Translations between bases in Base-extension Semantics

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Goals for this talk

- Introduce the concept of bases, atomic rules and inference figures
- Introduce two notions of atomic derivability
- Show how one may relate these notions of atomic derivability

Presentation root directory

1. Bases and atomic rules
2. Atomic derivability
3. Comparing relations

Notation

- $\mathbb{A}$ represents a fixed, countably infinite set of propositional atoms.
- Lower case latin letters represent propositional atoms.
- Upper case latin letters represent finite multisets of propositional atoms.
- The sum of two multisets P and Q is denoted $P \fatsemi Q$.

Atomic rules

Atomic rules take the form:

$$(P_1 \Rightarrow p_1), \ldots, (P_n \Rightarrow p_n) \Rightarrow p$$

Atomic rules

Pictorially, we can represent this as:

$$\dfrac{\begin{matrix} [P_1] & & [P_n] \\ p_1 & \dots & p_n \end{matrix}}{p}$$

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Any idea what these figures are supposed to look like?

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Any idea what these figures are supposed to look like?

Natural deduction!

Contextual atomic rules

Contextual atomic rules are rules which may have contextual brackets distributed across them.

Contextual atomic rules

Pictorially, we can represent such rules as:

$$\dfrac{\left\{\begin{matrix} [P_{1_1}] & & [P_{1_{l_n}}] \\ p_{1_1} & \ldots & p_{1_{l_1}} \end{matrix}\right\} \qquad \ldots \qquad \left\{\begin{matrix} [P_{n_1}] & & [P_{n_{l_n}}] \\ p_{n_1} & \ldots & p_{n_{l_n}} \end{matrix}\right\}}{p}$$

Contextual atomic rules

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or linearly, like this:

$$\{(P_{1_i} \Rightarrow p_{1_i})\}_{i=1}^{l_1}, \dots, \{(P_{n_i} \Rightarrow p_{n_i})\}_{i=1}^{l_n} \Rightarrow p$$

Examples of atomic rules

- $\{\Rightarrow a\} \Rightarrow c$
- $\{\Rightarrow a\}, \{(c \fatcomma d \Rightarrow e), (f \Rightarrow g)\} \Rightarrow q$
- $\{a \Rightarrow b\}, \{\Rightarrow e\}, (f \Rightarrow g) \Rightarrow q$

- $(\Rightarrow a) \Rightarrow c$
- $(\Rightarrow b) \Rightarrow c$
- $(\Rightarrow c), (a \Rightarrow p), (b \Rightarrow p) \Rightarrow p$

Bases

Definition (Base)

A base is a set of atomic rules.

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- $\{((\Rightarrow a) \Rightarrow c), ((\Rightarrow b) \Rightarrow c), ((\Rightarrow c), (a \Rightarrow p), (b \Rightarrow p) \Rightarrow p)\}$
- $\{(\{\Rightarrow a\} \Rightarrow c), (\{a \Rightarrow c\} \Rightarrow b), (\{\Rightarrow a\}, \{\Rightarrow a\} \Rightarrow a)\}$
- $\{(\{\Rightarrow a\} \Rightarrow c), (\{a \Rightarrow c\} \Rightarrow b), (\{\Rightarrow d\}, \{\Rightarrow d\} \Rightarrow a), (\{\Rightarrow a\} \Rightarrow d)\}$

Presentation root directory

Definition (Context-free atomic derivability $\rhd_{\mathscr{B}}$)

The relation of derivability in a context-free base $\mathscr{B}$, is defined as so:

- Ref $S \rhd_{\mathscr{B}} p$ if $p \in S$.
- App For $((P_1 \Rightarrow p_1), \ldots, (P_n \Rightarrow p_n) \Rightarrow q) \in \mathscr{B}$ and $S \fatsemi P_i \rhd_{\mathscr{B}} p_i$ for each $i \in \{1, \ldots, n\}$ then $S \rhd_{\mathscr{B}} q$.

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Definition (Contextual atomic derivability $\vdash_{\mathscr{B}}$)

The relation of derivability in a contextual base $\mathscr{B}$, is defined as so:

Ref $p \vdash_{\mathscr{B}} p$.

