Transfer viva:
Proof-theoretic Semantics for ILL and beyond

Yll Buzoku

Department of Computer Science
University College London

May 30, 2024

Goals for this talk

- To cover the work done in obtaining a proof-theoretic semantics for ILL.
- To highlight some interesting aspects of the results obtained.
- To present an alternative semantics for ILL.
- To discuss my future plans for my PhD.

A natural deduction system for ILL

$$\frac{}{\varphi \vdash \varphi}\ \mathsf{Ax}$$

$$\frac{\Gamma, \varphi \vdash \psi}{\Gamma \vdash \varphi \multimap \psi}\ \multimap\text{-I} \qquad \frac{\Gamma \vdash \varphi \multimap \psi \qquad \Delta \vdash \varphi}{\Gamma, \Delta \vdash \psi}\ \multimap\text{-E}$$

$$\frac{\Gamma \vdash \varphi \qquad \Delta \vdash \psi}{\Gamma, \Delta \vdash \varphi \otimes \psi}\ \otimes\text{-I} \qquad \frac{\Gamma \vdash \varphi \otimes \psi \qquad \Delta, \varphi, \psi \vdash \chi}{\Gamma, \Delta \vdash \chi}\ \otimes\text{-E}$$

$$\frac{}{\vdash 1}\ 1\text{-I} \qquad \frac{\Gamma \vdash \varphi \qquad \Delta \vdash 1}{\Gamma, \Delta \vdash \varphi}\ 1\text{-E}$$

$$\frac{\Gamma \vdash \varphi \qquad \Gamma \vdash \psi}{\Gamma \vdash \varphi \,\&\, \psi}\ \&\text{-I} \qquad \frac{\Gamma \vdash \varphi_0 \,\&\, \varphi_1}{\Gamma \vdash \varphi_i}\ \&\text{-E}_i$$

$$\frac{\Gamma \vdash \varphi_i}{\Gamma \vdash \varphi_0 \oplus \varphi_1}\ \oplus\text{-I}_i \qquad \frac{\Gamma \vdash \varphi \oplus \psi \qquad \Delta, \varphi \vdash \chi \qquad \Delta, \psi \vdash \chi}{\Gamma, \Delta \vdash \chi}\ \oplus\text{-E}$$

$$\frac{\Gamma_1 \vdash \varphi_1 \qquad \ldots \qquad \Gamma_n \vdash \varphi_n}{\Gamma_1, \ldots, \Gamma_n \vdash \top}\ \top\text{-I} \qquad \frac{\Gamma \vdash 0}{\Gamma \vdash \varphi}\ 0\text{-E}$$

A natural deduction system for ILL (cont.)

$$\frac{\Gamma_1 \vdash\ !\psi_1 \qquad \ldots \qquad \Gamma_n \vdash\ !\psi_n \qquad !\psi_1, \ldots, !\psi_n \vdash \varphi}{\Gamma_1, \ldots, \Gamma_n \vdash\ !\varphi}\ \text{!-Promotion}$$

$$\frac{\Gamma \vdash\ !\varphi \qquad \Delta, \varphi \vdash \psi}{\Gamma, \Delta \vdash \psi}\ \text{!-Dereliction}$$

$$\frac{\Gamma \vdash\ !\varphi \qquad \Delta \vdash \psi}{\Gamma, \Delta \vdash \psi}\ \text{!-Weakening}$$

$$\frac{\Gamma \vdash\ !\varphi \qquad \Delta, !\varphi, !\varphi \vdash \psi}{\Gamma, \Delta \vdash \psi}\ \text{!-Contraction}$$

Alternative systems

- One may consider other natural deduction systems.

Alternative systems

- One may consider other natural deduction systems.
- The exponential has two rules only in the system of Negri:

$$\frac{(!\,\varphi)^m, \Gamma \vdash \chi \qquad \Delta_1 \vdash !\,\psi_1 \ldots \Delta_n \vdash !\,\psi_n \qquad !\,\psi_1, \ldots, !\,\psi_n \vdash \varphi}{\Gamma, \Delta_1, \ldots \Delta_n \vdash \chi}\ \text{!-I}$$

$$\frac{\Gamma \vdash !\,\varphi \qquad \Delta, \varphi \vdash \psi}{\Gamma, \Delta \vdash \psi}\ \text{!-E}$$

Substructural atomic derivability

Definition (Basic rules)

Basic rules take the following form:

$$\{(P_{1_i} \Rightarrow q_{1_i})\}_{i=1}^{l_1}, \ldots, \{(P_{n_i} \Rightarrow q_{n_i})\}_{i=1}^{l_n} \Rightarrow r$$

where

- Each P_i is an atomic multiset, called a premiss multiset.
- Each q_i and r is an atomic proposition.
- Each $(P_i \Rightarrow q_i)$ is a pair (P_i, q_i) called an atomic sequent.
- Each collection $\{(P_{i_1} \Rightarrow q_{i_1}), \ldots, (P_{i_{l_i}} \Rightarrow q_{i_{l_i}})\}$ is called an atomic box.

