p-adic analytic action on Fukaya categories and iterates of symplectomorphisms

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Overview

Motivation: Bell's theorems

Theorem (J. Bell, 2005)

Let X be an affine variety over a field of characteristic 0 and ϕ be an automorphism of X. Consider a subvariety $Y \subset X$ and a point $x \in X$. Then the set

$$\{k \in \mathbb{N} : \phi^k(x) \in Y\}$$

is a union of finitely many arithmetic progressions and finitely many other numbers.

This theorem has versions for coherent sheaves as well, describing similar results for $\{k \in \mathbb{N} : \mathcal{T}or(\mathcal{F}, (\phi^k)^*\mathcal{F}') \neq 0\}$. It is valid for surfaces.

Symplectic analogues?

Then one can ask if there is a symplectic analogue of this theorem. For instance:

Conjecture (Seidel)

Let L and L' be two Lagrangians in a symplectic manifold M with a symplectomorphism ϕ. Then the set

$$\{k \in \mathbb{N} : \phi^k(L) \text{ is "Floer theoretically isomorphic" to } L'\}$$

is a union of finitely many arithmetic progressions and finitely many other numbers.

For the heuristic relation of Bell's theorem to this conjecture, consider $X =$ "moduli of Lagrangians", $x = L \in X$, $Y = \{L'\} \subset X$.

Basics of Floer homology and Fukaya categories

To explain further, we review basics of Floer homology. Let $\Lambda = \mathbb{Q}((T^{\mathbb{R}}))$.

Definition

Let (M, ω_M) be a symplectic manifold (e.g. an oriented surface with an area form), let L, L' be two Lagrangians (e.g. non-separating curves on the surface) that intersect transversally (so $L \cap L'$ is finite).
Define the chain complex

$$CF(L, L') = \Lambda\langle L \cap L' \rangle$$

Its differential defined by counting 2-gons:

$$d(x) = \sum \pm T^{E(u)} \# \{ \quad \} \cdot y$$

formal variable

Area of the 2-gon

Basics of Floer homology and Fukaya categories

The cohomology of $CF(L, L')$ is denoted by $HF(L, L')$. There is also product structure $CF(L_1, L_2) \otimes CF(L_0, L_1) \to CF(L_0, L_2)$ defined by

$$\mu^2(y,x) = \sum \pm T^{E(u)} \# \Big\{$$

$$\Big\} \cdot z$$

Associative in cohomology

holomorphic triangles

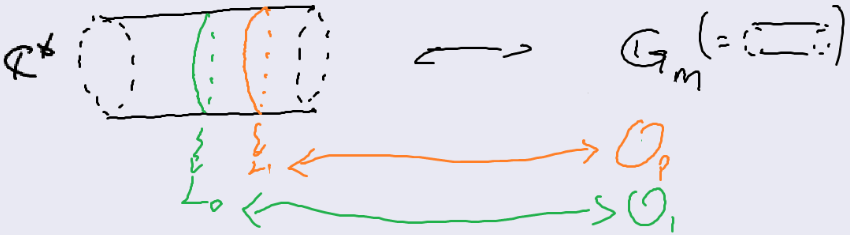
Basics of Floer homology and Fukaya categories

We obtain a category $\mathcal{F}(M, \Lambda)$ such that

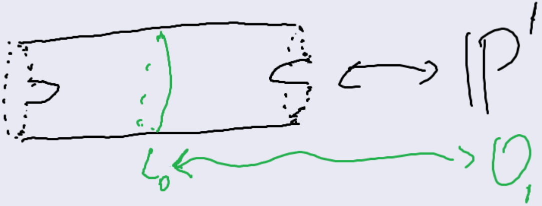
$$ob(\mathcal{F}(M,\Lambda)) = \{\text{Lagrangians}\}$$
$$hom(L, L') = CF(L, L')$$

This may be hard to compute, but in some cases it is equivalent to $Coh(M^\vee)$ for an algebraic variety $M^\vee$ (mirror dual)

Example



Example



Basic observation: Let $f \in Aut(\mathbb{P}^1)$ be the action of some element of "$\mathbb{R}_+ \subset \mathbb{G}_m$". Then, the rank of $Ext^*(\mathcal{F}, (f^*)^n(\mathcal{F}'))$ is constant in n, with finitely many exceptions.

This serves as a second motivation for the main result:

Main result

Theorem (K., 2020)

Let M be a monotone symplectic manifold and ϕ be a symplectomorphism isotopic to identity. Given Lagrangians $L, L' \subset M$, the rank of $HF(L, \phi^k(L'))$ is constant in $k \in \mathbb{Z}$, with finitely many exceptions.

