

A SURVEY OF RIEMANN SURFACES AND THEIR CLASSIFICATION

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Abstract

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ABSTRACT OF DISSERTATION submitted by **Y. Buzoku** and entitled **A survey of Riemann surfaces and their classification.**

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Declaration

This dissertation is submitted under the regulations of Birkbeck, University of London as part of the examination requirements for the M.Sc. degree in Mathematics. Any quotation or excerpt from the published or unpublished work of other persons is explicitly indicated and in each such instance a full reference of the source of such work is given. I have read and understood the Birkbeck College guidelines on plagiarism and in accordance with those requirements submit this work as my own.

I have found a beautiful proof, but this abstract is too short to contain it.

Chapter 1

Introduction

The aim of this dissertation is to investigate and gain a deeper understanding of Riemann Surfaces, their construction, their classification and an analysis of the geometry of some classes of these surfaces. Doing so will bring together ideas from different, seemingly unrelated fields of mathematics. Often times, Riemann Surfaces are studied solely due to these connections as they might not occur in higher dimensional analogues. We begin this dissertation with an introduction to Riemann surfaces as classical complex-analytical manifolds, and then showing that we can define them equivalently as algebraic varieties of complex curves. We then study briefly the automorphisms of a particular set of such Riemann Surfaces, and give a short review of topological notions that will be important in our classification later, including quotient spaces. We then show how can endow a Riemann surface structure on such spaces, and give examples of spaces arising in such a way, and their lifts to their universal covers.

From there we begin to classify Riemann surfaces by stating the uniformisation theorem, and classifying Riemann surfaces with universal cover the Riemann sphere. We then introduce Teichmüller spaces and discuss Riemann surfaces with the complex

plane as universal cover.

The remainder of this dissertation will then focus on the infinite family of Riemann surfaces with the hyperbolic plane as universal cover. A discussion of the geometry of some compact Riemann surfaces follows, with some anecdotal but interesting ideas introduced, such as the interplay between cubic graphs and Riemann surfaces, naturally arising hyperbolic Teichmüller spaces, and Hubers Theorem relating the length spectrum of hyperbolic Riemann surfaces and the eigenvalues of the Laplacian on said surfaces. We conclude with a discussion of the heat kernel on the sphere, plane and hyperbolic plane and prove the uniformisation theorem for compact Riemann surfaces and the claim that there exist exactly two non-compact simply connected Riemann surfaces.

Chapter 2

Riemann Surfaces

This chapter shows the basic definitions from Topology, (Complex) Analysis and Riemann Surface Theory to define Riemann Surfaces, their mappings and boilerplate stuff. Discuss what is needed and constructions in detail. I guess weird test yadda yadda

2.1 Basic Definitions

We begin by stating the basic definitions of Riemann Surfaces and the analogs of some important notions from Complex Analysis in Riemann Surface Theory.

Definition 2.1.1 (Riemann Surface). A Hausdorff topological space X is said to be a Riemann Surface if:

- There exists a collection of open sets $U_\alpha \subset X$, where α ranges over some index set such that $\bigcup_{\alpha} U_\alpha$ cover X .
- There exists for each α , a homeomorphism, called a chart map, $\psi_\alpha: U_\alpha \rightarrow \tilde{U}_\alpha$,

where \tilde{U}_α is an open set in \mathbb{C} , with the property that for all α, β , the composite map $\psi_\alpha \circ \psi_\beta^{-1}$ is holomorphic on its domain of definition. (These composite maps are sometimes called transition maps).

We call the triple $(\{U_\alpha\}, \{\tilde{U}_\alpha\}, \{\psi_\alpha\})$ an atlas of charts for the Riemann Surface X , though we also use the common notation (U, \tilde{U}, ψ) to denote this, where $U = \{U_\alpha\}$, $\tilde{U} = \{\tilde{U}_\alpha\}$ and $\psi = \{\psi_\alpha\}$.

2.2 Algebraic Curves

ALGBRAIC CURVES ARE COOOOOL MAN!

2.3 Proper Discontinuous Group Actions

Geometry and groups episode three, the revenge of the fundamental domain.

2.3.1 Fuchsian groupoids

Empty something.

Chapter 3

Calculus on Riemann Surfaces

3.1 Introduction

To gain a deeper understanding of Riemann Surfaces from a more analytical and geometric perspective, we wish to introduce notions from calculus to such surfaces. This involves defining the notions of integration, differential forms and vector fields on Riemann Surfaces. As we will see however, these notions give rise to certain groups and vector spaces which will be very important in the following sections, especially in the proof of the uniformisation theorem.

3.2 The Tangent and Cotangent Spaces

We begin our study of calculus on Riemann surfaces by defining notions of the tangent and cotangent spaces. The following definitions may be found in Chapter 5 of [1] unless otherwise noted.

