

Transfer viva: Proof-theoretic Semantics for ILL and beyond

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Goals for this talk

- To cover the work done in obtaining a proof-theoretic semantics for ILL.
- To highlight some interesting aspects of the results obtained.
- To present an alternative semantics for ILL.
- To discuss my future plans for my PhD.

A natural deduction system for ILL

$$\begin{array}{c}
 \frac{}{\varphi \vdash \varphi} \text{Ax} \\
 \frac{\Gamma, \varphi \vdash \psi}{\Gamma \vdash \varphi \multimap \psi} \multimap\text{-I} \\
 \frac{\Gamma \vdash \varphi \quad \Delta \vdash \psi}{\Gamma, \Delta \vdash \varphi \otimes \psi} \otimes\text{-I} \\
 \frac{}{\vdash 1} 1\text{-I} \\
 \frac{\Gamma \vdash \varphi \quad \Gamma \vdash \psi}{\Gamma \vdash \varphi \& \psi} \&\text{-I} \\
 \frac{\Gamma \vdash \varphi_i}{\Gamma \vdash \varphi_0 \oplus \varphi_1} \oplus\text{-I}_i \\
 \frac{\Gamma_1 \vdash \varphi_1 \quad \dots \quad \Gamma_n \vdash \varphi_n}{\Gamma_1, \dots, \Gamma_n \vdash \top} \top\text{-I} \\
 \frac{\Gamma \vdash \varphi \multimap \psi \quad \Delta \vdash \varphi}{\Gamma, \Delta \vdash \psi} \multimap\text{-E} \\
 \frac{\Gamma \vdash \varphi \otimes \psi \quad \Delta, \varphi, \psi \vdash \chi}{\Gamma, \Delta \vdash \chi} \otimes\text{-E} \\
 \frac{\Gamma \vdash \varphi \quad \Delta \vdash 1}{\Gamma, \Delta \vdash \varphi} 1\text{-E} \\
 \frac{\Gamma \vdash \varphi_0 \& \varphi_1}{\Gamma \vdash \varphi_i} \&\text{-E}_i \\
 \frac{\Gamma \vdash \varphi \oplus \psi \quad \Delta, \varphi \vdash \chi \quad \Delta, \psi \vdash \chi}{\Gamma, \Delta \vdash \chi} \oplus\text{-E} \\
 \frac{\Gamma \vdash 0}{\Gamma \vdash \varphi} 0\text{-E}
 \end{array}$$

A natural deduction system for ILL (cont.)

$$\frac{\Gamma_1 \vdash !\psi_1 \quad \dots \quad \Gamma_n \vdash !\psi_n \quad !\psi_1, \dots, !\psi_n \vdash \varphi}{\Gamma_1, \dots, \Gamma_n \vdash !\varphi} \text{!-Promotion}$$

$$\frac{\Gamma \vdash !\varphi \quad \Delta, \varphi \vdash \psi}{\Gamma, \Delta \vdash \psi} \text{!-Dereliction}$$

$$\frac{\Gamma \vdash !\varphi \quad \Delta \vdash \psi}{\Gamma, \Delta \vdash \psi} \text{!-Weakening}$$

$$\frac{\Gamma \vdash !\varphi \quad \Delta, !\varphi, !\varphi \vdash \psi}{\Gamma, \Delta \vdash \psi} \text{!-Contraction}$$

Alternative systems

- One may consider other natural deduction systems.

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- One may consider other natural deduction systems.
- The exponential has two rules only in the system of Negri:

$$\frac{(!\varphi)^m, \Gamma \vdash \chi \quad \Delta_1 \vdash !\psi_1 \dots \Delta_n \vdash !\psi_n \quad !\psi_1, \dots, !\psi_n \vdash \varphi}{\Gamma, \Delta_1, \dots, \Delta_n \vdash \chi} !-I$$

$$\frac{\Gamma \vdash !\varphi \quad \Delta, \varphi \vdash \psi}{\Gamma, \Delta \vdash \psi} !-E$$

Substructural atomic derivability

Definition (Basic rules)

Basic rules take the following form:

$$\{(P_{1_i} \Rightarrow q_{1_i})\}_{i=1}^{l_1}, \dots, \{(P_{n_i} \Rightarrow q_{n_i})\}_{i=1}^{l_n} \Rightarrow r$$

where

- Each P_i is an atomic multiset, called a premiss multiset.
- Each q_i and r is an atomic proposition.
- Each $(P_i \Rightarrow q_i)$ is a pair (P_i, q_i) called an atomic sequent.
- Each collection $\{(P_{1_i} \Rightarrow q_{1_i}), \dots, (P_{l_i} \Rightarrow q_{l_i})\}$ is called an atomic box.

