Homework 6. Due December 4, 9:59PM.

CS181: Fall 2022

GUIDELINES:

- Upload your assignments to Gradescope by 9:59 PM.
- Follow the instructions mentioned on the course webpage for uploading to Gradescope very carefully (including starting each problem on a new page and matching the pages with the assignments); this makes it easy and smooth for everyone. As the guidelines are simple enough, bad uploads will not be graded.
- You may use results proved in class without proofs as long as you state them clearly.
- Most importantly, make sure you adhere to the policies for academic honesty set out on the course webpage. The policies will be enforced strictly. Homework is a stepping stone for exams; keep in mind that reasonable partial credit will be awarded and trying the problems will help you a lot for the exams.
- Note that we have a **modified grading scheme for this assignment: Any attempt will get you 100% of the credit**. Nevertheless, please attempt the problems honestly and write down the solutions the best way you can - this is really the most helpful way to flex your neurons in preparation for the exam.
- All problem numbers correspond to our text 'Introduction to Theory of Computation' by Boaz Barak. So, exercise a.b refers to Chapter a, exercise b.

1. Which of the following are valid qubit states: [.5 points]
   - $0.1|0\rangle + 0.9|1\rangle$.
   - $0.6|0\rangle - 0.8|1\rangle$.
   - $-0.96|0\rangle - 0.28|1\rangle$.
2. Prove that $|u\rangle = 0.6|0\rangle + 0.8|1\rangle$ and $|v\rangle = 0.8|0\rangle - 0.6|1\rangle$ form an orthonormal basis and hence we can build a measurement device for measuring in this basis. [0.5 points]
3. Describe what happens when you measure the following states in the measurement device for the above basis. [1 point]
   - State $|+\rangle$.
   - State $|-\rangle$.

4. In class, we discussed two facets of the QM Law 3: a. Nature allows us to perform certain transformations on states. b. Nature allows us to build measurement devices for any orthonormal basis $|u\rangle, |v\rangle$. Note that devices that *rotate* are very different from devices that measure! Measurements always result in a collapse (so the outcome is only one of two possibilities - at least for the ones we are looking at).

   Suppose you are given a photon in state $|0\rangle$. Suppose you only have access to measurement devices (i.e., for any orthonormal basis, you can get a corresponding device), but do not have the ability to build devices for rotations. Describe a process for transforming the $|0\rangle$ photon to one in state $|1\rangle$ with very high probability.

   For example, can you use certain number of measuring devices (of any type) such that the end result of chaining these measurements when starting from $|0\rangle$ is a photon in $|1\rangle$ with 99.9% chance? [1 point]

   [Hint: This maybe a bit tricky but keep in mind the relaxed grading scheme. The idea is to use an idea similar to how we solved the EV bomb.]

5. Build a circuit that transforms an input three qubit state $|\psi\rangle = |000\rangle$ to $\frac{1}{\sqrt{2}}|000\rangle + \frac{1}{\sqrt{2}}|111\rangle$. You can use Hadamard gates and CNOT gates as disccused in class. [1 point].

Additional problems. Do not turn these in.

1. Consider the basis $|u\rangle = (1/\sqrt{3})|0\rangle + \sqrt{2/3}|1\rangle$ and $|v\rangle = -\sqrt{2/3}|0\rangle + (\sqrt{1/3})|1\rangle$. What happens when you measure $|+\rangle$ in this basis?

2. Draw a quantum circuit for computing XOR on two bits.

3. What does the circuit computed on problem 5 when you pass a state where the first qubit is in state $\alpha|0\rangle + \beta|1\rangle$ and you measure the first two bits of the output qubits in the standard basis?

4. Suppose you have a three qubit system in state $|111\rangle$. What system do you end up in if you apply Hadamrd gate to each of the qubits separately?

5. Build a circuit that transforms an input three qubit state $|\psi\rangle = |000\rangle$ to $(1/2)|000\rangle + (1/2)|110\rangle + (1/2)|011\rangle + (1/2)|101\rangle$. You can use Hadamard gates and CNOT gates as disccused in class.

