Base-extension semantics for Intuitionistic Linear Logic

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Goals for this talk

- To briefly introduce Intuitionistic Linear Logic.
- To give an overview of the setup of Base-Extension Semantics in the substructural setting.
- Present a Base-extension semantics for Intuitionistic Linear Logic.

Presentation root directory

Presentation root directory

Notation

- $\mathbb{A}$ represents a fixed, countably infinite set of propositional atoms.
- Lower case latin letters represent propositional atoms
- Upper case latin letters represent finite multisets of propositional atoms.
- The empty multiset is denoted $\varnothing$.
- The sum of two multisets P and Q is denoted P, Q.
- Lower case greek letters represent ILL formulas.
- Upper case greek letters represent ILL finite multisets thereof.
- Atomic multiset is taken to mean multiset of propositional atoms.

A natural deduction system for ILL

$$\frac{}{\varphi \vdash \varphi}\ \text{Ax}$$

$$\frac{\Gamma, \varphi \vdash \psi}{\Gamma \vdash \varphi \multimap \psi}\ \multimap\text{-I} \qquad \frac{\Gamma \vdash \varphi \multimap \psi \qquad \Delta \vdash \varphi}{\Gamma, \Delta \vdash \psi}\ \multimap\text{-E}$$

$$\frac{\Gamma \vdash \varphi \qquad \Delta \vdash \psi}{\Gamma, \Delta \vdash \varphi \otimes \psi}\ \otimes\text{-I} \qquad \frac{\Gamma \vdash \varphi \otimes \psi \qquad \Delta, \varphi, \psi \vdash \chi}{\Gamma, \Delta \vdash \chi}\ \otimes\text{-E}$$

$$\frac{}{\vdash 1}\ 1\text{-I} \qquad \frac{\Gamma \vdash \varphi \qquad \Delta \vdash 1}{\Gamma, \Delta \vdash \varphi}\ 1\text{-E}$$

$$\frac{\Gamma \vdash \varphi \qquad \Gamma \vdash \psi}{\Gamma \vdash \varphi \,\&\, \psi}\ \&\text{-I} \qquad \frac{\Gamma \vdash \varphi_0 \,\&\, \varphi_1}{\Gamma \vdash \varphi_i}\ \&\text{-E}_i$$

$$\frac{\Gamma \vdash \varphi_i}{\Gamma \vdash \varphi_0 \oplus \varphi_1}\ \oplus\text{-I}_i \qquad \frac{\Gamma \vdash \varphi \oplus \psi \qquad \Delta, \varphi \vdash \chi \qquad \Delta, \psi \vdash \chi}{\Gamma, \Delta \vdash \chi}\ \oplus\text{-E}$$

No 0 intro rule

$$\frac{\Gamma \vdash 0}{\Gamma \vdash \varphi}\ 0\text{-E}$$

A natural deduction system for ILL (cont.)

$$\frac{\Gamma_1 \vdash \,!\,\psi_1 \qquad \ldots \qquad \Gamma_n \vdash \,!\,\psi_n \qquad !\,\psi_1, \ldots, !\,\psi_n \vdash \varphi}{\Gamma_1, \ldots, \Gamma_n \vdash \,!\,\varphi} \text{ !-Promotion}$$

$$\frac{\Gamma \vdash \,!\,\varphi \qquad \Delta, \varphi \vdash \psi}{\Gamma, \Delta \vdash \psi} \text{ !-Dereliction}$$

$$\frac{\Gamma \vdash \,!\,\varphi \qquad \Delta \vdash \psi}{\Gamma, \Delta \vdash \psi} \text{ !-Weakening}$$

$$\frac{\Gamma \vdash \,!\,\varphi \qquad \Delta, !\,\varphi, !\,\varphi \vdash \psi}{\Gamma, \Delta \vdash \psi} \text{ !-Contraction}$$

What is the general form of an inference figure?

