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Homework 5. Due November 28, 9:59PM.

CS181: Fall 2022

GUIDELINES:

- Upload your assignments to Gradescope by 9:59 PM.
- Follow the instructions mentioned on the course webpage for uploading to Gradescope very carefully (including starting each problem on a new page and matching the pages with the assignments); this makes it easy and smooth for everyone. As the guidelines are simple enough, bad uploads will not be graded.
- You may use results proved in class without proofs as long as you state them clearly.
- Most importantly, make sure you adhere to the policies for academic honesty set out on the course webpage. The policies will be enforced strictly. Homework is a stepping stone for exams; keep in mind that reasonable partial credit will be awarded and trying the problems will help you a lot for the exams.
- Note that we have a **modified grading scheme for this assignment**: A sincere attempt will get you 100% of the credit and a reasonable attempt will get you 50% for each problem. Nevertheless, please attempt the problems honestly and write down the solutions the best way you can - this is really the most helpful way to flex your neurons in preparation for the exam.
- All problem numbers correspond to our text 'Introduction to Theory of Computation' by Boaz Barak. So, exercise a.b refers to Chapter a, exercise b.

1. Exercise 9.7. [1 point]

2. Design a context-free grammar for the following language:

$$L = \{x \in \{0,1\}^* : \text{ x has an equal number of 1's and 0's}\}.$$

   You can assume L has the empty string. Write a few sentences to explain your reasoning. [1 point]

   [Hint: This is a tricky question but keep the relaxed grading scheme in mind. Think of cases like a) the first and last symbols of x are different; b) the first and last symbols of x are same. In case (a), you can do reduce back to the same question (by considering two cases). In case (b), what should you do? Can you conveniently split it up into two pieces?]

3. Show that the languages $L_1 = \{1^n 2^n 3^m | n, m \geq 0\}$, $L_2 = \{1^n 2^m 3^m | n, m \geq 0\}$ are context-free. What is the language $L_1 \cap L_2$? Is it contxt-free? [1 point]

4. Show that the following languages are not context-free:

   - $L = \{0^n; 1^{2n}; 2^{3n} | n \geq 0\}$. [.5 point]
   - $L = \{w; t | w$ is a substring of t where $w, t \in \{0, 1\}^* |$. [.5 point]

Additional problems. Do not turn these in.

1. Exercise 9.13. Replace NAND-TM with just plain TM in the entire problem. [1 point]

   [Hint: Part (2) is a bit advanced but a fascinating problem - you can read up on *busy beaver function*. For part (2), try to come up with a program whose description length is at most n but that takes $\omega(TOWER(n))$ steps to stop. I also highly recommend reading the two references in the problem.]

2. Design a context-free grammar for the following launguage: $L = \{0^m 1 0^n 1 0^{|m-n|} : m, n \geq 0\}$. Here $|m - n|$ denotes the absolute value of $m - n$.

3. Exercise 10.1.

4. Design a context-free grammar for the following launguages:

   (a) $L = \{x :$ 5'th bit from end is a 1$\}$.

   (b) $L = \{x : x$ has at least three 1's$\}$.

   (c) $L = \{0^m 1^n : m \neq n\}$.

   (d) $L = \{x : x$ is not of the form $0^n 1^n\}$.

1. Exercise 9.7. [1 point]

Let's say for the sake of contradiction, both TM's M and M' are computable such that $EQ(M, M') = 1$ iff M and M' are functionally equivalent.

Lets define a HALT function on M such that

- M halts on some x input, return 1 — if M halts, M' accepts $f(x) = 0$
- else M doesn't halt, return 0 — else, M' accepts non-constant function $f(x)$

This allows us to realize a reduction by checking whether $M'(x) \Rightarrow f(x) = 0$, implying that M did not HALT and is thus functionally equivalent to M! Henceforth, we can implicitly use Rice's theorem on $M'(x)$ to infer that M and M' are functionally equivalent.

However, we know that HALT is uncomputable, therefore EQ must also be uncomputable because EQ is computed by HALT.

2. Design a context-free grammar for the following language:

$$L = \{x \in \{0,1\}^* : \text{ x has an equal number of 1's and 0's}\}.$$

You can assume L has the empty string. Write a few sentences to explain your reasoning. [1 point]

[Hint: This is a tricky question but keep the relaxed grading scheme in mind. Think of cases like a) the first and last symbols of x are different; b) the first and last symbols of x are same. In case (a), you can do reduce back to the same question (by considering two cases). In case (b), what should you do? Can you conveniently split it up into two pieces?]

Let G be a context-free grammar $G = (\{A\}, \{0,1\}, R, A)$ where R must have the same number of 1s and 0s.

