Translations between bases in Base-extension Semantics

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Goals for this talk

- Introduce the concept of bases and general inference figures
- Introduce two notions of atomic derivability
- Show how one may relate these notions of atomic derivability

Presentation root directory

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Notation

- $\mathbb{A}$ represents a fixed, countably infinite set of propositional atoms.
- Lower case latin letters represent propositional atoms.
- Upper case latin letters represent finite multisets of propositional atoms.
- Atomic multiset is taken to mean multiset of propositional atoms.
- The sum of two multisets P and Q is denoted $P \fatsemi Q$.

Atomic rules

Atomic rules take the form:

$$\langle\langle P_1, p_1\rangle, \ldots, \langle P_n, p_n\rangle, p\rangle$$

Atomic rules

Pictorially, we can represent this as:

$$\dfrac{\begin{matrix} [P_1] & & [P_n] \\ p_1 & \dots & p_n \end{matrix}}{p}$$

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Any idea what these figures are supposed to look like?

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Natural deduction!

Contextual atomic rules

Contextual atomic rules are rules which may have contextual brackets distributed across them.

Contextual atomic rules

Pictorially, we can represent such rules as:

$$\dfrac{\left\{\begin{matrix}[P_{1_1}] & & [P_{1_{l_n}}] \\ p_{1_1} & \dots & p_{1_{l_1}}\end{matrix}\right\} \qquad \dots \qquad \left\{\begin{matrix}[P_{n_1}] & & [P_{n_{l_n}}] \\ p_{n_1} & \dots & p_{n_{l_n}}\end{matrix}\right\}}{p}$$

Contextual atomic rules

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or linearly, like this:

$$\langle\{\langle P_{1_i}, p_{1_i}\rangle\}_{i=1}^{l_1}, \dots, \{\langle P_{n_i}, p_{n_i}\rangle\}_{i=1}^{l_n}, p\rangle$$

Examples of atomic rules

- $\langle\{\langle\varnothing, a\rangle\}, c\rangle$
- $\langle\{\langle\varnothing, a\rangle\}, \{\langle c, d, e\rangle, \langle f, g\rangle\}, q\rangle$
- $\langle\{\langle a, b\rangle\}, \{\langle\varnothing, e\rangle\}, \langle f, g\rangle, q\rangle$

- $\langle\langle\varnothing, a\rangle, c\rangle$
- $\langle\langle\varnothing, b\rangle, c\rangle$
- $\langle\langle\varnothing, c\rangle, \langle a, p\rangle, \langle b, p\rangle, p\rangle$

Bases

Definition (Base)

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- $\{(\{\Rightarrow a\} \Rightarrow c), (\{a \Rightarrow c\} \Rightarrow b)\}$
- $\{(\{\Rightarrow a\} \Rightarrow c), (\{a \Rightarrow c\} \Rightarrow b), (\{\Rightarrow a\}, \{\Rightarrow a\} \Rightarrow a)\}$
- $\{(\{\Rightarrow a\} \Rightarrow c), (\{a \Rightarrow c\} \Rightarrow b), (\{\Rightarrow d\}, \{\Rightarrow d\} \Rightarrow a), (\{\Rightarrow a\} \Rightarrow d)\}$

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Atomic derivability

Definition (Context-free atomic derivability $\rhd_{\mathscr{B}}$)

The relation of derivability in a context-free base $\mathscr{B}$, is defined as so:

Ref $S \rhd_{\mathscr{B}} p$ if $p \in S$.

App For $\langle \langle P_1, p_1 \rangle, \ldots, \langle P_n, p_n \rangle, q \rangle \in \mathscr{B}$ and $S \fatsemi P_i \rhd_{\mathscr{B}} p_i$ for each $i \in \{1, \ldots, n\}$ then $S \rhd_{\mathscr{B}} q$.

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Definition (Contextual atomic derivability $\vdash_{\mathscr{B}}$)

The relation of derivability in a contextual base $\mathscr{B}$, is defined as so:

Ref $p \vdash_{\mathscr{B}} p$.

