

Base-extension semantics for Intuitionistic Linear Logic

Yli Buzoku

Department of Computer Science
University College London

August 2, 2024

Goals for this talk

- To introduce Intuitionistic Linear Logic and a natural deduction system for it.
- Present a Base-extension semantics for Intuitionistic Linear Logic.
- Talk about some of the difficulties involved the process of developing such a semantics.

Presentation root directory

- 1 Overview of Intuitionistic Linear Logic
- 2 Base-extension Semantics for ILL
- 3 Including the exponential

Presentation root directory

- 1 Overview of Intuitionistic Linear Logic
- 2 Base-extension Semantics for ILL
- 3 Including the exponential

Notation

- \mathbb{A} represents a fixed, countably infinite set of propositional atoms.
- Lower case latin letters represent propositional atoms.
- Upper case latin letters represent finite multisets of propositional atoms.
- Atomic multiset is taken to mean multiset of propositional atoms.
- The sum of two multisets P and Q is denoted $P \uplus Q$.
- Lower case greek letters represent ILL formulas.
- Upper case greek letters represent ILL finite multisets thereof.

Formulas of Intuitionistic Linear Logic

Definition

Formulas of ILL are defined inductively as follows:

$$\text{Form} \ni \varphi, \psi ::= p \in \mathbb{A} \mid \top \mid 0 \mid 1 \mid \varphi \multimap \psi \mid \varphi \otimes \psi \mid \varphi \& \psi \mid \varphi \oplus \psi \mid !\varphi$$

Formulas of Intuitionistic Linear Logic

Definition

Formulas of ILL are defined inductively as follows:

$$\text{Form} \ni \varphi, \psi ::= p \in \mathbb{A} \mid \top \mid 0 \mid 1 \mid \varphi \multimap \psi \mid \varphi \otimes \psi \mid \varphi \& \psi \mid \varphi \oplus \psi \mid !\varphi$$

Definition (Sequent)

A sequent is a pair (Γ, φ) .

Formulas of Intuitionistic Linear Logic

Definition

Formulas of ILL are defined inductively as follows:

Form $\ni \varphi, \psi ::= p \in \mathbb{A} \mid \top \mid 0 \mid 1 \mid \varphi \multimap \psi \mid \varphi \otimes \psi \mid \varphi \& \psi \mid \varphi \oplus \psi \mid !\varphi$

Definition (Sequent)

A sequent is a pair (Γ, φ) .

An example ILL sequent is $(\{\varphi, \varphi \multimap \psi\}, \psi \otimes \chi)$

A natural deduction system for ILL

$$\begin{array}{c}
 \frac{}{\varphi \vdash \varphi} \text{Ax} \\
 \frac{\Gamma, \varphi \vdash \psi}{\Gamma \vdash \varphi \multimap \psi} \multimap\text{-I} \\
 \frac{\Gamma \vdash \varphi \quad \Delta \vdash \psi}{\Gamma, \Delta \vdash \varphi \otimes \psi} \otimes\text{-I} \\
 \frac{}{\vdash 1} 1\text{-I} \\
 \frac{\Gamma \vdash \varphi \quad \Gamma \vdash \psi}{\Gamma \vdash \varphi \& \psi} \&\text{-I} \\
 \frac{\Gamma \vdash \varphi_i}{\Gamma \vdash \varphi_0 \oplus \varphi_1} \oplus\text{-I}_i \\
 \frac{\Gamma_1 \vdash \varphi_1 \quad \dots \quad \Gamma_n \vdash \varphi_n}{\Gamma_1, \dots, \Gamma_n \vdash \top} \top\text{-I} \\
 \frac{\Gamma \vdash \varphi \multimap \psi \quad \Delta \vdash \varphi}{\Gamma, \Delta \vdash \psi} \multimap\text{-E} \\
 \frac{\Gamma \vdash \varphi \otimes \psi \quad \Delta, \varphi, \psi \vdash \chi}{\Gamma, \Delta \vdash \chi} \otimes\text{-E} \\
 \frac{\Gamma \vdash \varphi \quad \Delta \vdash 1}{\Gamma, \Delta \vdash \varphi} 1\text{-E} \\
 \frac{\Gamma \vdash \varphi_0 \& \varphi_1}{\Gamma \vdash \varphi_i} \&\text{-E}_i \\
 \frac{\Gamma \vdash \varphi \oplus \psi \quad \Delta, \varphi \vdash \chi \quad \Delta, \psi \vdash \chi}{\Gamma, \Delta \vdash \chi} \oplus\text{-E} \\
 \frac{\Gamma \vdash 0}{\Gamma \vdash \varphi} 0\text{-E}
 \end{array}$$

A natural deduction system for ILL (cont.)

$$\frac{\Gamma_1 \vdash !\psi_1 \quad \dots \quad \Gamma_n \vdash !\psi_n \quad !\psi_1, \dots, !\psi_n \vdash \varphi}{\Gamma_1, \dots, \Gamma_n \vdash !\varphi} \text{!-Promotion}$$

$$\frac{\Gamma \vdash !\varphi \quad \Delta, \varphi \vdash \psi}{\Gamma, \Delta \vdash \psi} \text{!-Dereliction}$$

$$\frac{\Gamma \vdash !\varphi \quad \Delta \vdash \psi}{\Gamma, \Delta \vdash \psi} \text{!-Weakening}$$

$$\frac{\Gamma \vdash !\varphi \quad \Delta, !\varphi, !\varphi \vdash \psi}{\Gamma, \Delta \vdash \psi} \text{!-Contraction}$$

What is the general form of an inference figure?