Definition 3.2.1 (Smooth path on a Riemann Surface). Let X be a Riemann Surface

and let $\epsilon > 0$. Then we say $\gamma : (-\epsilon, \epsilon) \rightarrow X$ is a smooth path on X if we can define $\frac{d\gamma}{dt}$ for all $t \in (-\epsilon, \epsilon)$.

Definition 3.2.2 (Tangent Space of a Riemann Surface). Let X be a Riemann surface. Let $p \in X$ be a point on the Riemann surface. We define $T_p X$, the tangent space of X at p , to be the space of equivalence classes of smooth paths γ through p such that $\gamma(0) = p$. Two paths γ_1, γ_2 are said to be equivalent if $\frac{d\gamma_1}{dt} = \frac{d\gamma_2}{dt}$.

Definition 3.2.3 (Cotangent Space of a Riemann Surface). Let X be a Riemann surface. For all $p \in X$ we define the real cotangent space at p to be $T_p^* X = \text{Hom}_{\mathbb{R}}(T_p X, \mathbb{R})$ and the complex cotangent space at p to be $T_p^* X^{\mathbb{C}} = \text{Hom}_{\mathbb{R}}(T_p X, \mathbb{C})$.

Chapter 4

The Uniformisation Theorem

4.1 Preliminaries

4.1.1 The Δ operator and Harmonic functions

We begin our treatment of the uniformisation theorem, with a discussion of a particular and extremely important differential operator on Riemann Surfaces; the Laplacian. The majority of definitions can be found in [1] unless explicitly stated.

Definition 4.1.1 (The Laplacian). Let X be a Riemann Surface. We define a linear map $\Delta : \Omega^0(X) \rightarrow \Omega^2(X)$, where $\Delta = 2i\bar{\partial}\partial$. We call this linear map the Laplacian.

Note that though, we have said that Δ is a map, it is a map from a space of functions to a space of functions, and hence is technically an operator; in particular, since it defines a differential equation in local coordinates, it is a differential operator and hence will be referred to as so.

In local co-ordinates, Δ simply recovers the expected Laplacian of a function since

for a given function $f \in \Omega^0(X)$ we get that

$$\begin{aligned}\Delta f &= 2i\bar{\partial}\partial f = 2i\left(\frac{\partial}{\partial\bar{z}}\right)\left(\frac{\partial}{\partial z}\right)f(d\bar{z} \wedge dz) \\ &= \frac{i}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right)f(2i dx \wedge dy) \\ &= -\left(\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2}\right)dx \wedge dy\end{aligned}$$

which when we equate $dx \wedge dy$ with the usual area from $dx dy$ becomes the usual Laplacian from vector calculus, up to a minus sign. We usually do this so we can identify functions with two-forms. For example, when given a function $f(x, y)$, it has an associated two-form $\rho = f(x, y)dx \wedge dy$, which can be integrated. Though we can define a notion of the Laplacian without the minus sign, as we will shortly see, that minus sign is important to our setup for the proof of the uniformisation theorem. Hence, we will leave it so.

A special class of function that is also important to our theory is that of the Harmonic functions. They play a crucial role in studying the Laplacian operator generally and have a number of very nice properties that we will be exploiting in the coming section.

Definition 4.1.2 (Harmonic function). Let X be a Riemann surface. A function $f \in \Omega^0(X)$ is said to be harmonic if $\Delta f = 0$. A function $g : X \rightarrow \mathbb{C}$ is said to be harmonic if both its real and imaginary parts are real valued harmonic functions.

We note that we can relax the differentiability condition considerably to make the functions only twice differentiable. Thus harmonic functions need to be at least twice differentiable. Furthermore, the domain of definition is not restricted to just Riemann surfaces but can also be any open subset of \mathbb{R}^n or \mathbb{C}^n for any $n \geq 1$.

Let us give some examples of Harmonic functions.

Example 4.1.3 (Examples of various Harmonic functions on different domains). Over open sets in \mathbb{C} we have:

- Constant functions are trivially harmonic.
- Any holomorphic function is automatically a harmonic function since its real and imaginary parts satisfy the Cauchy-Riemann equations.
- $f(z) = e^z = e^{x+iy} = e^x \sin(y)$

Taking our domain of definition to be open subsets of \mathbb{R}^3 , we have:

- $f(x, y, z) = \frac{1}{\sqrt{x^2+y^2+z^2}}$

Finally, on the punctured plane $\mathbb{C} \setminus \{0\}$, we also have a nice family of examples:

- $f(x + iy) = \ln(x^2 + y^2)$ and hence $g(x + iy) = K \ln(x^2 + y^2)$ for all $K \in \mathbb{R}$.

All of the aforementioned functions satisfy the equation $\Delta f = 0$ on their domain of definition. This concludes the example.