App For $\langle\{\langle P_{1_i}, p_{1_i}\rangle\}_{i=1}^{l_1}, \ldots, \{\langle P_{n_i}, p_{n_i}\rangle\}_{i=1}^{l_n}, q\rangle \in \mathscr{B}$ and $C_i \fatsemi P_{i_j} \vdash_{\mathscr{B}} p_{i_j}$ for each $i \in \{1, \ldots, n\}$ and $j \in \{1, \ldots, l_i\}$ then $C_1 \fatsemi \cdots \fatsemi C_n \vdash_{\mathscr{B}} q$.

Example derivations

Example

Let $\mathscr{B} = \{\langle \varnothing, a\rangle, \langle\langle \varnothing, a\rangle, \langle \varnothing, b\rangle, c\rangle\}$

$$\dfrac{\dfrac{}{b \rhd_{\mathscr{B}} b}\,\text{Ref} \qquad \dfrac{}{\rhd_{\mathscr{B}} a}\,\langle \varnothing, a\rangle}{b \rhd_{\mathscr{B}} c}\,\langle\langle \varnothing, a\rangle, \langle \varnothing, b\rangle, c\rangle$$

Example derivations

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Let $\mathscr{B} = \{\langle \varnothing, a \rangle, \langle \{\langle \varnothing, a \rangle\}, \{\langle \varnothing, b \rangle\}, c \rangle\}$

$$
\dfrac{\dfrac{}{b \vdash_{\mathscr{B}} b}\ \text{Ref} \qquad \dfrac{}{\vdash_{\mathscr{B}} a}\ \langle \varnothing, a \rangle}{b \vdash_{\mathscr{B}} c}\ \langle \{\langle \varnothing, a \rangle\}, \{\langle \varnothing, b \rangle\}, c \rangle
$$

Example derivations

Example

By the definition of $\vdash_{\mathscr{B}}$, deriving a from $a \fatsemi a$ in the empty base is not possible, i.e. $a \fatsemi a \vdash_{\varnothing} a$ is not possible. But $a \fatsemi a \rhd_{\varnothing} a$ is possible.

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Example

Let $\mathscr{B} = \{\langle\{\langle\varnothing, a\rangle\}, c\rangle, \langle\{\langle a, c\rangle\}, b\rangle\}$

$$\cfrac{\cfrac{\cfrac{\times}{a \fatsemi a \vdash_{\mathscr{B}} a}}{a \fatsemi a \vdash_{\mathscr{B}} c}\ \langle\{\langle\varnothing, a\rangle\}, c\rangle}{a \vdash_{\mathscr{B}} b}\ \langle\{\langle a, c\rangle\}, b\rangle$$

We see that in this base, a is not derivable from $a \fatsemi a$.

Example derivations

Example

Let $\mathscr{B} = \{\langle\langle\varnothing, a\rangle, c\rangle, \langle\langle a, c\rangle, b\rangle\}$

$$\dfrac{\dfrac{\dfrac{}{a \fatsemi a \rhd_{\mathscr{B}} a}\ \mathrm{Ref}}{a \fatsemi a \rhd_{\mathscr{B}} c}\ \langle\langle\varnothing, a\rangle, c\rangle}{a \rhd_{\mathscr{B}} b}\ \langle\langle a, c\rangle, b\rangle$$

