Transfer viva:
Proof-theoretic Semantics for ILL and beyond

Yll Buzoku

Department of Computer Science
University College London

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Goals for this talk

- To cover what work I have already done in the direction obtaining a BeS for IMALL.
- To cover some issues with the current approach to dealing with the modality of ILL.
- To discuss a (currently promising) approach to dealing with the modality of ILL.

A natural deduction system for IMALL

$$\frac{}{\varphi \vdash \varphi}\ \text{Ax}$$

$$\frac{\Gamma, \varphi \vdash \psi}{\Gamma \vdash \varphi \multimap \psi}\ \multimap\text{-I} \qquad \frac{\Gamma \vdash \varphi \multimap \psi \qquad \Delta \vdash \varphi}{\Gamma, \Delta \vdash \psi}\ \multimap\text{-E}$$

$$\frac{\Gamma \vdash \varphi \qquad \Delta \vdash \psi}{\Gamma, \Delta \vdash \varphi \otimes \psi}\ \otimes\text{-I} \qquad \frac{\Gamma \vdash \varphi \otimes \psi \qquad \Delta, \varphi, \psi \vdash \chi}{\Gamma, \Delta \vdash \chi}\ \otimes\text{-E}$$

$$\frac{}{\vdash 1}\ 1\text{-I} \qquad \frac{\Gamma \vdash \varphi \qquad \Delta \vdash 1}{\Gamma, \Delta \vdash \varphi}\ 1\text{-E}$$

$$\frac{\Gamma^{\gamma} \vdash \varphi \qquad \Gamma^{\gamma} \vdash \psi}{\Gamma \vdash \varphi \,\&\, \psi}\ \&\text{-I} \qquad \frac{\Gamma \vdash \varphi_0 \,\&\, \varphi_1}{\Gamma \vdash \varphi_i}\ \&\text{-E}_i$$

$$\frac{\Gamma \vdash \varphi_i}{\Gamma \vdash \varphi_0 \oplus \varphi_1}\ \oplus\text{-I}_i \qquad \frac{\Gamma \vdash \varphi \oplus \psi \qquad \Delta^{\gamma}, \varphi \vdash \chi \qquad \Delta^{\gamma}, \psi \vdash \chi}{\Gamma, \Delta \vdash \chi}\ \oplus\text{-E}$$

No 0 intro rule

$$\frac{\Gamma \vdash 0}{\Gamma \vdash \varphi}\ 0\text{-E}$$

Definition

Atomic rules Basic rules take the following form:

$$\{(P_{1_i} \rightsquigarrow q_{1_i})\}_{i=1}^{l_1}, \ldots, \{(P_{n_i} \rightsquigarrow q_{n_i})\}_{i=1}^{l_n} \Rightarrow r$$

where

- Each P_i is an atomic multiset, called a premiss multiset.
- Each q_i and r is an atomic proposition.
- Each $(P_i \rightsquigarrow q_i)$ is a pair (P_i, q_i) called an atomic sequent.
- Each collection $\{(P_{i_1} \rightsquigarrow q_{i_1}), \ldots, (P_{i_{l_i}} \rightsquigarrow q_{i_{l_i}})\}$ is called an atomic box.

Basic rules are defined as before except now they are not allowed to take take the values 1 or 0.

Definition (Basic derivability relation)

The relation of derivability in a base $\mathscr{B}$, is defined inductively as so:

Ref $p \vdash_{\mathscr{B}} p$

App Given that $(\{(P_{1_i} \rightsquigarrow q_{1_i})\}_{i=1}^{l_1}, \ldots, \{(P_{n_i} \rightsquigarrow q_{n_i})\}_{i=1}^{l_n} \Rightarrow r) \in \mathscr{B}$ and atomic multisets C_i such that the following hold:

$$C_i, P_{i_j} \vdash_{\mathscr{B}} q_{i_j} \text{ for all } i = 1, \ldots, n \text{ and } j = 1, \ldots, l_i$$

Then $C_1, \ldots, C_n \vdash_{\mathscr{B}} r$.

Base-extension Semantics for IMALL

(At) $\Vdash^{L}_{\mathscr{B}} p$ iff $L \vdash_{\mathscr{B}} p$.

($\otimes$) $\Vdash^{L}_{\mathscr{B}} \varphi \otimes \psi$ iff for every $\mathscr{C} \supseteq \mathscr{B}$, atomic multiset K, and atom p, if $\varphi \fatcomma \psi \Vdash^{K}_{\mathscr{C}} p$ then $\Vdash^{L,K}_{\mathscr{C}} p$.

(1) $\Vdash^{L}_{\mathscr{B}} 1$ iff for every $\mathscr{C} \supseteq \mathscr{B}$, atomic multiset K, and atom p, if $\Vdash^{K}_{\mathscr{C}} p$, then $\Vdash^{L,K}_{\mathscr{C}} p$.