What is the general form of an inference figure?

$$\frac{\Gamma \vdash \varphi_1 \quad \Delta \vdash \varphi_2}{\Gamma, \Delta \vdash \varphi_1 \otimes \varphi_2} \otimes\text{-I}$$

What is the general form of an inference figure?

$$\frac{\Gamma \vdash \varphi_1 \quad \Gamma \vdash \varphi_2}{\Gamma \vdash \varphi_1 \& \varphi_2} \&-I$$

What is the general form of an inference figure?

What is the general form of an inference figure?

$$\frac{\Gamma_1 \vdash \varphi_1 \quad \dots \quad \Gamma_n \vdash \varphi_n}{\Gamma_1, \dots, \Gamma_n \vdash \psi}$$

$$\frac{\Gamma \vdash \varphi_1 \quad \dots \quad \Gamma \vdash \varphi_n}{\Gamma \vdash \psi}$$

What is the general form of an inference figure?

$$\frac{\Gamma_1 \vdash \varphi_1 \quad \dots \quad \Gamma_n \vdash \varphi_n}{\Gamma_1, \dots, \Gamma_n \vdash \psi}$$

$$\frac{\Gamma \vdash \varphi_1 \quad \dots \quad \Gamma \vdash \varphi_n}{\Gamma \vdash \psi}$$

$$\frac{\Gamma \vdash \varphi_1 \oplus \varphi_2 \quad \Delta, \varphi_1 \vdash \chi \quad \Delta, \varphi_2 \vdash \chi}{\Gamma, \Delta \vdash \chi} \oplus\text{-E}$$

What is the general form of an inference figure?

$$\frac{\Gamma \vdash \varphi_1 \oplus \varphi_2 \quad \Delta, \varphi_1 \vdash \chi \quad \Delta, \varphi_2 \vdash \chi}{\Gamma, \Delta \vdash \chi} \oplus\text{-E}$$

What is the general form of an inference figure?

$$\frac{\Gamma \vdash \varphi_1 \oplus \varphi_2 \quad \Delta \{ \varphi_1 \vdash \chi \quad \varphi_2 \vdash \chi \}}{\Gamma, \Delta \vdash \chi} \oplus\text{-E}$$

What is the general form of an inference figure?

$$\frac{\Gamma\{\emptyset \vdash \varphi_1 \oplus \varphi_2\} \quad \Delta\{\varphi_1 \vdash \chi \quad \varphi_2 \vdash \chi\}}{\Gamma, \Delta \vdash \chi} \oplus\text{-E}$$

What is the general form of an inference figure?

$$\frac{\Gamma\{\emptyset \vdash \varphi_1 \oplus \varphi_2\} \quad \Delta\{\varphi_1 \vdash \chi \quad \varphi_2 \vdash \chi\}}{\Gamma, \Delta \vdash \chi} \oplus\text{-E}$$

What is the general form of an inference figure?

$$\frac{\Gamma\{\emptyset \vdash \varphi_1\} \quad \Delta\{\emptyset \vdash \varphi_2\}}{\Gamma, \Delta \vdash \varphi_1 \otimes \varphi_2} \otimes\text{-I}$$

$$\frac{\Gamma\{\emptyset \vdash \varphi_1 \quad \emptyset \vdash \varphi_2\}}{\Gamma \vdash \varphi_1 \& \varphi_2} \&\text{-I}$$

What is the general form of an inference figure?

$$\frac{\dots \quad \Gamma_i \{ \Delta_{i_1} \vdash \varphi_{i_1} \quad \dots \quad \Delta_{i_{l_i}} \vdash \varphi_{i_{l_i}} \} \quad \dots}{\Gamma_1, \dots, \Gamma_n \vdash \psi}$$

What is the general form of an inference figure?

What is the general form of an inference figure?

