A SURVEY OF RIEMANN SURFACES AND THEIR CLASSIFICATION, WITH PARTICULAR FOCUS ON THEIR GEOMETRY

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Abstract

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ABSTRACT OF DISSERTATION submitted by **Y. Buzoku** and entitled **A survey of Riemann surfaces and their classification, with particular focus on their geometry**.

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Declaration

This dissertation is submitted under the regulations of Birkbeck, University of London as part of the examination requirements for the M.Sc. degree in Mathematics. Any quotation or excerpt from the published or unpublished work of other persons is explicitly indicated and in each such instance a full reference of the source of such work is given. I have read and understood the Birkbeck College guidelines on plagiarism and in accordance with those requirements submit this work as my own.

I have found a beautiful proof, but this abstract is too short to contain it.

Chapter 1

Introduction

The aim of this dissertation is to investigate and gain a deeper understanding of Riemann Surfaces, their construction, their classification and an analysis of the geometry of some classes of these surfaces. Doing so will bring together ideas from different, seemingly unrelated fields of mathematics. Often times, Riemann Surfaces are studied solely due to these connections as they might not occur in higher dimensional analogues. We begin this dissertation with an introduction to Riemann surfaces as classical complex-analytical manifolds, and then showing that we can define them equivalently as algebraic varieties of complex curves. We then study briefly the automorphisms of a particular set of such Riemann Surfaces, and give a short review of topological notions that will be important in our classification later, including quotient spaces. We then show how can endow a Riemann surface structure on such spaces, and give examples of spaces arising in such a way, and their lifts to their universal covers.

From there we begin to classify Riemann surfaces by stating the uniformisation theorem, and classifying Riemann surfaces with universal cover the Riemann sphere. We then introduce Teichmüller spaces and discuss Riemann surfaces with the complex plane as universal cover.

The remainder of this dissertation will then focus on the infinite family of Riemann surfaces with the hyperbolic plane as universal cover. A discussion of the geometry of some compact Riemann surfaces follows, with some anecdotal but interesting ideas introduced, such as the interplay between cubic graphs and Riemann surfaces, naturally arising hyperbolic Teichmüller spaces, and Hubers Theorem relating the length spectrum of hyperbolic Riemann surfaces and the eigenvalues of the Laplacian on said surfaces. We conclude with a discussion of the Heat Kernel on the Sphere, Plane and H and prove the uniformisation theorem for compact Riemann surfaces and the claim that there exist exactly two non-compact simply connected Riemann surfaces.

Chapter 2

Riemann Surfaces

This chapter shows the basic definitions from Topology, (Complex) Analysis and Riemann Surface Theory to define Riemann Surfaces, their mappings and boilerplate stuff. Discuss what is needed and constructions in detail. I guess weird test yadda yadda

2.1 Basic Definitions

We begin by stating the basic definitions of Riemann Surfaces and the analogs of some important notions from Complex Analysis in Riemann Surface Theory.

Definition 2.1.1 (Riemann Surface)**.** A Hausdorff topological space X is said to be a Riemann Surface if:

- There exists a collection of open sets $U_\alpha \subset X$, where α ranges over some index set such that $\bigcup\limits_{\alpha} U_\alpha$ cover X.
- There exists for each α, a homeomorphism, called a chart map, $\psi_\alpha \colon U_\alpha \to \tilde{U}_\alpha$,

where $\tilde{U}_\alpha$ is an open set in $\mathbb{C}$, with the property that for all α, β, the composite map $\psi_\alpha \circ \psi_\beta^{-1}$ is holomorphic on its domain of definition. (These composite maps are sometimes called transition maps).

We call the triple $(\{U_\alpha\}, \{\tilde{U}_\alpha\}, \{\psi_\alpha\})$ an atlas of charts for the Riemann Surface X, though we also use the common notation $(U, \tilde{U}, \psi)$ to denote this, where $U = \{U_\alpha\}$, $\tilde{U} = \{\tilde{U}_\alpha\}$ and $\psi = \{\psi_\alpha\}$.

2.2 Algebraic Curves

ALGBRAIC CURVES ARE COOOOOL MAN!

2.3 Proper Discontinuous Group Actions

Geometry and groups episode three, the revenge of the fundamental domain.

2.3.1 Fuchsian grouppos

Empty something.

Example I have a vase; it is a cuboid with a square base. I don't want to paint the base, just the four sides. I can colour each of the sides red, blue, green or yellow. How many different ways can the vase be coloured?

To answer this we let G be the rotation group of the vase, and X be the set of all colourings. Each of the four sides can be any one of four colours, so $|X| = 4^4 = 256$.

The only rotational symmetries of G are those about the vertical axis of symmetry through the vase. If we let α be a rotation of $90°$, then $G = \langle\alpha\rangle = \{1, \alpha, \alpha^2, \alpha^{-1}\}$. Firstly note that $\mathrm{Fix}(1) = X$.

What is $\mathrm{Fix}(\alpha)$? Let's label the faces A, B, C, D and the colours r, g, b, y (for red, green, blue, yellow). Then a string (r, r, g, b) for instance will mean that A is red, B is red, C is green and D is blue. Let $(c_1, c_2, c_3, c_4) \in \mathrm{Fix}(\alpha)$, where $c_i \in \{r, g, b, y\}$. Then $(c_1, c_2, c_3, c_4) = \alpha \cdot (c_1, c_2, c_3, c_4) = (c_2, c_3, c_4, c_1)$. Hence $c_1 = c_2 = c_3 = c_4$. There are four possible such colourings, (r, r, r, r), (g, g, g, g), (b, b, b, b) and (y, y, y, y). So $|\mathrm{Fix}(\alpha)| = 4$. Similarly $|\mathrm{Fix}(\alpha^{-1})| = 4$.

Finally, suppose $(c_1, c_2, c_3, c_4) \in \mathrm{Fix}(\alpha^2)$. This happens precisely when $c_1 = c_3$ and $c_2 = c_4$. There are $4^2 = 16$ such colourings, for example

$$(r, b, r, b), (b, r, b, r), (r, r, r, r), (y, r, y, r)$$

and so on. Hence $|\mathrm{Fix}(\alpha^2)| = 16$. We are now in a position to count the colourings, using the Orbit Counting Lemma. We get

$$\begin{aligned}
\#\text{ colourings} &= \frac{1}{|G|}\sum_{g\in G}|\mathrm{Fix}(g)| \\
&= \tfrac{1}{4}\left(|\mathrm{Fix}(1)| + |\mathrm{Fix}(\alpha)| + |\mathrm{Fix}(\alpha^2)| + |\mathrm{Fix}(\alpha^{-1})|\right) \\
&= \tfrac{1}{4}(256 + 4 + 16 + 4) \\
&= 70.
\end{aligned}$$

Chapter 3

Die Wende

Ich liebe doch alle menschen

References

[1] J. Smith. *My favourite Theorems*, Madeup University Press (2026).

[2] MacTutor History of Mathematics Archive, at https://mathshistory.st-andrews.ac.uk/ [accessed 9 May 2030]