Substructural atomic derivability

Definition (Basic derivability relation)

The relation of derivability in a base $\mathscr{B}$, is defined inductively as so:

Ref $p \vdash_{\mathscr{B}} p$

App Given that $(\{(P_{1_i} \Rightarrow q_{1_i})\}_{i=1}^{l_1}, \ldots, \{(P_{n_i} \Rightarrow q_{n_i})\}_{i=1}^{l_n} \Rightarrow r) \in \mathscr{B}$ and atomic multisets C_i such that the following hold:

$$C_i \fatsemi P_{i_j} \vdash_{\mathscr{B}} q_{i_j} \text{ for all } i = 1, \ldots, n \text{ and } j = 1, \ldots, l_i$$

Then $C_1 \fatsemi \cdots \fatsemi C_n \vdash_{\mathscr{B}} r$.

$$
\begin{array}{rlll}
\text{(At)} & \Vdash^{L}_{\mathscr{B}} p & \text{iff} & L \vdash_{\mathscr{B}} p \\
(\multimap) & \Vdash^{L}_{\mathscr{B}} \varphi \multimap \psi & \text{iff} & \varphi \Vdash^{L}_{\mathscr{B}} \psi \\
(\otimes) & \Vdash^{L}_{\mathscr{B}} \varphi \otimes \psi & \text{iff} & \text{for any } \mathscr{C} \text{ such that } \mathscr{B} \subseteq \mathscr{C}, \text{ atomic multisets } K \\
& & & \text{and any } p \in \mathbb{A}, \text{ if } \varphi \,⨟\, \psi \Vdash^{K}_{\mathscr{B}} p \text{ then } \Vdash^{L⨟K}_{\mathscr{C}} p \\
(1) & \Vdash^{L}_{\mathscr{B}} 1 & \text{iff} & \text{for any } \mathscr{C} \text{ such that } \mathscr{B} \subseteq \mathscr{C}, \text{ atomic multisets } K \\
& & & \text{and any } p \in \mathbb{A}, \text{ if } \Vdash^{K}_{\mathscr{B}} p \text{ then } \Vdash^{L⨟K}_{\mathscr{C}} p \\
(\top) & \Vdash^{L}_{\mathscr{B}} \top & \text{iff} & \text{always} \\
(\&) & \Vdash^{L}_{\mathscr{B}} \varphi \,\&\, \psi & \text{iff} & \Vdash^{L}_{\mathscr{B}} \varphi \text{ and } \Vdash^{L}_{\mathscr{B}} \psi \\
(\oplus) & \Vdash^{L}_{\mathscr{B}} \varphi \oplus \psi & \text{iff} & \text{for any } \mathscr{C} \text{ such that } \mathscr{B} \subseteq \mathscr{C}, \text{ atomic multisets } K \\
& & & \text{and any } p \in \mathbb{A}, \text{ if } \varphi \Vdash^{K}_{\mathscr{C}} p \text{ and } \psi \Vdash^{K}_{\mathscr{C}} p, \text{ then } \Vdash^{L⨟K}_{\mathscr{C}} p \\
(0) & \Vdash^{L}_{\mathscr{B}} 0 & \text{iff} & \Vdash^{L}_{\mathscr{B}} p \text{ for any } p \in \mathbb{A} \\
(!) & \Vdash^{L}_{\mathscr{B}} !\,\varphi & \text{iff} & \text{for any } \mathscr{C} \text{ such that } \mathscr{B} \subseteq \mathscr{C}, \text{ atomic multisets } K \\
& & & \text{and any } p \in \mathbb{A}, \text{ if for any } \mathscr{D} \text{ such that } \mathscr{C} \subseteq \mathscr{D}, \\
& & & (\text{if } \Vdash^{\varnothing}_{\mathscr{D}} \varphi \text{ then } \Vdash^{L}_{\mathscr{D}} p) \text{ then } \Vdash^{L⨟K}_{\mathscr{C}} p \\
(⨟) & \Vdash^{L}_{\mathscr{B}} \Gamma \,⨟\, \Delta & \text{iff} & \text{there exists multisets } K \text{ and } M \text{ such that } L = K \,⨟\, M \\
& & & \text{and } \Vdash^{K}_{\mathscr{B}} \Gamma \text{ and } \Vdash^{M}_{\mathscr{B}} \Delta \\
\text{(Inf)} & !\,\Delta \,⨟\, \Theta \Vdash^{L}_{\mathscr{B}} \varphi & \text{iff} & \text{for any } \mathscr{C} \text{ such that } \mathscr{B} \subseteq \mathscr{C}, \text{ atomic multisets } K, \\
& & & \text{if } \Vdash^{\varnothing}_{\mathscr{C}} \Delta \text{ and } \Vdash^{K}_{\mathscr{C}} \Theta \text{ then } \Vdash^{L⨟K}_{\mathscr{C}} \varphi
\end{array}
$$

