ p > 0 \implies \Gamma_{\rm geo}(T) > 0.$$

Moreover, for $T > 0$, equality holds only in the degenerate cases $u_0 = 0$ (one-dimensional reduction) or $p = 0$ (no longer an irrelevant direction), see Appendix B, (B.19)*.*

This shows that, within the same physical system and the same threshold-free window, merely including one EFT irrelevant direction in the model class and making it readable at the interface produces a structural difference between endpoint compression and full-trajectory accumulation.

7.5 Small-quantity approximation: when $\Gamma_{\rm geo}$ is negligible

To make "negligible / non-negligible" an operational criterion, introduce the slope ratio

$$s(\ell) := \frac{\dot y(\ell)}{\dot x(\ell)} = \frac{(\dot u/\sigma_u)}{\left(\dot\theta/\sigma_\theta\right)} = -\frac{p\,\sigma_\theta}{b\,\sigma_u}\, u_0\, e^{-p\ell}, \qquad (\dot{} := \mathrm{d}/\mathrm{d}\ell). \tag{7.20}$$

When $|s(\ell)| \ll 1$ (i.e. the irrelevant direction is uniformly weak at the given resolution), one may perform a second-order expansion of the arc length and obtain the approximate structure

$$\Gamma_{\mathrm{geo}} \approx \frac{b}{2\sigma_\theta}\left[\int_0^T s(\ell)^2\,\mathrm{d}\ell - \frac{1}{T}\left(\int_0^T s(\ell)\,\mathrm{d}\ell\right)^2\right] = \frac{bT}{2\sigma_\theta}\,\mathrm{Var}_{[0,T]}\big[s(\ell)\big] \;\geq 0. \tag{7.21}$$

Proposition 7.1 (Quadratic regime: variance control and small-T behavior)**.** *Assume $|s(\ell)| \ll 1$ on $[0,T]$ so that the second-order expansion leading to (7.21) is valid. Then the leading contribution to Γ_{geo} is controlled by the scale variance of the slope ratio $s(\ell)$, namely (7.21). For the exponential trajectory (7.20), writing $s(\ell) = -K\,u_0\,e^{-p\ell}$ with $K := p\sigma_\theta/(b\sigma_u)$, one obtains*

$$\Gamma_{\mathrm{geo}} \;\approx\; \frac{bT}{2\sigma_\theta}\,K^2 u_0^2\left[\frac{1-e^{-2pT}}{2pT} - \left(\frac{1-e^{-pT}}{pT}\right)^2\right] \;\geq 0. \tag{7.22}$$

In particular, as $T \to 0$,

$$\Gamma_{\mathrm{geo}} \;=\; \frac{p^4\,\sigma_\theta}{24\,b\,\sigma_u^2}\,u_0^2\,T^3 \;+\; \mathrm{O}(T^4) \qquad (\textit{within the quadratic regime}). \tag{7.23}$$

Equation (7.21) provides a direct criterion: under this viewpoint, the leading contribution to Γ_{geo} is controlled by the scale fluctuation (variance) of $s(\ell)$. If $s(\ell)$ is approximately constant (or approximately 0), then Γ_{geo} can be treated as negligible; if $s(\ell)$ varies significantly with ℓ (e.g. exponential decay), then Γ_{geo} should not be compressed to 0 by an endpoint description.

7.6 Reproducible procedure (baseline–nontrivial contrast with Sec. 6)

Note: this section uses a single joint interface that reads out the statistics (Y_θ, Y_u) for (θ, u), and therefore does not use the $\sup_{O\in\mathcal{O}}$ aggregation rule defined in Sec. 3. The error structure of this joint interface is represented by a two-dimensional covariance matrix (diagonal approximation in this section); the aggregation rule is locked as part of the viewpoint V. For reproducibility, one should fully report the interface choice, resolution parameters, and the structure of the error model. Unless explicitly stated, we use "gap" for the strict residual Γ_{path}; we report Γ_{geo} only as a computable proxy discrepancy.

Under a fixed viewpoint V, the minimal reproducible procedure corresponding to this example is:

1. Fix the window: choose a threshold-free window and fix the scheme and the regulator-handling convention;
2. Fix the model class: take $\mathcal{M}_2 = \{\alpha, u\}$, and lock the definition $u = \mu^p C_d(\mu)$;
3. Fix the interface: choose statistical directions that can read out both θ and u, and specify the resolution $(\sigma_\theta, \sigma_u)$;
4. Fit the scale flow: within the window, estimate the linear drift of $\theta(\ell)$ and the decay of $u(\ell)$ (or its more general form);
5. Compute: compute Δ_{I}, d_{end}, L_{II}, Γ_{geo} by (7.12)–(7.16) and report error propagation;

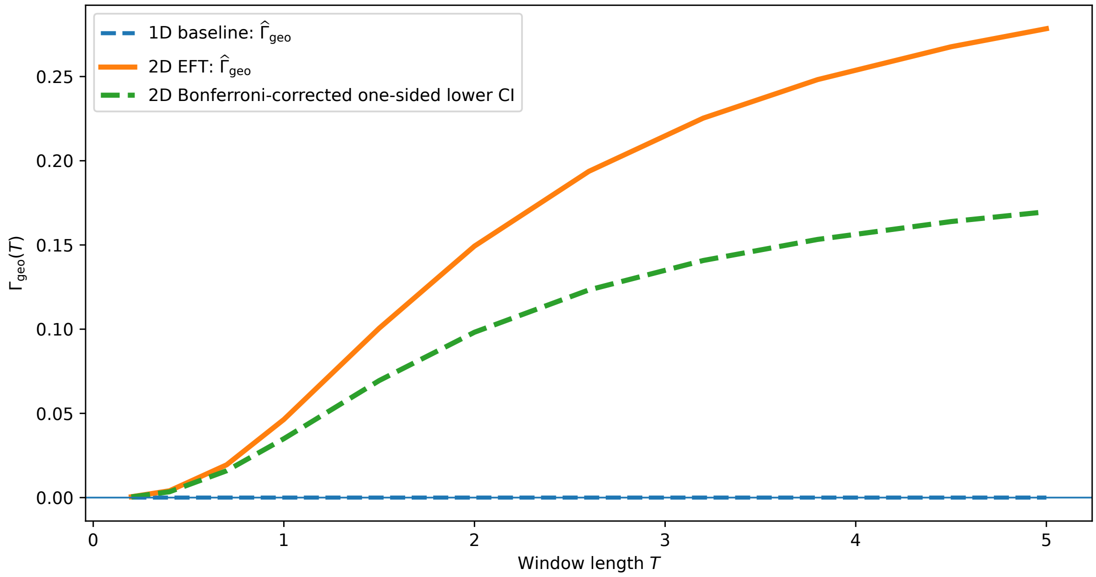

Figure 2: **Run-through of the decision rule on the QED/EFT calibration.** We plot the point estimates $\widehat{\Gamma}_{\rm geo}(T)$ for the 1D baseline and the 2D EFT truncation, together with the Bonferroni-corrected one-sided lower confidence bound $L_{\rm corr}(T)$ defined in Sec. 8.4. In this calibration run, $L_{\rm corr}(T) > 0$ over the scanned window lengths, hence the reported output is MISMATCH.

6. Criterion: test whether $\Gamma_{\rm geo}$ is negligible at the given resolution; if it is not negligible, then "endpoint $\approx$ trajectory" does not hold under this viewpoint, and intermediate-scale structure must enter the model class via explicit directions (e.g. u).

Summary Section 6 provides a one-dimensional baseline case with $\Gamma_{\rm geo} = 0$, corresponding to endpoint-compressibility; this section provides a minimal physical example with $\Gamma_{\rm geo} > 0$ in the same window via the two-dimensional truncation $\mathcal{M}_2 = \{\alpha, u\}$, corresponding to intermediate-scale structure not being fully representable by endpoints. This contrast yields a structural statement: the nonzero gap term is not rhetorical, but a testable output determined jointly by the viewpoint resolution, the model-class dimension, and the geometry of the scale flow.

Fig. 2 runs the Sec. 8.4 decision rule on the QED/EFT calibration, contrasting the 1D baseline ($\Gamma_{\rm geo} \equiv 0$) with the 2D EFT truncation (generically $\Gamma_{\rm geo}(T) > 0$).

8 Discussion: Measurement Semantics and Constraint Norms for the Gap Diagnostics $(\Gamma_{\text{path}}, \Gamma_{\text{geo}})$

This section does only three things: (i) state the strict semantics of the gap diagnostics $(\Gamma_{\text{path}}, \Gamma_{\text{geo}})$; (ii) provide a reproducible estimation and decision protocol; and (iii) write "$\Gamma_{\text{path}} \approx_\tau 0$ / $\Gamma_{\text{path}} > 0$" (and analogously for Γ_{geo}) as falsifiable statements, where $\approx_\tau 0$ denotes the locked numerical decision under the reported tolerance τ (Subsec. 8.4). Under the locked viewpoint V_ε (Sec. 2), we fix the semantics of the gap diagnostics $(\Gamma_{\text{path}}, \Gamma_{\text{geo}})$, provide a reproducible domain-gated reporting protocol, and state falsifiable gap claims.

8.1 Domain-gated estimation and reporting protocol

Output format (structured schema). All reported diagnostics are returned together with a *domain status* under the locked viewpoint V_ε. We use four mutually exclusive statuses:

$$\texttt{STATUS} \in \{\texttt{ADMISSIBLE}, \texttt{MISMATCH}, \texttt{ABYSS}, \texttt{REC-CRITICAL}\}.$$

In addition, each report carries a (possibly empty) set of diagnostic flags,

$$\texttt{FLAGS} \subseteq \{\texttt{FLAG:GEO\_OUT\_OF\_DOMAIN},\ \texttt{FLAG:PROTO\_TYPEII\_NONFIN}\},$$

together with trigger metadata (gate name, location in the window, and any non-finiteness classification).

ABYSS kind (non-finiteness / geometry). Whenever a required divergence is non-finite, we record

$$\texttt{ABYSS\_KIND} \in \{\texttt{TYPE-I},\ \texttt{TYPE-II},\ \texttt{GEOMETRY}\}$$

Non-finiteness events are treated as domain-boundary labels (ABYSS) under the declared viewpoint, not as numerical mismatch of otherwise comparable objects. in the metadata. `TYPE-I` denotes a structural failure (e.g. absolute continuity/support inclusion under the protocol); `TYPE-II` denotes a technical convention/normalization incompletion (treated as a protocol-completion flag by default); `GEOMETRY` denotes a Route-II geometry-certificate failure (Condition $\mathcal{G}_{\text{bridge}}$).

Field gating (what is reported). Numerical fields are reported only when the corresponding gates permit them; otherwise they are omitted (or marked `NA`) and the blocking reason is recorded in `FLAGS`. Concretely:

Field	Meaning	Reported iff
$\widehat{\Delta}_I$	Route-I endpoint divergence	finite (else `STATUS=ABYSS`)
$\widehat{L}_{\text{II}}$	Route-II length proxy	`STATUS`$\in$ {`ADMISSIBLE`, `MISMATCH`} and Condition $\mathcal{G}_{\text{bridge}}$ holds
$\widehat{d_{\text{end}}}$	induced geodesic endpoint distance	Condition $\mathcal{G}_{\text{geo}}$ holds
$\widehat{\Gamma}_{\text{geo}}$	geometric proxy gap $\widehat{L}_{\text{II}} - \widehat{d_{\text{end}}}$	Condition $\mathcal{G}_{\text{geo}}$ holds

Notation convention (estimates vs. target objects): $(\Delta_I, L_{\text{II}}, d_{\text{end}}, \Gamma_{\text{geo}}) \longleftrightarrow (\widehat{\Delta}_I, \widehat{L}_{\text{II}}, \widehat{d_{\text{end}}}, \widehat{\Gamma}_{\text{geo}})$.

A failure of Condition $\mathcal{G}_{\text{geo}}$ is recorded as `FLAG:GEO_OUT_OF_DOMAIN` and $\widehat{d_{\text{end}}}, \widehat{\Gamma}_{\text{geo}}$ are not reported.

Status precedence (conflict resolution). A window may trigger multiple gates. We enforce a fixed precedence order for the final report:

$$\texttt{ABYSS} > \texttt{REC-CRITICAL} > \texttt{MISMATCH} > \texttt{ADMISSIBLE}.$$

Flags do not define additional status values. Once a higher-precedence status is triggered, lower-precedence quantities are not forced to be computed, and the output is a label + trigger metadata rather than a numerical surrogate.

Algorithmic protocol. The default viewpoint-locked reporting procedure is given in Appendix H and is split into two parts: Algorithm 1 specifies the construction of protocol readouts and the Route-I / Π_{rec}-critical gates (Steps 0–2), while Algorithm 2 specifies the Route-II geometric gates and the final mismatch decision (Steps 3–6). All experiments in this paper follow these steps under the locked viewpoint V_ε.

ε-scan for abyss detection (optional). When a fixed ε yields admissible finite divergences, we may optionally run the protocol over a decreasing sequence $\varepsilon_1 > \varepsilon_2 > \cdots > \varepsilon_m$ (all declared as part of the report) and record $\widehat{\Delta}_I^{(\varepsilon_j)}$ and/or $\widehat{\Delta}_{\text{path}}^{(\varepsilon_j)}$. A Type-I abyss is indicated by growth without stabilization as $\varepsilon \downarrow 0$, whereas Type-II behavior typically correlates with normalization/regulator bookkeeping and can be removed by protocol completion.

Viewpoint Card (required for reproducibility). Each reported experiment must include a *Viewpoint Card* summarizing the locked specification:

Interfaces $\mathcal{O}$: list of interfaces/observables and the readout space for each $O \in \mathcal{O}$.

Protocol protocol$_\varepsilon$: explicit mapping $T_\mu \mapsto X_{T_\mu}^{(O)} \mapsto X_\mu^{(O,\varepsilon)}$, including the resolution channel family N_ε, the meaning/units of ε, and any discretization/binning.

Regulator/standardization (R, S): normalization convention (densitization/differencing/explicit retention) and regulator details.

Divergence D: KL / other classical divergence / quantum divergence, and whether D is applied to classical readouts or states.

Aggregation Agg: e.g. mean / weighted sum / sup; if sup is used, note that Condition $\mathcal{G}_{\text{geo}}$ may fail (set `FLAG:GEO_OUT_OF_DOMAIN`).

Model class $\mathcal{M}$: parameterization $\mathbf{M}(g)$, domain of g, and any identifiability constraints.

Projection Π_{rec}: objective $\mathcal{F}_\mu(g)$, optimization method, and whether any tie-breaking rule is declared (default: none).

Threshold rule: numerical tolerances $(\tau_{\text{fin}}, \tau_{\text{crit}}, \tau)$; statistical decision clause α_{sig} (with any multiple-testing correction and one-sided lower CI construction as in Subsec. 8.4); and the reporting precedence.

ε-scan (optional): whether performed; if yes, list $\{\varepsilon_j\}$ and the tracked quantities.

