Lecture 11. Kőnig's theorem and Tutte's theorem

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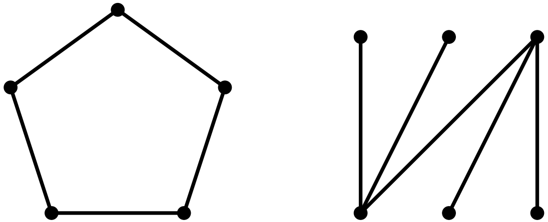
Theorem

If $G = (A \cup B, E)$ is a bipartite graph, then the maximum size of a matching in G equals the minimum size of a vertex cover of G.



Kőnig's theorem

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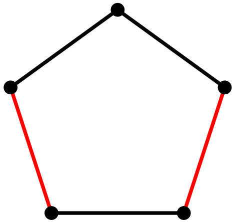

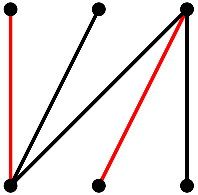

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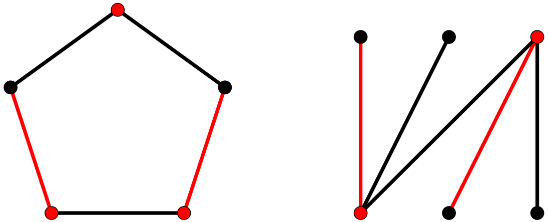

Theorem

If $G = (A \cup B, E)$ is a bipartite graph, then the maximum size of a matching in G equals the minimum size of a vertex cover of G.

Proof. Step 1. $\tau(G) \geq \nu(G)$.

We have already seen that a minimum cover has at least the size of a maximum matching, i.e. $\tau(G) \geq \nu(G)$. Now take a minimum vertex cover U of G. We construct a matching of size $|U|$ to prove that equality can always be achieved, i.e. $\tau(G) = \nu(G)$.

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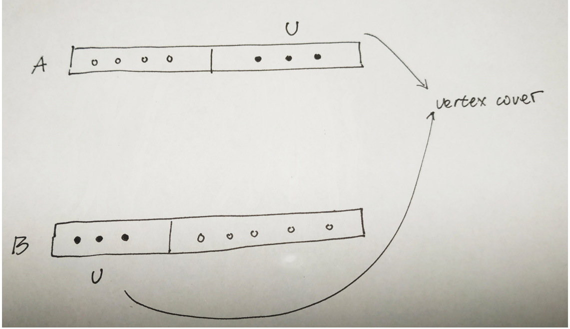

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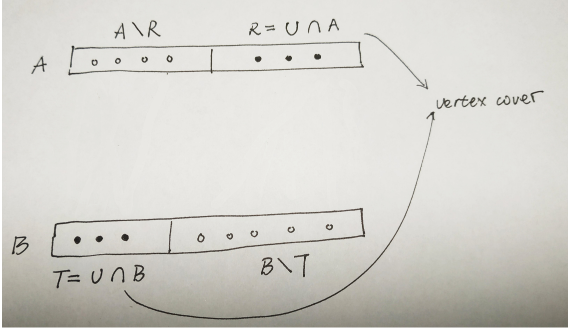

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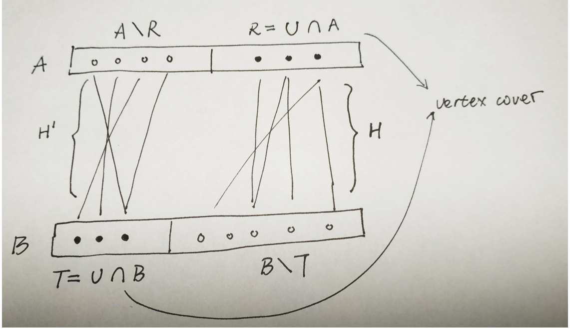


Theorem

If $G = (A \cup B, E)$ is a bipartite graph, then the maximum size of a matching in G equals the minimum size of a vertex cover of G.

Proof. Step 2. Consider two subgraphs.

Let $R = U \cap A$ and $T = U \cap B$. Let H, H' be the subgraphs of G induced by $R \cup (B \setminus T)$ and $T \cup (A \setminus R)$. We use Hall's theorem to show that H has a matching covering R, and H' has a matching covering T. Since these subgraphs are disjoint, the two matchings together form a matching of size $|U|$ in G.

Theorem

If $G = (A \cup B, E)$ is a bipartite graph, then the maximum size of a matching in G equals the minimum size of a vertex cover of G.

