Translations between bases in Base-extension Semantics

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Goals for this talk

- Introduce a more generalised framework of atomic rules and bases.
- Introduce two notions of atomic derivability.
- Show how one may relate these notions of atomic derivability.
- Recall support relations for IPL and ILL.
- Show how one may thus relate these notions of support.

Presentation root directory

Notation

- $\mathbb{A}$ represents a fixed, countably infinite set of propositional atoms.
- Lower case latin letters represent propositional atoms.
- Upper case latin letters represent finite multisets of propositional atoms.
- The sum of two multisets P and Q is denoted $P \fatsemi Q$.

Atomic rules

Atomic rules take the form:

$$(P_1 \Rightarrow p_1), \ldots, (P_n \Rightarrow p_n) \Rightarrow p$$

Atomic rules

Pictorially, we can represent this as:

$$\dfrac{\begin{matrix} [P_1] & & [P_n] \\ p_1 & \dots & p_n \end{matrix}}{p}$$

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or linearly, like this:

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Contextual atomic rules

Contextual atomic rules are rules which may have contextual brackets distributed across them.

Contextual atomic rules

Pictorially, we can represent such rules as:

$$\dfrac{\left\{\begin{matrix}[P_{1_1}] & & [P_{1_{l_n}}] \\ p_{1_1} & \ldots & p_{1_{l_1}}\end{matrix}\right\} \qquad \ldots \qquad \left\{\begin{matrix}[P_{n_1}] & & [P_{n_{l_n}}] \\ p_{n_1} & \ldots & p_{n_{l_n}}\end{matrix}\right\}}{p}$$

Contextual atomic rules

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or linearly, like this:

$$\{(P_{1_i} \Rightarrow p_{1_i})\}_{i=1}^{l_1}, \dots, \{(P_{n_i} \Rightarrow p_{n_i})\}_{i=1}^{l_n} \Rightarrow p$$

Examples of atomic rules

- $\{\Rightarrow a\} \Rightarrow c$
- $\{\Rightarrow a\}, \{(c \fatsemi d \Rightarrow e), (f \Rightarrow g)\} \Rightarrow q$
- $\{a \Rightarrow b\}, \{\Rightarrow e\}, (f \Rightarrow g) \Rightarrow q$

- $(\Rightarrow a) \Rightarrow c$
- $(\Rightarrow b) \Rightarrow c$
- $(\Rightarrow c), (a \Rightarrow p), (b \Rightarrow p) \Rightarrow p$

Bases

Definition (Base)

A base is a set of atomic rules.

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Else it is said to be context-free.

- $\{((\Rightarrow a) \Rightarrow c), ((\Rightarrow b) \Rightarrow c), ((\Rightarrow c), (a \Rightarrow p), (b \Rightarrow p) \Rightarrow p)\}$
- $\{(\{\Rightarrow a\} \Rightarrow c), (\{a \Rightarrow c\} \Rightarrow b), (\{\Rightarrow a\}, \{\Rightarrow a\} \Rightarrow a)\}$
- $\{(\{\Rightarrow a\} \Rightarrow c), (\{a \Rightarrow c\} \Rightarrow b), (\{\Rightarrow d\}, \{\Rightarrow d\} \Rightarrow a), (\{\Rightarrow a\} \Rightarrow d)\}$

Presentation root directory

Definition (Context-free atomic derivability $\rhd_{\mathscr{B}}$)

The relation of derivability in a context-free base $\mathscr{B}$, is defined as so:

- Ref $S \rhd_{\mathscr{B}} p$ if $p \in S$.
- App For $((P_1 \Rightarrow p_1), \ldots, (P_n \Rightarrow p_n) \Rightarrow q) \in \mathscr{B}$ and $S \fatsemi P_i \rhd_{\mathscr{B}} p_i$ for each $i \in \{1, \ldots, n\}$ then $S \rhd_{\mathscr{B}} q$.