8.2 Strict objects: endpoint–path bridge (minimal recap)

Formal definitions and proofs are given in Sec. 5 and Appendix A. Here we record only the bridge identity and its semantics.

Fix a discrete scale grid and let $Z = (Y_0, \dots, Y_N)$ be the protocol-induced path variable under the locked viewpoint. Write $E := (Y_0, Y_N)$ for the endpoint projection and $\mathcal{H} := (Y_1, \dots, Y_{N-1})$ for the history. For two compared scale flows (hence two path-process objects P_Z, Q_Z), the chain rule yields the exact decomposition

$$\boxed{\Delta_{\text{path}}(0 \to N) = \Delta_{\text{pair}}(0 \to N) + \Gamma_{\text{path}}(0 \to N), \qquad \Gamma_{\text{path}}(0 \to N) \geq 0,} \tag{8.1}$$

where $\Delta_{\text{pair}}(0 \to N) := D(P_E \| Q_E)$ and $\Gamma_{\text{path}}(0 \to N) := \mathbb{E}_{P_E}[D(P_{H|E} \| Q_{H|E})]$. The equality condition is sharp: $\Gamma_{\text{path}} = 0$ iff $D(P_{H|E} \| Q_{H|E}) = 0$ P_E-a.s., i.e. endpoints are sufficient statistics for distinguishing the two path processes under the locked viewpoint. **Notation.** Here H denotes the *history* variable in the chain-rule decomposition (conditioned on the endpoint variable E); it is unrelated to the null/alternative hypotheses H_0/H_1 used in Subsec. 8.4.

We use Γ as a collective name for endpoint–trajectory discrepancy. The strict object is Γ_{path}. The computable proxy used in this paper is $\Gamma_{\text{geo}} := L_{\text{II}} - d_{\text{end}}$ (Sec. 8.3), which is only meaningful on its Route-II domain gates and is not asserted to be unconditionally equivalent to Γ_{path}.

8.3 Computable proxy: domain and decision semantics of Γ_{geo} (locked)

To avoid ambiguous sign behavior caused by domain violations or numerical artifacts, we use the geometric gap Γ_{geo} only when it is *well-defined as an object* under the locked viewpoint V_ε and the corresponding reporting gates pass.

Domain gate for Γ_{geo} (when the geometric proxy is reportable). The reporting protocol for Γ_{geo} begins with a domain check:

1. **Admissibility:** the window is V_ε-admissible (Sec. 2, A1–A3), so that all required divergences are finite under the protocol, and the local quadratic expansion needed for Route II is certified on the window (e.g. by Condition $\mathcal{G}_{\text{bridge}}$).

2. **Non-critical reconstruction:** the window is not Π_{rec}-critical, so that the reconstructed coordinate map is single-valued and a C^1 trajectory $g(\gamma)$ is available on the window without viewpoint drift.

3. **Geometric realizability (Riemannian, locked):** the aggregation rule and protocol induce a single, positive-definite *Riemannian* length structure on the model manifold on the window (Condition $\mathcal{G}_{\text{geo}}$), i.e. a (piecewise) C^1 matrix field $G(g) \succ 0$ locked as part of V_ε. (Generalizations beyond the Riemannian setting are outside the scope of this paper and are not used in the reporting protocol.)

If any of the above checks fails, we do not assign a numerical value to Γ_{geo} on that window. Instead we report `STATUS` (and trigger metadata) as follows:

- If admissibility or the local quadratic certificate fails, we return `STATUS=ABYSS` for Route-II quantities (with `ABYSS_KIND=GEOMETRY` and the triggered gate).
- If reconstruction is critical, we return `STATUS=REC-CRITICAL` (with Π_{rec} diagnostics).
- If Condition $\mathcal{G}_{\mathrm{geo}}$ fails while the window remains admissible and non-critical, we return `STATUS=ADMISSIBLE`, add `FLAG:GEO_OUT_OF_DOMAIN`, and omit both $\widehat{d_{\mathrm{end}}}$ and Γ_{geo} (other permitted Route-I / Route-II surrogates remain reportable under the protocol).

In particular, a failure of Condition $\mathcal{G}_{\mathrm{geo}}$ is treated as an indication that the chosen aggregation/interface family does not support a single smooth quadratic metric surrogate under the declared resolution, not as a contradiction of the endpoint–path framework.

Condition $\mathcal{G}_{\mathrm{geo}}$ (domain of validity for the geometric proxy). There exist parameter coordinates $g(\gamma) \in \mathbb{R}^d$ and a (piecewise) C^1 positive-definite matrix field $G(g)$ on the window, equivalently a local quadratic form $Q_g(\mathrm{d}g) := \mathrm{d}g^\top G(g)\,\mathrm{d}g$, such that

$$\mathrm{d}s^2 = Q_g(\mathrm{d}g) = \mathrm{d}g^\top G(g)\,\mathrm{d}g, \qquad L_{\mathrm{II}}(\gamma_0 \to \gamma_1) := \int_{\gamma_0}^{\gamma_1} \sqrt{\dot{g}^\top G(g)\,\dot{g}}\,d\gamma. \tag{8.2}$$

If G is induced via an aggregated local quadratic form (e.g. by $\mathrm{Agg} = \sup_{O\in\mathcal{O}}$ over interfaces), the aggregation rule must select a *single* positive-definite matrix field $G(g)$ on the window (at least piecewise C^1); otherwise Condition $\mathcal{G}_{\mathrm{geo}}$ fails on the window and Γ_{geo} is not reported. Such failures commonly indicate that the induced local surrogate is non-quadratic (e.g. Finsler-type) or multi-valued under aggregation (e.g. sup), rather than a breakdown of the strict endpoint–path framework; we therefore treat them as out-of-domain for the Route-II proxy. *Interpretation (viewpoint-relative structural reorganization).* Within the declared viewpoint V_ε,

failure of Condition $\mathcal{G}_{\mathrm{geo}}$ indicates that the coarse-grained description has exited the single-SPD (quadratic/Riemannian) proxy class selected by the reporting protocol. We interpret this as a systematic reorganization of the effective structural/logic representation under V_ε (a representation-class change), not as a mere numerical pathology or a "trivial" case. Accordingly, Route-II geometric proxies are intentionally left undefined on that window and we report `FLAG:GEO_OUT_OF_DOMAIN`.
Under Condition $\mathcal{G}_{\mathrm{geo}}$, L_{II} is the arc length of the observed trajectory segment $g(\gamma)$ under G, while d_{end} is the endpoint distance under the *same* structure:

$$d_{\mathrm{end}}(\gamma_0 \to \gamma_1) := \mathrm{dist}_G\big(g(\gamma_0), g(\gamma_1)\big). \tag{8.3}$$

Write $g_0 := g(\gamma_0)$ and $g_1 := g(\gamma_1)$ for the endpoints on the window. Here dist_G denotes the *global* geodesic distance (the infimum over all connecting curves), and is not defined as the length of (nor restricted to) the observed discrete trajectory segment $\{g(\gamma_k)\}$ on the window.

Under Condition $\mathcal{G}_{\mathrm{geo}}$, define

$$\boxed{\Gamma_{\mathrm{geo}} := L_{\mathrm{II}} - d_{\mathrm{end}}.} \tag{8.4}$$

Since $d_{\mathrm{end}} = \mathrm{dist}_G(g_0, g_1)$ is the infimum of G-lengths over all curves connecting g_0 to g_1, while L_{II} is the G-length of the specific observed segment $g(\gamma)$, one always has $L_{\mathrm{II}} \geq d_{\mathrm{end}}$, hence

$\Gamma_{\text{geo}} \geq 0$. Moreover, the equality case has the following strict semantics:

$$\boxed{\begin{aligned} \Gamma_{\text{geo}} &\geq 0, \\ \Gamma_{\text{geo}} = 0 &\iff g(\gamma) \text{ is a minimizing } G\text{-geodesic from } g_0 \text{ to } g_1. \end{aligned}} \tag{8.5}$$

Equivalently, $g(\gamma)$ realizes $d_{\text{end}} = \text{dist}_G(g_0, g_1)$; the statement is invariant under monotone reparameterization (and under arc-length parametrization the speed is constant).

The QED/EFT implementations in this paper satisfy Condition $\mathcal{G}_{\text{geo}}$. Sections 6–7 use equal-covariance Gaussian interfaces (or an equivalent Fisher implementation), so that G is a constant diagonal metric and the endpoint distance is a scaled Euclidean distance. In this implementation, $\sqrt{2\Delta_{\text{I}}}$ coincides exactly with dist_G, hence the relation

$$d_{\text{end}} = \sqrt{2\Delta_{\text{I}}}$$

appearing in Secs. 6–7 is an identity rather than an approximation. This identity is implementation-specific; in general one should use the definition $d_{\text{end}} = \text{dist}_G$ in (8.3) and treat $\sqrt{2\Delta_{\text{I}}}$ as an optional approximation only when it is explicitly declared as part of the locked viewpoint and certified on the same domain gates.

Realization clause (theoretical objects vs. computed estimates). Equations (8.2)–(8.5) define the *theoretical* objects L_{II}, $d_{\text{end}} = \text{dist}_G$, and $\Gamma_{\text{geo}} := L_{\text{II}} - d_{\text{end}}$, where dist_G is the global geodesic distance induced by G:

$$\text{dist}_G(u, v) := \inf_{x(0)=u,\, x(1)=v} \int_0^1 \sqrt{x'(s)^\top G(x(s)) x'(s)}\, ds.$$

In computations, one reports $\widehat{L}_{\text{II}}$, $\widehat{d_{\text{end}}}$, and $\widehat{\Gamma}_{\text{geo}} := \widehat{L}_{\text{II}} - \widehat{d_{\text{end}}}$ as *estimates* of the corresponding theoretical quantities under a locked viewpoint V. The numerical procedures used to produce these estimates are part of V (hence must be reported), but they do not redefine the objects.
Decision semantics (numerical consistency): due to discretization/optimization error, $\widehat{\Gamma}_{\text{geo}}$ may be slightly negative even when $\Gamma_{\text{geo}} \geq 0$ holds theoretically. We treat values in $[-\tau, 0]$ as numerical zero and report $\widehat{\Gamma}_{\text{geo}} := 0$; if $\widehat{\Gamma}_{\text{geo}} < -\tau$, we report the value and attach `FLAG:NUMERIC_NEGATIVE`. The tolerance τ is part of the locked viewpoint V_ε and must be reported. Equivalently, we write $\widehat{\Gamma}_{\text{geo}} \approx_\tau 0$ for this "treated as zero" decision.
The `ADMISSIBLE/MISMATCH` decision is made solely via the corrected one-sided lower confidence bound in Subsec. 8.4, not via a separate threshold on $\widehat{\Gamma}_{\text{geo}}$; the tolerance τ is used only for numerical-validity bookkeeping and reporting conventions.
From this point onward, all reported numerics are V_ε-conditioned estimates; altering workflow conventions changes the reported scalar even when the symbolic expression is unchanged. Details of a reproducible instantiation are given in Appendix D.

8.4 Estimation and Decision Protocol (turning "$\approx_\tau 0$" into a decidable statement)

This subsection provides a *reproducible* protocol that is *comparable across works*. Here the notation $\approx_\tau 0$ denotes a numerical decision under a reported tolerance τ; it is distinct from the analytic approximation symbol $\simeq$ used elsewhere to denote local second-order approximations. All decisions are made under the locked viewpoint V_ε defined in Sec. 2; changing any component changes the diagnostic specification. When the geometric proxy is used, the reported quantities are *estimates* of the corresponding theoretical objects defined in (8.2)–(8.5); the numerical procedures that produce these estimates belong to V and must be reported, but they do not redefine the objects (see Appendix D for a reproducible instantiation template).

Inputs (must be reported; otherwise cross-work comparison as the same scalar is not allowed)

1. **Window and coordinates:** the window $[\mu_0, \mu_1]$ and the choice of scale coordinate ($\gamma = \ln(\mu/\mu_0)$ or $\ell = -\gamma$). If a scan over window length T is performed (e.g. $\ell \in [0, T]$ or $\gamma_1 - \gamma_0 = T$), one must report the rule that generates the window family $\{W(T)\}$ (e.g. fixing μ_0 and varying $\mu_1(T)$), and the discrete grid $\{T_j\}_{j=1}^m$ (or the equivalent $\{\mu_{1,j}\}$).
2. **Complete list of viewpoint clauses:** $\mathcal{O}$ and protocol, R, S, the threshold rule, the model class $\mathcal{M}$, reconstruction Π_{rec}, the divergence/distance function D, and the interface aggregation rule Agg (including an explicit statement of how regulator dependence is handled: density-normalization / differencing / explicit retention, and the corresponding implementation convention).
3. **If using the geometric proxy:** one must (i) declare that the local quadratic expansion / geometry certificate holds on the window (Condition $\mathcal{G}_{\mathrm{bridge}}$, or an equivalent auditable certificate), (ii) declare that Condition $\mathcal{G}_{\mathrm{geo}}$ holds (see Sec. 8.3, (8.2)–(8.5)), and (iii) report the viewpoint-locked procedures used to produce the estimates $\widehat{L}_{\mathrm{II}}$ and $\widehat{d_{\mathrm{end}}}$ (hence $\widehat{\Gamma}_{\mathrm{geo}}$). These procedures are part of V: changing them changes the scalar being reported, even if the symbolic formula $\Gamma_{\mathrm{geo}} = L_{\mathrm{II}} - d_{\mathrm{end}}$ is unchanged. (For avoidance of doubt: $\widehat{d_{\mathrm{end}}}$ is intended as an approximation to the geodesic distance dist_G defined by the induced metric G, rather than a path-length restricted to the observed discrete trajectory.)
4. **Uncertainty model (locked):** one must provide a fixed method for generating uncertainty that propagates to $\widehat{\Gamma}_{\mathrm{geo}}$ (choose one and lock it):
   (a) analytic error propagation (must report the covariance matrix / linearization rule); or
   (b) bootstrap / resampling (must report the resampling unit, the number of resamples B, and the interval construction rule) [23].
5. **Significance level (locked):** lock a global one-sided significance level $\alpha_{\mathrm{sig}} \in (0, 1)$. If a window scan (multiple testing) is performed, one must also lock a multiple-comparison correction rule; this paper uses Bonferroni by default[24, 25]:

$$\alpha_{\mathrm{sig},j} := \alpha_{\mathrm{sig}}/m, \qquad j = 1, \dots, m. \tag{8.6}$$

(If other corrections such as Holm are used, they must also be locked in the inputs and kept consistent throughout.)

Outputs (must be reported) For each window (or each T_j), one must report

$$\widehat{\Delta}_I, \qquad \widehat{L}_{\mathrm{II}}, \qquad \widehat{d_{\mathrm{end}}}, \qquad \widehat{\Gamma}_{\mathrm{geo}}, \qquad \mathrm{CI}_L\big(\widehat{\Gamma}_{\mathrm{geo}}; 1-\alpha_{\mathrm{sig}}\big), \tag{8.7}$$

where $\mathrm{CI}_L(\cdot; 1-\alpha_{\mathrm{sig}})$ denotes a *one-sided lower confidence bound.* If a window scan is performed, replace it by $\mathrm{CI}_L(\cdot; 1-\alpha_{\mathrm{sig},j})$ (see (8.6)).

