Base-extension semantics for Intuitionistic Linear Logic

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Goals for this talk

- To introduce Intuitionistic Linear Logic and a natural deduction system for it.
- Present a Base-extension semantics for Intuitionistic Linear Logic.
- Talk about some of the difficulties involved the process of developing such a semantics.

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- Present a Base-extension semantics for Intuitionistic Linear Logic.
- Talk about some of the difficulties involved the process of developing such a semantics.
- Is Tammy really a fox?

Presentation root directory

Some Terminology

Definition (Region)

Given a topological space X, a region is a non-empty, connected, open subset of X. The closure of a region is called a closed region.

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Given a topological space X, a region is a non-empty, connected, open subset of X. The closure of a region is called a closed region.

For the following definitions, let $f : U \subseteq \mathbb{C} \to \mathbb{C}$, where U is a region.

Definition (Holomorphic)

A function f is said to be holomorphic if it differentiable at all points $z \in U$.

Presentation root directory

Notation

- $\mathbb{A}$ represents a fixed, countably infinite set of propositional atoms.
- Lower case latin letters represent propositional atoms
- Upper case latin letters represent finite multisets of propositional atoms.
- The empty multiset is denoted $\varnothing$.
- The sum of two multisets P and Q is denoted P, Q.
- Lower case greek letters represent ILL formulas.
- Upper case greek letters represent ILL finite multisets thereof.
- Atomic multiset is taken to mean multiset of propositional atoms.

A natural deduction system for ILL

$$\frac{}{\varphi \vdash \varphi}\ \mathrm{Ax}$$

$$\frac{\Gamma, \varphi \vdash \psi}{\Gamma \vdash \varphi \multimap \psi}\ \multimap\text{-I} \qquad \frac{\Gamma \vdash \varphi \multimap \psi \qquad \Delta \vdash \varphi}{\Gamma, \Delta \vdash \psi}\ \multimap\text{-E}$$

$$\frac{\Gamma \vdash \varphi \qquad \Delta \vdash \psi}{\Gamma, \Delta \vdash \varphi \otimes \psi}\ \otimes\text{-I} \qquad \frac{\Gamma \vdash \varphi \otimes \psi \qquad \Delta, \varphi, \psi \vdash \chi}{\Gamma, \Delta \vdash \chi}\ \otimes\text{-E}$$

$$\frac{}{\vdash 1}\ 1\text{-I} \qquad \frac{\Gamma \vdash \varphi \qquad \Delta \vdash 1}{\Gamma, \Delta \vdash \varphi}\ 1\text{-E}$$

$$\frac{\Gamma \vdash \varphi \qquad \Gamma \vdash \psi}{\Gamma \vdash \varphi \,\&\, \psi}\ \&\text{-I} \qquad \frac{\Gamma \vdash \varphi_0 \,\&\, \varphi_1}{\Gamma \vdash \varphi_i}\ \&\text{-E}_i$$

$$\frac{\Gamma \vdash \varphi_i}{\Gamma \vdash \varphi_0 \oplus \varphi_1}\ \oplus\text{-I}_i \qquad \frac{\Gamma \vdash \varphi \oplus \psi \qquad \Delta, \varphi \vdash \chi \qquad \Delta, \psi \vdash \chi}{\Gamma, \Delta \vdash \chi}\ \oplus\text{-E}$$

No 0 intro rule

$$\frac{\Gamma \vdash 0}{\Gamma \vdash \varphi}\ 0\text{-E}$$

A natural deduction system for ILL (cont.)

$$\frac{\Gamma_1 \vdash\, !\psi_1 \qquad \ldots \qquad \Gamma_n \vdash\, !\psi_n \qquad !\psi_1, \ldots, !\psi_n \vdash \varphi}{\Gamma_1, \ldots, \Gamma_n \vdash\, !\varphi}\ \text{!-Promotion}$$

$$\frac{\Gamma \vdash\, !\varphi \qquad \Delta, \varphi \vdash \psi}{\Gamma, \Delta \vdash \psi}\ \text{!-Dereliction}$$

$$\frac{\Gamma \vdash\, !\varphi \qquad \Delta \vdash \psi}{\Gamma, \Delta \vdash \psi}\ \text{!-Weakening}$$

$$\frac{\Gamma \vdash\, !\varphi \qquad \Delta, !\varphi, !\varphi \vdash \psi}{\Gamma, \Delta \vdash \psi}\ \text{!-Contraction}$$

What is the general form of an inference figure?