($\multimap$) $\Vdash^{L}_{\mathscr{B}} \varphi \multimap \psi$ iff $\varphi \Vdash^{L}_{\mathscr{B}} \psi$.

(&) $\Vdash^{L}_{\mathscr{B}} \varphi \mathbin{\&} \psi$ iff $\Vdash^{L}_{\mathscr{B}} \varphi$ and $\Vdash^{L}_{\mathscr{B}} \psi$.

($\oplus$) $\Vdash^{L}_{\mathscr{B}} \varphi \oplus \psi$ iff for every $\mathscr{C} \supseteq \mathscr{B}$, atom p and atomic multiset K such that $\varphi \Vdash^{K}_{\mathscr{C}} p$ and $\psi \Vdash^{K}_{\mathscr{C}} p$ hold, then $\Vdash^{L,K}_{\mathscr{C}} p$.

(0) $\Vdash^{L}_{\mathscr{B}} 0$ iff $\Vdash^{L}_{\mathscr{B}} p$, for all atomic p.

($\fatcomma$) $\Vdash^{L}_{\mathscr{B}} \Gamma \fatcomma \Delta$ iff there are atomic multisets U and V such that $L = K \fatcomma M$ and that $\Vdash^{K}_{\mathscr{B}} \Gamma$ and $\Vdash^{M}_{\mathscr{B}} \Delta$.

(Inf) For Θ a nonempty multiset: $\Theta \Vdash^{L}_{\mathscr{B}} \varphi$ iff for all $\mathscr{C} \supseteq \mathscr{B}$, atomic multisets K and atoms p if $\Vdash^{K}_{\mathscr{C}} \Theta$ then $\Vdash^{L,K}_{\mathscr{C}} \varphi$.

What of the exponential?

The exponential behaves very weirdly! Consider the rules in the sequent calculus:

$$\frac{\Gamma, \varphi \supset \psi}{\Gamma, !\,\varphi \supset \psi}\ \text{!-d} \quad \frac{\Gamma, !\,\varphi, !\,\varphi \supset \psi}{\Gamma, !\,\varphi \supset \psi}\ \text{!-c} \qquad \frac{!\,\Gamma \supset \varphi}{!\,\Gamma \supset !\,\varphi}\ \text{!-r} \qquad \frac{\Gamma \supset \psi}{\Gamma, !\,\varphi \supset \psi}\ \text{!-w}$$

- All "structural" aspects of the exponential are on the *left* hand side.
- The exponential right rule, still depends fully on the structure of the left hand side.
- None of these rules are invertible, so none are particularly nice candidates for giving a definition.

However, this means that to pinpoint the behaviour of the exponential, we need to pinpoint something about the resources which can be used to support it!

What of the exponential?

- The only known good natural deduction system for ILL has an introduction rule for the exponential which is closer to being a necessitation rule.
- In this system, *all* connectives have generalised elimination rules.

$$\frac{(!\,\varphi)^m, \Gamma \vdash \chi \qquad \Delta_1 \vdash !\,\psi_1 \ldots \Delta_n \vdash !\,\psi_n \qquad !\,\psi_1, \ldots, !\,\psi_n \vdash \varphi}{\Gamma, \Delta_1, \ldots \Delta_n \vdash \chi}\ \text{!-I}$$

$$\frac{\Gamma \vdash !\,\varphi \qquad \Delta, \varphi \vdash \psi}{\Gamma, \Delta \vdash \psi}\ \text{!-E}$$

Weakening and Contraction derivability

Proof.

It follows immediately by setting $n = 1$ and $m = 2$ for contraction and $m = 0$ for weakening in the introduction rule, $\psi = \varphi$ and $\Delta = !\,\varphi$ that

$$\dfrac{(!\,\varphi)^m, \Gamma \vdash \chi \qquad !\,\varphi \vdash !\,\varphi \qquad \dfrac{!\,\varphi \vdash !\,\varphi \qquad \varphi \vdash \varphi}{!\,\varphi \vdash \varphi}\,!\mathrm{E}}{\Gamma, !\,\varphi \vdash \chi}\,!\mathrm{I}$$

$\square$

Expanding what we know

$! \varphi \Vdash \varphi \otimes \ldots \otimes \varphi$ iff

for all bases $\mathscr{B}$ and atomic multisets L, such that $\Vdash^{L}_{\mathscr{B}} ! \varphi$ then $\Vdash^{L}_{\mathscr{B}} \varphi \otimes \ldots \otimes \varphi$

Expanding what we know

- So when does $\Vdash^{L}_{\mathscr{B}} \varphi \otimes \ldots \otimes \varphi$ hold?

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  - $\mathscr{B} = \{\varnothing \Rightarrow p\}$
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Note that if $L \neq \varnothing$ then the only way that this relation could hold is if the atoms were derivable from within the base, i.e. if were valid $\Vdash_{\mathscr{B}} L$.
So by cut, $\Vdash_{\mathscr{B}} p \otimes \ldots \otimes p$.