Substructural atomic derivability

Definition (Basic derivability relation)

The relation of derivability in a base \mathcal{B} , is defined inductively as so:

Ref $p \vdash_{\mathcal{B}} p$

App Given that $(\{(P_{1_i} \Rightarrow q_{1_i})\}_{i=1}^{l_1}, \dots, \{(P_{n_i} \Rightarrow q_{n_i})\}_{i=1}^{l_n} \Rightarrow r) \in \mathcal{B}$ and atomic multisets C_i such that the following hold:

$$C_i \text{ , } P_{i_j} \vdash_{\mathcal{B}} q_{i_j} \text{ for all } i = 1, \dots, n \text{ and } j = 1, \dots, l_i$$

Then $C_1 \text{ , } \dots \text{ , } C_n \vdash_{\mathcal{B}} r$.

| | | | |
|---------------|--|-----|---|
| (At) | $\Vdash_{\mathcal{B}}^L p$ | iff | $L \vdash_{\mathcal{B}} p$ |
| (\multimap) | $\Vdash_{\mathcal{B}}^L \varphi \multimap \psi$ | iff | $\varphi \Vdash_{\mathcal{B}}^L \psi$ |
| (\otimes) | $\Vdash_{\mathcal{B}}^L \varphi \otimes \psi$ | iff | for any \mathcal{C} such that $\mathcal{B} \subseteq \mathcal{C}$, atomic multisets K and any $p \in \mathbb{A}$, if $\varphi, \psi \Vdash_{\mathcal{B}}^K p$ then $\Vdash_{\mathcal{C}}^{L_2 K} p$ |
| (1) | $\Vdash_{\mathcal{B}}^L 1$ | iff | for any \mathcal{C} such that $\mathcal{B} \subseteq \mathcal{C}$, atomic multisets K and any $p \in \mathbb{A}$, if $\Vdash_{\mathcal{B}}^K p$ then $\Vdash_{\mathcal{C}}^{L_2 K} p$ |
| (\top) | $\Vdash_{\mathcal{B}}^L \top$ | iff | always |
| $(\&)$ | $\Vdash_{\mathcal{B}}^L \varphi \& \psi$ | iff | $\Vdash_{\mathcal{B}}^L \varphi$ and $\Vdash_{\mathcal{B}}^L \psi$ |
| (\oplus) | $\Vdash_{\mathcal{B}}^L \varphi \oplus \psi$ | iff | for any \mathcal{C} such that $\mathcal{B} \subseteq \mathcal{C}$, atomic multisets K and any $p \in \mathbb{A}$, if $\varphi \Vdash_{\mathcal{C}}^K p$ and $\psi \Vdash_{\mathcal{C}}^K p$, then $\Vdash_{\mathcal{C}}^{L_2 K} p$ |
| (0) | $\Vdash_{\mathcal{B}}^L 0$ | iff | $\Vdash_{\mathcal{B}}^L p$ for any $p \in \mathbb{A}$ |
| $(!)$ | $\Vdash_{\mathcal{B}}^L !\varphi$ | iff | for any \mathcal{C} such that $\mathcal{B} \subseteq \mathcal{C}$, atomic multisets K and any $p \in \mathbb{A}$, if for any \mathcal{D} such that $\mathcal{C} \subseteq \mathcal{D}$, (if $\Vdash_{\mathcal{D}}^{\emptyset} \varphi$ then $\Vdash_{\mathcal{D}}^L p$) then $\Vdash_{\mathcal{C}}^{L_2 K} p$ |
| $(,)$ | $\Vdash_{\mathcal{B}}^L \Gamma, \Delta$ | iff | there exists multisets K and M such that $L = K, M$ and $\Vdash_{\mathcal{B}}^K \Gamma$ and $\Vdash_{\mathcal{B}}^M \Delta$ |
| (Inf) | $!\Delta, \Theta \Vdash_{\mathcal{B}}^L \varphi$ | iff | for any \mathcal{C} such that $\mathcal{B} \subseteq \mathcal{C}$, atomic multisets K , if $\Vdash_{\mathcal{C}}^{\emptyset} \Delta$ and $\Vdash_{\mathcal{C}}^K \Theta$ then $\Vdash_{\mathcal{C}}^{L_2 K} \varphi$ |

Notes on $\Vdash_{\mathcal{B}}^L$

- The sequent $\langle \Gamma, \varphi \rangle$ is said to be valid if and only if $\Gamma \Vdash_{\emptyset}^{\emptyset} \varphi$ holds.
- We frequently write this as $\Gamma \Vdash \varphi$.