Notes on $\Vdash^{L}_{\mathscr{B}}$

- The sequent $\langle \Gamma, \varphi \rangle$ is said to be valid if and only if $\Gamma \Vdash^{\varnothing}_{\varnothing} \varphi$ holds.
- We frequently write this as $\Gamma \Vdash \varphi$.

Notes on $\Vdash^{L}_{\mathscr{B}}$

- If $\Vdash^{L}_{\mathscr{B}} \varphi$ then for all $\mathscr{C} \supseteq \mathscr{B}$ we have $\Vdash^{L}_{\mathscr{C}} \varphi$.
- Given $\Gamma \Vdash^{L}_{\mathscr{B}} \varphi$ and $\Vdash^{K}_{\mathscr{B}} \Gamma$, then it holds that $\Vdash^{L \fatsemi K}_{\mathscr{B}} \varphi$.

Why the extension to Inf?

- Up until now, all Base-extension Semantics have had a simple (Inf) clause.
- For example in IMALL, it suffices to take the following:
  $\Gamma \Vdash^{L}_{\mathscr{B}} \varphi$ iff for all $\mathscr{C} \supseteq \mathscr{B}$ and K, $\Vdash^{K}_{\mathscr{C}} \Gamma$ implies $\Vdash^{L,K}_{\mathscr{C}} \varphi$

Why the extension to Inf?

Question: Is this still an inferentialist semantics?

Can we better understand the exponential?

- $! \varphi \Vdash \varphi \otimes \ldots \otimes \varphi$
- $\Vdash^{L}_{\mathscr{B}} !(\varphi \mathbin{\&} \psi)$ iff $\Vdash^{L}_{\mathscr{B}} ! \varphi \otimes ! \psi$

Expanding what we know

$! \varphi \Vdash \varphi \otimes \ldots \otimes \varphi$ iff

for all bases $\mathscr{B}$ and atomic multisets L, such that $\Vdash^{L}_{\mathscr{B}} ! \varphi$ then $\Vdash^{L}_{\mathscr{B}} \varphi \otimes \ldots \otimes \varphi$

Expanding what we know

- So when does $\Vdash^{L}_{\mathscr{B}} \varphi \otimes \ldots \otimes \varphi$ hold?

Expanding what we know

Considering the case when $\varphi = p$ for simplicity we get the following:

Expanding what we know

Considering the case when $\varphi = p$ for simplicity we get the following:

- Given a base $\mathscr{B}$ such that $\vdash_{\mathscr{B}} p$ and $L = \varnothing$.

Expanding what we know

Considering the case when $\varphi = p$ for simplicity we get the following:

- Given a base $\mathscr{B}$ such that $\vdash_{\mathscr{B}} p$ and $L = \varnothing$.
- Then we have that $\Vdash^{L}_{\mathscr{B}} p \otimes \ldots \otimes p$.

Expanding what we know

Considering the case when $\varphi = p$ for simplicity we get the following:

- Given a base $\mathscr{B}$ such that $\vdash_{\mathscr{B}} p$ and $L = \varnothing$.
- Then we have that $\Vdash^{L}_{\mathscr{B}} p \otimes \ldots \otimes p$.
- Examples of such bases:
  - $\mathscr{B} = \{\varnothing \Rightarrow p\}$
  - $\mathscr{B} = \{\varnothing \Rightarrow q, q \Rightarrow p\}$

Expanding what we know

Considering the case when $\varphi = p$ for simplicity we get the following:

- Given a base $\mathscr{B}$ such that $\vdash_{\mathscr{B}} p$ and $L = \varnothing$.
- Then we have that $\Vdash^{L}_{\mathscr{B}} p \otimes \ldots \otimes p$.
- Examples of such bases:
  - $\mathscr{B} = \{\varnothing \Rightarrow p\}$
  - $\mathscr{B} = \{\varnothing \Rightarrow q,\ q \Rightarrow p\}$

Note that if $L \neq \varnothing$ then the only way that this relation could hold is if the atoms were derivable from within the base, i.e. if were valid $\Vdash^{\varnothing}_{\mathscr{B}} L$.
So ultimately, $\Vdash^{\varnothing}_{\mathscr{B}} p \otimes \ldots \otimes p$.

