# Base-extension semantics for Intuitionistic Linear Logic

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# Goals for this talk

- To introduce Intuitionistic Linear Logic and a natural deduction system for it.
- Present a Base-extension semantics for Intuitionistic Linear Logic.
- Talk about some of the difficulties involved the process of developing such a semantics.

# Presentation root directory

# Presentation root directory

# Notation

- $\mathbb{A}$ represents a fixed, countably infinite set of propositional atoms.
- Lower case latin letters represent propositional atoms.
- Upper case latin letters represent finite multisets of propositional atoms.
- Atomic multiset is taken to mean multiset of propositional atoms.
- The sum of two multisets $P$ and $Q$ is denoted $P \fatsemi Q$.
- Lower case greek letters represent ILL formulas.
- Upper case greek letters represent ILL finite multisets thereof.

# Formulas of Intuitionistic Linear Logic

## Definition

Formulas of ILL are defined inductively as follows:

$$\text{Form} \ni \varphi, \psi ::= p \in \mathbb{A} \mid \top \mid 0 \mid 1 \mid \varphi \multimap \psi \mid \varphi \otimes \psi \mid \varphi \,\&\, \psi \mid \varphi \oplus \psi \mid !\,\varphi$$

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A sequent is a pair $(\Gamma, \varphi)$.

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## Definition (Sequent)

A sequent is a pair $(\Gamma, \varphi)$.

An example ILL sequent is $(\{\varphi, \varphi \multimap \psi\}, \psi \otimes \chi)$

# A natural deduction system for ILL

$$\frac{}{\varphi \vdash \varphi}\ \mathsf{Ax}$$

$$\frac{\Gamma \,,\, \varphi \vdash \psi}{\Gamma \vdash \varphi \multimap \psi}\ \multimap\text{-I} \qquad \frac{\Gamma \vdash \varphi \multimap \psi \qquad \Delta \vdash \varphi}{\Gamma \,,\, \Delta \vdash \psi}\ \multimap\text{-E}$$

$$\frac{\Gamma \vdash \varphi \qquad \Delta \vdash \psi}{\Gamma \,,\, \Delta \vdash \varphi \otimes \psi}\ \otimes\text{-I} \qquad \frac{\Gamma \vdash \varphi \otimes \psi \qquad \Delta \,,\, \varphi \,,\, \psi \vdash \chi}{\Gamma \,,\, \Delta \vdash \chi}\ \otimes\text{-E}$$

$$\frac{}{\vdash 1}\ 1\text{-I} \qquad \frac{\Gamma \vdash \varphi \qquad \Delta \vdash 1}{\Gamma \,,\, \Delta \vdash \varphi}\ 1\text{-E}$$

$$\frac{\Gamma \vdash \varphi \qquad \Gamma \vdash \psi}{\Gamma \vdash \varphi \,\&\, \psi}\ \&\text{-I} \qquad \frac{\Gamma \vdash \varphi_0 \,\&\, \varphi_1}{\Gamma \vdash \varphi_i}\ \&\text{-E}_i$$

$$\frac{\Gamma \vdash \varphi_i}{\Gamma \vdash \varphi_0 \oplus \varphi_1}\ \oplus\text{-I}_i \qquad \frac{\Gamma \vdash \varphi \oplus \psi \qquad \Delta \,,\, \varphi \vdash \chi \qquad \Delta \,,\, \psi \vdash \chi}{\Gamma \,,\, \Delta \vdash \chi}\ \oplus\text{-E}$$

$$\frac{\Gamma_1 \vdash \varphi_1 \qquad \ldots \qquad \Gamma_n \vdash \varphi_n}{\Gamma_1 \,,\, \cdots \,,\, \Gamma_n \vdash \top}\ \top\text{-I} \qquad \frac{\Gamma \vdash 0}{\Gamma \vdash \varphi}\ 0\text{-E}$$

# A natural deduction system for ILL (cont.)

$$\frac{\Gamma_1 \vdash\, !\,\psi_1 \qquad \ldots \qquad \Gamma_n \vdash\, !\,\psi_n \qquad !\,\psi_1 \,⨾\, \ldots \,⨾\, !\,\psi_n \vdash \varphi}{\Gamma_1 \,⨾\, \ldots \,⨾\, \Gamma_n \vdash\, !\,\varphi}\ \text{!-Promotion}$$

$$\frac{\Gamma \vdash\, !\,\varphi \qquad \Delta \,⨾\, \varphi \vdash \psi}{\Gamma \,⨾\, \Delta \vdash \psi}\ \text{!-Dereliction}$$

$$\frac{\Gamma \vdash\, !\,\varphi \qquad \Delta \vdash \psi}{\Gamma \,⨾\, \Delta \vdash \psi}\ \text{!-Weakening}$$

$$\frac{\Gamma \vdash\, !\,\varphi \qquad \Delta \,⨾\, !\,\varphi \,⨾\, !\,\varphi \vdash \psi}{\Gamma \,⨾\, \Delta \vdash \psi}\ \text{!-Contraction}$$