Example derivations

Let
$\mathscr{B} = \{\langle\{\langle\varnothing, b\rangle\}, c\rangle, \langle\{\langle\varnothing, a\rangle\}, \{\langle b, c\rangle\}, d\rangle, \langle\{\langle\varnothing, d\rangle, \langle\varnothing, a\rangle\}, e\rangle, \langle\{\langle a, e\rangle\}, f\rangle\}$

$$
\dfrac{\dfrac{\dfrac{\dfrac{}{a \vdash_{\mathscr{B}} a}\,\text{Ref} \qquad \dfrac{\dfrac{}{b \vdash_{\mathscr{B}} b}\,\text{Ref}}{b \vdash_{\mathscr{B}} c}\,\langle\{\langle\varnothing, b\rangle\}, c\rangle}{a \vdash_{\mathscr{B}} d}\,\langle\{\langle\varnothing, a\rangle\}, \{\langle b, c\rangle\}, d\rangle \qquad \dfrac{}{a \vdash_{\mathscr{B}} a}\,\text{Ref}}{a \vdash_{\mathscr{B}} e}\,\langle\{\langle\varnothing, d\rangle, \langle\varnothing, a\rangle\}, e\rangle}{\vdash_{\mathscr{B}} f}\,\langle\{\langle a, e\rangle\}, f\rangle
$$

Presentation root directory

Comparing relations

Definition (Structural rules)

We define two rules:

- $\mathsf{Wk}_q^p = \langle \{\langle \varnothing, p \rangle\}, \{\langle \varnothing, q \rangle\}, q \rangle$
- $\mathsf{Ctn}_q^p = \langle \{\langle \varnothing, p \rangle\}, \{\langle p \fatcomma p, q \rangle\}, q \rangle$.

Mappings between bases

Mappings between bases

Definition

Let $\mathscr{B}$ be a context-free base. We define structural contextualisation of that base $(\mathscr{B})^\star$ as follows:

$$(\mathscr{B})^\star = \{\langle\{\langle P_1, p_1\rangle\}, \ldots, \{\langle P_n, p_n\rangle\}, q\rangle \mid \langle\langle P_1, p_1\rangle, \ldots, \langle P_n, p_n\rangle, q\rangle \in \mathscr{B}\} \\ \cup \{\mathsf{Wk}^p_q, \mathsf{Ctn}^p_q \mid \forall p, q \in \mathbb{A}\}$$

We define the decontextualisation of a contextual base $(\mathscr{B})_\star$ as:

$$(\mathscr{B})_\star = \{\langle\langle P_{1_1}, p_{1_1}\rangle, \ldots, \langle P_{n_{l_n}}, p_{n_{l_n}}\rangle, q\rangle \mid \langle\{(P_{1_i}, p_{1_i})\}_{i=1}^{l_1}, \ldots, \{(P_{n_i}, p_{n_i})\}_{i=1}^{l_n}, q\rangle$$

Properties of these mappings

Let $\mathscr{B}$ be a context-free base. Then the following hold:

- $((\mathscr{B})^{\star})_{\star} \supseteq \mathscr{B}$.

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Properties of these mappings

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- For all $\mathscr{C} \supseteq \mathscr{B}$ we have that $(\mathscr{C})^{\star} \supseteq (\mathscr{B})^{\star}$.

Properties of these mappings

Let $\mathscr{B}$ be a context-free base. Then the following hold:

- $((\mathscr{B})^\star)_\star \supseteq \mathscr{B}$.
- $(((\mathscr{B})^\star)_\star)^\star = (\mathscr{B})^\star$.
- For all $\mathscr{C} \supseteq \mathscr{B}$ we have that $(\mathscr{C})^\star \supseteq (\mathscr{B})^\star$.
- For all $\mathscr{C} \supseteq (\mathscr{B})^\star$ there exists an extension $\mathscr{X} \supseteq \mathscr{B}$ such that $(\mathscr{X})^\star = \mathscr{C}$.

Structural admissibility in a structurally contextualised base

Lemma

Suppose $L \vdash_{(\mathscr{B})^\star} p$ holds. Then $S \fatcomma L \vdash_{(\mathscr{B})^\star} p$ also holds for any atomic multiset S.

Proof.