Decision rule (null-hypothesis test, locked) Define the null and alternative:

$$H_0: \ \Gamma_{\mathrm{geo}} = 0, \qquad H_1: \ \Gamma_{\mathrm{geo}} > 0. \tag{8.8}$$

One-sided decision rule:

$$\text{if } \mathrm{CI}_L\big(\widehat{\Gamma}_{\mathrm{geo}}; 1-\alpha_{\mathrm{sig}}\big) > 0, \ \text{reject } H_0; \qquad \text{otherwise do not reject } H_0. \tag{8.9}$$

Notation. H_0/H_1 are statistical hypotheses and should not be confused with the history variable H in the chain-rule decomposition (Eq. (8.1) and Appendix A).

If a window scan is performed, replace α_{sig} by $\alpha_{\mathrm{sig},j}$.

Break location (first-detection window) Under a fixed viewpoint V, fixed resolution clauses (including the uncertainty model / interval construction), and a locked significance level (including the multiple-testing correction), parameterize the window length by $T > 0$ and take a discrete grid $0 < T_1 < \cdots < T_m$. For each T_j, compute $\widehat{\Gamma}_{\mathrm{geo}}(T_j)$ and $\mathrm{CI}_L\big(\widehat{\Gamma}_{\mathrm{geo}}(T_j); 1-\alpha_{\mathrm{sig},j}\big)$.

Define "a proxy-gap is detected at T_j" by

$$\mathrm{Det}_{\mathrm{geo}}(T_j): \quad \mathrm{CI}_L\big(\widehat{\Gamma}_{\mathrm{geo}}(T_j); 1-\alpha_{\mathrm{sig},j}\big) > 0.$$

Then the discrete implementation of the first-detection scale is defined by

$$\widehat{T}_{\delta,\mathrm{geo}} \ := \ \min\{T_j : \ \mathrm{Det}_{\mathrm{geo}}(T_j)\}, \tag{8.10}$$

and one reports a resolution-limited bracketing interval using adjacent grid points (e.g. $[T_{j-1}, T_j]$). If bootstrap is used to obtain an empirical distribution of $\widehat{T}_{\delta,\mathrm{geo}}$, one may report $[T^-_{\delta,\mathrm{geo}}, T^+_{\delta,\mathrm{geo}}]$ via quantiles, but this rule must be locked in the inputs.

Endpoint–path formulation of the structural consistency axiom: an intrinsic residual across viewpoints

This subsection formulates "whether endpoint compression can be removed by choosing a viewpoint" as a reproducible structural statement. Recall: under a fixed viewpoint V, the strict gap $\Gamma^{(\mathsf{V})}_{\mathrm{path}} \geq 0$ is given by the chain-rule decomposition of relative entropy, and $\Gamma^{(\mathsf{V})}_{\mathrm{path}} = 0$ if and only if the endpoint variable $E = (Y_0, Y_N)$ is a sufficient statistic for the path difference under that viewpoint (Appendix A).

Admissible viewpoint family $\mathfrak{V}$ (admissible viewpoints) Let $\mathfrak{V}$ denote a set of *admissible* viewpoints. Here "admissible" only means:

1. it satisfies the normative clauses in Sec. 2 (interface/protocol, regulator and scheme, threshold rule, model class and reconstruction, divergence function and aggregation operator, etc. are all reportable and locked within each viewpoint);

2. it has the *readability* stipulated by the research goal (it can read out the claimed parameter directions / statistical directions);

3. it excludes degenerate resolutions: it does not allow an artificially zero gap by making the interface arbitrarily coarse (e.g. one may require a uniform lower bound $G(\gamma) \succeq G_{\min} \succ 0$, or an equivalent resolution lower-bound clause).

Changing to a viewpoint $\mathsf{V} \notin \mathfrak{V}$ is treated as changing the problem itself, not as a different writing of the same object.

Definition 8.1 (Intrinsic residual across viewpoints)**.** Under a fixed scale window $[\mu_0, \mu_1]$ and a fixed admissible viewpoint family $\mathfrak{V}$, define the cross-viewpoint infimum of the strict gap by

$$\Gamma_*(\mu_0 \to \mu_1) \;:=\; \inf_{\mathsf{V}\in\mathfrak{V}} \Gamma^{(\mathsf{V})}_{\text{path}}(\mu_0 \to \mu_1) \;\geq 0. \tag{8.11}$$

When the main text uses the geometric proxy gap $\Gamma^{(\mathsf{V})}_{\text{geo}}$ for computable diagnostics, one may correspondingly define

$$\Gamma^{\text{geo}}_*(\mu_0 \to \mu_1) \;:=\; \inf_{\mathsf{V}\in\mathfrak{V}} \Gamma^{(\mathsf{V})}_{\text{geo}}(\mu_0 \to \mu_1) \;\geq 0, \tag{8.12}$$

and emphasize that Γ^{geo}_* serves only as a computable proxy for Γ_*; its interpretation should be anchored to the chain-rule decomposition for the strict object Γ_{path}.

Proposition 8.1 (Cross-viewpoint lower-bound criterion (endpoint–path version))**.** *Under a fixed scale window and a fixed admissible viewpoint family $\mathfrak{V}$:*

1. *If there exists a viewpoint $\mathsf{V} \in \mathfrak{V}$ such that $\Gamma^{(\mathsf{V})}_{\text{path}}(\mu_0 \to \mu_1) = 0$, then $\Gamma_*(\mu_0 \to \mu_1) = 0$.*

2. *If $\Gamma_*(\mu_0 \to \mu_1) > 0$, then for every $\mathsf{V} \in \mathfrak{V}$ one has $\Gamma^{(\mathsf{V})}_{\text{path}}(\mu_0 \to \mu_1) \geq \Gamma_*(\mu_0 \to \mu_1) > 0$; in this case, endpoint compression cannot eliminate the full-trajectory historical information under any admissible viewpoint.*

3. *If $\Gamma_*(\mu_0 \to \mu_1) = 0$ but attainability of the infimum is not claimed, then its strict meaning is: for every $\eta > 0$ there exists $\mathsf{V}_\eta \in \mathfrak{V}$ such that $\Gamma^{(\mathsf{V}_\eta)}_{\text{path}}(\mu_0 \to \mu_1) \leq \eta$; that is, the gap can be made* arbitrarily small *within the viewpoint family, but there need not exist a single viewpoint for which it is strictly zero.*

Remark (comparison with Secs. 6–7) The baseline case $\Gamma_{\text{geo}} = 0$ in Sec. 6 corresponds to an admissible viewpoint implementation under which, at the given resolution and model class, the system effectively degenerates to a one-dimensional readable coordinate (geodesic degeneration), so endpoint-only reporting is *proxy-consistent* at the declared resolution/model class (i.e. the diagnostic returns $\Gamma_{\text{geo}} = 0$). In Sec. 7, after including an EFT irrelevant direction in the model class and making it readable at the interface, one typically obtains $\Gamma_{\rm geo} > 0$, meaning endpoint compression is no longer sufficient. This distinction emphasizes that once the viewpoint clauses are locked, the decision "negligible / non-negligible" for Γ becomes a testable output.

8.5 Falsifiable statements

This subsection lists several statements. Each statement is phrased so that, once objects are locked, it can be decided as true/false; unless otherwise stated, viewpoint superscripts are omitted below.

T0 (cross-viewpoint lower bound: falsifiability of intrinsic residual) Lock an allowed viewpoint family $\mathfrak{V}_{\rm adm} \subseteq \mathfrak{V}$, and impose *hard constraints* to exclude degenerate resolutions (e.g. require $G(\gamma) \succeq G_{\min} \succ 0$). Define the (proxy-level) minimal gap

$$\Gamma^{\rm geo}_{*,\rm adm}(\mu_0 \to \mu_1) := \inf_{\mathsf{V}\in\mathfrak{V}_{\rm adm}} \Gamma^{(\mathsf{V})}_{\rm geo}(\mu_0 \to \mu_1). \tag{8.13}$$

If $\Gamma^{\rm geo}_{*,\rm adm}(\mu_0 \to \mu_1) > 0$, then the window is said to have an *intrinsic residual lower bound* under $\mathfrak{V}_{\rm adm}$ and the resolution lower-bound constraint; i.e. for every allowed viewpoint $\mathsf{V} \in \mathfrak{V}_{\rm adm}$, one has $\Gamma^{(\mathsf{V})}_{\rm geo}(\mu_0 \to \mu_1) \geq \Gamma^{\rm geo}_{*,\rm adm}(\mu_0 \to \mu_1) > 0$. Conversely, if there exists an allowed viewpoint such that $\Gamma^{(\mathsf{V})}_{\rm geo}(\mu_0 \to \mu_1) = 0$, then necessarily $\Gamma^{\rm geo}_{*,\rm adm}(\mu_0 \to \mu_1) = 0$.

T1 (geodesic-consistency window: proxy criterion) Under a locked viewpoint V and assuming Condition $\mathcal{G}_{\rm geo}$ holds,

$$\begin{gathered}\Gamma^{(\mathsf{V})}_{\rm geo} = 0 \iff g(\gamma) \text{ is a minimizing } G\text{-geodesic from } g(\gamma_0) \text{ to } g(\gamma_1),\\ (\text{equivalently: } L_{\rm II} = d_{\rm end} = \mathrm{dist}_G\big(g(\gamma_0), g(\gamma_1)\big)).\end{gathered} \tag{8.14}$$

Therefore, testing T1 is exactly the one-sided test $H_0 : \Gamma^{(\mathsf{V})}_{\rm geo} = 0$ described in Subsec. 8.4.

T2 (two-dimensional EFT irrelevant direction: necessary and sufficient condition for $\Gamma_{\rm geo} > 0$ and closed-form output) In a threshold-free window, Sec. 7 takes the two-dimensional truncation $\mathcal{M}_2 = \{\alpha, u\}$ and induces a constant diagonal metric under an equal-covariance Gaussian interface,

$$\mathrm{d}s^2 = \frac{\mathrm{d}\theta^2}{\sigma_\theta^2} + \frac{\mathrm{d}u^2}{\sigma_u^2}. \tag{8.15}$$

Let $\ell \in [0, T]$ be the UV→IR coarse-graining parameter, and take the scale flow

$$\frac{\mathrm{d}\theta}{\mathrm{d}\ell} = b, \quad b > 0; \qquad \frac{\mathrm{d}u}{\mathrm{d}\ell} = -p\,u, \quad p > 0; \qquad \theta(\ell) = \theta_0 + b\ell, \quad u(\ell) = u_0 e^{-p\ell}. \tag{8.16}$$

Then under this viewpoint implementation,

$$\Gamma_{\rm geo}(T) = 0 \iff (u_0 = 0) \text{ or } (p = 0); \qquad u_0 \neq 0,\ p > 0,\ T > 0 \implies \Gamma_{\rm geo}(T) > 0. \tag{8.17}$$

For a closed-form, reproducible output, define

$$a := \frac{b}{\sigma_\theta} > 0, \qquad c := \frac{p\,u_0}{\sigma_u}.$$

Then

$$\begin{aligned} L_{\mathrm{II}}(T) &= \int_0^T \sqrt{a^2 + c^2 e^{-2p\ell}}\, \mathrm{d}\ell \\ &= aT + \frac{1}{p}\left[\sqrt{a^2+c^2} - \sqrt{a^2+c^2e^{-2pT}} + a\ln\frac{a+\sqrt{a^2+c^2e^{-2pT}}}{a+\sqrt{a^2+c^2}}\right], \end{aligned} \tag{8.18}$$

and the endpoint geodesic distance is

$$d_{\mathrm{end}}(T) = \sqrt{(aT)^2 + \left(\frac{u_0}{\sigma_u}(1-e^{-pT})\right)^2}, \tag{8.19}$$

hence

$$\Gamma_{\mathrm{geo}}(T) = L_{\mathrm{II}}(T) - d_{\mathrm{end}}(T) \tag{8.20}$$

is a closed-form, reproducible object under this implementation.

Moreover, the two limits of $\Gamma_{\mathrm{geo}}(T)$ can be given directly:

$$\Gamma_{\mathrm{geo}}(T) = \frac{a^2}{24}\cdot\frac{\left(\frac{u_0}{\sigma_u}\right)^2 p^4}{\left(a^2+\left(\frac{u_0}{\sigma_u}\right)^2 p^2\right)^{3/2}}\, T^3 + \mathrm{O}(T^4), \qquad (T\downarrow 0), \tag{8.21}$$

$$\lim_{T\to\infty}\Gamma_{\mathrm{geo}}(T) = \frac{1}{p}\left[\sqrt{a^2+\left(\frac{u_0}{\sigma_u}\right)^2 p^2} - a + a\ln\frac{2a}{a+\sqrt{a^2+\left(\frac{u_0}{\sigma_u}\right)^2 p^2}}\right] > 0 \quad (u_0\neq 0, p>0). \tag{8.22}$$

T3 (threshold crossing stitching: if used, it must be tested by this clause) If the window crosses a threshold and the threshold rule is locked, one must also report the gaps on the three windows:

$$\Gamma_{\mathrm{geo}}([\mu_0,\mu_*])\ , \quad \Gamma_{\mathrm{geo}}([\mu_*,\mu_1])\ , \quad \Gamma_{\mathrm{geo}}([\mu_0,\mu_1]), \tag{8.23}$$

where μ_* is the threshold point (or stitching point). The test requirement of T3 is: all three must be computed under the same viewpoint V, and the threshold matching conditions must be reported; if any item is missing, one must not claim a "Γ behavior on threshold-crossing windows."

8.6 Checklist (minimum clauses for reproducibility and cross-work comparability)

Any work applying the divergence and gap quantities of this paper must at least report:

1. the window $[\mu_0,\mu_1]$ and coordinates (γ or ℓ), and the window-scan grid (if used);

2. the concrete implementation of $\mathcal{O}$ and the protocol (statistics, energy/momentum points, normalization, sampling);
3. the regulator R and scheme S, and the choice among density-normalization / differencing / explicit retention of regulator dependence;
4. the threshold rule and matching conditions (if thresholds are involved);
5. the model class $\mathcal{M}$ and reconstruction Π_{rec} (including the uniqueness criterion);
6. the divergence/distance function D and the interface aggregation rule Agg;
7. if using Γ_{geo}: one must provide G, $g(\gamma)$, $d_{\text{end}} = \text{dist}_G$, and declare Condition $\mathcal{G}_{\text{geo}}$ holds;
8. the uncertainty model and interval construction (including the locked α_{sig} and multiple-testing correction rule, if a scan is used).

If any item is missing, the result must be classified as a cross-viewpoint output and must not be used for direct comparison as the same scalar with other works.