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Definition (Contextual atomic derivability $\vdash_{\mathscr{B}}$)

The relation of derivability in a contextual base $\mathscr{B}$, is defined as so:

- Ref $p \vdash_{\mathscr{B}} p$.
- App For $(\{(P_{1_i} \Rightarrow p_{1_i})\}_{i=1}^{l_1}, \ldots, \{(P_{n_i} \Rightarrow p_{n_i})\}_{i=1}^{l_n} \Rightarrow q) \in \mathscr{B}$ and $C_i \fatsemi P_{i_j} \vdash_{\mathscr{B}} p_{i_j}$ for each $i \in \{1, \ldots, n\}$ and $j \in \{1, \ldots, l_i\}$ then $C_1 \fatsemi \cdots \fatsemi C_n \vdash_{\mathscr{B}} q$.

Example derivations

Example

Let $\mathscr{B} = \{(\Rightarrow a), ((\Rightarrow a), (\Rightarrow b) \Rightarrow c)\}$

$$\dfrac{\dfrac{}{b \rhd_{\mathscr{B}} b}\,\text{Ref} \qquad \dfrac{}{b \rhd_{\mathscr{B}} a} \Rightarrow a}{b \rhd_{\mathscr{B}} c}\,(\Rightarrow a), (\Rightarrow b) \Rightarrow c$$

Example derivations

Example

Let $\mathscr{B} = \{(\Rightarrow a), (\{\Rightarrow a\}, \{\Rightarrow b\} \Rightarrow c)\}$

$$
\dfrac{\dfrac{}{b \vdash_{\mathscr{B}} b}\,\text{Ref} \qquad \dfrac{}{\vdash_{\mathscr{B}} a} \Rightarrow a}{b \vdash_{\mathscr{B}} c}\,\{\Rightarrow a\}, \{\Rightarrow b\} \Rightarrow c
$$

Example derivations

Example

By the definition of $\vdash_{\mathscr{B}}$, deriving a from $a \,⨾\, a$ in the empty base is not possible, i.e. $a \,⨾\, a \vdash_{\varnothing} a$ is not possible. But $a \,⨾\, a \rhd_{\varnothing} a$ is possible.

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By the definition of $\vdash_{\mathscr{B}}$, deriving a from $a \,⨾\, a$ in the empty base is not possible, i.e. $a \,⨾\, a \vdash_{\varnothing} a$ is not possible. But $a \,⨾\, a \rhd_{\varnothing} a$ is possible.

Example

Let $\mathscr{B} = \{(\{\Rightarrow a\} \Rightarrow c), (\{a \Rightarrow c\} \Rightarrow b)\}$

$$\dfrac{\dfrac{\dfrac{\times}{a \,⨾\, a \vdash_{\mathscr{B}} a}}{a \,⨾\, a \vdash_{\mathscr{B}} c}\{\Rightarrow a\} \Rightarrow c}{a \vdash_{\mathscr{B}} b}\{a \Rightarrow c\} \Rightarrow b$$

We see that in this base, a is not derivable from $a \,⨾\, a$.

Example derivations

Example (A possible fix)

Let $(\mathscr{B})_\star = \{((\Rightarrow a) \Rightarrow c), ((a \Rightarrow c) \Rightarrow b)\}$

$$
\dfrac{\dfrac{\dfrac{}{a \,\fatsemi\, a \rhd_{(\mathscr{B})_\star} a}\ \text{Ref}}{a \,\fatsemi\, a \rhd_{(\mathscr{B})_\star} c}\ (\Rightarrow a) \Rightarrow c}{a \rhd_{(\mathscr{B})_\star} b}\ (a \Rightarrow c) \Rightarrow b
$$

Example derivations

Example (Another possible fix)

Let $\mathscr{C} = \{(\{\Rightarrow a\} \Rightarrow c), (\{a \Rightarrow c\} \Rightarrow b), (\{\Rightarrow a\}, \{\Rightarrow a\} \Rightarrow a)\}$

$$\dfrac{\dfrac{\dfrac{\dfrac{}{a \vdash_{\mathscr{C}} a}\,\text{Ref} \quad \dfrac{}{a \vdash_{\mathscr{C}} a}\,\text{Ref}}{a \,⨾\, a \vdash_{\mathscr{C}} a}\,\{\Rightarrow a\}, \{\Rightarrow a\} \Rightarrow a}{a \,⨾\, a \vdash_{\mathscr{C}} c}\,\{\Rightarrow a\} \Rightarrow c}{a \vdash_{\mathscr{C}} b}\,\{a \Rightarrow c\} \Rightarrow b$$

