Complex Scaling

James Leng

July 21, 2020

Motivation

- $R_V : L^2_c \to H^2_{loc}$. Want an operator from L^2 to H^2 without the "c" or the "loc." While $R_V(\lambda) : L^2_c \to H^2_{loc}$ only on $\mathrm{Im}(\lambda) > 0$, we want to achieve more.
- To do this, we focus on a class of operators $P_{V,\Gamma}(\lambda) : L^2(\Gamma) \to H^2(\Gamma)$ on $\lambda \in \Lambda_\Gamma$ and show that on Λ_Γ, the resonances of P_V and $P_{V,\Gamma}$ agree. We can then study the resonances of P_V by studying the family $P_{V,\Gamma}$.
- The upshot to this is that taking the trace gives the rank of the projection. Also, we can assume that there are no resonances on Γ by perturbing Γ.
- The downside is $P_{V,\Gamma}(\lambda)$ still has problems at $\lambda = 0$ and that $P_{V,\Gamma}$ is no longer self-adjoint.

Notation and Definitions

Let Γ be a simple curve with $\gamma : \mathbb{R} \to \gamma$ a parameterization with $\gamma \in C^2$ at least. For $f \in C^1(\Gamma)$ (e.g. $f \circ \gamma \in C^1(\mathbb{R})$ define

$$\partial_z^\Gamma f(z_0) = \gamma'(t_0)\partial_t(f \circ \gamma)(t_0), \quad \gamma(t_0) = z_0$$

$$D_z^\Gamma = \frac{1}{i}\partial_z^\Gamma$$

(does not depend on parameterization) If f is holomorphic in a neighborhood of Γ, then ∂_z^Γ coincides with the holomorphic differential operator. To integrate along Γ, we define $dz = \gamma(t)dt$.

Notation and Definitions 2

Given potential $V \in L^\infty_c(\mathbb{R}, \mathbb{C})$, assume that

$$\Gamma \cap \mathbb{R} \supset [-L, L], \mathrm{supp}(V) \subset (-L, L)$$

so we can define

$$P_{V,\Gamma} = (D_z^\Gamma)^2 + V.$$

Also assume that there exists $\theta \in (0, \pi)$, $a_\pm \in \mathbb{C}$, $K \subset \mathbb{C}$

$$\Gamma \setminus K = ((a_+ + e^{i\theta}(0, \infty)) \cup (a_- - e^{i\theta}(0, \infty))) \setminus K$$

Also, orient Γ such that $\mathrm{Im}(\gamma(t)) \to \infty$ as $t \to \infty$. Define

$$\Lambda_\Gamma = \{\lambda \in \mathbb{C} \setminus \mathbb{R}_{\leq 0}, -\theta < \arg(\lambda) < \pi - \theta\}$$

where $\arg \in (-\pi, \pi)$.

Notation and Definitions 3

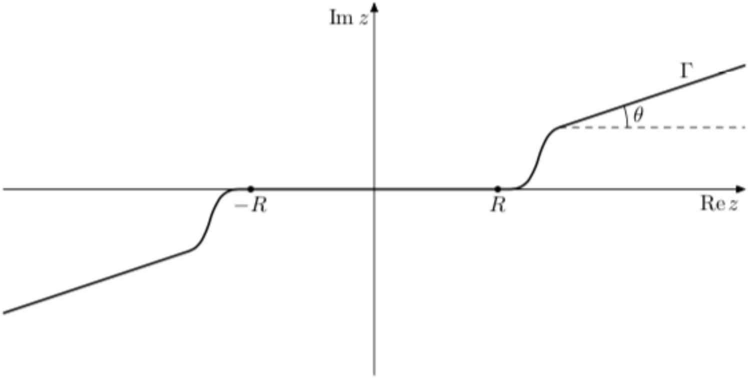

Figure 2.7. Curve Γ used in complex scaling. The curve is given by $x \mapsto x + ig(x)$ for a C^∞ function g satisfying $g(x) = 0$ for $-R \leq x \leq R$ and $g(x) = x \tan\theta$ for $|x|$ sufficiently large, where θ is a given constant.

