

# Translations between bases in Base-extension Semantics

Yll Buzoku

Department of Computer Science  
University College London

November 22, 2024

# Goals for this talk

- Introduce the concept of bases, atomic rules and inference figures
- Introduce two notions of atomic derivability
- Show how one may relate these notions of atomic derivability

# Presentation root directory

- 1 Bases and atomic rules
- 2 Atomic derivability
- 3 Comparing relations

# Presentation root directory

1 Bases and atomic rules

2 Atomic derivability

3 Comparing relations

# Notation

- $\mathbb{A}$  represents a fixed, countably infinite set of propositional atoms.
- Lower case latin letters represent propositional atoms.
- Upper case latin letters represent finite multisets of propositional atoms.
- Atomic multiset is taken to mean multiset of propositional atoms.
- The sum of two multisets  $P$  and  $Q$  is denoted  $P \uplus Q$ .

# Atomic rules

Atomic rules take the form:

$$(P_1 \Rightarrow p_1), \dots, (P_n \Rightarrow p_n) \Rightarrow p$$

# Atomic rules

Pictorially, we can represent this as:

$$\frac{[P_1] \quad \dots \quad [P_n]}{p}$$

# Atomic rules

Pictorially, we can represent this as:

$$\frac{[P_1] \quad \dots \quad [P_n]}{p}$$

*Any idea what these figures are supposed to look like?*



# Atomic rules

Pictorially, we can represent this as:

$$\frac{\begin{array}{ccc} [P_1] & & [P_n] \\ p_1 & \dots & p_n \end{array}}{p}$$

*Any idea what these figures are supposed to look like?*

**Natural deduction!**

# Contextual atomic rules

Contextual atomic rules are rules which may have contextual brackets distributed across them.

## Contextual atomic rules

Pictorially, we can represent such rules as:

$$\frac{\left\{ \begin{array}{ccc} [P_{1_1}] & & [P_{1_{l_1}}] \\ p_{1_1} & \dots & p_{1_{l_1}} \end{array} \right\} \quad \dots \quad \left\{ \begin{array}{ccc} [P_{n_1}] & & [P_{n_{l_n}}] \\ p_{n_1} & \dots & p_{n_{l_n}} \end{array} \right\}}{p}$$

# Contextual atomic rules

Pictorially, we can represent such rules as:

$$\frac{\left\{ \begin{array}{ccc} [P_{1_1}] & & [P_{1_{l_1}}] \\ p_{1_1} & \dots & p_{1_{l_1}} \end{array} \right\} \quad \dots \quad \left\{ \begin{array}{ccc} [P_{n_1}] & & [P_{n_{l_n}}] \\ p_{n_1} & \dots & p_{n_{l_n}} \end{array} \right\}}{p}$$

or linearly, like this:

$$\{(P_{1_i} \Rightarrow p_{1_i})\}_{i=1}^{l_1}, \dots, \{(P_{n_i} \Rightarrow p_{n_i})\}_{i=1}^{l_n} \Rightarrow p$$

## Examples of atomic rules

- $\{\Rightarrow a\} \Rightarrow c$
- $\{\Rightarrow a\}, \{(c, d \Rightarrow e), (f \Rightarrow g)\} \Rightarrow q$
- $\{a \Rightarrow b\}, \{\Rightarrow e\}, (f \Rightarrow g) \Rightarrow q$
  
- $(\Rightarrow a) \Rightarrow c$
- $(\Rightarrow b) \Rightarrow c$
- $(\Rightarrow c), (a \Rightarrow p), (b \Rightarrow p) \Rightarrow p$

# Bases

## Definition (Base)

A base is a set of atomic rules.

# Bases

## Definition (Base)

A base is a set of atomic rules.

A base is said to be contextual if it contains contextual rules.

Else it is said to be context-free.

# Bases

## Definition (Base)

A base is a set of atomic rules.

A base is said to be contextual if it contains contextual rules.