# What is the general form of an inference figure?

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$$\frac{\Gamma \vdash \varphi_1 \qquad \Delta \vdash \varphi_2}{\Gamma \,⨟\, \Delta \vdash \varphi_1 \otimes \varphi_2} \otimes\text{-I}$$

# What is the general form of an inference figure?

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$$\frac{\Gamma_1 \vdash \varphi_1 \qquad \dots \qquad \Gamma_n \vdash \varphi_n}{\Gamma_1 \,, \cdots , \Gamma_n \vdash \psi} \qquad\qquad \frac{\Gamma \vdash \varphi_1 \qquad \dots \qquad \Gamma \vdash \varphi_n}{\Gamma \vdash \psi}$$

# What is the general form of an inference figure?

$$\frac{\Gamma_1 \vdash \varphi_1 \qquad \dots \qquad \Gamma_n \vdash \varphi_n}{\Gamma_1 \,,\, \cdots \,,\, \Gamma_n \vdash \psi} \qquad\qquad \frac{\Gamma \vdash \varphi_1 \qquad \dots \qquad \Gamma \vdash \varphi_n}{\Gamma \vdash \psi}$$

$$\frac{\Gamma \vdash \varphi_1 \oplus \varphi_2 \qquad \Delta \,,\, \varphi_1 \vdash \chi \qquad \Delta \,,\, \varphi_2 \vdash \chi}{\Gamma \,,\, \Delta \vdash \chi}\ \oplus\text{-E}$$

# What is the general form of an inference figure?

$$\frac{\Gamma \vdash \varphi_1 \oplus \varphi_2 \qquad \Delta \mathbin{;} \varphi_1 \vdash \chi \qquad \Delta \mathbin{;} \varphi_2 \vdash \chi}{\Gamma \mathbin{;} \Delta \vdash \chi} \oplus\text{-E}$$

# What is the general form of an inference figure?

$$\frac{\Gamma \vdash \varphi_1 \oplus \varphi_2 \qquad \Delta\{\varphi_1 \vdash \chi \qquad \varphi_2 \vdash \chi\}}{\Gamma \mathbin{⨟} \Delta \vdash \chi} \oplus\text{-E}$$

# What is the general form of an inference figure?

$$\frac{\Gamma\{\varnothing \vdash \varphi_1 \oplus \varphi_2\} \qquad \Delta\{\varphi_1 \vdash \chi \qquad \varphi_2 \vdash \chi\}}{\Gamma \mathbin{⨟} \Delta \vdash \chi} \oplus\text{-E}$$

# What is the general form of an inference figure?

$$\frac{\Gamma\{\varnothing \vdash \varphi_1 \oplus \varphi_2\} \qquad \Delta\{\varphi_1 \vdash \chi \qquad \varphi_2 \vdash \chi\}}{\Gamma \fatsemi \Delta \vdash \chi} \oplus\text{-E}$$

# What is the general form of an inference figure?

$$\frac{\Gamma\{\varnothing \vdash \varphi_1\} \qquad \Delta\{\varnothing \vdash \varphi_2\}}{\Gamma \,⨾\, \Delta \vdash \varphi_1 \otimes \varphi_2} \otimes\text{-I}$$

$$\frac{\Gamma\{\varnothing \vdash \varphi_1 \qquad \varnothing \vdash \varphi_2\}}{\Gamma \vdash \varphi_1 \mathbin{\&} \varphi_2} \mathbin{\&}\text{-I}$$

# What is the general form of an inference figure?

$$\frac{\ldots \qquad \Gamma_i \; \{\Delta_{i_1} \vdash \varphi_{i_1} \qquad \ldots \qquad \Delta_{i_{l_i}} \vdash \varphi_{i_{l_i}}\} \qquad \ldots}{\Gamma_1 \,{}_9\, \cdots \,{}_9\, \Gamma_n \vdash \psi}$$

# What is the general form of an inference figure?

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Almost! What about:

$$\frac{\Gamma_1 \vdash \,!\,\psi_1 \qquad \ldots \qquad \Gamma_n \vdash \,!\,\psi_n \qquad !\,\psi_1 \,,\, \ldots \,,\, !\,\psi_n \vdash \varphi}{\Gamma_1 \,,\, \ldots \,,\, \Gamma_n \vdash \,!\,\varphi}\ \text{!-Promotion}$$

