Lecture 7. Edge connectivity

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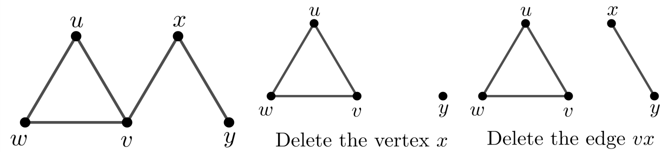
Delete a vertex is to delete this vertex and all the edges incident to this vertex.

Delete an edge is to delete only this edge.



Vertex cut $\longrightarrow$ Disconnecting set

k-connected $\longrightarrow$ k-edge-connected

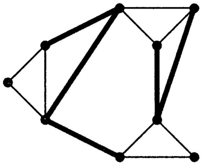
connectivity $\kappa(G)$ $\longrightarrow$ edge-connectivity $\kappa'(G)$

Cut vertex $\longrightarrow$ Bridge(Cut edge)

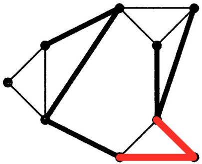
Edge connectivity

Definition

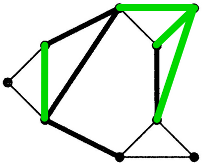
A disconnecting set of edges is a set $F \subseteq E(G)$ such that $G \setminus F$ has more than one component. A graph is k-edge-connected if every disconnecting set has at least k edges. The edge-connectivity of G, written $\kappa'(G)$, is the minimum size of a disconnecting set. One edge disconnecting G is called a bridge (cut edge).



disconnecting set



disconnecting set

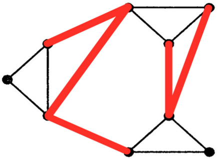


disconnecting set

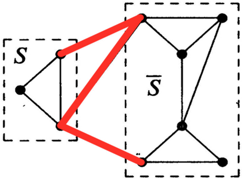
Edge cut

Definition

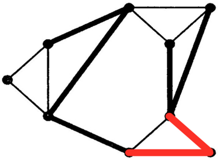
Given $S, T \subseteq V(G)$, the notation $[S, T]$ specifies the set of edges having one endpoint in S and the other in T. An edge cut is an edge set of the form $[S, \overline{S}]$, where S is a non-empty proper subset of $V(G)$.



disconnecting set but not an edge cut



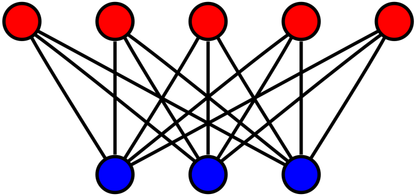
edge cut



edge cut

Example

$G = K_n$: $\kappa'(G) = n-1$. $\kappa'(G) = \kappa(G) = \delta(G) = n-1$.

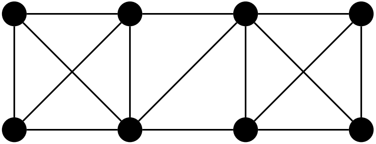


$G = K_{m,n}$, $m \le n$: $\kappa'(G) = \kappa(G) = \delta(G) = m$.

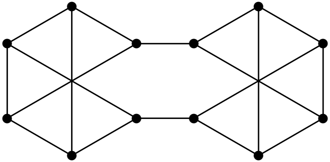
Remark. If G is $k+1$-edge connected, then G is k-edge connected.

An edge cut is a disconnecting set, but a disconnecting set may not be an edge cut.

Example



$\kappa'(G) = 3$, $\kappa(G) = 2$, $\delta(G) = 3$.



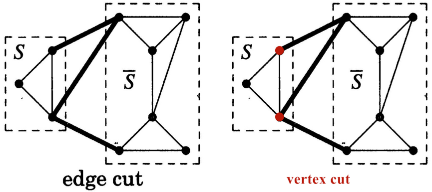
$\kappa'(G) = 2$, $\kappa(G) = 2$, $\delta(G) = 3$.

Theorem

$\kappa(G) \leq \kappa'(G) \leq \delta(G)$.

Proof.

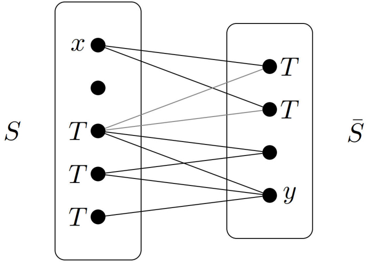
The edges incident to a vertex v of minimum degree form a disconnecting set, hence $\kappa'(G) \leq \delta(G)$. It remains to show $\kappa(G) \leq \kappa'(G)$. Suppose $|G| > 1$ and $[S, \overline{S}]$ is a minimum edge cut, having size $\kappa'(G)$. Every edge of $[S, \overline{S}]$ has an end in S. Do all the ends of the edge cut $[S, \overline{S}]$ form a vertex cut?