There are some assumption on M such as:

- (technical) $\mathcal{F}(M; \Lambda)$ is finitely generated and smooth
- $\exists$ finite set $\{L_i\}$ of generators such that each L_i is **Bohr-Sommerfeld monotone**

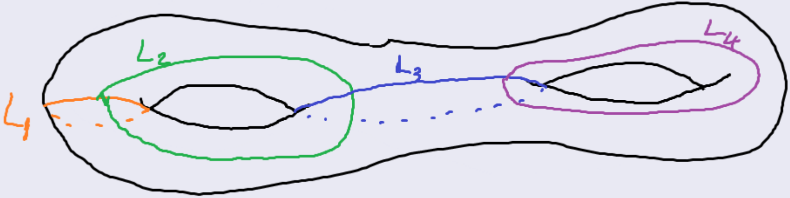
Some explanation, remarks

- B-S monotone $\Longrightarrow$ the coefficients of the differential and product on $CF(L_i, L_j)$ are finite sums (i.e. $\sum \pm T^{E(u)}.y$ is a finite sum)
- if $\{L_i\}$ is a set of generators, can see Fukaya category as an algebra $\bigoplus_{i,j} CF(L_i, L_j)$
- L can be seen as a right module $h_L = \bigoplus_j CF(L_j, L)$ or as a left module $h^L = \bigoplus_i CF(L, L_i)$
- $hom(L, L') = CF(L, L') \simeq h_L \otimes_{\mathcal{F}(M,\Lambda)} h^{L'}$

Examples

Example

Let $M = \Sigma_2$ be a genus 2 surface. Every non-separating curve has a B-S monotone representative in their isotopy class.



One can let L, L' be any pair of non-separating curves.

Main tool

- Bell proves his theorem by interpolating the orbit $\{\phi^k(x)\}$ by a p-adic analytic arc
- Analogous main tool for us: interpolate iterates of ϕ by a p-adic analytic action

Local action on $\mathcal{F}(M,\Lambda)$

- $\phi \in Symp^0(M) \rightsquigarrow$Autoequivalence on $\mathcal{F}(M,\Lambda) \rightsquigarrow \mathcal{F}(M,\Lambda)$-bimodule (bimodule defined by $\phi \rightsquigarrow \bigoplus_{i,j} CF(L_i, \phi(L_j))$)
- $1_M \rightsquigarrow 1_{\mathcal{F}(M,\Lambda)} \rightsquigarrow$ diagonal bimodule $\mathcal{F}(M,\Lambda)$
- Composition $\rightsquigarrow$tensor product of bimodules
- As ϕ is isotopic to identity, this suggests to define "the local action" as a deformation of the digonal bimodule.

Recall that closed 1-forms on M can be used to construct symplectic isotopy ϕ_α^t. WLOG assume $\phi = \phi_\alpha^1$.

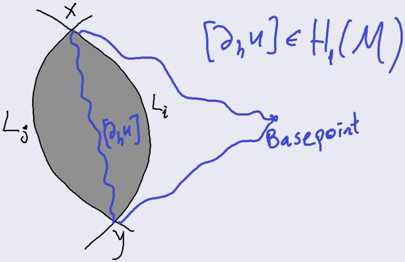
Local action on $\mathcal{F}(M,\Lambda)$

Definition

Let $\mathfrak{M}_\alpha^\Lambda|_t = \bigoplus_{i,j} CF(L_i, L_j)$. Define the differential via:

$$x \mapsto \sum \pm T^{E(u)} T^{t\alpha([\partial_h u])}.y$$

u varies among the holomorphic strips as in figure below:



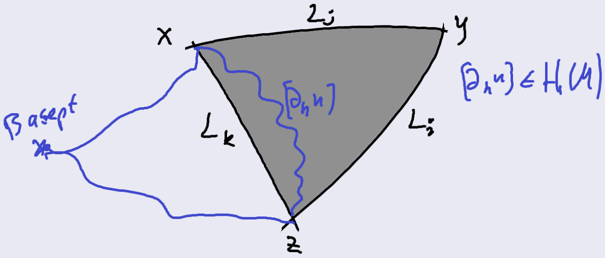
Local action on $\mathcal{F}(M, \Lambda)$

Definition (cont'd)

Define right multiplication by elements of $\mathcal{F}(M, \Lambda)$ via

$$\mu^1(|x|y) = \sum \pm T^{E(u)} T^{t\alpha([\partial_h u])}.z$$

where the count is as in:



Define left multiplication (and higher multiplications) similarly.

Local action on $\mathcal{F}(M, \Lambda)$

Remark

Plugging $t = 0$ gives us the usual Floer differential and multiplication (defined earlier).

Lemma

The bimodule $\mathfrak{M}_\alpha^\Lambda|_t$ is "geometric" for small $|t|$, i.e. it corresponds to action of ϕ_α^t.