Example derivations

Example (A very interesting derivation)

Let $\mathscr{B}$ be the following set of rules:

$\{(\{\Rightarrow b\} \Rightarrow c), (\{\Rightarrow a\}, \{b \Rightarrow c\} \Rightarrow d), (\{\Rightarrow d, \Rightarrow a\} \Rightarrow e), (\{a \Rightarrow e\} \Rightarrow f)\}$

$$
\begin{prooftree}
\AxiomC{}
\RightLabel{Ref}
\UnaryInfC{$a \vdash_{\mathscr{B}} a$}
\AxiomC{}
\RightLabel{Ref}
\UnaryInfC{$b \vdash_{\mathscr{B}} b$}
\RightLabel{$\{\Rightarrow b\} \Rightarrow c$}
\UnaryInfC{$b \vdash_{\mathscr{B}} c$}
\RightLabel{$\{\Rightarrow a\}, \{b \Rightarrow c\} \Rightarrow d$}
\BinaryInfC{$a \vdash_{\mathscr{B}} d$}
\AxiomC{}
\RightLabel{Ref}
\UnaryInfC{$a \vdash_{\mathscr{B}} a$}
\RightLabel{$\{\Rightarrow d, \Rightarrow a\} \Rightarrow e$}
\BinaryInfC{$a \vdash_{\mathscr{B}} e$}
\RightLabel{$\{a \Rightarrow e\} \Rightarrow f$}
\UnaryInfC{$\vdash_{\mathscr{B}} f$}
\end{prooftree}
$$

Presentation root directory

Comparing relations

Definition (Structural rules)

We define two rules:

- $\mathsf{Wk}^p_q = (\{\Rightarrow p\}, \{\Rightarrow q\} \Rightarrow q)$
- $\mathsf{Ctn}^p_q = (\{\Rightarrow p\}, \{p \mathbin{,} p \Rightarrow q\} \Rightarrow q)$

Mappings between bases

Mappings between bases

Definition

Let $\mathscr{B}$ be a context-free base. We define structural contextualisation of that base $(\mathscr{B})^\star$ as follows:

$$
\begin{aligned}
(\mathscr{B})^\star = \{&\{P_1 \Rightarrow p_1\}, \ldots, \{P_n \Rightarrow p_n\} \Rightarrow q \mid ((P_1 \Rightarrow p_1), \ldots, (P_n \Rightarrow p_n) \Rightarrow q) \in \mathscr{B}\} \\
&\cup \{\mathsf{Wk}^p_q, \mathsf{Ctn}^p_q \mid \forall p, q \in \mathbb{A}\}
\end{aligned}
$$

Mappings between bases

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$$(\mathscr{B})^\star = \{\{P_1 \Rightarrow p_1\}, \ldots, \{P_n \Rightarrow p_n\} \Rightarrow q \mid ((P_1 \Rightarrow p_1), \ldots, (P_n \Rightarrow p_n) \Rightarrow q) \in \mathscr{B}\}$$
$$\cup \{\mathsf{Wk}_q^p, \mathsf{Ctn}_q^p \mid \forall p, q \in \mathbb{A}\}$$

We define the decontextualisation of a contextual base as:

$$(\mathscr{B})_\star = \{(P_{1_1} \Rightarrow p_{1_1}), \ldots, (P_{n_{l_n}} \Rightarrow p_{n_{l_n}}) \Rightarrow q \mid (\{P_{1_i} \Rightarrow p_{1_i}\}_{i=1}^{l_1}, \ldots, \{P_{n_i} \Rightarrow p_{n_i}\}_{i=1}^{l_n} \Rightarrow q) \in \mathscr{B}\}$$