# What is the general form of an inference figure?

$$\frac{\Gamma_1\{\varnothing \vdash\, !\,\psi_1\} \qquad \ldots \qquad \Gamma_n\{\varnothing \vdash\, !\,\psi_n\} \qquad !\,\psi_1 \,\fatsemi \ldots \fatsemi\, !\,\psi_n \vdash \varphi}{\Gamma_1 \,\fatsemi \ldots \fatsemi\, \Gamma_n \vdash\, !\,\varphi}\ \text{!-Promotion}$$

# Presentation root directory

# Substructural atomic derivability

## Definition (Basic rule)

Basic rules take the following form:

$$\{(P_{1_i} \Rightarrow q_{1_i})\}_{i=1}^{l_1}, \ldots, \{(P_{n_i} \Rightarrow q_{n_i})\}_{i=1}^{l_n} \Rightarrow r$$

- Each $(P_i \Rightarrow q_i)$ is a pair $(P_i, q_i)$ called an atomic sequent.
- Each collection $\{(P_{i_1} \Rightarrow q_{i_1}), \ldots, (P_{i_{l_i}} \Rightarrow q_{i_{l_i}})\}$ is called an atomic box.

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**Definition (Base)**

A base $\mathscr{B}$ is a **SET** of basic rules.

# Substructural atomic derivability

## Definition (Basic derivability relation)

The relation of derivability in a base $\mathscr{B}$, is defined inductively as so:

Ref $p \vdash_{\mathscr{B}} p$

App Given that $(\{(P_{1_i} \Rightarrow q_{1_i})\}_{i=1}^{l_1}, \ldots, \{(P_{n_i} \Rightarrow q_{n_i})\}_{i=1}^{l_n} \Rightarrow r) \in \mathscr{B}$ and atomic multisets $C_i$ such that the following hold:

$$C_i \fatcomma P_{i_j} \vdash_{\mathscr{B}} q_{i_j} \text{ for all } i = 1, \ldots, n \text{ and } j = 1, \ldots, l_i$$

Then $C_1 \fatcomma \ldots \fatcomma C_n \vdash_{\mathscr{B}} r$.

# Example derivations

**Example (Derivation terminations)**

Let $\mathscr{B} = \{(\Rightarrow a), (\{\Rightarrow a\}, \{\Rightarrow b\} \Rightarrow c)\}$

$$\dfrac{\dfrac{}{b \vdash_{\mathscr{B}} b}\,\text{Ref} \qquad \dfrac{}{\vdash_{\mathscr{B}} a} \Rightarrow a}{b \vdash_{\mathscr{B}} c}\,\{\Rightarrow a\}, \{\Rightarrow b\} \Rightarrow c$$

Therefore, in $\mathscr{B}$, the atom $c$ is derivable from $b$.

# Example derivations

## Example (Invalid derivation)

By the definition of $\vdash_{\mathscr{B}}$, deriving $a$ from $a \mathbin{⨟} a$ in the empty base is not possible, i.e. $a \mathbin{⨟} a \vdash_{\varnothing} a$ is not possible.

# Example derivations

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By the definition of $\vdash_{\mathscr{B}}$, deriving $a$ from $a \mathbin{⨟} a$ in the empty base is not possible, i.e. $a \mathbin{⨟} a \vdash_{\varnothing} a$ is not possible.

## Example (Invalid derivation 2)

Let $\mathscr{B} = \{(\{\Rightarrow a\} \Rightarrow c), (\{a \Rightarrow c\} \Rightarrow b)\}$

$$\cfrac{\cfrac{\cfrac{\times}{a \mathbin{⨟} a \vdash_{\mathscr{B}} a}}{a \mathbin{⨟} a \vdash_{\mathscr{B}} c}\ \{\Rightarrow a\} \Rightarrow c}{a \vdash_{\mathscr{B}} b}\ \{a \Rightarrow c\} \Rightarrow b$$

We see that in this base, $a$ is not derivable from $a \mathbin{⨟} a$.

# Example derivations

**Example (A possible fix for invalid derivation 2)**

Let $\mathscr{C} = \{(\{\Rightarrow a\} \Rightarrow c), (\{a \Rightarrow c\} \Rightarrow b), (\{\Rightarrow a\}, \{\Rightarrow a\} \Rightarrow a)\}$

$$\dfrac{\dfrac{\dfrac{\dfrac{}{a \vdash_{\mathscr{C}} a}\,\text{Ref} \qquad \dfrac{}{a \vdash_{\mathscr{C}} a}\,\text{Ref}}{a \,\fatsemi\, a \vdash_{\mathscr{C}} a}\,\{\Rightarrow a\}, \{\Rightarrow a\} \Rightarrow a}{a \,\fatsemi\, a \vdash_{\mathscr{C}} c}\,\{\Rightarrow a\} \Rightarrow c}{a \vdash_{\mathscr{C}} b}\,\{a \Rightarrow c\} \Rightarrow b$$

Note that $\mathscr{C} \supset \mathscr{B}$ !!