App For $(\{(P_{1_i} \Rightarrow p_{1_i})\}_{i=1}^{l_1}, \ldots, \{(P_{n_i} \Rightarrow p_{n_i})\}_{i=1}^{l_n} \Rightarrow q) \in \mathscr{B}$ and $C_i \fatsemi P_{i_j} \vdash_{\mathscr{B}} p_{i_j}$ for each $i \in \{1, \ldots, n\}$ and $j \in \{1, \ldots, l_i\}$ then $C_1 \fatsemi \cdots \fatsemi C_n \vdash_{\mathscr{B}} q$.

Example derivations

Example

Let $\mathscr{B} = \{(\Rightarrow a), ((\Rightarrow a), (\Rightarrow b) \Rightarrow c)\}$

$$\dfrac{\dfrac{}{b \rhd_{\mathscr{B}} b}\ \mathrm{Ref} \qquad \dfrac{}{\rhd_{\mathscr{B}} a} \Rightarrow a}{b \rhd_{\mathscr{B}} c}\ (\Rightarrow a), (\Rightarrow b) \Rightarrow c$$

Example derivations

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$$\dfrac{\dfrac{}{b \vdash_{\mathscr{B}} b}\,\text{Ref} \qquad \dfrac{}{\vdash_{\mathscr{B}} a} \Rightarrow a}{b \vdash_{\mathscr{B}} c} \{\Rightarrow a\}, \{\Rightarrow b\} \Rightarrow c$$

Example derivations

Example

By the definition of $\vdash_{\mathscr{B}}$, deriving a from $a \fatsemi a$ in the empty base is not possible, i.e. $a \fatsemi a \vdash_{\varnothing} a$ is not possible. But $a \fatsemi a \rhd_{\varnothing} a$ is possible.

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Example

Let $\mathscr{B} = \{(\{\Rightarrow a\} \Rightarrow c), (\{a \Rightarrow c\} \Rightarrow b)\}$

$$\dfrac{\dfrac{\dfrac{\times}{a \fatsemi a \vdash_{\mathscr{B}} a}}{a \fatsemi a \vdash_{\mathscr{B}} c}\{\Rightarrow a\} \Rightarrow c}{a \vdash_{\mathscr{B}} b}\{a \Rightarrow c\} \Rightarrow b$$

We see that in this base, a is not derivable from $a \fatsemi a$.

Example derivations

Example (A possible fix)

Let $(\mathscr{B})_\star = \{((\Rightarrow a) \Rightarrow c), ((a \Rightarrow c) \Rightarrow b)\}$

$$\dfrac{\dfrac{\dfrac{}{a \,;\, a \rhd_{(\mathscr{B})_\star} a}\,\text{Ref}}{a \,;\, a \rhd_{(\mathscr{B})_\star} c}\,(\Rightarrow a) \Rightarrow c}{a \rhd_{(\mathscr{B})_\star} b}\,(a \Rightarrow c) \Rightarrow b$$

Example derivations

Example (Another possible fix)

Let $\mathscr{C} = \{(\{\Rightarrow a\} \Rightarrow c), (\{a \Rightarrow c\} \Rightarrow b), (\{\Rightarrow a\}, \{\Rightarrow a\} \Rightarrow a)\}$

$$\dfrac{\dfrac{\dfrac{\dfrac{}{a \vdash_{\mathscr{C}} a}\text{Ref} \qquad \dfrac{}{a \vdash_{\mathscr{C}} a}\text{Ref}}{a \,\mathbin{;}\, a \vdash_{\mathscr{C}} a}\{\Rightarrow a\}, \{\Rightarrow a\} \Rightarrow a}{a \,\mathbin{;}\, a \vdash_{\mathscr{C}} c}\{\Rightarrow a\} \Rightarrow c}{a \vdash_{\mathscr{C}} b}\{a \Rightarrow c\} \Rightarrow b$$

Note that $\mathscr{C} \supset \mathscr{B}$!!