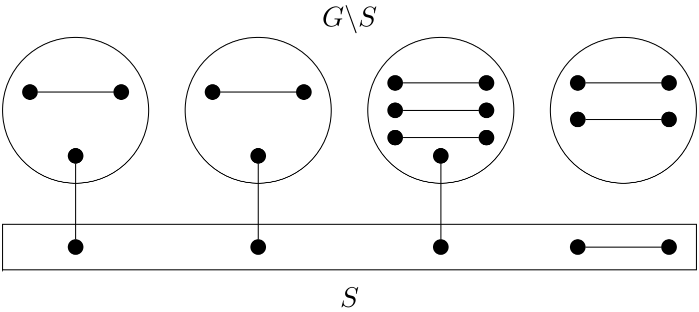
Proof. Step 3. Each subgraph satisfies Hall's condition.

Since $R \cup T$ is a vertex cover, G has no edge from $B \setminus T$ to $A \setminus R$. Suppose $S \subseteq R$, and consider $N_H(S) \subseteq B \setminus T$. If $|N_H(S)| < |S|$, then we can substitute $N_H(S)$ for S in U and obtain a smaller vertex cover, since $N_H(S)$ covers all edges incident to S that are not covered by T. The minimality of U thus implies that Hall's condition holds in H, and hence H has a matching covering R. Applying the same argument to H' yields the rest of the matching. $\square$

Tutte's theorem

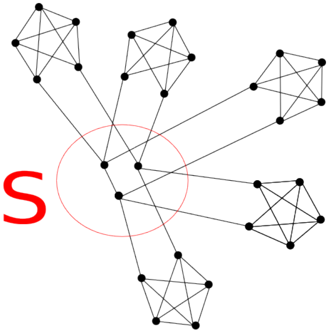
Definition

Given a graph G, let $q(G)$ denote the number of its odd components, i.e. the ones of odd order.



Theorem (Tutte, 1947)

A graph G has a perfect matching if and only if $q(G \setminus S) \leq |S|$ for all $S \subseteq V(G)$.



Theorem (Tutte, 1947)

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Proof. Step 1. Necessity.

If G has a perfect matching then clearly

$$q(G \setminus S) \leq |S| \text{ for all } S \subseteq V(G),$$

since every odd component of $G \setminus S$ will send an edge of the matching to S, and each such edge covers a different vertex in S.

Tutte's theorem

Theorem (Tutte, 1947)

A graph G has a perfect matching if and only if $q(G \setminus S) \leq |S|$ for all $S \subseteq V(G)$.

Proof. Step 2. Consider a maximal G with no perfect matching.

Tutte's condition is preserved by addition of edges: if $G' = G \cup \{e\}$ and $S \subseteq V(G)$, then $q(G' \setminus S) \leq q(G \setminus S)$, because when the addition of e combines two components of $G \setminus S$ into one, the number of components that have odd order does not increase.

Suppose to the contrary that G contains no perfect matching. Then there exists a simple graph G' which is obtained from G by adding edges, such that G' satisfies the condition, has no perfect matching, but adding any edge to G' creates a perfect matching. We will obtain a contradiction in every case by constructing a perfect matching in G'.

Theorem (Tutte, 1947)

A graph G has a perfect matching if and only if $q(G \setminus S) \leq |S|$ for all $S \subseteq V(G)$.

Proof. Step 3. $|G|$ is even.

Consider a simple graph G such that G satisfies the condition, has no perfect matching, but adding any edge to G creates a perfect matching.

By considering $S = \emptyset$, we know that G has an even number of vertices, since a graph of odd order must have a component of odd order.

Theorem (Tutte, 1947)

A graph G has a perfect matching if and only if $q(G \setminus S) \le |S|$ for all $S \subseteq V(G)$.

Proof. Step 4. Case 1.

Let U be the set of vertices in G that are connected to all other vertices. Suppose $G \setminus U$ consists of disjoint complete graphs; we build a perfect matching for such a G. The vertices in each component of $G \setminus U$ can be paired up arbitrarily, with one left over in the odd components. Since $q(G \setminus U) \le |U|$ and each vertex of U is adjacent to all of $G \setminus U$, we can match these leftover vertices arbitrarily to vertices in U to complete a matching.

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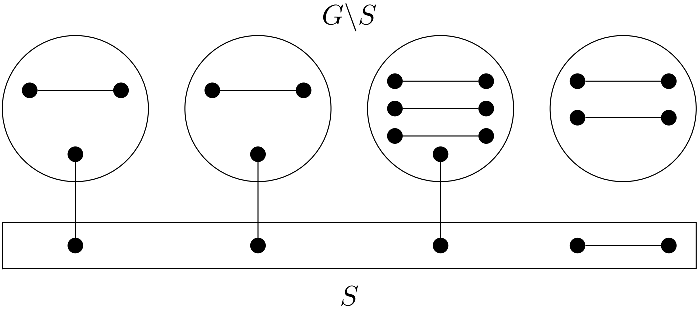


Theorem (Tutte, 1947)

A graph G has a perfect matching if and only if $q(G \setminus S) \le |S|$ for all $S \subseteq V(G)$.