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$$\frac{\Gamma \vdash \varphi \qquad \Delta \vdash \psi}{\Gamma, \Delta \vdash \varphi \otimes \psi} \otimes\text{-I}$$

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$$\frac{\Gamma_1 \vdash \varphi_1 \qquad \dots \qquad \Gamma_n \vdash \varphi_n}{\Gamma_1 \dots \Gamma_n \vdash \psi} \qquad\qquad \frac{\Gamma \vdash \varphi_1 \qquad \dots \qquad \Gamma \vdash \varphi_n}{\Gamma \vdash \psi}$$

What is the general form of an inference figure?

$$\frac{\Gamma_1 \vdash \varphi_1 \qquad \dots \qquad \Gamma_n \vdash \varphi_n}{\Gamma_1 \dots \Gamma_n \vdash \psi}$$

$$\frac{\Gamma \vdash \varphi_1 \qquad \dots \qquad \Gamma \vdash \varphi_n}{\Gamma \vdash \psi}$$

$$\frac{\Gamma \vdash \varphi \oplus \psi \qquad \Delta, \varphi \vdash \chi \qquad \Delta, \psi \vdash \chi}{\Gamma, \Delta \vdash \chi} \oplus\text{-E}$$

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What is the general form of an inference figure?

$$\frac{\Gamma\{\varnothing \vdash \varphi\} \qquad \Delta\{\varnothing \vdash \psi\}}{\Gamma, \Delta \vdash \varphi \otimes \psi} \otimes\text{-I}$$

$$\frac{\Gamma\{\varnothing \vdash \varphi \qquad \varnothing \vdash \psi\}}{\Gamma \vdash \varphi \,\&\, \psi} \&\text{-I}$$

What is the general form of an inference figure?

$$\frac{\ldots \qquad \Gamma_i \ \{\Delta_{i_1} \vdash \varphi_{i_1} \qquad \ldots \qquad \Delta_{i_{l_i}} \vdash \varphi_{i_{l_i}}\} \qquad \ldots}{\Gamma_1 \ldots \Gamma_n \vdash \psi}$$

What is the general form of an inference figure?

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Almost! What about:

$$\frac{\Gamma_1 \vdash\ !\psi_1 \qquad \ldots \qquad \Gamma_n \vdash\ !\psi_n \qquad !\psi_1, \ldots, !\psi_n \vdash \varphi}{\Gamma_1, \ldots, \Gamma_n \vdash\ !\varphi}\ \text{!-Promotion}$$

What is the general form of an inference figure?

$$\frac{\Gamma_1 \{\varnothing \vdash\, !\,\psi_1\} \quad \ldots \quad \Gamma_n \{\varnothing \vdash\, !\,\psi_n\} \quad !\,\psi_1, \ldots, !\,\psi_n \vdash \varphi}{\Gamma_1, \ldots, \Gamma_n \vdash\, !\,\varphi}\ !\text{-Promotion}$$

Presentation root directory

Substructural atomic derivability

Definition

Basic rules take the following form:

$$\{(P_{1_i} \Rightarrow q_{1_i})\}_{i=1}^{l_1}, \ldots, \{(P_{n_i} \Rightarrow q_{n_i})\}_{i=1}^{l_n} \Rightarrow r$$

where

- Each P_i is an atomic multiset, called a premiss multiset.
- Each q_i and r is an atomic proposition.
- Each $(P_i \Rightarrow q_i)$ is a pair (P_i, q_i) called an atomic sequent.
- Each collection $\{(P_{i_1} \Rightarrow q_{i_1}), \ldots, (P_{i_{l_i}} \Rightarrow q_{i_{l_i}})\}$ is called an atomic box.

Substructural atomic derivability

Definition (Basic derivability relation)

The relation of derivability in a base $\mathscr{B}$, is defined inductively as so:

Ref $p \vdash_{\mathscr{B}} p$

App Given that $(\{(P_{1_i} \Rightarrow q_{1_i})\}_{i=1}^{l_1}, \ldots, \{(P_{n_i} \Rightarrow q_{n_i})\}_{i=1}^{l_n} \Rightarrow r) \in \mathscr{B}$ and atomic multisets C_i such that the following hold:

$$C_i, P_{i_j} \vdash_{\mathscr{B}} q_{i_j} \text{ for all } i = 1, \ldots, n \text{ and } j = 1, \ldots, l_i$$

Then $C_1, \ldots, C_n \vdash_{\mathscr{B}} r$.