It is enlightening to see the claim that any holomorphic function is harmonic using our language of differential forms.

Lemma 4.1.4. *Holomorphic functions are harmonic functions, ie they have harmonic real and imaginary parts.*

Proof. Let X be a Riemann surface and f a holomorphic function defined on X . Let us consider the Laplacian applied to the sum $\frac{1}{2i}(f \pm \bar{f})$.

$$\frac{1}{2i}\Delta(f \pm \bar{f}) = \bar{\partial}\partial(f \pm \bar{f}) = -\partial(\bar{\partial}f) \pm \bar{\partial}(\overline{\partial f}) = -\partial(0) \pm \bar{\partial}(0) = 0 \pm 0 = 0$$

where both brackets $\bar{\partial}f = 0$ because f is holomorphic. □

Note however, that whilst holomorphic implies harmonic, the converse isn't always true. However we can say the following.

Lemma 4.1.5. *Let X be a Riemann Surface, U an open set around a point $p \in X$ and let $\phi \in \Omega^0(U)$ be a harmonic function. Then there exists an open neighbourhood $V \subset U$ of the point p and a holomorphic function $f: V \rightarrow \mathbb{C}$ with $\phi = \operatorname{Re}(f)$.*

Since this is a local result on X , then the proof follows exactly as in classical complex analysis. However, we wish to showcase how using calculus on Riemann surfaces can help solve problems efficiently and as such we provide a proof using the tools we have established.

Proof. Let $A \in \Omega^1(U)$ be a real one-form such that:

$$\begin{aligned} A &= -\frac{\partial \phi}{\partial y} dx + \frac{\partial \phi}{\partial x} dy \\ &= i\bar{\partial}\phi + \overline{(i\bar{\partial}\phi)} \end{aligned}$$

Since ϕ is harmonic, so $\bar{\partial}\partial\phi = 0$ and $d = \partial + \bar{\partial}$, we get that $dA = 0$. Hence, if V is an open, simply-connected set (such that $H^1(V) = 0$), then we can find a function $\psi \in \Omega^0(V)$ such that $d\psi = A$. By equating coefficients we get that $\partial\psi = -i\bar{\partial}\psi$ and $\bar{\partial}\psi = i\bar{\partial}\phi$. So if we construct a function $f: V \rightarrow \mathbb{C}$ where $f = \phi + i\psi$, we see that $\bar{\partial}f = \bar{\partial}(\phi + i\psi) = \bar{\partial}\phi + i(i\bar{\partial}\phi) = 0$ and hence, f is a holomorphic function whose real part is ϕ . \square

Harmonic functions are clearly quite well behaved and have nice properties that are not generally exhibited by other functions. These properties, such as the maximum principle which we will define and prove below, play a crucial role in our proof of the setup for the uniformisation theorem.

To do this, we need a version of the mean value theorem for harmonic functions.

Lemma 4.1.6 (The Mean Value Theorem for Harmonic Functions). *Let U be a domain in \mathbb{C} around the a point $a \in U$ and let $\phi \in \Omega^0(U)$ be harmonic on U . Let γ be a closed circle of radius $R > 0$ contained within U that encircles the point a . Then, the mean value of ϕ over the set bounded by γ is equal to the value of ϕ at the point a .*

We note that while the statement of this lemma talks about domains of \mathbb{C} , we could equally talk about domains on any Riemann surface as by definition of a Riemann Surface, we have chart maps that map open sets of a Riemann Surface to domains in \mathbb{C} .

Proof. Since ϕ is a real-valued harmonic function, then by lemma 4.1.5 then we have a holomorphic function f such that the real part of f is ϕ . We now consider the statement of Cauchys Integral formula about γ .

$$f(a) = \frac{1}{2\pi i} \int_{\gamma} \frac{f(z)}{z} dz$$

Parameterising by γ gives us that $z = a + Re^{i\theta}$ for $\theta \in [0, 2\pi)$ on γ and that $dz = iRe^{i\theta} d\theta$. Plugging these into the above equation gives us the relation that

$$\begin{aligned} f(a) &= \frac{1}{2\pi i} \int_{\gamma} \frac{f(z)}{z} dz = \frac{1}{2\pi i} \int_0^{2\pi} \frac{f(z)}{Re^{i\theta}} iRe^{i\theta} d\theta \\ &= \frac{1}{2\pi} \int_0^{2\pi} f(z) d\theta \end{aligned}$$

Since ϕ is the real part of f , we can split f into real and imaginary parts with the integrals splitting into real and imaginary parts. Hence, we get that

$$\phi(a) = \frac{1}{2\pi} \int_0^{2\pi} \phi(z) d\theta$$

which implies that the average value of $\phi(z)$ along γ equals $\phi(a)$ giving us our mean value theorem. \square

