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Numerical Simulation of Two-Dimensional Diffusion Equation with Non Local Boundary Conditions

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Abstract. This paper is devoted to the decomposition method which is applied to solve problems with non local boundary conditions. The analytic solution of the problem is calculated in a series form with easily computable components. The comparison of the methodology with some known techniques shows that the present approach is powerful, efficient and reliable.

Keywords: Decomposition method, Non local boundary conditions, Partial differential equations, Accuracy

1. Introduction

Partial differential equations with non local boundary conditions and partial integro-differential equations arise in many fields of science and engineering such as chemical diffusion, heat conduction processes, population dynamics, thermo-elasticity, medical science, electrochemistry and control theory [5-20]. A detailed description of the occurrence of such equations is given in [7]. In this paper, we consider a two-dimensional diffusion equation with non local boundary conditions. This type of problem was solved by many searchers using traditional numerical methods. For example, M. Siddique [13] proposed a fourth-order Padè-scheme. The purpose of this work is to study and use Adomian decomposition method [2,3,4,21] for this important problem. We show that this approach allows us to obtain an analytical solution. These results demonstrate that the decomposition method is more accurate, efficient and reliable in comparison with the conventional methods,like finite difference method etc…

We consider the two-dimensional diffusion equation given by

$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2},\ 0 < x,\ y < 1, t > 0$ (1)

Initial conditions are assumed to be of the form:

$$u(x, y, 0) = f(x, y), (x, y) \in \Omega \cup \partial\Omega$$

And the Dirichelet time-dependent boundary conditions are: $u(0, y, t) = \psi_0(y, t), 0 \leq t \leq T, 0 \leq y \leq 1$

$u(1, y, t) = \psi_1(y, t), 0 \leq t \leq T, 0 \leq y \leq 1$ (2)

$$u(x, 0, t) = \varphi_0(x)\gamma(t), 0 \leq t \leq T, 0 \leq x \leq 1$$

$$u(x, 1, t) = \varphi_1(x, t), 0 \leq t \leq T, 0 \leq x \leq 1$$

And non local boundary condition:

$\int_0^1 \int_0^1 u(x, y, t)dxdy = m(t),\ (x, y) \in \Omega \cup \partial\Omega$ (3)

Where $f, \psi_0, \psi_1, \varphi_0, \varphi_1$ and m are known functions and $\gamma(t)$ is to be determined.

2. Adomian decomposition method

2.1 The operator form

In this section, we outline the steps to obtain a solution the above problem using the Adomian decomposition method, which was initiated by G.Adomian [1, 2, 3]. For this purpose, it is convenient to rewrite the problem in the standard form:

$L_t(u(x, y, t)) = L_{xx}(u(x, y, t)) + L_{yy}(u(x, y, t))$(4)

Where the differential operators $L_t(.) = \frac{\partial}{\partial t}(.)$, $L_{xx} = \frac{\partial^2}{\partial x^2}$ and $L_{yy} = \frac{\partial^2}{\partial y^2}$.

Assuming that the inverse L_t^{-1} exists and is defined as:

$L_T^{-1} = \int_0^t (.)dt$ (5)

2.2 Application to the problem

Applying inverse operator on both the sides of (4) and using the initial condition, yields:

$u(x, y, t) = L_t^{-1}\left(L_{xx}(u(x, y, t)) + L_{yy}(u(x, y, t))\right)$

or

$u(x, y, t) = u(x, y, 0) + L_t^{-1}\left(L_{xx}(u(x, y, t)) + L_{yy}(u(x, y, t))\right)$ (6)

Now, we decompose the unknown function *u(x,y,t)* as a sum of components defined by the series :

$u(x, y, t) = \sum_{k=0}^{\infty} u_k(x, y, t)$(7)

Where $u_0(x, y, t)$ is identified as *u(x,y, 0)*. Substituting equation (7) into equation (6) one obtains