Else it is said to be context-free.

- $\{((\Rightarrow a) \Rightarrow c), ((\Rightarrow b) \Rightarrow c), ((\Rightarrow c), (a \Rightarrow p), (b \Rightarrow p) \Rightarrow p)\}$
- $\{(\{ \Rightarrow a \} \Rightarrow c), (\{ a \Rightarrow c \} \Rightarrow b), (\{ \Rightarrow a \}, \{ \Rightarrow a \} \Rightarrow a)\}$
- $\{(\{ \Rightarrow a \} \Rightarrow c), (\{ a \Rightarrow c \} \Rightarrow b), (\{ \Rightarrow d \}, \{ \Rightarrow d \} \Rightarrow a), (\{ \Rightarrow a \} \Rightarrow d)\}$



# Presentation root directory

1 Bases and atomic rules

2 Atomic derivability

3 Comparing relations

## Definition (Context-free atomic derivability $\triangleright_{\mathcal{B}}$ )

The relation of derivability in a context-free base  $\mathcal{B}$ , is defined as so:

**Ref**  $S \triangleright_{\mathcal{B}} p$  if  $p \in S$ .

**App** For  $((P_1 \Rightarrow p_1), \dots, (P_n \Rightarrow p_n) \Rightarrow q) \in \mathcal{B}$  and  $S, P_i \triangleright_{\mathcal{B}} p_i$  for each  $i \in \{1, \dots, n\}$  then  $S \triangleright_{\mathcal{B}} q$ .

## Definition (Context-free atomic derivability $\triangleright_{\mathcal{B}}$ )

The relation of derivability in a context-free base  $\mathcal{B}$ , is defined as so:

**Ref**  $S \triangleright_{\mathcal{B}} p$  if  $p \in S$ .

**App** For  $((P_1 \Rightarrow p_1), \dots, (P_n \Rightarrow p_n) \Rightarrow q) \in \mathcal{B}$  and  $S, P_i \triangleright_{\mathcal{B}} p_i$  for each  $i \in \{1, \dots, n\}$  then  $S \triangleright_{\mathcal{B}} q$ .

## Definition (Contextual atomic derivability $\vdash_{\mathcal{B}}$ )

The relation of derivability in a contextual base  $\mathcal{B}$ , is defined as so:

**Ref**  $p \vdash_{\mathcal{B}} p$ .

**App** For  $((\{P_{1_i} \Rightarrow p_{1_i}\}_{i=1}^{l_1}, \dots, \{P_{n_i} \Rightarrow p_{n_i}\}_{i=1}^{l_n} \Rightarrow q) \in \mathcal{B}$  and  $C_i, P_{ij} \vdash_{\mathcal{B}} p_{ij}$  for each  $i \in \{1, \dots, n\}$  and  $j \in \{1, \dots, l_i\}$  then  $C_1, \dots, C_n \vdash_{\mathcal{B}} q$ .

# Example derivations

## Example

Let  $\mathcal{B} = \{(\Rightarrow a), ((\Rightarrow a), (\Rightarrow b) \Rightarrow c)\}$

$$\frac{\frac{}{b \triangleright_{\mathcal{B}} b} \text{Ref} \quad \frac{}{\triangleright_{\mathcal{B}} a} \Rightarrow a}{b \triangleright_{\mathcal{B}} c} (\Rightarrow a), (\Rightarrow b) \Rightarrow c$$

# Example derivations

## Example

Let  $\mathcal{B} = \{(\Rightarrow a), (\{\Rightarrow a\}, \{\Rightarrow b\} \Rightarrow c)\}$

$$\frac{\frac{}{b \vdash_{\mathcal{B}} b} \text{Ref} \quad \frac{}{\vdash_{\mathcal{B}} a} \Rightarrow a}{b \vdash_{\mathcal{B}} c} \{\Rightarrow a\}, \{\Rightarrow b\} \Rightarrow c$$

# Example derivations

## Example

By the definition of  $\vdash_{\mathcal{B}}$ , deriving  $a$  from  $a, a$  in the empty base is not possible, i.e.  $a, a \vdash_{\emptyset} a$  is not possible. But  $a, a \triangleright_{\emptyset} a$  is possible.