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$$\frac{\Gamma \vdash \varphi \qquad \Delta \vdash \psi}{\Gamma, \Delta \vdash \varphi \otimes \psi} \otimes\text{-I}$$

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$$\frac{\Gamma \vdash \varphi \qquad \Gamma \vdash \psi}{\Gamma \vdash \varphi \,\&\, \psi}\ \&\text{-I}$$

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$$\frac{\Gamma_1 \vdash \varphi_1 \qquad \dots \qquad \Gamma_n \vdash \varphi_n}{\Gamma_1 \dots \Gamma_n \vdash \psi} \qquad\qquad \frac{\Gamma \vdash \varphi_1 \qquad \dots \qquad \Gamma \vdash \varphi_n}{\Gamma \vdash \psi}$$

What is the general form of an inference figure?

$$\frac{\Gamma_1 \vdash \varphi_1 \qquad \dots \qquad \Gamma_n \vdash \varphi_n}{\Gamma_1 \dots \Gamma_n \vdash \psi}$$

$$\frac{\Gamma \vdash \varphi_1 \qquad \dots \qquad \Gamma \vdash \varphi_n}{\Gamma \vdash \psi}$$

$$\frac{\Gamma \vdash \varphi \oplus \psi \qquad \Delta, \varphi \vdash \chi \qquad \Delta, \psi \vdash \chi}{\Gamma, \Delta \vdash \chi} \oplus\text{-E}$$

What is the general form of an inference figure?

$$\frac{\Gamma \vdash \varphi \oplus \psi \qquad \Delta, \varphi \vdash \chi \qquad \Delta, \psi \vdash \chi}{\Gamma, \Delta \vdash \chi} \oplus\text{-E}$$

What is the general form of an inference figure?

$$\frac{\Gamma \vdash \varphi \oplus \psi \qquad \Delta\{\varphi \vdash \chi \qquad \psi \vdash \chi\}}{\Gamma, \Delta \vdash \chi} \oplus\text{-E}$$

What is the general form of an inference figure?

$$\frac{\Gamma\{\varnothing \vdash \varphi \oplus \psi\} \qquad \Delta\{\varphi \vdash \chi \qquad \psi \vdash \chi\}}{\Gamma, \Delta \vdash \chi}\ \oplus\text{-E}$$

What is the general form of an inference figure?

$$\frac{\Gamma\{\varnothing \vdash \varphi \oplus \psi\} \qquad \Delta\{\varphi \vdash \chi \qquad \psi \vdash \chi\}}{\Gamma, \Delta \vdash \chi}\ \oplus\text{-E}$$

What is the general form of an inference figure?

$$\frac{\Gamma\{\varnothing \vdash \varphi\} \qquad \Delta\{\varnothing \vdash \psi\}}{\Gamma, \Delta \vdash \varphi \otimes \psi}\ \otimes\text{-I}$$

$$\frac{\Gamma\{\varnothing \vdash \varphi \qquad \varnothing \vdash \psi\}}{\Gamma \vdash \varphi \,\&\, \psi}\ \&\text{-I}$$

What is the general form of an inference figure?

$$\frac{\ldots \qquad \Gamma_i \ \{\Delta_{i_1} \vdash \varphi_{i_1} \qquad \ldots \qquad \Delta_{i_{l_i}} \vdash \varphi_{i_{l_i}}\} \qquad \ldots}{\Gamma_1 \ldots \Gamma_n \vdash \psi}$$

What is the general form of an inference figure?

What is the general form of an inference figure?

Almost! What about:

$$\frac{\Gamma_1 \vdash\ !\,\psi_1 \qquad \ldots \qquad \Gamma_n \vdash\ !\,\psi_n \qquad !\,\psi_1, \ldots, !\,\psi_n \vdash \varphi}{\Gamma_1, \ldots, \Gamma_n \vdash\ !\,\varphi}\ \text{!-Promotion}$$

What is the general form of an inference figure?

$$\frac{\Gamma_1\{\varnothing \vdash\ !\,\psi_1\} \qquad \ldots \qquad \Gamma_n\{\varnothing \vdash\ !\,\psi_n\} \qquad !\,\psi_1, \ldots, !\,\psi_n \vdash \varphi}{\Gamma_1, \ldots, \Gamma_n \vdash\ !\,\varphi}\ \text{!-Promotion}$$

Presentation root directory

Substructural atomic derivability

Definition

Basic rules take the following form:

$$\{(P_{1_i} \Rightarrow q_{1_i})\}_{i=1}^{l_1}, \ldots, \{(P_{n_i} \Rightarrow q_{n_i})\}_{i=1}^{l_n} \Rightarrow r$$

where

Each P_i is an atomic multiset, called a premiss multiset.

Each q_i and r is an atomic proposition.