9 Conclusion

This paper proposes a *viewpoint-relative* cross-scale gap framework. Under fixed observational interfaces, regulator choices, threshold rules, and a model class, it rewrites "structural differences along scale evolution" into distinguishable information differences, and establishes a reproducible structural relationship between the endpoint-type (Route I) and the full-trajectory-type (Route II) descriptions. The basic convention is: once the viewpoint

$$\mathsf{V} = \begin{pmatrix} \mathcal{O}, \text{protocol}, R, S, \text{threshold rule}, \mathcal{M}, \\ \Pi_{\text{rec}}, D, \text{Agg} \end{pmatrix}$$

is locked, all subsequent gap quantities, length quantities, and inequalities are defined under this viewpoint.

Under a fixed viewpoint V, this paper defines the endpoint gap Δ_{I} (Route I). Meanwhile, when the local second-order bridging is valid and Condition $\mathcal{G}_{\text{geo}}$ holds (or an equivalent single-interface/Fisher implementation), an information speed Ψ is generated from the induced quadratic form/metric and the RG flow field, and the computable path length (Route II) is defined by

$$L_{\text{II}} = \int_{\gamma_0}^{\gamma_1} \Psi \, d\gamma,$$
$$\Psi = \sqrt{\dot{g}^\top G(g)\, \dot{g}}, \qquad \text{under Condition } \mathcal{G}_{\text{bridge}}.$$

At the strict level, this paper defines the full-trajectory gap via a path process object,

$$\Delta_{\text{path}} := D(\mathbb{P}_Z \| \mathbb{Q}_Z),$$

and uses the monotonicity of relative-entropy-type distances under coarse-graining maps and the chain-rule decomposition to obtain the endpoint–path relationship: the endpoint-pair marginal divergence does not exceed the full-trajectory gap, and there exists a nonnegative gap term such that

$$\Delta_{\text{path}} = \Delta_{\text{pair}} + \Gamma_{\text{path}}, \qquad \Gamma_{\text{path}} \geq 0.$$

At the computable level, on the domain where Condition $\mathcal{G}_{\text{geo}}$ applies, this paper introduces a geometric proxy

$$\Gamma_{\text{geo}} = L_{\text{II}} - d_{\text{end}}, \qquad d_{\text{end}} = \text{dist}_G\big(g(\gamma_0), g(\gamma_1)\big),$$

and in the implementations of Secs. 6–7 takes $d_{\text{end}} = \sqrt{2\Delta_{\text{I}}}$ (an identity rather than an approximation within those implementations), thereby elevating "whether endpoint compression suffices to represent full-trajectory accumulation" to an estimable and falsifiable structural statement. This use of $d_{\text{end}} = \sqrt{2\Delta_{\text{I}}}$ is not assumed outside the declared constant-metric Gaussian/Fisher instantiations; in general settings the proxy must be constructed with $d_{\text{end}} = \text{dist}_G$ as defined in (8.3), and any endpoint approximation must be declared as a viewpoint clause and audited on-domain. It should be emphasized that $\Gamma \approx_\tau 0$ only means that, under the locked viewpoint and the reported tolerance τ, the endpoint description is sufficiently informative (the gap is compressible at that resolution), and does not correspond to the disappearance of an underlying gap.

On the physical alignment side, this paper provides two complementary examples in a threshold-free window in one-loop QED, thereby completing a precise calibration of the framework and a minimal nontrivial test. First, under a one-dimensional truncation and a stable interface resolution, it obtains the baseline behavior $\Gamma_{\mathrm{geo}} = 0$, demonstrating consistency between endpoints and the full trajectory at that viewpoint resolution. Second, within the same window, after extending to a two-dimensional truncation that includes an EFT irrelevant direction, one generally obtains a nontrivial gap $\Gamma_{\mathrm{geo}} > 0$, demonstrating the mechanism by which, when intermediate-scale structure is readable under the given interface, endpoint compression systematically underestimates full-trajectory accumulation. Accordingly, the contribution is to provide an operational, viewpoint-locked gap diagnostic together with (i) an invariant meaning and a sharp zero-gap criterion for the geometric proxy (Theorems 5.1–5.2); and (ii) closed-form and asymptotic calibrations in standard QED/EFT windows (Secs. 6–7 and Appendix B).

Operationally, the endpoint/path diagnostics are not meant to be forced outside their domain of validity. Accordingly, the paper is organized as a two-layer protocol: a viewpoint-locked, domain-gated procedure (Sec. 8 and Appendix H) returns a status (`ADMISSIBLE`, `MISMATCH`, `ABYSS`, `REC-CRITICAL`) together with the minimal numerical surrogates permitted by that status, while Appendix D supplies a reproducibility workflow that locks the numerical conventions required to instantiate those surrogates and to audit numerical validity near boundaries. In typical use, the status-level output is primary; the detailed workflow is invoked when one wishes to reproduce or audit the diagnostic under a fully specified viewpoint.

Future work can proceed along three directions. First, replace the minimal Gaussian implementation with more specific physical interfaces (e.g. real scattering distributions, lattice observables, or constructible process states) to obtain direct estimates or bounds for the strict gap Γ_{path}. Second, incorporate threshold-crossing stitching and multi-coupling flows into a unified workflow, so that the gap term becomes a diagnostic tool for "piecewise-structure incompressibility," and promote the threshold rule from a technical detail to a reportable and testable structural input. Third, under higher-order corrections or richer EFT bases, systematically study scaling laws of Γ as functions of window length and resolution, thereby advancing the gap diagnostic into a comparable ruler and providing an operational boundary characterization for the "validity domain of endpoint descriptions" in cross-scale modeling.

Appendix A: Relative-Entropy Chain Rule, DPI, $\Gamma_{\text{path}} \geq 0$, and the Equality Condition

This appendix states two structural facts used in the main text: (i) the data processing inequality (DPI), and (ii) the chain-rule decomposition (chain rule) for relative-entropy-type divergences (KL-type). Together they imply the conclusions used in Sec. 5 and Sec. 8: the strict gap term is nonnegative ($\Gamma_{\text{path}} \geq 0$) and the corresponding equality condition. All statements are made under the locked viewpoint V defined in Sec. 2; viewpoint superscripts are omitted for brevity.

A.1 Data processing inequality (DPI)

Let X, Y be two predictive objects on the same space (in the classical case, probability distributions; in the quantum case, density operators/states), and let Φ be a coarse-graining map (in the classical case, a Markov kernel/stochastic map; in the quantum case, a CPTP map). For a relative-entropy-type divergence $D(\cdot\|\cdot)$ (classical KL or quantum relative entropy), one has

$$D(X\|Y) \;\geq\; D\big(\Phi(X)\|\Phi(Y)\big). \tag{A.1}$$

Equation (A.1) is DPI: coarse-graining does not increase distinguishable information difference.

Endpoint–path upper bound (source of $\Delta_{\text{pair}} \leq \Delta_{\text{path}}$ in the main text) Let the joint path variable be (with Y_k denoting the fixed-viewpoint protocol readout variable, or its sufficient statistic, at scale μ_k)

$$Z := (Y_0, Y_1, \ldots, Y_N),$$

and fix the endpoint projection (coarse-graining) map Π_{end} that retains only the two endpoints:

$$\Pi_{\text{end}}(Z) := (Y_0, Y_N) =: E.$$

Applying (A.1) to two path-process objects $\mathbb{P}_Z, \mathbb{Q}_Z$ with $\Phi = \Pi_{\text{end}}$ yields

$$D(\mathbb{P}_Z\|\mathbb{Q}_Z) \;\geq\; D(\mathbb{P}_E\|\mathbb{Q}_E). \tag{A.2}$$

In the notation of the main text, the left-hand side is the strict path divergence $\Delta_{\text{path}}(0 \to N)$, and the right-hand side is the endpoint-pair marginal divergence $\Delta_{\text{pair}}(0 \to N) := D(\mathbb{P}_E\|\mathbb{Q}_E)$, hence the endpoint–path bound $\Delta_{\text{pair}}(0 \to N) \leq \Delta_{\text{path}}(0 \to N)$.

A.2 Relative-entropy chain rule (classical form)

Decompose the same joint variable Z into endpoints and history:

$$E := (Y_0, Y_N), \qquad \mathcal{H} := (Y_1, \ldots, Y_{N-1}), \qquad Z \equiv (E, \mathcal{H}).$$

In the classical case, if the relative entropy is finite and conditional distributions are well-defined (e.g. $\mathbb{P}_{E,\mathcal{H}} \ll \mathbb{Q}_{E,\mathcal{H}}$), then the KL relative entropy satisfies the chain-rule decomposition

$$D(\mathbb{P}_{E,\mathcal{H}}\|\mathbb{Q}_{E,\mathcal{H}}) = D(\mathbb{P}_E\|\mathbb{Q}_E) + \mathbb{E}_{\mathbb{P}_E}\Big[D(\mathbb{P}_{\mathcal{H}|E}\|\mathbb{Q}_{\mathcal{H}|E})\Big]. \tag{A.3}$$

Accordingly, define the strict gap term by

$$\Gamma_{\mathrm{path}}(0 \to N) := \mathbb{E}_{\mathbb{P}_E}\Big[D(\mathbb{P}_{\mathcal{H}|E} \| \mathbb{Q}_{\mathcal{H}|E})\Big] \;\geq\; 0, \tag{A.4}$$

and obtain the exact identity

$$\Delta_{\mathrm{path}}(0 \to N) = \Delta_{\mathrm{pair}}(0 \to N) + \Gamma_{\mathrm{path}}(0 \to N), \tag{A.5}$$

where

$$\Delta_{\mathrm{path}}(0 \to N) := D(\mathbb{P}_{E,\mathcal{H}} \| \mathbb{Q}_{E,\mathcal{H}}) = D(\mathbb{P}_Z \| \mathbb{Q}_Z), \qquad \Delta_{\mathrm{pair}}(0 \to N) := D(\mathbb{P}_E \| \mathbb{Q}_E).$$

The nonnegativity in (A.4) follows from the nonnegativity of relative entropy and the fact that expectation preserves nonnegativity.

How the quantum case is used (structural note; no additional technical burden in the main text) The main text allows quantum predictive objects. In that case, the "interface + protocol" in the viewpoint V specifies a fixed readable statistical object (e.g. a measurement-outcome distribution or its sufficient statistics), so that at the interface level one can reduce to classical distributions and apply the same structural chain (A.1)–(A.5). The main text uses Γ_{path} only through this structural meaning (whether endpoint compression discards distinguishable history information) and does not require additional technical details of the quantum chain rule.

A.3 Equality condition: $\Gamma_{\mathrm{path}} = 0$ and endpoint sufficiency

From (A.4), one immediately has

$$\Gamma_{\mathrm{path}}(0 \to N) = 0 \quad \Longleftrightarrow \quad D(\mathbb{P}_{\mathcal{H}|E} \| \mathbb{Q}_{\mathcal{H}|E}) = 0 \;\text{ holds in the } \mathbb{P}_E\text{-almost-sure sense.} \tag{A.6}$$

In the classical case, by the uniqueness of the zero point of $D(\cdot\|\cdot)$, (A.6) is equivalent to

$$\mathbb{P}_{\mathcal{H}|E} = \mathbb{Q}_{\mathcal{H}|E} \quad (\mathbb{P}_E\text{-a.s.}). \tag{A.7}$$

This means that, conditional on the endpoints $E = (Y_0, Y_N)$, the two path processes are indistinguishable on the history component $\mathcal{H}$. Therefore, under the fixed viewpoint V, the endpoint variable E is sufficient for distinguishing the two scale flows: the history $\mathcal{H}$ provides no additional distinguishable information.

Equality condition for DPI (recoverability statement; not required) For DPI, equality typically corresponds to the endpoint coarse-graining being a (recoverable) sufficient compression on the relevant object family: there exists a recovery map $\mathcal{R}$ such that $(\mathcal{R} \circ \Phi)$ recovers (or approximately recovers) the joint object on the relevant family, hence DPI saturates. The structural judgement in the main text only needs the semantics of "endpoint sufficiency / recoverability" and does not rely on further technical characterization of such recovery maps.

Appendix B: Closed Forms in 1D/2D for the Gaussian Interface / Fisher Metric (Computational Details for Sec. 6 and Sec. 7)

Under the fixed viewpoint V locked in the main text, this appendix records closed-form or directly computable expressions for the Gaussian interface (equivalently, a Fisher realization) used in Sec. 6 and Sec. 7: the Route-I endpoint divergence Δ_{I}, the induced metric G, the information speed Ψ, the path length L_{II}, and the geometric gap Γ_{geo}. Viewpoint superscripts are omitted throughout.

B.1 One-dimensional interface: $Y \mid \theta \sim \mathcal{N}(\theta, \sigma^2)$

Assume the observation protocol outputs a scalar statistic Y at a fixed resolution, modeled under parameter θ by a Gaussian family $P_\theta = \mathcal{N}(\theta, \sigma^2)$, where $\sigma > 0$ is the resolution parameter locked by the protocol (and hence part of the viewpoint terms).

Endpoint KL (Route I) For two endpoints θ_0, θ_1,

$$D(P_{\theta_0} \| P_{\theta_1}) = \frac{(\theta_0 - \theta_1)^2}{2\sigma^2}. \tag{B.1}$$

Hence, the Route-I endpoint divergence (single-interface case) is

$$\Delta_{\mathrm{I}}(\theta_0 \to \theta_1) = \frac{(\theta_0 - \theta_1)^2}{2\sigma^2}. \tag{B.2}$$

Induced metric (Fisher) and line element From the second-order expansion of the equal-covariance Gaussian family (equivalently, Fisher information), the induced metric is constant:

$$G(\theta) = \frac{1}{\sigma^2}, \qquad \mathrm{d}s^2 = \frac{\mathrm{d}\theta^2}{\sigma^2}. \tag{B.3}$$

Information speed and path length (Route II) Let the scale flow be parametrized by a scale parameter t (in the main text, one may take $t = \gamma$ or $t = \ell$). Then

$$\Psi(t) = \frac{\mathrm{d}s}{\mathrm{d}t} = \frac{1}{\sigma}\left|\frac{\mathrm{d}\theta}{\mathrm{d}t}\right|, \qquad L_{\mathrm{II}}(t_0 \to t_1) = \int_{t_0}^{t_1} \Psi(t)\,\mathrm{d}t = \frac{1}{\sigma}\int_{t_0}^{t_1} \left|\frac{\mathrm{d}\theta}{\mathrm{d}t}\right| \mathrm{d}t. \tag{B.4}$$

Endpoint distance and geometric gap The endpoint-equivalent distance (as defined in the main text) is

$$d_{\mathrm{end}}(\theta_0 \to \theta_1) = \sqrt{2\Delta_{\mathrm{I}}} = \frac{|\theta_1 - \theta_0|}{\sigma}. \tag{B.5}$$

(Identity in this constant-metric Gaussian/Fisher realization; in general $d_{\mathrm{end}} := \mathrm{dist}_G$ (see equation (5.11)), and $d_{\mathrm{end}} = \sqrt{2\Delta_{\mathrm{I}}}$ is at most a local/implementation-specific approximation (see equation (5.12)).)