Properties of these mappings

Let $\mathscr{B}$ be a context-free base. Then the following hold:

- $((\mathscr{B})^{\star})_{\star} \supseteq \mathscr{B}$

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Let $\mathscr{B}$ be a context-free base. Then the following hold:

- $((\mathscr{B})^{\star})_{\star} \supseteq \mathscr{B}$
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Properties of these mappings

Let $\mathscr{B}$ be a context-free base. Then the following hold:

- $((\mathscr{B})^{\star})_{\star} \supseteq \mathscr{B}$
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- For all $\mathscr{C} \supseteq \mathscr{B}$ we have that $(\mathscr{C})^{\star} \supseteq (\mathscr{B})^{\star}$

Properties of these mappings

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- $((\mathscr{B})^{\star})_{\star} \supseteq \mathscr{B}$
- $(((\mathscr{B})^{\star})_{\star})^{\star} = (\mathscr{B})^{\star}$
- For all $\mathscr{C} \supseteq \mathscr{B}$ we have that $(\mathscr{C})^{\star} \supseteq (\mathscr{B})^{\star}$
- For all $\mathscr{C} \supseteq (\mathscr{B})^{\star}$ there exists an extension $\mathscr{X} \supseteq \mathscr{B}$ such that $(\mathscr{X})^{\star} = \mathscr{C}$

Properties of these mappings

Let $\mathscr{B}$ be a context-free base. Then the following hold:

- $((\mathscr{B})^\star)_\star \supseteq \mathscr{B}$
- $(((\mathscr{B})^\star)_\star)^\star = (\mathscr{B})^\star$
- For all $\mathscr{C} \supseteq \mathscr{B}$ we have that $(\mathscr{C})^\star \supseteq (\mathscr{B})^\star$
- For all $\mathscr{C} \supseteq (\mathscr{B})^\star$ there exists an extension $\mathscr{X} \supseteq \mathscr{B}$ such that $(\mathscr{X})^\star = \mathscr{C}$
- For all $\mathscr{C} \supseteq (\mathscr{B})^\star$, it follows immediately that $\mathscr{C} = ((\mathscr{C})_\star)^\star$.

Structural admissibility in a structurally contextualised base

Lemma

Suppose $L \vdash_{(\mathscr{B})^\star} p$ holds. Then $R \fatsemi L \vdash_{(\mathscr{B})^\star} p$ also holds for any atomic multiset R.

Proof.

Let $R = \{r_1, \ldots, r_n\}$ for some n. Then we have that we can effectively weaken S away as follows:

$$
\dfrac{\dfrac{}{r_1 \vdash_{(\mathscr{B})^\star} r_1}\,\text{Ref} \qquad \dfrac{\dfrac{\dfrac{}{r_{n-1} \vdash_{(\mathscr{B})^\star} r_{n-1}}\,\text{Ref} \qquad \dfrac{\dfrac{}{r_n \vdash_{(\mathscr{B})^\star} r_n}\,\text{Ref} \qquad L \vdash_{(\mathscr{B})^\star} p}{r_n \,⨾\, L \vdash_{(\mathscr{B})^\star} p}\,\text{Wk}_p^{r_n}}{\vdots}\,\text{Wk}_p^{r_{n-1}}}{r_2 \,⨾\, \cdots \,⨾\, r_n \,⨾\, L \vdash_{(\mathscr{B})^\star} p}}{R \,⨾\, L \vdash_{(\mathscr{B})^\star} p}\,\text{Wk}_p^{r_1}
$$

$\square$

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Structural admissibility in a structurally contextualised base

Lemma

Suppose $R \fatsemi R \vdash_{(\mathscr{B})^\star} p$ *holds. Then* $R \vdash_{(\mathscr{B})^\star} p$ *also holds.*

Corollary

For arbitrary $m \geqslant 1$, *if* $R^m \vdash_{(\mathscr{B})^\star} p$ *then* $R \vdash_{(\mathscr{B})^\star} p$.

Proof.