## Example derivations

### Example

By the definition of  $\vdash_{\mathcal{B}}$ , deriving  $a$  from  $a, a$  in the empty base is not possible, i.e.  $a, a \vdash_{\emptyset} a$  is not possible. But  $a, a \triangleright_{\emptyset} a$  is possible.

### Example

Let  $\mathcal{B} = \{(\{ \Rightarrow a \} \Rightarrow c), (\{ a \Rightarrow c \} \Rightarrow b)\}$

$$\frac{\frac{\frac{\times}{a, a \vdash_{\mathcal{B}} a} \quad \{ \Rightarrow a \} \Rightarrow c}{a, a \vdash_{\mathcal{B}} c} \quad \{ a \Rightarrow c \} \Rightarrow b}{a \vdash_{\mathcal{B}} b}$$

We see that in this base,  $a$  is not derivable from  $a, a$ .

# Example derivations

## Example (A possible fix)

Let  $\mathcal{C} = \{(\{ \Rightarrow a \Rightarrow c \}), (\{ a \Rightarrow c \} \Rightarrow b), (\{ \Rightarrow a \}, \{ \Rightarrow a \} \Rightarrow a)\}$

$$\begin{array}{c}
 \frac{}{a \vdash_{\mathcal{C}} a} \text{Ref} \quad \frac{}{a \vdash_{\mathcal{C}} a} \text{Ref} \\
 \hline
 \frac{}{a, a \vdash_{\mathcal{C}} a} \{ \Rightarrow a \}, \{ \Rightarrow a \} \Rightarrow a \\
 \hline
 \frac{}{a, a \vdash_{\mathcal{C}} c} \{ \Rightarrow a \} \Rightarrow c \\
 \hline
 \frac{}{a \vdash_{\mathcal{C}} b} \{ a \Rightarrow c \} \Rightarrow b
 \end{array}$$



# Example derivations

## Example (Another possible fix)

Let  $(\mathcal{B})_\star = \{((\Rightarrow a) \Rightarrow c), ((a \Rightarrow c) \Rightarrow b)\}$

$$\begin{array}{c}
 \frac{}{a, a \triangleright_{(\mathcal{B})_\star} a} \text{Ref} \\
 \frac{}{a, a \triangleright_{(\mathcal{B})_\star} c} (\Rightarrow a) \Rightarrow c \\
 \frac{}{a \triangleright_{(\mathcal{B})_\star} b} (a \Rightarrow c) \Rightarrow b
 \end{array}$$

# Example derivations

## Example (A very interesting derivation)

Let  $\mathcal{B}$  be the following set of rules:

$\{(\{ \Rightarrow b \} \Rightarrow c), (\{ \Rightarrow a \}, \{ b \Rightarrow c \} \Rightarrow d), (\{ \Rightarrow d, \Rightarrow a \} \Rightarrow e), (\{ a \Rightarrow e \} \Rightarrow f)\}$

$$\begin{array}{c}
 \frac{}{a \vdash_{\mathcal{B}} a} \text{Ref} \quad \frac{\frac{}{b \vdash_{\mathcal{B}} b} \text{Ref}}{b \vdash_{\mathcal{B}} c} \{ \Rightarrow b \} \Rightarrow c}{\frac{}{a \vdash_{\mathcal{B}} d} \{ \Rightarrow a \}, \{ b \Rightarrow c \} \Rightarrow d} \text{Ref} \quad \frac{}{a \vdash_{\mathcal{B}} a} \text{Ref} \\
 \frac{\frac{}{a \vdash_{\mathcal{B}} d} \{ \Rightarrow a \}, \{ b \Rightarrow c \} \Rightarrow d \quad \frac{}{a \vdash_{\mathcal{B}} a} \{ \Rightarrow d, \Rightarrow a \} \Rightarrow e}{\frac{a \vdash_{\mathcal{B}} e}{\vdash_{\mathcal{B}} f} \{ a \Rightarrow e \} \Rightarrow f} \text{Ref}
 \end{array}$$