Therefore,

$$\Gamma_{\mathrm{geo}} = L_{\mathrm{II}} - d_{\mathrm{end}} = \frac{1}{\sigma}\left(\int_{t_0}^{t_1} \left|\frac{\mathrm{d}\theta}{\mathrm{d}t}\right| \mathrm{d}t - |\theta_1 - \theta_0|\right) \;\geq\; 0, \tag{B.6}$$

with equality if and only if $\theta(t)$ does not "turn back" on the window (i.e., no sign reversal of $\mathrm{d}\theta/\mathrm{d}t$, equivalently monotonicity on the window). In Sec. 6, one-loop QED on a threshold-free window gives a linear monotone drift of θ, hence $\Gamma_{\rm geo} = 0$ in that baseline case.

B.2 Two-dimensional interface: independent Gaussians $(Y_\theta, Y_u) \mid (\theta, u)$ (the minimal implementation locked in the main text)

Assume the observation protocol outputs a two-dimensional statistic (Y_θ, Y_u) satisfying

$$Y_\theta \mid \theta \sim \mathcal{N}(\theta, \sigma_\theta^2), \qquad Y_u \mid u \sim \mathcal{N}(u, \sigma_u^2), \qquad \text{and independent given } (\theta, u), \tag{B.7}$$

where $\sigma_\theta, \sigma_u > 0$ are resolution parameters locked by the protocol (and hence part of the viewpoint terms).

Endpoint KL and endpoint distance (Route I) For endpoint differences $\Delta\theta = \theta_1 - \theta_0$ and $\Delta u = u_1 - u_0$, the KL divergence for equal-covariance Gaussians gives

$$\Delta_{\rm I} = \frac{1}{2}\left(\frac{(\Delta\theta)^2}{\sigma_\theta^2} + \frac{(\Delta u)^2}{\sigma_u^2}\right), \qquad d_{\rm end} = \sqrt{2\Delta_{\rm I}} = \sqrt{\frac{(\Delta\theta)^2}{\sigma_\theta^2} + \frac{(\Delta u)^2}{\sigma_u^2}}. \tag{B.8}$$

Induced metric and line element The Fisher information yields a constant diagonal metric

$$G = \mathrm{diag}\left(\sigma_\theta^{-2}, \sigma_u^{-2}\right), \qquad \mathrm{d}s^2 = \frac{\mathrm{d}\theta^2}{\sigma_\theta^2} + \frac{\mathrm{d}u^2}{\sigma_u^2}. \tag{B.9}$$

Information speed and path length (Route II) Parametrize the scale flow by the coarse-graining parameter ℓ (dots denote $\mathrm{d}/\mathrm{d}\ell$). Then

$$\Psi(\ell) = \sqrt{\frac{\dot\theta(\ell)^2}{\sigma_\theta^2} + \frac{\dot u(\ell)^2}{\sigma_u^2}}, \qquad L_{\rm II}(0 \to T) = \int_0^T \Psi(\ell)\,\mathrm{d}\ell. \tag{B.10}$$

B.3 Minimal 2D flow in Sec. 7: closed forms for $L_{\rm II}$ and $\Gamma_{\rm geo}$

For the minimal scale flow used in Sec. 7,

$$\dot\theta = b, \qquad \dot u = -p\,u, \qquad u(\ell) = u_0 e^{-p\ell}, \qquad b > 0,\ p > 0,\ \ell \in [0, T], \tag{B.11}$$

equation (B.10) gives the information speed

$$\Psi(\ell) = \sqrt{\frac{b^2}{\sigma_\theta^2} + \frac{p^2 u_0^2 e^{-2p\ell}}{\sigma_u^2}}. \tag{B.12}$$

Hence the path length is

$$L_{\rm II}(0 \to T) = \int_0^T \sqrt{\frac{b^2}{\sigma_\theta^2} + \frac{p^2 u_0^2 e^{-2p\ell}}{\sigma_u^2}}\,\mathrm{d}\ell. \tag{B.13}$$

Closed form (identity) Define

$$a := \frac{b}{\sigma_\theta} > 0, \qquad c := \frac{p\,u_0}{\sigma_u} \in \mathbb{R}, \tag{B.14}$$

then (B.13) integrates to

$$L_{\mathrm{II}}(0 \to T) = aT + \frac{1}{p}\left[\sqrt{a^2+c^2} - \sqrt{a^2+c^2e^{-2pT}} + a\ln\frac{a+\sqrt{a^2+c^2e^{-2pT}}}{a+\sqrt{a^2+c^2}}\right]. \tag{B.15}$$

This expression degenerates continuously to $L_{\mathrm{II}} = aT$ at $c = 0$ (consistent with the 1D baseline), with no need for case splitting.

Endpoint distance and geometric gap (closed form) From (B.11), the endpoint differences are

$$\Delta\theta = bT, \qquad \Delta u = u_0(e^{-pT} - 1).$$

Substituting into (B.8) gives the endpoint distance

$$d_{\mathrm{end}}(T) = \sqrt{\left(\frac{bT}{\sigma_\theta}\right)^2 + \left(\frac{u_0}{\sigma_u}(1-e^{-pT})\right)^2}. \tag{B.16}$$

Therefore,

$$\Gamma_{\mathrm{geo}}(T) = L_{\mathrm{II}}(0 \to T) - d_{\mathrm{end}}(T), \tag{B.17}$$

where $L_{\mathrm{II}}(0 \to T)$ is given by (B.15) and $d_{\mathrm{end}}(T)$ is given by (B.16).

Nonnegativity and equality condition (locked) Under the constant metric $G = \mathrm{diag}(\sigma_\theta^{-2}, \sigma_u^{-2})$, L_{II} is the arc length of the trajectory and d_{end} is the endpoint geodesic distance. Hence for any $T \geq 0$,

$$\Gamma_{\mathrm{geo}}(T) \geq 0. \tag{B.18}$$

Moreover, equality holds if and only if the trajectory is a straight line without turn-back in the scaled coordinates

$$x(\ell) := \theta(\ell)/\sigma_\theta, \qquad y(\ell) := u(\ell)/\sigma_u.$$

For the exponentially decaying trajectory in (B.11), for $T > 0$ this straight-line condition holds only in the degenerate cases:

$$\Gamma_{\mathrm{geo}}(T) = 0 \iff (u_0 = 0) \text{ or } (p = 0). \tag{B.19}$$

Hence under the nontrivial setting $u_0 \neq 0,\ p > 0$ in the main text, one has $\Gamma_{\mathrm{geo}}(T) > 0$ for any $T > 0$.

Appendix C: Choice of EFT Direction and the Dimensionless Convention ($p = d - 4$ and the Conditions for Neglecting Mixing Terms)

This appendix explains the origin of the dimensionless normalization and the scaling exponent p for the irrelevant direction u used in Sec. 7, and clarifies the scope and testable conditions under which one may neglect mixing terms with the principal direction (e.g. α or the equivalent coordinate $\theta = \alpha^{-1}$). All statements are made under the fixed viewpoint V (including the scale-coordinate convention) locked in the main text; viewpoint superscripts are omitted for brevity.

C.1 Dimensionless normalization and the canonical scaling exponent

Consider a local operator $\mathcal{O}_d$ in a four-dimensional EFT, with engineering dimension $d > 4$. Suppose the effective action in some basis contains the term

$$S_{\text{EFT}} \supset \int d^4x\, C_d(\mu)\, \mathcal{O}_d(x).$$

Since $[d^4x] = -4$ and $[\mathcal{O}_d] = d$, the coefficient must satisfy

$$[C_d] = 4 - d$$

so that the action is dimensionless. Hence define the dimensionless Wilson coordinate

$$u(\mu) := \mu^{\,d-4} C_d(\mu), \qquad p := d - 4 > 0. \tag{C.1}$$

In Sec. 7, the coarse-graining parameter is

$$\ell := \ln(\mu_0/\mu), \qquad \ell \uparrow \iff \mu \downarrow,$$

so that increasing ℓ corresponds to UV→IR coarse-graining. In the minimal approximation that retains only canonical scaling and neglects anomalous dimensions and basis mixing, the irrelevant direction obeys

$$\frac{du}{d\ell} = -p\, u, \qquad \Rightarrow \qquad u(\ell) = u_0\, e^{-p\ell}, \tag{C.2}$$

which encodes the Wilsonian structure that irrelevant directions are suppressed in the IR when $p > 0$.

C.2 A more general form with mixing and a testable condition for "neglecting mixing terms"

More generally, in a fixed operator basis an irrelevant direction typically receives anomalous-dimension corrections and mixes with other directions. In the semantics of a truncation of dimension ≥ 2 (principal direction $\theta = \alpha^{-1}$ together with an EFT family $\{u, u_j\}$), the scale flow of u can be written as

$$\frac{du}{d\ell} = -\big(p + \gamma_u(\alpha)\big)\, u + r(\alpha, \theta) + \sum_{j \neq u} m_{uj}(\alpha)\, u_j + \cdots, \tag{C.3}$$

where $\gamma_u(\alpha)$ denotes the anomalous-dimension correction, $m_{uj}(\alpha)$ denotes linear mixing with other EFT directions u_j, and $r(\alpha, \theta)$ denotes a possible inhomogeneous source term (e.g. induced by matching/redefinitions). Here u_j is of the same type as u: dimensionless Wilson coordinates. The main text does not expand their physical meaning, and uses them only as placeholders for "additional structure directions".

The meaning of using the minimal flow (C.2) in Sec. 7 is that, on the chosen window and at the chosen accuracy, all terms in (C.3) other than $-p\,u$ contribute to $du/d\ell$ at a controllably negligible level. To state this as a reproducible condition, collect the neglected part in (C.3) into a residual term

$$\mathcal{R}_u(\ell) := -\gamma_u(\alpha(\ell))\, u(\ell) + r(\alpha(\ell), \theta(\ell)) + \sum_{j \neq u} m_{uj}(\alpha(\ell))\, u_j(\ell) + \cdots.$$

A sufficient, "hard" criterion that does not degenerate as $u(\ell) \to 0$ is: on the window $\ell \in [0, T]$ there exists a small parameter $\varepsilon \ll 1$ such that

$$\sup_{\ell \in [0,T]} \frac{\big|\mathcal{R}_u(\ell)\big|}{p\,|u(\ell)| + \sigma_u} \;\leq\; \varepsilon. \tag{C.4}$$

Here $\sigma_u > 0$ is the interface-resolution parameter in Sec. 7 (see (7.9)); it provides a normalization floor in the indistinguishable regime $|u(\ell)| \lesssim \sigma_u$, avoiding denominator degeneracy that would occur with a purely relative-error criterion. This term is fully fixed by the "protocol/resolution" clauses of the viewpoint V, and does not introduce new physical degrees of freedom.

It should be emphasized that even if (C.4) fails, the structural conclusion in Sec. 7 that "a generic 2D trajectory is non-geodesic $\Rightarrow \Gamma_{\text{geo}} > 0$" typically still holds. What changes is the specific functional form and numerical magnitude of Ψ and L_{II}, not the mechanism that a non-affine / non-straight scale process in dimension ≥ 2 triggers a gap. In that case, one should treat the actual flow in (C.3) as part of the viewpoint clauses and recompute $\Psi, L_{\text{II}}, \Gamma_{\text{geo}}$ accordingly.

C.3 Physical options: Euler–Heisenberg and four-fermion contact terms (illustrative)

To make the irrelevant direction u in Sec. 7 more physically explicit, we list two common EFT directions as optional illustrations. They are not required for the derivations; their role is to instantiate the "additional structure direction at intermediate scales" as standard EFT coefficients, so that the 2D trajectory and the mechanism for $\Gamma_{\text{geo}} > 0$ do not rely on an abstract parameter.

Option 1: Euler–Heisenberg-type F^4 direction ($d = 8$) In a low-energy effective description of QED, one may take a representative $d = 8$ F^4-type operator, e.g.

$$\mathcal{O}_8 \sim (F_{\mu\nu}F^{\mu\nu})^2 \quad \text{or an equivalent gauge-invariant basis.}$$

If its Wilson coefficient is denoted by $C_8(\mu)$, then the dimensionless coordinate is

$$u(\mu) = \mu^4 C_8(\mu), \qquad p = d - 4 = 4.$$

The advantage is that it connects directly to QED effective nonlinear responses (e.g. vacuum nonlinearity / light–light scattering), making it straightforward to express "interface sensitivity to u" in terms of concrete observable combinations.

Option 2: four-fermion contact term ($d = 6$) In a more general effective theory, four-fermion operators are among the most common irrelevant directions, e.g.

$$\mathcal{O}_6 \sim (\bar{\psi}\gamma_\mu\psi)(\bar{\psi}\gamma^\mu\psi) \quad \text{or a problem-matched four-fermion basis.}$$

Notation. Here γ^μ denotes Dirac gamma matrices and ψ denotes a fermion field; both are unrelated to the RG scale coordinate γ (denoted by γ in this paper) and to the information speed Ψ.

If the coefficient is $C_6(\mu)$, then

$$u(\mu) = \mu^2 C_6(\mu), \qquad p = d - 4 = 2.$$

Even if it is not dominant in some pure-QED windows, it still provides a clear example of an "irrelevant but readable" direction, enabling a reproducible numerical contrast for the nonzero behavior of Γ.

Synchronized locking of viewpoint clauses The common feature of these options is that they instantiate the "extra structure direction" as explicit EFT coefficients. The paper allows replacing the concrete meaning of u across different physical problems; once replaced, it must be synchronized and locked in the viewpoint V, including (i) the statistics (or sufficient statistics) through which the interface reads u, and (ii) the error model/covariance (thereby locking the induced metric G). Otherwise, the magnitude of Γ becomes a cross-viewpoint output and loses reproducibility.

Appendix D: Operational Workflow Pseudocode (Input–Output Interface Style)

Scope and precedence (domain-gated). This appendix provides a reproducible *implementation-level* workflow for producing numerical estimates $\widehat{\Delta}_I, \widehat{L}_{\mathrm{II}}, \widehat{d_{\mathrm{end}}}, \widehat{\Gamma}_{\mathrm{geo}}$ *only when such quantities are reportable*. It does *not* define the status semantics, admissibility conditions, or gate logic. All status decisions—including the admissible/mismatch regimes, abyss events (Type-I/Type-II), π-criticality, and domain flags—are defined by the domain-gated reporting protocol in Sec. 8 and Appendix H (Algorithms 1–2).

Concretely, the present workflow is executed only when the domain-gated protocol returns `STATUS` $\in$ {`ADMISSIBLE`, `MISMATCH`} and *not* `REC-CRITICAL`. In all other cases, the paper reports viewpoint-locked labels (`ABYSS` / `REC-CRITICAL` and, when applicable, `FLAG:GEO_OUT_OF_DOMAIN`) together with trigger metadata, and this appendix must *not* be used to force numerical surrogates. Any apparent conflict between an implementation detail below and the domain-gated protocol is resolved in favor of Sec. 8 and Appendix H.

Unless stated otherwise, viewpoint superscripts are omitted.

Implementation-to-notation map (locked). The workflow pseudocode uses ASCII-style variable names, with a suffix `_hat` for numerical estimates (hat notation in the main text). We fix the following correspondences: `Delta_I_hat` $\leftrightarrow \widehat{\Delta}_I$, `L_II_hat` $\leftrightarrow \widehat{L}_{\mathrm{II}}$, `d_end_hat` $\leftrightarrow \widehat{d_{\mathrm{end}}}$, `Gamma_geo_hat` $\leftrightarrow \widehat{\Gamma}_{\mathrm{geo}}$, and `V_eps` $\leftrightarrow V_\varepsilon$.

