RECOVERING GENERALIZED COHOMOLOGY FROM FLOER HOMOLOGY: THE COMPLEX ORIENTED CASE

LAURENT CÔTÉ AND YUSUF BARIŞ KARTAL

Abstract. We associate an invariant called the completed Tate cohomology to a filtered circle-equivariant spectrum and a complex oriented cohomology theory. We show that when the filtered spectrum is the spectral symplectic cohomology of a Liouville manifold, this invariant depends only on the stable homotopy type of the underlying manifold. We make explicit computations for several complex oriented cohomology theories, including Eilenberg–Maclane spectra, Morava K-theories, their integral counterparts, and complex K-theory. We show that the result for Eilenberg–Maclane spectra depends only on the rational cohomology, and we use the computations for Morava K-theory to recover the integral cohomology (as an ungraded group). In a different direction, we use the completed Tate cohomology computations for the complex K-theory to recover the complex K-theory of the underlying manifold from its equivariant filtered Floer homotopy type.

1. Introduction

The symplectic cohomology of a Liouville manifold is not sensitive to its homotopy type or homology. For instance, any subcritical Weinstein manifold has $SH^*(M) = 0$. More generally, attaching subcritical handles does not change the symplectic cohomology; however, it changes the homology. Symplectic cohomology carries a circle action; however, this does not help as the subcritical handle attachment does not effect equivariant quasi-isomorphism type of $SH^*(M)$ either.

Therefore, to recover more information about the homology of M, one needs to use extra structures on $SH^*(M)$. In [Zha14], Zhao proved that if one further recalls the filtration on the symplectic cochains, one can recover the rational cohomology of M. The construction, is a version of Tate cohomology of $SH^*(M)$ with respect to S^1-action, defined as a colimit of Tate cohomologies of each filtered piece. The dependence of the construction on the filtration is pretty loose though: for any equivalent filtration one gets the same answer. However, the construction in [Zha14] is insensitive to torsion information in $H^*(M, \mathbb{Z})$. The goal of this paper is to use equivariant Floer homotopy theory to recover further information about the stable homotopy type of M, and in particular its $\mathbb{F}_p$ Betti numbers.

In [Lar21], Large defined symplectic cohomology with coefficients in the sphere spectrum $\mathbb{S}$ as a colimit of Hamiltonian Floer homology over $\mathbb{S}$. In [CK23], the authors defined a genuine S^1-equivariant model $SH_S(M, \mathbb{S})$, where the S^1-action corresponds to "loop rotation". Analogously to [Lar21], we first construct a genuine S^1-equivariant spectrum $HF_S(H, \mathbb{S})$ associated to a (family of) Hamiltonian(s) that is linear at infinity, and take a colimit as the slope of H goes to infinity. Observe that by choosing a cofinal sequence of such Hamiltonians, one can endow $SH_S(M, \mathbb{S})$ with the structure of a *filtered equivariant spectrum*. The filtration depends on the choice of the sequence, but two such sequences produce equivalent filtrations (in a sense made precise in Section 2).

In the current work, we associate an invariant $\widehat{R}^*_{S^1}(X)$ which we call the *completed Tate cohomology* to a complex oriented cohomology theory R and a filtered S^1-equivariant spectrum. The central result of this paper is the following:

Theorem 4.1. *$\widehat{R}^*_{S^1}(SH_S(M,\mathbb{S}))$ is equivalent to $R^*(M)[[u]]$ localized at elements $[n]_R(u) \in R^*[[u]] = R^*(BS^1)$, for every $n \geq 1$.*

In other words, $\widehat{R}^*_{S^1}(SH_S(M,\mathbb{S}))$ only depends on $R^*(M)$, and as we will see, it is a good approximation to $R^*(M)$ for many R, including Morava K-theories. To explain the notation, we first recall that the complex orientation on R determines an isomorphism $R^*[[u]] = R^*(BS^1)$ (where $R^* = R^*(pt)$), and a formal group law $F_R \in R^*[[u_1,u_2]]$. The element $[n]_R(u) \in R^*[[u]]$ is obtained by iterating u under F_R, n times. Note that localization here is in a u-adically complete sense.

Given an S^1-equivariant filtered spectrum X with filtration $X_1 \to X_2 \to \dots X$, the completed Tate cohomology $\widehat{R}^*_{S^1}(X)$ is defined to be the limit of the $R^*((X_i)_{hS^1})$ localized at each $[n]_R(u)$. To prove Theorem 4.1, we find a cofinal sequence of Hamiltonians such that $HF_S(H,\mathbb{S})$ has building blocks that are up to equivalence given by $\Sigma^\infty M_+$ and copies of S^1/C_k. The localization kills the latter.

We compute this invariant concretely in a variety of examples. For the Eilenberg–Maclane spectra $R = H\mathbb{Q}$, $H\mathbb{Z}$, $H\mathbb{F}_p, \dots$, $[n]_R(u) = nu$. Hence, we have the following analogue of [Zha14, Thm. 1.1]:

Corollary 4.2. *$\widehat{H\mathbb{Q}}^*_{S^1}(SH_S(M,\mathbb{S})) \simeq \widehat{H\mathbb{Z}}^*_{S^1}(SH_S(M,\mathbb{S})) \simeq H^*(M,\mathbb{Q}((u)))$. On the other hand, $\widehat{H\mathbb{F}_p}^*_{S^1}(SH_S(M,\mathbb{S})) \simeq 0$.*

To our knowledge, [Zha14] makes the analogous computation in this generality for the rational cohomology only. Note that our result can possibly be interpreted as a symplectic homology version of [Zha14] as well.

To access the torsion in the homology of M, we compute $\widehat{R}^*_{S^1}(SH_S(M,\mathbb{S}))$ for the (generalized) Morava K-theories $K_{p^k}(n)$:

Corollary 4.3. *$\widehat{K_{p^k}(n)}^*_{S^1}(SH_S(M,\mathbb{S})) \simeq K_{p^k}(n)^*(M)((u))$*

Corollary 4.2 and Corollary 4.3 also explain the term completed "Tate cohomology". Normally, to define Tate cohomology, one would localize $R^*((X_i)_{hS^1})$ at $[1]_R(u) = u$ only. For $R = H\mathbb{Q}$, $K_{p^k}(n)$, this produces the same answer.

This can be used to show

Theorem 4.4. *For $2(p^n-1) > dim_{\mathbb{R}}(M)$ (or for $4(p^n-1) > dim_{\mathbb{R}}(M)$ for Weinstein M), the groups $\widehat{K_{p^k}(n)}^*_{S^1}(SH_S(M,\mathbb{S}))$ and $H^*(M,\mathbb{Z}[v_n,v_n^{-1}]/p^k)((u))$ are isomorphic.*

Combining Theorem 4.4 with the universal coefficients theorem, we show that the integral cohomology can be fully recovered:

Corollary 4.5. *The filtered S^1-equivariant homotopy type of $SH_S(M,\mathbb{S})$ determines $H^*(M,\mathbb{Z})$.*

As noted, the invariant $\widehat{R}^*_{S^1}(SH(M,\mathbb{S}))$ depends on the filtration in a rather loose sense. Also, we choose to work with $SH(M,\mathbb{S})$ for simplicity, but the invariant $\widehat{R}^*_{S^1}(SH(M,\mathbb{S}))$ can be defined under milder assumptions on M, when R is complex oriented. Our results can likely be extended to this generality but we will not pursue this.

We also show we can recover complex K-theory:

Theorem 4.6. *After simultaneously completing at every prime p, $\widehat{KU}^*_{S^1}(SH_S(M,\mathbb{S}))$ and $KU^*(M)((u))$ become isomorphic. In other words, $\widehat{KU}^*_{S^1}(SH_S(M,\mathbb{S}))^\wedge \simeq KU^*(M)((u))^\wedge$. As a result, the filtered S^1-equivariant homotopy type of $SH_S(M,\mathbb{S})$ determines $KU^*(M)$.*

This result can be interpreted as the (co)filtered equivariant homotopy type of "symplectic K-homology" determines the K-theory of the underlying manifold. Recall that for an Abelian group A, $A^\wedge := \lim_{n\in\mathbb{N}} A/nA$.