# Example derivations

## Example (Another possible fix for invalid derivation 2)

Let

$\mathscr{D} = \{(\{\Rightarrow a\} \Rightarrow c), (\{a \Rightarrow c\} \Rightarrow b), (\{\Rightarrow d\}, \{\Rightarrow d\} \Rightarrow a), (\{\Rightarrow a\} \Rightarrow d)\}$

$$\dfrac{\dfrac{\dfrac{\dfrac{}{a \vdash_{\mathscr{D}} a}\,\text{Ref}}{a \vdash_{\mathscr{D}} d}\,\{\Rightarrow a\} \Rightarrow d \qquad \dfrac{\dfrac{}{a \vdash_{\mathscr{D}} a}\,\text{Ref}}{a \vdash_{\mathscr{D}} d}\,\{\Rightarrow a\} \Rightarrow d}{a \,\fatcomma\, a \vdash_{\mathscr{D}} a}\,\{\Rightarrow d\}, \{\Rightarrow d\} \Rightarrow a}{\dfrac{a \,\fatcomma\, a \vdash_{\mathscr{D}} c}{a \vdash_{\mathscr{D}} b}\,\{a \Rightarrow c\} \Rightarrow b}\,\{\Rightarrow a\} \Rightarrow c$$

# Example derivations

## Example (A live complicated derivation!)

Consider a base $\mathscr{B}$ with only the following rules:

- $\{\Rightarrow c\}, \{\Rightarrow y\} \Rightarrow f$
- $\Rightarrow z$
- $\{\Rightarrow a\}, \{\Rightarrow b\}, \{\Rightarrow z\} \Rightarrow h$
- $\{\Rightarrow h\} \Rightarrow g$
- $\{\Rightarrow x\}, \{\Rightarrow g\} \Rightarrow e$
- $\{x \Rightarrow e\}, \{y \Rightarrow f\} \Rightarrow d$

is there a derivation of $d$ from the multiset $a \mathbin{,} b \mathbin{,} c$ in $\mathscr{B}$?

# Example derivations

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is there a derivation of $d$ from the multiset $a \,⹁\, b \,⹁\, c$ in $\mathscr{B}$?

*Disclaimer: The fact I couldn't fit this derivation on the slide has nothing to do with why we are doing this derivation live :-)*

# Example derivations

## Example (A very interesting derivation)

Let $\mathscr{B} = \{(\{\Rightarrow b\} \Rightarrow c), (\{\Rightarrow a\}, \{b \Rightarrow c\} \Rightarrow d), (\{\Rightarrow d, \Rightarrow a\} \Rightarrow e), (\{a \Rightarrow e\} \Rightarrow f)\}$

$$
\dfrac{\dfrac{\dfrac{\dfrac{}{a \vdash_{\mathscr{B}} a}\,\text{Ref} \qquad \dfrac{\dfrac{}{b \vdash_{\mathscr{B}} b}\,\text{Ref}}{b \vdash_{\mathscr{B}} c}\,\{\Rightarrow b\} \Rightarrow c}{a \vdash_{\mathscr{B}} d}\,\{\Rightarrow a\}, \{b \Rightarrow c\} \Rightarrow d \qquad \dfrac{}{a \vdash_{\mathscr{B}} a}\,\text{Ref}}{a \vdash_{\mathscr{B}} e}\,\{\Rightarrow d, \Rightarrow a\} \Rightarrow e}{\vdash_{\mathscr{B}} f}\,\{a \Rightarrow e\} \Rightarrow f
$$

# An interesting property of $\vdash_{\mathscr{B}}$

**Lemma (Substitution lemma)**

*The following are equivalent for arbitrary atomic multisets $P \fatcomma S$, atom $q$, and base $\mathscr{B}$, where we assume $P = \{p_1, \ldots, p_n\}$:*