Free Laplacian

Theorem

For $\lambda \in \mathbb{C} \setminus \{0\}$ and $f \in C_c^1(\Gamma)$ define

$$R_{0,\Gamma}(\lambda)f(z) := \frac{i}{2\lambda} \int_\Gamma e^{i\lambda\varphi(z,w)} f(w)dw$$

$$\varphi(\gamma(t), \gamma(s)) := \pm(\gamma(t) - \gamma(s)), \quad \pm(t - s) \geq 0.$$

Then $R_{0,\Gamma}$ extends to an operator $L^2(\Gamma) \to H^2(\Gamma)$ which is a two-sided inverse of $D_\Gamma^2 - \lambda^2$ for $\lambda \in \Lambda_\Gamma$.

First check that it's two-sided inverse in $C_c^2(\Gamma)$:

$$(D_\Gamma^2 - \lambda^2)R_{0,\Gamma} = (D_\Gamma^2 - \lambda^2)\frac{e^{i\lambda\varphi(z,w)}}{2\lambda} = \frac{\lambda}{2i}e^{i\lambda\varphi(z,w)} - \frac{\lambda}{2i}e^{i\lambda\varphi(z,w)} = 0$$

at $z \neq w$ and at $z = w$, it's equal to ∞. Other direction: check via integration by parts.

Proof

Suffices to show that $R_{0,\Gamma}$ is bounded on $L^2(\Gamma)$ since $C_c^2(\Gamma)$ is dense (since $D_\Gamma^2 R_{0,\Gamma} = I + \lambda^2 R_{0,\Gamma}$ and taking norms, we see that $R_{0,\Gamma}$ has range in H^2 since by Sobolev interpolation inequality suffices to only consider $\|\cdot\|_2$ and $\|D^2\cdot\|_2$). To do this, we use Schur's lemma:

Theorem

Suppose T is a integral operator

$$Tf = \int_Y K(x,y)f(y)dy$$

Then

$$\|T\|_{L^2}^2 \leq \sup_x \int |K(x,y)|dy \times \sup_y \int |K(x,y)|dx.$$

Proof

By reparametrization, can assume that $\gamma(t) = b_\pm + e^{i\theta} t$ for $\pm t \geq C_0$. Then

$$\mathrm{Re}(i\lambda\varphi(\gamma(t), \gamma(s))) = -\sin(\theta + \arg\lambda)|\lambda||t-s| + O(1) \leq -\epsilon|\lambda||t-s| + O(1)$$

so

$$\sup_t \int_{\mathbb{R}} |e^{i\lambda\varphi(\gamma(t),\gamma(s))}||\gamma'(s)| ds < \infty$$

Complex Scaling in Dim 1

Theorem

- *$P_{V,\Gamma} - \lambda^2 : H^2(\Gamma) \to L^2(\Gamma)$ is a Fredholm operator and $Spec(P_{V,\Gamma}) \cap \Lambda_\Gamma$ is discrete*
- $$m_R(\lambda) = tr_{L^2(\Gamma)} \frac{1}{2\pi i} \oint (\zeta^2 - P_{V,\Gamma})^{-1} 2\zeta d\zeta, \quad \lambda \in \Lambda_\Gamma$$

  (where m_R is the usual scattering resonance)
- $$m_D(\lambda) = tr_{L^2(\Gamma)} \frac{1}{2\pi i} \oint (\zeta^2 - P_{V,\Gamma})^{-1} 2\zeta d\zeta$$

  where
  $$m_D(\lambda) = \frac{1}{2\pi i} \oint \frac{D'(\zeta)}{D(\zeta)} d\zeta$$
  $$D(\lambda) = \det(I + VR_{0,\Lambda}\rho)$$

  $\rho \equiv 1$ *on* $supp(V)$

Remarks

Theorem

Let $\Omega \subset \mathbb{C}$ be an open set. Suppose X_1, X_2 Banach spaces, $X_1 \subset X_2$. If $P - z$ is a Fredholm operator and $P - z_0$ is invertible, then $(P-z)^{-1}$ is a meromorphic family of Fredholm operators. In addition,

$$\Pi_z := \frac{1}{2\pi i} \oint_z (w-P)^{-1} dw$$

is a projection of finite rank from X_2 to X_1.

