

Translations between bases in Base-extension Semantics

Yll Buzoku

Department of Computer Science
University College London

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Goals for this talk

- Introduce the concept of bases, atomic rules and inference figures
- Introduce two notions of atomic derivability
- Show how one may relate these notions of atomic derivability

Presentation root directory

- 1 Bases and atomic rules
- 2 Atomic derivability
- 3 Comparing relations

Presentation root directory

1 Bases and atomic rules

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Notation

- \mathbb{A} represents a fixed, countably infinite set of propositional atoms.
- Lower case latin letters represent propositional atoms.
- Upper case latin letters represent finite multisets of propositional atoms.
- Atomic multiset is taken to mean multiset of propositional atoms.
- The sum of two multisets P and Q is denoted $P \uplus Q$.

Atomic rules

Atomic rules take the form:

$$(P_1 \Rightarrow p_1), \dots, (P_n \Rightarrow p_n) \Rightarrow p$$

Atomic rules

Pictorially, we can represent this as:

$$\frac{\begin{array}{ccc} [P_1] & & [P_n] \\ p_1 & \dots & p_n \end{array}}{p}$$

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Any idea what these figures are supposed to look like?

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Natural deduction!

Contextual atomic rules

Contextual atomic rules are rules which may have contextual brackets distributed across them.

Contextual atomic rules

Pictorially, we can represent such rules as:

$$\frac{\left\{ \begin{array}{ccc} [P_{1_1}] & & [P_{1_{l_1}}] \\ p_{1_1} & \dots & p_{1_{l_1}} \end{array} \right\} \quad \dots \quad \left\{ \begin{array}{ccc} [P_{n_1}] & & [P_{n_{l_n}}] \\ p_{n_1} & \dots & p_{n_{l_n}} \end{array} \right\}}{p}$$

Contextual atomic rules

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or linearly, like this:

$$\{(P_{1_i} \Rightarrow p_{1_i})\}_{i=1}^{l_1}, \dots, \{(P_{n_i} \Rightarrow p_{n_i})\}_{i=1}^{l_n} \Rightarrow p$$

Examples of atomic rules

- $\{\Rightarrow a\} \Rightarrow c$
- $\{\Rightarrow a\}, \{(c, d \Rightarrow e), (f \Rightarrow g)\} \Rightarrow q$
- $\{a \Rightarrow b\}, \{\Rightarrow e\}, (f \Rightarrow g) \Rightarrow q$

- $(\Rightarrow a) \Rightarrow c$
- $(\Rightarrow b) \Rightarrow c$
- $(\Rightarrow c), (a \Rightarrow p), (b \Rightarrow p) \Rightarrow p$

Bases

Definition (Base)

A base is a set of atomic rules.

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Else it is said to be context-free.

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- $\{((\Rightarrow a) \Rightarrow c), ((\Rightarrow b) \Rightarrow c), ((\Rightarrow c), (a \Rightarrow p), (b \Rightarrow p) \Rightarrow p)\}$
- $\{\{\{\Rightarrow a\} \Rightarrow c\}, (\{a \Rightarrow c\} \Rightarrow b), (\{\Rightarrow a\}, \{\Rightarrow a\} \Rightarrow a)\}$
- $\{\{\{\Rightarrow a\} \Rightarrow c\}, (\{a \Rightarrow c\} \Rightarrow b), (\{\Rightarrow d\}, \{\Rightarrow d\} \Rightarrow a), (\{\Rightarrow a\} \Rightarrow d)\}$

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Definition (Context-free atomic derivability $\triangleright_{\mathcal{B}}$)

The relation of derivability in a context-free base \mathcal{B} , is defined as so:

Ref $S \triangleright_{\mathcal{B}} p$ if $p \in S$.

App For $((P_1 \Rightarrow p_1), \dots, (P_n \Rightarrow p_n) \Rightarrow q) \in \mathcal{B}$ and $S, P_i \triangleright_{\mathcal{B}} p_i$ for each $i \in \{1, \dots, n\}$ then $S \triangleright_{\mathcal{B}} q$.

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Definition (Contextual atomic derivability $\vdash_{\mathcal{B}}$)

The relation of derivability in a contextual base \mathcal{B} , is defined as so:

Ref $p \vdash_{\mathcal{B}} p$.

App For $((\{P_{1_i} \Rightarrow p_{1_i}\}_{i=1}^{l_1}, \dots, \{P_{n_i} \Rightarrow p_{n_i}\}_{i=1}^{l_n} \Rightarrow q) \in \mathcal{B}$ and $C_i, P_{ij} \vdash_{\mathcal{B}} p_{ij}$ for each $i \in \{1, \dots, n\}$ and $j \in \{1, \dots, l_i\}$ then $C_1, \dots, C_n \vdash_{\mathcal{B}} q$.