- $P \fatcomma S \vdash_{\mathscr{B}} q$
- *For every $\mathscr{C} \supseteq \mathscr{B}$, atomic multisets $T_1, \ldots, T_m$ where $T_1 \vdash_{\mathscr{C}} p_1, \ldots, T_n \vdash_{\mathscr{C}} p_m$, then $T_1 \fatcomma \cdots \fatcomma T_m \fatcomma S \vdash_{\mathscr{C}} q$*

$$
\begin{array}{rlll}
(\mathrm{At}) & \Vdash^{L}_{\mathscr{B}} p & \text{iff} & L \vdash_{\mathscr{B}} p \\
(\multimap) & \Vdash^{L}_{\mathscr{B}} \varphi \multimap \psi & \text{iff} & \varphi \Vdash^{L}_{\mathscr{B}} \psi \\
(\otimes) & \Vdash^{L}_{\mathscr{B}} \varphi \otimes \psi & \text{iff} & \text{for any } \mathscr{C} \text{ such that } \mathscr{B} \subseteq \mathscr{C}, \text{ atomic multisets } K \\
 & & & \text{and any } p \in \mathbb{A}, \text{ if } \varphi \,,\, \psi \Vdash^{K}_{\mathscr{B}} p \text{ then } \Vdash^{L,K}_{\mathscr{C}} p \\
(1) & \Vdash^{L}_{\mathscr{B}} 1 & \text{iff} & \text{for any } \mathscr{C} \text{ such that } \mathscr{B} \subseteq \mathscr{C}, \text{ atomic multisets } K \\
 & & & \text{and any } p \in \mathbb{A}, \text{ if } \Vdash^{K}_{\mathscr{B}} p \text{ then } \Vdash^{L,K}_{\mathscr{C}} p \\
(\top) & \Vdash^{L}_{\mathscr{B}} \top & \text{iff} & \text{always} \\
(\&) & \Vdash^{L}_{\mathscr{B}} \varphi \,\&\, \psi & \text{iff} & \Vdash^{L}_{\mathscr{B}} \varphi \text{ and } \Vdash^{L}_{\mathscr{B}} \psi \\
(\oplus) & \Vdash^{L}_{\mathscr{B}} \varphi \oplus \psi & \text{iff} & \text{for any } \mathscr{C} \text{ such that } \mathscr{B} \subseteq \mathscr{C}, \text{ atomic multisets } K \\
 & & & \text{and any } p \in \mathbb{A}, \text{ if } \varphi \Vdash^{K}_{\mathscr{C}} p \text{ and } \psi \Vdash^{K}_{\mathscr{C}} p, \text{ then } \Vdash^{L,K}_{\mathscr{C}} p \\
(0) & \Vdash^{L}_{\mathscr{B}} 0 & \text{iff} & \Vdash^{L}_{\mathscr{B}} p \text{ for any } p \in \mathbb{A} \\
(!) & \Vdash^{L}_{\mathscr{B}} !\,\varphi & \text{iff} & \text{for any } \mathscr{C} \text{ such that } \mathscr{B} \subseteq \mathscr{C}, \text{ atomic multisets } K \\
 & & & \text{and any } p \in \mathbb{A}, \text{ if for any } \mathscr{D} \text{ such that } \mathscr{C} \subseteq \mathscr{D}, \\
 & & & (\text{if } \Vdash^{\varnothing}_{\mathscr{D}} \varphi \text{ then } \Vdash^{L}_{\mathscr{D}} p) \text{ then } \Vdash^{L,K}_{\mathscr{C}} p \\
(,) & \Vdash^{L}_{\mathscr{B}} \Gamma \,,\, \Delta & \text{iff} & \text{there exists multisets } K \text{ and } M \text{ such that } L = K \,,\, M \\
 & & & \text{and } \Vdash^{K}_{\mathscr{B}} \Gamma \text{ and } \Vdash^{M}_{\mathscr{B}} \Delta \\
(\mathrm{Inf}) & !\,\Delta \,,\, \Theta \Vdash^{L}_{\mathscr{B}} \varphi & \text{iff} & \text{for any } \mathscr{C} \text{ such that } \mathscr{B} \subseteq \mathscr{C}, \text{ atomic multisets } K, \\
 & & & \text{if } \Vdash^{\varnothing}_{\mathscr{C}} \Delta \text{ and } \Vdash^{K}_{\mathscr{C}} \Theta \text{ then } \Vdash^{L,K}_{\mathscr{C}} \varphi
\end{array}
$$

# Base-extension Semantics for ILL

Note the clause for (!) could be written as:

- $\Vdash^{L}_{\mathscr{B}} !\,\varphi$ iff for any $\mathscr{C}$ such that $\mathscr{C} \supseteq \mathscr{B}$, atomic multisets $K$ and any $p \in \mathbb{A}$, if $!\,\varphi \Vdash^{K}_{\mathscr{C}} p$ then $\Vdash^{L,K}_{\mathscr{C}} p$.

# Notes on $\Vdash^{L}_{\mathscr{B}}$

- The sequent $(\Gamma, \varphi)$ is said to be valid if and only if $\Gamma \Vdash^{\varnothing}_{\varnothing} \varphi$ holds.