Theorem 4.6 provides a possible approach for a conjecture of Treumann that is inspired by string theory and mirror symmetry. See Remark 4.7 for further discussion.

Recovering more information about the stable homotopy type of M. We believe the information one can obtain from the filtered equivariant stable homotopy type of $SH(M,\mathbb{S})$ is limited. In particular, we do not expect that the stable homotopy type of M can be fully recovered without appealing to an extra structure on $SH(M,\mathbb{S})$. Up to many Floer theoretic obstacles, there is a heuristic genuine cyclotomic structure on $SH(M,\mathbb{S})$ with geometric fixed points $\Phi^g_{S^1}SH(M,\mathbb{S}) \simeq \Sigma^\infty M_+$ (the genuine action is different from above). Therefore, by statements such as [NS18, Theorem 1.4], one expects to recover the stable homotopy type of M form $SH(M,\mathbb{S})$, considered as a filtered cyclotomic spectrum. We plan to explore this and the related question of obtaining $R_*(M)$ from $SH(M,R)$ for a general R in subsequent work.

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2. Filtered equivariant spectra and completed Tate cohomology

Throughout this paper we use the word *G-equivariant spectra* to refer to Borel equivariant spectra, i.e. to the functors $BG \to Sp$. Given such a spectrum X, one can define its *homotopy quotient* X_{hG} (resp. *homotopy fixed points* X^{hG}) as the colimit (resp. limit) of the corresponding functor $BG \to Sp$. More concretely, X_{hG} can be defined as $X \wedge_G \Sigma^\infty EG_+$, and X^{hG} can be defined as $Map^G(\Sigma^\infty EG_+, X)$. A *filtration* on an equivariant spectrum is the data of a sequence $X_1 \to X_2 \to \dots$ of equivariant spectra and an equivalence $hocolim X_n \xrightarrow{\simeq} X$. For a given sequence $1 \le i_1 < i_2 < \dots$, the corresponding *subfiltration* is given by $X_{i_1} \to X_{i_2} \to \dots$, $hocolim X_{i_n} \xrightarrow{\simeq} X$. There is a natural map of filtrations from $X_1 \to X_2 \to \dots$ to any subfiltration. This map will induce an equivalence on the invariants we define. We call two filtrations *equivalent*, if they are weakly equivalent in the category of filtrations localized at the natural maps from $X_1 \to X_2 \to \dots$ to $X_{i_1} \to X_{i_2} \to \dots$. A concrete way to show two filtrations $X_1 \to X_2 \to \dots \to X$ and $X'_1 \to X'_2 \to \dots \to X$ are equivalent is to find sequences $1 \le i_1 \le i_2 \dots$, $1 \le j_1 \le j_2 \dots$, and maps $X'_k \to X_{i_k}$, $X_k \to X'_{j_k}$ of filtrations such that the two sided compositions are equivalent to the natural subfiltration maps $X_k \to X_{i_{j_k}}$ and $X'_k \to X'_{j_{i_k}}$. One can also show that, for any sequence $1 \le i_1 \le i_2 \le \dots$ that is not strictly monotone, but $i_k \to \infty$, $X_{i_1} \to X_{i_2} \to \dots$ is still equivalent to $X_1 \to X_2 \to \dots$ (by a zigzag through a common subfiltration). We call a filtration *trivial*, resp. *finite* if all of $X_k \to X_{k+1}$, resp. all but finitely many of $X_k \to X_{k+1}$ are equivalences. Observe that all finite filtrations are equivalent to the trivial one. Throughout the paper, unless stated otherwise, we only consider the filtrations $X_1 \to X_2 \to \dots X$ such that all of X_i are of finite type.

Similarly, we call the data $\dots \to X_3 \to X_2 \to X_1$, $X \xrightarrow{\simeq} \lim X_i$ a cofiltration. We will not discuss their equivalences.

Let R be a complex oriented ring spectrum. In other words, the pull-back $R^*(BS^1) = R^*(\mathbb{CP}^\infty) \to R^*(\mathbb{CP}^1)$ is surjective. The complex orientation is a lift $u \in R^*(BS^1)$ of $1 \in \pi_0(\mathbb{CP}^1) \cong \tilde{R}^2(\mathbb{CP}^1)$. Then $R^*(BS^1) \cong R^*[[u]]$, where $R^* = R^*(\mathbb{S})$.

It is standard that a complex orientation defines a formal group law $F_R \in R^*[[u_1, u_2]]$. Concretely, F_R is the image of u under the pull-back $R^*[[u]] = R^*(BS^1) \to R[[u_1, u_2]] = R^*(BS^1 \times BS^1)$ of the classifying map $BS^1 \times BS^1 \to BS^1$ corresponding to the tensor product of universal line bundles. One can think of $u \in R^*(BS^1)$ as the analogue of the first Chern class in the generalized cohomology theory R, and F_R is the formula for the Chern class of tensor products of line bundles.

Let $[n]_R(u) \in R^*[[u]]$ denote the n^{th} iterate of u under the formal group law F_R (i.e. $[2]_R(u) = F_R(u, u)$, $[3]_R(u) = F_R([2]_R(u), u)$, $\dots$).

Given an S^1-equivariant spectrum X, there is a natural map $X_{hS^1} = X \wedge_{S^1} \Sigma^\infty ES^1_+ \to \mathbb{S} \wedge_{S^1} \Sigma^\infty ES^1_+ = \Sigma^\infty BS^1_+$, and thus a ring homomorphism

$$R^*[[u]] = R^*(BS^1) \to R^*(X_{hS^1}) \tag{2.1}$$

Note that one can prove using the adjunctions that $R^*(X_{hS^1}) \simeq R^*(X)^{hS^1}$.

Definition 2.1. Let $X_1 \to X_2 \to \dots X$ be a filtered S^1-equivariant spectrum. Define the *completed Tate cohomology* of X to be the homotopy limit

$$\widehat{R}^*_{S^1}(X) := \mathit{holim}_k R^*((X_k)_{hS^1})[[1]_R(u)^{-1}, [2]_R(u)^{-1}, [3]_R(u)^{-1}, \dots] \tag{2.2}$$

Here we use $R^*((X_k)_{hS^1})[[1]_R(u)^{-1}, [2]_R(u)^{-1}, [3]_R(u)^{-1}, \dots]$ to denote u-adically completed localization. In other words, to obtain $\widehat{R}^*_{S^1}(X)$, we take the inverse limit of $R^*((X_k)_{hS^1})$ localized at $[n]_R(u)$ for every $n \geq 1$ and then u-adically completed. Observe

- $\widehat{R}^*_{S^1}(X)$ carries the structure of a u-adically complete $R^*((u))$-algebra and it only depends on the equivalence class of the filtration.
- $\widehat{R}^*_{S^1}(X)$ is functorial in X and preserves finite colimits.
- $\widehat{R}^*_{S^1}(X)$ is functorial in R. In other words, if $R \to R'$ is an homomorphism of complex oriented ring spectra (respecting complex orientations), then there is a natural map $\widehat{R}^*_{S^1}(X) \to \widehat{R'}^*_{S^1}(X)$.