1. Which of the following are valid qubit states: [.5 points]

- $0.1|0\rangle + 0.9|1\rangle$.
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- $-0.96|0\rangle - 0.28|1\rangle$. VALID

$0.1|0\rangle + 0.9|1\rangle$

$0.1^2 + 0.9^2 = 0.82 \neq 1$ NOT VALID

$0.6|0\rangle - 0.8|1\rangle$

$0.6^2 + 0.8^2 = 1$ VALID

$-0.96|0\rangle - 0.28|1\rangle$

$(-0.96)^2 + (-0.28)^2 = 1$ VALID

2. Prove that $|u\rangle = 0.6|0\rangle + 0.8|1\rangle$ and $|v\rangle = 0.8|0\rangle - 0.6|1\rangle$ form an orthonormal basis and hence we can build a measurement device for measuring in this basis. [0.5 points]

$$|u\rangle = 0.6|0\rangle + 0.8|1\rangle = 0.6\begin{bmatrix}1 & 0\end{bmatrix} + 0.8\begin{bmatrix}0 & 1\end{bmatrix} = \begin{bmatrix}0.6 & 0.8\end{bmatrix}$$

$$|v\rangle = 0.8|0\rangle - 0.6|1\rangle = 0.8\begin{bmatrix}1\\0\end{bmatrix} - 0.6\begin{bmatrix}0\\1\end{bmatrix} = \begin{bmatrix}0.8\\-0.6\end{bmatrix}$$

We use the dot product to evaluate orthogonality

$$|u\rangle \cdot |v\rangle = \begin{bmatrix}0.6 & 0.8\end{bmatrix} \cdot \begin{bmatrix}0.8\\-0.6\end{bmatrix}$$

$$= (0.6)(0.8) + (0.8)(-0.6)$$

$$= 0$$

Since the dot product is 0, $|u\rangle$ and $|v\rangle$ are orthogonal, and therefore $|u\rangle$ and $|v\rangle$ form an orthonormal basis. Hence, by QM Law 3c, for the orthonormal vectors $|u\rangle$ and $|v\rangle$, we can build a measurement device for measuring in this basis.

3. Describe what happens when you measure the following states in the measurement device for the above basis. [1 point]

- State $|+\rangle$.
- State $|-\rangle$.

When you measure the state $|+\rangle$, the measurement device returns $|u\rangle$ with the probability of the (length of the projection of $|+\rangle$ onto $u)^2$.

For the state $|-\rangle$, the measurement device returns $|v\rangle$ with the probability of the (length of the projection of $|-\rangle$ onto $v)^2$.

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   Suppose you are given a photon in state $|0\rangle$. Suppose you only have access to measurement devices (i.e., for any orthonormal basis, you can get a corresponding device), but do not have the ability to build devices for rotations. Describe a process for transforming the $|0\rangle$ photon to one in state $|1\rangle$ with very high probability.

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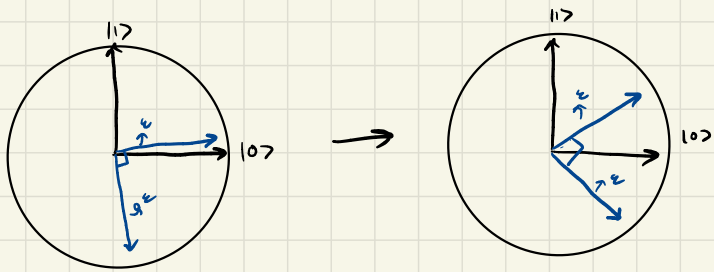
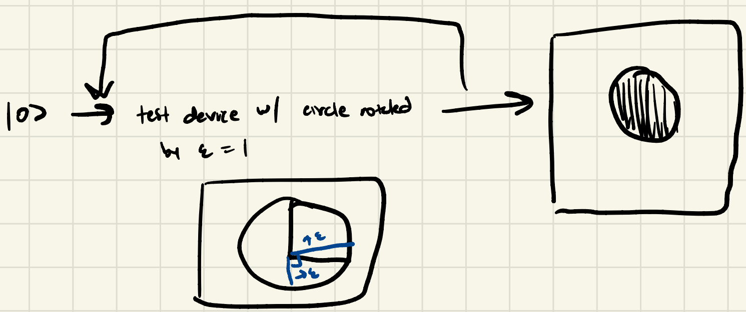


The process is that for each succession in the model, the measuring device rotates by some degree ε, say 1, until the state |1> is hit.



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