D.1 Input specification (must be locked and reported)

(I) Scale window and discrete grid

- Window: $[\mu_0, \mu_1]$;
- Grid: $\{\mu_k\}_{k=0}^N$ (strictly monotone), together with a locked choice of scale coordinate.

The main text allows two admissible coordinate conventions; you must choose *exactly one* and keep it consistent throughout the full pipeline:

- Choose $\gamma := \ln(\mu/\mu_0)$: then

$$\gamma_k := \ln(\mu_k/\mu_0), \qquad \Delta\gamma_k := \gamma_{k+1} - \gamma_k > 0.$$

- Choose $\ell := -\gamma = \ln(\mu_0/\mu)$: then

$$\ell_k := \ln(\mu_0/\mu_k), \qquad \Delta\ell_k := \ell_{k+1} - \ell_k > 0.$$

In the pseudocode below we use a unified symbol t for the locked scale coordinate (either $t =$ *gscale* or $t =$
ell), and we keep the sign convention consistent across all derivatives/integrals. We write t_k and $\Delta t_k := t_{k+1} - t_k > 0$.

(II) Locked viewpoint object V_ε (full tuple, including resolution and aggregation) The viewpoint must be locked and reported as the full tuple (see the Viewpoint Card in Sec. 8.1; include the resolution channel family N_ε, the meaning/units of $\varepsilon > 0$, and the aggregation rule Agg). Here protocol$_\varepsilon$ includes an explicit resolution/measurement-noise channel family N_ε and the meaning/units of $\varepsilon > 0$. The aggregation operator Agg (e.g. sup or a fixed weighted aggregation) is treated as a component of V_ε and must remain fixed across comparable runs.

(III) Interface outputs / predictive objects (protocol readouts) For each scale point k and each interface $O \in \mathcal{O}$, obtain or generate the protocol readout object

$$X_{\mu_k}^{(O,\varepsilon)}.$$

The concrete form of $X_{\mu_k}^{(O,\varepsilon)}$ may be a distribution, a quantum state, or a sufficient-statistics representation under the protocol; however, it must be consistent throughout the pipeline and consistent with the clauses in V_ε.

(IV) Applicability domain for Route II and for Γ_{geo} Route-II objects $(\dot{g}, \Psi, L_{\mathrm{II}})$ and the geometric-proxy gap $\Gamma_{\mathrm{geo}} = L_{\mathrm{II}} - d_{\mathrm{end}}$ are reported only on windows declared admissible by the domain-gated protocol (A1–A3 / Condition $\mathcal{G}_{\mathrm{bridge}}$ / Condition $\mathcal{G}_{\mathrm{geo}}$, as used in Sec. 8). If Condition $\mathrm{G}_{\mathrm{geo}}$ is not satisfied under the locked protocol/aggregation, the output must be flagged `FLAG:GEO_OUT_OF_DOMAIN` and $\widehat{d_{\mathrm{end}}}, \widehat{\Gamma}_{\mathrm{geo}}$ must not be reported.

(V) Workflow-augmented conventions (must be locked if numerical values are reported) When a *computed* value of $\widehat{\Gamma}_{\mathrm{geo}}$ is *reported* as a diagnostic output (or used for hypothesis testing), the locked viewpoint V_ε must be treated as *workflow-augmented*. In addition to the conceptual clauses of the viewpoint (Sec. 8.1), V_ε includes a locked set of numerical conventions required to make reported estimates well-defined and comparable, including (at minimum): divergence normalization/smoothing conventions; reconstruction solver conventions; metric estimation and any SPD enforcement; the discrete rule for $\widehat{L}_{\mathrm{II}}$; the computation rule for $d_{\mathrm{end}} = \mathrm{dist}_G$; tolerances and random seeds. Once fixed, these conventions are treated as part of the locked viewpoint and must be reported (e.g. via the Viewpoint Card in Sec. 8.1).

D.2 Workflow: conditional execution under the domain-gated protocol

Mandatory precedence rule. The domain-gated protocol (Sec. 8.1 and Appendix H) is executed first. This appendix specifies what to do *after* that protocol returns a status that permits numerical reporting.

Naming convention (pseudocode). In the pseudocode below, a suffix `_hat` denotes a numerical estimate corresponding to the hat notation in the main text (e.g. `Delta_I_hat` corresponds to $\widehat{\Delta}_I$).

```
# WORKFLOW (SUBORDINATE TO DOMAIN-GATED REPORTING; NUMERICS ONLY WHEN PERMITTED)

Input (ALL items below are part of the locked viewpoint V_eps and must be reported):
  window [muScale_0, muScale_1]
  grid {muScale_k}_{k=0..N} (strictly monotone; N fixed)
```

```
  scale coordinate choice: t = gamma or t = ell
  locked viewpoint V_eps = (O_set, protocol_eps incl. N_eps, Reg, S, threshold_rule,
                            model_class M, Rec, D, Agg)
  protocol readouts X_{mu_k}^{O,eps} for each k and O in O_set

  workflow conventions (MUST be locked as part of V_eps for reproducibility):
    divergence_conventions (argument ordering, normalization, log base, eps_D if
    used, support bookkeeping)
    reconstruction_solver (multi-start budget, optimizer, stopping rule, seeds)
    metric_spec (G_rule or estimator, SPD_enforcement_rule if used, eps_SPD)
    L_II_quadrature (midpoint rule / quadrature rule)

  conventions required ONLY if d_end_hat / Gamma_geo_hat are reportable (i.e., if
  not FLAG:GEO_OUT_OF_DOMAIN):
    d_end_implementation (e.g., global geodesic solver under dist_G, or a
    closed-form/direct mapping from Delta_I
                        in special constant-metric implementations where such an
                        identity holds)
    geodesic_distance_spec (DistAlg, discretization_L, multi-start schedule,
                            K_starts, optimizer, tau_dist, max_iters, seed_global)
                            # if a solver is used
    numerical_validity (grid refinement rule; acceptance tolerances tau_refine,
    tau_zero)  # tau_zero = tau (Sec. 8.3), not a mismatch threshold

# Step 0: Domain-gated protocol (authoritative; see Sec.8 / Appendix Algorithm)
(STATUS, payload, metadata) = DomainGatedReport( V_eps, window, grid )

if STATUS == ABYSS:
    return (STATUS, metadata, payload_optional)

if STATUS == REC-CRITICAL:
    return (STATUS, metadata, payload_optional)

# From here on, STATUS is ADMISSIBLE or MISMATCH.

# Initialize numeric variables defensively (avoid undefined references in edge
cases)
L_II_hat  = None
d_end_hat       = None
Gamma_geo_hat = None

# Step 0': Extract authoritative in-domain flags (Writeup A)
RouteII_in_domain = (STATUS in {ADMISSIBLE, MISMATCH})
Ggeo_in_domain    = (not metadata.has_flag("FLAG:GEO_OUT_OF_DOMAIN"))

# Step 1: Reconstruct trajectory (ONLY if not already provided by DomainGatedReport)
if payload.contains("g_hat"):
    g_hat = payload["g_hat"]
else:
    for k = 0..N:
        g_hat[k] = Rec( V_eps, {X_{mu_k}^{O,eps}}_{O in O_set} )
```

```
# Step 2: Local metric along the path (ONLY if Route-II is in-domain and a metric is
required)
# NOTE: This is used to build L_II_hat; it is NOT gated by Ggeo.
if RouteII_in_domain:
    for k = 0..N:
        A = G_rule( g_hat[k] )
        A = 0.5 * (A + A^T)
        G_hat[k] = SPD_Project( A; eps_SPD )

# Step 3: Route-II discrete length proxy (L_II_hat)
# If DomainGatedReport already returns L_II_hat, it MUST have been computed under
the same
# locked L_II_quadrature conventions recorded in V_eps; otherwise ignore payload and
recompute here.
if payload.contains("L_II_hat"):
    L_II_hat = payload["L_II_hat"]
    # Consistency check (engineering/audit): verify payload quadrature conventions
    match V_eps.L_II_quadrature.
    # If mismatch is detected (or conventions are not declared), discard payload
    L_II_hat and recompute below.
else:
    if RouteII_in_domain:
        L_II_hat = 0
        for k = 0..N-1:
            G_mid   = MidpointRule( G_hat[k], G_hat[k+1] )
            Delta_g = g_hat[k+1] - g_hat[k]
            Psi_ds  = sqrt( Delta_g^T * G_mid * Delta_g )
            L_II_hat += Psi_ds

# Step 4--6: Endpoint distance and gap (ONLY if Ggeo is in-domain AND L_II_hat is
available)
# d_end_hat must be instantiated according to the locked d_end_implementation:
#   - either via a global geodesic solver under dist_G (with locked
geodesic_distance_spec), or
#   - via a closed-form/direct mapping from Delta_I in special constant-metric
implementations where such an identity holds.
if Ggeo_in_domain and (L_II_hat is not None):
    g0 = g_hat[0]; gN = g_hat[N]
    d_end_hat = dist_G_global( g0, gN; G_rule, SPD_Project, MidpointRule,
                         L, K_starts, optimizer, tau_dist, max_iters, seed_global )

    Gamma_geo_hat = L_II_hat - d_end_hat

    # Mandatory numerical validity checks (do NOT clip Gamma_geo_hat)
    Validate_or_Refine( V_eps, Gamma_geo_hat, L_II_hat, d_end_hat )

# Output (conditional, viewpoint-locked)
Output:
  Always:
    STATUS, metadata (triggered gates; eps; channel family; Type-I/II if applicable;
```

```
                    Rec-diagnostics if applicable and provided by
                    DomainGatedReport),
  plus the full Viewpoint Card for V_eps.
If STATUS in {ADMISSIBLE, MISMATCH}:
  report Delta_I_hat (from DomainGatedReport)
  if L_II_hat is not None:
      report L_II_hat
  if Ggeo_in_domain and (Gamma_geo_hat is not None):
      report d_end_hat and Gamma_geo_hat, plus numerical validity report and (if
      computed) uncertainty report
  else:
      flag FLAG:GEO_OUT_OF_DOMAIN; do not report d_end_hat or Gamma_geo_hat.
```

Implementation notes (hard constraints; no extra degrees of freedom).

- Agg must not be switched across different windows / different experiments intended to be comparable.
- The model-class dimension d must not vary with scale within a single reported result chain.
- If $\dot{g}$ is obtained by finite differences, the differencing rule (forward/central/smoothing/fitting) must be written into the protocol and fixed.
- If Γ_{geo} is reported, then the implementations of $g(t)$, $G(g)$, and dist_G must be locked and reported (Condition G_{geo}).
- One must not clip negative $\widehat{\Gamma}_{\text{geo}}$ to zero. If $\widehat{\Gamma}_{\text{geo}} < 0$ (or near 0 within tolerance), a locked numerical-validity check must be applied (grid refinement for $\widehat{L}_{\text{II}}$ and solver tightening/refinement for $\widehat{d_{\text{end}}}$).
- If the window crosses a threshold, segment the window according to the locked threshold rule, run the workflow within each segment, and report the stitching/matching conditions (including which quantities are or are not comparable across segments).

D.3 Minimum reproducibility package

Any reported result must follow the conditional output logic of Sec. 8 and Appendix H. At minimum, every experiment report must include:

1. **Viewpoint Card (required).** A complete locked specification of V_ε as in Sec. 6 and Appendix A, including (at minimum):
   - interface family O (and whether $O = \{O\}$ is enforced);
   - readout protocol $\text{protocol}_\varepsilon$, including the resolution parameters and N_ε;
   - the endpoint divergences (R, S) used in Route I;
   - any declared domain D and aggregation operator Agg;
   - model class $\mathcal{M}$ and the reconstruction map Π_{rec};
   - the threshold rule used for domain-gated reporting.

Include any workflow-augmented conventions used to instantiate reported numerics (e.g. random seeds, solver tolerances, and implementation choices).

2. **Status and trigger metadata (required).** Report:

   - the `STATUS` value in $\{$`ADMISSIBLE`, `MISMATCH`, `ABYSS`, `REC-CRITICAL`$\}$;
   - the triggered gate(s) and trigger location (endpoint/path/conditional term);
   - any domain flags (e.g. `FLAG:GEO_OUT_OF_DOMAIN`).

   If `ABYSS` occurs, include the Type-I/Type-II classification and the ε-scan/audit information when performed.

3. **Numerics (conditional).** If `STATUS`$\in$ $\{$`ADMISSIBLE`, `MISMATCH`$\}$, report only the numerical surrogates permitted by the protocol. If `FLAG:GEO_OUT_OF_DOMAIN` is flagged, do not report $\widehat{d_{\mathrm{end}}}$ or $\widehat{\Gamma}_{\mathrm{geo}}$.

4. **Numerical validity and uncertainty (conditional).** Whenever $\widehat{\Gamma}_{\mathrm{geo}}$ is reported, include the numerical validity audit (Step 6) under the locked refinement/tolerance rules, and state the uncertainty sources included (at least: reconstruction, discretization, and solver tolerances), or explicitly declare that uncertainty was not computed.

In particular, under `ABYSS` or `REC-CRITICAL`, the report prioritizes structural labels and trigger metadata, and does not force numerical estimators beyond what the domain-gated protocol explicitly permits.

Appendix E: Window Dependence of the QED One-Loop Coefficient b and the Piecewise Rule for "Active Species" (Normalization Note)

This appendix provides no derivations. It only gives a standardized way of writing "how to choose a window / how to splice across thresholds" for Sec. 6, so that threshold handling is promoted from an implicit assumption to an explicit clause of the viewpoint V. Unless stated otherwise, viewpoint superscripts are omitted; the scale variable is written uniformly as μ.

E.1 Window-wise expression of the one-loop coefficient (piecewise by the active set)

In the QED one-loop approximation, the running of the electromagnetic coupling α can be written as

$$\frac{d\alpha}{d\ln\mu} = b(\mu)\,\alpha^2, \qquad b(\mu) = \frac{2}{3\pi}\sum_{f\in\mathrm{active}(\mu)} w_f\,Q_f^2. \tag{E.1}$$

Here active(μ) denotes the set of electrically charged Dirac fermions treated as "active" at scale μ (locked by the threshold rule in the viewpoint V); Q_f is the charge in units of e; and w_f is an optional multiplicity factor (e.g. color degeneracy). If not needed, one sets $w_f \equiv 1$ and *locks this choice* in the viewpoint clauses. Under the normalization used in Sec. 6, one may identify this multiplicity with the degeneracy factor in the main text, i.e. $w_f \equiv n_f$; we keep w_f here only to emphasize that the weight is fixed by the chosen threshold rule in the locked viewpoint V.