Lemma

The family of bimodules $\mathfrak{M}_\alpha^\Lambda|_t$ behave like a "local group action", i.e. for small t_1, t_2

$$\mathfrak{M}_\alpha^\Lambda|_{t_2} \otimes_{\mathcal{F}(M,\Lambda)} \mathfrak{M}_\alpha^\Lambda|_{t_1} \simeq \mathfrak{M}_\alpha^\Lambda|_{t_1+t_2}$$

Review of p-adics

Let $p > 2$ be a prime. Recall:

- $\mathbb{Z}_p$ = completion of $\mathbb{Z}$ with respect to norm $|x|_p := p^{-val_p(x)}$
- $\mathbb{Q}_p$ = field of fractions of $\mathbb{Z}_p$, normed field

Upshot: One can do analytic geometry over $\mathbb{Q}_p$

- $\mathbb{D}_1$ = closed unit disc = $\mathbb{Z}_p$
- $\mathbb{Q}_p\langle t\rangle = \{\sum a_i t^i : a_i \in \mathbb{Q}_p, |a_i|_p \to 0\}$ = analytic functions on $\mathbb{D}_1$
- $\mathbb{D}_{p^{-n}}$ = closed disc of radius $p^{-n} = p^n\mathbb{Z}_p$
- $\mathbb{Q}_p\langle t/p^n\rangle$ = analytic functions on $\mathbb{D}_{p^{-n}}$

Some strange features of p-adic analytic disc

- $1, 2, 3, \cdots \in \mathbb{D}_1$= unit disc
- Unit disc is an additive group
- (Strassman's theorem) if $f(t) \in \mathbb{Q}_p\langle t\rangle$ has infinitely many 0's, $f(t) = 0$
- Coherent sheaves on $\mathbb{D}_{p^{-n}}$ are locally free outside finitely many points

Fukaya category over smaller fields and over $\mathbb{Q}_p$

- B-S monotone $\Rightarrow$ the coefficients $\sum \pm T^{E(u)}$ are finite
- Fukaya category is defined over $\mathbb{Q}(T^{\mathbb{R}})$
- $E(u) \in \omega_M(H_2(M, \bigcup L_i \cup L \cup L'))$ and the latter is a finitely generated additive subgroup of $\mathbb{R}$
- Given finitely generated $G \supset \omega_M(H_2(M, \bigcup L_i \cup L \cup L'))$ with basis $g_1, \dots, g_k \subset G$, Fukaya category is defined over $\mathbb{Q}(T^G) = \mathbb{Q}(T^{g_1}, \dots, T^{g_k})$ (denote it by $\mathcal{F}(M, \mathbb{Q}(T^G))$)
- Any embedding $\mu : \mathbb{Q}(T^G) \to \mathbb{Q}_p$ defines a category $\mathcal{F}(M, \mathbb{Q}_p)$
- Assume $\alpha(H_1(M)) \subset G$

Local p-adic action

Want: p-adic family of bimodules
Suggestion: Replace previous formula by

$$x \mapsto \sum \pm\mu(T^{E(u)})\mu(T^{\alpha([\partial_h u])})^t.y$$

(analytic in $t \in \mathbb{D}_1$) and same with other structure maps. To define $\mu(T^{\alpha([\partial_h u])})^t \in \mathbb{Q}_p\langle t\rangle$, we need $\mu(T^{\alpha([\partial_h u])}) \equiv 1 \ (mod\ p)$

Definition (Poonen, Bell)

Given $v \in 1 + p\mathbb{Z}_p$, define $v^t := \sum \binom{t}{i}(v-1)^i \in \mathbb{Q}_p\langle t\rangle$

We can choose $\mu : \mathbb{Q}(T^G) \to \mathbb{Q}_p$ such that $\mu(T^g) \equiv 1 \ (mod\ p)$

Local p-adic action

Definition

Let $\mathfrak{M}_\alpha^{\mathbb{Q}_p} = \bigoplus_{i,j}(\mathbb{Q}_p\langle t\rangle)\langle L_i \cap L_j\rangle$, with bimodule structure as above.

Proposition

$\mathfrak{M}_\alpha^{\mathbb{Q}_p}$ also behaves like a "local group action", i.e.

$$\mathfrak{M}_\alpha^{\mathbb{Q}_p}|_{t=t_2} \otimes_{\mathcal{F}(M,\mathbb{Q}_p)} \mathfrak{M}_\alpha^{\mathbb{Q}_p}|_{t=t_1} \simeq \mathfrak{M}_\alpha^{\mathbb{Q}_p}|_{t=t_1+t_2}$$

for small $t_1, t_2 \in \mathbb{Z}_p$.

Idea of the proof.