Each $(P_i \Rightarrow q_i)$ is a pair (P_i, q_i) called an atomic sequent.

Each collection $\{(P_{i_1} \Rightarrow q_{i_1}), \ldots, (P_{i_{l_i}} \Rightarrow q_{i_{l_i}})\}$ is called an atomic box.

Substructural atomic derivability

Definition (Basic derivability relation)

The relation of derivability in a base $\mathscr{B}$, is defined inductively as so:

Ref $p \vdash_{\mathscr{B}} p$

App Given that $(\{(P_{1_i} \Rightarrow q_{1_i})\}_{i=1}^{l_1}, \ldots, \{(P_{n_i} \Rightarrow q_{n_i})\}_{i=1}^{l_n} \Rightarrow r) \in \mathscr{B}$ and atomic multisets C_i such that the following hold:

$$C_i, P_{i_j} \vdash_{\mathscr{B}} q_{i_j} \text{ for all } i = 1, \ldots, n \text{ and } j = 1, \ldots, l_i$$

Then $C_1, \ldots, C_n \vdash_{\mathscr{B}} r$.

Base-extension Semantics for ILL

(At) $\Vdash^{L}_{\mathscr{B}} p$ iff $L \vdash_{\mathscr{B}} p$.

($\otimes$) $\Vdash^{L}_{\mathscr{B}} \varphi \otimes \psi$ iff for every $\mathscr{C} \supseteq \mathscr{B}$, atomic multiset K, and atom p, if $\varphi \,,\, \psi \Vdash^{K}_{\mathscr{C}} p$ then $\Vdash^{L,K}_{\mathscr{C}} p$.

(1) $\Vdash^{L}_{\mathscr{B}} 1$ iff for every $\mathscr{C} \supseteq \mathscr{B}$, atomic multiset K, and atom p, if $\Vdash^{K}_{\mathscr{C}} p$, then $\Vdash^{L,K}_{\mathscr{C}} p$.

($\multimap$) $\Vdash^{L}_{\mathscr{B}} \varphi \multimap \psi$ iff $\varphi \Vdash^{L}_{\mathscr{B}} \psi$.

$\vdots$

Base-extension Semantics for ILL

$\vdots$

(&) $\Vdash^{L}_{\mathscr{B}} \varphi \,\&\, \psi$ iff $\Vdash^{L}_{\mathscr{B}} \varphi$ and $\Vdash^{L}_{\mathscr{B}} \psi$.

($\oplus$) $\Vdash^{L}_{\mathscr{B}} \varphi \oplus \psi$ iff for every $\mathscr{C} \supseteq \mathscr{B}$, atomic multiset K and atom $p \in \mathbb{A}$ such that $\varphi \Vdash^{K}_{\mathscr{C}} p$ and $\psi \Vdash^{K}_{\mathscr{C}} p$ hold, then $\Vdash^{L,K}_{\mathscr{C}} p$.

(0) $\Vdash^{L}_{\mathscr{B}} 0$ iff $\Vdash^{L}_{\mathscr{B}} p$, for all $p \in \mathbb{A}$.

$\vdots$

Base-extension Semantics for ILL

$\vdots$

($\fatcomma$) $\Vdash^{L}_{\mathscr{B}} \Gamma \fatcomma \Delta$ iff there are atomic multisets K and M such that $L = K, M$ and that $\Vdash^{K}_{\mathscr{B}} \Gamma$ and $\Vdash^{M}_{\mathscr{B}} \Delta$.

(!) $\Vdash^{L}_{\mathscr{B}} !\,\varphi$ iff for any $\mathscr{C}$ such that $\mathscr{C} \supseteq \mathscr{B}$, atomic multisets K and any $p \in \mathbb{A}$, if for any $\mathscr{D}$ such that $\mathscr{D} \supseteq \mathscr{C}$, (if $\Vdash^{\varnothing}_{\mathscr{D}} \varphi$ then $\Vdash^{L}_{\mathscr{D}} p$) then $\Vdash^{L,K}_{\mathscr{C}} p$.

(Inf) For $\Gamma = !\,\Delta \fatcomma \Theta$ being a nonempty multiset, $\Gamma \Vdash^{L}_{\mathscr{B}} \varphi$ iff for all $\mathscr{C} \supseteq \mathscr{B}$, atomic multisets K, if $\Vdash^{K}_{\mathscr{C}} \Theta$ and $\Vdash^{\varnothing}_{\mathscr{C}} \Delta$ then $\Vdash^{L,K}_{\mathscr{C}} \varphi$.