Introduce the coordinate consistent with the main text,

$$\theta(\mu) := \alpha^{-1}(\mu),$$

then (E.1) is equivalent to

$$\frac{d\theta}{d\ln\mu} = -b(\mu). \tag{E.2}$$

No-threshold window (constant within a segment) If, on the window $[\mu_0, \mu_1]$, the set active(μ) does not change with μ, then $b(\mu) \equiv b$ is a constant, and hence

$$\theta(\mu_1) - \theta(\mu_0) = -b\,\ln\frac{\mu_1}{\mu_0}. \tag{E.3}$$

Equation (E.3) is the "within-segment endpoint linear law" used in Sec. 6: in the θ coordinate, the one-loop flow is a constant-speed straight line.

Interface with the ℓ coordinate (if the main text uses ℓ) If the main text uses $\ell := \ln(\mu_0/\mu)$, then $d/d\ell = -\,d/d\ln\mu$, and (E.2) becomes

$$\frac{d\theta}{d\ell} = b(\mu(\ell)), \tag{E.4}$$

which yields the within-segment affine form $\theta(\ell) = \theta_0 + b\,\ell$ on a no-threshold segment. (The piecewise rules below are coordinate-independent; one only needs to lock the coordinate choice in V and keep it consistent throughout.)

E.2 Piecewise rule and matching clauses (threshold splicing)

When the scale window crosses mass thresholds, threshold handling must be written as an explicit clause of the viewpoint V. We adopt the following normalized writing. We do not claim any particular decoupling scheme is "optimal"; we only require that the clause be *locked* and *checkable*.

(T1) Threshold segmentation: partition the window into intervals with a fixed active set Given a collection of threshold scales $\{m_j\}_{j=1}^{J}$ (typically of the same order as particle masses), take the subset that falls in the open interval (μ_0, μ_1) and sort by scale:

$$\mu_0 < \tau_1 < \tau_2 < \cdots < \tau_{J'} < \mu_1, \qquad \{\tau_j\} = \{m_j\} \cap (\mu_0, \mu_1).$$

Define the segment endpoints by

$$\mu_{(0)} := \mu_0, \qquad \mu_{(j)} := \tau_j \ (j = 1, \ldots, J'), \qquad \mu_{(J'+1)} := \mu_1,$$

and the segments

$$I_j := [\mu_{(j-1)}, \mu_{(j)}], \qquad j = 1, \ldots, J' + 1.$$

The core requirement of the threshold rule is: on each segment I_j, the active set active(μ) is constant, so that

$$b(\mu) \equiv b_j \quad (\mu \in I_j), \qquad b_j = \frac{2}{3\pi} \sum_{f \in \text{active}_j} w_f Q_f^2, \tag{E.5}$$

and each segment must report active$_j$ (or equivalently report b_j together with the locked conventions for w_f, Q_f).

(T2) Matching point and matching rule: lock the change of degrees of freedom as a clause For adjacent segments I_j and I_{j+1}, choose a matching point near their interface,

$$\mu_j^{\text{match}} \sim \tau_j,$$

and impose a matching rule at that point. Both the selection rule for μ_j^{match} and the matching form must be written (and locked) in the threshold clause of V, for example:

- Continuous matching (minimal convention): $\alpha_j(\mu_j^{\text{match}}) = \alpha_{j+1}(\mu_j^{\text{match}})$, equivalently $\theta_j(\mu_j^{\text{match}}) = \theta_{j+1}(\mu_j^{\text{match}})$;
- Matching with a correction: $\alpha_{j+1}(\mu_j^{\text{match}}) = \alpha_j(\mu_j^{\text{match}}) + \delta\alpha_j$, where the definition, perturbative order, and negligibility conditions of $\delta\alpha_j$ must be locked; equivalently, in the θ coordinate one may write

  $$\theta_{j+1}(\mu_j^{\text{match}}) = \theta_j(\mu_j^{\text{match}}) + \Delta\theta_j^{\text{match}},$$

  where $\Delta\theta_j^{\text{match}}$ is fully determined by the chosen matching rule.

The "no-threshold window" baseline used in the main text corresponds to $J' = 0$, so there are no matching points and no matching rule.

(T3) Within-segment endpoint relations and full-window splicing On each segment $I_j = [\mu_{(j-1)}, \mu_{(j)}]$, (E.2) and (E.5) give the within-segment endpoint relation

$$\theta(\mu_{(j)}) - \theta(\mu_{(j-1)}) = -b_j \ln \frac{\mu_{(j)}}{\mu_{(j-1)}}, \qquad j = 1, \ldots, J' + 1. \tag{E.6}$$

By connecting adjacent segments using the matching clause (T2), the full-window endpoint relation can be written in the normalized form "sum of segment contributions + matching corrections":

$$\theta(\mu_1) - \theta(\mu_0) = -\sum_{j=1}^{J'+1} b_j \ln \frac{\mu_{(j)}}{\mu_{(j-1)}} + \sum_{j=1}^{J'} \Delta\theta_j^{\mathrm{match}}. \tag{E.7}$$

Under continuous matching, $\Delta\theta_j^{\mathrm{match}} = 0$, and (E.7) reduces to a pure sum over piecewise-constant coefficients b_j.

Within-segment "constant-speed straight line" vs. cross-threshold "kinks" Equation (E.6) shows that within each segment, θ is affine in $\ln \mu$ (or in ℓ). Across thresholds, changes in b_j (and possible $\Delta\theta_j^{\mathrm{match}}$) produce kinks/jumps in the trajectory. Therefore, "constant-speed straight line in θ at one loop" should be understood as *piecewise* constant-speed straight lines, rather than a single global straight line on a cross-threshold window.

E.3 Relation to the gap Γ (thresholds as viewpoint input, not an implicit assumption)

Once segmentation and matching are introduced, additional structural inputs enter at the viewpoint level: the window is no longer described by a single constant b, but is determined jointly by

$$\{I_j\}, \quad \{\mathrm{active}_j\} \text{ (equivalently } \{b_j\}), \quad \{\mu_j^{\mathrm{match}}, \text{ matching rule}\}.$$

Structural meaning (kept strictly within scope)

- At the level of strict objects: if your "history variable / process object" treats the segment labels, changes of active sets, or matching corrections (equivalently: process information from the locked threshold rule) as distinguishable information, then endpoint compression may discard such process information, providing a structural source for $\Gamma_{\mathrm{path}} > 0$.

- At the level of the geometric proxy: if the geometric coordinate is chosen as one-dimensional $g(t) = \theta(t)$ and θ is monotone on the window, then in the minimal one-dimensional constant-metric implementation one may still obtain $\Gamma_{\mathrm{geo}} = 0$ (the degenerate case "arc length = endpoint distance"). To make threshold structure explicit in Γ_{geo}, the viewpoint clauses must encode the "threshold structure" into the readable coordinates themselves (e.g. by enlarging $g(t)$ to include additional readable directions/effective degrees of freedom, or by following the main-text T3 rule to report segment-wise gaps alongside the full-window result).

Reporting requirement (mandatory for cross-threshold windows) Therefore, if the main text reports Γ on a cross-threshold window (either the strict Γ_{path} or the geometric proxy Γ_{geo}), one must simultaneously report the three clause-level inputs above:

$$\{I_j\}, \quad \{\text{active}_j\} \text{ (or equivalently } \{b_j\}), \quad \{\mu_j^{\text{match}}, \text{ matching rule}\}.$$

Missing any of these items makes the numerical comparison of Γ degenerate into a cross-viewpoint output and hence non-checkable (consistent with the viewpoint-locking clauses in Sec. 2).

Appendix F: Domain, Absolute Continuity, and Regularization by ε

Purpose. This appendix formalizes the operational domain assumptions used throughout the paper: (i) when divergences are finite (classical and quantum), (ii) how finite resolution is encoded by a viewpoint-locked channel N_ε, and (iii) how we classify non-finiteness events into `Type-I` (structural abyss) vs `Type-II` (protocol-incomplete divergence). The goal is reproducibility and consistent reporting under a locked viewpoint V_ε.

F.1 Classical finiteness: absolute continuity and KL divergence

Let $(\Omega, \mathcal{A})$ be a measurable space and let P, Q be probability measures on it. Write $P \ll Q$ when P is absolutely continuous with respect to Q.

Definition F.1 (KL divergence and finiteness)**.** If $P \ll Q$, define

$$D_{\mathrm{KL}}(P\|Q) := \int_\Omega \log\left(\frac{dP}{dQ}\right) dP,$$

with the convention $0 \log 0 := 0$. If $P \not\ll Q$, set $D_{\mathrm{KL}}(P\|Q) := \infty$.

Remark F.1 (Operational meaning)*.* In the endpoint–path framework, any occurrence of $D(P\|Q) = \infty$ indicates that the protocol readouts are not comparable under the declared viewpoint. In the main text this is treated as a domain boundary label (`ABYSS`, Type-I unless bookkeeping indicates Type-II; see Sec. 8.1).

F.2 Quantum finiteness: support inclusion for quantum relative entropy

Let ρ, σ be density operators on a finite-dimensional Hilbert space. Denote by $\mathrm{supp}(\rho)$ the support of ρ (span of eigenvectors with nonzero eigenvalues).

Definition F.2 (Quantum relative entropy and finiteness)**.** The quantum relative entropy is

$$S(\rho\|\sigma) := \mathrm{Tr}\big(\rho(\log\rho - \log\sigma)\big),$$

whenever $\mathrm{supp}(\rho) \subseteq \mathrm{supp}(\sigma)$. If $\mathrm{supp}(\rho) \nsubseteq \mathrm{supp}(\sigma)$, set $S(\rho\|\sigma) := \infty$.

Remark F.2 (Protocol decides the object)*.* The divergence D in the viewpoint V_ε is applied to the *protocol readout object*. If the protocol includes a measurement/POVM mapping states to outcome distributions, then D is evaluated on the induced classical distributions. If the protocol declares state-level divergences (e.g. $S(\rho\|\sigma)$), the support condition above is the finiteness gate. This paper defaults to protocol-induced classical readouts unless stated otherwise.

F.3 Finite resolution as a viewpoint-locked regularization channel

Resolution channel family. A viewpoint V_ε includes a resolution channel N_ε as part of the locked protocol clause. The channel family/type and the meaning/units of ε must be reported.

Classical examples. Common choices include:

- **Additive noise / convolution:** for continuous readouts $x \in \mathbb{R}^n$, N_ε is convolution with a strictly positive density kernel (e.g. Gaussian) of width ε.
- **Discrete smoothing:** for categorical readouts with probabilities p_i, N_ε may implement Laplace smoothing, e.g. $p_i \mapsto (1-\varepsilon)p_i + \varepsilon/K$.

Quantum examples. A standard full-rank mixing channel is the depolarizing family:

$$\mathsf{N}_\varepsilon(\rho) := (1-\lambda(\varepsilon))\rho + \lambda(\varepsilon)\frac{I}{d}, \qquad \lambda(\varepsilon) \in (0,1),$$

which ensures $\mathsf{N}_\varepsilon(\rho)$ is full rank for any ρ.

Remark F.3 (What ε does and does not guarantee)*.* A strictly positive (classical) kernel or full-rank (quantum) mixing typically prevents support-mismatch infinities at fixed $\varepsilon > 0$. However, $\varepsilon > 0$ does *not* guarantee non-degeneracy of local geometry: excessive smoothing can destroy identifiability by collapsing distinct parameters into nearly identical readouts (see App. F). Therefore, ε is a resolution declaration, not a universal cure-all.

F.4 Protocol completion and Type-II divergences (normalization options)

In field-theoretic or continuum settings, divergences may arise from bookkeeping conventions: volume factors, regulator dependence, reference-measure choices, or unreported densitization/differencing steps. These are treated as `Type-II` (protocol-incomplete) until audited.

Normalization options (summary). The paper allows three protocol-level standardizations (declared in V_ε):

Densitization: compare densities per unit volume / per degree of freedom rather than extensive quantities.

Differencing: compare differences relative to a baseline (e.g. subtract a reference divergence common to both objects).

Explicit retention: keep regulator/volume dependence explicit and compare within matched regulator settings.

Remark F.4 (Reporting requirement)*.* If a divergence becomes finite after applying a declared protocol completion step without changing the readout space, the event is classified as `Type-II`. The report must state which completion step was used and why it applies.

F.5 Operational classification: Type-I vs Type-II non-finiteness

Whenever a required divergence is non-finite, we classify it as `Type-I` (structural abyss) or `Type-II` (technical/protocol-incomplete) using a viewpoint-locked procedure.

1. **Normalization/measure audit.** Check whether the non-finiteness is attributable to bookkeeping: volume factors, regulators, reference measures, or an omitted densitization/differencing/explicit-retention clause (App. F). If a protocol-completion operation restores finiteness *without changing the declared readout space*, label as `Type-II`.

2. **ε-scan test (fixed channel family).** If the event persists after protocol completion, run an ε-scan with the *same* channel family N_ε: choose $\varepsilon_1 > \varepsilon_2 > \cdots > \varepsilon_m$ and record the relevant $\widehat{\Delta}_I^{(\varepsilon_j)}$ and/or $\widehat{\Delta}_{\text{path}}^{(\varepsilon_j)}$. If growth does not stabilize as $\varepsilon \downarrow 0$, this is evidence of structural incompatibility under V and is labeled `Type-I` (`ABYSS` boundary). If the scan stabilizes and the blow-up correlates with bookkeeping, retain `Type-II`.

3. **Report label + trigger metadata.** The report must include: trigger location (endpoint/path/conditional term), the interface subset (if applicable), the resolution channel family, and the ε values used in the scan (if performed).

Conservativeness. This procedure intentionally reserves `Type-I` for failures that remain after protocol completion and exhibit structural divergence in the $\varepsilon \downarrow 0$ idealization.

F.6 Non-degeneracy and identifiability under finite resolution

Even when divergences are finite, the induced local geometry may be degenerate if the protocol is too coarse.

Identifiability check (recommended). Let $g \mapsto \mathbf{M}(g)$ be the model class and let the protocol readouts be $P^{(O,\varepsilon)}_{\mathbf{M}(g)}$. A practical non-degeneracy check is to test local sensitivity of the readouts: the induced quadratic form (e.g. Fisher information metric for classical readouts, or the local second-order surrogate in Sec. 4) should have no near-zero directions on the window beyond declared symmetries/parameter redundancies. Operationally, report the smallest eigenvalue (or a proxy) and treat vanishing directions as a warning for `FLAG:GEO_OUT_OF_DOMAIN` or for Π_{rec}-critical instability if minimizers flatten.

Remark F.5 (Why this matters). Without identifiability, ε-smoothing can make distinct g values observationally indistinguishable, leading to Π_{rec}-criticality or metric degeneracy even though divergences are finite. This is not a contradiction; it is a domain limitation of the chosen V_ε.

F.7 Viewpoint Card checklist (for Appendix / reporting)

For reproducibility, each experiment should include a Viewpoint Card (Sec. 8) specifying: interfaces $\mathcal{O}$, protocol mapping including N_ε and units of ε, regulator/standardization (R, S), divergence D and whether applied to classical readouts or states, aggregation Agg, model class $\mathcal{M}$, projection rule Π_{rec} (including tie-breaking policy, default: none), thresholds and precedence order, and (optional) ε-scan details.