Almost! What about:

$$\frac{\Gamma_1 \vdash !\psi_1 \quad \dots \quad \Gamma_n \vdash !\psi_n \quad !\psi_1 \text{ , } \dots \text{ , } !\psi_n \vdash \varphi}{\Gamma_1 \text{ , } \dots \text{ , } \Gamma_n \vdash !\varphi} \text{!-Promotion}$$

What is the general form of an inference figure?

$$\frac{\Gamma_1 \{ \emptyset \vdash !\psi_1 \} \quad \dots \quad \Gamma_n \{ \emptyset \vdash !\psi_n \} \quad !\psi_1, \dots, !\psi_n \vdash \varphi}{\Gamma_1, \dots, \Gamma_n \vdash !\varphi} \text{!-Promotion}$$

Presentation root directory

- 1 Overview of Intuitionistic Linear Logic
- 2 Base-extension Semantics for ILL
- 3 Including the exponential

Substructural atomic derivability

Definition (Basic rule)

Basic rules take the following form:

$$\{(P_{1_i} \Rightarrow q_{1_i})\}_{i=1}^{l_1}, \dots, \{(P_{n_i} \Rightarrow q_{n_i})\}_{i=1}^{l_n} \Rightarrow r$$

- Each $(P_i \Rightarrow q_i)$ is a pair (P_i, q_i) called an atomic sequent.
- Each collection $\{(P_{i_1} \Rightarrow q_{i_1}), \dots, (P_{i_{l_i}} \Rightarrow q_{i_{l_i}})\}$ is called an atomic box.

Substructural atomic derivability

Definition (Basic rule)

Basic rules take the following form:

$$\{(P_{1_i} \Rightarrow q_{1_i})\}_{i=1}^{l_1}, \dots, \{(P_{n_i} \Rightarrow q_{n_i})\}_{i=1}^{l_n} \Rightarrow r$$

- Each $(P_i \Rightarrow q_i)$ is a pair (P_i, q_i) called an atomic sequent.
- Each collection $\{(P_{1_i} \Rightarrow q_{1_i}), \dots, (P_{l_i} \Rightarrow q_{l_i})\}$ is called an atomic box.

Definition (Base)

A base \mathcal{B} is a **SET** of basic rules.

Substructural atomic derivability

Definition (Basic derivability relation)

The relation of derivability in a base \mathcal{B} , is defined inductively as so:

Ref $p \vdash_{\mathcal{B}} p$

App Given that $(\{(P_{1_i} \Rightarrow q_{1_i})\}_{i=1}^{l_1}, \dots, \{(P_{n_i} \Rightarrow q_{n_i})\}_{i=1}^{l_n} \Rightarrow r) \in \mathcal{B}$ and atomic multisets C_i such that the following hold:

$$C_i \text{ , } P_{i_j} \vdash_{\mathcal{B}} q_{i_j} \text{ for all } i = 1, \dots, n \text{ and } j = 1, \dots, l_i$$

Then $C_1 \text{ , } \dots \text{ , } C_n \vdash_{\mathcal{B}} r$.

Example derivations

Example (Derivation terminations)

Let $\mathcal{B} = \{(\Rightarrow a), (\{\Rightarrow a\}, \{\Rightarrow b\} \Rightarrow c)\}$

$$\frac{\frac{}{b \vdash_{\mathcal{B}} b} \text{Ref} \quad \frac{}{\vdash_{\mathcal{B}} a} \Rightarrow a}{b \vdash_{\mathcal{B}} c} \{\Rightarrow a\}, \{\Rightarrow b\} \Rightarrow c$$

Therefore, in \mathcal{B} , the atom c is derivable from b .

Example derivations

Example (Invalid derivation)

By the definition of $\vdash_{\mathcal{B}}$, deriving a from a, a in the empty base is not possible, i.e. $a, a \vdash_{\emptyset} a$ is not possible.

Example derivations

Example (Invalid derivation)

By the definition of $\vdash_{\mathcal{B}}$, deriving a from a, a in the empty base is not possible, i.e. $a, a \vdash_{\emptyset} a$ is not possible.

Example (Invalid derivation 2)

Let $\mathcal{B} = \{(\{ \Rightarrow a \} \Rightarrow c), (\{ a \Rightarrow c \} \Rightarrow b)\}$

$$\begin{array}{c}
 \times \\
 \hline
 \frac{a, a \vdash_{\mathcal{B}} a}{a, a \vdash_{\mathcal{B}} c} \{ \Rightarrow a \} \Rightarrow c \\
 \hline
 \frac{a \vdash_{\mathcal{B}} b}{a \vdash_{\mathcal{B}} b} \{ a \Rightarrow c \} \Rightarrow b
 \end{array}$$

We see that in this base, a is not derivable from a, a .

Example derivations

Example (A possible fix for invalid derivation 2)

Let $\mathcal{C} = \{(\{ \Rightarrow a \} \Rightarrow c), (\{ a \Rightarrow c \} \Rightarrow b), (\{ \Rightarrow a \}, \{ \Rightarrow a \} \Rightarrow a)\}$

$$\frac{\frac{\frac{a \vdash_{\mathcal{L}} a}{a \vdash_{\mathcal{L}} a} \text{Ref} \quad \frac{a \vdash_{\mathcal{L}} a}{\{\Rightarrow a\}, \{\Rightarrow a\} \Rightarrow a} \text{Ref}}{\frac{a, a \vdash_{\mathcal{L}} a}{a, a \vdash_{\mathcal{L}} c} \{\Rightarrow a\} \Rightarrow c} \frac{a \vdash_{\mathcal{L}} b}{\{a \Rightarrow c\} \Rightarrow b}$$

Note that $\mathcal{C} \supset \mathcal{B}$!!