Notes on $\Vdash_{\mathcal{B}}^L$

- If $\Vdash_{\mathcal{B}}^L \varphi$ then for all $\mathcal{C} \supseteq \mathcal{B}$ we have $\Vdash_{\mathcal{C}}^L \varphi$.
- Given $\Gamma \Vdash_{\mathcal{B}}^L \varphi$ and $\Vdash_{\mathcal{B}}^K \Gamma$, then it holds that $\Vdash_{\mathcal{B}}^{L,K} \varphi$.

Why the extension to Inf?

- Up until now, all Base-extension Semantics have had a simple (Inf) clause.
- For example in IMALL, it suffices to take the following:
 $\Gamma \Vdash_{\mathcal{B}}^L \varphi$ iff for all $\mathcal{C} \supseteq \mathcal{B}$ and K , $\Vdash_{\mathcal{C}}^K \Gamma$ implies $\Vdash_{\mathcal{C}}^{L,K} \varphi$

Why the extension to Inf?

Question: Is this still an inferentialist semantics?

Can we better understand the exponential?

- $! \varphi \Vdash \varphi \otimes \dots \otimes \varphi$
- $\Vdash_{\mathcal{B}}^L !(\varphi \& \psi) \text{ iff } \Vdash_{\mathcal{B}}^L ! \varphi \otimes ! \psi$

Expanding what we know

$! \varphi \Vdash \varphi \otimes \dots \otimes \varphi$ iff

for all bases \mathcal{B} and atomic multisets L , such that $\Vdash_{\mathcal{B}}^L ! \varphi$ then

$\Vdash_{\mathcal{B}}^L \varphi \otimes \dots \otimes \varphi$

Expanding what we know

- So when does $\Vdash_{\mathcal{B}}^L \varphi \otimes \dots \otimes \varphi$ hold?

Expanding what we know

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Considering the case when $\varphi = p$ for simplicity we get the following:

- Given a base \mathcal{B} such that $\vdash_{\mathcal{B}} p$ and $L = \emptyset$.
- Then we have that $\Vdash_{\mathcal{B}}^L p \otimes \dots \otimes p$.
- Examples of such bases:
 - $\mathcal{B} = \{\emptyset \Rightarrow p\}$
 - $\mathcal{B} = \{\emptyset \Rightarrow q, q \Rightarrow p\}$

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- Examples of such bases:
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 - $\mathcal{B} = \{\emptyset \Rightarrow q, q \Rightarrow p\}$

Note that if $L \neq \emptyset$ then the only way that this relation could hold is if the atoms were derivable from within the base, i.e. if were valid $\Vdash_{\mathcal{B}}^{\emptyset} L$.

So ultimately, $\Vdash_{\mathcal{B}}^{\emptyset} p \otimes \dots \otimes p$.

Expanding what we know

Inferring from $! \varphi$ should imply that $\Vdash_{\mathcal{B}}^{\emptyset} \varphi$.

A proof-theoretic flavour of the !

We now consider the second identity: $\Vdash_{\mathcal{B}}^L !(\varphi \& \psi)$ iff $\Vdash_{\mathcal{B}}^L !\varphi \otimes !\psi$ with $\psi = \top$. Furthermore since $\varphi \& \top \equiv \varphi$ we go as follows:

$$\Vdash_{\mathcal{B}}^L !\varphi \text{ iff } \Vdash_{\mathcal{B}}^L !\varphi \otimes !\top$$

iff for all $\mathcal{C} \supseteq \mathcal{B}$, K and $p \in \mathbb{A}$, $!\varphi, !\top \Vdash_{\mathcal{C}}^K p$ implies $\Vdash_{\mathcal{C}}^{L,K} p$

iff for all $\mathcal{C} \supseteq \mathcal{B}$, K and $p \in \mathbb{A}$,

if (for all $\mathcal{D} \supseteq \mathcal{C}$, if $\Vdash_{\mathcal{D}}^{\emptyset} \varphi$ and $\Vdash_{\mathcal{D}}^{\emptyset} \top$ then $\Vdash_{\mathcal{D}}^K p$) then $\Vdash_{\mathcal{C}}^{L,K} p$

iff for all $\mathcal{C} \supseteq \mathcal{B}$, K and $p \in \mathbb{A}$,

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Some failed attempts

- The most convincing hypothesis was that $\Vdash_{\mathcal{B}}^L \varphi$ iff $L = \emptyset$ and for all $\mathcal{C} \supseteq \mathcal{B}$, atomic multisets K and atoms p , if $\varphi \Vdash_{\mathcal{C}}^K p$ then $\Vdash_{\mathcal{C}}^K p$.