Proof. Step 5. Case 2.

This leaves the case where $G \setminus U$ is not a disjoint union of cliques. We can therefore find two vertices in the same component which are not adjacent, and on a shortest path between them there are two nonadjacent vertices x, z at distance 2 (which have a common neighbour y). Furthermore, $G \setminus U$ has another vertex w not adjacent to y, since $y \notin U$.

Theorem (Tutte, 1947)

A graph G has a perfect matching if and only if $q(G \setminus S) \le |S|$ for all $S \subseteq V(G)$.

Proof.

By the maximality of G, adding any edge to G produces a perfect matching. Let M_1 and M_2 be perfect matchings in $G \cup (x,z)$ and $G \cup (y,w)$, respectively. It suffices to show that in $M_1 \cup M_2$ we can find a perfect matching avoiding (x,z) and (y,w), because that would be contained in G.

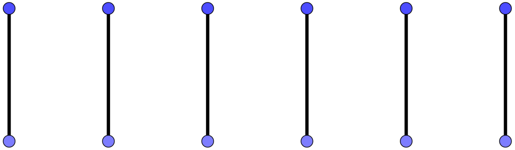
Tutte's theorem

Theorem (Tutte, 1947)

A graph G has a perfect matching if and only if $q(G \setminus S) \leq |S|$ for all $S \subseteq V(G)$.

Proof.

Let F be the graph on $V(G)$ with the edges that belong to exactly one of M_1 and M_2. Note that F contains (x,z) and (y,w) . Since every vertex of G has degree 1 in each of M_1 and M_2, every vertex of G has degree 0 or 2 in F. Hence F is a collection of disjoint even cycles (alternating between edges of M_1 and M_2) and isolated vertices.



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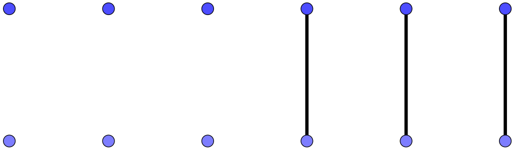


Tutte's theorem

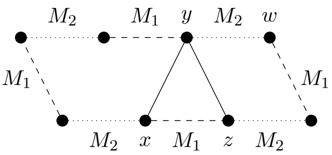
Theorem (Tutte, 1947)

A graph G has a perfect matching if and only if $q(G \setminus S) \leq |S|$ for all $S \subseteq V(G)$.

Proof.

Let C be the cycle of F containing (x,z) . If C does not also contain (y,w) , then the desired matching consists of the edges of M_2 from C and all of M_1 not in C.

If C contains both (y,w) and (x,z) , as illustrated below, then we use the edge (y,x) or the edge (y,z) to obtain a matching of $V(C)$ using only edges of G (avoiding both (x,z) and (y,w)).



Proof.

Specifically, we use (y,x) if the distance between y and x in C is odd, and we use (y,z) otherwise (then the distance between y and z in C is odd). In the above illustration, this second case applies. The remaining vertices of C form two paths of even order. We use the edges of M_1 in one of these paths and the edges of M_2 in the other to produce a matching in C that does not use (x,z) or (y,w). (In the above illustration, we use the edges of M_1 on the right side of (y,z) and the edges of M_2 on the left). Combined with M_1 or M_2 outside C, we have a perfect matching of G. $\square$

Petersen's corollary

Corollary (Petersen, 1891)

Every 3-regular graph with no cut edge has a perfect matching.

Proof.

Let $S \subseteq V(G)$. Let H be an odd component of $G \setminus S$. The number of edges between S and H cannot be 1, since G has no cut edge. It also cannot be even, since otherwise the sum of the vertex degrees in H would be odd. Hence there are at least three edges from H to S.

Corollary (Petersen, 1891)

Every 3-regular graph with no cut edge has a perfect matching.

Proof.

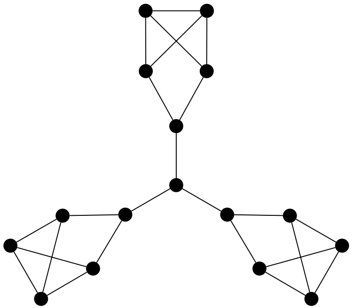
Since G is 3-regular, each vertex of S is incident to at most three edges between S and $G \setminus S$. Combining this fact with the previous paragraph, we have

$$3q(G \setminus S) \leq |\{e \mid e \text{ is between } S \text{ and } G \setminus S\}| \leq 3|S|$$

and hence $q(G \setminus S) \leq |S|$, which proves the corollary. □

Example

The condition that the graph has no cut edge is necessary. The graph below is 3-regular but has no perfect matching. Deleting the central vertex leaves 3 odd components.