If the filtration is trivial/finite, there is not need to pass to the limit. Hence we will also use the notation $\widehat{R}^*_{S^1}(X_k)$ to refer to the terms in the limit in (2.2). The following is crucial

Lemma 2.2. *If X is the suspension spectrum of S^1/C_k, for $k = 1, 2, \dots$ (with the trivial filtration), then $\widehat{R}^*_{S^1}(X) = 0$.*

Proof. Note that $X_{hS^1} = \Sigma^\infty(BC_k)_+$ and $R^*(BC_k) = R^*[[u]]/[k]_R(u)$. Therefore, inverting $[k]_R(u)$ kills $R^*(X_{hS^1})$. □

We will also use

Lemma 2.3. *If X is endowed with a trivial action and a trivial/finite filtration, then $\widehat{R}^*_{S^1}(X)$ is $R^*(X)[[u]]$ localized at $[n]_R(u)$ for every $n \geq 1$.*

Proof. This follows immediately from the equivalence $X_{hS^1} \simeq X \wedge \Sigma^\infty BS^1$. □

Example 2.4. Assume $R = HA$, the Eilenberg–Maclane spectrum of a commutative ring A. Then, $R^* = A$ and $F_R(u_1, u_2) = u_1 + u_2$, the *additive formal group law*. Therefore, $[n]_R(u) = nu$, and inverting it is the same as inverting n and u. As a result, if X is endowed with a trivial action and filtration, then $\widehat{R}^*_{S^1}(X) = H^*(X, A \otimes_{\mathbb{Z}} \mathbb{Q})((u))$. In particular, if $n = 0$ in A, for some $n \geq 2$, then $\widehat{R}^*_{S^1}(X) = 0$. On the other hand, if $R = H\mathbb{Q}$ or $H\mathbb{Z}$, then $\widehat{R}^*_{S^1}(X) = H^*(X, \mathbb{Q})((u))$.

If in (2.2) we were not taking the u-adically completed localization, $\widehat{H\mathbb{Z}}^*_{S^1}(X)$ and $H^*(X, \mathbb{Q})((u))$ would match only after u-adically completing the former.

The most important example for us is the following:

Example 2.5. Let $R = K_p(n)$ be a Morava K-theory. Then, $R^* = \mathbb{F}_p[v_n, v_n^{-1}]$. As $[k]_R(u)$ is always of the form $ku + h.o.t.$, when k is coprime with p, $[k]_R(u)$ is u multiplied with a unit. Moreover, $[p]_R(u) = v_n u^{p^n}$ (see [Wur91, Theorem 1.3] for instance). Hence, by iteration, we see that $[k]_R(u)$ is the same as a power of u multiplied by a unit (if $p^l || k$, $[k]_R(u) = \frac{k}{p^l} v_n^{\frac{p^{nl}-1}{p^n-1}} u^{p^{nl}} + h.o.t.$). Therefore, inverting all $[k]_R(u)$ is equivalent to inverting u. In other words, for X with a trivial action and trivial filtration $\widehat{K_p(n)}^*_{S^1}(X) = K_p(n)^*(X)((u))$.

Example 2.6. More generally, if p is nilpotent in R^* and F_R has height exactly k, i.e. $[p]_R(u)$ is u^{p^k} multiplied by a unit, then $\widehat{R}^*_{S^1}(X) = R^*(X)((u))$.

Example 2.7. Let $R = k_p(n)$ denote the connective Morava K-theory, which satisfies $R^* = \mathbb{F}_p[v_n]$ and $[p]_R(u) = v_n u^{p^n}$. The calculation of $[k]_R(u)$ in Example 2.5, but this time inverting these elements require inverting u and v_n. Therefore, $\widehat{k_p(n)}^*_{S^1}(X) = K_p(n)^*(X)((u))$, which is analogous to $H\mathbb{Z}$ leading to rational cohomology in Example 2.4.

The following example will also be important in the subsequent sections:

Example 2.8. Let $\widetilde{K}_p(n)$ be the integral Morava K-theory, i.e. the complex oriented spectrum satisfying $\widetilde{K}_p(n)^* = \mathbb{Z}_p[v_n, v_n^{-1}]$ and $[-p]_{\widetilde{K}_p(n)}(u) = -pu + v_n u^{p^n}$. Let $K_{p^k}(n)$ denote its quotient by p^k. See [AMS21, Proposition 5.14]. For $R = K_{p^k}(n)$, the same formula $[-p]_R(u) = -pu + v_n u^{p^n}$ holds. For $p \not| m$, $[mp^l]_R(u)$ is of the form $u^{p^{nl}} + O(p) + o(u^{p^{nl}})$ times a unit, where $O(p)$ denotes a multiple of p and $o(u^{p^{nl}})$ denotes the sum of the terms such that the exponent of u is larger than p^{nl}. One can also write this as $u^{p^{nl}} + O(p)$ times a unit (as $u^{p^{nl}} + o(u^{p^{nl}})$ is $u^{p^{nl}}$ times a unit). As p is nilpotent after inverting $u = [1]_R(u)$, elements of the form $u^{p^{nl}} + O(p)$ automatically become units. As a result, for X with a trivial action and filtration, $\widehat{R}^*_{S^1}(X) = R^*(X)((u))$. The same holds for $R = \widetilde{K}_p(n)$ after p-completing both sides, i.e. $\widehat{R}^*_{S^1}(X)^\wedge_p = R^*(X)((u))^\wedge_p$.

The following example will be important in recovering information about the complex K-theory from Floer theory:

Example 2.9. Recall that KU, the complex K-theory spectrum, has $KU^* = \mathbb{Z}[\beta, \beta^{-1}]$, where $|\beta| = 2$ and $F_{KU}(u_1, u_2) = u_1 + u_2 + u_1u_2 = (1 + u_1)(1 + u_2) - 1$, the *multiplicative formal group law*. Let p be a prime and $R = KU/p^k$ so that $R^* = \mathbb{Z}/p^k[\beta, \beta^{-1}]$ and $[p^l]_R(u) = \binom{p^l}{1}u + \binom{p^l}{2}u^2 + \cdots + u^{p^l}$, which implies $[mp^l]_R(u) = u^{p^l} + O(p)$ times a unit as in Example 2.8, for $p \not| m$. As a result, inverting u suffices as before and for X with a trivial action and filtration, $\widehat{R}^*_{S^1}(X) = R^*(X)((u))$. The same holds for KU after simultaneous completion of both sides at each p, i.e. $\widehat{KU}^*_{S^1}(X)^\wedge = KU^*(X)((u))^\wedge$, where for an Abelian group A, $A^\wedge$ denotes the inverse limit of A/nA, $n \in \mathbb{N}$.

One can make such concrete calculations for other complex oriented spectra, such as Brown–Peterson spectrum (or its variants such as its localization at $v_1 \in BP^*$). This could possibly let one to recover information about MU from Floer theory.

3. S^1-equivariant spectral symplectic cohomology

In this section, we recall the basics of the S^1-equivariant symplectic cohomology $SH_S(M, \mathbb{S})$ with coefficients in $\mathbb{S}$ following [CK23]. As we find it conceptually easier, we first recall the non-equivariant version $SH(M, \mathbb{S})$, following [Lar21, CK23].

Fix a *Liouville manifold* (M^{2N}, λ). In other words, we fix an exact symplectic manifold M and a primitive λ for the symplectic form ω_M such that the Liouville vector field Z defined by the property $\iota_Z d\lambda = \lambda$ points outward at infinity. By choosing a closed contact hypersurface $\partial\overline{M} \subset M$ that is transverse to Z, one obtains a decomposition

$$M = \overline{M} \cup_{\partial\overline{M}} [1, \infty) \times \partial\overline{M} \tag{3.1}$$

We denote the coordinate of the $[1, \infty)$ component by r and refer to it as the Liouville parameter.

Throughout the paper, we assume M is *stably framed*, i.e. we fix an equivalence $T_M \oplus \mathbb{C}^k \cong \mathbb{C}^{N+k}$ of symplectic vector bundles for some $k \geq 0$. This is required to define $SH(M, \mathbb{S})$ and $SH_S(M, \mathbb{S})$. We make this assumption for simplicity, as we only need the symplectic cohomology with coefficients in complex oriented spectra, which is well-defined under weaker assumptions.