Example derivations

Example (Another possible fix for invalid derivation 2)

Let

$$\mathcal{D} = \{(\{\Rightarrow a\} \Rightarrow c), (\{a \Rightarrow c\} \Rightarrow b), (\{\Rightarrow d\}, \{\Rightarrow d\} \Rightarrow a), (\{\Rightarrow a\} \Rightarrow d)\}$$

$$\frac{\frac{\frac{}{a \vdash_{\mathcal{D}} a} \text{Ref}}{a \vdash_{\mathcal{D}} d} \{\Rightarrow a\} \Rightarrow d \quad \frac{\frac{}{a \vdash_{\mathcal{D}} a} \text{Ref}}{a \vdash_{\mathcal{D}} d} \{\Rightarrow a\} \Rightarrow d}{\frac{a, a \vdash_{\mathcal{D}} a}{a, a \vdash_{\mathcal{D}} c} \{\Rightarrow d\}, \{\Rightarrow d\} \Rightarrow a} \{\Rightarrow a\} \Rightarrow c$$

$$\frac{a, a \vdash_{\mathcal{D}} c}{a \vdash_{\mathcal{D}} b} \{a \Rightarrow c\} \Rightarrow b$$

Example derivations

Example (A live complicated derivation!)

Consider a base \mathcal{B} with only the following rules:

- $\{\Rightarrow c\}, \{\Rightarrow y\} \Rightarrow f$
- $\Rightarrow z$
- $\{\Rightarrow a\}, \{\Rightarrow b\}, \{\Rightarrow z\} \Rightarrow h$
- $\{\Rightarrow h\} \Rightarrow g$
- $\{\Rightarrow x\}, \{\Rightarrow g\} \Rightarrow e$
- $\{x \Rightarrow e\}, \{y \Rightarrow f\} \Rightarrow d$

is there a derivation of d from the multiset a, b, c in \mathcal{B} ?

Example derivations

Example (A live complicated derivation!)

Consider a base \mathcal{B} with only the following rules:

- $\{\Rightarrow c\}, \{\Rightarrow y\} \Rightarrow f$
- $\Rightarrow z$
- $\{\Rightarrow a\}, \{\Rightarrow b\}, \{\Rightarrow z\} \Rightarrow h$
- $\{\Rightarrow h\} \Rightarrow g$
- $\{\Rightarrow x\}, \{\Rightarrow g\} \Rightarrow e$
- $\{x \Rightarrow e\}, \{y \Rightarrow f\} \Rightarrow d$

is there a derivation of d from the multiset a, b, c in \mathcal{B} ?

Disclaimer: The fact I couldn't fit this derivation on the slide has nothing to do with why we are doing this derivation live :-)

Example derivations

Example (A very interesting derivation)

Let $\mathcal{B} = \{(\{ \Rightarrow b \} \Rightarrow c), (\{ \Rightarrow a \}, \{ b \Rightarrow c \} \Rightarrow d), (\{ \Rightarrow d, \Rightarrow a \} \Rightarrow e), (\{ a \Rightarrow e \} \Rightarrow f)\}$

$$\begin{array}{c}
 \frac{}{a \vdash_{\mathcal{B}} a} \text{Ref} \quad \frac{\frac{}{b \vdash_{\mathcal{B}} b} \text{Ref}}{b \vdash_{\mathcal{B}} c} \{ \Rightarrow b \} \Rightarrow c}{a \vdash_{\mathcal{B}} d} \{ \Rightarrow a \}, \{ b \Rightarrow c \} \Rightarrow d \quad \frac{}{a \vdash_{\mathcal{B}} a} \text{Ref} \{ \Rightarrow d, \Rightarrow a \} \Rightarrow e \\
 \hline
 \frac{a \vdash_{\mathcal{B}} e}{\vdash_{\mathcal{B}} f} \{ a \Rightarrow e \} \Rightarrow f
 \end{array}$$

An interesting property of $\vdash_{\mathcal{B}}$

Lemma (Substitution lemma)

The following are equivalent for arbitrary atomic multisets P , S , atom q , and base \mathcal{B} , where we assume $P = \{p_1, \dots, p_n\}$:

- $P, S \vdash_{\mathcal{B}} q$
- *For every $\mathcal{C} \supseteq \mathcal{B}$, atomic multisets T_1, \dots, T_m where $T_1 \vdash_{\mathcal{C}} p_1, \dots, T_m \vdash_{\mathcal{C}} p_m$, then $T_1, \dots, T_m, S \vdash_{\mathcal{C}} q$*

| | | | |
|-----------------|--------------------------------------------------|-----|------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|
| (At) | $\Vdash_{\mathcal{B}}^L p$ | iff | $L \vdash_{\mathcal{B}} p$ |
| (\multimap) | $\Vdash_{\mathcal{B}}^L \varphi \multimap \psi$ | iff | $\varphi \Vdash_{\mathcal{B}}^L \psi$ |
| (\otimes) | $\Vdash_{\mathcal{B}}^L \varphi \otimes \psi$ | iff | for any \mathcal{C} such that $\mathcal{B} \subseteq \mathcal{C}$, atomic multisets K and any $p \in \mathbb{A}$, if $\varphi, \psi \Vdash_{\mathcal{B}}^K p$ then $\Vdash_{\mathcal{C}}^{L, K} p$ |
| (1) | $\Vdash_{\mathcal{B}}^L 1$ | iff | for any \mathcal{C} such that $\mathcal{B} \subseteq \mathcal{C}$, atomic multisets K and any $p \in \mathbb{A}$, if $\Vdash_{\mathcal{B}}^K p$ then $\Vdash_{\mathcal{C}}^{L, K} p$ |
| (\top) | $\Vdash_{\mathcal{B}}^L \top$ | iff | always |
| ($\&$) | $\Vdash_{\mathcal{B}}^L \varphi \& \psi$ | iff | $\Vdash_{\mathcal{B}}^L \varphi$ and $\Vdash_{\mathcal{B}}^L \psi$ |
| (\oplus) | $\Vdash_{\mathcal{B}}^L \varphi \oplus \psi$ | iff | for any \mathcal{C} such that $\mathcal{B} \subseteq \mathcal{C}$, atomic multisets K and any $p \in \mathbb{A}$, if $\varphi \Vdash_{\mathcal{C}}^K p$ and $\psi \Vdash_{\mathcal{C}}^K p$, then $\Vdash_{\mathcal{C}}^{L, K} p$ |
| (0) | $\Vdash_{\mathcal{B}}^L 0$ | iff | $\Vdash_{\mathcal{B}}^L p$ for any $p \in \mathbb{A}$ |
| (!) | $\Vdash_{\mathcal{B}}^L !\varphi$ | iff | for any \mathcal{C} such that $\mathcal{B} \subseteq \mathcal{C}$, atomic multisets K and any $p \in \mathbb{A}$, if for any \mathcal{D} such that $\mathcal{C} \subseteq \mathcal{D}$, (if $\Vdash_{\mathcal{D}}^{\emptyset} \varphi$ then $\Vdash_{\mathcal{D}}^L p$) then $\Vdash_{\mathcal{C}}^{L, K} p$ |
| (,) | $\Vdash_{\mathcal{B}}^L \Gamma, \Delta$ | iff | there exists multisets K and M such that $L = K, M$ and $\Vdash_{\mathcal{B}}^K \Gamma$ and $\Vdash_{\mathcal{B}}^M \Delta$ |
| (Inf) | $!\Delta, \Theta \Vdash_{\mathcal{B}}^L \varphi$ | iff | for any \mathcal{C} such that $\mathcal{B} \subseteq \mathcal{C}$, atomic multisets K , if $\Vdash_{\mathcal{C}}^{\emptyset} \Delta$ and $\Vdash_{\mathcal{C}}^K \Theta$ then $\Vdash_{\mathcal{C}}^{L, K} \varphi$ |

Base-extension Semantics for ILL

Note the clause for (!) could be written as:

- $\Vdash_{\mathcal{B}}^L !\varphi$ iff for any \mathcal{C} such that $\mathcal{C} \supseteq \mathcal{B}$, atomic multisets K and any $p \in \mathbb{A}$, if $!\varphi \Vdash_{\mathcal{C}}^K p$ then $\Vdash_{\mathcal{C}}^{L,K} p$.

Notes on $\Vdash_{\mathcal{B}}^L$

- The sequent (Γ, φ) is said to be valid if and only if $\Gamma \Vdash_{\emptyset}^{\emptyset} \varphi$ holds.

Notes on $\Vdash_{\mathcal{B}}^L$

- The sequent (Γ, φ) is said to be valid if and only if $\Gamma \Vdash_{\emptyset}^{\emptyset} \varphi$ holds.
- We frequently write this as $\Gamma \Vdash \varphi$.

Notes on $\Vdash_{\mathcal{B}}^L$

- If $\Vdash_{\mathcal{B}}^L \varphi$ then for all $\mathcal{C} \supseteq \mathcal{B}$ we have $\Vdash_{\mathcal{C}}^L \varphi$.
- Given $\Gamma \Vdash_{\mathcal{B}}^L \varphi$ and $\Vdash_{\mathcal{B}}^K \Gamma$, then it holds that $\Vdash_{\mathcal{B}}^{L,K} \varphi$.

Soundness

Theorem (Soundness)

If $\Gamma \vdash \varphi$ then $\Gamma \Vdash \varphi$

Soundness

Theorem (Soundness)

If $\Gamma \vdash \varphi$ then $\Gamma \Vdash \varphi$

- The strategy is to show every rule of the natural deduction system presented previously is semantically expressible.
- This suffices by induction to prove this theorem as now we can encode any deduction into a series of semantic proofs.