Now we are in a position to discuss the (strong) maximum principle for harmonic functions. Some authors may call following the strong maximum principle though since we don't need the "weak" maximum principle, we will simply refer to the following as the maximum principle. The following can be found in [2]

Lemma 4.1.7 (The Maximum Principle). *Let X be a Riemann Surface, let U be a domain on X and ϕ be a real-valued harmonic function on U . Then ϕ has no extrema in U . In other words, if for some $w \in U$ we have that $\phi(w) = \max\{\phi(z) : \forall z \in U\}$ then ϕ must be a constant.*

Proof. Let w_0 be the point at which the maximum of ϕ occurs in U . Consider a small circular neighbourhood of w_0 , with w_0 at the centre with the closed neighbourhood bounded by a circle called γ . We pick a point w_1 on the circle γ and suppose that $\phi(w_0) > \phi(w_1)$. Let us connect w_0 and w_1 with a straight line δ so that $\delta : [0, 1] \rightarrow U$ with $\delta(0) = w_0$ and $\delta(1) = w_1$.

Since ϕ is continuous, that implies that for all $t > 0$, we must have that $\phi(\delta(0)) > \phi(\delta(t))$. But this violates lemma 4.1.6, the mean value theorem for harmonic functions, for if this is the case, then we would have

$$\phi(w_0) > \frac{1}{2\pi} \int_0^{2\pi} \phi(\theta) d\theta$$

This clearly cannot be so we must make it so that $\phi(w_0) = \phi(\delta(t))$ for all $t > 0$. So ϕ is constant in this small neighbourhood, which is closed. However, for any point in this neighbourhood, there is an open ball contained in U surrounding it. Hence this neighbourhood is also open. This leads to the conclusion that the set of points maximising ϕ must either be empty, or must be the entire component of the domain containing w_0 . Hence, if ϕ has a maximum in U , ϕ must be constant across the whole of U . □

So we see that harmonic functions are indeed quite special. We now move on to our

next topics of interest.

4.1.2 Hilbert Spaces and the space $C^\infty(X)/\mathbb{R}$

Hilbert spaces are in some ways, the most familiar class of space that one may study. They crop up everywhere in Mathematics and more and more in Theoretical Physics and indeed they play a major part in the proof of the Uniformisation theorem. Specifially, we will want to construct a particular Hilbert space that will contain a function that we will use to prove the Uniformisation theorem. Let us provide some definitions first.

Definition 4.1.8 (Inner Product Space). Let V be a vector space over a field \mathbb{F} and $\langle \cdot, \cdot \rangle: V \times V \rightarrow \mathbb{F}$ be a positive definite sesquilinear form on V . Then the pair $\{V, \langle \cdot, \cdot \rangle\}$ (frequently abbreviated as just V or $V_{\mathbb{F}}$ if the underlying field is known) is called an Inner Product Space or Pre-Hilbert Space. Furthermore, such spaces have a norm associated with them, defined as a function $\|\cdot\|: V \rightarrow \mathbb{F}$ which can be computed by $\|v\| = \sqrt{\langle v, v \rangle}$ for all $v \in V$.

We note that inner product spaces are in fact metric spaces. We note that we can define on any inner product space a metric as follows.

Proposition 4.1.9. *Let V be an inner product space. Then V is a metric space.*

Proof. Let $\{V, \langle \cdot, \cdot \rangle\}$ be an inner product space. Pick two elements of $a, v \in V$. We define the metric $d: V \times V \rightarrow \mathbb{F}$ with the formula $d(a, b) = \|a - b\|$. By the properties of the inner product, we see that $d(a, b)$ is symmetric, and positive definite. The triangle inequality follows by applying the Cauchy-Schwartz inequality. \square

We can now define a Hilbert space.

Definition 4.1.10 (Hilbert Space). An inner product space \mathcal{H} is called a Hilbert Space if it is a complete metric space with a norm induced by the inner product.

We see that Hilbert spaces are very similar to just normal inner product spaces and as such a lot of our examples are familiar vector spaces. Examples include

Example 4.1.11. • \mathbb{R}^n with the usual Euclidean inner product.

However, the fact Hilbert spaces are complete, allows us to find the limits of all Cauchy sequences. This completeness property allows us to generalise our notions of calculus as we have well defined limits.

We are now in a position to introduce a space of critical importance to our study of the laplacian on a Riemann surface. This space is a Hilbert space, though we prove this in steps, first showing it is a vector space, then that given a particular norm, it is an inner product space, concluding with

Definition 4.1.12 (The space $C^\infty(X)/\mathbb{R}$). Let X be a Riemann surface. We define the space $C^\infty(X)/\mathbb{R}$ to be the space of all infinitely differentiable real-valued functions on X quotiented by constant functions. Two functions $\phi, \psi \in C^\infty(X)$ are equivalent if they differ by a constant function.

We can also equip this space with a special inner product.

References

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