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Scattering Theory: Odd dimension meromorphic continuation

I. Free resolvent when n is odd.

$R_0(\lambda) := (-\Delta_{\mathbb{R}^n} - \lambda^2)^{-1} : L^2 \to L^2, \quad \operatorname{Im}\lambda > 0.$

The Fourier Transform gives the formula:

$$R_0(\lambda)\varphi(x) := \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n} \frac{e^{ix\cdot\xi}}{|\xi|^2-\lambda^2}\hat{\varphi}(\xi)\,d\xi, \quad \operatorname{Im}\lambda > 0,$$

the fourier multiplier $\frac{1}{|\xi|^2-\lambda^2}$ can be written as $\int_0^\infty \frac{\sin t|\xi|}{|\xi|} e^{i\lambda t}\,dt$.

Let $U(t) := \frac{\sin t\sqrt{-\Delta}}{\sqrt{-\Delta}}$ (defined by functional calculus)

$$U(t)\varphi(x) := \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n} e^{ix\cdot\xi}\frac{\sin t|\xi|}{|\xi|}\hat{\varphi}(\xi)\,d\xi,$$

then $R_0(\lambda) = \int_0^\infty e^{i\lambda t}U(t)\,dt$.

$U(t)$ is related to the initial value problem:

$$(1)\ \begin{cases} \Box u = 0, & t \geq 0, \ \varphi_0, \varphi_1 \in C^\infty \\ u(0,x) = \varphi_0(x), & u_t(0,x) = \varphi_1(x). \end{cases}$$

($\cos t|\xi|$ as the Fourier multiplier)

by the Fourier Transform, $u(t,x) = U(t)\varphi_1(x) + \partial_t U(t)\varphi_0(x), \quad t \geq 0.$

(1) can also be solved using the fundamental solution $E_+(t)$,

$$u(t,x) = E_+(t) * \varphi_1(x) + \partial_t E_+(t) * \varphi_0(x), \quad t \geq 0.$$

When $n \geq 3$ odd, there is a nice expression of the distribution E_+:

$\langle E_+(t), \psi \rangle := \frac{1}{(n-2)!!}\left(\frac{1}{t}\frac{d}{dt}\right)^{\frac{n-3}{2}}\left(t^{n-2}\fint_{\partial B(0,t)} \psi \, dS\right)$. (see [Evans]).

(The strong Huyghens principle in odd dimensions)

$$\operatorname{supp} E_+ = \{(x,t) : |x| = |t|, \ t \geq 0\}.$$

Then a comparison gives the Schwartz kernel of $U(t)$:

$$U(t,x,y) = E_+(t, x-y), \ t \geq 0,$$

and (2) $U(t)\varphi(x) = 0, \ t > \sup\{|x-y|, \ y \in \operatorname{supp}\varphi\}$.

<u>Thm</u> (Meromorphic continuation of free resolvent in odd dimensions)

Suppose that $n \geq 3$ odd. Then the resolvent $R_0(\lambda)$, $\operatorname{Im}\lambda > 0$, extends analytically to $\lambda \in \mathbb{C}$, $R_0(\lambda) : L^2_c(\mathbb{R}^n) \to L^2_{loc}(\mathbb{R}^n)$. For any $\rho \in C_c^\infty(\mathbb{R}^n)$, we have the following estimates:

$$\|\rho R_0(\lambda) \rho\|_{L^2 \to H^j} \lesssim (1+|\lambda|)^{j-1} e^{L(\operatorname{Im}\lambda)_-}, \quad j = 0,1,2,$$

where $L > \operatorname{diam} \operatorname{supp}\rho$.

<u>Proof</u>: For analyticity it suffices to show that for any $\rho \in C_c^\infty(\mathbb{R}^n)$, $\rho R_0(\lambda)\rho : L^2 \to L^2$ continues from $\operatorname{Im}\lambda > 0$ to $\lambda \in \mathbb{C}$.

By (2), $L > \operatorname{diam} \operatorname{supp}\rho$ implies $\rho U(t)\rho = 0$ for $t \geq L$.

Then for $\operatorname{Im}\lambda > 0$ at first, $\rho R_0(\lambda)\rho = \int_0^L e^{i\lambda t}\rho U(t)\rho \, dt$,

the right hand side is analytic for $\lambda \in \mathbb{C}$ as an operator $L^2 \to L^2$.

Since $U(t) = \dfrac{\sin t\sqrt{-\Delta}}{\sqrt{-\Delta}}$, $\dfrac{|\sin t|\xi||}{|\xi|} \leq |t|$ for all $\xi \in \mathbb{R}^n$,

$$\|U(t)\|_{L^2 \to H^1} \lesssim \|U(t)\|_{L^2 \to L^2} + \|\sqrt{-\Delta}U(t)\|_{L^2 \to L^2} \lesssim 1 + |t|,$$

then $\|\rho R_0(\lambda)\rho\|_{L^2 \to H^1} \leq \displaystyle\int_0^L e^{(\operatorname{Im}\lambda)_- t}(1+|t|)\,dt \leq (1+L^2) e^{L(\operatorname{Im}\lambda)_-} \lesssim e^{L(\operatorname{Im}\lambda)_-}$.