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- This however fails to be complete as it requires us to be able to prove from $L \vdash_{\mathcal{B}} (!\varphi)^b$ that $L = \emptyset$, to pass Proposition 31.

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Some failed attempts

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- This however fails to be complete as it requires us to be able to prove from $L \vdash_{\mathcal{B}} (!\varphi)^b$ that $L = \emptyset$, to pass Proposition 31.
- Attempts to generalise this by replacing $L = \emptyset$ with $\Vdash_{\mathcal{B}}^{\emptyset} L$ fail for the same reason.
- A different attempt, based on a notion of closure was to define $\Vdash_{\mathcal{B}}^L !\varphi$ iff for every K, p and $\mathcal{C} \supseteq \mathcal{B}$ if we have $\varphi \Vdash_{\mathcal{C}}^K p$ then $\Vdash_{\mathcal{C}}^{L,K} p$ and $\ulcorner \forall \ulcorner M (\Vdash_{\mathcal{B}}^M \varphi \urcorner \Rightarrow \ulcorner \Vdash_{\mathcal{B}}^{\emptyset} M) \urcorner$ also fails as there is no way to prove the conclusion of the second conditional.

The idea of a zoned semantics

- Not absurd to distinguish basic sentences based on whether they are to be used as a hypothesis in an intuitionistic or a resourceful way.

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- Not absurd to distinguish basic sentences based on whether they are to be used as a hypothesis in an intuitionistic or a resourceful way.
- Thus we define a new type of basic rule that looks something like this:

$$\begin{array}{ccc}
 [P_1; C_1] & & [P_n; C_n] \\
 \vdots & & \vdots \\
 q_1 & \dots & q_n
 \end{array}
 \frac{}{r} \mathcal{R}$$

Exploring the zoned semantics

Definition (Zoned basic derivability relation)

The relation of derivability in a base \mathcal{B} , is defined inductively as so:

Ref $S; T \vdash_{\mathcal{B}} p$ iff $(p \in S \text{ and } T = \emptyset)$ or $T = [p]$.

App Given that

$(\{(P_{1_i}; C_{1_i} \Rightarrow q_{1_i})\}_{i=1}^{l_1}, \dots, \{(P_{n_i}; C_{n_i} \Rightarrow q_{n_i})\}_{i=1}^{l_n} \Rightarrow r) \in \mathcal{B}$ and atomic multisets G and C_i such that the following hold:

$$G, P_{ij}; L_i, C_{ij} \vdash_{\mathcal{B}} q_{ij} \text{ for all } i = 1, \dots, n \text{ and } j = 1, \dots, l_i$$

Then $G; L_1, \dots, L_n \vdash_{\mathcal{B}} r$.

Exploring the zoned semantics

- $\Vdash_{\mathcal{B}}^{G;L} p$ iff $G; L \vdash_{\mathcal{B}} p$
- $\Vdash_{\mathcal{B}}^{G;L} \Gamma, \Delta$ iff there exists $L = U, V$ such that $\Vdash_{\mathcal{B}}^{G;U} \Gamma$ and $\Vdash_{\mathcal{B}}^{G;V} \Delta$
- $\Vdash_{\mathcal{B}}^{G;L} !\varphi$ iff for every base $\mathcal{C} \supseteq \mathcal{B}$, atomic multisets h, K and atom p , such that $\varphi; \cdot \Vdash_{\mathcal{C}}^{H;K} p$ hold, then $\Vdash_{\mathcal{C}}^{G;H;L,K} p$.
- $\Gamma; \Delta \Vdash_{\mathcal{B}}^{G;L} \varphi$ iff for every $\mathcal{C} \supseteq \mathcal{B}$ and atomic multisets H, K such that $\Vdash_{\mathcal{C}}^{H;\emptyset} \Gamma$ and $\Vdash_{\mathcal{C}}^{G;H;L,K} \Delta$ then $\Vdash_{\mathcal{C}}^{G;H;L,K} \varphi$

Some other work undertaken

- Investigated alternative IPL support relations that are more “resource”-aware.
- Started investigating translations between Sandqvist’s semantics for IPL and the semantics presented here for ILL.
- Very recently, I have been working on a Base-extension semantics for Lambek Calculus.

Future directions of work

- Study how rules are generally translated from Sandqvists bases for IPL to the bases I presented here.
- Give a Proof-theoretic Semantics for Classical Linear Logic.
- Similarly, study other similar logics such as DILL, FILL and what happens when we have subexponentials.
- Give a categorical interpretation of the semantics presented herein.
- Investigate the connection to Linear Logic programming.
- Develop a connection to the Kripke semantics of Miller for his fragment of ILL.

Thank you!

Thank you for listening!