Transfer viva: Proof-theoretic Semantics for ILL and beyond

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Goals for this talk

- To cover the work done in obtaining a proof-theoretic semantics for ILL.
- To discuss my future plans for my PhD.

A natural deduction system for ILL

$$\frac{}{\varphi \vdash \varphi}\ \mathsf{Ax}$$

$$\frac{\Gamma, \varphi \vdash \psi}{\Gamma \vdash \varphi \multimap \psi}\ \multimap\text{-I} \qquad \frac{\Gamma \vdash \varphi \multimap \psi \qquad \Delta \vdash \varphi}{\Gamma, \Delta \vdash \psi}\ \multimap\text{-E}$$

$$\frac{\Gamma \vdash \varphi \qquad \Delta \vdash \psi}{\Gamma, \Delta \vdash \varphi \otimes \psi}\ \otimes\text{-I} \qquad \frac{\Gamma \vdash \varphi \otimes \psi \qquad \Delta, \varphi, \psi \vdash \chi}{\Gamma, \Delta \vdash \chi}\ \otimes\text{-E}$$

$$\frac{}{\vdash 1}\ 1\text{-I} \qquad \frac{\Gamma \vdash \varphi \qquad \Delta \vdash 1}{\Gamma, \Delta \vdash \varphi}\ 1\text{-E}$$

$$\frac{\Gamma \vdash \varphi \qquad \Gamma \vdash \psi}{\Gamma \vdash \varphi \,\&\, \psi}\ \&\text{-I} \qquad \frac{\Gamma \vdash \varphi_0 \,\&\, \varphi_1}{\Gamma \vdash \varphi_i}\ \&\text{-E}_i$$

$$\frac{\Gamma \vdash \varphi_i}{\Gamma \vdash \varphi_0 \oplus \varphi_1}\ \oplus\text{-I}_i \qquad \frac{\Gamma \vdash \varphi \oplus \psi \qquad \Delta, \varphi \vdash \chi \qquad \Delta, \psi \vdash \chi}{\Gamma, \Delta \vdash \chi}\ \oplus\text{-E}$$

No 0 intro rule

$$\frac{\Gamma \vdash 0}{\Gamma \vdash \varphi}\ 0\text{-E}$$

A natural deduction system for ILL (cont.)

$$\frac{\Gamma_1 \vdash \,!\,\psi_1 \qquad \ldots \qquad \Gamma_n \vdash \,!\,\psi_n \qquad !\,\psi_1, \ldots, !\,\psi_n \vdash \varphi}{\Gamma_1, \ldots, \Gamma_n \vdash \,!\,\varphi}\ \text{!-Promotion}$$

$$\frac{\Gamma \vdash \,!\,\varphi \qquad \Delta, \varphi \vdash \psi}{\Gamma, \Delta \vdash \psi}\ \text{!-Dereliction}$$

$$\frac{\Gamma \vdash \,!\,\varphi \qquad \Delta \vdash \psi}{\Gamma, \Delta \vdash \psi}\ \text{!-Weakening}$$

$$\frac{\Gamma \vdash \,!\,\varphi \qquad \Delta, !\,\varphi, !\,\varphi \vdash \psi}{\Gamma, \Delta \vdash \psi}\ \text{!-Contraction}$$

Alternative systems

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- The exponential has two rules only in the system of Negri:

$$\frac{(!\,\varphi)^m, \Gamma \vdash \chi \qquad \Delta_1 \vdash !\,\psi_1 \ldots \Delta_n \vdash !\,\psi_n \qquad !\,\psi_1, \ldots, !\,\psi_n \vdash \varphi}{\Gamma, \Delta_1, \ldots \Delta_n \vdash \chi}\ !\text{-I}$$

$$\frac{\Gamma \vdash !\,\varphi \qquad \Delta, \varphi \vdash \psi}{\Gamma, \Delta \vdash \psi}\ !\text{-E}$$

Weakening and Contraction derivability

Proof.