From Yonah's talk (?), near a pole λ,

$$(P_{V,\Gamma} - \lambda^2)^{-1} = \sum_{j=1}^{m} \frac{(P_{V,\Lambda} - \lambda^2)^{j-1}\Pi}{(\lambda^2 - \zeta^2)^j} + A(\zeta, \lambda)$$

where π is the projection.

Proof Step 1

$\rho R_{0,\Gamma} : L^2(\Gamma) \to H^2([-L, L]) \to L^2(\Gamma)$. $V\rho R_{0,\Gamma} : L^2(\Gamma) \to L^2(\Gamma)$ is compact.

$$(P_{V,\Gamma} - \lambda^2)R_{0,\Gamma} = I + VR_{0,\Gamma}(\lambda)$$

As before, $(I + VR_{0,\Gamma}(\lambda))^{-1}$ is a meromorphic family of operators so

$$R_{V,\Gamma} = (I + VR_{0,\Gamma}(\lambda))^{-1}R_{0,\Gamma}$$

Implies that $\mathrm{Spec}(P_{V,\Gamma}) \cap \Lambda_\Gamma$ is discrete.

Proof Step 2

Suppose λ is a pole with $m_R(\lambda) = m$. Arguing as before, this implies that there exists u_m such that

$$(P_V - \lambda^2)^m u_m(x) = 0$$

and

$$(P_V - \lambda^2)^{m-1} u_m(x) = \begin{cases} Ae^{i\lambda x} & x \geq L \\ Be^{-i\lambda x} & x \leq -L \end{cases}$$

so

$$u_m = \begin{cases} P(x)e^{i\lambda x} & x \geq L \\ Q(x)e^{-i\lambda x} & x \leq -L \end{cases}$$

with $\deg(P) = \deg(Q) = m - 1$. Put

$$\tilde{u_m} = \begin{cases} e^{i\lambda x} & x \in \Gamma^+ \\ Q(x)e^{-i\lambda x} & x \in \Gamma^- \\ u_m(x) & x \in [-L, L] \end{cases}$$

Proof Step 2 cont.

where $\Gamma = \Gamma^+ \cup [-L, L] \cup \Gamma^-$. Then

$$(P_{V,\Gamma} - \lambda^2)^m \tilde{u_m} = 0, \quad (P_{V,\Gamma} - \lambda^2)^{m-1} \tilde{u_m} \neq 0.$$

Arguing as before, we see that $\mathrm{Re}(i\lambda z) < -\epsilon|\lambda||z| + O(1)$ so

$$\|\tilde{u}_m\|_2^2 \leq \int_{-L}^{L} |u_m(x)|^2 dx + C \sum \int_{\pm} |z|^{m-1} e^{-\epsilon|\lambda||z|} d|z| < \infty.$$

hence $\tilde{u}_m$ is unique L^2 solution. Hence, for each eigenvector of λ of P_V with order m, we obtain an eigenvector of λ of $P_{V,\Gamma}$ of order m.

Proof Step 2 cont.

Conversely, suppose λ is an eigenvalue of $P_{V,\Gamma}$ order m with $\tilde{u_m}$ as the eigenvector, then the only possible value of $(P_{V,\Gamma} - \lambda^2)^{m-1}\tilde{u_m}$ on Γ^+ are $e^{i\lambda z}$ or $e^{-i\lambda z}$. Only one of them is L^2. Similar argumnt for Γ^-. Hence

$$\tilde{u_m} = \begin{cases} e^{i\lambda x} & x \in \Gamma^+ \\ Q(x)e^{-i\lambda x} & x \in \Gamma^- \\ u_m(x) & x \in [-L, L] \end{cases}$$

for some u_m. Then can define u_m as before.