Consider an Hamiltonian $S^1 \times M \to \mathbb{R}$ that is non-degenerate and linear at infinity. In other words, outside a compact subset of M, H is of the form $ar + b$ for some constants $a \geq 0, b$. We refer to a as the slope of H. By [Lar21], for a generic family of cylindrical almost complex structures $J : S^1 \to \mathcal{J}(M)$, there exists a category $\mathcal{M}_{H,J}$ such that

- $ob(\mathcal{M}_{H,J})$ is the set of 1-periodic orbits of H
- the morphisms $\mathcal{M}_{H,J}(x, y)$ is given by the moduli of Floer trajectories from x to y and it forms a manifold with corners with a stable trivialization of its tangent bundle
- the composition maps are smooth embeddings into boundary strata, which is covered by the images of the composition maps

In other words, $\mathcal{M}_{H,J}$ is a *framed flow category*. To this data, one associates a spectrum $|\mathcal{M}_{H,J}|$ called the *geometric realization*. We denote this spectrum by $HF(H, \mathbb{S})$ (omitting J from the notation). To define this, one uses Pontryagin–Thom collapse and obtain a "chain complex in spectra". Then, there is a realization construction for such complexes. See [CK23] for more details.

Similar to classical Floer theory, for $H' \geq H$, one can define continuation maps $HF(H,\mathbb{S}) \to HF(H',\mathbb{S})$. More precisely, one assembles the continuation trajectories into a *framed flow bimodule* over $\mathcal{M}_{H,J}$-$\mathcal{M}_{H',J'}$, inducing a map of geometric realizations. Define $SH(M,\mathbb{S})$ as the homotopy colimit of $HF(H,\mathbb{S})$ as the slope goes to infinity. By choosing a sequence of Hamiltonians H_n with slope converging to infinity, one can equip $SH(M,\mathbb{S})$ with a filtration in the sense of Section 2. Different choices of cofinal sequences of Hamiltonians lead to equivalent filtrations.

Remark 3.1. Recall that in classical Floer theory, one can define a finite filtration on Floer cochains by action. In other words, one considers the subcomplex of $CF(H,J)$ generated by orbits of action less than p. One can similarly consider the subcategory spanned by these orbits, and there is a map from its geometric realization to $HF(H,\mathbb{S})$, whose quotient is the realization of the subcategory spanned by other orbits. Note that for such an inclusion to induce a map into $HF(H,\mathbb{S})$, one needs the former subcategory to be final, i.e. there are no morphisms from its objects to other objects (i.e. no Floer trajectories from low action orbits to high action orbits). This endows $HF(H,\mathbb{S})$ with a filtration, which is equivalent to the trivial filtration as it is finite. This differs slightly from the filtration on $SH(M,\mathbb{S})$ we consider above. Nevertheless, we will use this finite filtration to compute $\widehat{R}^*_{S^1}(SH(M,\mathbb{S}))$.

In [CK23], the authors extended the realization construction for flow categories to Morse–Bott and equivariant setting, and used this to define S^1-equivariant versions of $HF(H,\mathbb{S})$ and $SH(M,\mathbb{S})$ using Borel construction. A *framed Morse–Bott flow category* $\mathcal{M}$ is defined similarly, but one allows $ob(\mathcal{M})$ to be a finite disjoint union of smooth closed manifolds. If $X, Y \subset ob(\mathcal{M})$ denote two components and $\mathcal{M}(X,Y)$ denotes the morphisms from a point on X to one Y, we assume the domain and target maps $\mathcal{M}(X,Y) \to X, Y$ are smooth. The motivating example is the category associated to a Morse–Bott function $f : N \to \mathbb{R}$: the object space is the union of critical manifolds of f and the morphisms are given by the broken negative gradient trajectories (with respect to a generic metric). The stable framing condition is generalized as a framing of the relative tangent bundle $T_{\mathcal{M}(X,Y)} - T_X$ twisted by virtual bundles on X and Y. See [CK23] for more details.

To this data, one associates a spectrum $|\mathcal{M}|$, also called the *geometric realization*. If the category and the framings are equivariant with respect to a compact group action $|\mathcal{M}|$ is also a genuine equivariant spectrum. As before, the inclusion of a subcategory $\mathcal{M}' \subset \mathcal{M}$ spanned by a union of components of $ob(\mathcal{M})$ and such that there are no morphisms from $ob(\mathcal{M}')$ to $ob(\mathcal{M}) \setminus ob(\mathcal{M}')$ defines a map $|\mathcal{M}'| \to |\mathcal{M}|$ with cofiber $|\mathcal{M} \setminus \mathcal{M}'|$.

Let $S = S^\infty = ES^1$ and fix the Morse–Bott function $\tilde{f} : S \to \mathbb{R}$ given by $\sum_{i=1}^\infty i|z_i|^2$ in complex coordinates. Note that $\tilde{f}$ lifts a Morse function on $S^\infty/S^1 = \mathbb{CP}^\infty$. The critical set of $\tilde{f}$ consists of infinite copies of circle with Morse index $0, 2, 4, \dots$. By lifting a generic metric on $\mathbb{CP}^\infty$ to S^∞, one obtains an equivariant framed Morse–Bott flow category.

Consider a pair (H,J), where $H : S \times S^1 \times M \to \mathbb{R}$, $J : S \times S^1 \to \mathcal{J}(M)$ such that

- $H_s = H|_{\{s\}\times S^1 \times M}$ is linear and J_s is cylindrical outside a compact subset of M that does not depend on s. Moreover, the slope of H_s does not depend on s.
- H_s is increasing along the negative gradient trajectories of $\tilde{f}$.
- For every $s \in crit(\tilde{f})$, H_s is non-degenerate, J_s is regular.
- (H,J) is equivariant with respect to the diagonal S^1-action on $S \times S^1$. More concretely, (H_s, J_s) and (H_{zs}, J_{zs}) on $S^1 \times M$ are related by the action of $z \in S^1$.

For generic J there exists a framed S^1-equivariant Morse–Bott flow category $\mathcal{M}_{\tilde{f},H,J}$ such that

- $ob(\mathcal{M}_{\tilde{f},H,J})$ is the space of pairs (s,x), where $s \in \tilde{f}$, $x \in orb(H_s)$
- morphisms from (s,x) to (s',x') are given by pairs (γ,u) (up to translation) where γ is a negative gradient trajectory of $\tilde{f}$ from s to s' and u is a J-holomorphic cylinder from x to x'

Notice that $orb(H_s)$ and $orb(H_{zs})$ are naturally identified by rotation, and $ob(\mathcal{M}_{\tilde{f},H,J})$ is a disjoint union of copies of S^1-torsors. Also note that the Floer data for the equation u satisfies depends on γ. See [CK23] for the precise setup and the framings. Let $HF_S(H,\mathbb{S})$ denote the geometric realization $|\mathcal{M}_{\tilde{f},H,J}|$. This is a genuine S^1-equivariant spectrum. For $H' \geq H$, there are equivariant continuation maps $HF_S(H,\mathbb{S}) \to HF_S(H',\mathbb{S})$. We define $SH_S(M,\mathbb{S})$ to be the homotopy colimit of $HF_S(H,\mathbb{S})$ as the slope goes to infinity. Analogous to $SH(M,\mathbb{S})$, $SH_S(M,\mathbb{S})$ is a filtered S^1-equivariant spectrum, where the filtration is well-defined up to equivalence.

4. Approximately autonomous Hamiltonians and the completed Tate cohomology of $SH_S(M,\mathbb{S})$

The goal of this section is to prove the following theorem:

Theorem 4.1. *For a given complex oriented spectrum R, $\widehat{R}^*_{S^1}(SH_S(M,\mathbb{S}))$ is equivalent to $R^*(M)[[u]]$ localized at $[n]_R(u)$, for every $n \geq 1$.*

Recall that for us localized means in an u-adically complete sense. By Lemma 2.3, one can phrase Theorem 4.1 as $\widehat{R}^*_{S^1}(SH_S(M,\mathbb{S})) \simeq \widehat{R}^*_{S^1}(\Sigma^\infty M_+)$, where $\Sigma^\infty M_+$ is endowed with the trivial filtration and trivial action. The importance of Theorem 4.1 is that one can recover purely topological information from the S^1-equivariant Floer homotopy type endowed with a natural filtration.