It follows immediately by setting $n = 1$ and $m = 2$ for contraction and $m = 0$ for weakening in the introduction rule, $\psi = \varphi$ and $\Delta = !\,\varphi$ that

$$\dfrac{(!\,\varphi)^m, \Gamma \vdash \chi \qquad !\,\varphi \vdash !\,\varphi \qquad \dfrac{!\,\varphi \vdash !\,\varphi \qquad \varphi \vdash \varphi}{!\,\varphi \vdash \varphi}\,!\mathrm{E}}{\Gamma, !\,\varphi \vdash \chi}\,!\mathrm{I}$$

□

Substructural atomic derivability

Definition

Atomic rules Basic rules take the following form:

$$\{(P_{1_i} \Rightarrow q_{1_i})\}_{i=1}^{l_1}, \ldots, \{(P_{n_i} \Rightarrow q_{n_i})\}_{i=1}^{l_n} \Rightarrow r$$

where

- Each P_i is an atomic multiset, called a premiss multiset.
- Each q_i and r is an atomic proposition.
- Each $(P_i \Rightarrow q_i)$ is a pair (P_i, q_i) called an atomic sequent.
- Each collection $\{(P_{i_1} \Rightarrow q_{i_1}), \ldots, (P_{i_{l_i}} \Rightarrow q_{i_{l_i}})\}$ is called an atomic box.

Substructural atomic derivability

Definition (Basic derivability relation)

The relation of derivability in a base $\mathscr{B}$, is defined inductively as so:

Ref $p \vdash_{\mathscr{B}} p$

App Given that $(\{(P_{1_i} \Rightarrow q_{1_i})\}_{i=1}^{l_1}, \ldots, \{(P_{n_i} \Rightarrow q_{n_i})\}_{i=1}^{l_n} \Rightarrow r) \in \mathscr{B}$ and atomic multisets C_i such that the following hold:

$$C_i \fatsemi P_{i_j} \vdash_{\mathscr{B}} q_{i_j} \text{ for all } i = 1, \ldots, n \text{ and } j = 1, \ldots, l_i$$

Then $C_1 \fatsemi \cdots \fatsemi C_n \vdash_{\mathscr{B}} r$.

(At)	$\Vdash^{L}_{\mathscr{B}} p$	iff	$L \vdash_{\mathscr{B}} p$
$(\multimap)$	$\Vdash^{L}_{\mathscr{B}} \varphi \multimap \psi$	iff	$\varphi \Vdash^{L}_{\mathscr{B}} \psi$
$(\otimes)$	$\Vdash^{L}_{\mathscr{B}} \varphi \otimes \psi$	iff	for any $\mathscr{C}$ such that $\mathscr{B} \subseteq \mathscr{C}$, atomic multisets K and any $p \in \mathbb{A}$, if $\varphi \fatcomma \psi \Vdash^{K}_{\mathscr{B}} p$ then $\Vdash^{L \fatcomma K}_{\mathscr{C}} p$
(1)	$\Vdash^{L}_{\mathscr{B}} 1$	iff	for any $\mathscr{C}$ such that $\mathscr{B} \subseteq \mathscr{C}$, atomic multisets K and any $p \in \mathbb{A}$, if $\Vdash^{K}_{\mathscr{B}} p$ then $\Vdash^{L \fatcomma K}_{\mathscr{C}} p$
$(\top)$	$\Vdash^{L}_{\mathscr{B}} \top$	iff	always
$(\&)$	$\Vdash^{L}_{\mathscr{B}} \varphi \,\&\, \psi$	iff	$\Vdash^{L}_{\mathscr{B}} \varphi$ and $\Vdash^{L}_{\mathscr{B}} \psi$
$(\oplus)$	$\Vdash^{L}_{\mathscr{B}} \varphi \oplus \psi$	iff	for any $\mathscr{C}$ such that $\mathscr{B} \subseteq \mathscr{C}$, atomic multisets K and any $p \in \mathbb{A}$, if $\varphi \Vdash^{K}_{\mathscr{C}} p$ and $\psi \Vdash^{K}_{\mathscr{C}} p$, then $\Vdash^{L \fatcomma K}_{\mathscr{C}} p$
(0)	$\Vdash^{L}_{\mathscr{B}} 0$	iff	$\Vdash^{L}_{\mathscr{B}} p$ for any $p \in \mathbb{A}$
$(!)$	$\Vdash^{L}_{\mathscr{B}} !\,\varphi$	iff	for any $\mathscr{C}$ such that $\mathscr{B} \subseteq \mathscr{C}$, atomic multisets K and any $p \in \mathbb{A}$, if for any $\mathscr{D}$ such that $\mathscr{C} \subseteq \mathscr{D}$, (if $\Vdash^{\varnothing}_{\mathscr{D}} \varphi$ then $\Vdash^{L}_{\mathscr{D}} p$) then $\Vdash^{L \fatcomma K}_{\mathscr{C}} p$
$(\fatcomma)$	$\Vdash^{L}_{\mathscr{B}} \Gamma \fatcomma \Delta$	iff	there exists multisets K and M such that $L = K \fatcomma M$ and $\Vdash^{K}_{\mathscr{B}} \Gamma$ and $\Vdash^{M}_{\mathscr{B}} \Delta$
(Inf)	$!\,\Delta \fatcomma \Theta \Vdash^{L}_{\mathscr{B}} \varphi$	iff	for any $\mathscr{C}$ such that $\mathscr{B} \subseteq \mathscr{C}$, atomic multisets K, if $\Vdash^{\varnothing}_{\mathscr{C}} \Delta$ and $\Vdash^{K}_{\mathscr{C}} \Theta$ then $\Vdash^{L \fatcomma K}_{\mathscr{C}} \varphi$