Proof Step 3

Let ρ be a cutoff function supported on $[-L, L]$ and is 1 on the support of V. Then

$$VR_{0,\Gamma}(\lambda)\rho = VR_0(\lambda)\rho.$$

As before

$$R_{V,\Gamma}(\lambda) = R_{0,\Gamma}(\lambda)(I + VR_{0,\Gamma}\rho)^{-1}(I - VR_{0,\Gamma}(1-\rho))$$

just replace $R_{0,\Gamma}$ with R_0. Then apply Theorem C.11.

Continuity of Resonances

Theorem

Suppose that $V_0 \in L^\infty_c([-L, L])$ and $\Omega \subset\subset \mathbb{C}$ with C^1 boundary such that P_{V_0} has no resonances in $\partial\Omega$. Then for V near V_0

$$\sum_{\lambda \in \Omega} m_V(\lambda) = \sum_{\lambda \ in \Omega} m_{V_0}(\lambda).$$

Let Γ be a curve as we were considering. Suppose $0 \notin \Omega$. Writing Ω as a union of piecewise C^1 boundary with no resonances in its boundary, we may assume that $\Omega \subset \Lambda_\Gamma$ or $\Omega \subset \Lambda_{\overline{\Gamma}}$. Assume $\Omega \subset \Lambda_\Gamma$. Resonances of P_{V_0} in Ω are the same as eigenvalues of $P_{V_0,\Gamma}$ in Ω:

$$\sum_{\lambda \in \Omega} m_{V_0}(\lambda) = \frac{1}{2\pi i} \operatorname{tr} \oint_{\partial\Omega} (\zeta - P_{V_0,\Gamma})^{-1} d\zeta$$

Continuity of Resonances

Want to show that

$$\frac{1}{2\pi}\left|\operatorname{tr}\oint_{\partial\Omega}(\zeta - P_{V_0,\Gamma})^{-1}d\zeta - \operatorname{tr}\oint_{\partial\Omega}(\zeta - P_{V,\Gamma})^{-1}d\zeta\right| < 1$$

for V sufficiently close to V_0. First, note that

$$\|(\zeta - P_{V_0,\Gamma})^{-1}\|_{L^2\to H^2} \leq M$$

Note that

$$\zeta - P_{V,\Gamma} = (\zeta - P_{V_0,\Gamma})(I - (\zeta - P_{V_0,\Gamma})^{-1}(V - V_0))$$

If $\|V - V_0\| < \epsilon$, the last factor can be inverted into a Neumann series so

$$\|(\zeta - P_{V,\Gamma})^{-1}\|_{L^2\to H^2} \lesssim M.$$

Continuity of Resonances

Let ρ be a cutoff. Then by theorem B.21,

$$\|\rho(\zeta - P_{V_0,\Gamma})^{-1}\|_1 \leq \|(\zeta - P_{V_0,\Gamma})^{-1}\|_{L^2\to H^2} \lesssim M.$$

The bound then follows from this

$$\frac{1}{2\pi}\left|\operatorname{tr}\oint_{\partial\Omega}(\zeta - P_{V_0,\Gamma})^{-1}d\zeta - \operatorname{tr}\oint_{\partial\Omega}(\zeta - P_{V,\Gamma})^{-1}d\zeta\right| =$$

$$\frac{1}{2\pi}\left|\operatorname{tr}\oint((\zeta - P_{V_0,\Gamma})^{-1}(V - V_0)\rho(\zeta - P_{V,\Gamma})^{-1}d\zeta\right| \leq CM\|V-V_0\|_\infty < 1.$$

Continuity of Resonances

If $0 \in \Omega$, then can assume that $\Omega = B(0, r)$ since we can apply argument with $\Omega \setminus B(0, r)$. Also, assume that V_0 has resonance at 0. Since $m_D(0) = m_R(0) - 1$, we have

$$m_R(0) - 1 = \operatorname{tr} \int_{|\zeta|=r} (I + V_0 R_0(\zeta)\rho)^{-1} V_0 \partial_\zeta (\rho R_0(\zeta)\rho) d\zeta.$$

Also, since R_0 has no resonances on $\partial B(0, r)$, we have

$$I + V\rho R_0(\lambda)\rho = (I + V_0\rho R_0(\lambda)\rho)(I + (I + V_0\rho R_0(\lambda)\rho)^{-1}(V - V_0)R_0(\lambda)\rho).$$

Hence, we can assume

$$\|(I + V\rho R_0(\lambda)\rho)^{-1}\|_{L^2} \lesssim \|(I + V\rho R_0(\lambda)\rho)^{-1}\|_{L^2} \lesssim M$$

for λ on the boundary.