Base-extension Semantics for ILL

(At) $\Vdash^{L}_{\mathscr{B}} p$ iff $L \vdash_{\mathscr{B}} p$.

($\otimes$) $\Vdash^{L}_{\mathscr{B}} \varphi \otimes \psi$ iff for every $\mathscr{C} \supseteq \mathscr{B}$, atomic multiset K, and atom p, if $\varphi \,⹁\, \psi \Vdash^{K}_{\mathscr{C}} p$ then $\Vdash^{L,K}_{\mathscr{C}} p$.

(1) $\Vdash^{L}_{\mathscr{B}} 1$ iff for every $\mathscr{C} \supseteq \mathscr{B}$, atomic multiset K, and atom p, if $\Vdash^{K}_{\mathscr{C}} p$, then $\Vdash^{L,K}_{\mathscr{C}} p$.

($\multimap$) $\Vdash^{L}_{\mathscr{B}} \varphi \multimap \psi$ iff $\varphi \Vdash^{L}_{\mathscr{B}} \psi$.

$\vdots$

Base-extension Semantics for ILL

$$\vdots$$

(&) $\Vdash^{L}_{\mathscr{B}} \varphi \,\&\, \psi$ iff $\Vdash^{L}_{\mathscr{B}} \varphi$ and $\Vdash^{L}_{\mathscr{B}} \psi$.

($\oplus$) $\Vdash^{L}_{\mathscr{B}} \varphi \oplus \psi$ iff for every $\mathscr{C} \supseteq \mathscr{B}$, atomic multiset K and atom $p \in \mathbb{A}$ such that $\varphi \Vdash^{K}_{\mathscr{C}} p$ and $\psi \Vdash^{K}_{\mathscr{C}} p$ hold, then $\Vdash^{L,K}_{\mathscr{C}} p$.

(0) $\Vdash^{L}_{\mathscr{B}} 0$ iff $\Vdash^{L}_{\mathscr{B}} p$, for all $p \in \mathbb{A}$.

$$\vdots$$

Base-extension Semantics for ILL

$\vdots$

($\fatcomma$) $\Vdash^{L}_{\mathscr{B}} \Gamma \fatcomma \Delta$ iff there are atomic multisets K and M such that $L = K, M$ and that $\Vdash^{K}_{\mathscr{B}} \Gamma$ and $\Vdash^{M}_{\mathscr{B}} \Delta$.

(!) $\Vdash^{L}_{\mathscr{B}} !\,\varphi$ iff for any $\mathscr{C}$ such that $\mathscr{C} \supseteq \mathscr{B}$, atomic multisets K and any $p \in \mathbb{A}$, if for any $\mathscr{D}$ such that $\mathscr{D} \supseteq \mathscr{C}$, (if $\Vdash^{\varnothing}_{\mathscr{D}} \varphi$ then $\Vdash^{L}_{\mathscr{D}} p$) then $\Vdash^{L,K}_{\mathscr{C}} p$.

(Inf) For $\Gamma = !\,\Delta \fatcomma \Theta$ being a nonempty multiset, $\Gamma \Vdash^{L}_{\mathscr{B}} \varphi$ iff for all $\mathscr{C} \supseteq \mathscr{B}$, atomic multisets K, if $\Vdash^{K}_{\mathscr{C}} \Theta$ and $\Vdash^{\varnothing}_{\mathscr{C}} \Delta$ then $\Vdash^{L,K}_{\mathscr{C}} \varphi$.