Notes on $\Vdash^{L}_{\mathscr{B}}$

- The sequent $\langle \Gamma, \varphi \rangle$ is said to be valid if and only if $\Gamma \Vdash^{\varnothing}_{\varnothing} \varphi$ holds.
- We frequently write this as $\Gamma \Vdash \varphi$.

Notes on $\Vdash^{L}_{\mathscr{B}}$

- If $\Vdash^{L}_{\mathscr{B}} \varphi$ then for all $\mathscr{C} \supseteq \mathscr{B}$ we have $\Vdash^{L}_{\mathscr{C}} \varphi$.
- $\Vdash^{L}_{\mathscr{B}} \varphi$ iff $\Vdash^{L}_{\mathscr{B}} \varphi \otimes 1$ iff $\Vdash^{L}_{\mathscr{B}} \varphi \,{}_{9}\, 1$
- Given $\Gamma \Vdash^{L}_{\mathscr{B}} \varphi$ and $\Vdash^{K}_{\mathscr{B}} \Gamma$, then it holds that $\Vdash^{L {}_{9} K}_{\mathscr{B}} \varphi$.

Why the extension to Inf?

- Up until now, all Base-extension Semantics have had a simple (Inf) clause.
- For example in IMALL, it suffices to take the following:
  $\Gamma \Vdash^{L}_{\mathscr{B}} \varphi$ iff for all $\mathscr{C} \supseteq \mathscr{B}$ and K, $\Vdash^{K}_{\mathscr{C}} \Gamma$ implies $\Vdash^{L,K}_{\mathscr{C}} \varphi$

Why the extension to Inf?

Are we still inferentialist?

Can we understand the exponential better?

- $!\,\varphi \Vdash \varphi \otimes \ldots \otimes \varphi$
- $\Vdash^{L}_{\mathscr{B}} !(\varphi \mathbin{\&} \psi)$ iff $\Vdash^{L}_{\mathscr{B}} !\,\varphi \otimes !\,\psi$

Expanding what we know

$! \varphi \Vdash \varphi \otimes \ldots \otimes \varphi$ iff

for all bases $\mathscr{B}$ and atomic multisets L, such that $\Vdash^{L}_{\mathscr{B}} ! \varphi$ then $\Vdash^{L}_{\mathscr{B}} \varphi \otimes \ldots \otimes \varphi$

Expanding what we know

- So when does $\Vdash^{L}_{\mathscr{B}} \varphi \otimes \ldots \otimes \varphi$ hold?