# Example derivations

## Example (A live complicated derivation!)

Consider a base  $\mathcal{B}$  with only the following rules:

- $\{\Rightarrow c\}, \{\Rightarrow y\} \Rightarrow f$
- $\Rightarrow z$
- $\{\Rightarrow a\}, \{\Rightarrow b\}, \{\Rightarrow z\} \Rightarrow h$
- $\{\Rightarrow h\} \Rightarrow g$
- $\{\Rightarrow x\}, \{\Rightarrow g\} \Rightarrow e$
- $\{x \Rightarrow e\}, \{y \Rightarrow f\} \Rightarrow d$

is there a derivation of  $d$  from the multiset  $a, b, c$  in  $\mathcal{B}$ ?

# Presentation root directory

1 Bases and atomic rules

2 Atomic derivability

3 Comparing relations

# Comparing relations

## Definition (Structural rules)

We define two rules:

- $\text{Wk}_q^p = (\{\Rightarrow p\}, \{\Rightarrow q\} \Rightarrow q)$
- $\text{Ctn}_q^p = (\{\Rightarrow p\}, \{p, p \Rightarrow q\} \Rightarrow q)$

# Mappings between bases

# Mappings between bases

## Definition

Let  $\mathcal{B}$  be a context-free base. We define structural contextualisation of that base  $(\mathcal{B})^*$  as follows:

$$(\mathcal{B})^* = \{ \{P_1 \Rightarrow p_1\}, \dots, \{P_n \Rightarrow p_n\} \Rightarrow q \mid ((P_1 \Rightarrow p_1), \dots, (P_n \Rightarrow p_n) \Rightarrow q) \in \mathcal{B} \} \\ \cup \{ \text{Wk}_q^p, \text{Ctn}_q^p \mid \forall p, q \in \mathbb{A} \}$$

# Mappings between bases

## Definition

Let  $\mathcal{B}$  be a context-free base. We define structural contextualisation of that base  $(\mathcal{B})^*$  as follows:

$$(\mathcal{B})^* = \{ \{P_1 \Rightarrow p_1\}, \dots, \{P_n \Rightarrow p_n\} \Rightarrow q \mid ((P_1 \Rightarrow p_1), \dots, (P_n \Rightarrow p_n) \Rightarrow q) \in \mathcal{B} \} \\ \cup \{ \text{Wk}_q^p, \text{Ctn}_q^p \mid \forall p, q \in \mathbb{A} \}$$

We define the decontextualisation of a contextual base as:

$$(\mathcal{B})_\star = \{ (P_{1_1} \Rightarrow p_{1_1}), \dots, (P_{n_{l_n}} \Rightarrow p_{n_{l_n}}) \Rightarrow q \\ \mid (\{P_{1_i} \Rightarrow p_{1_i}\}_{i=1}^{l_1}, \dots, \{P_{n_i} \Rightarrow p_{n_i}\}_{i=1}^{l_n} \Rightarrow q) \in \mathcal{B} \}$$



# Properties of these mappings

Let  $\mathcal{B}$  be a context-free base. Then the following hold:

- $((\mathcal{B})^*)_{\star} \supseteq \mathcal{B}.$

# Properties of these mappings

Let  $\mathcal{B}$  be a context-free base. Then the following hold:

- $((\mathcal{B})^*)_{\star} \supseteq \mathcal{B}$ .
- $((\mathcal{B})^*)_{\star}^* = (\mathcal{B})^*$ .