Base-extension Semantics for ILL

$$\vdots$$

- ($\fatsemi$) $\Vdash^{L}_{\mathscr{B}} \Gamma \fatsemi \Delta$ iff there are atomic multisets K and M such that $L = K, M$ and that $\Vdash^{K}_{\mathscr{B}} \Gamma$ and $\Vdash^{M}_{\mathscr{B}} \Delta$.
- (!) $\Vdash^{L}_{\mathscr{B}} !\,\varphi$ iff for any $\mathscr{C}$ such that $\mathscr{C} \supseteq \mathscr{B}$, atomic multisets K and any $p \in \mathbb{A}$, if for any $\mathscr{D}$ such that $\mathscr{D} \supseteq \mathscr{C}$, (if $\Vdash^{\varnothing}_{\mathscr{D}} \varphi$ then $\Vdash^{L}_{\mathscr{D}} p$) then $\Vdash^{L,K}_{\mathscr{C}} p$.
- (Inf) For $\Gamma = !\,\Delta \fatsemi \Theta$ being a nonempty multiset, $\Gamma \Vdash^{L}_{\mathscr{B}} \varphi$ iff for all $\mathscr{C} \supseteq \mathscr{B}$, atomic multisets K, if $\Vdash^{K}_{\mathscr{C}} \Theta$ and $\Vdash^{\varnothing}_{\mathscr{C}} \Delta$ then $\Vdash^{L,K}_{\mathscr{C}} \varphi$.

Note the clause for (!) could be written as:

$\Vdash^{L}_{\mathscr{B}} !\,\varphi$ iff for any $\mathscr{C}$ such that $\mathscr{C} \supseteq \mathscr{B}$, atomic multisets K and any $p \in \mathbb{A}$, if $!\,\varphi \Vdash^{K}_{\mathscr{C}} p$ then $\Vdash^{L,K}_{\mathscr{C}} p$.

Notes on $\Vdash^{L}_{\mathscr{B}}$

The sequent $\langle \Gamma, \varphi \rangle$ is said to be valid if and only if $\Gamma \Vdash^{\varnothing}_{\varnothing} \varphi$ holds.

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The sequent $\langle \Gamma, \varphi \rangle$ is said to be valid if and only if $\Gamma \Vdash^{\varnothing}_{\varnothing} \varphi$ holds.
We frequently write this as $\Gamma \Vdash \varphi$.

Notes on $\Vdash^{L}_{\mathscr{B}}$

If $\Vdash^{L}_{\mathscr{B}} \varphi$ then for all $\mathscr{C} \supseteq \mathscr{B}$ we have $\Vdash^{L}_{\mathscr{C}} \varphi$.

$\Vdash^{L}_{\mathscr{B}} \varphi$ iff $\Vdash^{L}_{\mathscr{B}} \varphi \otimes 1$ iff $\Vdash^{L}_{\mathscr{B}} \varphi \,;\, 1$

$\Gamma \Vdash^{L}_{\mathscr{B}} \varphi$ iff for all $\mathscr{C} \supseteq \mathscr{B}$ and atomic multisets K, $\Vdash^{K}_{\mathscr{C}} \Gamma$ implies $\Vdash^{L,K}_{\mathscr{C}} \varphi$.

Soundness

Theorem (Soundness)

If $\Gamma \vdash \varphi$ *then* $\Gamma \Vdash \varphi$

Soundness

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The strategy is to show every rule of the natural deduction system presented previously is semantically expressible.

This suffices by induction to prove this theorem as now we can encode any deduction into a series of semantic proofs.

Example

$$\text{If } \Gamma \,,\, \varphi \Vdash \psi \text{ then } \Gamma \Vdash \varphi \multimap \psi.$$

Proof.

The hypothesis iff all bases $\mathscr{B}$ and atomic multisets $M = K, L$ where $\Vdash^{K}_{\mathscr{B}} \varphi$ and $\Vdash^{L}_{\mathscr{B}} \Gamma$ implies $\Vdash^{M}_{\mathscr{B}} \psi$.

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Monotonicity gives that this holds in all $\mathscr{C} \supseteq \mathscr{B}$.

Example

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Thus we have that $\varphi \Vdash^{L}_{\mathscr{B}} \psi$.

Example

$$\text{If } \Gamma \,\text{,}\, \varphi \Vdash \psi \text{ then } \Gamma \Vdash \varphi \multimap \psi.$$

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Monotonicity gives that this holds in all $\mathscr{C} \supseteq \mathscr{B}$.

Thus we have that $\varphi \Vdash^{L}_{\mathscr{B}} \psi$.

$\Vdash^{L}_{\mathscr{B}} \varphi \multimap \psi$ if and only if $\varphi \Vdash^{L}_{\mathscr{B}} \psi$.

Example

If $\Gamma \, , \, \varphi \Vdash \psi$ then $\Gamma \Vdash \varphi \multimap \psi$.

Proof.