# Notes on $\Vdash^{L}_{\mathscr{B}}$

- The sequent $(\Gamma, \varphi)$ is said to be valid if and only if $\Gamma \Vdash^{\varnothing}_{\varnothing} \varphi$ holds.
- We frequently write this as $\Gamma \Vdash \varphi$.

# Notes on $\Vdash^{L}_{\mathscr{B}}$

- If $\Vdash^{L}_{\mathscr{B}} \varphi$ then for all $\mathscr{C} \supseteq \mathscr{B}$ we have $\Vdash^{L}_{\mathscr{C}} \varphi$.
- Given $\Gamma \Vdash^{L}_{\mathscr{B}} \varphi$ and $\Vdash^{K}_{\mathscr{B}} \Gamma$, then it holds that $\Vdash^{L \fatsemi K}_{\mathscr{B}} \varphi$.

# Soundness

**Theorem (Soundness)**

*If* $\Gamma \vdash \varphi$ *then* $\Gamma \Vdash \varphi$

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- The strategy is to show every rule of the natural deduction system presented previously is semantically expressible.
- This suffices by induction to prove this theorem as now we can encode any deduction into a series of semantic proofs.

# A reminder of the rules of natural deduction for ILL

$$\frac{}{\varphi \vdash \varphi}\ \mathsf{Ax}$$

$$\frac{\Gamma , \varphi \vdash \psi}{\Gamma \vdash \varphi \multimap \psi}\ \multimap\text{-I} \qquad \frac{\Gamma \vdash \varphi \multimap \psi \qquad \Delta \vdash \varphi}{\Gamma , \Delta \vdash \psi}\ \multimap\text{-E}$$

$$\frac{\Gamma \vdash \varphi \qquad \Delta \vdash \psi}{\Gamma , \Delta \vdash \varphi \otimes \psi}\ \otimes\text{-I} \qquad \frac{\Gamma \vdash \varphi \otimes \psi \qquad \Delta , \varphi , \psi \vdash \chi}{\Gamma , \Delta \vdash \chi}\ \otimes\text{-E}$$

$$\frac{}{\vdash 1}\ 1\text{-I} \qquad \frac{\Gamma \vdash \varphi \qquad \Delta \vdash 1}{\Gamma , \Delta \vdash \varphi}\ 1\text{-E}$$

$$\frac{\Gamma \vdash \varphi \qquad \Gamma \vdash \psi}{\Gamma \vdash \varphi \,\&\, \psi}\ \&\text{-I} \qquad \frac{\Gamma \vdash \varphi_0 \,\&\, \varphi_1}{\Gamma \vdash \varphi_i}\ \&\text{-E}_i$$

$$\frac{\Gamma \vdash \varphi_i}{\Gamma \vdash \varphi_0 \oplus \varphi_1}\ \oplus\text{-I}_i \qquad \frac{\Gamma \vdash \varphi \oplus \psi \qquad \Delta , \varphi \vdash \chi \qquad \Delta , \psi \vdash \chi}{\Gamma , \Delta \vdash \chi}\ \oplus\text{-E}$$

$$\frac{\Gamma_1 \vdash \varphi_1 \qquad \ldots \qquad \Gamma_n \vdash \varphi_n}{\Gamma_1 , \cdots , \Gamma_n \vdash \top}\ \top\text{-I} \qquad \frac{\Gamma \vdash 0}{\Gamma \vdash \varphi}\ 0\text{-E}$$

# Reminder (cont.)

$$\frac{\Gamma_1 \vdash \,!\,\psi_1 \qquad \ldots \qquad \Gamma_n \vdash \,!\,\psi_n \qquad !\,\psi_1 \,,\, \ldots \,,\, !\,\psi_n \vdash \varphi}{\Gamma_1 \,,\, \ldots \,,\, \Gamma_n \vdash \,!\,\varphi} \text{ !-Promotion}$$

$$\frac{\Gamma \vdash \,!\,\varphi \qquad \Delta \,,\, \varphi \vdash \psi}{\Gamma \,,\, \Delta \vdash \psi} \text{ !-Dereliction}$$

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# Completeness

**Theorem (Completeness)**

*If* $\Gamma \Vdash \varphi$ *then* $\Gamma \vdash \varphi$.

# Completeness

**Theorem (Completeness)**

*If* $\Gamma \Vdash \varphi$ *then* $\Gamma \vdash \varphi$.

- Define two functions $(\cdot)^\flat$ : Form $\to \mathbb{A}$ and $(\cdot)^\natural$ : $\mathbb{A} \to$ Form.
- Show that any valid sequent remains valid when “atomised”.
- Construct a base $\mathscr{N}$ whose rules simulate the natural deduction rules of ILL.
- Show that every derivation in this simulation base is equivalent to a corresponding derivation in ILL.
- Show we can go from a valid derivation to a derivation in ILL.