Continuity of Resonances

Hence V has no resonances on $\partial B(0,r)$. Hence

$$\sum_{|\lambda|<r} m_V(\lambda) = 1 + \frac{1}{2\pi i}\mathrm{tr}\oint_{|\zeta|=r}(I + V\rho R_0(\zeta)\rho)^{-1}V\partial_\zeta(\rho R_0(\zeta)\rho)d\zeta.$$

By lemma 2.18 and Cauchy's integral theorem, we have that

$$\|\partial_\zeta \rho R_0(\lambda)\rho\|_1 \leq C.$$

Multiplicity ≤ 1 is Generic

Theorem

For the generic (Baire category) V, $m_R(\lambda) \leq 1$ for all λ

- Order the resonances with $z \leq w$ if $|z| < |w|$ or if $|z| = |w|$ and $\arg(z) \leq \arg(w)$.
- Define $\mathcal{V}_n$ as the set of all potentials where the first n resonances are simple.
- By continuity of Resonances, $\mathcal{V}_n$ is open.
- Use Rouche's theorem to prove that a small perturbation of a function with a zero with multiplicity m has m simple zeros near that zero.
- Do some Grushin problem stuff (invertibility) to reduce an eigenvector of order m to the subspace generated by that eigenvector.
- Show that $\mathcal{V}_n$ is dense.

Manifolds

An n dimensional submanifold of $\mathbb{C}^n$ is (maximally) totally real if for all m, $T_m M \cap i T_m M = 0$.

Example

Clifford Torus $\mathbb{T}^n \subset \mathbb{C}^n$. Any Lagrangian submanifold is totally real since $\langle u, v \rangle = \omega(u, iv)$ is the standard inner product.

If $u \in C^\infty(M)$, then $\tilde{u} \in C^\infty(\mathbb{C}^n)$ is an almost analytic extension if

$$\bar{\partial}_{z_j} \tilde{u}(z) = O(d(z, M)^\infty)$$

For $0 \le \theta < \pi$, let Γ_θ be a deformation of $\mathbb{R}^n$ in $\mathbb{C}^n$ with

$$\Gamma_\theta \cap B_{\mathbb{C}^n}(0, R_1) = B_{\mathbb{R}^n}(0, R_1)$$

$$\Gamma_\theta \cap \mathbb{C}^n \setminus B_{\mathbb{C}^n})(0, R_2) = e^{i\theta}\mathbb{R}^n \cap \mathbb{C}^n \setminus B_{\mathbb{C}^n}(0, R_2)$$

$$\Gamma_\theta = f_\theta(\mathbb{R}^n), f_\theta : \mathbb{R}^n \to \mathbb{C}^n$$

injective, $R_0 < R_1 < R_2$.

Lemmas

Lemma

Γ_θ is totally real if and only if $\det(\partial_x f_\theta) \neq 0$. In particular, it is totally real for $f_\theta(x) = x + i\partial_x F_\theta$ where F_θ is convex.

Exercise

Prove the above lemma.

Lemma

Every $u \in C^\infty(M)$ has an almost analytic extension $\tilde{u}$. Furthermore, if $\tilde{P}$ is a holomorphic differential operator near M, then it defines a differential operator P_M on M such that $P_M(u) = \tilde{P}(\tilde{U})$.