The hypothesis iff all bases $\mathscr{B}$ and atomic multisets $M = K, L$ where $\Vdash^{K}_{\mathscr{B}} \varphi$ and $\Vdash^{L}_{\mathscr{B}} \Gamma$ implies $\Vdash^{M}_{\mathscr{B}} \psi$.

Monotonicity gives that this holds in all $\mathscr{C} \supseteq \mathscr{B}$.

Thus we have that $\varphi \Vdash^{L}_{\mathscr{B}} \psi$.

$\Vdash^{L}_{\mathscr{B}} \varphi \multimap \psi$ if and only if $\varphi \Vdash^{L}_{\mathscr{B}} \psi$.

So we obtain that $\Vdash^{L}_{\mathscr{B}} \varphi \multimap \psi$ and thus $\Gamma \Vdash \varphi \multimap \psi$

$\square$

Completeness

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- Define two functions $(\cdot)^\flat$: Form $\to \mathbb{A}$ and $(\cdot)^\natural : \mathbb{A} \to$ Form.
- Show that any valid sequent remains valid when “atomised”.
- Construct a base $\mathscr{N}$ whose rules simulate the natural deduction rules of ILL.
- Show that every derivation in this simulation base is equivalent to a corresponding derivation in ILL.
- Show we can go from a valid atomic derivation to a derivation in ILL.

Presentation root directory

What of the exponential?

The exponential is hard...

What of the exponential?

The exponential is hard...

But I finally got it!

Why was it so hard?

$$\frac{\Gamma_1 \vdash \,!\,\psi_1 \qquad \ldots \qquad \Gamma_n \vdash \,!\,\psi_n \qquad !\,\psi_1, \ldots, !\,\psi_n \vdash \varphi}{\Gamma_1, \ldots, \Gamma_n \vdash \,!\,\varphi} \text{ !-Promotion}$$

$$\frac{\Gamma \vdash \,!\,\varphi \qquad \Delta, \varphi \vdash \psi}{\Gamma, \Delta \vdash \psi} \text{ !-Dereliction}$$

$$\frac{\Gamma \vdash \,!\,\varphi \qquad \Delta \vdash \psi}{\Gamma, \Delta \vdash \psi} \text{ !-Weakening}$$

$$\frac{\Gamma \vdash \,!\,\varphi \qquad \Delta, !\,\varphi, !\,\varphi \vdash \psi}{\Gamma, \Delta \vdash \psi} \text{ !-Contraction}$$

Why was it so hard?

$$\frac{!\,\psi \vdash !\,\psi \qquad !\,\psi \vdash \varphi}{!\,\psi \vdash !\,\varphi}$$

What do we have now?

(!) $\Vdash^{L}_{\mathscr{B}} !\,\varphi$ iff for any $\mathscr{C}$ such that $\mathscr{C} \supseteq \mathscr{B}$, atomic multisets K and any $p \in \mathbb{A}$, if $!\,\varphi \Vdash^{K}_{\mathscr{C}} p$ then $\Vdash^{L,K}_{\mathscr{C}} p$.

(Inf) For $\Gamma = !\,\Delta \,\text{,}\, \Theta$ being a nonempty multiset, $\Gamma \Vdash^{L}_{\mathscr{B}} \varphi$ iff for all $\mathscr{C} \supseteq \mathscr{B}$, atomic multisets K, if $\Vdash^{K}_{\mathscr{C}} \Theta$ and $\Vdash^{\varnothing}_{\mathscr{C}} \Delta$ then $\Vdash^{L,K}_{\mathscr{C}} \varphi$.

What do we know?

An important identity from Intuitionistic Linear Logic:

$$!\,\varphi \Vdash \varphi \otimes \ldots \otimes \varphi$$

Expanding what we know

$! \varphi \Vdash \varphi \otimes \ldots \otimes \varphi$ iff

for all bases $\mathscr{B}$ and atomic multisets L, such that $\Vdash^{L}_{\mathscr{B}} ! \varphi$ then $\Vdash^{L}_{\mathscr{B}} \varphi \otimes \ldots \otimes \varphi$

Expanding what we know

So when does $\Vdash^{L}_{\mathscr{B}} \varphi \otimes \ldots \otimes \varphi$ hold?