For $|\lambda|$ large, $\lambda \rho R_0(\lambda)\rho = \displaystyle\int_0^L D_t(e^{i\lambda t})\rho U(t)\rho \, dt = i\int_0^L e^{i\lambda t}\rho \cos t\sqrt{-\Delta}\rho \, dt$.

thus $\|\rho R_0(\lambda)\rho\|_{L^2 \to L^2} \lesssim |\lambda|^{-1} e^{L(\operatorname{Im}\lambda)_-}$, one can conclude that

$$\|\rho R_0(\lambda)\rho\|_{L^2 \to L^2} \lesssim (1+|\lambda|)^{-1} e^{L(\operatorname{Im}\lambda)_-}.$$

The bound for $j=2$ follows from $j=0,1$ and $(-\Delta-\lambda^2)R_0(\lambda) = I$. $\square$

The Schwartz kernel of $R_0(\lambda)$ can be computed explicitly using

$$R_0(\lambda, x, y) = \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n} \frac{e^{i(x-y)\cdot\xi}}{|\xi|^2 - \lambda^2}\, d\xi$$

$$= \frac{1}{(2\pi)^n}\int_0^\infty \int_{S^{n-1}} \frac{e^{ir\omega\cdot(x-y)}}{r^2 - \lambda^2}\, d\omega\, r^{n-1} dr,$$

we have the following:

<u>Thm</u> (Schwartz kernel of $R_0(\lambda)$ in odd dimensions)

$$R_0(\lambda, x, y) = \frac{e^{i\lambda|x-y|}}{|x-y|^{n-2}} P_n(\lambda|x-y|), \text{ for } n \geq 3 \text{ odd}.$$

$P_n(k) := i\, 2^{-n+1}\pi^{-\frac{n-1}{2}} k^{n-2} e^{-2ik}(-\partial_k)^{\frac{n-3}{2}}(e^{2ik}/ik^{\frac{n-1}{2}})$ is a polynomial of degree $(n-3)/2$ with the leading term $\frac{1}{4\pi}\cdot\frac{k^{\frac{n-3}{2}}}{(2\pi i)^{\frac{n-3}{2}}}$ and the constant term $\frac{(n-3)!}{\pi^{\frac{n-1}{2}}2^{n-1}(\frac{n-3}{2})!}$.

In particular, when $n=3$, $R_0(\lambda,x,y)=\frac{e^{i\lambda|x-y|}}{4\pi|x-y|}$.

For n even, $R_0(\lambda,x,y)$ can be expressed by the Hankel functions of the first kind, see "Galkowski & Smith (2015)" for details.

The Taylor expansion of $|x-y|$ as $|x|\to\infty$ gives the following:

<u>Thm</u> (Outgoing asymptotics)

For $n\geq 3$ odd, $\lambda\in\mathbb{C}$ and $\varphi\in\mathcal{E}'(\mathbb{R}^n)$,

$$R_0(\lambda)\varphi(r\omega)=e^{i\lambda r}r^{-\frac{n-1}{2}}h(r,\omega),$$

$$h(r,\omega)\sim\sum_{j=0}^{\infty}h_j(\omega)r^{-j},\quad h_0(\omega)=\frac{1}{4\pi}\left(\frac{\lambda}{2\pi i}\right)^{\frac{n-3}{2}}\hat{\varphi}(\lambda\omega),\quad\text{as } r\to\infty.$$

<u>Proof</u>: Use $|r\omega-y|=r(1-2r^{-1}\omega\cdot y+r^{-2}|y|^2)^{1/2}$

$$=r-\omega\cdot y+(|y|^2/2-(\omega\cdot y)^2/2)r^{-1}+\cdots$$

and $|r\omega-y|^{-p}=r^{-p}(1-2r^{-1}\omega\cdot y+r^{-2}|y|^2)^{-p/2}$

$$=r^{-p}(1-p\,\omega\cdot y\,r^{-1}+\cdots)$$

then use the Thm above. $\square$

II. Meromorphic continuation

Assumptions: $n \geq 3$ odd, $V \in L^\infty(\mathbb{R}^n; \mathbb{C})$,

$R_V(\lambda) := (P_V - \lambda^2)^{-1}$, $P_V = -\Delta + V$, defined at first for $\operatorname{Im}\lambda \gg 0$.

<u>Thm</u> (Meromorphic continuation of the resolvent)

$R_V(\lambda): L^2 \to L^2$, $\operatorname{Im}\lambda > 0$, is meromorphic with finitely many poles. It extends meromorphically to $\lambda \in \mathbb{C}$ as operators $R_V(\lambda): L^2_c \longrightarrow L^2_{loc}$.