Define a map

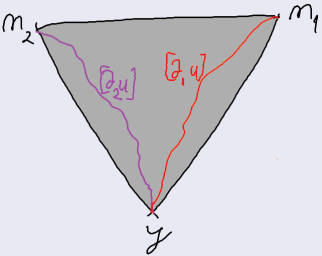
$$\mathfrak{M}_\alpha^{\mathbb{Q}_p}|_{t=t_2} \otimes_{\mathcal{F}(M,\mathbb{Q}_p)} \mathfrak{M}_\alpha^{\mathbb{Q}_p}|_{t=t_1} \to \mathfrak{M}_\alpha^{\mathbb{Q}_p}|_{t=t_1+t_2}$$

□

Local p-adic action

Idea of the proof (cont'd).

by $m_2 \otimes m_1 \mapsto \sum \pm \mu(T^{E(u)})\mu(T^{\alpha([\partial_1 u])})^{t_1}\mu(T^{\alpha([\partial_2 u])})^{t_2}.y$



This induces a (quasi)-isomorphism at $t_1 = t_2 = 0$. Therefore, it induces a (quasi)-isomorphism at a small (p-adic) neighborhood of $(0, 0)$. □

Local p-adic action

Observe that being p-adically small is equivalent to being in $p^n\mathbb{Z}_p$ for some $n \gg 0$. Hence,

Corollary

$\mathfrak{M}_\alpha^{\mathbb{Q}_p}|_{t=t_2} \otimes_{\mathcal{F}(M,\mathbb{Q}_p)} \mathfrak{M}_\alpha^{\mathbb{Q}_p}|_{t=t_1} \simeq \mathfrak{M}_\alpha^{\mathbb{Q}_p}|_{t=t_1+t_2}$ *for* $t_1, t_2 \in p^n\mathbb{Z}_p$.

We can combine this corollary with quasi-isomorphism preserving property of base change to show:

Corollary

$\mathfrak{M}_\alpha^{\Lambda}|_{t_2} \otimes_{\mathcal{F}(M,\Lambda)} \mathfrak{M}_\alpha^{\Lambda}|_{t_1} \simeq \mathfrak{M}_\alpha^{\Lambda}|_{t_1+t_2}$ *for* $t_1, t_2 \in p^n\mathbb{Z}_{(p)} := p^n\mathbb{Z}_p \cap \mathbb{Q} \subset \mathbb{R}$.

Proof of the theorem

Recall that $\mathfrak{M}_\alpha^\Lambda|_t$ corresponds to action of ϕ_α^t for small t. By the last corollary, the bimodule $\mathfrak{M}_\alpha^\Lambda|_t$ corresponds to action of ϕ_α^t for all $t \in p^n\mathbb{Z}_{(p)}$. Therefore,

Proposition

$h_{L'} \otimes_{\mathcal{F}(M,\Lambda)} \mathfrak{M}_\alpha^\Lambda|_{p^n k} \simeq h_{\phi_\alpha^{p^n k}(L')}$ *for all* $k \in \mathbb{Z}$ *and*

$$HF(L, \phi_\alpha^{p^n k}(L')) = Hom(L, \phi_\alpha^{p^n k}(L')) \cong H^*(h_{L'} \otimes_{\mathcal{F}(M,\Lambda)} \mathfrak{M}_\alpha^\Lambda|_{p^n k} \otimes_{\mathcal{F}(M,\Lambda)} h^L)$$

.

We observe that the rank of $H^*(h_{L'} \otimes_{\mathcal{F}(M,\Lambda)} \mathfrak{M}_\alpha^\Lambda|_{p^n k} \otimes_{\mathcal{F}(M,\Lambda)} h^L)$ is the same as the rank of $H^*(h_{L'} \otimes_{\mathcal{F}(M,\mathbb{Q}_p)} \mathfrak{M}_\alpha^{\mathbb{Q}_p} \otimes_{\mathcal{F}(M,\mathbb{Q}_p)} h^L)$ at $t = p^n k \in p^n\mathbb{Z}_p$ (a coherent sheaf over $\mathbb{D}_{p^{-n}}$).

Proof of the theorem

As explained, the rank of a analytic coherent sheaf over $\mathbb{D}_{p^{-n}} \subset \mathbb{D}_1$ is constant with finitely many exceptions. Therefore,

- Rank of $HF(L, \phi_\alpha^{p^n k}(L'))$ is constant in k with finitely many exceptions
- Replace L' by $\phi_\alpha^i(L')$, $i = 0, \dots, p^n - 1$ to obtain constancy of rank of $HF(L, \phi_\alpha^{p^n k+i}(L'))$ in k
- Hence the rank of $HF(L, \phi_\alpha^k(L'))$ is p^n periodic. Replace p by p' to obtain $(p')^{n'}$-periodicity as well

Hence, we obtain

Theorem (K., 2020)

The rank of $HF(L, \phi_\alpha^k(L'))$ is constant in $k \in \mathbb{Z}$, with finitely many exceptions.

Thank you!