Berge's corollary

Theorem (Berge 1958)

The largest matching in an n-vertex graph G covers $n + \min_{S \subseteq V(G)} (|S| - q(G \setminus S))$ vertices.

This result is a defect version of Tutte's theorem. The proof technique is to construct a new graph G' which satisfies Tutte's condition. Thus, G' contains a perfect matching. From this we can find a largest matching with the desired size in G.

Berge's corollary

Theorem (Berge 1958)

The largest matching in an n-vertex graph G covers

$n + \min_{S \subseteq V(G)} (|S| - q(G \setminus S))$ *vertices.*

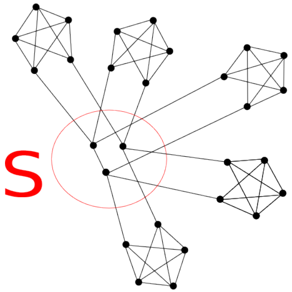
Proof. Step 1. New notation.

Note that when $S = \emptyset$, $|S| - q(G \setminus S) = 0 - q(G \setminus S) \le 0$.

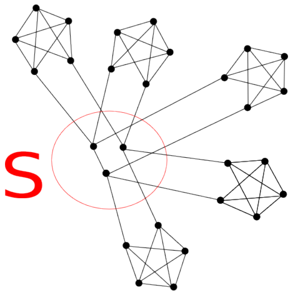
Thus we have $\min_{S \subseteq V(G)} (|S| - q(G \setminus S)) \le 0$.

Let $d(S) = q(G \setminus S) - |S|$ and $d = \max_{S \subseteq V} d(S)$. Thus, $-d = \min_{S \subseteq V}(-d(S))$.

We need to prove that the largest matching in an n-vertex graph G covers $n - d$ vertices.



In this graph, if we delete this S, the number of odd components is 5, i.e. $q(G \setminus S) = 5$. Thus, $|S| - q(G \setminus S) = -2$. We can check that $\min\limits_{S \subseteq V(G)} (|S| - q(G \setminus S)) = -2$. The largest matching in this graph covers at most $n - 2$ vertices.



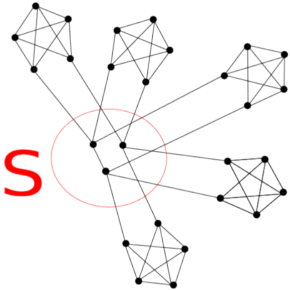
Proof. Step 2. Upper bound.

Given $S \subseteq V(G)$, at most $|S|$ edges can match vertices of S to vertices in odd components of $G \setminus S$, so every matching has at least $q(G \setminus S) - |S|$ unmatched vertices.

Proof. Step 3. Construct a new graph.

We have shown that no matching can have more than $n-d$ vertices; we want to achieve this bound. Considering the case $S=\emptyset$ shows $d \geq 0$.

Let G' be obtained by adding a set D of d vertices to G, each of which are adjacent to every other vertex.



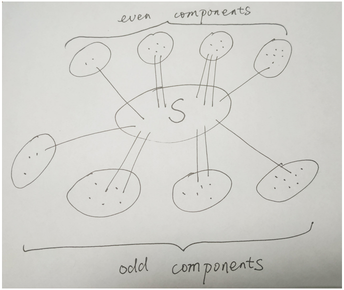
Proof. Step 4. Prove $q(G' \setminus S') \le |S'|$ for $S' = \emptyset$.

For each S, $d(S) = q(G \setminus S) - |S|$ has the same parity as $|G|$.

Thus, $d = \max_{S \subseteq V} d(S)$ has the same parity as $|G|$.

Therefore, $|G'| = |G| + d$ is even.

The condition $q(G' \setminus S') \le |S'|$ holds for $S' = \emptyset$ because $|G'|$ is even.



Proof. Step 5. Prove $q(G' \setminus S') \le |S'|$ for $S' \neq \emptyset$.

If S' is nonempty but does not contain all of D, then $G' \setminus S'$ has only one component, and $1 \le |S'|$. Finally, if $D \subseteq S'$, let $S = S' \setminus D$. We have $G' \setminus S' = G \setminus S$, so $q(G' \setminus S') = q(G \setminus S) \le |S| + d = |S'|$, and G' indeed satisfies Tutte's condition.

Because G' satisfies Tutte's condition, then we can obtain a matching of the desired size in G from a perfect matching in G', because deleting D eliminates edges that match at most d vertices of G. $\square$

Thank you!