Expanding what we know

The meaning of $\Vdash^{L}_{\mathscr{B}} !\,\varphi$ *should imply that* $\Vdash^{\varnothing}_{\mathscr{B}} \varphi$*.*

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- The most convincing hypothesis was that $\Vdash^{L}_{\mathscr{B}} ! \, \varphi$ iff $L = \varnothing$ and for all $\mathscr{C} \supseteq \mathscr{B}$, atomic multisets K and atoms p, if $\varphi \Vdash^{K}_{\mathscr{C}} p$ then $\Vdash^{K}_{\mathscr{C}} p$.

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- This however fails to be complete as it requires us to be able to prove from $L \vdash_{\mathscr{B}} (!\,\varphi)^{\flat}$ that $L = \varnothing$, to pass the $\dagger$ lemma.

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- Add a side condition?

Generalisations of the suggestion

- Attempts to generalise this by replacing $L = \varnothing$ with $\Vdash^{\varnothing}_{\mathscr{B}} L$ fail for the same reason.
- Attempts at sidestepping this by adding $\Vdash^{\varnothing}_{\mathscr{B}} L$ as the hypothesis of a conditional similarly fail.
- A different attempt, based on a notion of closure was to define $\Vdash^{L}_{\mathscr{B}} !\,\varphi$ iff for every K, p and $\mathscr{C} \supseteq \mathscr{B}$ if we have $\varphi \Vdash^{K}_{\mathscr{C}} p$ then $\Vdash^{L,K}_{\mathscr{C}} p$ and $\ulcorner\forall\urcorner M(\Vdash^{M}_{\mathscr{B}} \varphi \ulcorner\Rightarrow\urcorner \Vdash^{\varnothing}_{\mathscr{B}} M)$ also fails as there is no way to prove the conclusion of the second conditional.

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- We want to re-introduce, in a controlled manner, the ability for normal intuitionistic inferences to be held valid.
- Not absurd to distinguish basic sentences based on whether they are to be used as a hypothesis in an intuitionistic or a resourceful way.
- The distinction *must* however be made by the formula consequence relation.
- Thus a basic rule might look something like the following:

$$\dfrac{\begin{matrix}[P_1; C_1] \\ \vdots \\ q_1\end{matrix} \qquad \dots \qquad \begin{matrix}[P_n; C_n] \\ \vdots \\ q_n\end{matrix}}{r}\,\mathcal{R}$$

Exploring the alternative approach

Definition (Extended basic derivability relation)

The relation of derivability in a base $\mathscr{B}$, is defined inductively as so:

Ref $S; T \vdash_{\mathscr{B}} p$ iff ($p \in S$ and $T = \varnothing$) or $T = [p]$.

App Given that $(\{(P_{1_i}; C_{1_i} \rightsquigarrow q_{1_i})\}_{i=1}^{l_1}, \ldots, \{(P_{n_i}; C_{n_i} \rightsquigarrow q_{n_i})\}_{i=1}^{l_n} \Rightarrow r) \in \mathscr{B}$ and atomic multisets G and C_i such that the following hold:

$$F, P_{i_j}; L_i, C_{i_j} \vdash_{\mathscr{B}} q_{i_j} \text{ for all } i = 1, \ldots, n \text{ and } j = 1, \ldots, l_i$$

Then $F; L_1, \ldots, L_n \vdash_{\mathscr{B}} r$.

Exploring the alternative approach

- $\Vdash_{\mathscr{B}}^{F;L} p$ iff $F; L \vdash_{\mathscr{B}} p$
- $\Vdash_{\mathscr{B}}^{F;L} \Gamma \fatsemi \Delta$ iff there exists $L = U \fatsemi V$ such that $\Vdash_{\mathscr{B}}^{F;U} \Gamma$ and $\Vdash_{\mathscr{B}}^{F;V} \Delta$

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- Perhaps something like this:
  $\Vdash^{F;\cdot}_{\mathscr{B}} !\,\varphi$ iff for all $\mathscr{C} \supseteq \mathscr{B}$, atomic multisets L, K and atoms p, if $\varphi \Vdash^{L;K}_{\mathscr{C}} p$ then $\Vdash^{F,L;K}_{\mathscr{C}} p$

Exploring the alternative approach

- This approach seems to provide a nice generalisation.
- Unknown if the atomic system is sufficiently well behaved yet.
- Seems to suggest a want for a Lolli-like encoding of sequents i.e. $(!\,\Gamma, \Delta : \varphi) \mapsto \Gamma; \Delta \Vdash \varphi$, as it is still difficult to encode the !I rule.
- Change to a ND system with explicit structurality or a ND-like system with no explicit structural rules?

Thank you!

Thank you for listening!