Example derivations

Example

Let $\mathcal{B} = \{(\Rightarrow a), ((\Rightarrow a), (\Rightarrow b) \Rightarrow c)\}$

$$\frac{\frac{}{b \triangleright_{\mathcal{B}} b} \text{Ref} \quad \frac{}{\triangleright_{\mathcal{B}} a} \Rightarrow a}{b \triangleright_{\mathcal{B}} c} (\Rightarrow a), (\Rightarrow b) \Rightarrow c$$

Example derivations

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Let $\mathcal{B} = \{(\Rightarrow a), (\{\Rightarrow a\}, \{\Rightarrow b\} \Rightarrow c)\}$

$$\frac{\frac{}{b \vdash_{\mathcal{B}} b} \text{Ref} \quad \frac{}{\vdash_{\mathcal{B}} a} \Rightarrow a}{b \vdash_{\mathcal{B}} c} \{\Rightarrow a\}, \{\Rightarrow b\} \Rightarrow c$$

Example derivations

Example

By the definition of $\vdash_{\mathcal{B}}$, deriving a from a, a in the empty base is not possible, i.e. $a, a \vdash_{\emptyset} a$ is not possible. But $a, a \triangleright_{\emptyset} a$ is possible.

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Example

Let $\mathcal{B} = \{(\{ \Rightarrow a \} \Rightarrow c), (\{ a \Rightarrow c \} \Rightarrow b)\}$

$$\frac{\frac{\frac{\times}{a, a \vdash_{\mathcal{B}} a} \quad \{ \Rightarrow a \} \Rightarrow c}{a, a \vdash_{\mathcal{B}} c} \quad \{ a \Rightarrow c \} \Rightarrow b}{a \vdash_{\mathcal{B}} b}$$

We see that in this base, a is not derivable from a, a .

Example derivations

Example

Let $\mathcal{B} = \{((\Rightarrow a) \Rightarrow c), ((a \Rightarrow c) \Rightarrow b)\}$

$$\frac{\frac{a, a \triangleright_{\mathcal{B}} a}{a, a \triangleright_{\mathcal{B}} c} \text{Ref}}{a \triangleright_{\mathcal{B}} b} \begin{array}{l} (\Rightarrow a) \Rightarrow c \\ (a \Rightarrow c) \Rightarrow b \end{array}$$

Example derivations

Example (A very interesting derivation)

Let \mathcal{B} be the following set of rules:

$\{(\{ \Rightarrow b \} \Rightarrow c), (\{ \Rightarrow a \}, \{ b \Rightarrow c \} \Rightarrow d), (\{ \Rightarrow d, \Rightarrow a \} \Rightarrow e), (\{ a \Rightarrow e \} \Rightarrow f)\}$

$$\begin{array}{c}
 \frac{}{a \vdash_{\mathcal{B}} a} \text{Ref} \quad \frac{\frac{}{b \vdash_{\mathcal{B}} b} \text{Ref}}{b \vdash_{\mathcal{B}} c} \{ \Rightarrow b \} \Rightarrow c}{\frac{}{a \vdash_{\mathcal{B}} d} \{ \Rightarrow a \}, \{ b \Rightarrow c \} \Rightarrow d} \\
 \frac{}{a \vdash_{\mathcal{B}} a} \text{Ref} \quad \frac{}{a \vdash_{\mathcal{B}} e} \{ \Rightarrow d, \Rightarrow a \} \Rightarrow e \\
 \frac{}{\vdash_{\mathcal{B}} f} \{ a \Rightarrow e \} \Rightarrow f
 \end{array}$$

Example derivations

Example (A live complicated derivation!)

Consider a base \mathcal{B} with only the following rules:

- $\{\Rightarrow c\}, \{\Rightarrow y\} \Rightarrow f$
- $\Rightarrow z$
- $\{\Rightarrow a\}, \{\Rightarrow b\}, \{\Rightarrow z\} \Rightarrow h$
- $\{\Rightarrow h\} \Rightarrow g$
- $\{\Rightarrow x\}, \{\Rightarrow g\} \Rightarrow e$
- $\{x \Rightarrow e\}, \{y \Rightarrow f\} \Rightarrow d$

is there a derivation of d from the multiset a, b, c in \mathcal{B} ?

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Comparing relations

Definition (Structural rules)

We define two rules:

- $\text{Wk}_q^p = (\{\Rightarrow p\}, \{\Rightarrow q\} \Rightarrow q)$
- $\text{Ctn}_q^p = (\{\Rightarrow p\}, \{p, p \Rightarrow q\} \Rightarrow q)$

Mappings between bases

Mappings between bases

Definition

Let \mathcal{B} be a context-free base. We define structural contextualisation of that base $(\mathcal{B})^*$ as follows:

$$(\mathcal{B})^* = \{ \{P_1 \Rightarrow p_1\}, \dots, \{P_n \Rightarrow p_n\} \Rightarrow q \mid ((P_1 \Rightarrow p_1), \dots, (P_n \Rightarrow p_n) \Rightarrow q) \in \mathcal{B} \} \\ \cup \{ \text{Wk}_q^p, \text{Ctn}_q^p \mid \forall p, q \in \mathbb{A} \}$$