Expanding what we know

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- Given a base $\mathscr{B}$ such that $\vdash_{\mathscr{B}} p$ and $L = \varnothing$.
- Then we have that $\Vdash^{L}_{\mathscr{B}} p \otimes \ldots \otimes p$.
- Examples of such bases:
  - $\mathscr{B} = \{\varnothing \Rightarrow p\}$
  - $\mathscr{B} = \{\varnothing \Rightarrow q,\ q \Rightarrow p\}$

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Note that if $L \neq \varnothing$ then the only way that this relation could hold is if the atoms were derivable from within the base, i.e. if were valid $\Vdash^{\varnothing}_{\mathscr{B}} L$.
So ultimately, $\Vdash^{\varnothing}_{\mathscr{B}} p \otimes \ldots \otimes p$.

Expanding what we know

Inferring from $! \varphi$ *should imply that* $\Vdash_{\mathscr{B}}^{\varnothing} \varphi$.

A proof-theoretic flavour of the !

We now consider the second identity: $\Vdash^{L}_{\mathscr{B}} !(\varphi \,\&\, \psi)$ iff $\Vdash^{L}_{\mathscr{B}} !\,\varphi \otimes !\,\psi$ with $\psi = \top$. Furthermore since $\varphi \,\&\, \top \equiv \varphi$ we go as follows:

$$\begin{aligned}
\Vdash^{L}_{\mathscr{B}} !\,\varphi \text{ iff } & \Vdash^{L}_{\mathscr{B}} !\,\varphi \otimes !\,\top && (1)\\
\text{iff } & \Vdash^{L}_{\mathscr{B}} !\,\varphi \otimes 1 && (2)\\
\text{iff } & \text{for all } \mathscr{C} \supseteq \mathscr{B}, K \text{ and } p \in \mathbb{A}, !\,\varphi \fatsemi 1 \Vdash^{K}_{\mathscr{C}} p \text{ implies } \Vdash^{L \fatsemi K}_{\mathscr{C}} p && (3)\\
\text{iff } & \text{for all } \mathscr{C} \supseteq \mathscr{B}, K \text{ and } p \in \mathbb{A}, !\,\varphi \Vdash^{K}_{\mathscr{C}} p \text{ implies } \Vdash^{L \fatsemi K}_{\mathscr{C}} p && (4)
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\text{iff } & \Vdash^{L}_{\mathscr{B}} !\,\varphi \otimes 1 & (2)\\
\text{iff } & \text{for all } \mathscr{C} \supseteq \mathscr{B}, K \text{ and } p \in \mathbb{A}, !\,\varphi \,,\, 1 \Vdash^{K}_{\mathscr{C}} p \text{ implies } \Vdash^{L,K}_{\mathscr{C}} p & (3)\\
\text{iff } & \text{for all } \mathscr{C} \supseteq \mathscr{B}, K \text{ and } p \in \mathbb{A}, !\,\varphi \Vdash^{K}_{\mathscr{C}} p \text{ implies } \Vdash^{L,K}_{\mathscr{C}} p & (4)
\end{aligned}
$$

Disclaimer: This is not a proof of anything! Just a somewhat convincing argument!

Some failed attempts

- The most convincing hypothesis was that $\Vdash^{L}_{\mathscr{B}} !\varphi$ iff $L = \varnothing$ and for all $\mathscr{C} \supseteq \mathscr{B}$, atomic multisets K and atoms p, if $\varphi \Vdash^{K}_{\mathscr{C}} p$ then $\Vdash^{K}_{\mathscr{C}} p$.