Example derivations

Example (A very interesting derivation)

Let $\mathscr{B}$ be the following set of rules:

$\{(\{\Rightarrow b\} \Rightarrow c), (\{\Rightarrow a\}, \{b \Rightarrow c\} \Rightarrow d), (\{\Rightarrow d, \Rightarrow a\} \Rightarrow e), (\{a \Rightarrow e\} \Rightarrow f)\}$

$$
\begin{prooftree}
\AxiomC{}
\RightLabel{Ref}
\UnaryInfC{$a \vdash_{\mathscr{B}} a$}
\AxiomC{}
\RightLabel{Ref}
\UnaryInfC{$b \vdash_{\mathscr{B}} b$}
\RightLabel{$\{\Rightarrow b\} \Rightarrow c$}
\UnaryInfC{$b \vdash_{\mathscr{B}} c$}
\RightLabel{$\{\Rightarrow a\}, \{b \Rightarrow c\} \Rightarrow d$}
\BinaryInfC{$a \vdash_{\mathscr{B}} d$}
\AxiomC{}
\RightLabel{Ref}
\UnaryInfC{$a \vdash_{\mathscr{B}} a$}
\RightLabel{$\{\Rightarrow d, \Rightarrow a\} \Rightarrow e$}
\BinaryInfC{$a \vdash_{\mathscr{B}} e$}
\RightLabel{$\{a \Rightarrow e\} \Rightarrow f$}
\UnaryInfC{$\vdash_{\mathscr{B}} f$}
\end{prooftree}
$$

Presentation root directory

Comparing relations

Definition (Structural rules)

We define two rules:

- $\mathsf{Wk}^p_q = (\{\Rightarrow p\}, \{\Rightarrow q\} \Rightarrow q)$
- $\mathsf{Ctn}^p_q = (\{\Rightarrow p\}, \{p \,⹁\, p \Rightarrow q\} \Rightarrow q)$

Mappings between bases

Mappings between bases

Definition

Let $\mathscr{B}$ be a context-free base. We define structural contextualisation of that base $(\mathscr{B})^\star$ as follows:

$$(\mathscr{B})^\star = \{\{P_1 \Rightarrow p_1\}, \ldots, \{P_n \Rightarrow p_n\} \Rightarrow q \mid ((P_1 \Rightarrow p_1), \ldots, (P_n \Rightarrow p_n) \Rightarrow q) \in \mathscr{B}\}$$
$$\cup \{\mathsf{Wk}^p_q, \mathsf{Ctn}^p_q \mid \forall p, q \in \mathbb{A}\}$$

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$$\cup \{\mathsf{Wk}^p_q, \mathsf{Ctn}^p_q \mid \forall p, q \in \mathbb{A}\}$$

We define the decontextualisation of a contextual base as:

$$(\mathscr{B})_\star = \{(P_{1_1} \Rightarrow p_{1_1}), \ldots, (P_{n_{l_n}} \Rightarrow p_{n_{l_n}}) \Rightarrow q$$
$$\mid (\{P_{1_i} \Rightarrow p_{1_i}\}_{i=1}^{l_1}, \ldots, \{P_{n_i} \Rightarrow p_{n_i}\}_{i=1}^{l_n} \Rightarrow q) \in \mathscr{B}\}$$

Properties of these mappings

Let $\mathscr{B}$ be a context-free base. Then the following hold:

- $((\mathscr{B})^\star)_\star \supseteq \mathscr{B}$.

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Properties of these mappings

Let $\mathscr{B}$ be a context-free base. Then the following hold:

- $((\mathscr{B})^{\star})_{\star} \supseteq \mathscr{B}$.
- $(((\mathscr{B})^{\star})_{\star})^{\star} = (\mathscr{B})^{\star}$.
- For all $\mathscr{C} \supseteq \mathscr{B}$ we have that $(\mathscr{C})^{\star} \supseteq (\mathscr{B})^{\star}$.

Properties of these mappings

Let $\mathscr{B}$ be a context-free base. Then the following hold:

- $((\mathscr{B})^\star)_\star \supseteq \mathscr{B}$.
- $(((\mathscr{B})^\star)_\star)^\star = (\mathscr{B})^\star$.
- For all $\mathscr{C} \supseteq \mathscr{B}$ we have that $(\mathscr{C})^\star \supseteq (\mathscr{B})^\star$.
- For all $\mathscr{C} \supseteq (\mathscr{B})^\star$ there exists an extension $\mathscr{X} \supseteq \mathscr{B}$ such that $(\mathscr{X})^\star = \mathscr{C}$.