$\sum_{k=0}^{\infty} u_k (x, y, t) = f(x, y) + L_t^{-1}\{L_{xx}(\sum_{k=0}^{\infty} u_k(x, y, t)) + L_{yy}(\sum_{k=0}^{\infty} u_k(x, y, t))\}$(8)

The components $u_k(x, y, t)$ are obtained by the recursive formula:

$u_0(x,y,t) = f(x,y)$ (9) $u_{k+1}(x,t) = L_t^{-1}\left(L_{xx}\big(u_k(x,y,t)\big) + L_{yy}\big(u_k(x,y,t)\big)\right),\ k \geq 0$ (10)

From the equations (9) and (10) we obtain the first few terms as:

$u_0 = f(x,y)$

$$u_1 = L_t^{-1}(L_{xx}\big(u_0(x,y,t)\big) + L_{yy}\big(u_0(x,y,t)\big))$$

$$u_2 = L_t^{-1}(L_{xx}\big(u_1(x,y,t)\big) + L_{yy}\big(u_1(x,y,t)\big))$$

$u_3 = L_t^{-1}(L_{xx}\big(u_2(x,y,t)\big) + L_{yy}\big(u_2(x,y,t)\big))$

and so on. As a result, the components u_0 ,$u_1, u_2,$ ...are identified and the series solution is thus entirely determined. However, in many cases the exact solution in a closed form may be obtained as we can see in our examples.

3. Numerical examples

3.1 Example 1

We consider the two-dimensional diffusion equation (1):

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}$$

In which $u = u(x,y,t)$.

The Dirichelet time-dependent boundary conditions on the boundary $\partial\Omega$ of the square Ω defined by the line *x=0, y=0, x=1 y=1* are given by:

$$u(0,y,t) = e^{(y+2t)}\ 0 \leq t \leq T, 0 \leq y \leq 1$$

$u(x,0,t) = e^{(x+2t)}\ 0 \leq t \leq T, 0 \leq x \leq 1$ (11)

$$u(x,1,t) = e^{(1+x+2t)}\ 0 \leq t \leq T, \quad 0 \leq x \leq 1$$

And the non local boundary condition:

$\int_0^1 \int_0^1 u(x,y,t)dxdy = (e-1)^2 e^{2t}$ (12)

With the initial conditions:

$u(x,y,0) = e^{(x+y)}$ (13)

Theoretical solution is given by :

$u(x,y,t) = e^{(x+y+2t)}$ (14)

Using the Adomian's method described above, equation (9) gives the first component:

$u_0(x,y,t) = f(x,y) = e^{(x+y)}$ (15)

And equation (10) gives the following components of the series:

$u_1 = L_t^{-1}\left(L_{xx}(u_0) + L_{yy}(u_0)\right) = 2\int_0^t e^{x+y}dt = 2te^{x+y}$ (16)

$u_2 = L_t^{-1}\left(L_{xx}(u_1) + L_{yy}(u_1)\right) = 4\int_0^t te^{x+y}dt = 2\,t^2 e^{x+y}$(17)

$u_3 = L_t^{-1}\left(L_{xx}(u_2) + L_{yy}(u_2)\right) = 4\int_0^t t^2 e^{x+y}dt = \frac{4}{3}t^3 e^{x+y}$(18)

$u_4 = L_t^{-1}\left(L_{xx}(u_3) + L_{yy}(u_3)\right) = \frac{8}{3}\int_0^t t^3 e^{x+y} dt = \frac{2}{3} t^4 e^{x+y}$(19)
$u_5 = L_t^{-1}\left(L_{xx}(u_4) + L_{yy}(u_4)\right) = \frac{4}{3}\int_0^t t^4 e^{x+y} dt = \frac{4}{3\times 5} t^5 e^{x+y}$(20)
$u_6 = L_t^{-1}\left(L_{xx}(u_5) + L_{yy}(u_5)\right) = \frac{8}{3\times 5}\int_0^t t^5 e^{x+y} dt = \frac{8}{3\times 5\times 6} t^6 e^{x+y}$(21)
$u_7 = L_t^{-1}\left(L_{xx}(u_6) + L_{yy}(u_6)\right) = \frac{16}{3\times 5\times 6}\int_0^t t^6 e^{x+y} dt = \frac{16}{3\times 5\times 6\times 7} t^7 e^{x+y}$(22)