# The Simulation Base

Consider the valid sequent $\Gamma \Vdash \varphi$. Let $\alpha$, $\beta$ range over all subformulas of this sequent.

- To construct $\mathscr{N}$ we simulate all possible ND-rules of our system.
- This means $\mathscr{N}$ contains the introduction and elimination rules governing $\alpha \circ \beta$, where $\circ \in \{\otimes, \&, \oplus, \multimap\}$, governing $\alpha$ where $\alpha \in \{1, 0, \top\}$ and all rules governing $\alpha$ where $\alpha = !\,\beta$.
- Thus, rules include but, of course, are not limited to:
  - $\{\alpha^\flat \Rightarrow \beta^\flat\} \Rightarrow (\alpha \multimap \beta)^\flat$
  - $\{\Rightarrow (\alpha \multimap \beta)^\flat\}, \{\Rightarrow \alpha^\flat\} \Rightarrow \beta^\flat$
  - $\{\Rightarrow \alpha^\flat\}, \{\Rightarrow \beta^\flat\} \Rightarrow (\alpha \otimes \beta)^\flat$
  - $\{\Rightarrow (\alpha \otimes \beta)^\flat\}, \{\alpha^\flat \fatsemi \beta^\flat \Rightarrow p\} \Rightarrow p$

# The Simulation Base: An explaination

Recall the *flattened* form of the introduction and elimination rules for the tensor:

- $\{\Rightarrow \alpha^\flat\}, \{\Rightarrow \beta^\flat\} \Rightarrow (\alpha \otimes \beta)^\flat$
- $\{\Rightarrow (\alpha \otimes \beta)^\flat\}, \{\alpha^\flat \mathbin{,} \beta^\flat \Rightarrow p\} \Rightarrow p$

# The Simulation Base: An explaination

Recall the *flattened* form of the introduction and elimination rules for the tensor:

- $\{\Rightarrow \alpha^\flat\}, \{\Rightarrow \beta^\flat\} \Rightarrow (\alpha \otimes \beta)^\flat$
- $\{\Rightarrow (\alpha \otimes \beta)^\flat\}, \{\alpha^\flat \,⨾\, \beta^\flat \Rightarrow p\} \Rightarrow p$

Another way of writing these rules would be as follows:
Let $\alpha^\flat = x$, $\beta^\flat = y$ and $(\alpha \otimes \beta)^\flat = z$. Then:

- $\{\Rightarrow x\}, \{\Rightarrow y\} \Rightarrow z$
- $\{\Rightarrow z\}, \{x \,⨾\, y \Rightarrow p\} \Rightarrow p$

## The Simulation Base: A quiz

### Example

Given that $a = \alpha^\flat$ and $b = \beta^\flat$, which ILL formula is being simulated by the atom $x$ in the following rules?

- $\{\Rightarrow a, \Rightarrow b\} \Rightarrow x$
- $\{\Rightarrow x\} \Rightarrow a$
- $\{\Rightarrow x\} \Rightarrow b$

# The Simulation Base: A quiz

## Example

Given that $a = \alpha^\flat$ and $b = \beta^\flat$, which ILL formula is being simulated by the atom $x$ in the following rules?

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- $\{\Rightarrow x\} \Rightarrow a$
- $\{\Rightarrow x\} \Rightarrow b$

*Thats right! It's* $\alpha \,\&\, \beta$

# Presentation root directory

# What of the exponential?

*The exponential is hard...*

# What of the exponential?

*The exponential is hard...*

*But we finally understand it!*

# Why was it so hard?

$$\frac{\Gamma_1 \vdash !\psi_1 \qquad \ldots \qquad \Gamma_n \vdash !\psi_n \qquad !\psi_1 , \ldots , !\psi_n \vdash \varphi}{\Gamma_1 , \ldots , \Gamma_n \vdash !\varphi} \text{ !-Promotion}$$

$$\frac{\Gamma \vdash !\varphi \qquad \Delta , \varphi \vdash \psi}{\Gamma , \Delta \vdash \psi} \text{ !-Dereliction}$$

$$\frac{\Gamma \vdash !\varphi \qquad \Delta \vdash \psi}{\Gamma , \Delta \vdash \psi} \text{ !-Weakening}$$

$$\frac{\Gamma \vdash !\varphi \qquad \Delta , !\varphi , !\varphi \vdash \psi}{\Gamma , \Delta \vdash \psi} \text{ !-Contraction}$$

# Why was it so hard?

$$\frac{!\,\psi \vdash !\,\psi \qquad !\,\psi \vdash \varphi}{!\,\psi \vdash !\,\varphi}$$

# What do we have now?

(!) $\Vdash^{L}_{\mathscr{B}} \,!\,\varphi$ iff for any $\mathscr{C}$ such that $\mathscr{C} \supseteq \mathscr{B}$, atomic multisets $K$ and any $p \in \mathbb{A}$, if $!\,\varphi \Vdash^{K}_{\mathscr{C}} p$ then $\Vdash^{L,K}_{\mathscr{C}} p$.