Let $R = \{r_1, \dots, r_n\}$ for some n. Then we have that we can effectively contract on R as follows:

$$
\dfrac{
\dfrac{}{r_1 \vdash_{(\mathscr{B})^\star} r_1}\,\text{Ref}
\qquad
\begin{array}{c}
\dfrac{
\dfrac{}{r_{n-1} \vdash_{(\mathscr{B})^\star} r_{n-1}}\,\text{Ref}
\qquad
\dfrac{
\dfrac{}{r_n \vdash_{(\mathscr{B})^\star} r_n}\,\text{Ref}
\qquad
R \,, R \vdash_{(\mathscr{B})^\star} p
}{r_1 \,, \cdots \,, r_{n-1} \,, R \vdash_{(\mathscr{B})^\star} p}\,\text{Ctn}_p^{r_n}
}{}\,\text{Ctn}_p^{r_{n-1}} \\
\vdots \\
r_1 \,, R \vdash_{(\mathscr{B})^\star} p
\end{array}
}{R \vdash_{(\mathscr{B})^\star} p}\,\text{Ctn}_p^{r_1}
$$

$\square$

Key results under this base translation

- If $R \rhd_{\mathscr{B}} p$ then $R \vdash_{(\mathscr{B})^\star} p$.

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- If $R \rhd_{\mathscr{B}} p$ then $R \vdash_{(\mathscr{B})^\star} p$.
- $R \vdash_{(\mathscr{B})^\star} p$ iff for all bases $\mathscr{X} \supseteq (\mathscr{B})^\star$ where for each $r \in R$ we have $\vdash_{\mathscr{X}} r$ then it follows that $\vdash_{\mathscr{X}} p$.

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- $R \vdash_{(\mathscr{B})^\star} p$ iff $! R \Vdash^{\varnothing}_{(\mathscr{B})^\star} p$

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- $R \vdash_{(\mathscr{B})^\star} p$ iff $!\, R \Vdash^{\varnothing}_{(\mathscr{B})^\star} p$
- $\Vdash^{R}_{(\mathscr{B})^\star} p$ iff $!\, R \Vdash^{\varnothing}_{(\mathscr{B})^\star} p$

Presentation root directory

BeS for IPL: Summary

- $\Vdash_{\mathscr{B}} p$ iff $\vartriangleright_{\mathscr{B}} p$.
- $\Vdash_{\mathscr{B}} \varphi \wedge \psi$ iff $\Vdash_{\mathscr{B}} \varphi$ and $\Vdash_{\mathscr{B}} \psi$.
- $\Vdash_{\mathscr{B}} \varphi \vee \psi$ iff for all $\mathscr{C} \supseteq \mathscr{B}$, p, if $\varphi \Vdash_{\mathscr{C}} p$ and $\psi \Vdash_{\mathscr{C}} p$ then $\Vdash_{\mathscr{C}} p$.
- $\Vdash_{\mathscr{B}} \varphi \to \psi$ iff $\varphi \Vdash_{\mathscr{B}} \psi$.
- $\Vdash_{\mathscr{B}} \bot$ iff $\Vdash_{\mathscr{B}} p$ for all p.
- $\Vdash_{\mathscr{B}} \top$ iff always.
- $\Gamma \Vdash_{\mathscr{B}} \varphi$ iff for all $\mathscr{C} \supseteq \mathscr{B}$ if $\Vdash_{\mathscr{C}} \gamma$ for all $\gamma \in \Gamma$ then $\Vdash_{\mathscr{C}} \varphi$.
- $\Gamma \Vdash \varphi$ iff $\Gamma \Vdash_{\mathscr{B}} \varphi$ for all $\mathscr{B}$.