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We define the decontextualisation of a contextual base as:

$$(\mathcal{B})_\star = \{ (P_{1_1} \Rightarrow p_{1_1}), \dots, (P_{n_{l_n}} \Rightarrow p_{n_{l_n}}) \Rightarrow q \\ \mid (\{P_{1_i} \Rightarrow p_{1_i}\}_{i=1}^{l_1}, \dots, \{P_{n_i} \Rightarrow p_{n_i}\}_{i=1}^{l_n} \Rightarrow q) \in \mathcal{B} \}$$

Properties of these mappings

Let \mathcal{B} be a context-free base. Then the following hold:

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- $((\mathcal{B})^*)_{\star}^* = (\mathcal{B})^*$.
- For all $\mathcal{C} \supseteq \mathcal{B}$ we have that $(\mathcal{C})^* \supseteq (\mathcal{B})^*$.
- For all $\mathcal{C} \supseteq (\mathcal{B})^*$ there exists an extension $\mathcal{X} \supseteq \mathcal{B}$ such that $(\mathcal{X})^* = \mathcal{C}$.

Structural admissibility in a structurally contextualised base

Lemma

Suppose $L \vdash_{(\mathcal{B})^} p$ holds. Then $R, L \vdash_{(\mathcal{B})^*} p$ also holds for any atomic multiset R .*

Proof.

Let $R = \{r_1, \dots, r_n\}$ for some n . Then we have that we can effectively weaken S away as follows:

$$\begin{array}{c}
 \frac{\frac{\frac{}{r_1 \vdash_{(\mathcal{B})^*} r_1} \text{Ref} \quad \frac{\frac{\frac{}{r_n \vdash_{(\mathcal{B})^*} r_n} \text{Ref} \quad \frac{L \vdash_{(\mathcal{B})^*} p}{r_n, L \vdash_{(\mathcal{B})^*} p} \text{Wk}_p^{r_n}}{r_n, L \vdash_{(\mathcal{B})^*} p} \text{Wk}_p^{r_{n-1}}}{r_{n-1} \vdash_{(\mathcal{B})^*} r_{n-1}} \text{Ref} \quad \vdots \quad \frac{\frac{}{r_1 \vdash_{(\mathcal{B})^*} r_1} \text{Ref} \quad \frac{r_2, \dots, r_n, L \vdash_{(\mathcal{B})^*} p}{R, L \vdash_{(\mathcal{B})^*} p} \text{Wk}_p^{r_1}}{R, L \vdash_{(\mathcal{B})^*} p}
 \end{array}$$

□

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Suppose $R, R \vdash_{(\mathcal{B})^} p$ holds. Then $R \vdash_{(\mathcal{B})^*} p$ also holds.*

Corollary

For arbitrary $m \geq 1$, if $R^m \vdash_{(\mathcal{B})^} p$ then $R \vdash_{(\mathcal{B})^*} p$.*

Proof.

Let $R = \{r_1, \dots, r_n\}$ for some n . Then we have that we can effectively contract on R as follows:

$$\begin{array}{c}
 \frac{\frac{\frac{}{r_1 \vdash_{(\mathcal{B})^*} r_1} \text{Ref} \quad \vdots \quad \frac{\frac{\frac{}{r_n \vdash_{(\mathcal{B})^*} r_n} \text{Ref} \quad R, R \vdash_{(\mathcal{B})^*} p}{r_1, \dots, r_{n-1}, R \vdash_{(\mathcal{B})^*} p} \text{Ctn}_p^{r_n}}{r_{n-1} \vdash_{(\mathcal{B})^*} r_{n-1}} \text{Ref}}{R \vdash_{(\mathcal{B})^*} p} \text{Ctn}_p^{r_1}
 \end{array}
 \quad \square$$

Key results under this base translation

- If $R \triangleright_{\mathcal{B}} p$ then $R \vdash_{(\mathcal{B})^*} p$.

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- If $R \triangleright_{\mathcal{B}} p$ then $R \vdash_{(\mathcal{B})^*} p$.
- $R \vdash_{(\mathcal{B})^*} p$ iff for all bases $\mathcal{X} \supseteq (\mathcal{B})^*$ where for each $r \in R$ we have $\vdash_{\mathcal{X}} r$ then it follows that $\vdash_{\mathcal{X}} p$.

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- If $R \triangleright_{\mathcal{B}} p$ then $R \vdash_{(\mathcal{B})^*} p$.
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- $R \vdash_{(\mathcal{B})^*} p$ iff $!R \Vdash_{(\mathcal{B})^*}^{\emptyset} p$

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- $R \vdash_{(\mathcal{B})^*} p$ iff $!R \Vdash_{(\mathcal{B})^*}^{\emptyset} p$
- $\Vdash_{(\mathcal{B})^*}^R p$ iff $!R \Vdash_{(\mathcal{B})^*}^{\emptyset} p$

Thank you!





```
C:\>ver  
  
SCP/DOS Version 0.98  
  
C:\>          Thank you for listening!\n          Comments? Observations? Please ask and/or feel free\n          to email me at y.buzoku@ucl.ac.uk
```

Figure: Thank you from DOS! :D

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