A reminder of the rules of natural deduction for ILL

$$\begin{array}{c}
 \frac{}{\varphi \vdash \varphi} \text{Ax} \\
 \frac{\Gamma, \varphi \vdash \psi}{\Gamma \vdash \varphi \multimap \psi} \multimap\text{-I} \qquad \frac{\Gamma \vdash \varphi \multimap \psi \quad \Delta \vdash \varphi}{\Gamma, \Delta \vdash \psi} \multimap\text{-E} \\
 \frac{\Gamma \vdash \varphi \quad \Delta \vdash \psi}{\Gamma, \Delta \vdash \varphi \otimes \psi} \otimes\text{-I} \qquad \frac{\Gamma \vdash \varphi \otimes \psi \quad \Delta, \varphi, \psi \vdash \chi}{\Gamma, \Delta \vdash \chi} \otimes\text{-E} \\
 \frac{}{\vdash 1} 1\text{-I} \qquad \frac{\Gamma \vdash \varphi \quad \Delta \vdash 1}{\Gamma, \Delta \vdash \varphi} 1\text{-E} \\
 \frac{\Gamma \vdash \varphi \quad \Gamma \vdash \psi}{\Gamma \vdash \varphi \& \psi} \&\text{-I} \qquad \frac{\Gamma \vdash \varphi_0 \& \varphi_1}{\Gamma \vdash \varphi_i} \&\text{-E}_i \\
 \frac{\Gamma \vdash \varphi_i}{\Gamma \vdash \varphi_0 \oplus \varphi_1} \oplus\text{-I}_i \qquad \frac{\Gamma \vdash \varphi \oplus \psi \quad \Delta, \varphi \vdash \chi \quad \Delta, \psi \vdash \chi}{\Gamma, \Delta \vdash \chi} \oplus\text{-E} \\
 \frac{\Gamma_1 \vdash \varphi_1 \quad \dots \quad \Gamma_n \vdash \varphi_n}{\Gamma_1, \dots, \Gamma_n \vdash \top} \top\text{-I} \qquad \frac{\Gamma \vdash 0}{\Gamma \vdash \varphi} 0\text{-E}
 \end{array}$$

Reminder (cont.)

$$\frac{\Gamma_1 \vdash !\psi_1 \quad \dots \quad \Gamma_n \vdash !\psi_n \quad !\psi_1, \dots, !\psi_n \vdash \varphi}{\Gamma_1, \dots, \Gamma_n \vdash !\varphi} \text{!-Promotion}$$

$$\frac{\Gamma \vdash !\varphi \quad \Delta, \varphi \vdash \psi}{\Gamma, \Delta \vdash \psi} \text{!-Dereliction}$$

$$\frac{\Gamma \vdash !\varphi \quad \Delta \vdash \psi}{\Gamma, \Delta \vdash \psi} \text{!-Weakening}$$

$$\frac{\Gamma \vdash !\varphi \quad \Delta, !\varphi, !\varphi \vdash \psi}{\Gamma, \Delta \vdash \psi} \text{!-Contraction}$$

Completeness

Theorem (Completeness)

If $\Gamma \models \varphi$ then $\Gamma \vdash \varphi$.

Completeness

Theorem (Completeness)

If $\Gamma \Vdash \varphi$ then $\Gamma \vdash \varphi$.

- Define two functions $(\cdot)^b : \text{Form} \rightarrow \mathbb{A}$ and $(\cdot)^\natural : \mathbb{A} \rightarrow \text{Form}$.
- Show that any valid sequent remains valid when “atomised”.
- Construct a base \mathcal{N} whose rules simulate the natural deduction rules of ILL.
- Show that every derivation in this simulation base is equivalent to a corresponding derivation in ILL.
- Show we can go from a valid derivation to a derivation in ILL.

The Simulation Base

Consider the valid sequent $\Gamma \vdash \varphi$. Let α, β range over all subformulas of this sequent.

- To construct \mathcal{N} we simulate all possible ND-rules of our system.
- This means \mathcal{N} contains the introduction and elimination rules governing $\alpha \circ \beta$, where $\circ \in \{\otimes, \&, \oplus, \multimap\}$, governing α where $\alpha \in \{1, 0, \top\}$ and all rules governing α where $\alpha = !\beta$.
- Thus, rules include but, of course, are not limited to:
 - $\{\alpha^b \Rightarrow \beta^b\} \Rightarrow (\alpha \multimap \beta)^b$
 - $\{\Rightarrow (\alpha \multimap \beta)^b\}, \{\Rightarrow \alpha^b\} \Rightarrow \beta^b$
 - $\{\Rightarrow \alpha^b\}, \{\Rightarrow \beta^b\} \Rightarrow (\alpha \otimes \beta)^b$
 - $\{\Rightarrow (\alpha \otimes \beta)^b\}, \{\alpha^b, \beta^b \Rightarrow p\} \Rightarrow p$

The Simulation Base: An explanation

Recall the *flattened* form of the introduction and elimination rules for the tensor:

- $\{\Rightarrow \alpha^b\}, \{\Rightarrow \beta^b\} \Rightarrow (\alpha \otimes \beta)^b$
- $\{\Rightarrow (\alpha \otimes \beta)^b\}, \{\alpha^b, \beta^b \Rightarrow p\} \Rightarrow p$

The Simulation Base: An explanation

Recall the *flattened* form of the introduction and elimination rules for the tensor:

- $\{\Rightarrow \alpha^b\}, \{\Rightarrow \beta^b\} \Rightarrow (\alpha \otimes \beta)^b$
- $\{\Rightarrow (\alpha \otimes \beta)^b\}, \{\alpha^b, \beta^b \Rightarrow p\} \Rightarrow p$

Another way of writing these rules would be as follows:

Let $\alpha^b = x$, $\beta^b = y$ and $(\alpha \otimes \beta)^b = z$. Then:

- $\{\Rightarrow x\}, \{\Rightarrow y\} \Rightarrow z$
- $\{\Rightarrow z\}, \{x, y \Rightarrow p\} \Rightarrow p$

The Simulation Base: A quiz

Example

Given that $a = \alpha^b$ and $b = \beta^b$, which ILL formula is being simulated by the atom x in the following rules?

- $\{\Rightarrow a, \Rightarrow b\} \Rightarrow x$
- $\{\Rightarrow x\} \Rightarrow a$
- $\{\Rightarrow x\} \Rightarrow b$

The Simulation Base: A quiz

Example

Given that $a = \alpha^b$ and $b = \beta^b$, which ILL formula is being simulated by the atom x in the following rules?

- $\{\Rightarrow a, \Rightarrow b\} \Rightarrow x$
- $\{\Rightarrow x\} \Rightarrow a$
- $\{\Rightarrow x\} \Rightarrow b$

That's right! It's α & β

Presentation root directory

- 1 Overview of Intuitionistic Linear Logic
- 2 Base-extension Semantics for ILL
- 3 Including the exponential

What of the exponential?

The exponential is hard...

What of the exponential?

The exponential is hard...

But we finally understand it!

Why was it so hard?

$$\frac{\Gamma_1 \vdash !\psi_1 \quad \dots \quad \Gamma_n \vdash !\psi_n \quad !\psi_1, \dots, !\psi_n \vdash \varphi}{\Gamma_1, \dots, \Gamma_n \vdash !\varphi} \text{!-Promotion}$$

$$\frac{\Gamma \vdash !\varphi \quad \Delta, \varphi \vdash \psi}{\Gamma, \Delta \vdash \psi} \text{!-Dereliction}$$

$$\frac{\Gamma \vdash !\varphi \quad \Delta \vdash \psi}{\Gamma, \Delta \vdash \psi} \text{!-Weakening}$$

$$\frac{\Gamma \vdash !\varphi \quad \Delta, !\varphi, !\varphi \vdash \psi}{\Gamma, \Delta \vdash \psi} \text{!-Contraction}$$

Why was it so hard?

$$\frac{! \psi \vdash ! \psi \quad ! \psi \vdash \varphi}{! \psi \vdash ! \varphi}$$

What do we have now?

(!) $\Vdash_{\mathcal{B}} !\varphi$ iff for any \mathcal{C} such that $\mathcal{C} \supseteq \mathcal{B}$, atomic multisets K and any $p \in \mathbb{A}$, if $!\varphi \Vdash_{\mathcal{C}}^K p$ then $\Vdash_{\mathcal{C}}^{L,K} p$.

(Inf) For $\Gamma = !\Delta, \Theta$ being a nonempty multiset, $\Gamma \Vdash_{\mathcal{B}} \varphi$ iff for all $\mathcal{C} \supseteq \mathcal{B}$, atomic multisets K , if $\Vdash_{\mathcal{C}}^K \Theta$ and $\Vdash_{\mathcal{C}}^{\emptyset} \Delta$ then $\Vdash_{\mathcal{C}}^{L,K} \varphi$.

What we had last year

(Inf) $\Gamma \Vdash_{\mathcal{B}}^L \varphi$ iff for all $\mathcal{C} \supseteq \mathcal{B}$, atomic multisets K , if $\Vdash_{\mathcal{C}}^K \Gamma$ then $\Vdash_{\mathcal{C}}^{L,K} \varphi$.

What do we know?

Two important identities from Intuitionistic Linear Logic:

- $! \varphi \Vdash \varphi \otimes \dots \otimes \varphi$

- $!(\varphi \& \psi) \equiv !\varphi \otimes !\psi$

Expanding what we know

$! \varphi \Vdash \varphi \otimes \dots \otimes \varphi$ iff

for all bases \mathcal{B} and atomic multisets L , such that $\Vdash_{\mathcal{B}}^L ! \varphi$ then

$\Vdash_{\mathcal{B}}^L \varphi \otimes \dots \otimes \varphi$

Expanding what we know

- So when does $\Vdash_{\mathcal{B}}^L \varphi \otimes \dots \otimes \varphi$ hold?