Before moving onto the proof, we discuss major corollaries in the light of examples given in Section 2.

Corollary 4.2 (c.f. [Zha14, Thm. 1.1,§8.1,§8.3])**.** $\widehat{H\mathbb{Q}}^*_{S^1}(SH_S(M,\mathbb{S})) \simeq \widehat{H\mathbb{Z}}^*_{S^1}(SH_S(M,\mathbb{S})) \simeq H^*(M,\mathbb{Q}((u)))$. *On the other hand,* $\widehat{H\mathbb{F}_p}^*_{S^1}(SH_S(M,\mathbb{S})) \simeq 0$.

This follows immediately from Theorem 4.1 applied to Example 2.4. Note that for a commutative ring A, $\widehat{HA}^*_{S^1}(SH_S(M,\mathbb{S}))$ depends only on the S^1-equivariant (co)filtered homotopy type of $HA^*(SH_S(M,\mathbb{S}))$, which does not require Floer homotopy theory to define. This reaffirms the conclusion of [Zha14] that the symplectic cohomology with its S^1-action and filtration recovers the rational cohomology. On the other hand, the torsion information is lost.

Similarly, by applying Theorem 4.1 to Example 2.5 and Example 2.8, we obtain

Corollary 4.3. $\widehat{K_{p^k}(n)}^*_{S^1}(SH_S(M,\mathbb{S})) \simeq K_{p^k}(n)^*(M)((u))$.

Our second major statement follows from Corollary 4.3. Due to its importance, we state it as a theorem:

Theorem 4.4. *For $2(p^n-1) > dim_{\mathbb{R}}(M)$ (or for $4(p^n-1) > dim_{\mathbb{R}}(M)$ for Weinstein M), the groups $\widehat{K_{p^k}(n)}^*_{S^1}(SH_S(M,\mathbb{S}))$ and $H^*(M,\mathbb{Z}[v_n,v_n^{-1}]/p^k)((u))$ are isomorphic.*

Proof. We borrow the idea from [AMS21, Lemma 5.15]. If $2(p^n-1) > dim_{\mathbb{R}}(M)$ or $4(p^n-1) > dim_{\mathbb{R}}(M)$ and M is Weinstein, $|v_n| = 2(p^n-1)$ is larger than the range of the cohomology of M. Therefore, Atiyah–Hirzebruch spectral sequence degenerates and $K_{p^k}(n)^*(M) \simeq H^*(M, \mathbb{Z}[v_n, v_n^{-1}]/p^k)$. The theorem now follows from Corollary 4.3. □

In particular, $\widehat{K_p(n)}^*_{S^1}(SH_S(M,\mathbb{S})) \cong H^*(M, \mathbb{F}_p[v_n, v_n^{-1}])((u))$, whose rank over $\mathbb{F}_p[v_n, v_n^{-1}]((u))$ gives the total dimension of $H^*(M,\mathbb{F}_p)$. However, one can recover more information: the isomorphism in Theorem 4.4 is natural in k; hence, the inverse limits of $\widehat{K_{p^k}(n)}^*_{S^1}(SH_S(M,\mathbb{S}))$ and $H^*(M, \mathbb{Z}[v_n, v_n^{-1}]/p^k)((u))$ in k are also isomorphic. Express $H^*(M,\mathbb{Z})$ as a direct sum of cyclic groups. In $\lim\limits_{k} H^*(M, \mathbb{Z}[v_n, v_n^{-1}]/p^k)((u))$, $\mathbb{Z}$ summands turn into $\mathbb{Z}_p[v_n, v_n^{-1}]((u))^\wedge$ and $\mathbb{Z}/p^l$ summands turn into $\mathbb{Z}_p[v_n, v_n^{-1}]((u))^\wedge/p^l \oplus \mathbb{Z}_p[v_n, v_n^{-1}]((u))^\wedge/p^l$, as modules over $\mathbb{Z}_p[v_n, v_n^{-1}]((u))^\wedge$ (the p-adic completion of $\mathbb{Z}_p[v_n, v_n^{-1}]((u))$). As a result

Corollary 4.5. *The filtered S^1-equivariant homotopy type of $SH_S(M,\mathbb{S})$ determines $H^*(M,\mathbb{Z})$.*

Notice that, we only need the (cofiltered, S^1-equivariant) generalized cohomology of $SH_S(M,\mathbb{S})$ with respect to Morava K-theories. Therefore, less information than $SH_S(M,\mathbb{S})$ is actually required.

Another major implication of Theorem 4.1 is the following:

Theorem 4.6. *After simultaneously completing at every prime p, $\widehat{KU}^*_{S^1}(SH_S(M,\mathbb{S}))$ and $KU^*(M)((u))$ become isomorphic. In other words, $\widehat{KU}^*_{S^1}(SH_S(M,\mathbb{S}))^\wedge \simeq KU^*(M)((u))^\wedge$. As a result, the filtered S^1-equivariant homotopy type of $SH_S(M,\mathbb{S})$ determines $KU^*(M)$.*

This result can be interpreted as the (co)filtered equivariant homotopy type of "symplectic K-homology" determines the K-theory of the underlying manifold. Recall that for an Abelian group A, $A^\wedge := \lim\limits_{n\in\mathbb{N}} A/nA$.

Proof. The isomorphism statement follows from Theorem 4.1 and Example 2.9. Moreover, as M is of finite type, each $KU^i(M)$ is a finitely generated Abelian group. $KU^*(M)((u))$ can be identified with $KU^0(M)((\beta^{-1}u))$ at even degrees and with $KU^1(M)((\beta^{-1}u))$ at odd degrees. Each of these groups split as $(T\oplus F)((\beta^{-1}u))$, where T is the torsion part of $KU^0(M)$, resp. $KU^1(M)$, and F is a finitely generated free Abelian group. The completion procedure leaves $T((\beta^{-1}u))$ the same, and $F((\beta^{-1}u))$ turns into a finitely generated free module over $\mathbb{Z}((\beta^{-1}u))^\wedge$ of the same rank (and the former is still the torsion part). It is clear that T and F are uniquely determined. □

Remark 4.7. As we mentioned, Theorem 4.6 provides evidence for a string theory and mirror symmetry inspired conjecture of Treumann, which states that the Fukaya category determines the complex K-theory, see [Tre19]. This is clearly false for wrapped Fukaya categories, as there can be subcritical Weinstein manifolds with different complex K-theories; however, Treumann further conjectures filtered versions of the same claim. In other words, the filtered wrapped Fukaya category recovers the complex K-theory. Theorem 4.6 provides evidence for this conjecture. When the Atiyah–Hirzebruch spectral sequence $H^*(SH_S(M,\mathbb{S}), KU^*) \Rightarrow KU^*(SH_S(M,\mathbb{S}))$ degenerates (e.g. when the ordinary symplectic cohomology is supported in even degrees), $KU^*(SH_S(M,\mathbb{S}))$ is uniquely determined by the symplectic cohomology. Assuming the filtration can be chosen so that the same degeneration claim holds uniformly at every level, $KU^*(SH_S(M,\mathbb{S}))$ is likely to be determined as a (co)filtered equivariant spectrum by the filtered equivariant symplectic cohomology. It is reasonable to expect that a filtered equivariant version of [Gan12, Gan19] holds, i.e. the filtered equivariant symplectic cohomology is determined by the filtered wrapped Fukaya category, and this would prove Treumann's conjecture.