Base-extension Semantics for ILL

$\vdots$

- ($\fatcomma$) $\Vdash^{L}_{\mathscr{B}} \Gamma \fatcomma \Delta$ iff there are atomic multisets K and M such that $L = K, M$ and that $\Vdash^{K}_{\mathscr{B}} \Gamma$ and $\Vdash^{M}_{\mathscr{B}} \Delta$.
- (!) $\Vdash^{L}_{\mathscr{B}} !\,\varphi$ iff for any $\mathscr{C}$ such that $\mathscr{C} \supseteq \mathscr{B}$, atomic multisets K and any $p \in \mathbb{A}$, if for any $\mathscr{D}$ such that $\mathscr{D} \supseteq \mathscr{C}$, (if $\Vdash^{\varnothing}_{\mathscr{D}} \varphi$ then $\Vdash^{L}_{\mathscr{D}} p$) then $\Vdash^{L,K}_{\mathscr{C}} p$.
- (Inf) For $\Gamma = !\,\Delta \fatcomma \Theta$ being a nonempty multiset, $\Gamma \Vdash^{L}_{\mathscr{B}} \varphi$ iff for all $\mathscr{C} \supseteq \mathscr{B}$, atomic multisets K, if $\Vdash^{K}_{\mathscr{C}} \Theta$ and $\Vdash^{\varnothing}_{\mathscr{C}} \Delta$ then $\Vdash^{L,K}_{\mathscr{C}} \varphi$.

Note the clause for (!) could be written as:

$\Vdash^{L}_{\mathscr{B}} !\,\varphi$ iff for any $\mathscr{C}$ such that $\mathscr{C} \supseteq \mathscr{B}$, atomic multisets K and any $p \in \mathbb{A}$, if $!\,\varphi \Vdash^{K}_{\mathscr{C}} p$ then $\Vdash^{L,K}_{\mathscr{C}} p$.

Notes on $\Vdash^{L}_{\mathscr{B}}$

The sequent $\langle \Gamma, \varphi \rangle$ is said to be valid just in case when $\Gamma \Vdash^{\varnothing}_{\varnothing} \varphi$.

Notes on $\Vdash^{L}_{\mathscr{B}}$

If $\Vdash^{L}_{\mathscr{B}} \varphi$ then for all $\mathscr{C} \supseteq \mathscr{B}$ we have $\Vdash^{L}_{\mathscr{C}} \varphi$.

$\Vdash^{L}_{\mathscr{B}} \varphi$ iff $\Vdash^{L}_{\mathscr{B}} \varphi \otimes 1$ iff $\Vdash^{L}_{\mathscr{B}} \varphi \mathbin{,} 1$

$\Vdash^{L}_{\mathscr{B}} \varphi$ iff for all $\mathscr{C} \supseteq \mathscr{B}$, atomic multisets K and atoms p such that $\varphi \Vdash^{K}_{\mathscr{C}} p$ then $\Vdash^{L,K}_{\mathscr{C}} p$.

Soundness

Completeness

Presentation root directory

What of the exponential?

The exponential is hard...

What of the exponential?

The exponential is hard...

But I finally got it!

Why was it so hard?

Alternative approach

We want to re-introduce in a controlled manner the ability for multiple and vacuous discharge of hypotheses.

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Alternative approach

- We want to re-introduce in a controlled manner the ability for multiple and vacuous discharge of hypotheses.
- It is not absurd to distinguish basic sentences based on their content.
- The idea is to define a consequence relation which allows for simultaneous inference from both sentences which are ”persistent” and ”consumable”, thus extending the definition of an basic rule to allow derivations to occur from both collections of sentences.
- Thus our basic rule might looks something like the following:

$$\dfrac{\begin{array}{c}[P_1; C_1] \\ \vdots \\ q_1\end{array} \qquad \ldots \qquad \begin{array}{c}[P_n; C_n] \\ \vdots \\ q_n\end{array}}{r}\ \mathcal{R}$$

Thank you!

Thank you for listening!
Questions? Comments? Observations? Please ask and/or feel free to email me at `y.buzoku@ucl.ac.uk`.

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