Expanding what we know

Considering the case when $\varphi = p$ for simplicity we get the following:

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Given a base $\mathscr{B}$ such that $\vdash_{\mathscr{B}} p$ and $L = \varnothing$.

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Considering the case when $\varphi = p$ for simplicity we get the following:

Given a base $\mathscr{B}$ such that $\vdash_{\mathscr{B}} p$ and $L = \varnothing$.

Then we have that $\Vdash^{L}_{\mathscr{B}} p \otimes \ldots \otimes p$.

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Considering the case when $\varphi = p$ for simplicity we get the following:

Given a base $\mathscr{B}$ such that $\vdash_{\mathscr{B}} p$ and $L = \varnothing$.

Then we have that $\Vdash^{L}_{\mathscr{B}} p \otimes \ldots \otimes p$.

Examples of such bases:

$\mathscr{B} = \{(\Rightarrow p)\}$

$\mathscr{B} = \{(\Rightarrow q),\ (q \Rightarrow p)\}$

Expanding what we know

Considering the case when $\varphi = p$ for simplicity we get the following:

Given a base $\mathscr{B}$ such that $\vdash_{\mathscr{B}} p$ and $L = \varnothing$.

Then we have that $\Vdash^{L}_{\mathscr{B}} p \otimes \ldots \otimes p$.

Examples of such bases:

$$\mathscr{B} = \{(\Rightarrow p)\}$$
$$\mathscr{B} = \{(\Rightarrow q),\ (q \Rightarrow p)\}$$

Note that if L were non-empty then all of the atoms in L, call them r_i, would have to be derivable purely from $\mathscr{B}$.

Expanding what we know

Inferring from $!\,\varphi$ *should imply that* $\Vdash^{\varnothing}_{\mathscr{B}} \varphi$.

Alternative approach

$$\dfrac{\begin{array}{c}[P_1; C_1]\\ \vdots \\ q_1\end{array} \qquad \dots \qquad \begin{array}{c}[P_n; C_n]\\ \vdots \\ q_n\end{array}}{r}\ \mathcal{R}$$

Is Tammy really a fox?

Is Tammy really a fox?

Maybe?

Is Tammy really a fox?

Maybe?

But I actually now think she is a Cat . . .

Is Tammy really a fox?

Maybe?

But I actually now think she is a Cat . . .

Specifically a symmetric monoidal cat!

Thank you!

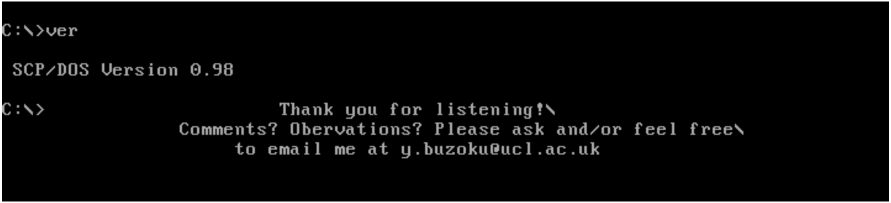

Figure: Thank you from DOS! :D

References I

- G.M. Bierman, *On intuitionistic linear logic*, Tech. Report UCAM-CL-TR-346, University of Cambridge, Computer Laboratory, August 1994.
- Yll Buzoku, *A proof-theoretic semantics for intuitionistic linear logic*, 2024.
- Alexander V. Gheorghiu, Tao Gu, and David J. Pym, *Proof-theoretic semantics for intuitionistic multiplicative linear logic*, Automated Reasoning with Analytic Tableaux and Related Methods (Cham) (Revantha Ramanayake and Josef Urban, eds.), Springer Nature Switzerland, 2023, pp. 367–385.

References II

- J.-Y. Girard, *Linear logic: its syntax and semantics*, London Mathematical Society Lecture Note Series, p. 1–42, Cambridge University Press, 1995.
- Alexander V. Gheorghiu and David J. Pym, *From proof-theoretic validity to base-extension semantics for intuitionistic propositional logic*, 2022.
- Sara Negri, *A normalizing system of natural deduction for intuitionistic linear logic*, Archive for Mathematical Logic **41** (2002), no. 8, 789–810.
- Tor Sandqvist, *An inferentialist interpretation of classical logic*, Ph.D. thesis, Uppsala universitet, 2005.

References III

______, *Base-extension semantics for intuitionistic sentential logic*, Log. J. IGPL **23** (2015), 719–731.