Complex Scaling Operator

Let P be a black box Hamiltonian with P equal to the Laplacian outside $B(0, R_0)$ defined on a Hilbert space $\mathcal{H}_{R_0} \oplus L^2(\mathbb{C} \setminus B(R_0))$. Define

$$\mathcal{H}_\theta = \mathcal{H}_{R_0} \oplus L^2(\Gamma_\theta \setminus B(0, R_0))$$

$$D_\theta = \{u \in \mathcal{H}_\theta : \chi u \in \mathcal{D}, (1-\chi)u \in H^2(\Gamma_\theta)\}$$

$$P_\theta u = P(\chi u) + \Delta_\theta((1-\chi)u)$$

$$\Delta_\theta = \Delta_{\Gamma_\theta}$$

What is Δ_θ?

$$-\Delta_\theta = -\partial_z \cdot \partial_z$$

$$z = x + i\partial_x F_\theta(x)$$

$$\frac{\partial}{\partial z} = \frac{\partial x}{\partial z}\frac{\partial}{\partial x} = (I + i\partial_x^2 F_\theta(x))^{-1}\frac{\partial}{\partial x}$$

$$-\Delta_\theta = (I + i\partial_x^2 F_\theta(x))^{-1}\partial_x \cdot (I + i\partial_x^2 F_\theta(x))^{-1}\partial_x$$

Lemma

$-\Delta_\theta$ *is elliptic of order* 2*, or in other words:*

$$|\xi|^2/C \leq \sigma(\Delta_\theta) \leq |\xi|^2/C$$

Comment on proof in the book: note that for x large enough $f_\theta(x) = x + e^{i\theta}x$ so $\partial_x^2 F_\theta$ is bounded.

Resolvent of Δ_θ

Exercise (Exercise 4.8)

Show that $-\Delta_\theta - \lambda^2 : H^2(\Gamma) \to L^2(\Gamma)$ is a Fredholm operator if $\mathrm{Im}(e^{i\theta}\lambda) > 0$.

Hint: invert $e^{-2i\theta}\Delta - \lambda^2 = e^{-2i\theta}(\Delta - (e^{-i\theta}\lambda)^2)$ and use ellipticity of Δ_θ. Free resolvent of the Laplacian:

$$R_0(\lambda, x, y) = \frac{e^{i\lambda|x-y|}}{|x-y|^{n-2}} P_n(\lambda|x-y|)$$

We want to define

$$R_\theta(\lambda, x, y) = \frac{e^{i\lambda}((x-y)\cdot(x-y))^{1/2}}{((x-y)\cdot(x-y))^{n-2/2}} P_n(\lambda((x-y)\cdot(x-y))^{1/2})$$

Defining the Square Root

the function $z \mapsto \sqrt{z}$ has a branch point at 0. Want a branch cut along the negative real ray (the one that gives the principle square root). Hence

Lemma

For $0 \leq \theta < \frac{\pi}{2}$, $z, w \in \Gamma_\theta$, $(z-w) \cdot (z-w) = 0 \implies z = w$ *and the principle square root is well defined for* $\sqrt{(z-w) \cdot (z-w)}$ *and satisfies*

$$Im, Re((z-w) \cdot (z-w)) \geq 0$$

Comment on proof: The identity

$$f(x) - f(y) = \int_0^1 f(tx + (1-t)y)dt$$

("Taylor's integral remainder theorem") follows from the fundamental theorem of calculus.

Resolvent

Theorem

Define

$$R_\theta(\lambda)\varphi(x) = \int_{-\infty}^{\infty} R_\theta(\lambda, x, y)\varphi(y) det(I + i\partial^2 F_\theta(y))dy.$$

Then $R_\theta : H^2(\Gamma) \to L^2(\Gamma)$ *is the two-sided inverse of* $\Delta_\theta - \lambda^2$*.*
Moreover, for $\delta > 0$*,* $Im(\lambda) > \delta Re(\lambda) \geq 0$

$$R_\theta(\lambda) = O_\delta(Im(\lambda)^{-2}) : L^2(\Gamma) \to L^2(\Gamma).$$

We will show later that $R_\theta(\lambda)$ is a two sided inverse for $C_c^\infty(\Gamma)$ functions. Assuming that, we show that it is true for $R_\theta(\lambda) : L^2(\Gamma) \to L^2(\Gamma)$. Then by elliptic regularity and arguing as before, we have the desired result.