Structural admissibility in a structurally contextualised base

Lemma

Suppose $L \vdash_{(\mathscr{B})^\star} p$ holds. Then $R \fatsemi L \vdash_{(\mathscr{B})^\star} p$ also holds for any atomic multiset R.

Proof.

Let $R = \{r_1, \ldots, r_n\}$ for some n. Then we have that we can effectively weaken S away as follows:

$$\dfrac{\dfrac{}{r_n \vdash_{(\mathscr{B})^{\star}} r_n}\,\mathrm{Ref} \qquad L \vdash_{(\mathscr{B})^{\star}} p}{r_n \,;\, L \vdash_{(\mathscr{B})^{\star}} p}\,\mathrm{Wk}_p^{r_n}$$

$$\dfrac{\dfrac{}{r_{n-1} \vdash_{(\mathscr{B})^{\star}} r_{n-1}}\,\mathrm{Ref} \qquad r_n \,;\, L \vdash_{(\mathscr{B})^{\star}} p}{\vdots}\,\mathrm{Wk}_p^{r_{n-1}}$$

$$\dfrac{\dfrac{}{r_1 \vdash_{(\mathscr{B})^{\star}} r_1}\,\mathrm{Ref} \qquad r_2 \,;\, \cdots \,;\, r_n \,;\, L \vdash_{(\mathscr{B})^{\star}} p}{R \,;\, L \vdash_{(\mathscr{B})^{\star}} p}\,\mathrm{Wk}_p^{r_1}$$

□

Structural admissibility in a structurally contextualised base

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Structural admissibility in a structurally contextualised base

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Corollary

For arbitrary $m \geqslant 1$, if $R^m \vdash_{(\mathscr{B})^\star} p$ then $R \vdash_{(\mathscr{B})^\star} p$.

Proof.

Let $R = \{r_1, \ldots, r_n\}$ for some n. Then we have that we can effectively contract on R as follows:

$$
\dfrac{\dfrac{}{r_1 \vdash_{(\mathscr{B})^\star} r_1}\,\text{Ref} \qquad \dfrac{\dfrac{\dfrac{}{r_{n-1} \vdash_{(\mathscr{B})^\star} r_{n-1}}\,\text{Ref} \qquad \dfrac{\dfrac{}{r_n \vdash_{(\mathscr{B})^\star} r_n}\,\text{Ref} \qquad R\,,\,R \vdash_{(\mathscr{B})^\star} p}{r_1\,,\,\cdots\,,\,r_{n-1}\,,\,R \vdash_{(\mathscr{B})^\star} p}\,\text{Ctn}_p^{r_n}}{\vdots}\,\text{Ctn}_p^{r_{n-1}}}{r_1\,,\,R \vdash_{(\mathscr{B})^\star} p}}{R \vdash_{(\mathscr{B})^\star} p}\,\text{Ctn}_p^{r_1}
$$

□

Key results under this base translation

- If $R \rhd_{\mathscr{B}} p$ then $R \vdash_{(\mathscr{B})^\star} p$.

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- If $R \rhd_{\mathscr{B}} p$ then $R \vdash_{(\mathscr{B})^{\star}} p$.
- $R \vdash_{(\mathscr{B})^{\star}} p$ iff for all bases $\mathscr{X} \supseteq (\mathscr{B})^{\star}$ where for each $r \in R$ we have $\vdash_{\mathscr{X}} r$ then it follows that $\vdash_{\mathscr{X}} p$.

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- If $R \rhd_{\mathscr{B}} p$ then $R \vdash_{(\mathscr{B})^\star} p$.
- $R \vdash_{(\mathscr{B})^\star} p$ iff for all bases $\mathscr{X} \supseteq (\mathscr{B})^\star$ where for each $r \in R$ we have $\vdash_{\mathscr{X}} r$ then it follows that $\vdash_{\mathscr{X}} p$.
- $R \vdash_{(\mathscr{B})^\star} p$ iff $!\, R \Vdash^{\varnothing}_{(\mathscr{B})^\star} p$

Thank you!

Thank you for listening!
Comments? Observations? Please ask and/or feel free to email me at y.buzoku@ucl.ac.uk

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