Let $S = \{s_1 \fatsemi \cdots \fatsemi s_n\}$ for some n. Then we have that we can effectively weaken S away as follows:

$$
\dfrac{\dfrac{}{s_1 \vdash_{(\mathscr{B})^\star} s_1}\,\text{Ref} \qquad
\begin{array}{c}
\dfrac{\dfrac{}{s_{n-1} \vdash_{(\mathscr{B})^\star} s_{n-1}}\,\text{Ref} \qquad \dfrac{\dfrac{}{s_n \vdash_{(\mathscr{B})^\star} s_n}\,\text{Ref} \qquad L \vdash_{(\mathscr{B})^\star} p}{s_n \fatsemi L \vdash_{(\mathscr{B})^\star} p}\,\mathrm{Wk}_p^{s_n}}{\vdots}\,\mathrm{Wk}_p^{s_{n-1}} \\
s_2 \fatsemi \cdots \fatsemi s_n \vdash_{(\mathscr{B})^\star} p
\end{array}}{S \fatsemi L \vdash_{(\mathscr{B})^\star} p}\,\mathrm{Wk}_p^{s_1}
$$

$\square$

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Corollary

For arbitrary $m \geqslant 1$*, if* $L^m \vdash_{(\mathscr{B})^\star} p$ *then* $L \vdash_{(\mathscr{B})^\star} p$*.*

Proof.

Let $L = \{l_1 \fatcomma \cdots \fatcomma l_n\}$ for some n. Then we have that we can effectively contract on L as follows:

$$\begin{array}{c}
\dfrac{\dfrac{}{l_{n-1} \vdash_{(\mathscr{B})^\star} l_{n-1}}\,\text{Ref} \qquad \dfrac{\dfrac{}{l_n \vdash_{(\mathscr{B})^\star} l_n}\,\text{Ref} \qquad L \fatcomma L \vdash_{(\mathscr{B})^\star} p}{l_1 \fatcomma \cdots \fatcomma l_{n-1} \fatcomma L \vdash_{(\mathscr{B})^\star} p}\,\text{Ctn}_p^{l_n}}{}\,\text{Ctn}_p^{l_{n-1}} \\
\vdots \\
\dfrac{\dfrac{}{l_1 \vdash_{(\mathscr{B})^\star} l_1}\,\text{Ref} \qquad l_1 \fatcomma L \vdash_{(\mathscr{B})^\star} p}{L \vdash_{(\mathscr{B})^\star} p}\,\text{Ctn}_p^{l_1}
\end{array}$$

$\square$

Key results under this base translation

- If $L \rhd_{\mathscr{B}} p$ then $L \vdash_{(\mathscr{B})^{\star}} p$.

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- $L \vdash_{(\mathscr{B})^\star} p$ iff for all bases $\mathscr{X} \supseteq (\mathscr{B})^\star$ where for each $l \in L$ we have $\vdash_{\mathscr{X}} l$ then it follows that $\vdash_{\mathscr{X}} p$.

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- $L \vdash_{(\mathscr{B})^\star} p$ iff $!\, L \Vdash^{\varnothing}_{(\mathscr{B})^\star} p$

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- $L \vdash_{(\mathscr{B})^\star} p$ iff $!\,L \Vdash^{\varnothing}_{(\mathscr{B})^\star} p$
- $\Vdash^{L}_{(\mathscr{B})^\star} p$ iff $!\,L \Vdash^{\varnothing}_{(\mathscr{B})^\star} p$

Thank you!

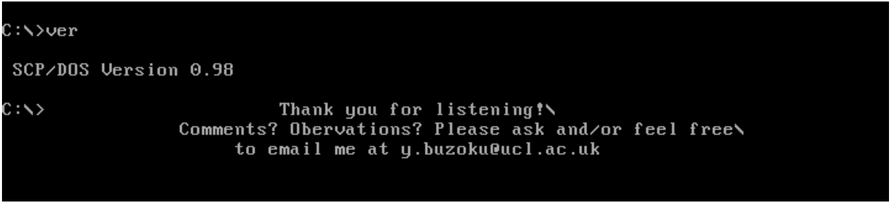

Figure: Thank you from DOS! :D

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