# Properties of these mappings

Let  $\mathcal{B}$  be a context-free base. Then the following hold:

- $((\mathcal{B})^*)_{\star} \supseteq \mathcal{B}$ .
- $((\mathcal{B})^*)_{\star}^* = (\mathcal{B})^*$ .
- For all  $\mathcal{C} \supseteq \mathcal{B}$  we have that  $(\mathcal{C})^* \supseteq (\mathcal{B})^*$ .

# Properties of these mappings

Let  $\mathcal{B}$  be a context-free base. Then the following hold:

- $((\mathcal{B})^*)_{\star} \supseteq \mathcal{B}$ .
- $((\mathcal{B})^*)_{\star}^* = (\mathcal{B})^*$ .
- For all  $\mathcal{C} \supseteq \mathcal{B}$  we have that  $(\mathcal{C})^* \supseteq (\mathcal{B})^*$ .
- For all  $\mathcal{C} \supseteq (\mathcal{B})^*$  there exists an extension  $\mathcal{X} \supseteq \mathcal{B}$  such that  $(\mathcal{X})^* = \mathcal{C}$ .

# Structural admissibility in a structurally contextualised base

## Lemma

*Suppose  $L \vdash_{(\mathcal{B})^*} p$  holds. Then  $R, L \vdash_{(\mathcal{B})^*} p$  also holds for any atomic multiset  $R$ .*

Proof.

Let  $R = \{r_1, \dots, r_n\}$  for some  $n$ . Then we have that we can effectively weaken  $S$  away as follows:

$$\begin{array}{c}
 \frac{\frac{\frac{}{r_1 \vdash_{(\mathcal{B})^*} r_1} \text{Ref} \quad \frac{\frac{\frac{}{r_n \vdash_{(\mathcal{B})^*} r_n} \text{Ref} \quad \frac{L \vdash_{(\mathcal{B})^*} p}{r_n, L \vdash_{(\mathcal{B})^*} p} \text{Wk}_p^{r_n}}{r_{n-1}, L \vdash_{(\mathcal{B})^*} p} \text{Wk}_p^{r_{n-1}}}{r_{n-1} \vdash_{(\mathcal{B})^*} r_{n-1}} \text{Ref} \quad \vdots \quad \frac{\frac{}{r_2, \dots, r_n, L \vdash_{(\mathcal{B})^*} p} \text{Wk}_p^{r_1}}{R, L \vdash_{(\mathcal{B})^*} p} \text{Wk}_p^{r_1}}{R, L \vdash_{(\mathcal{B})^*} p}
 \end{array}$$

□

# Structural admissibility in a structurally contextualised base

## Lemma

*Suppose  $R, R \vdash_{(\mathcal{B})^*} p$  holds. Then  $R \vdash_{(\mathcal{B})^*} p$  also holds.*

# Structural admissibility in a structurally contextualised base

## Lemma

*Suppose  $R, R \vdash_{(\mathcal{B})^*} p$  holds. Then  $R \vdash_{(\mathcal{B})^*} p$  also holds.*

## Corollary

*For arbitrary  $m \geq 1$ , if  $R^m \vdash_{(\mathcal{B})^*} p$  then  $R \vdash_{(\mathcal{B})^*} p$ .*



Proof.

Let  $R = \{r_1, \dots, r_n\}$  for some  $n$ . Then we have that we can effectively contract on  $R$  as follows:

$$\begin{array}{c}
 \frac{\frac{\frac{}{r_1 \vdash_{(\mathcal{B})^*} r_1} \text{Ref} \quad \vdots \quad \frac{\frac{\frac{}{r_n \vdash_{(\mathcal{B})^*} r_n} \text{Ref} \quad R, R \vdash_{(\mathcal{B})^*} p}{r_1, \dots, r_{n-1}, R \vdash_{(\mathcal{B})^*} p} \text{Ctn}_p^{r_n}}{r_1, \dots, r_{n-1}, R \vdash_{(\mathcal{B})^*} p} \text{Ctn}_p^{r_{n-1}}}{R \vdash_{(\mathcal{B})^*} p} \text{Ctn}_p^{r_1}
 \end{array}
 \quad \square$$

## Key results under this base translation

- If  $R \triangleright_{\mathcal{B}} p$  then  $R \vdash_{(\mathcal{B})^*} p$ .