The rules that satisfy the grammar as such:

- $A \to A10$ — ends w/ 10
- $A \to A01$ — end with 01
- $A \to 01A$ — start with 01
- $A \to 10A$ — start with 10
- $A \to 1A0$ — start with 1 and end with 0
- $A \to 0A1$ — start with 0 and end with 1
- $A \to \varepsilon$ → empty string
- $A \to AA$ → palindrome or end with same bit as initial bit

Words generated by this grammar satisfy L:

- $A \to \varepsilon$ satisfies an empty string
- if $x, x' \in L \Rightarrow A \to 10A \Rightarrow x = 10x'$ can be generated for some x' since x' has equal numbers of 1s and 0s and is $\in L$. Therefore x must be too. The same property or proof is true for the other rules: $A \to 01A \Rightarrow x = 01x'$

  $A \to 0A1 \Rightarrow x = 0x'1$

  $A \to 1A0 \Rightarrow x = 1x'0$

  $A \to A10 \Rightarrow x = x'10$

  $A \to A01 \Rightarrow x = x'01$

- we are left with the case that A is a palindrome or at least a string with the same starting and ending bit, that is, $A \to AA$.

3. Show that the languages $L_1 = \{1^n 2^n 3^m | n, m \geq 0\}$, $L_2 = \{1^n 2^m 3^m | n, m \geq 0\}$ are context-free. What is the language $L_1 \cap L_2$? Is it contxt-free? [1 point]

To prove that the languages L_1 and L_2 are context-free, we will show that there exist context-free grammars that satisfy L_1 and L_2.

$G_1 = (\{A, B\}, \{1, 2, 3\}, R, X)$

Rules:

$X \to AB$ combine A & B

$A \to 1A2 \mid \varepsilon$ empty string or as many 1s with same number of 2s

$B \to B3 \mid \varepsilon$ empty string or as many 3s

Since L_1 can be represented with the Grammar G_1, L_1 by definition is context free.

$G_2 = (\{A, B\}, \{1, 2, 3\}, R, X)$

Rules:

$X \to AB$ combine A & B

$A \to 1A \mid \varepsilon$ as many 1s or empty string

$B \to 2B3 \mid \varepsilon$ as many 2s with equal amounts of 3s or empty string

Since L_2 can be represented with the Grammar G_2, L_2 by definition is context free

The language defined by $L_1 \cap L_2$, call it L_3, is $L_3 = \{1^{2n} 2^{n+m} 3^{2m}, n, m \geq 0\}$

Let the Grammar G_3 define the language L_3: $G_3 = (\{A, B\}, \{1, 2, 3\}, R, X\}$

Rules:

$X \to AB$ combine A & B

$A \to 11A2 \mid \varepsilon$ for every 2 1s, there is a single 2 n times $\Rightarrow 1^{2n} 2^n$

$B \to 2B33 \mid \varepsilon$ for every 2 3s, there is a single 2 m times $\Rightarrow 2^m 3^{2m}$

Henceforth, since L_3 can be represented by the Grammar G_3, L_3 by definition is context free.

4. Show that the following languages are not context-free:

a) • $L = \{0^n; 1^{2n}; 2^{3n} | n \geq 0\}$. [.5 point]

b) • $L = \{w; t | w$ is a substring of t where $w, t \in \{0,1\}^*|$. [.5 point]

a) For the sake of contradiction, assume L is context-free. Then there exists a p s.t. pumping lemma holds.

Let $x = 0^p; 1^p; 2^p$, for some value p s.t. the conditions of pumping lemma hold true.

There are two cases of a and c to see if PL holds:

case 1) a and c span the semicolon, that is, the pumping of the string at most 2 of the 3 characters in the alphabet $\{0,1,2\}$. However since all 3 characters are to the value of some multiple of n, the 2 characters that are pumped will not exist in L.

case 2) a and c don't span the semicolon $\Rightarrow ba^2dc^2 \notin L \Rightarrow |ac| \nleq p$

Henceforth, the language is not context-free, because it doesn't hold

b) For the sake of contradiction, assume L is context-free. Then there exists a p s.t. pumping lemma holds.

Let $x = 0^p1^p; 0^p1^{p+1}$. There are 2 cases of a and c to see if PL holds:

is a substring of

case 1) a and c span semicolon $\Rightarrow$ a, c has some number of 1s and 0s because $|ac| \leq p \Rightarrow ba^2dc^2 \notin L$ cannot be pumped because if the first half appended more 0s or 1s would make it not a substring of the other half.

case 2) a and c are on left side of ';'. pumping it would again mean the left half is not a substring of the second half, as $|LH| > |RH|$

case 3) a and c are on right side of ';'. once again, number of 0s or 1s would be appended in the middle or at the end, which means the LH string won't be a substring of the RH.

Henceforth, the language is not context-free by proof of pumping lemma.