Appendix G: $\Pi_{\rm rec}$-Critical diagnostics and operational criteria

Purpose. This appendix specifies practical, reproducible diagnostics for detecting $\Pi_{\rm rec}$-criticality (set-valued reconstruction) under a locked viewpoint V_ε, and clarifies when such events may be interpreted as a *structural reorganization critical window* rather than as a numerical artifact. The main text treats $\Pi_{\rm rec}$-criticality as a label/output status; here we provide a minimal operational procedure.

G.1 Setup: reconstruction objective and minimizer set

Recall the reconstruction objective (Sec. 2.5):

$$\mathcal{F}_\mu(g) \ := \ \mathrm{Agg}_{O\in\mathcal{O}}\, D\Big(P^{(O,\varepsilon)}_{T_\mu}\Big\|P^{(O,\varepsilon)}_{\mathbf{M}(g)}\Big), \qquad \mathcal{A}(\mu) \ := \ \arg\min_{g\in\mathbb{R}^d} \mathcal{F}_\mu(g).$$

A scale point μ is called $\Pi_{\rm rec}$-*critical* if $\mathcal{A}(\mu)$ is not a singleton (Sec. 2.5).

G.2 Numerical evidence for non-uniqueness: multi-start + flatness

Multi-start evidence. A necessary (but not sufficient) indication of $\Pi_{\rm rec}$-criticality is the presence of multiple distinct minimizers found by independent initializations. Operationally, we run a multi-start solver producing candidate minimizers $\{g_\mu^{(i)}\}_{i=1}^K$ and record:

$$\Delta\mathcal{F}_\mu^{(i)} := \mathcal{F}_\mu(g_\mu^{(i)}) - \min_j \mathcal{F}_\mu(g_\mu^{(j)}), \qquad \mathrm{sep}_\mu^{(i,j)} := \|g_\mu^{(i)} - g_\mu^{(j)}\|.$$

If there exist $i \neq j$ such that $\Delta\mathcal{F}_\mu^{(i)} \leq \tau_{\rm crit}$, $\Delta\mathcal{F}_\mu^{(j)} \leq \tau_{\rm crit}$, and $\mathrm{sep}_\mu^{(i,j)} \geq \tau_{\rm sep}$, we declare *candidate* non-uniqueness at μ.

Flatness evidence via Hessian. To distinguish true degeneracy from accidental numerical multiplicity, we test local flatness around the best candidate minimizer $g_\mu^\star$. Let H_μ be an estimated Hessian of $\mathcal{F}_\mu$ at $g_\mu^\star$ (exact, automatic, or quasi-Newton). Define the flatness indicator

$$\kappa_\mu \ := \ \lambda^+_{\min}(H_\mu), \tag{G.1}$$

where we compute eigenvalues of the symmetrized Hessian $\frac{1}{2}(H_\mu + H_\mu^\top)$, and $\lambda^+_{\min}$ denotes its smallest nonnegative eigenvalue. If $\lambda_{\min}(\frac{1}{2}(H_\mu + H_\mu^\top)) < -\tau_{\rm neg}$ for a declared numerical tolerance $\tau_{\rm neg} > 0$, we treat $g_\mu^\star$ as failing a second-order local-minimum check (numerical saddle / solver or estimation failure) and do not interpret this as flatness evidence. If $\kappa_\mu \leq \tau_{\rm flat}$, the objective is locally non-strictly convex (or nearly so), consistent with a set-valued minimizer. In this case, together with multi-start evidence, we label μ as `REC-CRITICAL`.

Remark G.1 (Why Hessian is a diagnostic rather than a theorem). The criteria above are operational diagnostics: they provide reproducible evidence for degeneracy under a fixed implementation budget. They do not claim a universal mathematical equivalence between $\kappa_\mu \leq \tau_{\rm flat}$ and $\mathcal{A}(\mu)$ having positive dimension, but they are sufficient for the paper's reporting protocol: when degeneracy is detected, we avoid forcing a single-valued trajectory.

G.3 A tension index $\mathcal{T}(\mu)$ (proxy diagnostics)

When a local geometric surrogate $G(g)$ is available (i.e. within the Route-II domain and under Condition $\mathrm{G}_{\mathrm{geo}}$, so that a single SPD metric field is instantiated), a convenient proxy for proximity to degeneracy is the smallest eigenvalue of the induced quadratic form. We define a *tension index* by

$$\mathcal{T}(\mu) \;:=\; \frac{1}{\lambda_{\min}(G(g_\mu^\star))}, \tag{G.2}$$

interpreting large $\mathcal{T}(\mu)$ as approaching a near-degenerate direction. Here $\mathcal{T}(\mu)$ is unrelated to the window length T used elsewhere in the paper. In settings where G is not explicitly constructed, an alternative proxy is any viewpoint-locked flatness measure used in the implementation, e.g. the variance of a slope-ratio statistic in the two-dimensional example (Subsec. 8.4), or the condition number of a local quadratic fit.

Remark G.2 (Relationship to Π_{rec}-criticality). A large $\mathcal{T}(\mu)$ indicates loss of identifiability in some direction and often co-occurs with Π_{rec}-criticality, but it does not imply Π_{rec}-criticality by itself. Conversely, Π_{rec}-criticality can occur without a well-defined G (if Route-II prerequisites fail), in which case Hessian/flatness tests on $\mathcal{F}_\mu$ remain applicable.

G.4 Diameter of the minimizer set (optional reporting)

When multiple minimizers are detected, we recommend reporting a scale-local degeneracy size. Given candidate minimizers $\{g_\mu^{(i)}\}$ satisfying $\Delta\mathcal{F}_\mu^{(i)} \le \tau_{\mathrm{crit}}$, define the diameter

$$\mathrm{diam}(\mathcal{A}(\mu)) \;\approx\; \max_{i,j} \|g_\mu^{(i)} - g_\mu^{(j)}\|.$$

If a metric $G(g)$ is available, one may instead report a metric-adjusted diameter using the local quadratic form at $g_\mu^\star$.

G.5 When to interpret Π_{rec}-criticality as a structural reorganization window

In the main text, `REC-CRITICAL` is a structural label meaning: under the locked viewpoint V_ε, the reconstruction is not single-valued and Route-II objects requiring a C^1 trajectory should not be forced. Interpreting such an event as a *structural reorganization critical window* requires additional evidence.

Stability requirement (against viewpoint drift). A candidate reorganization window should persist under small perturbations that do *not* change the viewpoint in substance: e.g. modest changes in numerical tolerance, multi-start budget, or small ε adjustments within the same channel family. If Π_{rec}-criticality disappears under such perturbations, it is more consistent with numerical instability than with a structural transition.

Co-occurrence requirement (independent diagnostics). A reorganization interpretation is supported when Π_{rec}-criticality co-occurs with at least one independent signal under the same locked viewpoint, such as: (i) a sharp change in endpoint/path behavior (e.g. sudden growth of Δ_I or Δ_{path}), (ii) emergence of a near-degenerate direction in the tension index $\mathcal{T}(\mu)$, or (iii) a robust increase in the diameter $\mathrm{diam}(\mathcal{A}(\mu))$ across a scale neighborhood. Absent such co-occurrence, the default interpretation is *identifiability breakdown* or *model insufficiency* under the chosen protocol.

Reporting rule. The paper therefore uses the following conservative rule:

- Always report `REC-CRITICAL` when the operational diagnostics indicate non-uniqueness/flatness.
- Only use the phrase "critical window" (or analogous structural-transition language) when the stability and co-occurrence requirements above are satisfied and documented in the report metadata.

Minimal metadata for reproducibility. When reporting `REC-CRITICAL`, include: the multi-start budget K, the tolerance thresholds $(\tau_{\text{crit}}, \tau_{\text{sep}}, \tau_{\text{neg}}, \tau_{\text{flat}})$, whether Hessian is exact or approximate, and (if used) the tension proxy $\mathcal{T}(\mu)$.

Appendix H: Domain-gated reporting protocol (full pseudocode)

Algorithmic protocol. The following viewpoint-locked pseudocode is the default reporting procedure for all experiments in this paper. For stability of typesetting, we split the full procedure into two parts: Steps 0–2 and Steps 3–6.

Algorithm 1: Domain-gated reporting of $(\Delta_I, L_{\mathrm{II}}, d_{\mathrm{end}}, \Gamma_{\mathrm{geo}})$ under a locked viewpoint V_ε (Part I: Steps 0–2)

Input: Locked viewpoint $V_\varepsilon = (\mathcal{O}, \mathrm{protocol}_\varepsilon, R, S, \mathrm{threshold}, \mathcal{M}, \Pi_{\mathrm{rec}}, D, \mathrm{Agg})$
scale window $[\mu_0, \mu_1]$ and grid $\{\mu_k\}_{k=0}^N$
baseline model T_μ^{base} (if used)
tolerances $(\tau_{\mathrm{fin}}, \tau_{\mathrm{crit}}, \tau)$, significance rule α_{sig} (with any multiple-testing correction), and estimation budgets
Output: `STATUS` in {`ADMISSIBLE`, `MISMATCH`, `ABYSS`, `REC-CRITICAL`}
(optional) numerical tuple $(\widehat{\Delta}_I, \widehat{L}_{\mathrm{II}}, \widehat{d_{\mathrm{end}}}, \widehat{\Gamma}_{\mathrm{geo}})$
metadata: ε, triggered gate(s), and (if applicable) abyss type or Π-degeneracy diagnostics

Step 0: Build protocol readouts.
foreach $O \in \mathcal{O}$ **do**
  for $k \leftarrow 0$ **to** N **do**
    compute $X_{\mu_k}^{(O,\varepsilon)}$ (including the resolution channel N_ε)

Step 1: Finite-divergence gate (A1–A2).
Compute Route-I endpoint divergence under Agg:
$\widehat{\Delta}_I \leftarrow \mathrm{Agg}_{O\in\mathcal{O}}\, D\Big(X_{\mu_0}^{(O,\varepsilon)} \Big\| X_{\mu_N}^{(O,\varepsilon)}\Big)$
for $k \leftarrow 0$ **to** $N-1$ **do**
  $\widehat{\delta}_k \leftarrow \mathrm{Agg}_{O\in\mathcal{O}}\, D\Big(X_{\mu_k}^{(O,\varepsilon)} \Big\| X_{\mu_{k+1}}^{(O,\varepsilon)}\Big)$
if *any required divergence is non-finite* **then**
  Classify as `Type-I` vs `Type-II` using the protocol bookkeeping
  return *`ABYSS` (with abyss type and trigger location)*

Step 2: Reconstruction gate (set-valued $\mathcal{A}(\mu)$).
for $k \leftarrow 0$ **to** N **do**
  Solve the reconstruction objective
  $\mathcal{F}_{\mu_k}(g) \leftarrow \mathrm{Agg}_{O\in\mathcal{O}}\, D\Big(X_{\mu_k}^{(O,\varepsilon)} \Big\| X_{\mathbf{M}(g)}^{(O,\varepsilon)}\Big)$
  `// Notation:` $X_{\mu_k}^{(O,\varepsilon)} = P_{T_{\mu_k}}^{(O,\varepsilon)}$ `and` $X_{\mathbf{M}(g)}^{(O,\varepsilon)} = P_{\mathbf{M}(g)}^{(O,\varepsilon)}$.
  Estimate the minimizer set $\mathcal{A}(\mu_k)$ (e.g. multi-start + local flatness tests)
  Pick $g_{\mu_k}^\star \in \mathcal{A}(\mu_k)$ as the candidate minimizer used for local diagnostics (Hessian/flatness and $\mathcal{T}$).
  if $|\mathcal{A}(\mu_k)| > 1$ ***or*** $\dim \mathcal{A}(\mu_k) > 0$ **then**
    $\deg(\mu_k) \leftarrow 1$
  else
    $\deg(\mu_k) \leftarrow 0$
if $\max_k \deg(\mu_k) = 1$ **then**
  return *`REC-CRITICAL` (with Π_{rec}-diagnostics, e.g. diameter / flat-direction eigenvalues)*

Algorithm 2: Domain-gated reporting of $(\Delta_I, L_{\mathrm{II}}, d_{\mathrm{end}}, \Gamma_{\mathrm{geo}})$ under a locked viewpoint V_ε (Part II: Steps 3–6)

Input: Same as Part I (Algorithm 1).

Output: Same as Part I (Algorithm 1).

Step 3: Geometry certificate gate (A3 / Condition $\mathcal{G}_{\mathrm{bridge}}$).

Check Condition $\mathcal{G}_{\mathrm{bridge}}$ on the window (or on a declared segmentation)

if *Condition $\mathcal{G}_{\mathrm{bridge}}$ fails* ***and*** *no segmentation is declared* **then**

  return *`ABYSS` (`ABYSS_KIND=GEOMETRY`; gate=`Gbridge`; Route-II not reportable)*

Step 4: Construct Route-II length proxy.

Using the single-valued reconstructed trajectory $g(\gamma)$, compute $\widehat{\Psi}$ and $\widehat{L}_{\mathrm{II}}$ per the definitions in Sec. 4–5

Step 5: Geometric proxy gate (Condition $\mathcal{G}_{\mathrm{geo}}$).

Check Condition $\mathcal{G}_{\mathrm{geo}}$ (reportable length structure under the locked V_ε)

Gate semantics: Condition $\mathcal{G}_{\mathrm{geo}}$ controls reportability of $\widehat{d_{\mathrm{end}}}$ and $\widehat{\Gamma}_{\mathrm{geo}}$; it is not a correctness test of Route I.

if *Condition $\mathcal{G}_{\mathrm{geo}}$ fails* **then**

  return *`ADMISSIBLE` (report $(\widehat{\Delta}_I, \widehat{L}_{\mathrm{II}})$ only; flag `FLAG:GEO_OUT_OF_DOMAIN`)*

Step 6: Report Γ_{geo} and classify mismatch.

Compute $\widehat{d_{\mathrm{end}}}$ (geodesic distance under the induced length structure)

$\widehat{\Gamma}^{\mathrm{raw}}_{\mathrm{geo}} \leftarrow \widehat{L}_{\mathrm{II}} - \widehat{d_{\mathrm{end}}}$

Compute the corrected one-sided lower confidence bound $\mathrm{CI}_L\big(\widehat{\Gamma}^{\mathrm{raw}}_{\mathrm{geo}}; 1-\alpha_{\mathrm{sig}}\big)$ using the locked uncertainty model (and, if a window scan is performed, the locked multiple-testing correction as in Subsec. 8.4)

Apply the locked numerical-consistency rule (Sec. 8.3) to obtain the reported $\widehat{\Gamma}_{\mathrm{geo}}$ from $\widehat{\Gamma}^{\mathrm{raw}}_{\mathrm{geo}}$

if $\mathrm{CI}_L\big(\widehat{\Gamma}^{\mathrm{raw}}_{\mathrm{geo}}; 1-\alpha_{\mathrm{sig}}\big) > 0$ **then**

  return *`MISMATCH` (full numeric tuple and metadata)*

return *`ADMISSIBLE` (full numeric tuple and metadata)*

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