Expanding what we know

Inferring from $!\,\varphi$ *should imply that* $\Vdash^{\varnothing}_{\mathscr{B}} \varphi$*.*

A proof-theoretic flavour of the !

We now consider the second identity: $\Vdash^{L}_{\mathscr{B}} !(\varphi \,\&\, \psi)$ iff $\Vdash^{L}_{\mathscr{B}} !\varphi \otimes !\psi$ with $\psi = \top$. Furthermore since $\varphi \,\&\, \top \equiv \varphi$ we go as follows:

$\Vdash^{L}_{\mathscr{B}} !\varphi$ iff $\Vdash^{L}_{\mathscr{B}} !\varphi \otimes !\top$

iff for all $\mathscr{C} \supseteq \mathscr{B}$, K and $p \in \mathbb{A}$, $!\varphi \,,\, !\top \Vdash^{K}_{\mathscr{C}} p$ implies $\Vdash^{L,K}_{\mathscr{C}} p$

iff for all $\mathscr{C} \supseteq \mathscr{B}$, K and $p \in \mathbb{A}$,

if (for all $\mathscr{D} \supseteq \mathscr{C}$, if $\Vdash^{\varnothing}_{\mathscr{D}} \varphi$ and $\Vdash^{\varnothing}_{\mathscr{D}} \top$ then $\Vdash^{K}_{\mathscr{D}} p$) then $\Vdash^{L,K}_{\mathscr{C}}$

iff for all $\mathscr{C} \supseteq \mathscr{B}$, K and $p \in \mathbb{A}$,

if (for all $\mathscr{D} \supseteq \mathscr{C}$, if $\Vdash^{\varnothing}_{\mathscr{D}} \varphi$ then $\Vdash^{K}_{\mathscr{D}} p$) then $\Vdash^{L,K}_{\mathscr{C}}$

Some failed attempts

- The most convincing hypothesis was that $\Vdash^{L}_{\mathscr{B}} \,!\,\varphi$ iff $L = \varnothing$ and for all $\mathscr{C} \supseteq \mathscr{B}$, atomic multisets K and atoms p, if $\varphi \Vdash^{K}_{\mathscr{C}} p$ then $\Vdash^{K}_{\mathscr{C}} p$.

Some failed attempts

- The most convincing hypothesis was that $\Vdash^{L}_{\mathscr{B}} \, ! \, \varphi$ iff $L = \varnothing$ and for all $\mathscr{C} \supseteq \mathscr{B}$, atomic multisets K and atoms p, if $\varphi \Vdash^{K}_{\mathscr{C}} p$ then $\Vdash^{K}_{\mathscr{C}} p$.
- This however fails to be complete as it requires us to be able to prove from $L \vdash_{\mathscr{B}} (! \, \varphi)^{\flat}$ that $L = \varnothing$, to pass Proposition 31.

Some failed attempts

- The most convincing hypothesis was that $\Vdash^{L}_{\mathscr{B}} !\,\varphi$ iff $L = \varnothing$ and for all $\mathscr{C} \supseteq \mathscr{B}$, atomic multisets K and atoms p, if $\varphi \Vdash^{K}_{\mathscr{C}} p$ then $\Vdash^{K}_{\mathscr{C}} p$.
- This however fails to be complete as it requires us to be able to prove from $L \vdash_{\mathscr{B}} (!\,\varphi)^{\flat}$ that $L = \varnothing$, to pass Proposition 31.
- Attempts to generalise this by replacing $L = \varnothing$ with $\Vdash^{\varnothing}_{\mathscr{B}} L$ fail for the same reason.

