A SURVEY OF RIEMANN SURFACES AND THEIR CLASSIFICATION, WITH PARTICULAR FOCUS ON THEIR GEOMETRY

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Abstract

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ABSTRACT OF DISSERTATION submitted by **Y. Buzoku** and entitled **A survey of Riemann surfaces and their classification, with particular focus on their geometry**.

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Declaration

This dissertation is submitted under the regulations of Birkbeck, University of London as part of the examination requirements for the M.Sc. degree in Mathematics. Any quotation or excerpt from the published or unpublished work of other persons is explicitly indicated and in each such instance a full reference of the source of such work is given. I have read and understood the Birkbeck College guidelines on plagiarism and in accordance with those requirements submit this work as my own.

I have found a beautiful proof, but this abstract is too short to contain it.

Chapter 1

Introduction

The aim of this dissertation is to investigate and gain a deeper understanding of Riemann Surfaces, their construction, their classification and an analysis of the geometry of some classes of these surfaces. Doing so will bring together ideas from different, seemingly unrelated fields of mathematics. Often times, Riemann Surfaces are studied solely due to these connections as they might not occur in higher dimensional analogues. We begin this dissertation with an introduction to Riemann surfaces as classical complex-analytical manifolds, and then showing that we can define them equivalently as algebraic varieties of complex curves. We then study briefly the automorphisms of a particular set of such Riemann Surfaces, and give a short review of topological notions that will be important in our classification later, including quotient spaces. We then show how can endow a Riemann surface structure on such spaces, and give examples of spaces arising in such a way, and their lifts to their universal covers.

From there we begin to classify Riemann surfaces by stating the uniformisation theorem, and classifying Riemann surfaces with universal cover the Riemann sphere. We then introduce Teichmüller spaces and discuss Riemann surfaces with the complex plane as universal cover.

The remainder of this dissertation will then focus on the infinite family of Riemann surfaces with the hyperbolic plane as universal cover. A discussion of the geometry of some compact Riemann surfaces follows, with some anecdotal but interesting ideas introduced, such as the interplay between cubic graphs and Riemann surfaces, naturally arising hyperbolic Teichmüller spaces, and Hubers Theorem relating the length spectrum of hyperbolic Riemann surfaces and the eigenvalues of the Laplacian on said surfaces. We conclude with a discussion of the heat kernel on the sphere, plane and hyperbolic plane and prove the uniformisation theorem for compact Riemann surfaces and the claim that there exist exactly two non-compact simply connected Riemann surfaces.

Chapter 2

Riemann Surfaces

This chapter shows the basic definitions from Topology, (Complex) Analysis and Riemann Surface Theory to define Riemann Surfaces, their mappings and boilerplate stuff. Discuss what is needed and constructions in detail. I guess weird test yadda yadda

2.1 Basic Definitions

We begin by stating the basic definitions of Riemann Surfaces and the analogs of some important notions from Complex Analysis in Riemann Surface Theory.

Definition 2.1.1 (Riemann Surface)**.** A Hausdorff topological space X is said to be a Riemann Surface if:

- There exists a collection of open sets $U_\alpha \subset X$, where α ranges over some index set such that $\bigcup\limits_\alpha U_\alpha$ cover X.
- There exists for each α, a homeomorphism, called a chart map, $\psi_\alpha \colon U_\alpha \to \tilde{U}_\alpha$,

where $\tilde{U}_\alpha$ is an open set in $\mathbb{C}$, with the property that for all α, β, the composite map $\psi_\alpha \circ \psi_\beta^{-1}$ is holomorphic on its domain of definition. (These composite maps are sometimes called transition maps).

We call the triple $(\{U_\alpha\}, \{\tilde{U}_\alpha\}, \{\psi_\alpha\})$ an atlas of charts for the Riemann Surface X, though we also use the common notation $(U, \tilde{U}, \psi)$ to denote this, where $U = \{U_\alpha\}$, $\tilde{U} = \{\tilde{U}_\alpha\}$ and $\psi = \{\psi_\alpha\}$.

2.2 Algebraic Curves

ALGBRAIC CURVES ARE COOOOOL MAN!

2.3 Proper Discontinuous Group Actions

Geometry and groups episode three, the revenge of the fundamental domain.

2.3.1 Fuchsian grouppos

Empty something.

Chapter 3

Calculus on Riemann Surfaces

3.1 Introduction

To gain a deeper understanding of Riemann Surfaces from a more analytical and geometric perspective, we wish to introduce notions from calculus to such surfaces. This involves defining the notions of integration, differential forms and vector fields on Riemann Surfaces. As we will see however, these notions give rise to certain groups and vector spaces which will be very important in the following sections, especially in the proof of the uniformisation theorem.

3.2 The Tangent and Cotangent Spaces

We begin our study of calculus on Riemann surfaces by defining notions of the tangent and cotangent spaces. The following definitions may be found in Chapter 5 of [1] unless otherwise noted.

Definition 3.2.1 (Smooth path on a Riemann Surface)**.** Let X be a Riemann Surface and let $\epsilon > 0$. Then we say $\gamma : (-\epsilon, \epsilon) \to X$ is a smooth path on X if we can define $\frac{d\gamma}{dt}$ for all $t \in (-\epsilon, \epsilon)$.

