Translations between bases in Base-extension Semantics

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Goals for this talk

- Introduce the concept of bases and general inference figures
- Introduce two notions of atomic derivability
- Show how one may relate these notions of atomic derivability

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Notation

- $\mathbb{A}$ represents a fixed, countably infinite set of propositional atoms.
- Lower case latin letters represent propositional atoms.
- Upper case latin letters represent finite multisets of propositional atoms.
- Atomic multiset is taken to mean multiset of propositional atoms.
- The sum of two multisets P and Q is denoted $P \fatsemi Q$.

Atomic rules

Atomic rules take the form:

$$\langle\langle P_1, p_1\rangle, \ldots, \langle P_n, p_n\rangle, p\rangle$$

Atomic rules

Pictorially, we can represent this as:

$$\dfrac{\begin{array}{c}[P_1] \\ p_1\end{array} \qquad \dots \qquad \begin{array}{c}[P_n] \\ p_n\end{array}}{p}$$

Atomic rules

Pictorially, we can represent this as:

$$\dfrac{\begin{matrix}[P_1] & & [P_n] \\ p_1 & \ldots & p_n\end{matrix}}{p}$$

Any idea what these figures are supposed to look like?

Atomic rules

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Any idea what these figures are supposed to look like?
Natural deduction!

Contextual atomic rules

Contextual atomic rules are rules which may have contextual brackets distributed across them.

Contextual atomic rules

Pictorially, we can represent this as:

$$\frac{\left\{\begin{matrix} [P_{1_1}] & & [P_{1_{l_n}}] \\ p_{1_1} & \ldots & p_{1_{l_1}} \end{matrix}\right\} \qquad \ldots \qquad \left\{\begin{matrix} [P_{n_1}] & & [P_{n_{l_n}}] \\ p_{n_1} & \ldots & p_{n_{l_n}} \end{matrix}\right\}}{p}$$

Contextual atomic rules

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or like this:

$$\langle \{\langle P_{1_i}, p_{1_i}\rangle\}_{i=1}^{l_1}, \dots, \{\langle P_{n_i}, p_{n_i}\rangle\}_{i=1}^{l_n}, p\rangle$$

Examples of atomic rules

- $\langle\{\langle\varnothing, a\rangle\}, c\rangle$
- $\langle\{\langle\varnothing, a\rangle\}, \{\langle c, d, e\rangle, \langle f, g\rangle\}, q\rangle$
- $\langle\{\langle a, b\rangle\}, \{\langle\varnothing, e\rangle\}, \langle f, g\rangle, q\rangle$
- $\langle\langle\varnothing, a\rangle, c\rangle$
- $\langle\langle\varnothing, b\rangle, c\rangle$
- $\langle\langle\varnothing, c\rangle, \langle a, p\rangle, \langle b, p\rangle, p\rangle$

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Context-free atomic derivability

Definition ($\rhd_{\mathscr{B}}$)

The relation of derivability in a context-free base $\mathscr{B}$, is defined as so:

Ref $S \rhd_{\mathscr{B}} p$ if $p \in S$.

App For $\langle\langle P_1, p_1\rangle, \ldots, \langle P_n, p_n\rangle, q\rangle \in \mathscr{B}$ and $S \fatsemi P_i \rhd_{\mathscr{B}} p_i$ for each $i \in \{1, \ldots, n\}$ then $S \rhd_{\mathscr{B}} q$.

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Definition ($\vdash_{\mathscr{B}}$)

The relation of derivability in a contextual base $\mathscr{B}$, is defined as so:

- Ref $p \vdash_{\mathscr{B}} p$.
- App For $\langle\{\langle P_{1_i}, p_{1_i}\rangle\}_{i=1}^{l_1}, \ldots, \{\langle P_{n_i}, p_{n_i}\rangle\}_{i=1}^{l_n}, q\rangle \in \mathscr{B}$ and $C_i \fatsemi P_{i_j} \vdash_{\mathscr{B}} p_{i_j}$ for each $i \in \{1, \ldots, n\}$ and $j \in \{1, \ldots, l_i\}$ then $C_1 \fatsemi \cdots \fatsemi C_n \vdash_{\mathscr{B}} q$.

Example derivations

Example (Derivation terminations)

Let $\mathscr{B} = \{(\Rightarrow a), (\{\Rightarrow a\}, \{\Rightarrow b\} \Rightarrow c)\}$

$$\dfrac{\dfrac{}{b \vdash_{\mathscr{B}} b}\,\mathrm{Ref} \qquad \dfrac{}{\vdash_{\mathscr{B}} a} \Rightarrow a}{b \vdash_{\mathscr{B}} c}\,\{\Rightarrow a\}, \{\Rightarrow b\} \Rightarrow c$$

Therefore, in $\mathscr{B}$, the atom c is derivable from b.

Example derivations

Example (Invalid derivation)

By the definition of $\vdash_{\mathscr{B}}$, deriving a from $a \fatsemi a$ in the empty base is not possible, i.e. $a \fatsemi a \vdash_{\varnothing} a$ is not possible.

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Example (Invalid derivation 2)

Let $\mathscr{B} = \{(\{\Rightarrow a\} \Rightarrow c), (\{a \Rightarrow c\} \Rightarrow b)\}$

$$\dfrac{\dfrac{\dfrac{\times}{a \fatsemi a \vdash_{\mathscr{B}} a}}{a \fatsemi a \vdash_{\mathscr{B}} c}\ \{\Rightarrow a\} \Rightarrow c}{a \vdash_{\mathscr{B}} b}\ \{a \Rightarrow c\} \Rightarrow b$$

We see that in this base, a is not derivable from $a \fatsemi a$.