Some failed attempts

- The most convincing hypothesis was that $\Vdash^{L}_{\mathscr{B}} !\,\varphi$ iff $L = \varnothing$ and for all $\mathscr{C} \supseteq \mathscr{B}$, atomic multisets K and atoms p, if $\varphi \Vdash^{K}_{\mathscr{C}} p$ then $\Vdash^{K}_{\mathscr{C}} p$.
- This however fails to be complete as it requires us to be able to prove from $L \vdash_{\mathscr{B}} (!\,\varphi)^{\flat}$ that $L = \varnothing$, to pass Proposition 31.
- Attempts to generalise this by replacing $L = \varnothing$ with $\Vdash^{\varnothing}_{\mathscr{B}} L$ fail for the same reason.
- A different attempt, based on a notion of closure was to define $\Vdash^{L}_{\mathscr{B}} !\,\varphi$ iff for every K, p and $\mathscr{C} \supseteq \mathscr{B}$ if we have $\varphi \Vdash^{K}_{\mathscr{C}} p$ then $\Vdash^{L,K}_{\mathscr{C}} p$ and $\ulcorner\forall\urcorner M(\Vdash^{M}_{\mathscr{B}} \varphi \ulcorner\Rightarrow\urcorner \Vdash^{\varnothing}_{\mathscr{B}} M)$ also fails as there is no way to prove the conclusion of the second conditional.

The idea of a zoned semantics

- Not absurd to distinguish basic sentences based on whether they are to be used as a hypothesis in an intuitionistic or a resourceful way.

The idea of a zoned semantics

- Not absurd to distinguish basic sentences based on whether they are to be used as a hypothesis in an intuitionistic or a resourceful way.
- Thus we define a new type of basic rule that lookds something like this:

$$\dfrac{\begin{matrix}[P_1; C_1] \\ \vdots \\ q_1\end{matrix} \qquad \dots \qquad \begin{matrix}[P_n; C_n] \\ \vdots \\ q_n\end{matrix}}{r}\,\mathcal{R}$$

Exploring the zoned semantics

Definition (Zoned basic derivability relation)

The relation of derivability in a base $\mathscr{B}$, is defined inductively as so:

Ref $S; T \vdash_{\mathscr{B}} p$ iff ($p \in S$ and $T = \varnothing$) or $T = [p]$.

App Given that
$(\{(P_{1_i}; C_{1_i} \Rightarrow q_{1_i})\}_{i=1}^{l_1}, \ldots, \{(P_{n_i}; C_{n_i} \Rightarrow q_{n_i})\}_{i=1}^{l_n} \Rightarrow r) \in \mathscr{B}$ and atomic multisets G and C_i such that the following hold:

$$G \fatsemi P_{i_j}; L_i \fatsemi C_{i_j} \vdash_{\mathscr{B}} q_{i_j} \text{ for all } i = 1, \ldots, n \text{ and } j = 1, \ldots, l_i$$

Then $G; L_1 \fatsemi \cdots \fatsemi L_n \vdash_{\mathscr{B}} r$.

Exploring the zoned semantics

- $\Vdash^{G;L}_{\mathscr{B}} p$ iff $G; L \vdash_{\mathscr{B}} p$
- $\Vdash^{G;L}_{\mathscr{B}} \Gamma \fatsemi \Delta$ iff there exists $L = U \fatsemi V$ such that $\Vdash^{G;U}_{\mathscr{B}} \Gamma$ and $\Vdash^{G;V}_{\mathscr{B}} \Delta$
- $\Vdash^{G;L}_{\mathscr{B}} !\,\varphi$ iff for every base $\mathscr{C} \supseteq \mathscr{B}$, atomic multisets h, K and atom p, such that $\varphi; \cdot \Vdash^{H;K}_{\mathscr{C}} p$ hold, then $\Vdash^{G \fatsemi H; L \fatsemi K}_{\mathscr{C}} p$.
- $\Gamma; \Delta \Vdash^{G;L}_{\mathscr{B}} \varphi$ iff for every $\mathscr{C} \supseteq \mathscr{B}$ and atomic multisets H, K such that $\Vdash^{H;\varnothing}_{\mathscr{C}} \Gamma$ and $\Vdash^{G \fatsemi H; L \fatsemi K}_{\mathscr{C}} \Delta$ then $\Vdash^{G \fatsemi H; L \fatsemi K}_{\mathscr{C}} \varphi$

Some other work undertaken

- Investigated alternative IPL support relations that are more "resource"-aware.
- Started investigating translations between Sandqvist's semantics for IPL and the semantics presented here for ILL.
- Very recently, I have been working on a Base-extension semantics for Lambek Calculus.

Future directions of work

- Study how rules are generally translated from Sandqvists bases for IPL to the bases I presented here.
- Give a Proof-theoretic Semantics for Classical Linear Logic.
- Similarly, study other similar logics such as DILL, FILL and what happens when we have subexponentials.
- Give a categorical interpretation of the semantics presented herein.
- Investigate the connection to Linear Logic programming.
- Develop a connection to the Kripke semantics of Miller for his fragment of ILL.

Thank you!

Thank you for listening!