Substituting (15)-(22) into equation (7), we obtain the solution u(x; y; t) of (1) with (11), and (12) in series form as:

$u(x,y,t) = e^{x+y}(1 + \frac{2}{1!}t + \frac{4}{2!}t^2 + \frac{4\times 2}{3!}t^3 + \frac{2\times 2\times 4}{4!}t^4 + \frac{2\times 4\times 4}{5!}t^5 + \frac{2\times 4\times 8}{6!}t^6 + \frac{2\times 4\times 16}{7!}t^7 + \cdots.)$(23)

Which can be rewritten as :

$u(x,y,t) = e^{x+y}(1 + \frac{2t}{1!} + \frac{2^2t^2}{2!} + \frac{2^3t^3}{3!} + \frac{2^4t^4}{4!} + \frac{2^5t^5}{5!} + \frac{2^6t^6}{6!} + \frac{2^7t^7}{7!} + \cdots)$ (24)

It can be easily observed that (24) is equivalent to the exact solution:

$u(x,y,t) = e^{(x+y)}e^{2t} = e^{(x+y+2t)}$ (25)

3. 2. Example 2

Consider the two-dimensional non homogeneous diffusion problem

$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} - e^{-t}(x^2 + y^2 + 4),\ t > 0, x > 0, y < 1$(26)

With Initial condition

$u(x,y,0) = 1 + x^2 + y^2$ (27)

And the boundary conditions:

$u(0,y,t) = 1 + y^2e^{-t}\ 0 \le t \le 1, 0 \le y \le 1$
$u(1,y,t) = 1 + (1 + y^2)e^{-t}\ 0 \le t \le 1, 0 \le y \le 1$(28)
$u(x,0,t) = 1 + x^2e^{-t}, 0 \le t \le 1, 0 \le x \le 1$
$u(x,1,t) = 1 + (1 + x^2)e^{-t}, 0 \le t \le 1, 0 \le x \le 1$

And the non local boundary condition:

$\int_0^1\int_0^1 u(x,y,t)dxdy = 1 + \frac{2}{3}e^{-t}, 0 \le x \le 1, 0 \le y \le 1$ (29)

The exact solution is:

$u(x,y,t) = 1 + e^{-t}(x^2 + y^2)$(30)

Writing the problem in operator form and applying the inverse operator one obtains;

$L_t^{-1}\left(L_t\big(u(x,y,t)\big)\right) = L_t^{-1}(L_{xx}u(x,y,t))) + L_t^{-1}(L_{yy}u(x,y,t))) + L_t^{-1}(-e^{-t}(x^2 + y^2 + 4)$ (31)

$L_t^{-1}\left(L_t\big(u(x,y,0)\big)\right) = u(x,y,0)$ (32)

From which we obtain:

$u(x,y,t) = u(x,y,0) + L_t^{-1}(L_{xx}u(x,y,t))) + L_t^{-1}(L_{yy}u(x,y,t))) + L_t^{-1}(-e^{-t}(x^2 + y^2 + 4)$(33)

Using Adomnian decomposition, the zeroth component is given by:

$u_0(x,y,t) = u(x,y,0) + L_t^{-1}(-e^{-t}(x^2 + y^2 + 4)$ (34)

and

$u_{k+1}(x,t) = L_t^{-1}\left(L_{xx}\big(u_k(x,y,t)\big) + L_{yy}\big(u_k(x,y,t)\big)\right),\ k \geq 0$ (35)