BeS for ILL: Structure and the Exponential

- $\Vdash^{L}_{\mathscr{B}} p$ iff $L \vdash_{\mathscr{B}} p$.
- $\Vdash^{L}_{\mathscr{B}} !\,\varphi$ iff
  $\forall \mathscr{C} \supseteq \mathscr{B},\ K,\ p$, if $(\forall \mathscr{D} \supseteq \mathscr{C}\ (\Vdash^{\varnothing}_{\mathscr{D}} \varphi$ implies $\Vdash^{K}_{\mathscr{D}} p)$ then $\Vdash^{L \fatcomma K}_{\mathscr{C}} p)$.
- $\Gamma \Vdash^{L}_{\mathscr{B}} \varphi$ iff
  $\forall \mathscr{C} \supseteq \mathscr{B},\ K$, if $\Vdash^{\varnothing}_{\mathscr{C}} \Delta$ and $\Vdash^{K}_{\mathscr{C}} \Theta$ where $!\,\Delta \fatcomma \Theta = \Gamma$ then $\Vdash^{L \fatcomma K}_{\mathscr{C}} \varphi$.
- $\Gamma \Vdash \varphi$ iff $\Gamma \Vdash^{\varnothing}_{\mathscr{B}} \varphi$ for all $\mathscr{B}$.

BeS for ILL: Multiplicatives

- $\Vdash^{L}_{\mathscr{B}} \varphi \otimes \psi$ iff for all $\mathscr{C} \supseteq \mathscr{B}$, K and p, if $\varphi \, ⨾ \, \psi \Vdash^{K}_{\mathscr{C}} p$ then $\Vdash^{L ⨾ K}_{\mathscr{C}} p$.
- $\Vdash^{L}_{\mathscr{B}} \varphi \multimap \psi$ iff $\varphi \Vdash^{L}_{\mathscr{B}} \psi$.
- $\Vdash^{L}_{\mathscr{B}} 1$ iff for all $\mathscr{C} \supseteq \mathscr{B}$, K and p, if $\Vdash^{K}_{\mathscr{C}} p$ then $\Vdash^{L ⨾ K}_{\mathscr{C}} p$.

BeS for ILL: Additives

- $\Vdash^{L}_{\mathscr{B}} \varphi \,\&\, \psi$ iff $\Vdash^{L}_{\mathscr{B}} \varphi$ and $\Vdash^{L}_{\mathscr{B}} \psi$.
- $\Vdash^{L}_{\mathscr{B}} \varphi \oplus \psi$ iff for all $\mathscr{C} \supseteq \mathscr{B}$, K and p, if $\varphi \Vdash^{K}_{\mathscr{C}} p$ and $\psi \Vdash^{K}_{\mathscr{C}} p$ then $\Vdash^{L,K}_{\mathscr{C}} p$.
- $\Vdash^{L}_{\mathscr{B}} 0$ iff $\Vdash^{L}_{\mathscr{B}} p$ for all p.
- $\Vdash^{L}_{\mathscr{B}} \top$ iff always.