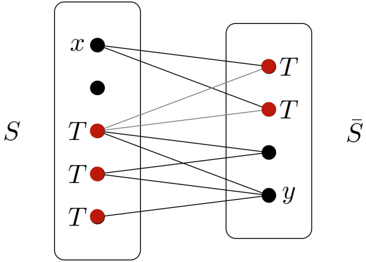


Sketch of the proof.

Case 1. Every vertex of S is adjacent to every vertex of $\overline{S}$.

Case 2. There exists $x \in S, y \in \overline{S}$ with x not adjacent to y.





Proof.

If every vertex of S is adjacent to every vertex of $\overline{S}$, then $\kappa'(G) = |S||\overline{S}| = |S|(|G| - |S|)$. This expression is minimized at $|S| = 1$. By definition, $\kappa(G) \leq |G| - 1 \leq \kappa'(G)$, so the inequality holds.

Hence we may assume there exists $x \in S, y \in \overline{S}$ with x not adjacent to y. Let T be the vertex set consisting of all neighbours of x in $\overline{S}$ and all vertices of $S \setminus x$ that have neighbours in $\overline{S}$. Deleting T destroys all the edges in the cut $[S, \overline{S}]$ (but does not delete x or y), so T is a separating set. Now, by the definition of T we can injectively associate at least one edge of $[S, \overline{S}]$ to each vertex in T, so $\kappa(G) \leq |T| \leq |[S, \overline{S}]| = \kappa'(G)$. □

Definition

Two paths are internally disjoint if neither contains a non-endpoint vertex of the other. That is, they have no common internal vertex. The length of the shortest path from u to v is called the distance from u to v, denoted by $d(u,v)$.

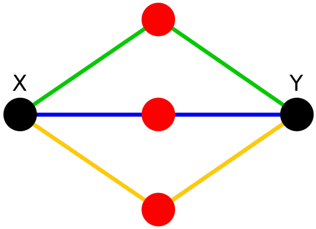


Figure: Three internally disjoint xy-paths

Definition

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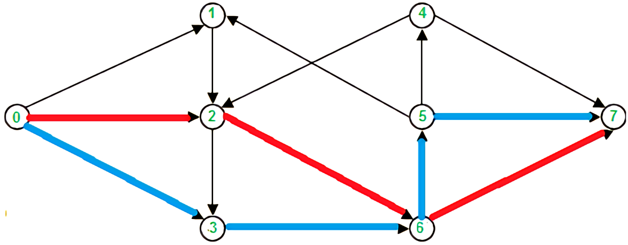


Figure: Two paths which are NOT internally disjoint

Whitney's theorem

Theorem (Whitney's theorem)

A graph G having at least three vertices is 2-connected if and only if each pair $u,v \in V(G)$ is connected by a pair of internally disjoint u,v-paths in G.

Proof.

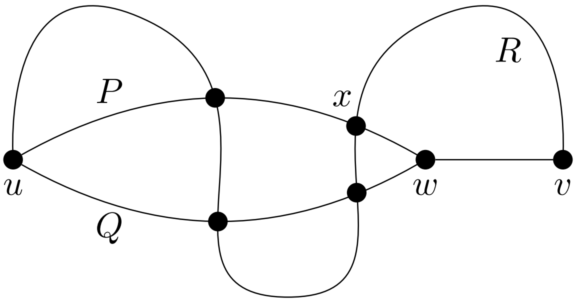
When G has internally disjoint u,v-paths, deletion of one vertex cannot separate u from v. Since this is given for every u,v, the condition is sufficient(G is 2-connected).

For the converse, suppose that G is 2-connected. We prove by induction on $d(u,v)$ that G has two internally disjoint u,v-paths. When $d(u,v) = 1$, the graph $G \setminus (u,v)$ is connected, since $\kappa'(G) \geq \kappa(G) \geq 2$. A u,v-path in $G \setminus (u,v)$ is internally disjoint in G from the u,v-path consisting of the edge (u,v) itself.

Proof.

For the induction step, we consider $d(u,v) = k > 1$ and assume that G has internally disjoint x,y-paths whenever $1 \leq d(x,y) < k$. Let w be the vertex before v on a shortest u,v-path. We have $d(u,w) = k-1$, and hence by the induction hypothesis G has internally disjoint u,w-paths P and Q. Since $G \setminus w$ is connected, $G \setminus w$ contains a u,v-path R. If this path avoids P or Q, we are finished. But R may share internal vertices with both P and Q. Let x be the last vertex of R belonging to $P \cup Q$. Without loss of generality, we may assume $x \in P$. We combine the u,x-subpath of P with the x,v-subpath of R to obtain a u,v-path internally disjoint from $Q \cup \{(w,v)\}$. $\square$

Whitney's theorem



Corollary

G is 2-connected and $|G| \geq 3$ if and only if every two vertices in G lie on a common cycle.

Thank you!