Applying these formula, we obtain the components of the series as :

$$u_0(x,y,t) = 1 + x^2 + y^2 - (x^2 + y^2 + 4)\int_0^t e^{-t}\,dt$$

$u_0(x,y,t) = -3 + (4 + x^2 + y^2)e^{-t}$ (36)

$$u_1(x,y,t) = L_t^{-1}\left(\big(L_{xx} + L_{yy}\big)\big(u_0(x,y,t)\big)\right) = \int_0^t 4e^{-t}dt$$

$u_1(x,y,t) = -4e^{-t} + 4$ (37)

$u_2(x,y,t) = L_t^{-1}\left(\big(L_{xx} + L_{yy}\big)\big(u_1(x,y,t)\big)\right) = \int_0^t 0dt = 0$(38)

And the remaining components are:

$u_k(x,y,t) = 0,\ \ k \geq 2$ (39)

Once the components are determined then, the series solution completely determined as follows:

$u(x,y,t) = u_0(x,y,t) + u_1(x,y,t) + \sum_{k=2}^{\infty} u_k\,(x,y,t)$(40)

Which yields:

$u(x,y,t) = 1 + (x^2 + y^2)e^{-2t}$ (41)

This solution coincides with the exact one.

3.3 Example 3

Consider the two-dimensional diffusion problem (1)

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}, \qquad 0 < x, y < 1, t > 0$$

Subject to the initial condition

$u(x,y,0) = (1-y)e^x,\ \ 0 \leq x,\ \ y \leq 1$ (42)

And the boundary conditions

$$u(0,y,t) = (1-y)e^t,\ \ 0 \leq t \leq 1, \qquad 0 \leq y \leq 1$$

$u(1,y,t) = (1-y)e^{1+t},\ \ 0 \leq t \leq 1,\ 0 \leq y \leq 1$ (43)

$$u(x,0,t) = e^{1+t},\ \ 0 \leq t \leq 1, \qquad 0 \leq x \leq 1$$

$$u(x,1,t) = 0,\ \ 0 \leq t \leq 1, \qquad 0 \leq x \leq 1$$

Rewriting the equation (42) in an operator form as the follows

$u_0(x,y,t) = u(x,y,0)$ (44)

$u_{k+1}(x,y,t) = L_t^{-1}\left(\big(L_{xx} + L_{yy}\big)\big(u_k(x,y,t)\big)\right),\ k \geq 0$ (45)

Using the Adomian's method, described above, equation (9) gives the first component of the series as:

$u_0(x,y,t) = (1-y)e^x$ (46)

And the remaining components are obtained using (10) which gives

$u_1(x,y,t) = \int_0^t (1-y)e^x \, dt = (1-y)te^x$ (45)

$u_2(x,y,t) = \int_0^t (1-y)te^x \, dt = (1-y)e^x(\frac{t^2}{2!})$ (46)

$u_3(x,y,t) = \int_0^t (1-y)(\frac{t^2}{2!})e^x \, dt = (1-y)e^x(\frac{t^3}{3!})$ (47)

…….

$u_k(x,y,t) = \int_0^t (1-y)(\frac{t^{k-1}}{(k-1)!})e^x \, dt = (1-y)e^x(\frac{t^k}{k!})$ (48)

Using these results, the solution in series form is given by;

$u(x,y,t) = (1-y)e^x(\sum_{k=0}^{\infty}(\frac{t^k}{k!})) = (1-y)e^{x+t}$ (49)

As we can verify by substitution, this solution is equivalent to the theoretical one.

6. Conclusion

In this work we applied the Adomian decomposition method for solving a two-dimensional diffusion equation, the method avoid the difficulties and massive computational work by determining the theoretical solution. The obtained solution is very rapidly convergent and provides a reliable technique that requires less effort and yields highly accurate results in comparison with the traditional techniques. The Adomian methodology is very powerful and efficient tool in finding exact solutions for a wide classe of problems. Through the illustrated problems, the obtained results are more accurate than those obtained by M. Siddique.

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