BeS for ILL: Summary

- $\Vdash^{L}_{\mathscr{B}} p$ iff $L \vdash_{\mathscr{B}} p$.
- $\Vdash^{L}_{\mathscr{B}} \varphi \otimes \psi$ iff $\forall \mathscr{C} \supseteq \mathscr{B}, K, p\,(\varphi \,\text{;}\, \psi \Vdash^{K}_{\mathscr{C}} p$ implies $\Vdash^{L \text{;} K}_{\mathscr{C}} p)$.
- $\Vdash^{L}_{\mathscr{B}} \varphi \multimap \psi$ iff $\varphi \Vdash^{L}_{\mathscr{B}} \psi$.
- $\Vdash^{L}_{\mathscr{B}} 1$ iff $\forall \mathscr{C} \supseteq \mathscr{B}, K, p\,(\Vdash^{K}_{\mathscr{C}} p$ implies $\Vdash^{L \text{;} K}_{\mathscr{C}} p)$.
- $\Vdash^{L}_{\mathscr{B}} \varphi \,\&\, \psi$ iff $\Vdash^{L}_{\mathscr{B}} \varphi$ and $\Vdash^{L}_{\mathscr{B}} \psi$.
- $\Vdash^{L}_{\mathscr{B}} \varphi \oplus \psi$ iff $\forall \mathscr{C} \supseteq \mathscr{B}, K, p\,(\varphi \Vdash^{K}_{\mathscr{C}} p$ and $\psi \Vdash^{K}_{\mathscr{C}} p$ implies $\Vdash^{L \text{;} K}_{\mathscr{C}} p)$.
- $\Vdash^{L}_{\mathscr{B}} 0$ iff $\Vdash^{L}_{\mathscr{B}} p$ for all p.
- $\Vdash^{L}_{\mathscr{B}} \top$ iff always.
- $\Vdash^{L}_{\mathscr{B}} !\,\varphi$ iff
  $\forall \mathscr{C} \supseteq \mathscr{B}, K, p\,(\forall \mathscr{D} \supseteq \mathscr{C}\,(\Vdash^{\varnothing}_{\mathscr{D}} \varphi$ implies $\Vdash^{K}_{\mathscr{D}} p)$ implies $\Vdash^{L \text{;} K}_{\mathscr{C}} p)$.
- $\Gamma \Vdash^{L}_{\mathscr{B}} \varphi$ iff
  $\forall \mathscr{C} \supseteq \mathscr{B}, K, \Vdash^{\varnothing}_{\mathscr{C}} \delta$ and $\Vdash^{K}_{\mathscr{C}} \gamma$ for $!\,\delta \in \Gamma$ and $\gamma \in \Gamma$ then $\Vdash^{L \text{;} K}_{\mathscr{C}} \varphi$.
- $\Gamma \Vdash \varphi$ iff $\Gamma \Vdash^{\varnothing}_{\mathscr{B}} \varphi$ for all $\mathscr{B}$.

Presentation root directory

Comparing clauses

(IPL) $\Vdash_{\mathscr{B}} p$ iff $\rhd_{\mathscr{B}} p$.

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(IPL) $\Vdash_{\mathscr{B}} \varphi \vee \psi$ iff for all $\mathscr{C} \supseteq \mathscr{B}$, p, if $\varphi \Vdash_{\mathscr{C}} p$ and $\psi \Vdash_{\mathscr{C}} p$ then $\Vdash_{\mathscr{C}} p$.

(ILL) $\Vdash^{L}_{\mathscr{B}} \varphi \oplus \psi$ iff for all $\mathscr{C} \supseteq \mathscr{B}$, K, p, if $\varphi \Vdash^{K}_{\mathscr{C}} p$ and $\psi \Vdash^{K}_{\mathscr{C}} p$ then $\Vdash^{L,K}_{\mathscr{C}} p$.

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(IPL) $\Vdash_{\mathscr{B}} \varphi \to \psi$ iff $\varphi \Vdash_{\mathscr{B}} \psi$.

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(IPL) $\Vdash_{\mathscr{B}} \varphi \vee \psi$ iff for all $\mathscr{C} \supseteq \mathscr{B}$, p, if $\varphi \Vdash_{\mathscr{C}} p$ and $\psi \Vdash_{\mathscr{C}} p$ then $\Vdash_{\mathscr{C}} p$.

(ILL) $\Vdash^{L}_{\mathscr{B}} \varphi \oplus \psi$ iff for all $\mathscr{C} \supseteq \mathscr{B}$, K, p, if $\varphi \Vdash^{K}_{\mathscr{C}} p$ and $\psi \Vdash^{K}_{\mathscr{C}} p$ then $\Vdash^{L,K}_{\mathscr{C}} p$.

(IPL) $\Vdash_{\mathscr{B}} \varphi \to \psi$ iff $\varphi \Vdash_{\mathscr{B}} \psi$.

(ILL) $\Vdash^{L}_{\mathscr{B}} \varphi \multimap \psi$ iff $\varphi \Vdash^{L}_{\mathscr{B}} \psi$.

(IPL) $\Vdash_{\mathscr{B}} \bot$ iff $\Vdash_{\mathscr{B}} p$ for all p.

(ILL) $\Vdash^{L}_{\mathscr{B}} 0$ iff $\Vdash^{L}_{\mathscr{B}} p$ for all p.