Estimates for $P_n(x)/x^{n-2}$ and $e(i\lambda x)$

- First, we estimate $P_n(x)/x^{n-2}$:
  P_n is degree $\frac{n-3}{2}$ so we get an estimate

$$|\zeta^{2-n}P_n(\lambda\zeta)| \lesssim n\zeta^{2-n}(1+(\lambda\zeta)^{(n-3)/2})$$

- To estimate $e(i\lambda x)$, we estimate $\mathrm{Re}(i\lambda x)$:
  Let $\delta = \frac{\mathrm{Im}(e^{i\theta}\lambda)}{|\lambda|}$. Because
  $\Gamma_\theta \cap \mathbb{C}^n \setminus B(0,R_2) = e^{i\theta}\mathbb{R}^n \setminus B(0,R_2)$, we have

$$\mathrm{Re}((z-w)\cdot(z-w))^{1/2} = \mathrm{Re}(i\lambda e^{i\theta} + O((1+|w|+|z|)^{-1}))|w-z|$$

  and that is equal to

$$-\mathrm{Im}(e^{i\theta}\lambda)(1+O(\delta^{-1}(1+|w|+|z|)^{-1}))|w-z|$$

R_θ is bounded

$$\int_{\Gamma_\theta} |R(\lambda, z, w)||dw| \lesssim \int_{|z-w|\geq C} \frac{e^{-|z-w|/C}}{|z-w|^{(n-1)/2}} \lesssim \int_C^\infty e^{-r/C} r^{(n-1)/2} dt$$
$$< \infty$$

Then by Schur's lemma, R_θ is bounded. Hence by Elliptic regularity,

$$\|R_\theta\|_{H^2} \lesssim \|R_\theta\|_{L^2} + \|\Delta_\theta R_\theta\|_{L^2} \lesssim \|R_\theta\|_{L^2} + \|Id + \lambda^2 R_\theta\|_{L^2} \lesssim \|R_\theta\|_{L^2}$$

R_θ is a two-sided inverse on C_c^∞

Lemma

Let $\tilde{u}$ and $\tilde{v}$ be almost analytic extensions of u and v compactly supported smooth functions. Then

$$\int_{\Gamma_\theta} \partial_{z_j}\tilde{u}(x)v(x)dx = -\int_{\Gamma_\theta} u(x)\partial_{z_j}\tilde{v}(x)dx$$

$$\partial_{z_j}\tilde{u}dz_1 \wedge dz_2 \wedge \cdots \wedge dz_n = (-1)^{j-1}d(\tilde{u}dz_1 \wedge \cdots \wedge \hat{dz}_j \wedge \cdots \wedge dz_n)$$

$$-\sum_{k=1}^{n} \bar{\partial}_{z_k}\tilde{u}d\bar{z}_k \wedge \cdots \wedge dz_n$$

the latter term is 0 on Γ_θ. Hence

$$\int_{\Gamma_\theta} v\partial_{z_j}udz = (-1)^{j-1}\int_{\Gamma_\theta} vdu = -(-1)^{j-1}\int_{\Gamma_\theta} udv = -\int_{\Gamma_\theta} u\partial_{z_j}vdz.$$

R_θ is a two-sided inverse

By integration by parts $\int v\Delta_\theta u = \int u\Delta_\theta v$. Hence, it suffices to show

$$\int_{\Gamma_\theta} (\Delta_z - \lambda^2) R_\theta(\lambda, z, w)\varphi(w)dw = \varphi(z)$$

$$\int_{\Gamma_\theta} (\Delta_w - \lambda^2) R_\theta(\lambda, z, w)\varphi(w)dw = \varphi(z).$$

Since $\Delta_z R_\theta = \Delta_w R_\theta$, it suffices to prove the first identity. Since $R_\theta = R_0$ for z, w in $\mathbb{R}^n$ and $\Delta_z = -\sum \partial_{z_j}^2$, $(\Delta_z - \lambda^2)R_\theta(\lambda, z, w) = 0$ for $z \neq w$ in $\mathbb{R}^n$. Since $R_\theta(\lambda, z, w)$ is analytic away from the diagonal, it follows that $(\Delta_z - \lambda^2)R_\theta(\lambda, z, w) = 0$

R_θ two-sided inverse.