(Inf) For $\Gamma = \,!\,\Delta \,\fatsemi\, \Theta$ being a nonempty multiset, $\Gamma \Vdash^{L}_{\mathscr{B}} \varphi$ iff for all $\mathscr{C} \supseteq \mathscr{B}$, atomic multisets $K$, if $\Vdash^{K}_{\mathscr{C}} \Theta$ and $\Vdash^{\varnothing}_{\mathscr{C}} \Delta$ then $\Vdash^{L,K}_{\mathscr{C}} \varphi$.

# What we had last year

(Inf) $\Gamma \Vdash^{L}_{\mathscr{B}} \varphi$ iff for all $\mathscr{C} \supseteq \mathscr{B}$, atomic multisets $K$, if $\Vdash^{K}_{\mathscr{C}} \Gamma$ then $\Vdash^{L,K}_{\mathscr{C}} \varphi$.

# What do we know?

Two important identities from Intuitionistic Linear Logic:

- $!\,\varphi \Vdash \varphi \otimes \ldots \otimes \varphi$
- $!(\varphi \mathbin{\&} \psi) \equiv \,!\,\varphi \otimes \,!\,\psi$

# Expanding what we know

$! \varphi \Vdash \varphi \otimes \ldots \otimes \varphi$ iff

for all bases $\mathscr{B}$ and atomic multisets $L$, such that $\Vdash^{L}_{\mathscr{B}} ! \varphi$ then $\Vdash^{L}_{\mathscr{B}} \varphi \otimes \ldots \otimes \varphi$

# Expanding what we know

- So when does $\Vdash^{L}_{\mathscr{B}} \varphi \otimes \ldots \otimes \varphi$ hold?

# Expanding what we know

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- Given a base $\mathscr{B}$ such that $\vdash_{\mathscr{B}} p$ and $L = \varnothing$.
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- Examples of such bases:
  - $\mathscr{B} = \{(\Rightarrow p)\}$
  - $\mathscr{B} = \{(\Rightarrow q), (q \Rightarrow p)\}$

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Considering the case when $\varphi = p$ for simplicity we get the following:

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- Examples of such bases:
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  - $\mathscr{B} = \{(\Rightarrow q),\ (q \Rightarrow p)\}$

Note that if $L$ were non-empty then all of the atoms in $L$ would have to be derivable purely from $\mathscr{B}$.

# Expanding what we know

*Inferring from* $!\,\varphi$ *should imply that* $\Vdash^{\varnothing}_{\mathscr{B}} \varphi$.

# A proof-theoretic flavour of the !

We now consider the second identity: $\Vdash^{L}_{\mathscr{B}} !(\varphi \mathbin{\&} \psi)$ iff $\Vdash^{L}_{\mathscr{B}} !\varphi \otimes !\psi$ with $\psi = \top$. Furthermore since $\varphi \mathbin{\&} \top \equiv \varphi$ we go as follows:

$\Vdash^{L}_{\mathscr{B}} !\varphi$ iff $\Vdash^{L}_{\mathscr{B}} !\varphi \otimes !\top$

iff for all $\mathscr{C} \supseteq \mathscr{B}$, $K$ and $p \in \mathbb{A}$, $!\varphi \mathbin{⨾} !\top \Vdash^{K}_{\mathscr{C}} p$ implies $\Vdash^{L ⨾ K}_{\mathscr{C}} p$

iff for all $\mathscr{C} \supseteq \mathscr{B}$, $K$ and $p \in \mathbb{A}$,

if ( for all $\mathscr{D} \supseteq \mathscr{C}$, if $\Vdash^{\varnothing}_{\mathscr{D}} \varphi$ and $\Vdash^{\varnothing}_{\mathscr{D}} \top$ then $\Vdash^{K}_{\mathscr{D}} p$) then $\Vdash^{L ⨾ K}_{\mathscr{C}} p$

iff for all $\mathscr{C} \supseteq \mathscr{B}$, $K$ and $p \in \mathbb{A}$,

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# Muchas gracias!

Muchas gracias!
Questions? Comments? Observations? Please ask and/or feel free to email me at `y.buzoku@ucl.ac.uk`.

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