Now, we move on to the proof of Theorem 4.1. The proof strategy is as follows: we find a cofinal sequence of linear Hamiltonians such that $HF_S(H,\mathbb{S})$ has building blocks equivalent to copies of $\Sigma^\infty(S^1/C_k)_+$ for various k and a single copy of $\Sigma^\infty M_+$. By Lemma 2.2, $\widehat{R}^*_{S^1}(\Sigma^\infty(S^1/C_k)_+)\simeq 0$, which implies $\widehat{R}^*_{S^1}(HF_S(H,\mathbb{S}))\simeq \widehat{R}^*_{S^1}(\Sigma^\infty M_+)$ and by passing to the limit, the same holds for $\widehat{R}^*_{S^1}(SH_S(M,\mathbb{S}))$. Recall that, for a spectrum with a finite/trivial filtration (such as $HF_S(H,\mathbb{S})$ or $\Sigma^\infty M_+$), we use the notation $\widehat{R}^*_{S^1}(X)$ for $R^*(X_{hS^1})$ localized at every $[n]_R(u)$, which coincides with Definition 2.1.

Fix a generic large slope $a>0$. One is tempted to use autonomus Hamiltonians $H\to\mathbb{R}$ of slope a that is small in the interior of M. Assuming equivariant $HF_S(H,\mathbb{S})$ is defined, it can be filtered by action, where the constant orbits produce a building block equivalent to $\Sigma^\infty M_+$ and others produce subquotients equivalent to some $\Sigma^\infty(S^1/C_k)_+$. Even though our topological framework from [CK23] allows the use of such Hamiltonians, to avoid further gluing analysis, we will use approximately autonomous Hamiltonians.

We closely follow [CFHW96], and [Zha14, §5]. Assume the Reeb flow on $\partial\overline{M}$ is generic and fix a large slope $a>0$ that is different from the length of any periodic Reeb orbit. Let $H^a: M\to\mathbb{R}$ be an Hamiltonian such that

$$H^a(x)=\begin{cases} f(x), & \text{for } x\in int(\overline{M})\\ (r-1)^2/2, & x=(r,y)\in[1,a+1]\times\partial\overline{M}\\ a(r-1)-a^2/2, & x=(r,y)\in[a+1,\infty)\times\partial\overline{M}\end{cases} \tag{4.1}$$

Here, $f(x)$ is a negative C^2-small Morse function in $int(\overline{M})$, and without loss of generality we smooth H^a near $r=1$ and $r=a+1$. The 1-periodic orbits of H^a are either (i) the critical points of f, or (ii) of the form $\{\ell+1\}\times\gamma_0$, where $\ell\in(0,a)$ and γ_0 is a length ℓ Reeb orbit. In particular, non-constant orbits come in S^1-families. Without loss of generality, assume the 1-periodic orbits of H have different action.

We perturb H^a near these to obtain a non-degenerate Hamiltonian. Let $h_0: S^1\to[0,1]$ be a Morse function with a maximum at $1/2$ and minimum at 0. For a non-constant k-fold orbit γ of H^a, define a function h_t on $Im(\gamma)$ by $h_t(\gamma(t'))=h_0(k(t'-t))$ and extend h_t to a small tubular neighborhood U_γ of $Im(\gamma)$. Consider $H^a_\epsilon=H^a+\epsilon h_t$, for a small $\epsilon>0$. It is shown in [CFHW96, Proposition 2.2] that

- the Hamiltonian H^a_ϵ has two non-degenerate orbits within U_γ given by $\gamma^-(t)=\gamma(t)$ and $\gamma^+(t)=\gamma(t+1/(2k))$
- $ind(\gamma^+)=ind(\gamma^-)+1$
- there are exactly two Floer trajectories from γ^+ to γ^- within U_γ

We make this perturbation for every non-constant orbit, and denote the resulting Hamiltonian by H. More precisely, as non-constant orbits come in circle families, we make a choice of $\gamma: S^1\to M$ for each of them. Then, apply the procedure above for each chosen orbit parametrization. For small enough ϵ, perturbation will not create new 1-periodic orbits other than $\gamma^\pm$. In other words, the 1-periodic orbits of H consists of (i) the critical points of f, (ii) γ^+ and γ^- for every γ. In particular, the Hamiltonian H is non-degenerate and it has only finitely many orbits.

Choice of a generic almost complex structure $S^1 \to M$ allows one to define a flow category $\mathcal{M}_{H,J}$ and its geometric realization $HF(H,\mathbb{S})$. There is a finite action filtration on $\mathcal{M}_{H,J}$ and $HF(H,\mathbb{S})$. If p is the action of γ, there exists a $\delta > 0$ such that $\gamma^{\pm}$ have action within the window $(p-\delta, p+\delta)$. We assume ϵ is small enough, so that there exists a $\delta > 0$ that works for every orbit and the action windows $(p-\delta, p+\delta)$ corresponding to different orbits do not intersect. As a warm-up for the equivariant case, we show the following non-equivariant statements

1. $F^0HF(H,\mathbb{S}) \simeq \Sigma^\infty M_+$
2. if p is the action of an orbit γ, then the spectrum $F^{(p-\delta,p+\delta]}HF(H,\mathbb{S})$ defined as the homotopy cofiber of $F^{p-\delta}HF(H,\mathbb{S}) \to F^{p+\delta}HF(H,\mathbb{S})$ is equivalent to $\Sigma^\infty Im(\gamma)_+ \simeq \Sigma^\infty S^1_+$

Neither of these claims are hard. There are two ways to prove (1): namely to use bimodules over $\mathcal{M}_f$-$\mathcal{M}_{H,J}$, where $\mathcal{M}_f$ is the Morse flow category of f (for a generic metric) or to use M-valued modules over $F^0\mathcal{M}_{H,J}$. To expand the first option, given $x \in crit(f)$ and $y \in orb(H)$, (after choice of data) define a bimodule by associating to (x,y) the compactified moduli of spiked discs (i.e. a half gradient trajectory of $-f$ from x to a point in M and a half disc through that point with output y). This bimodule defines a map $\Sigma^\infty M_+ = |\mathcal{M}_f| \to |\mathcal{M}_{H,J}| = HF(H,\mathbb{S})$ that factors through $F^0HF(H,\mathbb{S})$. One can further filter $\Sigma^\infty M_+$ and $F^0HF(H,\mathbb{S})$ by the values of f and it is easy to see $\Sigma^\infty M_+ \to F^0HF(H,\mathbb{S})$ induces isomorphisms on the subquotients. This proves (1).

Remark 4.8. We skip lengthy discussions of framings; however, we note that to frame this bimodule, it is more natural to use the description of the framings on $\mathcal{M}_{H,J}$ in terms of negative caps (see [CK23, Lemma 5.6] and the preceding discussion). To explain this further in that context, if W_x denote the framing bundle over $\{x\} \subset crit(f)$, and V_y^- is the bundle over $\{y\} \subset orb(H)$ defined as in loc. cit. then a point in the moduli of spiked discs has tangent space given by $W_x + V_y^- - \mathbb{R}^{2k} - T_M$, where k is the stabilization constant (i.e. we fix $T_M \oplus \mathbb{C}^k \simeq \mathbb{C}^{n+k}$). To see this, notice the moduli of spiked discs is the fibered product of moduli of half gradient trajectories from x and moduli of half discs to x over their evaluation maps to M. The former has tangent space that can be identified with W_x and the tangent space of the discs can be identified with $V_y^- - \mathbb{C}^k$. By subtracting T_M pulled back along the evaluation map at the spike, we obtain the asserted equivalence. Combining this with [CK23, Lemma 5.6] gives us the framings.

Given the facts about moduli of Floer trajectories from γ^+ to γ^-, (2) is actually straightforward. Namely, it implies that $F^{(p-\delta,p+\delta]}HF(H,\mathbb{S})$ is the equivalent to the cofiber of $\mathbb{S} \xrightarrow{0} \mathbb{S}$, which is $\Sigma^\infty S^1_+$. However, this argument is harder to implement in the equivariant case, and instead we will define a map using the formalism of P-relative modules.