## Key results under this base translation

- If  $R \triangleright_{\mathcal{B}} p$  then  $R \vdash_{(\mathcal{B})^*} p$ .
- $R \vdash_{(\mathcal{B})^*} p$  iff for all bases  $\mathcal{X} \supseteq (\mathcal{B})^*$  where for each  $r \in R$  we have  $\vdash_{\mathcal{X}} r$  then it follows that  $\vdash_{\mathcal{X}} p$ .

## Key results under this base translation

- If  $R \triangleright_{\mathcal{B}} p$  then  $R \vdash_{(\mathcal{B})^*} p$ .
- $R \vdash_{(\mathcal{B})^*} p$  iff for all bases  $\mathcal{X} \supseteq (\mathcal{B})^*$  where for each  $r \in R$  we have  $\vdash_{\mathcal{X}} r$  then it follows that  $\vdash_{\mathcal{X}} p$ .
- $R \vdash_{(\mathcal{B})^*} p$  iff  $!R \Vdash_{(\mathcal{B})^*}^{\emptyset} p$

## Key results under this base translation




- If  $R \triangleright_{\mathcal{B}} p$  then  $R \vdash_{(\mathcal{B})^*} p$ .
- $R \vdash_{(\mathcal{B})^*} p$  iff for all bases  $\mathcal{X} \supseteq (\mathcal{B})^*$  where for each  $r \in R$  we have  $\vdash_{\mathcal{X}} r$  then it follows that  $\vdash_{\mathcal{X}} p$ .
- $R \vdash_{(\mathcal{B})^*} p$  iff  $!R \Vdash_{(\mathcal{B})^*}^{\emptyset} p$
- $\Vdash_{(\mathcal{B})^*}^R p$  iff  $!R \Vdash_{(\mathcal{B})^*}^{\emptyset} p$

# Thank you!





```
C:\>ver  
  
SCP/DOS Version 0.98  
  
C:\>          Thank you for listening!\n              Comments? Observations? Please ask and/or feel free\n              to email me at y.buzoku@ucl.ac.uk
```

Figure: Thank you from DOS! :D

# References I

-  G.M. Bierman, *On intuitionistic linear logic*, Tech. Report UCAM-CL-TR-346, University of Cambridge, Computer Laboratory, August 1994.
-  Yli Buzoku, *A proof-theoretic semantics for intuitionistic linear logic*, 2024.
-  Alexander V. Gheorghiu, Tao Gu, and David J. Pym, *Proof-theoretic semantics for intuitionistic multiplicative linear logic*, Automated Reasoning with Analytic Tableaux and Related Methods (Cham) (Revantha Ramanayake and Josef Urban, eds.), Springer Nature Switzerland, 2023, pp. 367–385.

## References II

-  J.-Y. Girard, *Linear logic: its syntax and semantics*, London Mathematical Society Lecture Note Series, p. 1–42, Cambridge University Press, 1995.
-  Alexander V. Gheorghiu and David J. Pym, *From proof-theoretic validity to base-extension semantics for intuitionistic propositional logic*, 2022.
-  Sara Negri, *A normalizing system of natural deduction for intuitionistic linear logic*, *Archive for Mathematical Logic* **41** (2002), no. 8, 789–810.
-  Tor Sandqvist, *An inferentialist interpretation of classical logic*, Ph.D. thesis, Uppsala universitet, 2005.



## References III



\_\_\_\_\_, *Base-extension semantics for intuitionistic sentential logic*,  
Log. J. IGPL **23** (2015), 719–731.