Definition 3.2.2 (Tangent Space of a Riemann Surface)**.** Let X be a Riemann surface. Let $p \in X$ be a point on the Riemann surface. We define T_pX, the tangent space of X at p, to be the space of equivalence classes of smooth paths γ through p such that $\gamma(0) = p$. Two paths γ_1, γ_2 are said to be equivalent if $\frac{d\gamma_1}{dt} = \frac{d\gamma_2}{dt}$.

Definition 3.2.3 (Cotangent Space of a Riemann Surface)**.** Let X be a Riemann surface. For all $p \in X$ we define the real cotangent space at p to be $T_p^*X = Hom_{\mathbb{R}}(T_pX, \mathbb{R})$ and the complex cotangent space at p to be $T_p^*X^{\mathbb{C}} = Hom_{\mathbb{R}}(T_pX, \mathbb{C})$.

Chapter 4

The Uniformisation Theorem

4.1 Compact Riemann Surfaces

4.1.1 Introduction

We begin this section by introducing some details about the main steps involved in our approach to understanding the uniformisation theorem for Riemann surfaces. In particular there is a single equation on which our approach hinges. This equation, though abstract, gives a powerful result for Riemann Surfaces. We will also investigate this result by looking at a particular Physical interpretation to try and give us some intuition into why such a statement might hold true.

4.1.2 The Main Theorem for Compact Riemann Surfaces

We begin by stating the main theorem for compact Riemann surfaces. This naming is non-standard though is adhered to by [1].

Theorem 4.1.1 (The main theorem for compact Riemann surfaces)**.**
Let X be a compact, closed Riemann Surface with $\rho \in \Omega^2(X)$. Then we have that $\Delta\phi = \rho \Leftrightarrow \int_X \rho = 0$, where $\phi \in \Omega^0(X)$ and ϕ is unique up to addition by a constant.

We note that our solution to the equation is a real valued function and our 2-form is also similarly identified with a real valued function. We do so because if we can prove the existance of a solution for real- valued functions and forms, then we can construct a complex-valued solution by taking a linear combination of two real-valued functions. Similarly, we can identify real 1-forms with $(1, 0)$-forms by mapping a real

Whilst seemingly complicated, this theorem says three things:

1. If ϕ is a solution to the equation $\Delta\phi = \rho$ then $\int_X \rho = 0$.
2. If ϕ is a solution of the equation, then adding a constant to ϕ does not change ρ
3. If $\rho \in \Omega^2(X)$ such that $\int_X \rho = 0$ then there exists a $\phi \in \Omega^0(X)$ such that $\Delta\phi = \rho$.

Points one and two are easy to prove and we will do so below:

Proof.
We start by proving point 1.
Let $\phi \in \Omega^0(X)$. Recall that $\Delta = 2i\bar{\partial}\partial$, so applying this directly to ϕ, yields a 2-form $\rho = 2i\frac{\partial^2\phi}{\partial\bar{z}\partial z} d\bar{z} \wedge dz = 2i\bar{\partial}\partial\phi$. Noting that $d(\partial\phi) = \partial^2\phi + \bar{\partial}\partial\phi = \bar{\partial}\partial\phi$, we can rewrite ρ as $\rho = 2id(\partial\phi)$ as $\partial^2 = \bar{\partial}^2 = 0$.

Integrating ρ over X, gives $\int_X \rho = 2i \int_X \bar{\partial}\partial\phi = 2i \int_X d(\partial\phi) = 2i \int_{\partial X} \partial\phi$, where we have changed our domain of integration from X to its boundary ∂X by Stokes' theorem. However, a closed, compact Riemann surface has no boundary, so such an integral would be over an empty set and as such $\int_X \rho = 2i \int_X d(\partial\phi) = 2i \int_{\partial X} \partial\phi = 0$.

The proof of point 2 is similarly elementary.
Let ϕ and ψ be such that $\Delta\phi = \rho$ and $\Delta\psi = \rho$. Letting $\theta = \psi - \phi$ allows us to write this as $\Delta\theta = 0$, implying that θ is a harmonic function.

Since $\psi, \phi \in \Omega^0(X) = C^\infty(X, \mathbb{R})$, θ is a real- valued function. Recall, that we also defined the notion of the Dirichlet norm. We can use the fact that it is a norm to show that θ is a constant.
Consider $||\theta||_D = ||d\theta|| = 2\int_X (\frac{\partial\theta}{\partial z})^2 dx \wedge dy = \int_X \theta\Delta\theta = 0$. Hence, $d\theta = 0$. Since θ is real-valued, this implies that $\theta = constant$.
This concludes our proof of points 1 and 2. □

Whilst points 1 and 2 were quite easy to prove, it is the final point to which we now turn our attention to. To try and understand why this, seemingly abstract, result is true, we momentarily turn to the theory of Electromagnetism. Maxwells first equation posits that for a region of space Ω with boundary, $\int_\Omega \rho dV = \int_{\partial\Omega} \nabla \cdot E dS$. We see this as a consequence of Stokes theorem.

References

[1] S. Donaldson, Riemann Surfaces

[2] J. Smith. *My favourite Theorems*, Madeup University Press (2026).

[3] MacTutor History of Mathematics Archive, at https://mathshistory.st-andrews.ac.uk/ [accessed 9 May 2030]