Let $X_\gamma \simeq S^1$ denote the image of γ. We define an X_γ-relative module on $F^{p+\delta}\mathcal{M}_{H,J}$ by using moduli of continuation trajectories from with output (some rotation of) γ. Note that, heuristically this is using continuation maps from $HF(H,\mathbb{S})$ to "$HF(H^a,\mathbb{S})$". However, we prefer to avoid defining $HF(H^a,\mathbb{S})$ (as H^a is autonomous), and instead work locally near orbits of H^a.

Choose generic Floer data parametrized by the cylinder that is equal to (H,J) on the input (positive) end and whose Hamiltonianian is the same as $H^a + 1$ on the output end. We also assume the Hamiltonian has negative derivative in the s-direction. Given $x \in ob(F^{p+\delta}\mathcal{M}_{H,J})$, define $\mathcal{N}(x)$ to be the moduli of broken trajectories from x

to an orbit of H^a supported at X_γ (i.e. some $\gamma(t+\theta)$). As we restrict ourselves to $F^{p+\delta}\mathcal{M}_{H,J}$, $\mathcal{N}(x)$ is compact. One can show using standard gluing methods that $\mathcal{N}$ defines a module over $F^{p+\delta}\mathcal{M}_{H,J}$. Moreover, there is a compatible evaluation map $\mathcal{N}(x)\to X_\gamma$: send a trajectory that is asymptotic to $\gamma(t+\theta)$ at the negative end to $\gamma(\theta)\in X_\gamma$ (one can identify X_γ with these orbits, we are sending a trajectory to the orbit its asymptotic to). The framings are similar to standard continuation maps, and this defines a map $|F^{p+\delta}\mathcal{M}_{H,J}|\to\Sigma^\infty(X_\gamma)_+$. For $x\in ob(F^{p-\delta}\mathcal{M}_{H,J})$, $\mathcal{N}(x)$ is empty; therefore, this map factors through $F^{(p-\delta,p+\delta]}HF(H,\mathbb{S})\to\Sigma^\infty(X_\gamma)_+$. Observe $dim(\mathcal{N}(\gamma^+))=1$ and $dim(\mathcal{N}(\gamma^-))=0$ for index reasons, and each of them contains a constant trajectory in the interior. The constant trajectory is mapped to $\gamma(1/(2k))$, resp. $\gamma(0)$ and the connected component containing it is equivalent to I, resp. $\{pt\}$. The boundary of I correspond to two broken trajectories given by combining a Floer trajectory from γ^+ to γ^- with the constant trajectory from γ^-. Assuming these components are entire $\mathcal{N}(\gamma^\pm)$, one sees that $|F^{p+\delta}\mathcal{M}_{H,J}|\to\Sigma^\infty(X_\gamma)_+$ is an equivalence. Both $F^{(p-\delta,p+\delta]}HF(H,\mathbb{S})$ and $\Sigma^\infty(X_\gamma)_+$ are filtered (by the value of the Morse function on X_γ) and the map between them respects the filtration. Even though there may be other components of $\mathcal{N}(\gamma^\pm)$, for energy reasons, the map induced on the associated graded will be insensitive to these, implying the desired equivalence statement.

Now we turn our attention to the equivariant case. The most important part is to write a map using the relative module formalism, as the rest follows from similar filtration arguments. Let H^a and h_0 be as above. Choose a critical point $x_i\in S$ of index $2i$ for every $i\geq 0$ (in other words, choose a point from each critical manifold), and choose a decreasing sequence of small $\epsilon_i>0$. For a given k-fold orbit γ of H^a, define a function on $\{x_i\}\times S^1\times X_\gamma\subset S\times S^1\times M$ by $h(x_i,t,\gamma(t'))=\epsilon_i h_0(k(t'-t))$. Extend it to a compactly supported function on $\{x_i\}\times S^1\times U_\gamma$ as before and by using the S^1-action on S extend it to $X_i\times S^1\times U_\gamma$. Apply this to all non-constant orbits γ to obtain a function on $X_i\times S^1\times M$, where $X_i\subset S$ is the orbit of x_i and then extend it to $h:S\times S^1\times M\to\mathbb{R}$ such that

- h is S^1-equivariant
- h is decreasing along the negative gradient trajectories of $\tilde{f}$

We consider the Hamiltonian $H:=H^a+h:S\times S^1\times M\to\mathbb{R}$ (the parameters ϵ_i are in h) and choose a generic family $S\times S^1\to\mathcal{J}(M)$ of almost complex structures as usual. Assuming ϵ_i are small enough, [CFHW96, Proposition 2.2] applies and the following holds

- the Hamiltonian $H|_{\{x_i\}\times S^1\times M}$ has two non-degenerate orbits $\gamma_i^\pm$ within U_γ for each γ and x_i of index difference 1
- there are exactly two Floer trajectories from γ_i^+ to γ_i^- within $\{x_i\}\times S^1\times U_\gamma$

As before, by assuming ϵ_i is small enough, we can assume no extra orbit appears over x_i. As a result, the corresponding flow category $\mathcal{M}_{\tilde{f},H,J}$ has object space given by (i) copies of each X_i for each critical point of the Morse function f, (ii) two copies of X_i for each non-constant orbit γ and for each i obtained by rotating $\gamma_i^\pm$.

There is a natural filtration by action on $\mathcal{M}_{\tilde{f},H,J}$. By assuming ϵ_i are small enough we can ensure the existence of $\delta>0$ such that for a given non-constant orbit γ, there exists $p>\delta>0$ such that $\{\gamma_i^\pm : i\geq 0\}$ (and the orbits obtained by rotating them) are all in the action window $(p-\delta,p+\delta)$, and no other orbit is. Also, F^0 is given by orbits corresponding to $crit(f)$. This induces an equivariant filtration on $HF_S(H,\mathbb{S})$. We prove

Proposition 4.9. *$F^0HF_S(H,\mathbb{S})\simeq\Sigma^\infty M_+$ and for any k-fold non-constant orbit γ and the corresponding p as above, $F^{(p-\delta,p+\delta]}HF_S(H,\mathbb{S})\simeq\Sigma^\infty(X_\gamma)_+\simeq\Sigma^\infty(S^1/C_k)_+$ as (Borel) equivariant spectra.*

Recall that Borel equivariant means as a functor $BS^1\to Sp$.

Proof. The proof combines the methods above with the techniques in [CK23]. To prove $F^0HF_S(H,\mathbb{S})\simeq\Sigma^\infty M_+$, first consider the product flow category $\mathcal{M}_{\tilde f,f}$, satisfying $ob(\mathcal{M}_{\tilde f,f})=crit(\tilde f)\times crit(f)$, and the morphisms are given by pairs of negative gradient trajectories of $\tilde f$ and f. This category is S^1-equivariant and it is a special case of the construction of [CK23, §4]. By Theorem 4.6 of loc. cit., it is Borel equivariantly equivalent to $\Sigma^\infty M_+$ (considered with the trivial action). To write an equivariant PSS map $\Sigma^\infty M_+\to HF_S(H,\mathbb{S})$, we need an equivariant $\mathcal{M}_{\tilde f,f}$-$\mathcal{M}_{\tilde f,H,J}$-bimodule similar to before. We choose:

- a function $S\times M\times(-\infty,0]\to\mathbb{R}$ that agrees with $\tilde f+f$ near $-\infty$ and a family of metrics on $S\times M$ parametrized by $(-\infty,0]$ that agrees with the product metric near $-\infty$
- Floer data on M parametrized by $S\times D$ (where $D=\mathbb{CP}^1\setminus\{0\}$ with fixed negative cylindrical end) that agrees with the $S\times S^1$-parametric data chosen to define $HF_S(H,\mathbb{S})$ outside a compact subset of D (in particular, outside a compact set, the data only depends on the argument of a point in D)