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- This however fails to be complete as it requires us to be able to prove from $L \vdash_{\mathscr{B}} (!\,\varphi)^{\flat}$ that $L = \varnothing$, to pass Proposition 31 from the report.

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- Attempts to generalise this by replacing $L = \varnothing$ with $\Vdash^{\varnothing}_{\mathscr{B}} L$ fail for the same reason.

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- This however fails to be complete as it requires us to be able to prove from $L \vdash_{\mathscr{B}} (!\,\varphi)^{\flat}$ that $L = \varnothing$, to pass Proposition 31 from the report.
- Attempts to generalise this by replacing $L = \varnothing$ with $\Vdash^{\varnothing}_{\mathscr{B}} L$ fail for the same reason.
- A different attempt, based on a notion of closure was to define $\Vdash^{L}_{\mathscr{B}} !\,\varphi$ iff for every K, p and $\mathscr{C} \supseteq \mathscr{B}$ if we have $\varphi \Vdash^{K}_{\mathscr{C}} p$ then $\Vdash^{L,K}_{\mathscr{C}} p$ and $\ulcorner\forall\urcorner M(\Vdash^{M}_{\mathscr{B}} \varphi \ulcorner\Rightarrow\urcorner \Vdash^{\varnothing}_{\mathscr{B}} M)$ also fails as there is no way to prove the conclusion of the second conditional.

The idea of a zoned semantics

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- We want to re-introduce, in a controlled manner, the ability for normal intuitionistic inferences to be held valid.
- Not absurd to distinguish basic sentences based on whether they are to be used as a hypothesis in an intuitionistic or a resourceful way.
- The distinction *must* however be made by the formula consequence relation.
- Thus a basic rule might look something like the following:

$$\dfrac{\begin{matrix}[P_1; C_1] \\ \vdots \\ q_1\end{matrix} \qquad \dots \qquad \begin{matrix}[P_n; C_n] \\ \vdots \\ q_n\end{matrix}}{r}\,\mathcal{R}$$

Exploring the zoned semantics

Definition (Zoned basic derivability relation)

The relation of derivability in a base $\mathscr{B}$, is defined inductively as so:

Ref $S; T \vdash_{\mathscr{B}} p$ iff ($p \in S$ and $T = \varnothing$) or $T = [p]$.

App Given that
$(\{(P_{1_i}; C_{1_i} \Rightarrow q_{1_i})\}_{i=1}^{l_1}, \ldots, \{(P_{n_i}; C_{n_i} \Rightarrow q_{n_i})\}_{i=1}^{l_n} \Rightarrow r) \in \mathscr{B}$ and atomic multisets G and C_i such that the following hold:

$$G, P_{i_j}; L_i, C_{i_j} \vdash_{\mathscr{B}} q_{i_j} \text{ for all } i = 1, \ldots, n \text{ and } j = 1, \ldots, l_i$$

Then $G; L_1, \ldots, L_n \vdash_{\mathscr{B}} r$.

Exploring the zoned semantics

- $\Vdash_{\mathscr{B}}^{G;L} p$ iff $G; L \vdash_{\mathscr{B}} p$
- $\Vdash_{\mathscr{B}}^{G;L} \Gamma \fatsemi \Delta$ iff there exists $L = U \fatsemi V$ such that $\Vdash_{\mathscr{B}}^{G;U} \Gamma$ and $\Vdash_{\mathscr{B}}^{G;V} \Delta$
- $\Vdash_{\mathscr{B}}^{G;L} !\,\varphi$ iff for every base $\mathscr{C} \supseteq \mathscr{B}$, atomic multiset K and atom p, such that $\varphi; \cdot \Vdash_{\mathscr{C}}^{G;K} p$ hold, then $\Vdash_{\mathscr{C}}^{G;L \fatsemi K} p$.
- Seems to suggest a want for a Lolli-like encoding of sequents i.e. $(!\,\Gamma, \Delta : \varphi) \mapsto \Gamma; \Delta \Vdash \varphi$, as it is still difficult to encode the !I rule.

Thank you!

Thank you for listening!