Translation of formulas

We seem to have a corrspondence between the clauses for the following pairs of connectives:

- $\wedge$ and $\&$
- $\vee$ and $\oplus$
- $\rightarrow$ and $\multimap$
- $\bot$ and 0

Translation of formulas

One possible translation of formulas between IPL and ILL is:

- $[\![p]\!] \equiv p$, where p is an atom.
- $[\![\varphi \wedge \psi]\!] \equiv [\![\varphi]\!] \,\&\, [\![\psi]\!]$
- $[\![\varphi \vee \psi]\!] \equiv ![\![\varphi]\!] \oplus ![\![\psi]\!]$
- $[\![\varphi \to \psi]\!] \equiv ![\![\varphi]\!] \multimap [\![\psi]\!]$
- $[\![\bot]\!] \equiv 0$

Translation of formulas

(IPL) $\Vdash_{\mathscr{B}} p$ iff $\rhd_{\mathscr{B}} p$.

(ILL) $\Vdash^{L}_{\mathscr{B}} p$ iff $L \vdash_{\mathscr{B}} p$.

(IPL) $\Vdash_{\mathscr{B}} \varphi \wedge \psi$ iff $\Vdash_{\mathscr{B}} \varphi$ and $\Vdash_{\mathscr{B}} \psi$.

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Sequents are translated as follows:

- If $\Gamma \vdash_{\mathrm{IPL}} \varphi$ then $![\![\Gamma]\!] \vdash_{\mathrm{ILL}} [\![\varphi]\!]$

Translation of formulas

One possible translation of formulas between IPL and ILL is:

- $[\![p]\!] \equiv p$, where p is an atom.
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- $[\![\bot]\!] \equiv 0$

Sequents are translated as follows:

- If $\Gamma \vdash_{\mathsf{IPL}} \varphi$ then $![\![\Gamma]\!] \vdash_{\mathsf{ILL}} [\![\varphi]\!]$

Semantically we should therefore have:

- If $\Gamma \Vvdash \varphi$ then $![\![\Gamma]\!] \Vdash [\![\varphi]\!]$

Immediate correspondence between bases

$$\Gamma \vdash_{\mathsf{IPL}} \varphi \quad \text{iff } \Gamma \Vdash \varphi \quad \text{iff for all } \mathscr{B}, p, \text{ if } \Vdash_{\mathscr{B}} \gamma \text{ then } \Vdash_{\mathscr{B}} \varphi$$

$$!\llbracket\Gamma\rrbracket \vdash_{\mathsf{ILL}} \llbracket\varphi\rrbracket \quad \text{iff } !\llbracket\Gamma\rrbracket \Vdash \llbracket\varphi\rrbracket \quad \text{iff for all } \mathscr{C}, p, \text{ if } \Vdash_{\mathscr{C}}^{\varnothing} \llbracket\gamma\rrbracket \text{ then } \Vdash_{\mathscr{C}}^{\varnothing} \llbracket\varphi\rrbracket$$

Immediate correspondence between bases

$$\Gamma \vdash_{\mathsf{IPL}} \varphi \quad \text{iff } \Gamma \Vdash \varphi \quad \text{iff for all } \mathscr{B}, p, \text{ if } \Vdash_{\mathscr{B}} \gamma \text{ then } \Vdash_{\mathscr{B}} \varphi$$

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Since we have that $\Gamma \vdash_{\mathsf{IPL}} \varphi$ implies $!\llbracket\Gamma\rrbracket \vdash_{\mathsf{ILL}} \llbracket\varphi\rrbracket$, then we have a mapping between bases $\mathscr{B}$ and bases $\mathscr{C}$.

But what about if we have $\Gamma \Vdash_{\mathscr{B}} \varphi$?

But what about if we have $\Gamma \Vvdash_{\mathscr{B}} \varphi$?

Theorem

If $\Gamma \Vvdash_{\mathscr{B}} \varphi$ *then* $![\![\Gamma]\!] \Vdash^{\varnothing}_{(\mathscr{B})^{\star}} [\![\varphi]\!]$

Thank you!

Thank you for listening!
Comments? Observations? Please ask and/or feel free to email me at y.buzoku@ucl.ac.uk

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