We further assume the choice is circle equivariant, with respect to the diagonal action on $S\times D$ for the latter. We define a bimodule $\mathcal{N}^{PSS}$ by assigning to $(p,x)\in ob(\mathcal{M}_{\tilde f,f})$ and $(p',y)\in(\mathcal{M}_{\tilde f,H,J})$ the set of (η,θ,u) such that

- η is a trajectory of $-\tilde f$ from p to p'
- $\theta:(-\infty,0]\to M$ such that $(\eta|_{(-\infty,0]},\theta):(-\infty,0]\to S\times M$ satisfies the gradient equation with respect to chosen Morse data above
- $u:D\to M$ satisfies the Cauchy–Riemann equation with respect to the chosen Floer data above, which, for each $q\in D$ evaluated at $(\eta(|q|^{-1}),q)\in S\times D$ (more precisely, use a smoothing of $|q|^{-1}:D\to[0,\infty)$ near 0)
- θ is asymptotic to x, u is asymptotic to y, and $\theta(0)=u(\infty)$

Generic choices ensure this defines a bimodule, and it carries a natural framing (c.f. Remark 4.8). It could look complicated at first glance, but the picture is very simple: we are counting pairs of Morse trajectories η from p to p' and PSS-trajectories within $S\times M$ from (p,x) to (p',y) "above η". As a result, we obtain an equivariant PSS-map $\Sigma^\infty M_+\simeq\Sigma^\infty(S\times M)_+\to HF_S(H,\mathbb{S})$, which induces an equivalence $\Sigma^\infty M_+\to F^0HF_S(H,\mathbb{S})$ by a filtration argument as above.

Next we write a map $F^{(p-\delta,p+\delta]}HF_S(H,\mathbb{S})\to\Sigma^\infty(S\times X_\gamma)_+\simeq\Sigma^\infty(X_\gamma)_+$ by defining an $S\times X_\gamma$-relative module over $F^{p+\delta}\mathcal{M}_{\tilde f,H,J}$ that is empty on objects of $F^{p-\delta}\mathcal{M}_{\tilde f,H,J}$. This module will assign to a $(p,x)\in ob(F^{p+\delta}\mathcal{M}_{\tilde f,H,J})$ the pairs (η,u) where $\eta:(-\infty,0]\to S$ is a half-gradient trajectory asymptotic to p and $u:\mathbb{R}\times S^1\to M$ is a solution to Cauchy–Riemann equation with input x and output given by an orbit of H^a with trace X_γ. To this end choose

- a function $S\times(-\infty,0]\to\mathbb{R}$ that agrees with $\tilde f$ near $-\infty$ and a family of metrics on S parametrized by $(-\infty,0]$ that agrees with the chosen metric near $-\infty$

- Floer data on M parametrized by $S \times \mathbb{R} \times S^1$ that agrees on the input (positive) end with the $S \times S^1$-parametric data chosen to define $HF_S(H, \mathbb{S})$, and whose Hamiltonian term agrees with H^a on the output end

We also assume circle-equivariance of this data. As promised to $(p, x) \in ob(F^{p+\delta}\mathcal{M}_{\tilde{f},H,J})$, we assign the pairs (η, u) where

- $\eta : (-\infty, 0] \to S$ solves the Morse equation with respect to chosen data and is asymptotic to p at $-\infty$
- $u : \mathbb{R} \times S^1 \to M$ satisfies the Cauchy–Riemann equation with respect to the data given at $(s,t) \in \mathbb{R} \times S^1$ by the chosen Floer data at $(-e^{\eta(s)}, s, t) \in S \times \mathbb{R} \times S^1$, and u is asymptotic to x at ∞ (input) and an orbit of H^a with trace X_γ at $-\infty$ (output)

Generic choices ensure this is a module over $F^{p+\delta}\mathcal{M}_{\tilde{f},H,J}$. To make it $S \times X_\gamma$-relative, we send (η, u) to $(\eta(0), \lim_{s\to-\infty} u(s,0))$. It is essentially a combination of the S-relative module on a Morse flow category that we defined in [CK23], and the X_γ-relative module discussed above. Framings are standard. As a result, we obtain a map $F^{p+\delta}HF_S(H, \mathbb{S}) \to \Sigma^\infty(S \times X_\gamma)_+$, which factors through $F^{(p-\delta,p+\delta]}HF_S(H, \mathbb{S})$. To show this is an equivalence, first filter in the S-direction, using the Morse function $\tilde{f}$, and then apply the previous argument. $\square$

Proof of Theorem 4.1. One can choose a sequence of increasing Hamiltonians H_n as above whose slopes go to infinity. As a result, $SH_S(M, \mathbb{S}) \simeq colim_n HF_S(H_n, \mathbb{S})$ and the filtration on $SH_S(M, \mathbb{S})$ is equivalent to $HF_S(H_1, \mathbb{S}) \to HF_S(H_2, \mathbb{S}) \to \dots SH_S(M, \mathbb{S})$. By Proposition 4.9 as well as Lemma 2.2, the PSS map $\Sigma^\infty M_+ \to HF_S(H_n, \mathbb{S})$ induces an isomorphism $\widehat{R}^*_{S^1}(HF_S(H_n, \mathbb{S})) \to \widehat{R}^*_{S^1}(\Sigma^\infty M_+)$ (recall that, for finitely filtered X, $\widehat{R}^*_{S^1}(X)$ is equivalent to $R^*(X_{hS^1})$ localized at $[n]_R(u)$, so this is just a notation). PSS maps also commute with the continuation maps (up to homotopy), and there is a map $\Sigma^\infty M_+ \to SH_S(M, \mathbb{S})$. By the remarks above, the induced map

$$\widehat{R}^*_{S^1}(SH_S(M, \mathbb{S})) = lim_n \widehat{R}^*_{S^1}(HF_S(H_n, \mathbb{S})) \to \widehat{R}^*_{S^1}(\Sigma^\infty M_+) \tag{4.2}$$

is an equivalence. Lemma 2.3 concludes the proof. $\square$

References

[AMS21] Mohammed Abouzaid, Mark McLean, and Ivan Smith. Complex cobordism, Hamiltonian loops and global Kuranishi charts. *arXiv e-prints*, page arXiv:2110.14320, October 2021.

[CFHW96] K. Cieliebak, A. Floer, H. Hofer, and K. Wysocki. Applications of symplectic homology. II. Stability of the action spectrum. *Math. Z.*, 223(1):27–45, 1996.

[CK23] Laurent Côté and Yusuf Barış Kartal. Equivariant Floer homotopy via Morse-Bott theory. *arXiv e-prints*, page arXiv:2309.15089, 2023.

[Gan12] Sheel Ganatra. *Symplectic Cohomology and Duality for the Wrapped Fukaya Category*. ProQuest LLC, Ann Arbor, MI, 2012. Thesis (Ph.D.)–Massachusetts Institute of Technology.

[Gan19] Sheel Ganatra. Cyclic homology, S^1-equivariant Floer cohomology, and Calabi-Yau structures. *arXiv e-prints*, page arXiv:1912.13510, December 2019.

[Lar21] Tim Large. Spectral fukaya categories for liouville manifolds. 2021.

[NS18] Thomas Nikolaus and Peter Scholze. On topological cyclic homology. 2018.

[Tre19] David Treumann. Complex k-theory of mirror pairs. *arXiv preprint arXiv:1909.03018*, 2019.

[Wur91] Urs Wurgler. Morava K-theories: a survey. In *Algebraic topology Poznań 1989*, volume 1474 of *Lecture Notes in Math.*, pages 111–138. Springer, Berlin, 1991.

[Zha14] Jingyu Zhao. Periodic symplectic cohomologies. *arXiv preprint arXiv:1405.2084*, 2014.

Max Planck Institute for Mathematics, Vivatsgasse 7, 53111 Bonn, Germany
Email address: cote@mpim-bonn.mpg.de

School of Mathematics, University of Edinburgh, Edinburgh, UK
Email address: ykartal@ed.ac.uk