<u>Proof</u>: Let $\rho \in C_c^\infty(\mathbb{R}^n)$ satisfy $\rho \equiv 1$ on $\operatorname{supp} V$. Then

$$R_V(\lambda) = R_0(\lambda)(I + VR_0(\lambda)\rho)^{-1}(I - VR_0(\lambda)(1-\rho)),$$

gives the meromorphic continuation samely as in the one dimensional case. $\square$

Scattering resonances are the poles of $R_V(\lambda)$.

<u>Defn</u> (multiplicities)

$$m_V(\lambda) := \dim \operatorname{span}\{A_1(L^2_c), \cdots, A_J(L^2_c)\},$$

where $R_V(\zeta) = \sum_{j=1}^{J} \frac{A_j}{(\zeta-\lambda)^j} + A(\zeta,\lambda)$, $\zeta \mapsto A(\zeta,\lambda)$ analytic near λ.

Thm (Singular part of $R_V(\lambda)$)

Suppose $m_V(\mu)>0$, $\mu\neq 0$. Then for some integer $K(\mu)\leq m_V(\mu)$,

$$R_V(\lambda) = -\sum_{k=1}^{K(\mu)} \frac{(P_V-\mu^2)^{k-1}}{(\lambda^2-\mu^2)^k}\Pi_\mu + A(\lambda,\mu),$$

where $\Pi_\mu = -\frac{1}{2\pi i}\oint_\mu R_V(\lambda)2\lambda\, d\lambda$, $(P_V-\mu^2)^{K(\mu)}\Pi_\mu = 0$.

Proof: $R_V(\lambda) = \sum_{k=1}^{K} \frac{A_k}{(\lambda^2-\mu^2)^k} + A(\lambda,\mu)$,

then $A_1 = \frac{1}{2\pi i}\oint_\mu R_V(\lambda)2\lambda\, d\lambda =: \Pi_\mu$,

modulo terms analytic near μ we have

$$I = (P_V-\lambda^2)R_V(\lambda) \equiv \sum_{k=1}^{K}\frac{(P_V-\mu^2)A_k - A_{k+1}}{(\lambda^2-\mu^2)^k}, \quad A_{K+1}=0.$$

It follows that $A_{k+1} = (P_V-\mu^2)A_k$ and $(P_V-\mu^2)^K\Pi_\mu = 0$.

$P_V-\mu^2$ commutes with Π_μ thus $P_V-\mu^2 : \operatorname{Ran}\Pi_\mu \to \operatorname{Ran}\Pi_\mu$

is nilpotent. Then $K\leq \dim\operatorname{Ran}\Pi_\mu$, one can also show that

$m_V(\mu) = \dim \operatorname{Ran}\Pi_\mu$, thus $K = K(\mu)\leq m_V(\mu)$. $\square$

Thm (Outgoing resonant states)

A complex number $\lambda_0\neq 0$ is a resonance iff $\exists\, 0\neq\varphi\in L^2_c(\mathbb{R}^n)$ s.t.

for $u = R_0(\lambda_0)\varphi$, we have $(P_V-\lambda_0^2)u = 0$.

Proof: $0 = (-\Delta + V - \lambda_0^2) R_0(\lambda_0)\varphi$

$= (I + VR_0(\lambda_0))\varphi$

$= (I + VR_0(\lambda_0)(1-\rho))(I + VR_0(\lambda_0)\rho)\varphi$

thus $I + VR_0(\lambda_0)\rho$ has non-trivial kernel at λ_0,

$(I + VR_0(\lambda_0)\rho)^{-1}$ has a pole there.

Note that $(I + VR_0(\lambda)\rho)^{-1} = I - VR_V(\lambda)\rho$. λ_0 is a resonance.

Conversely, $R_V(\lambda) = R_0(\lambda) - R_0(\lambda)VR_V(\lambda)$,

use $R_V(\lambda) = -\sum_{k=1}^{K(\mu)} \frac{(P_V - \mu^2)^{k-1}}{(\lambda^2-\mu^2)^k}\Pi_\mu + A(\lambda,\mu)$, then

$\frac{1}{2\pi i}\oint_{\lambda_0} R_V(\lambda)(\lambda^2-\lambda_0^2)^{K-1} 2\lambda d\lambda = \frac{-1}{2\pi i}\oint_{\lambda_0} R_0(\lambda)VR_V(\lambda)(\lambda^2-\lambda_0^2)^{K-1}2\lambda d\lambda$

$\Rightarrow \quad -(P_V - \lambda_0^2)^{K-1}\Pi_{\lambda_0} = R_0(\lambda_0)V(P_V-\lambda_0^2)^{K-1}\Pi_{\lambda_0}$,

chose ψ s.t. $(P_V - \lambda_0^2)^{K-1}\Pi_{\lambda_0}\psi \neq 0$,

let $\varphi := V(P_V - \lambda_0^2)^{K-1}\Pi_{\lambda_0}\psi \in L^2_c(\mathbb{R}^n)$, then

$R_0(\lambda_0)\varphi = -(P_V - \lambda_0^2)^{K-1}\Pi_{\lambda_0}\psi \in \ker(P_V - \lambda_0^2)$,

since $(P_V - \lambda_0^2)^K \Pi_{\lambda_0} = 0$.