Expanding what we know

Considering the case when $\varphi = p$ for simplicity we get the following:

Expanding what we know

Considering the case when $\varphi = p$ for simplicity we get the following:

- Given a base \mathcal{B} such that $\vdash_{\mathcal{B}} p$ and $L = \emptyset$.

Expanding what we know

Considering the case when $\varphi = p$ for simplicity we get the following:

- Given a base \mathcal{B} such that $\vdash_{\mathcal{B}} p$ and $L = \emptyset$.
- Then we have that $\Vdash_{\mathcal{B}}^L p \otimes \dots \otimes p$.

Expanding what we know

Considering the case when $\varphi = p$ for simplicity we get the following:

- Given a base \mathcal{B} such that $\vdash_{\mathcal{B}} p$ and $L = \emptyset$.
- Then we have that $\Vdash_{\mathcal{B}}^L p \otimes \dots \otimes p$.
- Examples of such bases:
 - $\mathcal{B} = \{(\Rightarrow p)\}$
 - $\mathcal{B} = \{(\Rightarrow q), (q \Rightarrow p)\}$

Expanding what we know

Considering the case when $\varphi = p$ for simplicity we get the following:

- Given a base \mathcal{B} such that $\vdash_{\mathcal{B}} p$ and $L = \emptyset$.
- Then we have that $\Vdash_{\mathcal{B}}^L p \otimes \dots \otimes p$.
- Examples of such bases:
 - $\mathcal{B} = \{(\Rightarrow p)\}$
 - $\mathcal{B} = \{(\Rightarrow q), (q \Rightarrow p)\}$

Note that if L were non-empty then all of the atoms in L would have to be derivable purely from \mathcal{B} .

Expanding what we know

Inferring from $! \varphi$ should imply that $\Vdash_{\mathcal{B}}^{\emptyset} \varphi$.

A proof-theoretic flavour of the !

We now consider the second identity: $\Vdash_{\mathcal{B}}^L !(\varphi \& \psi)$ iff $\Vdash_{\mathcal{B}}^L !\varphi \otimes !\psi$ with $\psi = \top$. Furthermore since $\varphi \& \top \equiv \varphi$ we go as follows:

$$\Vdash_{\mathcal{B}}^L !\varphi \text{ iff } \Vdash_{\mathcal{B}}^L !\varphi \otimes !\top$$

iff for all $\mathcal{C} \supseteq \mathcal{B}$, K and $p \in \mathbb{A}$, $!\varphi, !\top \Vdash_{\mathcal{C}}^K p$ implies $\Vdash_{\mathcal{C}}^{L,K} p$

iff for all $\mathcal{C} \supseteq \mathcal{B}$, K and $p \in \mathbb{A}$,

if (for all $\mathcal{D} \supseteq \mathcal{C}$, if $\Vdash_{\mathcal{D}}^{\emptyset} \varphi$ and $\Vdash_{\mathcal{D}}^{\emptyset} \top$ then $\Vdash_{\mathcal{D}}^K p$) then $\Vdash_{\mathcal{C}}^{L,K} p$

iff for all $\mathcal{C} \supseteq \mathcal{B}$, K and $p \in \mathbb{A}$,




if (for all $\mathcal{D} \supseteq \mathcal{C}$, if $\Vdash_{\mathcal{D}}^{\emptyset} \varphi$ then $\Vdash_{\mathcal{D}}^K p$) then $\Vdash_{\mathcal{C}}^{L,K} p$

Muchas gracias!





Muchas gracias!

Questions? Comments? Observations? Please ask and/or feel free to email me at y.buzoku@ucl.ac.uk.

References I

-  G.M. Bierman, *On intuitionistic linear logic*, Tech. Report UCAM-CL-TR-346, University of Cambridge, Computer Laboratory, August 1994.
-  Yli Buzoku, *A proof-theoretic semantics for intuitionistic linear logic*, 2024.
-  Alexander V. Gheorghiu, Tao Gu, and David J. Pym, *Proof-theoretic semantics for intuitionistic multiplicative linear logic*, Automated Reasoning with Analytic Tableaux and Related Methods (Cham) (Revantha Ramanayake and Josef Urban, eds.), Springer Nature Switzerland, 2023, pp. 367–385.

References II

-  J.-Y. Girard, *Linear logic: its syntax and semantics*, London Mathematical Society Lecture Note Series, p. 1–42, Cambridge University Press, 1995.
-  Alexander V. Gheorghiu and David J. Pym, *From proof-theoretic validity to base-extension semantics for intuitionistic propositional logic*, 2022.
-  Sara Negri, *A normalizing system of natural deduction for intuitionistic linear logic*, *Archive for Mathematical Logic* **41** (2002), no. 8, 789–810.
-  Tor Sandqvist, *An inferentialist interpretation of classical logic*, Ph.D. thesis, Uppsala universitet, 2005.

References III



_____, *Base-extension semantics for intuitionistic sentential logic*,
Log. J. IGPL **23** (2015), 719–731.