Exercise

If u is a distribution such that $|u(\phi)| \lesssim \sum_{|\alpha|\leq k} \sup |\partial^\alpha \phi|, \phi \in C_c^k(K)$ with K compact and $\mathrm{supp}(u) = \{y\}$, then $u(\phi) = \sum_\alpha c_\alpha \partial^\alpha \phi(y)$ for $c \in L^\infty$.

As $R_\theta(\lambda, z, w)$ is a smooth family of distributions. Hence

$$((\Delta_\theta)_z - \lambda^2) R_\theta(\lambda, z, w) = \sum_{|\alpha|\leq k} c_\alpha \partial^\alpha \delta(z).$$

We claim that $z \mapsto \int_{\Gamma_\theta} R_\theta(\lambda, z, w)\varphi(w)dw$ is can be holomorphically continuied in a neighborhood of Γ_θ (where φ is a rapidly decreasing holomorphic function). To see this, we have

$$\int_{\Gamma_\theta} R_\theta(\lambda, z, w)\varphi(w)dw$$

for $\varphi(z) = \exp(-e^{-2i\theta} z \cdot z)$.

R_θ two-sided inverse.

Then by Cauchy's theorem,

$$\int_{\Gamma_\theta} R_\theta(\lambda, z, w)\varphi(w)dw = \int_{-z+\Gamma_\theta} R_\theta(\lambda, w, 0)\varphi(z+w)dw$$

$$= \int_{\Gamma_\theta} R_\theta(\lambda, w, 0)\varphi(z+w)dw.$$

(both Γ_θ and $-z+\Gamma_\theta$ pass through 0) Hence,

$$z \mapsto \int_{\Gamma_\theta} R_\theta(\lambda, z, w)\varphi(w)dw$$

has a holomorphic continuation and in particular, $\Delta_\theta - \lambda^2$ of it is holomorphic. As $\Delta_\theta - \lambda^2$ of it is the delta function on $\mathbb{R}^n$, it follows that it is the δ function everywhere.

General P_θ

Theorem

Let P_θ be a black box Hamiltonian in Δ_θ. If $Im(\lambda e^{i\theta}) > 0$, then $P_\theta - \lambda^2$ is Fredholm with index 0. In particular, the spectrum of P_θ is decrete in $C \setminus e^{i\theta}[0, \infty)$

The spectrum of P_θ away from $\mathbb{C} \setminus e^{-2i\theta}[0, \infty)$ doesn't depend on the curve Γ_θ or θ:

Theorem

Let λ be a complex number with $Im(\lambda e^{i\theta}) > 0$. Then

$$m_{P_\theta}(\lambda) = m_P(\lambda).$$

Applications

Theorem (Smoothness of Resonances)

Suppose $P(s)$ is a family of black box hamiltonians and $\lambda \in \mathbb{R} \setminus 0$ is a simple eigenvalue of $P(0)$. Let u be an eigenvector corresponding to λ. Then there exists $s_1, \epsilon_1 > 0$ and

$$u(s) \in C^\infty((-s_1, s_1), \mathcal{D}_{loc}), \lambda(s) \in C^\infty((-s_1, s_1); \mathbb{C}), u(0) = \lambda(0)$$

and $\lambda(s)$ is the unique resonance of $P(s)$ in $D(\lambda, \epsilon_1)$ and $u(s)$ is the corresponding eigenvector.

Theorem

The generic potential $V \in C^\infty(B(0, R_1) \setminus B(0, R_0))$ has all its resonances λ of $P + V$ (where P is a black box Hamiltonian) with $0 > arg(\lambda) > -\frac{\pi}{2}$ simple.