Example derivations

Example (A possible fix for invalid derivation 2)

Let $\mathscr{C} = \{(\{\Rightarrow a\} \Rightarrow c), (\{a \Rightarrow c\} \Rightarrow b), (\{\Rightarrow a\}, \{\Rightarrow a\} \Rightarrow a)\}$

$$
\dfrac{\dfrac{\dfrac{\dfrac{}{a \vdash_{\mathscr{C}} a}\,\text{Ref} \qquad \dfrac{}{a \vdash_{\mathscr{C}} a}\,\text{Ref}}{a \,⨾\, a \vdash_{\mathscr{C}} a}\,\{\Rightarrow a\}, \{\Rightarrow a\} \Rightarrow a}{a \,⨾\, a \vdash_{\mathscr{C}} c}\,\{\Rightarrow a\} \Rightarrow c}{a \vdash_{\mathscr{C}} b}\,\{a \Rightarrow c\} \Rightarrow b
$$

Note that $\mathscr{C} \supset \mathscr{B}$!!

Example derivations

Example (Another possible fix for invalid derivation 2)

Let

$\mathscr{D} = \{(\{\Rightarrow a\} \Rightarrow c), (\{a \Rightarrow c\} \Rightarrow b), (\{\Rightarrow d\}, \{\Rightarrow d\} \Rightarrow a), (\{\Rightarrow a\} \Rightarrow d)\}$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{a \vdash_{\mathscr{D}} a}\,\text{Ref}}{a \vdash_{\mathscr{D}} d}\,\{\Rightarrow a\} \Rightarrow d \qquad \dfrac{\dfrac{}{a \vdash_{\mathscr{D}} a}\,\text{Ref}}{a \vdash_{\mathscr{D}} d}\,\{\Rightarrow a\} \Rightarrow d}{a \fatsemi a \vdash_{\mathscr{D}} a}\,\{\Rightarrow d\}, \{\Rightarrow d\} \Rightarrow a}{a \fatsemi a \vdash_{\mathscr{D}} c}\,\{\Rightarrow a\} \Rightarrow c}{a \vdash_{\mathscr{D}} b}\,\{a \Rightarrow c\} \Rightarrow b$$

Example derivations

Example (A live complicated derivation!)

Consider a base $\mathscr{B}$ with only the following rules:

- $\{\Rightarrow c\}, \{\Rightarrow y\} \Rightarrow f$
- $\Rightarrow z$
- $\{\Rightarrow a\}, \{\Rightarrow b\}, \{\Rightarrow z\} \Rightarrow h$
- $\{\Rightarrow h\} \Rightarrow g$
- $\{\Rightarrow x\}, \{\Rightarrow g\} \Rightarrow e$
- $\{x \Rightarrow e\}, \{y \Rightarrow f\} \Rightarrow d$

is there a derivation of d from the multiset $a \,\text{⁏}\, b \,\text{⁏}\, c$ in $\mathscr{B}$?

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is there a derivation of d from the multiset $a \,⹁\, b \,⹁\, c$ in $\mathscr{B}$?

Disclaimer: The fact I couldn't fit this derivation on the slide has nothing to do with why we are doing this derivation live :-)

Example derivations

Example (A very interesting derivation)

Let $\mathscr{B} = \{(\{\Rightarrow b\} \Rightarrow c), (\{\Rightarrow a\}, \{b \Rightarrow c\} \Rightarrow d), (\{\Rightarrow d, \Rightarrow a\} \Rightarrow e), (\{a \Rightarrow e\} \Rightarrow f)\}$

$$\dfrac{\dfrac{\dfrac{\dfrac{}{a \vdash_{\mathscr{B}} a}\,\text{Ref} \qquad \dfrac{\dfrac{}{b \vdash_{\mathscr{B}} b}\,\text{Ref}}{b \vdash_{\mathscr{B}} c}\,\{\Rightarrow b\} \Rightarrow c}{a \vdash_{\mathscr{B}} d}\,\{\Rightarrow a\}, \{b \Rightarrow c\} \Rightarrow d \qquad \dfrac{}{a \vdash_{\mathscr{B}} a}\,\text{Ref}}{a \vdash_{\mathscr{B}} e}\,\{\Rightarrow d, \Rightarrow a\} \Rightarrow e}{\vdash_{\mathscr{B}} f}\,\{a \Rightarrow e\} \Rightarrow f$$

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Comparing relations

Thank you!

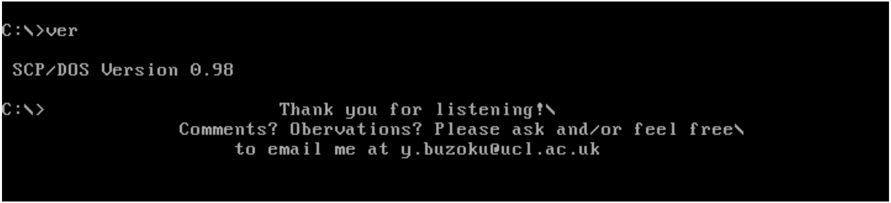

Figure: Thank you from DOS! :D

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